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Abstract. Most energy minimization-based restoration methods are developed for signal-independent Gaussian noise. The assumption of Gaussian noise distribution leads to a quadratic data fidelity term, which is appealing in optimization. When an image is acquired with a photon counting device, it contains signal-dependent Poisson or mixed Poisson–Gaussian noise. We quantify the loss in performance that occurs when a restoration method suited for Gaussian noise is utilized for mixed noise. Signal-dependent noise can be treated by methods based on either classical maximum *a posteriori* (MAP) probability approach or on a variance stabilization approach (VST). We compare performances of these approaches on a large image material and observe that VST-based methods outperform those based on MAP in both quality of restoration and in computational efficiency. We quantify improvement achieved by utilizing Huber regularization instead of classical total variation regularization. The conclusion from our study is a recommendation to utilize a VST-based approach combined with regularization by Huber potential for restoration of images degraded by blur and signal-dependent noise. This combination provides a robust and flexible method with good performance and high speed. © 2016 SPIE and IS&T [DOI: 10.1117/1.JEI.25.4.043020]

Keywords: image restoration; Poisson noise; mixed Poisson–Gaussian noise; variance stabilizing transform; total variation; Huber potential function.

Paper 16231 received Mar. 23, 2016; accepted for publication Jul. 12, 2016; published online Aug. 4, 2016.

1 Introduction

During acquisition, images are degraded by blur and different types of noise. Most imaging is performed with photon counting devices, e.g., CCD or CMOS sensors, which introduce signal-dependent Poisson noise in combination with signal-independent noise from the electronics. Image restoration aims to reverse the effects of imperfect imaging and to recover noise-free and blur-free images. Image restoration is commonly performed by regularized energy minimization due to its simplicity and generally good performance. The objective energy function consists of a data fidelity term, which models the image formation process and a regularization term, which imposes *a priori* knowledge on the solution.

If a Gaussian noise model is assumed, the data fidelity term is quadratic and easy to optimize. This makes such an assumption appealing and popular. However, in the presence of signal-dependent noise, this model is inappropriate and leads to significantly reduced performance. In this paper, we confirm the importance of a proper choice of noise model, also hinted at in Ref. 1, by empirical tests on a large image material. We show that a correct treatment of both signal-dependent and signal-independent components of the noise is very important, even when one of them constitutes as little as one percent of the overall noise.

The data fidelity term of the energy function should either be adjusted to the correct noise model, or alternatively, the image should be transformed to make the Gaussian model applicable. This latter approach is known as variance stabilization. For pure Poisson noise, the data fidelity term, which

maximizes *a posteriori* probability (MAP), is Kullback–Leibler divergence. Following the alternative path, a variance stabilizing transform (VST), such as Anscombe's,² is incorporated in the data term. A limited study, presented in Ref. 3, indicates that the VST approach does not fall behind the direct one. Our presented study here shows that the VST approach actually outperforms the direct approach. This is supported by a thorough evaluation on more than 3500 noisy and blurred images.

For the more general case, when considering a mixture of both Poisson and Gaussian noise components, the MAP approach leads to practical difficulties since the log-likelihood function involves infinite summation. This imposes a need for approximate solutions and leads to complicated and slow algorithms. An often used alternative is to ignore one of the noise components; this comes at a cost of reduced performance, as confirmed by our tests. The generalized VST approach is, on the other hand, straightforward; we demonstrate in this paper that it provides fast and efficient restoration of images degraded by mixed noise.

The regularization part of the energy term is, by definition, image dependent. Several options are proposed, including various sparsity promoting approaches. A popular regularizer is total variation (TV),^{4,5} imposing sparsity in the gradient domain. Additional improvement can be achieved by the use of a potential function, which modulates the regularization component. Potential functions are designed to enhance/preserve particular image features; preservation of sharp edges is typically targeted. Our previous study⁶

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shows that the Huber potential improves performance and outperforms other potential functions for the case of signal-independent Gaussian noise. Here, we evaluate, through a thorough empirical study, how large an improvement is achieved by using the Huber potential for signal-dependent noise.

The main message of our paper is signal-dependent and signal-independent components of image noise have to be treated appropriately in image restoration (Sec. 3.1). A VST-based approach, where the noise is transformed to be approximately signal independent, can successfully be used to handle Poisson and mixed noise. Our thorough empirical study (Sec. 3.2) demonstrates that this approach outperforms direct modeling of the signal-dependent noise.

