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Additive, abelian, and exact categories

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Contents

1 Modules

In this preliminary section, we will define modules over rings and explore the bare necessities for the sections to come. Most (if not all) definitions and results in this section can be found in any exposition of modules over a ring, e.g., [3] or [4].

1.1 Basic constructions

Definition 1.1. For any ring (with unity) R, a *(left) R-module* (or *module over R*) is an abelian group $(M, +)$ equipped with a compatible (left) R-action $\cdot\colon R \times M \to M$ by R. Spelled out, an R-module should satisfy the following eight axioms, the first four of which are for the abelian group and the last four of which are for the ring action:

1. Addition is associative, i.e., $(a + b) + c = a + (b + c)$ for any $a, b, c \in M$
2. Addition is commutative, i.e., $a + b = b + a$ for any $a, b \in M$
3. There is an additive identity, denoted by 0, such that $0 + a = a = a + 0$ for any $a \in M$
4. Any $a \in M$ has an additive inverse, denoted by $-a$, such that $a + (-a) = 0 = (-a) + a$
5. The ring identity 1_R acts trivially, i.e., $1_R \cdot m = m$ for any $m \in M$
6. A sum $r + s$ of ring elements acts as $(r + s) \cdot m = rm + sm$, i.e., in a way compatible with the group addition, for any $r, s \in R$ and $m \in M$
7. A product rs of ring elements acts as $(rs) \cdot m = r \cdot (s \cdot m)$, i.e., in a way compatible with a composition of sorts, for any $r, s \in R$ and $m \in M$
8. Any ring element acts additively, i.e., $r \cdot (m + n) = r \cdot m + r \cdot n$ for any $r \in R$ and $m, n \in M$

Remark 1.2. Readers familiar with linear algebra will note that the above definition with "ring R" replaced with "field K" would be a definition of a vector space; modules over fields are precisely vector spaces. Accordingly, one may think of modules over rings as being generalized vector spaces with many, but not all, of the properties of vector spaces (see Example 1.32).

Remark 1.3. We will concern ourselves only with left R-modules, but the reader should know that there are notions of a *right* R-module and even of so-called bimodules, which are equipped with both a left ring action and a right ring action (possibly by different rings!).

Example 1.4. For any fixed ring R, there is a trivial example of an R-module, namely when the underlying group is trivial (i.e., the group is a singleton $\{0\}$), in which case the R-action maps everything to 0. We will refer to this module as the *zero module* (though to be strict, it might be better to speak of it as *a* zero module) and denote it by $\{0\}$ or sometimes just 0.

More interestingly, we may turn any ring R into an R-module in a natural manner.

Example 1.5. Consider a ring R. It is by definition an abelian group with the additional structure of a multiplication $\cdot\colon R \times R \to R$. This multiplication may be viewed as an R-action, seeing as the axioms for the identity, associativity, and compatibility (with the ring addition) for the ring multiplication guarantee that the necessary module axioms hold. We call this module the *regular (left) R-module* and denote it by ${}_RR$ or sometimes just R.

Example 1.6. As we noted, any vector space is a module. For instance, the familiar Euclidean space $\mathbb{R}^3$ is an $\mathbb{R}$-module.

Similar to how K-modules (for a field K) are the familiar vector spaces over K, the following remark expresses that $\mathbb{Z}$-modules are essentially (this "essentially" is formalized in Definition 2.60) just the underlying abelian groups.

Remark 1.7. For any abelian group M, there is a unique ring action by $\mathbb{Z}$ endowed with which M is a $\mathbb{Z}$-module, namely the one defined for any $n \in \mathbb{Z}$ and $x \in M$ by

$$n \cdot x := \begin{cases} \overbrace{x + \cdots + x}^{n \text{ terms}} & \text{if } n > 0 \\ 0_M & \text{if } n = 0 \\ -(\underbrace{x + \cdots + x}_{-n \text{ terms}}) & \text{if } n < 0 \end{cases}$$

It follows fairly readily that this is the only possible candidate for a compatible $\mathbb{Z}$-action on M: suppose that we have a $\mathbb{Z}$-action on M. Then write any $n > 0$ as the sum $1 + \cdots + 1$ and use the module axioms repeatedly to obtain

$$\begin{aligned} n \cdot x &= (1 + \cdots + 1) \cdot x \\ &= (1 \cdot x) + \cdots + (1 \cdot x) \\ &= x + \cdots + x \end{aligned}$$

It is a straightforward task to show (using the module axioms) that the ring action of any module behaves as expected for the additive identity and all additive inverses of the ring, i.e., in this case that $0_{\mathbb{Z}} \cdot x = 0_M$ and $(-n) \cdot x = -(n \cdot x)$.

That the above definition satisfies the module axioms can be shown one axiom at a time through a painstaking and painful consideration of cases depending on the sign of the integers acting on the group.

Definition 1.8. Given an R-module M, an *R-submodule* (or just *submodule*) of M is a non-empty subset $U \subseteq M$ that is closed under both addition and the action of R, i.e., U is a subgroup of M (viewed as an abelian group) that is closed under the R-action. This makes U with structure inherited from M into a module in its own right. Explicitly, the requirement is that

(1) $U \neq \varnothing$,

(2) If $m, n \in U$, then $(m + n) \in U$, and

(3) For any $r \in R$, if $m \in U$, then $(r \cdot m) \in U$.

Example 1.9. Any R-module M has the submodules $\{0\}$ and M (which may coincide).

Example 1.10. The submodules of a regular left module ${}_RR$ are (as sets) precisely the left ideals of the ring R. Both submodules and ideals may be viewed as additive subgroups with some additional condition satisfied. Upon closer scrutiny, the additional condition is in both cases closure under ring multiplication by an element from the left.

Much like for vector spaces (in fact, the definition is the very same), we may construct from a module M and a submodule U the quotient module M/U.

Definition 1.11. Let R be a ring, M be an R-module, and $U \subseteq M$ be a submodule of M. Consider the equivalence relation $\sim$ on M under which two elements are equivalent if and only if they differ by an element of U:

$$x \sim y \iff (x - y) \in U$$

The *quotient module* M/U of M by U is the set $M/\!\sim$ of all equivalence classes (where $[x]$ denotes the equivalence class of x) of this relation

$$M/\!\sim\, = \{[x] \mid x \in M\}$$

endowed with the addition and R-action given by

$$\begin{aligned} [m] + [n] &:= [m+n] \\ r \cdot [m] &:= [rm] \end{aligned}$$

In other words, the quotient module is the corresponding quotient group equipped with the above R-action.

It is not immediately clear that the quotient module is a well-defined R-module by the above. We verify that it is in the proof of the following proposition.

Proposition 1.12. *The quotient module M/U is a well-defined R-module.*

Proof. Verify first that $\sim$ is an equivalence relation. For reflexivity,

$$x \sim x \iff (x - x) = 0 \in U$$

and every submodule contains the zero element. For symmetry, use the fact that U is closed under the R-action and that the ring action in any module respects negation in the ring (a fact that follows readily from the module axioms and the observation that the ring zero acts by sending everything to zero):

$$\begin{aligned} x \sim y &\iff (x-y) \in U \\ &\implies (-1_R) \cdot (x-y) = -(1_R \cdot (x-y)) = -(x-y) = (y-x) \in U \\ &\iff y \sim x \end{aligned}$$

For transitivity, use the fact that U is closed under addition:

$$\begin{aligned} x \sim y \text{ and } y \sim z &\iff (x-y) \in U \text{ and } (y-z) \in U \\ &\implies (x-y) + (y-z) = (x-z) \in U \\ &\iff x \sim z \end{aligned}$$

Thus, the underlying set $M/{\sim}$ of the quotient module is well-defined. Next, verify that the operations are well-defined. For addition,

$$\begin{aligned} [x] = [x'] \text{ and } [y] = [y'] &\iff (x - x'), (y - y') \in U \\ &\implies (x - x') + (y - y') = (x + y) - (x' + y') \in U \\ &\iff [x + y] = [x' + y'] \end{aligned}$$

which shows that the right-hand side in the definition of the addition does not depend on the representatives in the left-hand side. For the R-action, we similarly get

$$\begin{aligned} [x] = [x'] &\iff x - x' \in U \\ &\implies r \cdot (x - x') = (r \cdot x - r \cdot x') \in U \\ &\iff [r \cdot x] = [r \cdot x'] \end{aligned}$$

Thus, the quotient module at least consists of a well-defined underlying set and well-defined operations, and it only remains to show that the module axioms are satisfied. These are each inherited from the corresponding axiom for M in a straightforward way, with $[0_M]$ as the zero element and $[-x]$ as the inverse of $[x]$ in M/U. □

1.2 R-module morphisms

As is typical for algebraic structures, there is a notion of homomorphisms (or morphism for short) of modules. They are defined as one would expect from looking at the morphisms of vector spaces (i.e., linear maps).

Definition 1.13. A *morphism* of R-modules (for some fixed ring R) is a function $f\colon M \to N$ from an R-module M to another, N, that respects the addition and R-action of the modules involved. In other words, f is additive and R-homogeneous:

1. $f(m + m') = f(m) + f(m')$ for any $m, m' \in M$.
2. $f(rm) = rf(m)$ for any $r \in R$ and $m \in M$.

Prefixes may be used to specify certain properties of the R-module morphism as follows:

- an injective morphism is called a *monomorphism*,
- a surjective morphism is called an *epimorphism*, and
- a bijective morphism is called an *isomorphism*.

In the case that there is an isomorphism from a module M to another module N, we say that the modules are *isomorphic* and write $M \cong N$.

Example 1.14. The most simple examples of R-module morphisms are the *identity morphism* (for any R-module M), which is just the identity map,

$$\begin{aligned} \mathrm{id}_M\colon M &\to M \\ m &\mapsto m \end{aligned}$$

fixing every element of M and the *zero morphism* (for any R-modules M and N)

$$\begin{aligned} 0_{MN}\colon M &\to N \\ m &\mapsto 0_N \end{aligned}$$

which kills every element of M.

The identity morphisms are all trivially isomorphisms, whereas a zero morphism is a monomorphism if and only if its domain is the zero module and an epimorphism if and only if its codomain is the zero module.

Two more interesting examples of R-module morphisms are the so-called canonical inclusion and canonical projection morphisms.

Example 1.15. Given an R-submodule $U \subseteq M$ of a module M, the *canonical inclusion morphism* of U into M is the morphism

$$\begin{aligned} \iota\colon U &\to M \\ m &\mapsto m \end{aligned}$$

and the *canonical projection morphism* of M onto M/U is the morphism

$$\begin{aligned} \iota\colon M &\to M/U \\ m &\mapsto [m] \end{aligned}$$

One may readily verify that the canonical inclusion is a monomorphism and that the canonical projection is an epimorphism.

A morphism $f\colon M \to N$ of R-modules gives rise to several important modules related to M and N. In order to define them, we will need the following proposition.

Proposition 1.16. *Let $f\colon M \to N$ be a morphism of R-modules. Then,*

(1) for any submodule $U \subseteq M$, the image of U under f

$$f(U) = \{f(u) \mid u \in U\} \subseteq N$$

is a submodule of N, and

(2) for any submodule $V \subseteq N$, the preimage of V under f

$$f^{-1}(V) = \{m \in M \mid f(m) \in V\}$$

is a submodule of M.