2 Image Restoration by Energy Minimization

Energy minimization is commonly used to address inverse problems in image processing, such as image denoising, restoration, and inpainting. Typically, the energy function of regularized restoration is of the form

$$E(u) = D(u) + \lambda R(u), \quad (1)$$

where u is an (unknown) image. The energy function consists of a “data fidelity term,” $D(u)$, which drives the solution toward the observed data, and a “regularization term,” $R(u)$, which provides noise suppression. The “regularization parameter” λ controls the trade-off between the two terms, i. e., the level of smoothing versus faithful recovery of the (possibly noisy) image detail.

We consider that the unknown image u of size $r \times c$ is represented as a vector $u = [u_1, \dots, u_n]^T$ of length $n = r \times c$, where image rows are sequentially concatenated. With x and y , we denote original and degraded (observed) images, respectively. The convolution with a point spread function (PSF), which models degradation of the image with blur, is equal to multiplication of matrix $H_{n \times n}$ and image $x_{n \times 1}$. Blurred original image, Hx , is additionally corrupted with noise. In this way, degraded image y is formed. An argument which minimizes the energy function,

$$\hat{x} = \arg \min_u E(u), \quad (2)$$

is considered to be an estimate of the original image x .

2.1 Data Fidelity Term

When an image x is degraded by Poisson noise, the observed image y is described as

$$y_i \sim \mathcal{P}[(Hx)_i], \quad \forall i = 1, 2, \dots, n. \quad (3)$$

The corresponding data fidelity term, derived based on MAP approach, is Kullback–Leibler divergence:

$$D_{\text{MAP}}(u) = \sum_{i=1}^n \{(Hu)_i - y_i \cdot \log[(Hu)_i]\}. \quad (4)$$

A review of restoration methods for images corrupted with Poisson noise, based on MAP approach, is given in Ref. 7; two more recently developed methods are presented in Refs. 3 and 8.

A more general approach is to also take into account the signal-independent noise sources, modeling the noise with a mixed Poisson–Gaussian distribution. If image x is degraded by blur and signal-dependent mixed Poisson–Gaussian noise, the acquired image is such that

$$y_i = \theta_i + \eta_i, \quad \forall i = 1, 2, \dots, n, \quad (5)$$

where $\theta_i \sim \mathcal{P}[(Hx)_i]$ and $\eta_i \sim \mathcal{N}(0, \sigma_m^2)$. The corresponding data fidelity term, based on MAP approach, is

$$D_{\text{MAPPG}}(u) = - \sum_{i=1}^n \log \left(\sum_{k=0}^{+\infty} \frac{[(Hu)_i]^k e^{-(Hu)_i}}{k!} \cdot \frac{e^{-\left(\frac{y_i-k}{\sqrt{2}\sigma_m}\right)^2}}{\sqrt{2\pi}\sigma_m} \right). \quad (6)$$

Since this data fidelity term includes an infinite sum, some approximation is required. One way to address this problem is to approximate the infinite sum with a finite number of summands, such as in the methods presented in Refs. 9–12. This usually leads to slow and complicated algorithms.

Another way to overcome practical difficulties related to D_{MAPPG} is to use a VST-based approach. For signal-dependent noise, VST can be used to remove signal dependency and render the noise approximately Gaussian. For such a transformed image, the corresponding data fidelity term is quadratic, which is appealing for minimization. When an image x is degraded by mixed Poisson–Gaussian noise, generalized Anscombe VST^{13,14} is used to transform the observed image y into z :

$$z_i = 2\sqrt{\max\left\{y_i + \frac{3}{8} + \sigma_m^2, 0\right\}}, \quad \forall i = 1, 2, \dots, n, \quad (7)$$

where

$$z_i \approx 2\sqrt{\max\left\{(Hx)_i + \frac{3}{8} + \sigma_m^2, 0\right\}} + \epsilon, \quad \epsilon \sim \mathcal{N}(0, 1), \quad \forall i = 1, 2, \dots, n. \quad (8)$$

In this case, the data fidelity term is of the form:

$$D_{\text{VST}}(u) = \sum_{i=1}^n \left(z_i - 2\sqrt{\max\left\{(Hu)_i + \frac{3}{8} + \sigma_m^2, 0\right\}} \right)^2. \quad (9)$$

When $\sigma_m = 0$, the data fidelity term [Eq. (9)] treats pure Poisson noise, such as in the method presented in Ref. 1.

We introduce the following abbreviations: VSTP denotes a combination of the VST approach and pure Poisson noise, VSTPG stands for a combination of the VST approach and Poisson–Gaussian noise, while MAPP and MAPPG denote combinations of the MAP approach with Poisson and Poisson–Gaussian noise, respectively.