Proof. To show that $f(U)$ is a submodule of N, note first that $f(U)$ is non-empty seeing as U is non-empty. Then we may note that $f(U)$ is closed under addition and the R-action, because f is a morphism and U has the same closure properties: any $v, v' \in f(U)$ are, by the definition of the image, of the form $v = f(u)$ and $v' = f(u')$ for some $u, u' \in U$, and their sum may thus be rewritten as

$$\begin{aligned} v + v' &= f(u) + f(u') \\ &= f(u + u') \\ &\in f(U) \end{aligned}$$

which shows that $f(U)$ is closed under addition. Similarly for the R-action, we find that

$$\begin{aligned} r \cdot v &= r \cdot f(u) \\ &= f(r \cdot u) \\ &\in f(U) \end{aligned}$$

Thus $f(U)$ is a submodule of N.

To show that $f^{-1}(V)$ is a submodule of M, mostly the same argument can be applied. However, in order to prove the non-emptiness of $f^{-1}(V)$, we need work just slightly more than before: note that the $0_N \in V$ and that $f(0_M) = 0_N$, which is the case for any submodule and R-module morphism. We thus find that $f^{-1}(V)$ contains 0_M and must be non-empty. Let $u, u' \in f^{-1}(V)$ and show for closure under addition of $f^{-1}(V)$ that their sum $u + u'$ is in $f^{-1}(V)$.

$$f(u + u') = f(u) + f(u') \in V \implies (u + u') \in f^{-1}(V)$$

Let further $r \in R$ and show for closure under the R-action of $f^{-1}(V)$ that $r \cdot u \in f^{-1}(V)$:

$$f(r \cdot u) = r \cdot f(u) \in V \implies r \cdot u \in f^{-1}(V)$$

Thus $f^{-1}(V)$ is a submodule of M, which finishes the proof. ∎

Definition 1.17. Let $f \colon M \to N$ be a morphism of R-modules.

The *kernel* of f, denoted by $\ker f$, is the preimage of the smallest submodule:

$$\ker f := f^{-1}(\{0\}) = \{x \in M \mid f(x) = 0_N\} \subseteq M$$

The *image* of f, denoted by $\operatorname{im} f$, is the image of the entire module (the largest submodule if you will):

$$\operatorname{im} f := f(M) = \{f(m) \mid m \in M\} \subseteq N$$

The *cokernel* of f, denoted by $\operatorname{cok} f$, is the quotient $N / \operatorname{im} f$ of N by the image $\operatorname{im} f$.

The *coimage* of f, denoted by $\operatorname{coim} f$, is the quotient $M / \ker f$ of M by the kernel $\ker f$.

These definitions may be illustrated as

$$\begin{tikzcd} & M \arrow[rrr, "f"] \arrow[dr, "\pi_1"] & & & N \arrow[dr, "\pi_2"] & \\ \ker f \arrow[ur, "\iota_1"] & & \operatorname{coim} f & \operatorname{im} f \arrow[ur, "\iota_2"] & & \operatorname{cok} f \end{tikzcd}$$

where ι_1 and ι_2 are the canonical inclusions of the respective submodule, π_1 is the canonical projection of M onto $\operatorname{coim} f = M / \ker f$, and π_2 is the canonical projection of N onto $\operatorname{cok} = N / \operatorname{im} f$.

Remark 1.18. One may show that the coimage and image of an R-module morphism are isomorphic (see the corollary of the first isomorphism theorem: Corollary 1.25). In other words, the coimage is somewhat redundant and was defined above only for completeness. The remaining modules (the kernel, image, and cokernel), however, are in most cases not isomorphic to each other or either of M and N. Pictorially, we have

$$\begin{tikzcd} & M \arrow[rr, "f"] \arrow[dr, "\pi_1"] & & N \arrow[dr, "\pi_2"] & \\ \ker f \arrow[ur, "\iota_1"] & & \operatorname{coim} f \cong \operatorname{im} f \arrow[ur, "\iota_2"] & & \operatorname{cok} f \end{tikzcd}$$

The kernel and cokernel can be used to characterize monomorphisms and epimorphisms. Generalizing the below informally, we may think of the kernel of a morphism as measuring how far from injective the morphism is, and similarly for the cokernel and surjectivity [4, p. 13].

Proposition 1.19. *An R-module morphism $\varphi\colon M \to N$ is*

- *a monomorphism if and only if its kernel is trivial, i.e., if and only if* $\ker \varphi = \{0\}$
- *an epimorphism if and only if its cokernel is trivial, i.e., if and only if* $\operatorname{cok} \varphi = \{0\}$.

Proof. The "only if" part for injectivity follows readily from the definition of the kernel: we have observed that any morphism maps the zero element to the zero element, so the kernel surely contains the zero element of M. If φ is injective, then the kernel must consist of only this element, for any preimage is (at most) singleton.

Suppose for the converse (for injectivity) that the kernel is trivial and prove that $\varphi(m) = \varphi(m')$ implies $m = m'$:

$$\begin{aligned} \varphi(m) = \varphi(m') &\implies \varphi(m) - \varphi(m') = 0_N \\ &\implies \varphi(m - m') = 0_N \\ &\implies (m - m') \in \ker \varphi \\ &\implies m - m' = 0_M \\ &\implies m = m' \end{aligned}$$

For the surjectivity part, note that any quotient of modules A/B is singleton if and only if $A = B$. In this case, this is to say that $\operatorname{cok} \varphi = N/\operatorname{im} \varphi = \{0\}$ if and only if $N = \operatorname{im} \varphi$, i.e., if and only if φ is surjective. $\square$

There is also a characterization of isomorphisms, which might not be of as much practical use as the previous proposition but which will later serve to elucidate what an isomorphism of R-modules is from a categorical point of view. Seeing as any invertible function is bijective, the following proposition can be understood to characterize R-module isomorphisms as invertible morphisms whose inverse is also a morphism.

Proposition 1.20. *An R-module isomorphism is an invertible morphism and its inverse is an R-module (iso-)morphism.*

Proof. Let $\varphi\colon M \to N$ be an isomorphism of R-modules and $\psi\colon N \to M$ be its inverse, i.e., such that $\psi \circ \varphi = \mathrm{id}_M$ and $\varphi \circ \psi = \mathrm{id}_N$. Note that ψ is bijective, seeing as it is the inverse of a bijection, so we need to prove only that ψ is additive and homogeneous.

The additivity of ψ follows from the additivity of φ (in fact, this step proves that the inverse of an isomorphism of abelian groups is an isomorphism of abelian groups and even generalizes to arbitrary groups): for any

$m, n \in M$ and $r \in R$, we have

$$\begin{aligned}\psi(m+n) = \psi(m) + \psi(n) &\iff \psi(m) + \psi(n) - \psi(m+n) = 0 \\ &\iff \varphi(\psi(m) + \psi(n) - \psi(m+n)) = 0\end{aligned}$$

where the second equivalence follows from Proposition 1.19. Because φ is additive and ψ is an inverse of φ (in particular a right inverse), the left-hand side of the final equation simplifies to

$$\varphi(\psi(m)) + \varphi(\psi(n)) - \varphi(\psi(m+n)) = m + n - (m+n) = 0$$

and additivity of ψ follows.

The homogeneity of ψ follows from the additivity and homogeneity of φ in a similar fashion: for any $r \in R$ and $m \in M$, we have

$$\begin{aligned}\psi(rm) = r\psi(m) &\iff r\psi(m) - \psi(rm) = 0 \\ &\iff \varphi(r\psi(m) - \psi(rm)) = 0\end{aligned}$$

The left-hand simplifies to 0 as

$$\begin{aligned}\varphi(r\psi(m) - \psi(rm)) &= \varphi(r\psi(m)) - \varphi(\psi(rm)) \\ &= r\varphi(\psi(m)) - \varphi(\psi(rm)) \\ &= rm - rm \\ &= 0\end{aligned}$$

□

A fundamental notion in homological algebra is that of an exact sequence of morphisms. At its core, homological algebra may be thought of as the study of these exact sequences and how the exactness of a sequence is preserved under certain transformations.

Definition 1.21. [3, p. 393] A finite or infinite sequence

$$\cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \cdots$$

of R-module morphisms is said to be *exact at i* if $\operatorname{im} f_{i+1} = \ker f_i$ and just *exact* if it is exact at every i (endpoints excluded).

Many properties of morphisms may be expressed in terms of exactness of sequences, as is made apparent by the following example.

Example 1.22. Consider the sequence $0 \xrightarrow{0} M \xrightarrow{f} N$. It is exact if and only if $\ker f = \operatorname{im} 0 = 0$, i.e., if and only if f is a monomorphism (by Proposition 1.19). Similarly, the sequence $M \xrightarrow{f} N \xrightarrow{0} 0$ is exact if and only if f is an epimorphism. Thus, an isomorphism is precisely a morphism f such that $0 \xrightarrow{0} M \xrightarrow{f} N \xrightarrow{0} 0$ is exact.

Things get more involved with two rather than one morphism embraced by zero morphisms:

$$0 \xrightarrow{0} L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{0} 0$$

By the above, this sequence is exact precisely when f is a monomorphism, g is an epimorphism, and $\operatorname{im} f = \ker g$. Exact sequences of this form are called *short exact sequences*, a notion that will be generalized to the setting of so-called abelian and exact categories in Sections 4 and 5.

1.3 Induced morphisms

The following proposition provides a way of factorizing a morphism through a quotient module.

Proposition 1.23. *[3, p. 322] Let $f\colon M \to N$ be a morphism of R-modules. For any submodule $U \subseteq \ker f$ (which may be viewed as a submodule of M), there is a unique morphism $\hat{f}\colon M/U \to N$ such that $f = \hat{f} \circ \pi$, where $\pi\colon M \to M/U$ is the canonical projection morphism.*

Proof. Consider the map $\hat{f}$ defined by

$$\hat{f}([m]) = f(m)$$

for every $[m] \in M/U$. We need to verify that it is well-defined, i.e., that $[m] = [m']$ implies $\hat{f}([m]) = \hat{f}([m'])$. This follows readily from the assumption that $U \subseteq \ker f$:

$$\begin{aligned}
[m] = [m'] &\iff (m - m') \in U \\
&\implies (m - m') \in \ker f \\
&\iff f(m - m') = 0_N \\
&\iff f(m) - f(m') = 0_N \\
&\iff f(m) = f(m') \\
&\iff \hat{f}([m]) = \hat{f}([m'])
\end{aligned}$$

Next, we should verify that $\hat{f}$ is a morphism of R-modules, i.e., that it is additive and R-homogeneous. This follows immediately from the definition of the operations on the quotient module and the definition of $\hat{f}$ in terms of M and f: for any $m, m' \in M$ and $r \in R$, we have

$$\begin{aligned}
\hat{f}([m] + [m']) &= \hat{f}([m + m']) \\
&= f(m + m') \\
&= f(m) + f(m') \\
&= \hat{f}([m]) + \hat{f}([m'])
\end{aligned}$$

and

$$\begin{aligned}
\hat{f}(r[m]) &= \hat{f}([rm]) \\
&= f(rm) \\
&= rf(m) \\
&= r\hat{f}([m])
\end{aligned}$$

This shows the existence of the morphism.

Finally, note that the map is trivially unique with the required factorization property, seeing as the requirement reads $(\hat{f} \circ \pi)(m) = f(m)$ or equivalently $\hat{f}([m]) = f(m)$ for every $m \in M$. □

In the case that $U = \ker f$ in the above proposition, the induced morphism $\hat{f}$ turns out to be a monomorphism. By restricting its codomain to $\operatorname{im} f$, it is made surjective and hence an isomorphism. This is the essence of the next proposition, which goes by several names, including the "first isomorphism theorem (for modules)" and the "homomorphism theorem (for modules)" [3, p. 25, 322].

The theorem may be interpreted as stating that any R-module homomorphism factors into a canonical projection morphism, an isomorphism, and a canonical inclusion morphism [3, p. 322]. Another interpretation is that it factors into an epimorphism and a monomorphism, i.e., a surjective and an injective part.

Proposition 1.24 (The first isomorphism theorem for modules)**.** *[3, p. 322] For any morphism $f\colon M \to N$ of R-modules, there is a unique isomorphism $\hat{f}$ making the following diagram commute:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/\ker f \arrow[r, "\hat{f}"'] & \operatorname{im} f \arrow[u, "\iota"'] \end{tikzcd}$$

where $\pi\colon M \to M/\ker f$ is the canonical projection morphism and $\iota\colon \operatorname{im} f \to N$ is the canonical inclusion morphism. Explicitly, the condition is that $f = \iota \circ \hat{f} \circ \pi$.

Proof. Taking $U = \ker f$ in Proposition 1.23 gives a unique morphism, $f'\colon M/\ker f \to N$ say, making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/\ker f \arrow[ur, "f'"'] & \end{tikzcd}$$

Furthermore, this f' is defined by $f'([m]) = f(m)$ for every $m \in M$. Because of the uniqueness of f', we conclude that $\hat{f}$ must satisfy $\iota \circ \hat{f} = f'$, which is to say that $\hat{f}$ must be the restriction (with respect to the codomain) of f' to $\operatorname{im} f$. Note that this $\hat{f}$ is a well-defined morphism, seeing as f' by definition has the same image as f. We thus have the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/\ker f \arrow[ur, "f'"'] \arrow[r, "\hat{f}"'] & \operatorname{im} f \arrow[u, "\iota"'] \end{tikzcd}$$

where, in particular, the outer square commutes and $\hat{f}$ is the only morphism making said square commute. It remains to be shown that $\hat{f}$ is an isomorphism.

On one hand, $\operatorname{im} f' = \operatorname{im} f$ as noted above, and on the other hand $\operatorname{im} \hat{f} = \operatorname{im} f'$ seeing as $\hat{f}$ is just the restriction (with respect to the codomain) of f'. That is, $\operatorname{im} \hat{f} = \operatorname{im} f$ and $\hat{f}$ is surjective.

For injectivity, recall Proposition 1.19, and show that the kernel of $\hat{f}$ is trivial. Because $\hat{f}([m]) = f(m)$ for every $m \in M$, we have

$$\begin{aligned} \hat{f}([m]) = 0 &\iff f(m) = 0 \\ &\iff m \in \ker f \\ &\iff [m] = [0] \end{aligned}$$

and hence that the kernel is trivial. We conclude that $\hat{f}$ is bijective and hence an isomorphism, which finishes the proof. ∎

Corollary 1.25. *If $f\colon M \to N$ is an R-module morphism, then $M/\ker f \cong \operatorname{im} f$.*

1.4 Direct products and sums of modules

From a collection of R-modules, one may construct an R-module containing all the information of the constituent modules by equipping their Cartesian product with suitable module operations. It turns out to often be useful to add a finiteness constraint, which matches the inherent finiteness of linear combinations, to the resulting module. These constructions are given in the following definition.

Definition 1.26. [4, p. 19] Given a family of R-modules $\{M_j\}_{j\in J}$ (possibly infinite), we define their *direct product*, denoted by $\prod_{j\in J} M_j$, as the Cartesian product of the modules

$$\prod_{j\in J} M_j = \{(m_j)_{j\in J} \mid m_j \in M_j\}$$

equipped with componentwise operations, i.e., for arbitrary $m_j, n_j \in M_j$ and $r \in R$,

$$\begin{aligned}(m_j)_{j\in J} + (n_j)_{j\in J} &:= (m_j + n_j)_{j\in J}\\ r(m_j)_{j\in J} &:= (rm_j)_{j\in J}\end{aligned}$$

and their *direct sum*, denoted by $\bigoplus_{j\in J} M_j$, as the subset of their Cartesian product with finite support

$$\bigoplus_{j\in J} M_j = \{(m_j)_{j\in J} \mid m_j \in M_j \text{ and } m_j \neq 0 \text{ for only finitely many } j \in J\}$$

equipped with the very same operations.

It is straightforward to verify that the direct product $\prod_{j\in J} M_j$ really is a module. Each module axiom follows from the corresponding axiom for the modules M_j, with zero element $(0_{M_j})_{j\in J}$ and inverses given by $-(m_j)_{j\in J} = (-m_j)_{j\in J}$ in the product. The direct sum may then also be seen to be a module, as an immediate consequence of the following remark.

Remark 1.27. The direct sum of any family of R-modules is a submodule of their direct product: the zero tuple has finite support, which shows non-emptiness of the direct sum. Any elements $(m_j)_{j\in J}$ and $(n_j)_{j\in J}$ of $\bigoplus_{j\in J} M_j$ have $m < \infty$ and, respectively, $n < \infty$ nonzero components. Their sum has at most $m+n < \infty$ nonzero components and is hence an element of the direct sum. Similarly, $r \cdot (m_j)_{j\in J}$ has at most m nonzero components and is hence an element of the direct sum (for any $r \in R$).

Furthermore, the direct sum and direct product coincide for finite families, seeing as the requirement of finite support for the direct sum is vacuous in this case.

The direct product and sum each comes equipped with an important family of morphisms.

Definition 1.28. [4, p. 19] Consider a direct product $\prod_{j\in J} M_j$. The ith *projection morphism*, usually denoted by π_i, is the morphism from $\prod_{j\in J} M_j$ to M_i defined by projecting every tuple onto its ith component:

$$\pi_i((m_j)_{j\in J}) := m_i$$

For a direct sum $\bigoplus_{j\in J} M_j$, the ith *inclusion morphism*, usually denoted by ι_i, is the morphism from M_i to $\bigoplus_{j\in J} M_j$ defined by mapping every element $m_i \in M_i$ to the tuple whose ith component is m_i and the rest are zero:

$$\iota_i(m_i) := (n_j)_{j\in J}$$

where

$$n_j = \begin{cases} m_i & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

That these maps really are morphisms follows immediately from their definition and the componentwise operations on the direct product and sum.

Any morphism *from* a direct sum "decompose" very naturally into a family of morphisms from the constituent modules of the sum. Conversely, any such family is the decomposition of some morphism from the direct sum. Similarly for the direct product, any morphism *into* a direct product decompose into a family of morphisms into the constituent modules of the product and any such family is the decomposition of some morphism into the direct product. These properties are known as the universal property of the direct sum and the direct product, respectively.

Proposition 1.29 (The universal property of the direct product and direct sum)**.**

1. *Let M and N_i be R-modules, $N = \prod_{i \in I} N_i$, and $\varphi_i \colon M \to N_i$ be morphisms (for some set I and every $i \in I$). Then there is a unique morphism $\varphi \colon M \to N$ with the property that*

$$\varphi_i = \pi_i \circ \varphi, \quad \forall i \in I$$

i.e., making the following diagram commute for every $i \in I$:

$$\begin{tikzcd} M \arrow[r, dotted, "\varphi"] \arrow[dr, "\varphi_i"'] & N \arrow[d, "\pi_i"] \\ & N_i \end{tikzcd}$$

2. *Let M_i and N be R-modules, $M = \bigoplus_{i \in I} M_i$, and $\varphi_i \colon M_i \to N$ be morphisms (for some set I and every $i \in I$). Then there is a unique morphism $\varphi \colon M \to N$ with the property that*

$$\varphi_i = \varphi \circ \iota_i, \quad \forall i \in I$$

i.e., making the following diagram commute for every $i \in I$:

$$\begin{tikzcd} M \arrow[r, dotted, "\varphi"] & N \\ M_i \arrow[u, "\iota_i"] \arrow[ur, "\varphi_i"'] & \end{tikzcd}$$

Proof.

1. The condition that $\varphi_i = \pi_i \circ \varphi$ specifies φ uniquely as the function defined by

$$\varphi(m) = (\varphi_i(m))_{i \in I}$$

and it remains to show that this φ is an R-module morphism. Both additivity and homogeneity are straightforward consequences of the definition of the direct product, the module axioms, and the assumption that φ_i are morphisms:

$$\begin{aligned} \varphi(m+n) &= (\varphi_i(m+n))_{i \in I} \\ &= (\varphi_i(m) + \varphi_i(n))_{i \in I} \\ &= (\varphi_i(m))_{i \in I} + (\varphi_i(n))_{i \in I} \\ &= \varphi(m) + \varphi(n) \end{aligned}$$

and

$$\begin{aligned} \varphi(r \cdot m) &= (\varphi_i(r \cdot m))_{i \in I} \\ &= (r \cdot \varphi_i(m))_{i \in I} \\ &= r \cdot (\varphi_i(m))_{i \in I} \\ &= r \cdot \varphi(m) \end{aligned}$$

2. The condition that $\varphi_i = \varphi \circ \iota_i$ dictates that

$$\begin{aligned}
\varphi((m_i)_{i\in I}) &= \varphi(\sum_{\substack{i\in I\\ m_i\neq 0}} \iota_i(m_i)) \\
&= \sum_{\substack{i\in I\\ m_i\neq 0}} \varphi(\iota_i(m_i)) \\
&= \sum_{\substack{i\in I\\ m_i\neq 0}} \varphi_i(m_i)
\end{aligned}$$

which shows that φ is unique. It remains to show that this φ is an R-module morphism, which follows readily after some tinkering with the indices of summation from the definition of the direct sum, the module axioms, and the assumption that φ_i are morphisms:

$$\begin{aligned}
\varphi((m_i)_{i\in I} + (n_i)_{i\in I}) &= \varphi((m_i + n_i)_{i\in I}) \\
&= \sum_{\substack{i\in I\\ m_i+n_i\neq 0}} \varphi_i(m_i + n_i) \\
&= \sum_{\substack{i\in I\\ m_i \neq 0 \text{ or } n_i \neq 0}} \varphi_i(m_i + n_i) \\
&= \sum_{\substack{i\in I\\ m_i \neq 0 \text{ or } n_i \neq 0}} \varphi_i(m_i) + \varphi_i(n_i) \\
&= \sum_{\substack{i\in I\\ m_i \neq 0 \text{ or } n_i \neq 0}} \varphi_i(m_i) + \sum_{\substack{i\in I\\ m_i \neq 0 \text{ or } n_i \neq 0}} \varphi_i(n_i) \\
&= \sum_{\substack{i\in I\\ m_i\neq 0}} \varphi_i(m_i) + \sum_{\substack{i\in I\\ n_i\neq 0}} \varphi_i(n_i) \\
&= \varphi((m_i)_{i\in I}) + \varphi((n_i)_{i\in I})
\end{aligned}$$

and

$$\begin{aligned}
\varphi(r \cdot (m_i)_{i\in I}) &= \varphi((r \cdot m_i)_{i\in I}) \\
&= \sum_{\substack{i\in I\\ r\cdot m_i\neq 0}} \varphi_i(r \cdot m_i) \\
&= \sum_{\substack{i\in I\\ m_i\neq 0}} \varphi_i(r \cdot m_i) \\
&= \sum_{\substack{i\in I\\ m_i\neq 0}} r \cdot \varphi_i(m_i) \\
&= r \cdot \sum_{\substack{i\in I\\ m_i\neq 0}} \varphi_i(m_i) \\
&= r \cdot \varphi((m_i)_{i\in I})
\end{aligned}$$

$\square$

One of the main differences between modules over any ring and vector spaces (i.e., modules over a field) is that modules in general do not have bases.