2.2 Regularization

The role of the regularization term is to provide numerical stability and impose desired properties on the solution. Here,

we focus on the family of regularization methods based on TV,⁴ i.e., we observe

$$R(u) = \sum_{i=1}^n \Phi(|\nabla(u_i)|), \quad (10)$$

where ∇ stands for image gradient and $|\cdot|$ denotes ℓ_2 norm. The function Φ is referred to as “potential function.” Classical TV regularization is obtained when Φ is the identity function, $\Phi_{TV}(t) = t$.

In most cases, the potential function is designed so that small intensity changes (assumed to be noise) are penalized while large changes (assumed to be edges) are preserved. A number of potentials are studied and used in image restoration (see Refs. 6 and 15 and references therein). In Ref. 16, theoretical conditions for edge preserving potentials are given.

We have, in Ref. 6, evaluated the effectiveness of seven different potential functions in restoration of images degraded by Gaussian noise. This analysis showed that the largest improvement in image quality is achieved when the Huber¹⁷ potential function

$$\Phi_{\text{Huber}}(t) = \begin{cases} \frac{t^2}{2\omega}, & t \leq \omega \\ t - \frac{\omega}{2}, & t > \omega \end{cases} \quad (11)$$

is utilized. This is a convex function. When the parameter ω tends to zero, this function tends to classical TV. The regularization term [Eq. (10)] is differentiable at points where $|\nabla(u_i)| \neq 0$ for both Huber [Eq. (11)] and TV potential functions. However, at points where $|\nabla(u_i)| = 0$, Eq. (10) is differentiable only for Eq. (11).

2.3 Optimization

We consider grayscale images and represent them as vectors with intensity values from $[0,1]$ (for images in the range $[0, \text{peak}]$ intensity is divided by peak). All considered objective functions are of the form of Eq. (1), where data fidelity terms are given by Eqs. (4) or (9), and a regularization term is given by Eq. (10). Minimization of each objective function is seen as a constrained optimization problem:

$$\min_u E(u) \text{ s.t. } 0 \leq u_i \leq 1, \quad i = 1, 2, \dots, n, \quad (12)$$

which we solve utilizing spectral projected gradient (SPG) optimization.¹⁸ SPG is an efficient method for solving a constrained optimization problem $\min_{x \in \Omega} f(x)$, where Ω is a closed convex set in $\mathbb{R}^n$ and the objective function f has continuous partial derivatives on an open set containing Ω .

A wide variety of approaches and different algorithms for minimization of regularized energy functions are presented in the literature; a number of references on the topic are given in Ref. 19. We utilize SPG for the optimization of energy functions since we have previously experienced excellent performance of SPG on restoration of images degraded by Gaussian noise.⁶ This flexible method allows a variety of data fidelity terms and potentials to be used in the energy function. We appreciate the simplicity, flexibility, and robustness of SPG.

Pseudocode of the method is given in Algorithm 1. Equations for computing the gradients for all objective

Algorithm 1 Spectral Projected Gradient.

Choose values for parameters: $\theta_{\min}$, $\theta_{\max}$, γ , σ_1 , σ_2 , tol s.t.

$0 < \theta_{\min} < \theta_{\max}$, $\gamma \in (0,1)$, $0 < \sigma_1 < \sigma_2 < 1$, $\text{tol} > 0$.

Choose initial guess $x_0 \in \Omega$ and $\theta_0 = 1$. Compute x_{k+1} and θ_{k+1} as follows:

$d_k = P_{\Omega}[x_k - \theta_k \nabla f(x_k)] - x_k$; $x_{k+1} = x_k + d_k$

$\delta = \nabla f(x_k)^T d_k$; $\lambda_k = 1$

while $f(x_{k+1}) > f(x_k) + \gamma \lambda_k \delta$

$\lambda_{\text{temp}} = -\frac{1}{2} \lambda_k^2 \delta / [f(x_{k+1}) - f(x_k) - \lambda_k \delta]$

if $(\lambda_{\text{temp}} \geq \sigma_1 \wedge \lambda_{\text{temp}} \leq \sigma_2 \lambda_k)$

then $\lambda_k = \lambda_{\text{temp}}$

else $\lambda_k = \lambda_k / 2$; $x_{k+1} = x_k + \lambda_k d_k$

end while

$s_k = x_{k+1} - x_k$; $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$; $\beta_k = s_k^T y_k$

if $\beta_k \leq 0$

then $\theta_{k+1} = \theta_{\max}$

else $\theta_{k+1} = \min \left\{ \theta_{\max}, \max \left\{ \theta_{\min}, \frac{s_k^T s_k}{\beta_k} \right\} \right\}$

Repeat until: $\|x_{k+1} - x_k\|_{\infty} \leq \text{tol}$.

functions, required in SPG optimization, are given in the Appendix A.