Definition 1.30. [3, p. 330] Let R be a fixed ring. A *basis* of an R-module M is a subset $B \subseteq M$ that is R-linearly independent and generates M. In other words,

$$\sum_{b \in B} r_b \cdot b = 0 \implies \forall b \in B \colon r_b = 0$$

where only finitely many of the coefficients are nonzero (i.e., almost all zero), and any $m \in M$ can be expressed as an R-linear combination of elements from B.

A module with a basis is said to be *free*.

Another class of well-behaved modules is that of the finitely generated modules.

Definition 1.31. Let R be a fixed ring. An R-module M is said to be *finitely generated* if there is a finite subset $C \subseteq M$ that generates M.

Example 1.32. Consider the regular module $\mathbb{Z}$ and any non-trivial and proper ideal and submodule, $9\mathbb{Z}$ say. Then consider the quotient module $\mathbb{Z}/9\mathbb{Z}$. It is not the zero module, so any generating subset would have to be non-empty. On the other hand, any non-empty set is linearly dependent, because 9 acts on any element of $\mathbb{Z}/9\mathbb{Z}$ by sending it to the zero element. Therefore, $\mathbb{Z}/9\mathbb{Z}$ is a module without any basis; it is not free.

On the other hand, $\mathbb{Z}/9\mathbb{Z}$ is finite and hence in particular finitely generated (by itself).

The following proposition characterizes free modules as modules isomorphic to some number of copies of the regular module. The notation $R^{(i)}$ is informally to be understood as the ith copy of R but is formally interchangeable with just R.

Proposition 1.33. *[3, p. 330] An R-module M is free if and only if it is isomorphic to a direct sum $\bigoplus_{i \in I} R^{(i)}$ for some index set I.*

Proof. For the "only if" part, let M be a free R-module with basis B. Let $I = B$ and consider the map $\varphi \colon \bigoplus_{b \in B} R^{(b)} \to M$ defined by

$$\varphi((r_b)_{b \in B}) = \sum_{\substack{b \in B \\ r_b \neq 0}} r_b \cdot b$$

Intuitively, φ is the map sending a coordinate tuple to the vector with said coordinates in the basis B.

It is straightforward to verify from the module axioms that φ is an R-module morphism. For additivity,

$$\begin{aligned}
\varphi((r_b)_{b \in B} + (r_b)_{b \in B}) &= \varphi((r_b + s_b)_{b \in B}) \\
&= \sum_{\substack{b \in B \\ r_b + s_b \neq 0}} (r_b + s_b) \cdot b \\
&= \sum_{\substack{b \in B \\ r_b + s_b \neq 0}} (r_b \cdot b + s_b \cdot b) \\
&= \sum_{\substack{b \in B \\ r_b \neq 0}} r_b \cdot b + \sum_{\substack{b \in B \\ s_b \neq 0}} s_b \cdot b \\
&= \varphi((r_b)_{b \in B}) + \varphi((s_b)_{b \in B})
\end{aligned}$$

and for homogeneity,

$$\begin{aligned}
\varphi(r \cdot (r_b)_{b\in B}) &= \varphi((r \cdot r_b)_{b\in B}) \\
&= \sum_{\substack{b\in B \\ r\cdot r_b \neq 0}} (r \cdot r_b) \cdot b \\
&= \sum_{\substack{b\in B \\ r_b \neq 0}} (r \cdot r_b) \cdot b \\
&= \sum_{\substack{b\in B \\ r_b \neq 0}} r \cdot (r_b \cdot b) \\
&= r \cdot \sum_{\substack{b\in B \\ r_b \neq 0}} (r_b \cdot b) \\
&= r \cdot \varphi((r_b)_{b\in B})
\end{aligned}$$

where one may note that all the sums above are finite and hence well-defined. Furthermore, it is immediate that the morphism is surjective seeing as B generates M. Similarly, the linear independence of B implies that the kernel of φ is trivial, i.e., that φ is injective. Thus φ is an isomorphism and the "only if" part has been shown.

For the "if" part, let $\varphi\colon \bigoplus_{i\in I} R^{(i)} \to M$ (for some set I) be an isomorphism of R-modules. Note that the direct sum has a basis $\{e_i\}_{i\in I}$ (called the *standard basis* or the *canonical basis*), where e_i is the tuple $e_i = (\delta_{ij})_{j\in I}$ whose ith component is the unity 1_R and the other components are zero: $\{e_i\}_{i\in I}$ is linearly independent because the components of any linear combination (with almost all coefficients zero) are precisely the coefficients of the linear combination, so if the linear combination is zero then so are all the coefficients. Furthermore, $\{e_i\}_{i\in I}$ spans $\bigoplus_{i\in I} R^{(i)}$ seeing as any tuple $(r_i)_{i\in I}$ with finite support may be written as the linear combination

$$\sum_{\substack{i\in I \\ r_i \neq 0}} r_i \cdot e_i$$

Note further that the standard basis induces a basis of M via the isomorphism φ: $B' = \{\varphi(e_i)\}_{i\in I}$: That B' generates M follows from the surjectivity of φ by given an element $m \in M$ expressing its preimage $\varphi^{-1}(m)$ as a linear combination $\varphi^{-1}(m) = \sum r_i \cdot e_i$ of basis elements of B, applying φ to get $m = \varphi(\sum r_i \cdot e_i)$, and pulling out the addition and R-action to get $m = \sum r_i \cdot \varphi(e_i)$. The linear independence of B' follows from the injectivity of φ by given a linear combination $\sum r_i \cdot \varphi(e_i) = 0$, pushing in the addition and R-action to get $\varphi(\sum r_i \cdot e_i) = 0$, by injectivity deducing that $\sum r_i \cdot e_i = 0$, and by the linear independence of B concluding that all r_i are zero. □

Example 1.34. For any fixed non-trivial ring R and infinite index set I, the direct sum $\bigoplus_{i\in I} R^{(i)}$ of regular modules is free by Proposition 1.33 but not finitely generated: for any finite subset $C \subseteq \bigoplus_{i\in I} R^{(i)}$, the union of the support of every tuple is finite. Thus there is an index $i \in I$ whose component is 0 in every linear combination of the tuples in C, which shows that C fails to generate any element of the form $\iota_i(r) \in \bigoplus_{i\in I} R^{(i)}$ for $r \neq 0$.

2 Categories

In this section, we will define and explore the basic properties of categories – the main concept of study in this report.

A category is on one hand just an algebraic structure like any other, such as the group and the ring, but on the other hand, it is reasonable to think of categories as more abstract than most algebraic structures because categories are designed to house algebraic structures of other types (even categories themselves). Accordingly, category theory can be used to express the relationships between different types of algebraic structures formally.

There are uses for categories outside of abstract algebra, in particular in algebraic topology from where the notion of a category originated in the 1940s [5, p. 29], but we will concern ourselves mostly with purely algebraic uses.

2.1 Basic constructions

Definition 2.1. [4, p. 40] A *category* $\mathcal{C}$ consists of three pieces of data:

(1) a class ob $\mathcal{C}$ of *objects*,

(2) for each pair of objects $X, Y \in \text{ob } \mathcal{C}$, a set $\mathcal{C}(X, Y)$ of *morphisms*, and

(3) a *law of composition* (of morphisms) consisting of *composition functions*

$$\begin{aligned} \circ \colon \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) &\to \mathcal{C}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

for each triple of objects $X, Y, Z \in \text{ob } \mathcal{C}$,

subject to the following three axioms:

(C0) the morphism sets are pairwise disjoint, i.e., for any distinct pairs (X, Y) and (W, Z) of objects,

$$\mathcal{C}(X,Y) \cap \mathcal{C}(W,Z) = \varnothing,$$

(C1) the law of composition is associative, i.e.,

$$f \circ (g \circ h) = (f \circ g) \circ h$$

for any composable morphisms f, g, and h (i.e., where the compositions just above are defined), and

(C2) for each object X, there is a morphism $1_X \in \mathcal{C}(X, X)$, called the *identity morphism* on X, that is a left and right inverse of the law of composition, i.e., satisfying

$$g \circ 1_X = g$$

and

$$1_X \circ f = f$$

for any morphisms f and g where the compositions above are defined.

Remark 2.2. The morphisms in a category are in many cases functions and the law of composition is often given by function composition. This need not be the case though (see Example 2.10); strictly speaking, the objects, morphisms, and composition functions of the above definition are undefined terms, and none of the axioms require the morphisms to be functions.

Adhering to the common case, however, we shall use the notation $f\colon X \to Y$ for a morphism $f \in \mathcal{C}(X, Y)$ and speak of the *domain* X and *codomain* Y as if f were a function. Sometimes, we shall also write $X \xrightarrow{f} Y$ for the morphism $f\colon X \to Y$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ for the composition $g \circ f$ of the morphisms $f\colon X \to Y$ and $g\colon Y \to Z$.

Remark 2.3. We shall whenever deemed fit use the shorthand notations $X \in \mathcal{C}$ to mean $X \in \operatorname{ob} \mathcal{C}$, i.e., that X is an object of the category, and gf to mean the composition $g \circ f$ of the morphisms g and f.

At times, we will use the "longhand" notation $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ for the morphism set $\mathcal{C}(X, Y)$.

Remark 2.4. There are several definitions of a category in the literature. While they are similar in spirit, there are subtle and technical differences between them. The above definition is that given by Hilton and Stammbach ([4]) and relies on the notion of classes in addition to sets, whereas for instance the definition given by Mac Lane ([5]) assumes only the Zermelo-Fraenkel set theory and the existence of an appropriate "universe". The enquiring reader is referred to Mac Lane and Grillet ([5, 3]) for their preferred definitions of a category as well as lengthier discussions on alternative definitions.

Remark 2.5. [4, p. 41] It follows readily from the associativity of the composition that the identity morphism is unique for any object. We may thus speak of *the* identity morphism 1_X for an object X.

Example 2.6. The category of all sets is the canonical example of a category, and we denote it by Sets. Its objects are all sets, its morphisms are all the functions, i.e.,

$$\operatorname{Sets}(X, Y) = \{f \mid f \text{ is a function from } X \text{ to } Y\}$$

for any sets X and Y, and the law of composition is given by the usual function composition.

Axiom (C1) follows from the associativity of function composition and (C2) holds with the identity maps as identity morphisms. However, with the usual set-theoretic convention that functions are certain collections of ordered pairs, Sets as defined above fails to satisfy (C0): the empty function $\varnothing$, whose domain is the empty set, appears in more than one morphism set. By instead considering functions to be equipped with their domain and codomain, i.e., each as a tuple (X, f, Y) where f is the usual ordered pairs of the function, X is the domain of f, and Y is the codomain of f, the axiom (C0) is ensured to hold.

In light of equipping a morphism with its domain and codomain, we may in great generality view (C0) as more of a convention than an axiom [5, p. 27].

Example 2.7. For a fixed ring R, the category of all R-modules, which we denote by R-Mod, is the category whose objects are all the R-modules, morphisms are all the R-module morphisms, and law of composition is given by the usual composition of functions. We verify the axioms by noting that the law of composition is associative seeing as function composition is associative, and the identity R-module morphism is a two-sided identity with respect to function composition.

In the special cases that R is some field K and, respectively, R is $\mathbb{Z}$, we may (according to Remarks 1.2 and 1.7) speak of the category $\operatorname{Vec} K$ of all vector spaces over K and the category Ab of all abelian groups. To be pedantic, $\mathbb{Z}$-Mod and Ab are not equal (seeing as a $\mathbb{Z}$-module is not equal to its underlying abelian group) but, as we shall see in Example 2.62, they are isomorphic as categories (see Definition 2.60).

These categories will be investigated in detail throughout this report, for we shall that they have very useful properties with respect to homological algebra.

Example 2.8. The idea from the previous example of collecting all algebraic structures of a certain kind together with their homomorphisms and law of composition given by function composition gives rise to many categories, for instance

- the category Grp of all groups,
- the category Rings of all rings (with unity) with homomorphisms of rings (respecting the unity), and

- the category Rngs of all rings with or without unity and homomorphisms that may or may not respect the unity.

Categories are not limited to purely algebraic structures as seen in the next example.

Example 2.9. [5, p. 12] The category Top is the category of all topological spaces with continuous maps (equipped with their domain and codomain) as morphisms and the usual function composition as law of composition. It is well-known from topology that the (function) composition of continuous maps is continuous and that the identity map is continuous, which shows that the category axioms hold in Top.

Similarly, we may consider all pointed spaces (i.e., topological spaces, each together with one of its points as a distinguished basepoint) with continuous maps that preserve the basepoints (again equipped with the domain and codomain). The composition of basepoint-preserving maps preserve basepoints, for if $f\colon X \to Y$ and $g\colon Y \to Z$ are basepoint-preserving maps for the pointed spaces (X, x), (Y, y), and (Z, z), then the composition $g \circ f$ maps x to $g(f(x)) = g(y) = z$. This consideration thus gives rise to a category: the category of all pointed spaces, which we may denote by Top_*.

In contrast to the examples given so far, a category need not contain sets (possibly with additional structure) as objects and their structure-preserving functions as morphisms.

Example 2.10. [4, p. 44] Any group G may be viewed as a category $\mathcal{C}_G$ consisting of a single object $*$, a morphism $\hat{g}\colon * \to *$ for every group element $g \in G$, and composition given by the group multiplication:

$$\hat{g} \circ \hat{h} := \widehat{gh}$$

This is a well-defined category, seeing as the associativity of the group multiplication ensures that the composition is associative and the identity element e of the group corresponds to the identity morphism $1_* = \hat{e}$.

The category finds a use in group representation theory, where a representation of the group G may be defined as a functor (see Definition 2.52) from $\mathcal{C}_G$ to the category $\mathrm{Vec}\,K$ of all vector spaces over some field K.

Remark 2.11. [5, p. 11] Note that the existence of inverses in the group was not used in the previous example. In fact, any *monoid* can be represented by a category in this way. Moreover, there is a converse: any category with a single object induces a monoid of the morphisms with operation given by composition. If all the morphisms are isomorphisms, then the induced monoid is a group. This converse can be generalized: in a category $\mathcal{C}$ (with possibly several objects), every endomorphism set, i.e., morphism set of the form $\mathcal{C}(X, X)$, together with the morphism composition on the set is a monoid. If all the endomorphisms are isomorphisms, then the monoid is a group.

In other words, monoids may be thought of as precisely categories with a single object and groups may be thought of as precisely categories with a single object and all morphisms isomorphisms. From the opposite perspective, one may understand categories as generalized monoids.

For the algebraic structures of Example 2.8, isomorphisms give rise to the notion of "essentially" equal objects. Homeomorphisms fill the same role for topological spaces. This notion can be defined in the setting of categories as seen below.

Definition 2.12. [4, p. 42] A morphism $f\colon X \to Y$ in a category is called an *isomorphism* if it is invertible, i.e., if there is a morphism $g\colon Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, the morphism g is called an *inverse* of f.

If there is an isomorphism with domain X and codomain Y, we say that the objects X and Y are *isomorphic* and write $X \cong Y$.

Remark 2.13. [4, p. 42] The inverse $g\colon Y \to X$ of a morphism $f\colon X \to Y$, if it exists, is unique, for if $h\colon Y \to X$ is also an inverse of f, then

$$g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$$

Thus we may speak of *the* inverse of an invertible morphism f and denote it by f^{-1}.

Remark 2.14. A composition $g \circ f\colon X \to Z$ of isomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is an isomorphism. More specifically, its inverse is readily seen to be $(g \circ f)^{-1} = f^{-1} \circ g^{-1}\colon Z \to X$, as depicted below.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & Z \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

Example 2.15. In the category of all sets, the isomorphisms are precisely all the invertible functions, i.e., all the bijections.

The categorical notion of an isomorphism coincides with isomorphisms for the algebraic structures of the previous examples. That is, the isomorphisms of R-Mod (for some fixed ring R) are precisely the R-module isomorphisms, the isomorphisms of Grp are precisely the group isomorphisms, and so on. This follows readily from the fact that the inverse of an R-module morphism is an R-module morphism (see Proposition 1.20) and the corresponding results for the other algebraic structures.

In a similar spirit, the isomorphisms of Top are (by definition) precisely the homeomorphisms.

It should be noted that while isomorphisms for many algebraic structures are defined as bijective homomorphisms, this definition does not generalize to the setting of categories. In Top there are morphisms that, as functions, are bijective but whose inverse is not continuous, i.e., has no inverse morphism. Still worse, it does not even make sense to speak of *bijective* morphisms in a category whose morphisms are not functions. With this observation in mind, it would make sense to view Proposition 1.20 as the definition of an R-module isomorphism and the part of Definition 1.13 about isomorphisms as a consequence of the definition.

We may also generalize the notion of monomorphisms and epimorphisms to the setting of categories.

Definition 2.16. [4, p. 48] A morphism $f\colon X \to Y$ is called a

- *monomorphism* or is said to be *monic* if it is left-cancellative, i.e., if $f \circ g = f \circ h$ implies $g = h$ for any morphisms g and h composable thusly .
- *epimorphism* or is said to be *epic* if it is right-cancellative, i.e., if $g \circ f = h \circ f$ implies $g = h$ for any morphisms g and h composable thusly.

Example 2.17. In the category Sets of all sets with all functions, the monomorphisms and epimorphisms are precisely the injective and, respectively, surjective functions. That is, when all functions are available, injectivity and surjectivity is necessary and sufficient for left- and right-cancellativity (with respect to function composition).

In categories with functions and function composition as morphisms and law of composition but where not all functions are morphisms, it is plain that injectivity and surjectivity is still a sufficient condition for being monic and epic, respectively. In the case of R-Mod, injectivity and surjectivity are also necessary conditions, which we formulate in Proposition 2.20. For an example where surjectivity is not a necessary condition for a morphism to be epic, consider the category Rings and the inclusion morphism

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Q} \\ n &\mapsto n \end{aligned}$$

which is epic but not surjective [4, p. 50].

Remark 2.18. The composition of two monomorphisms i and j is readily shown to be monic:

$$\begin{aligned}(j \circ i) \circ g = (j \circ i) \circ h &\implies j \circ (i \circ g) = j \circ (i \circ h) \\ &\implies i \circ g = i \circ h \\ &\implies g = h\end{aligned}$$

Similarly, the composition of two epimorphisms is epic.

Conversely, if a composition $j \circ i$ is monic, then the first morphism, i, is necessarily monic:

$$\begin{aligned}i \circ g = i \circ h &\implies j \circ i \circ g = j \circ i \circ h \\ &\implies g = h\end{aligned}$$

Similarly, if a composition is epic, then the *last* morphism is necessarily epic.

Remark 2.19. Note that any isomorphism $f\colon X \to Y$ is monic and epic seeing as we may compose by the inverse to obtain the necessary equality:

$$f \circ g = f \circ h \implies f^{-1} \circ f \circ g = f^{-1} \circ f \circ h$$

where the latter equality simplifies to $1_X \circ g = 1_X \circ h$ and so to $g = h$. Right-cancellativity follows in the same manner.

The converse, that monic epimorphisms are isomorphisms, does not hold in general, as witnessed by the monic and epic inclusion morphism $\iota\colon \mathbb{Z} \to \mathbb{Q}$ in Example 2.17 that is not an isomorphism. That said, we will see that the converse does hold in particularly well-behaved categories (see Proposition 4.12).

Much like the categorical notion of an isomorphism in the case of R-Mod turned out to coincide with that of an R-module isomorphism as defined in Definition 1.13, the categorical notions of monomorphisms and epimorphisms agree with those defined previously for R-modules.

Proposition 2.20. *[4, p. 29] In R-Mod, a morphism f is*

- *a monomorphism (in the categorical sense) if and only if it is injective (i.e., a monomorphism in the module sense), and*
- *an epimorphism (in the categorical sense) if and only if it is surjective (i.e., an epimorphism in the module sense).*

Proof. We noted in the previous example that injectivity and surjectivity are sufficient conditions, which leaves to verify that monomorphisms are injective and epimorphisms are surjective.

Suppose for a proof by contraposition that $f\colon M \to N$ is a non-injective morphism of R-modules, and try to find two distinct morphisms g and h that disprove left-cancellativity of f, i.e., with $f \circ g = f \circ h$.

Because f is non-injective, its kernel $\ker f$, which we recall is a submodule of M, is non-trivial. This is to say that the zero morphism $0\colon \ker f \to M$ and the canonical inclusion morphism $\iota\colon \ker f \to M$ are distinct. Their compositions $f \circ 0$ and $f \circ \iota$ by f to the left are both the zero morphism however, seeing as ι maps every element into the kernel of f. In particular, the compositions are equal, which proves that f is not a monomorphism.

The corresponding statement for epimorphisms can be proved in a dual fashion. Suppose for a proof by contraposition that $f\colon M \to N$ is a non-surjective morphism of R-modules, and find two distinct morphisms g and h that disprove right-cancellativity of f, i.e., with $g \circ f = h \circ f$.

Because f is non-surjective, its cokernel $\operatorname{cok} f$ is non-trivial. This is to say that the zero morphism $0\colon N \to \operatorname{cok} f$ and the canonical projection morphism $\pi\colon N \to \operatorname{cok} f$ are distinct. Their compositions $0 \circ f$ and $\pi \circ f$ by f to the right are both the zero morphism however, seeing as f maps every element into $\operatorname{im} f = \ker \pi$. In particular, the compositions are equal, which proves that f is not an epimorphism. □

Definition 2.21. [4, p. 43] An object Y of a category $\mathcal{C}$ is said to be

- *initial* if $\mathcal{C}(Y, Z)$ is singleton (i.e., contains precisely one morphism) for every $Z \in \mathcal{C}$, and
- *terminal* if $\mathcal{C}(X, Y)$ is singleton for every $X \in \mathcal{C}$.