3 Evaluation

For evaluation, we use a dataset consisting of 30 nonblurred and noise-free test images (Fig. 1). We blur each test image x with 11 Gaussian PSFs H and obtain blurred observations Hx , which we further degrade with 11 levels of Poisson noise [Eq. (3)] and with 11×3 levels of mixed Poisson–Gaussian noise [Eq. (5)]. We obtain 30 degraded images y for each PSF and noise level (in total 3630 degraded images for Poisson noise and 10890 for mixed noise). Details about considered degradation levels can be found in Appendix B.

We measure the quality of restoration in terms of PSNR = $10 \log_{10} \left(\frac{[\max(x_i)]^2}{\text{MSE}} \right)$, where $\text{MSE} = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{x}_i)^2$. Original and restored images are denoted by x and $\hat{x}$, respectively, and restored image $\hat{x}$ is given by Eq. (2). In general, for a given method, we quantify the improvement in PSNR between noisy input and restored output images: $\Delta \text{PSNR}_{\text{method}} = \text{PSNR}_{\text{out}} - \text{PSNR}_{\text{in}}$. Positive ΔPSNR indicates that the resulting restored image has a higher PSNR, i.e., is of a higher quality, than the starting degraded one. We compare different approaches by comparing the improvements in PSNR achieved by them, observing the difference in improvements reached; positive value of $\Delta \text{PSNR}_{\text{method1}} - \Delta \text{PSNR}_{\text{method2}}$ indicates that “method1” outperforms “method2.” In some

Fig. 1 Used test images: 10 “classic” test images, 10 astronomical, and 10 texture images. All the images are 256×256 pixels.

experiments, we also consider signal-to-noise ratio (SNR), $\text{SNR} = 10 \log_{10} \left(\frac{\frac{1}{n} \sum_{i=1}^n x_i^2}{\text{MSE}} \right)$, as well as structural similarity index measure (SSIM)²⁰ as image quality measures.

To present and compare the performances of restoration methods in a fair way, it is important to select optimal regularization parameters [λ for methods using TV potential and (λ, ω) for methods with Huber potential]. Numerous parameter selection schemes are proposed in literature, e.g., SURE-based approaches^{21,22} suited for restoration in presence of Gaussian noise, L-curve,²³ generalized cross validation, discrepancy principle,²⁴ residual-based methods,^{25,26} method based on no-reference measure of image content,²⁷ which can be used for parameter estimation in presence of blur and not-necessarily Gaussian noise. Having ground truth available, we select optimal parameters empirically, and by that avoid the risk of bias from relying on possibly imperfect estimates from methods for parameter selection. We select the best performing parameters for each image and each observed degradation as an argument that maximizes its ΔPSNR . For this parameter optimization, we utilize Nelder–Mead simplex search.²⁸ To verify the generality of our conclusions, as a secondary measure of image quality, we compute the SSIM index of the images restored using parameters that maximize ΔPSNR .

3.1 Importance of Utilizing the Correct Noise Model

To quantify the importance of appropriate treatment of signal-dependent noise, we restore images degraded by mixed Poisson–Gaussian noise using restoration methods suited for (1) mixed noise (VSTPG) and (2) Gaussian noise (MAPG).⁶

Comparison of performances of VSTPG and MAPG (with TV potential) on the first five images in Fig. 1, degraded by 4×6 different blur and mixed noise levels, is presented in Fig. 2. It is clear that the use of an appropriate noise model is very important. We observe a consistent additional improvement in PSNR when assuming the correct noise model (indicated by positive difference in improvements achieved by VSTPG and MAPG in the plot), which goes up to 6 dB and reaches on average 2.42 dB.

Restoration methods require estimation of the noise present in the image. A number of methods for estimation of noise parameters exist in the literature, e.g., Refs. 29–31. We evaluate the sensitivity of VSTPG with respect to the

parameter σ_m and conclude that an inaccurate estimate of this value affects the restoration result far less than selection of the wrong noise model. On the same data as above, using a value of σ_m that is three times larger than the correct one leads to a reduction in performance by only 0.18 dB.

3.2 Comparison of Maximum A Posteriori and Variance Stabilization Approaches

For pure Poisson noise, the complexity of the MAP approach, with KL divergence as data term, is comparable with the complexity of the VST approach, which leads to a quadratic data term. It is interesting to study which of these two approaches gives a better performance in practice.