If Y is both initial and terminal, Y is said to be a *zero object*.

Remark 2.22. A category need not have initial or terminal objects (see Example 2.10 for instance), and even if it does, there might be no zero object. For an example of the latter, consider the category Sets, where the empty set is the only initial object and the singleton sets are the only terminal objects. However, we shall see in Section 3 that initial and terminal objects must coincide in so-called preadditive categories, which rules out the latter situation above.

Remark 2.23. [4, p. 43] It is readily shown from the definition of an isomorphism that any two initial objects are isomorphic and that any two terminal objects are isomorphic. In particular, any two zero objects are isomorphic. We may thus sloppily refer to a zero object as *the* zero object and denote it by 0 in situations where it is immaterial which zero object is referred to.

Example 2.24. Consider the category Rings of all rings with unity and recall that its morphisms are assumed to respect the unity.

The ring $\mathbb{Z}$ with the usual addition and multiplication is an initial object in Rings. It is simple to see that there is at most one morphism φ from $\mathbb{Z}$ to any ring R, by using the fact that $\mathbb{Z}$ is generated additively by its unity: such a φ would satisfy $\varphi(1_{\mathbb{Z}}) = 1_R$. Additivity of φ, which also implies that φ respects negation and the zero element, then forces φ to be defined by

$$\varphi(n) = \begin{cases} \overbrace{1_R + \cdots + 1_R}^{n \text{ terms}} & \text{if } n > 0 \\ 0_R & \text{if } n = 0 \\ -(\underbrace{1_R + \cdots + 1_R}_{-n \text{ terms}}) & \text{if } n < 0 \end{cases}$$

for every $n \in \mathbb{Z}$, much like what we had for the $\mathbb{Z}$-action on any abelian group in Remark 1.7. Verifying that this indeed defines a homomorphism of rings, in particular additivity and multiplicativity of φ, can be done through a painstaking and painful consideration of cases depending on the sign of the elements of $\mathbb{Z}$, also much like for the $\mathbb{Z}$-action.

As for a terminal object in Rings, consider the zero ring $\{0\}$. There is only a single map into $\{0\}$, namely the zero map, and it is trivially a homomorphism of rings (recall that the unity coincides with the zero element in the zero ring). Thus, $\{0\}$ is a terminal object in Rings.

Note that the initial object $\mathbb{Z}$ and the terminal object $\{0\}$ are not isomorphic. By Remark 2.23, we conclude that Rings has no zero object.

Zero objects give rise to a categorical definition of zero morphisms.

Definition 2.25. [4, p. 43] In a category with a zero object 0, we define for any two objects X and Y the *zero morphism* from X to Y as the unique morphism from X to Y via the zero object 0:

$$0_{XY} = X \to 0 \to Y$$

i.e., as the composition of the unique morphisms $X \to 0$ and $0 \to Y$.

Remark 2.26. The zero morphism 0_{XY} as defined above seems to depend on the zero object 0. One may see that it ultimately does not by writing the zero morphism $\hat{0}_{XY}$ with respect to another zero object $\hat{0}$ as

$$\begin{aligned}\hat{0}_{XY} &= X \to \hat{0} \to Y \\ &= (X \to 0) \to \hat{0} \to Y \\ &= X \to 0 \to (\hat{0} \to Y) \\ &= X \to 0 \to Y \\ &= 0_{XY}\end{aligned}$$

Thus we may indeed speak of *the* zero morphism between any two objects X and Y in a category with zero objects and adopt the notation 0_{XY} or sometimes just 0 for said morphism.

Remark 2.27. A zero morphism composes into a zero morphism in the sense that $0_{XY} \circ f = 0_{WY}$ and $g \circ 0_{XY} = 0_{XZ}$ for any morphisms $f \colon W \to Y$ and $g \colon Y \to Z$. This is readily seen diagrammatically as

$$\begin{aligned}W \xrightarrow{f} X \xrightarrow{0_{XY}} Y &= W \xrightarrow{f} (X \to 0 \to Y) \\ &= (W \xrightarrow{f} X \to 0) \to Y \\ &= W \to 0 \to Y \\ &= W \xrightarrow{0_{WY}} Y\end{aligned}$$

and

$$\begin{aligned}X \xrightarrow{f} Y \xrightarrow{0_{YZ}} Z &= X \xrightarrow{f} (Y \to 0 \to Z) \\ &= (X \xrightarrow{f} Y \to 0) \to Z \\ &= X \to 0 \to Z \\ &= X \xrightarrow{0_{XZ}} Z\end{aligned}$$

Example 2.28. The zero module $\{0\}$ is a zero object in R-Mod for any ring R. It is initial, because for any other module N the only element of the zero module must by additivity be mapped to the zero element of N, and this defines an R-module morphism. It is terminal because the only function from a module M into $\{0\}$ is the function mapping all of M to 0, which is an R-module morphism.

The zero morphism $0_{MN} \colon M \to N$ in the categorical sense is thus the function mapping all of M to the zero element of N, which is precisely the zero morphism in the module sense.

2.2 Duality

One may note that there is a sense of duality in some of the definitions in Section 2.1; the property of being a monomorphism is "dual" to that of being an epimorphism (and of course vice versa), and being initial is "dual" to being terminal. This notion of duality can be exploited to obtain properties about a "concept" (e.g., epimorphisms) given properties about the dual concept (monomorphisms). In an effort to make this more precise, we consider the opposite category.

Definition 2.29. [4, p. 46] Let $\mathcal{C}$ be a category. The *opposite category* (or *dual category*) of $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ with the same objects but with all morphisms reversed and the law of composition modified accordingly. In other words,

$$\mathrm{ob}\,\mathcal{C}^{\mathrm{op}} := \mathrm{ob}\,\mathcal{C}$$

and for any objects X and Y of the opposite category

$$\mathcal{C}^{\mathrm{op}}(X, Y) := \mathcal{C}(Y, X)$$

with composition in $\mathcal{C}^{\text{op}}$ given for any morphisms $g\colon Z \to Y$ and $f\colon Y \to X$ in $\mathcal{C}^{\text{op}}$ by

$$f \circ_{\mathcal{C}^{\text{op}}} g := g \circ_{\mathcal{C}} f$$

where the morphisms of the right-hand side are $f\colon X \to Y$ and $g\colon Y \to Z$ and the composition is done in the original category $\mathcal{C}$. Diagrammatically, we may view morphisms in the opposite category as going from right to left and get

$$X \xleftarrow{f} Y \xleftarrow{g} Z := X \xrightarrow{f} Y \xrightarrow{g} Z$$

describing the definition.

Remark 2.30. Consider the identity morphism 1_X for some object X in a category $\mathcal{C}$. By the definition of the composition in the opposite category, 1_X is an identity with respect to composition and hence the identity morphism for X in $\mathcal{C}^{\text{op}}$ as well. In other words, $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ have the same identity morphisms, and no ambiguity arises from the notation 1_X when passing between a category and its dual.

We may now informally define the notion of duality (see [5, p. 31] for an exposition using formal logic and [4, p. 48] for a lengthier informal description): Let $P(\mathcal{C})$ be a statement about a category $\mathcal{C}$, for instance "f is a monomorphism". Seeing as the opposite category $\mathcal{C}^{\text{op}}$ contains the same objects and morphisms, we may consider the same statement (in general with domains and codomains swapped as well as compositions reversed) in $\mathcal{C}^{\text{op}}$: "f (in $\mathcal{C}^{\text{op}}$) is a monomorphism". Now interpret what this statement $P(\mathcal{C}^{\text{op}})$ about the opposite category means for the original category $\mathcal{C}$. This is by definition the opposite or dual statement $P^{\text{op}}(\mathcal{C})$ for $\mathcal{C}$. In other words, the statement and its dual are linked by the following informal rule:

$$P^{\text{op}}(\mathcal{C}) \iff P(\mathcal{C}^{\text{op}})$$

By the morphism-reversing nature of the opposite category, this is to say that the dual statement is obtained by reversing all morphisms and adjusting the compositions accordingly.

Remark 2.31. Note that the opposite (as a "map" on categories) has the property of an involution in the sense that the opposite of an opposite category is the original category: $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$. It follows that the opposite of an opposite statement is the original statement:

$$(P^{\text{op}})^{\text{op}}(\mathcal{C}) \iff P^{\text{op}}(\mathcal{C}^{\text{op}}) \iff P((\mathcal{C}^{\text{op}})^{\text{op}}) \iff P(\mathcal{C})$$

A few examples are in order to clarify this notion of duality.

Example 2.32. Consider the property of an object Y to be initial in a category $\mathcal{C}$, and recall that this is to say that there is a unique morphism $Y \to Z$ for every object $Z \in \mathcal{C}$. In order to find the dual property, bethink what it means for Y to be initial in the opposite category $\mathcal{C}^{\text{op}}$ and rephrase this in terms of the original category $\mathcal{C}$: Y is initial in $\mathcal{C}^{\text{op}}$ if and only if there is a unique morphism $Y \to X$ (in $\mathcal{C}^{\text{op}}$) for every object $X \in \text{ob}\ \mathcal{C}^{\text{op}}$, which we may informally depict as

$$\forall X \xleftarrow{\exists!} Y$$

In the original category $\mathcal{C}$, where the arrows are reversed compared to the opposite category, this may be expressed equivalently as there being a unique morphism $X \to Y$ (in $\mathcal{C}$) for every object $X \in \text{ob}\ \mathcal{C}^{\text{op}} = \text{ob}\ \mathcal{C}$, which we may depict as

$$\forall X \xrightarrow{\exists!} Y$$

and recognize as the statement that Y is a terminal object in $\mathcal{C}$. We conclude that the properties of being initial and terminal are dual.

One may note that the property of being a zero object is self-dual, seeing as an initial and terminal object in the opposite category is a terminal and initial object of the original category; an object of a category is a zero object if and only if it is a zero object of the opposite category. Pictorially, this corresponds to the fact that the diagram expressing that an object Y is a zero object is symmetric with respect to the reversal of all arrows: compare

$$\forall X \xrightarrow{\exists!} Y \xrightarrow{\exists!} \forall Z$$

and

$$\forall X \xleftarrow{\exists!} Y \xleftarrow{\exists!} \forall Z$$

Example 2.33. Recall the definition of a morphism $f\colon X \to Y$ being a monomorphism in some category $\mathcal{C}$: for any two morphisms $g, h\colon W \to X$ (in $\mathcal{C}$), the equality

$$W \xrightarrow{g} X \xrightarrow{f} Y = W \xrightarrow{h} X \xrightarrow{f} Y$$

implies that $g = h$. Now to find the dual statement of f being a monomorphism (in $\mathcal{C}$), consider what it means for f to be a monomorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$ (where f is a morphism from Y to X): for any two morphisms $g, h\colon Z \to Y$ (in $\mathcal{C}^{\mathrm{op}}$), the equality

$$X \xleftarrow{f} Y \xleftarrow{g} Z = X \xleftarrow{f} Y \xleftarrow{h} Z$$

implies that $g = h$. Interpreted in the original category (i.e., with the morphisms reversed and the composition adjusted accordingly compared to above), this is the familiar statement that f is an epimorphism in $\mathcal{C}$: for any two morphisms $g, h\colon Y \to Z$ (in $\mathcal{C}$), the equality

$$X \xrightarrow{f} Y \xrightarrow{g} Z = X \xrightarrow{f} Y \xrightarrow{h} Z$$

implies that $g = h$. We conclude that monomorphisms and epimorphisms are dual concepts.

Example 2.34. The notion of an isomorphism is self-dual. To see this, recall that a morphism $f\colon Y \to X$ in the opposite $\mathcal{C}^{\mathrm{op}}$ of some category $\mathcal{C}$ is an isomorphism if and only if it has an inverse $g\colon X \to Y$, with $g \circ f = 1_Y$ and $f \circ g = 1_X$, which may be expressed as the commutative diagram

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, bend right, dotted, "\exists g"'] & Y \arrow[l, bend right, "f"'] \arrow[loop right, "1_Y"] \end{tikzcd}$$

This yields in the original category $\mathcal{C}$ the commutative diagram

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, dotted, "\exists g"] \arrow[loop right, "1_Y"] \end{tikzcd}$$

which is precisely to say that f is an isomorphism in $\mathcal{C}$. That is, a morphism is an isomorphism in some fixed category if and only if it is an isomorphism in the opposite category. Moreover, one may note that if a morphism g completes either of the two diagrams above, then it completes the other as well. That is, the inverse $g = f^{-1}$ of an isomorphism f in some category is the inverse of f in the opposite category as well.

Duality will be used extensively in the remainder of this document in order to minimize the redundancy in proofs. More concretely, propositions will typically consist of two parts: one part of the form "$P(\mathcal{C})$ for every category $\mathcal{C}$" about some property P and one part of the form "$P^{\mathrm{op}}(\mathcal{C})$ for every category $\mathcal{C}$" about the equally interesting dual property P^{op}. The first part will be proved directly. Rather than proving the second part directly (which would amount to repeating the same argument with morphisms and compositions reversed), one may note that $P^{\mathrm{op}}(\mathcal{C})$ and $P(\mathcal{C}^{\mathrm{op}})$ are equivalent for any category $\mathcal{C}$ and that the latter holds by the first part.

2.3 Kernels and cokernels

Yet another algebraic concept that may be formulated in category theory, albeit in a slightly different form than one might expect, is that of the kernel and cokernel of a morphism.

Definition 2.35. [4, p. 50, 61] In a category with zero objects, a *kernel* of a morphism $f\colon X \to Y$ is a pair (K, k) of an object K (a *kernel object*) and a morphism $k\colon K \to X$ (a *kernel morphism*) such that k

(1) composes with f to the zero morphism: $f \circ k = 0_{KY}$ and

(2) is a universal such morphism, in the sense that any other such morphism $l\colon L \to X$ factors through k as $l = k \circ g$ for some unique morphism $g\colon L \to K$.

Expressed in a commutative diagram, the situation is

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& K \arrow[ul, "k"] \arrow[ur, "0"'] & \\
& L \arrow[u, dotted, "\exists! g"] \arrow[uul, bend left, "l"] \arrow[uur, "0"'] &
\end{tikzcd}$$

Dually (i.e., by reversing all the arrows), a *cokernel* $f\colon Y \to X$ is a pair (C, c) of an object C (a *cokernel object*) and a morphism $c\colon X \to C$ (a *cokernel morphism*) such that c

(1) composes with f to the zero morphism (in the opposite order compared to above): $c \circ f = 0_{YC}$ and

(2) is a universal such morphism in the sense that any other such morphism $d\colon X \to D$ factors through c as $d = g \circ c$ for some unique morphism $g\colon C \to D$.

Expressed in a commutative diagram,

$$\begin{tikzcd}
X & & Y \arrow[ll, "f"'] \arrow[dl, "0"] \arrow[ddl, "0"] \\
& C \arrow[d, dotted, "\exists! g"'] & \\
& D &
\arrow[from=1-1, to=2-2, "c"'] \arrow[from=1-1, to=3-2, bend right, "d"']
\end{tikzcd}$$

Remark 2.36. We may for brevity refer to either of the kernel object and the kernel morphism as just the "kernel" whenever the context leaves no room for confusion (which is how Grillet defines the kernel [3, p. 602]) and similarly for cokernels.

Proposition 2.37. *[4, p. 61] Kernels are monomorphisms.*

Proof. Let $k\colon K \to X$ be a kernel of a morphism $f\colon X \to Y$. We need to prove that k is left-cancellative, i.e., that

$$k \circ g = k \circ h \implies g = h$$

for any morphisms $g, h\colon L \to K$ for some object L.

Note that f composes with both $k \circ g$ and $k \circ h$ to the zero morphism $0 = f \circ (k \circ g) = f \circ (k \circ h)$, seeing as $f \circ k = 0$ by the definition of the kernel (recall from Remark 2.27 that a zero morphism composes into zero morphisms). By setting $l = k \circ g = k \circ h$, we may depict the situation as the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& K \arrow[ul, "k"'] \arrow[ur, "0"] & \\
& L \arrow[u, "{g,h}"] \arrow[uul, bend left, "l"] \arrow[uur, "0"'] &
\end{tikzcd}$$

which is reminiscent of the diagram in the definition of the kernel. The uniqueness property of the kernel dictates that g and h are equal, and the proof is finished. $\square$

As the next proposition shows, kernels of a morphism are unique up to an isomorphism of sorts, which lets us speak of *the* kernel of a morphism.

Proposition 2.38. *If $k\colon K \to X$ and $k'\colon K' \to X$ are kernels of a morphism $f\colon X \to Y$, then there is a unique isomorphism $g\colon K' \to K$ such that $k' = k \circ g$. In other words, there is a unique isomorphism g making the following diagram commute:*

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& K \arrow[ul, "k"'] \arrow[ur, "0"] & \\
& K' \arrow[u, dotted, "\exists! g"] \arrow[uul, bend left, "k'"] \arrow[uur, "0"'] &
\end{tikzcd}$$

Proof. By the first criterion for the kernel, the kernels compose with f to the zero morphism: $fk = fk' = 0$. By the second criterion, the kernels then factor through each other as

$$k' = k \circ g, \quad k = k' \circ g'$$

for some unique morphisms $g\colon K' \to K$ and $g'\colon K \to K'$. This shows the uniqueness part for g of the proposition, but the existence is not quite established, seeing as g is required to be an isomorphism. We show this remaining part by proving that g and g' are mutual inverses. By the factorizations above, we have

$$k \circ g \circ g' = k' \circ g' = k = k \circ 1_K$$

and

$$k' \circ g' \circ g = k \circ g = k' = k' \circ 1_{K'}$$

We recall that the kernels are monomorphisms by Proposition 2.37 and conclude that

$$g \circ g' = 1_K, \quad g' \circ g = 1_{K'}$$

which finishes the proof. $\square$

Remark 2.39. As noted in the above proof, the direction of the isomorphism g of the proposition is immaterial: if $g\colon K' \to K$ is an isomorphism making the given diagram commute, then $g^{-1}\colon K \to K'$ is an isomorphism making the same diagram with g replaced by g^{-1} commute.

Given Propositions 2.37 and 2.38, we may exploit the duality between kernels and cokernels to obtain dual propositions with very little work.

Proposition 2.40. *Cokernels are epimorphisms.*

Proof. Consider a cokernel c of an arbitrary category $\mathcal{C}$. Seeing as cokernels are dual to kernels and epimorphisms are dual to monomorphisms, we have the following chain of implications

$$\begin{aligned} c \text{ is a cokernel in } \mathcal{C} &\iff c \text{ is a kernel in } \mathcal{C}^{\mathrm{op}} \\ &\implies c \text{ is a monomorphism in } \mathcal{C}^{\mathrm{op}} \\ &\iff c \text{ is an epimorphism in } \mathcal{C} \end{aligned}$$

where the second implication uses Proposition 2.37. $\square$

Proposition 2.41. *If $c\colon X \to C$ and $c'\colon X \to C'$ are cokernels of a morphism $f\colon Y \to X$, then there is a unique isomorphism $g\colon C \to C'$ making the following diagram commute:*

$$\begin{tikzcd} X \arrow[dr, "c"] \arrow[ddr, bend right, "c'"'] & & Y \arrow[ll, "f"'] \arrow[dl, "0"'] \arrow[ddl, "0"] \\ & C \arrow[d, dotted, "\exists! g"'] & \\ & C' & \end{tikzcd}$$

Proof. Let $\mathcal{C}$ denote the category implied in the statement of the proposition. In the opposite category $\mathcal{C}^{\mathrm{op}}$, c and c' are kernels of $f\colon X \to Y$. By Proposition 2.38 there is a unique isomorphism $g\colon C' \to C$ (in the opposite category) making the following diagram commute

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & C \arrow[ul, "c"'] \arrow[ur, "0"] & \\ & C' \arrow[u, dotted, "\exists! g"] \arrow[uul, bend left, "c'"] \arrow[uur, "0"'] & \end{tikzcd}$$

Because isomorphisms are self-dual, g is an isomorphism in the original category $\mathcal{C}$, and the condition that g in $\mathcal{C}^{\mathrm{op}}$ makes the above diagram commute is equivalent to g in $\mathcal{C}$ making the diagram in the proposition commute. We thus conclude that g (in $\mathcal{C}$) is the unique isomorphism making the diagram in the proposition commute. $\square$

Example 2.42. In R-Mod (for any ring R), any morphism admits both a kernel and a cokernel.

The kernel of a morphism $f\colon M \to N$ is essentially the kernel of the morphism in the module sense as defined in Section 1. To be specific, the kernel (in the categorical sense) is the inclusion morphism $\iota\colon \ker f \to M$ from the kernel in the module sense. One readily verifies that $f \circ \iota = 0$ and that any other morphism $l\colon L \to M$ with $f \circ l = 0$ factors through ι via the restriction of l (with respect to the codomain) $\hat{l}\colon L \to \ker f$.

The cokernel of a morphism $f\colon M \to N$ is essentially the cokernel of the morphism in the module sense. To be specific, the cokernel (in the categorical sense) is the quotient morphism $\pi\colon N \to \operatorname{cok} f$ to the cokernel in the module sense. One readily verifies that $\pi \circ f = 0$ and that any other morphism $d\colon N \to D$ with $d \circ f = 0$ (or equivalently, $\operatorname{im} f \subseteq \ker d$) factors through π via the unique morphism given by Proposition 1.23.

Remark 2.43. For the category Rings of all rings (with unity), there are no categorical kernels, seeing as Rings has no zero object (recall Example 2.24). That is to say that the ring-theoretic notion and the categorical notion of a kernel differ.