We compare the performances of MAPP and VSTP methods (used with TV potential) on 3630 images degraded by blur and pure Poisson noise. Figure 3 shows the difference in improvement in PSNR between the methods, i.e., $\Delta\text{PSNR}_{\text{VSTP}} - \Delta\text{PSNR}_{\text{MAPP}}$. As can be seen from the plot, VSTP method outperforms MAPP, especially for lower Poisson counts (corresponding to higher noise levels). VSTP gives on average 42.1% and 1.1% greater improvement in restoration quality as compared to MAPP, for lower

Fig. 2 Difference in improvements in PSNR achieved due to appropriate treatment of mixed Poisson–Gaussian noise. Improvement achieved by VSTPG is up to 6 dB higher than by MAPG.

Fig. 3 Difference in improvements achieved by VSTP and MAPP is presented. Its positive values indicate that VSTP outperforms MAPP.

(maximal pixel intensity ≤ 1000) and higher counting (maximal pixel intensity > 1000) regimes, respectively. The average improvement in PSNR achieved by the VST approach for all degraded images is 0.29 dB greater than that reached by the MAP approach. Our evaluation confirms that the two methods are of similar speed; the average time for recovery of one 256×256 pixels image is 1.84 s for VSTP and 1.88 s for the MAPP approach.

For mixed Poisson–Gaussian noise, the MAP approach is of much higher computational complexity. To make comparison of MAPPG and VSTPG feasible, we observe their performances on a small dataset given in Ref. 32; we are reporting the results of MAPPG stated there by the authors.

The dataset contains four images, the first of them is shown in Fig. 4(a). Each test image is degraded by one PSF and one Poisson–Gaussian noise level (corresponding to very low photon count); details are given in Table 1. Figure 4(b) shows the result of the described degradation applied to Fig. 4(a). We apply VSTPG with TV potential for their restoration. Table 1 presents the results. Presented restoration results for VSTPG in Table 1 are obtained with regularization parameters, which optimize the SNR value. The SSIM value is calculated for the restored image obtained with parameters which maximize SNR. The SNR and SSIM values of the restoration results achieved by utilizing MAPPG approach¹² with TV potential, included in Table 1, are taken from Ref. 32.

Table 1 Comparison of VSTPG and MAPPG¹² for four test images used in Ref. 32. VSTPG outperforms MAPPG in SNR, SSIM, and computational time.

		VSTPG	MAPPG ¹²
First image (350×350), peak = 20 PSF: Uniform 5×5 , $\sigma_m^2 = 9$ SNR = 7.64 dB, SSIM = 0.749	SNR (dB)	13.87	13.73
	SSIM	0.934	0.933
	Time (s)	4.35	48587
Second image (257×256), peak = 60 PSF: Gaussian 9×9 , std 0.5, $\sigma_m^2 = 36$ SNR = 9.40 dB, SSIM = 0.646	SNR (dB)	15.55	15.43
	SSIM	0.888	0.880
	Time (s)	2.43	351
Third image (256×256), peak = 100 PSF: Uniform 3×3 , $\sigma_m^2 = 36$ SNR = 10.68 dB, SSIM = 0.684	SNR (dB)	14.25	13.81
	SSIM	0.851	0.847
	Time (s)	2.30	8322
Fourth image (256×256), peak = 150 PSF: Gaussian 7×7 , std 1, $\sigma_m^2 = 40$ SNR = 15.77 dB, SSIM = 0.643	SNR (dB)	20.57	20.33
	SSIM	0.875	0.870
	Time (s)	2.27	43397

As can be seen, the VSTPG method outperforms MAPPG in terms of SNR, SSIM, and computational time. Restoration of one image by MAPPG takes up to 13.5 h, whereas the VSTPG method takes 4.35 s on the same image and reaches a better restoration quality. Figure 4(c) presents the result of restoration of Fig. 4(b) by VSTPG approach. The images restored by MAPPG can be found in Ref. 32.

3.3 Improvement Achieved from Utilizing Huber Potential Function

An additional way to improve performance of restoration methods is to utilize Huber potential function. We evaluate the difference in PSNR improvement, averaged over 30 images, obtained by using TV and Huber potentials for MAPP, VSTP, and VSTPG. For details about the data used, see Appendix C.

Figure 5(a) presents $\Delta\text{PSNR}_{\text{Huber}} - \Delta\text{PSNR}_{\text{TV}}$ obtained when VSTP is used for restoration of images degraded by Poisson noise. MAPP exhibits a similar behavior. Huber

Fig. 4 Performance of VSTPG: (a) original image, (b) degraded image (SNR = 7.64 dB, SSIM = 0.749), and (c) restored image (SNR = 13.87 dB, SSIM = 0.943).

Fig. 5 Difference in improvements achieved by the method utilizing Huber potential, compared to TV, is shown for (a) VSTP approach and (b) VSTPG approach.

potential gives on average 7.6% and 8.0% greater improvement in restoration quality as compared to TV regularization when used with VSTP and MAPP methods, respectively.

Figure 5(b) presents $\Delta \text{PSNR}_{\text{Huber}} - \Delta \text{PSNR}_{\text{TV}}$ for VSTPG method for mixed noise with the lowest considered Gaussian noise level. The other two Gaussian noise levels exhibit similar behaviors. Huber potential gives on average 7.8%, 6.8%, and 4.8% greater improvement in restoration quality as compared to TV regularization for considered Gaussian noise levels.

The improvement from using the Huber potential is consistent with the results of the study performed in Ref. 6, where a comparison was made for pure Gaussian noise on a smaller evaluation set (120 degraded images). We have extended this study to 3630 degraded images and observe that Huber gives on average 4.6% greater improvement in restoration quality, as compared to TV, for Gaussian noise.

The performance improvement from the Huber potential comes at the cost of one more parameter to tune. With a good optimization strategy, this cost can be kept reasonably low. Although the optimal values of λ and ω vary between images, we observe some consistency. The optimal value of λ is generally slightly larger for the Huber potential than for TV; on average, for all methods and all degraded images, optimal λ_{Huber} is 1.5 times larger than λ_{TV} . A typical example is shown in Fig. 6, where optimal $\lambda_{\text{TV}} = 0.0018$ and $\lambda_{\text{Huber}} = 0.0032$. If a good value of λ_{TV} is known, this provides a good starting guess for λ_{Huber} . The additional parameter, ω , controls the point of transition between ℓ_2 , Tikhonov regularization, and ℓ_1 , TV regularization. Choosing a too small value of ω makes the Huber potential approach TV, with the above observed reduced performance. Selecting a too large value, however, leads to quadratic regularization, which gives a rapid decay in performance and blurred edges as a result.

Fig. 6 Relation between regularization parameter λ for (a) TV and (b) Huber potentials on Peppers image (the second in Fig. 1) degraded by PSF with $\sigma_p = 1.5$ and Poisson noise with $\text{peak} = 398$, when VSTP is used for restoration. Optimal parameter $\lambda_{\text{Huber}} = 0.0032$ is slightly larger than $\lambda_{\text{TV}} = 0.0018$. Quality of restoration is less sensitive to choice of ω for the Huber potential (optimal $\omega = 0.0016$).

Fig. 7 Improvement in PSNR for (a) VSTP and (b) VSTPG when Huber potential is used, for different levels of blur and noise.

We find that a good initialization for the two-dimensional (2-D) search for optimal parameters for the Huber potential is the optimal λ from the one-dimensional search for the TV potential, combined with a small value of ω (e.g., 10^{-5} , giving a behavior of the Huber potential which is very similar to that of TV). Considering the logarithm of λ and ω , can help in addressing problems related to the difference in scale of these parameters.

3.4 Summary of Results

The performed evaluation shows that the VST-based approach together with the Huber potential function provides a very good combination for restoration of images degraded by signal-dependent noise. Figure 7 illustrates the level of improvement in PSNR obtained when VSTP and VSTPG methods are used in combination with Huber potential function. Table 2 summarizes the average improvement obtained when VST-based approach is applied with Huber potential for restoration of images degraded by signal-dependent noise and different levels of signal-independent noise.

A qualitative evaluation is given in Fig. 8, where restoration of three degraded images from the used dataset are presented. The images are degraded by a PSF with $\sigma_p = 2$ and Poisson and mixed Poisson–Gaussian noise. The restored images, obtained utilizing the VST method with Huber potential function, are presented together with the reached PSNR and SSIM values. The optimal parameters λ and ω that maximize PSNR are also given. SSIM values are calculated for restored images obtained with parameters which maximize PSNR. As can be seen, the presented method

reduces blur and suppresses noise and leads to good restoration results. Time for recovery of one 256×256 pixels image in MATLAB on an Intel Core i7-2600 CPU 3.40 GHz is on average 1.8 s. On a limited dataset, we observe that the execution time scales linearly with the image size. Parallel implementation can provide additional speeding-up.

3.5 Illustrative Example on a Naturally Degraded Image

To verify applicability of the recommended VSTPG method with the Huber potential function for restoration of images degraded by signal-dependent noise, we utilize it on a naturally degraded image, as shown in Fig. 9(a). We perform restoration on a raw-data 840×840 pixels image [Fig. 9(a)] from the CCD sensor of a Samsung digital camera WB550 at ISO 1600. It is generally considered that the main contributions to CCD sensor noise are photon noise (“shot noise”), which is Poisson distributed, and thermal noise, which is well modeled by a Gaussian distribution; our visual inspection confirms that a mixed Poisson–Gaussian noise model is appropriate.

We estimate PSF utilizing the blind restoration method proposed in Ref. 33. This method is suitable for images degraded by mixed noise and enables simultaneous estimation of unknown PSF and the high quality image. The parameter of mixed noise, σ_m , is estimated by a method presented in Ref. 29. This method estimates parameters of noise by using the selected weak textured patches. Since ground truth is now not available, we cannot select regularization parameters by optimizing PSNR. We estimate regularization parameters λ_{Huber} and ω as arguments which maximize the

Table 2 Average improvement in PSNR over all degraded images obtained by restoration using VST approach with Huber potential in regularization for low (peak ≤ 1000) and high (peak > 1000) photon counting regimes.

Photon count	Poisson noise		Mixed noise $\sigma_m/\sqrt{\text{peak}} = 0.01$		Mixed noise $\sigma_m/\sqrt{\text{peak}} = 0.1$		Mixed noise $\sigma_m/\sqrt{\text{peak}} = 1$	
	Low	High	Low	High	Low	High	Low	High
VST + Huber	2.21 dB	1.08 dB	2.21 dB	1.06 dB	2.23 dB	1.06 dB	3.69 dB	1.21 dB

	Poisson noise	Mixed noise	Poisson noise	Mixed noise	Poisson noise	Mixed noise

PSNR	21.77 dB	20.97 dB	19.86 dB	19.22 dB	22.44 dB	22.18 dB
SSIM	0.308	0.187	0.400	0.339	0.596	0.470

PSNR	23.90 dB	23.84 dB	23.36 dB	21.82 dB	24.32 dB	25.68 dB
SSIM	0.492	0.540	0.701	0.591	0.783	0.730
λ	$5.3 \cdot 10^{-4}$	$5.1 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	$1.3 \cdot 10^{-4}$	$1.8 \cdot 10^{-3}$	$5.0 \cdot 10^{-4}$
ω	$5.7 \cdot 10^{-3}$	$7.4 \cdot 10^{-3}$	$2.5 \cdot 10^{-6}$	$6.9 \cdot 10^{-8}$	$1.3 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$

Fig. 8 Restoration of blurred and noisy images using VST method with Huber potential. First row contains images degraded by PSF ($\sigma_p = 2$) and two types of noise: Poisson (peak = 631) and mixed (peak = 631, $\sigma_m/\sqrt{\text{peak}} = 1$). Second row contains restored images. Reached PSNR and SSIM values together with optimal parameters λ and ω that maximize PSNR are given below each restored image.

Fig. 9 Performance of VSTPG and MAPG with Huber potential on a part of a raw grayscale image acquired by a digital camera and naturally degraded by mixed Poisson–Gaussian noise: (a) Original 840×840 pixels raw-data image. (b) Zoomed in part of size 200×200 pixels. (c) Result of restoration of (b) by utilizing VSTPG approach. (d) Result of restoration of (b) by utilizing MAPG approach.

no-reference image content measure Q proposed in Ref. 27. Q -measure drops when the amount of noise or/and blur increases in the image; its maximum indicates optimal parameter values. To select optimal parameters in our example, we maximize Q -measure utilizing Nelder–Mead simplex search.²⁸

Results of restoration of Fig. 9(b) are shown in Fig. 9(c), when the VSTPG approach is applied, and in Fig. 9(d), when the MAPG approach is applied. It is clearly visible that VSTPG more efficiently removes noise than MAPG. This example once more confirms that signal dependency of noise should not be neglected in restoration processes.

4 Conclusion

We evaluate performance of energy minimization-based restoration methods for images degraded by signal-dependent noise on a large image material. We show the importance of using the appropriate noise model; if the signal-dependent component is neglected, we observe an average loss of performance of 2.42 dB. Signal-dependent noise can be treated following a direct MAP-based approach, or using a VST approach, which transforms the problem to a Gaussian case. We compare methods derived utilizing VST and MAP and show that the VST-based method outperforms the method based on MAP for Poisson noise with on

average 0.29 dB. For mixed Poisson–Gaussian noise, the difference is 0.24 dB in favor of the presented VSTPG method. A large difference in speed, in favor of the same method, is one more advantage of the VSTPG, which we, for all of these reasons, recommend in this paper.

An additional way to improve performances of TV-based restoration methods for signal-dependent noise is to utilize the Huber potential function in the regularization term. We show that an average improvement of 0.11 dB in restoration performances is achieved when the Huber potential is utilized instead of TV.

The presented VST-based method utilizing the Huber potential function and SPG for optimization is a fast and efficient method for restoration of images degraded by blur and Poisson or Poisson–Gaussian noise.

Appendix A: Gradients of Objective Functions

The gradient of the objective function $E(u)$ is

$$\nabla E(u) = [\nabla E(u)_i]_{i=1}^n = [\nabla D(u)_i]_{i=1}^n + \lambda [\nabla R(u)_i]_{i=1}^n. \quad (13)$$

The gradient of the data fidelity term of MAPP Eq. (4) is

$$\nabla D_{\text{MAPP}}(u) = H^T(1 - y./Hu). \quad (14)$$

Here, the vector whose elements are all equal to 1, is denoted by $\mathbf{1}$ and $\cdot$ is element-wise division.

The gradient of the data fidelity term of VST [Eq. (9)] is

$$\nabla D_{\text{VSTPG}}(u) = H^T G(u), \quad (15)$$

$$G(u)_i = \begin{cases} 2 - y_i / \sqrt{(Hu)_i + \frac{3}{8} + \sigma_m^2}, & (Hu)_i + \frac{3}{8} + \sigma_m^2 > 0 \\ 0, & (Hu)_i + \frac{3}{8} + \sigma_m^2 \leq 0 \end{cases}. \quad (16)$$

The gradient of the regularization term [Eq. (10)] is

$$\begin{aligned} \nabla R(u)_i &= \Phi'(|\nabla(u_i)|) \frac{2u_i - u_r - u_b}{|\nabla(u_i)|} + \Phi'(|\nabla(u_l)|) \frac{u_i - u_l}{|\nabla(u_l)|} \\ &\quad + \Phi'(|\nabla(u_a)|) \frac{u_i - u_a}{|\nabla(u_a)|}, \end{aligned} \quad (17)$$

where u_a and u_l denote the edge neighbors above and left of the pixel u_i , respectively. We compute the discrete image gradient at point u_i , $\nabla(u_i)$, as $\nabla(u_i) = (u_r - u_i, u_b - u_i)$, where r and b denote indices of the edge neighbors to the right and below the pixel u_i , respectively. The image edges are handled using a periodic boundary condition.

The gradient of the regularization term with TV potential is nondifferentiable at points where $|\nabla u_i| = 0$. To meet the requirements of SPG, we consider a smooth version of Eq. (17) in the case of TV, where $|\nabla u_i|$ is replaced with $\sqrt{|\nabla(u_i)|^2 + \varepsilon^2}$ and where ε is a small positive number (we use $\varepsilon = 10^{-5}$ throughout). The use of a relaxed gradient for TV could possibly lead to a less accurate solution; it was observed in Ref. 15 that the differences are negligible.

We assume periodic boundary conditions. All multiplications of vectors with H are performed in the frequency domain using 2-D fast Fourier transform.

Appendix B: Spectral Projected Gradient Parameters

We perform optimization using SPG with settings recommended in Ref. 34: $\theta_{\min} = 10^{-3}$, $\theta_{\max} = 10^3$, $\gamma = 10^{-4}$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.9$. Algorithm 1 terminates when the max-norm between two consecutive images is less than $\text{tol} = 10^{-5}$ or when the number of iterations reaches 200. We define the projection P_Ω of a vector $x \in \mathbb{R}^n$ to the feasible set $\Omega = [0, 1]^n$ as $[P_\Omega(x)]_i = \min\{1, \max\{0, x_i\}\}$, for all $i = 1, 2, \dots, n$.

Appendix C: Dataset Description

The dataset consists of 30 nonblurred and noise-free test images presented in Fig. 1. We blur each test image with Gaussian PSF, since they closely resemble real PSFs in many imaging systems. We observe PSFs with 11 different standard deviations σ_p between 0 and 5 pixels. We consider images with 11 maximal pixel intensities from 100 to 10,000 i.e., 11 different levels of Poisson noise. Low intensity/photon count corresponds to a high noise level. In addition, for Gaussian noise in the mixed noise model, we choose variance such that the ratio $\sigma_m/\sqrt{\text{peak}}$ is equal to $\{0.01, 0.1, 1\}$.

Acknowledgments

The authors would like to thank the anonymous reviewers for their helpful comments. The work was jointly supported by the Swedish Research Council, Vetenskapsrådet, and by the Ministry of Education, Science and Technological Development of the Republic of Serbia through Projects ON174008 and III44006 of MI-SANU. Nataša Sladoje is supported by the Swedish Governmental Agency for Innovation Systems (VINNOVA).

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