Example 2.44. As an example of a category with zero objects but where not all morphisms have a kernel, consider the category $\mathcal{C}$ of all even-dimensional vector spaces over some fixed field K with linear maps as morphisms and function composition as morphism composition. Let f denote the projection morphism $f\colon K^2 \to K^2$ of K^2 onto the subspace generated by the first standard basis vector $(1_K, 0_K) \in K^2$ (the fact that this subspace is not an object of $\mathcal{C}$ is of no matter; f is a morphism in $\mathcal{C}$ nevertheless), whose matrix in the standard bases is

$$[f] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This f has no categorical kernel. To see that this is so, note first that $\mathcal{C}$ and $\operatorname{Vec} K$ have the same zero objects and hence that a morphism in $\mathcal{C}$ is a zero morphism in $\mathcal{C}$ if and only if it is a zero morphism in $\operatorname{Vec} K$, which by Example 2.28 is precisely to say that the morphism is a zero map.

Suppose that $k\colon V \to K^2$ (for some even-dimensional vector space V) is a morphism that composes with f to zero, i.e., $f \circ k = 0$, and observe that k cannot be universal with this property. If V is zero-dimensional, then k would be a zero morphism and the nonzero morphisms composing with f to zero (e.g., the linear map $K^2 \to K^2$ with matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the standard bases) would not factor through k. If V is of dimension $\dim V = 2n$ for some $n \geq 1$, we may fix an arbitrary basis B for V and express f as an $2{\times}2n$ matrix in B and the standard basis for K^2:

$$[k] = \begin{pmatrix} a_{1,1} & \cdots & a_{1,2n} \\ a_{2,1} & \cdots & a_{2,2n} \end{pmatrix}$$

In order for k to compose with f to zero, the first row must be zero, so that the matrix is of the form:

$$[k] = \begin{pmatrix} 0 & \cdots & 0 \\ a_1 & \cdots & a_{2n} \end{pmatrix}$$

Now, because V is of dimension at least 2, we find that the zero map $K^2 \to K^2$ factors through k via some $g\colon K^2 \to V$ in more than one way. On one hand, $g = 0$ trivially works. On the other hand, we may take a non-trivial (i.e., some coefficient nonzero) K-linear combination $k_1 a_1 + \cdots + k_{2n} a_{2n} = 0$ of $a_1, \ldots, a_{2n}$ and consider the linear map g with matrix (in the standard basis for K^2, say, and B)

$$[g] = \begin{pmatrix} k_1 & k_1 \\ \vdots & \vdots \\ k_{2n} & k_{2n} \end{pmatrix}$$

Such a linear combination exists because $2n > 1$ so that $a_1, \ldots, a_{2n}$ are linearly dependent as vectors in K, its non-triviality guarantees that g is not the zero map, and g composes with k to zero by construction.

Thus f has no categorical kernel, and $\mathcal{C}$ is a category with zero objects where some morphisms have no kernel.

2.4 Pullbacks and pushouts

With commutative diagrams being of importance in category theory, we formalize the concept of completing a commutative square in a "most natural" way.

Definition 2.45. [5, p. 65, 71] [4, p. 59] Given two morphisms $\varphi\colon A \to Y$ and $\psi\colon B \to Y$ (in some category), which we may depict as

$$\begin{tikzcd} & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

a *pullback* of the pair (φ, ψ) is a pair of morphisms $\alpha\colon X \to A$ and $\beta\colon X \to B$ with a common domain X such that

(1) the corresponding square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes and

(2) the pair of morphisms is universal in the sense that any other commutative completion of the square

$$\begin{tikzcd} X' \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] & & \\ & & A \arrow[d, "\varphi"] \\ & B \arrow[r, "\psi"] & Y \end{tikzcd}$$

factors through X via a unique morphism $\chi\colon X' \to X$ to make the following diagram commute:

$$\begin{tikzcd} X' \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] \arrow[dr, dotted, "\chi"] & & \\ & X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ & B \arrow[r, "\psi"] & Y \end{tikzcd}$$

In other words, there should be a unique morphism $\chi\colon X' \to X$ satisfying $\alpha' = \alpha \circ \chi$ and $\beta' = \beta \circ \chi$. Dually, we define a *pushout* of a pair of morphisms

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \\ B & \end{tikzcd}$$

as a pair of morphisms $\varphi\colon A \to Y$ and $\psi\colon B \to Y$ with a common codomain Y such that

(1) the square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes and

(2) the pair of morphisms is universal in the sense that any other commutative completion of the square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[ddr, bend left, "\varphi'"] & \\ B \arrow[drr, bend right, "\psi'"'] & & \\ & & Y' \end{tikzcd}$$

factors through Y via a unique morphism $\upsilon\colon Y \to Y'$ to make the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \arrow[ddr, bend left, "\varphi'"] & \\
B \arrow[r, "\psi"] \arrow[drr, bend right, "\psi'"] & Y \arrow[dr, dotted, "\upsilon"] & \\
& & Y'
\end{tikzcd}$$

In other words, there should be a unique morphism $\upsilon\colon Y \to Y'$ satisfying $\varphi' = \upsilon \circ \varphi$ and $\psi' = \upsilon \circ \psi$.

Remark 2.46. Depending on the category, an arbitrary pair of morphisms (with common codomain and, respectively, domain) may or may not have a pullback or pushout. Consider for instance the category $\mathcal{C}$ of all even-dimensional vector spaces over some fixed field K and the morphism f from Example 2.44 that was shown to have no kernel. The morphisms

$$\begin{tikzcd}
& X \arrow[d, "f"] \\
0 \arrow[r, "0"] & Y
\end{tikzcd}$$

do not admit a pullback, because for any pullback completing the square

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "0"] & X \arrow[d, "f"] \\
0 \arrow[r, "0"] & Y
\end{tikzcd}$$

k is a kernel of f by Remark 2.49.

Like kernels (and cokernels), pullbacks and pushouts are unique up to an isomorphism of sorts.

Proposition 2.47. *[4, p. 60] If (α, β) and (α', β') are pullbacks of two morphisms $\varphi\colon A \to Y$ and $\psi\colon B \to Y$, then there is a unique isomorphism $\chi\colon X' \to X$ (where X is the domain of α and β and X' is the domain of α' and β') such that $\alpha' = \alpha \circ \chi$ and $\beta' = \beta \circ \chi$. In other words, there is a unique isomorphism χ making the following diagram commute:*

$$\begin{tikzcd}
X' \arrow[dr, dotted, "\chi"] \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] & & \\
& X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"] & Y
\end{tikzcd}$$

Dually, if (φ, ψ) and (φ', ψ') are pushouts of two morphisms $\alpha\colon X \to A$ and $\beta\colon X \to B$, then there is a unique isomorphism $\upsilon\colon Y \to Y'$ (where Y is the codomain of φ and ψ and Y' is the codomain of φ' and ψ') such that $\varphi' = \upsilon \circ \varphi$ and $\psi' = \upsilon \circ \psi$. In other words, there is a unique isomorphism υ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \arrow[ddr, bend left, "\varphi'"] & \\
B \arrow[r, "\psi"] \arrow[drr, bend right, "\psi'"] & Y \arrow[dr, dotted, "\upsilon"] & \\
& & Y'
\end{tikzcd}$$

Proof. Because pullbacks and pushouts are, per the definition, dual concepts and the statements of the proposition are dual to each other, it suffices to show only one of the statements; by duality, the other follows.

By the definition of the pullback, there is a unique morphism $\chi\colon X' \to X$ that makes the diagram commute. It remains to be shown that χ is an isomorphism.

Much like in the proof of the corresponding property for the kernel, we note that there is, by the definition of the pullback, also a (unique) morphism $\chi'\colon X \to X'$ and prove that χ and χ' are inverses of each other. Consider the composition $\chi\chi'\colon X \to X$. Composing by α to the left we obtain $\alpha\chi\chi' = \alpha'\chi' = \alpha$ by the definition of χ and χ'. That is to say that the diagram

$$\begin{tikzcd}
X \arrow[dr, "\chi\chi'"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"] & Y
\end{tikzcd}$$

commutes. By the definition of the pullback, there should be only one morphism from X to X making the diagram commute and the identity morphism is clearly that morphism. We conclude that $\chi\chi' = 1_X$. By similar reasoning, we find that $\chi'\chi = 1_{X'}$. Thus we have shown that χ is an isomorphism, which finishes the proof of the statement about pullbacks.

The statement about pushouts follows by duality. $\square$

Example 2.48. [5, p. 66, 72] In Sets, any pair of morphisms with common codomain admits a pullback and any pair of morphisms with common domain admits a pushout.

To show the former, take an arbitrary such morphism pair (i.e., just a function pair)

$$\begin{tikzcd}
& A \arrow[d, "\varphi"] \\
B \arrow[r, "\psi"] & Y
\end{tikzcd}$$

and consider the so-called fiber product (or fibered product [5, p. 71]) of A and B (with respect to φ and ψ):

$$A \times_Y B := \{(a, b) \in A \times B \mid \varphi(a) = \psi(b)\}$$

That is, we consider the subset of the Cartesian product consisting of the pairs such that φ and ψ coincide in the above sense. Seeing as it is a subset of the Cartesian product, we may consider the canonical projection functions $\pi'_A\colon A \times B \to A$ and $\pi'_B\colon A \times B \to B$ restricted to the fiber product. Let $\pi_A\colon A \times_Y B \to A$ and $\pi_B\colon A \times_Y B \to B$ denote these restrictions.

Claim. *(π_A, π_B) is a pullback of (φ, ψ).*

Proof. First verify that the morphisms π_A and π_B complete the square in a commutative way, i.e., that the following square commutes:

$$\begin{tikzcd}
A \times_Y B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "\varphi"] \\
B \arrow[r, "\psi"] & Y
\end{tikzcd}$$

Let $(a, b) \in A \times_Y B$ be any pair of the fiber product and traverse the diagram to the bottom right corner along the two paths:

$$(\varphi \circ \pi_A)((a, b)) = \varphi(a)$$
$$(\psi \circ \pi_B)((a, b)) = \psi(b)$$

By the definition of the fiber product, the right-hand sides above are equal. Because the pair (a, b) was picked arbitrarily, we conclude that the diagram commutes.

Next, consider another completion of the "square" (in a commutative way)

$$\begin{tikzcd}
X \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"] & Y
\end{tikzcd}$$

and show that there is a unique morphism $\chi \colon X \to A \times_Y B$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "\chi"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& A \times_Y B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"] & Y
\end{tikzcd}$$

The obvious choice is to define χ by

$$\chi(x) = (\alpha(x), \beta(x))$$

for any $x \in X$, for then

$$(\pi_A \circ \chi)(x) = \pi_A\big((\alpha(x), \beta(x))\big) = \alpha(x)$$
$$(\pi_B \circ \chi)(x) = \pi_B\big((\alpha(x), \beta(x))\big) = \beta(x)$$

which shows commutativity. Furthermore, this χ is the only morphism (function) from X to $A \times_Y B$ making the triangles commute. Any other function, χ' say, would map some $x \in X$ to a pair (a, b) different from $(\alpha(x), \beta(x))$ (i.e., with $a \neq \alpha(x)$ or $b \neq \beta(x)$), making sure that at least one of the triangles fail to commute:

$$\alpha(x) \neq a = \pi_A((a, b)) = (\pi_A \circ \chi')(x)$$

or

$$\beta(x) \neq b = \pi_B((a, b)) = (\pi_B \circ \chi')(x)$$

This shows that χ is unique with the above factorization property and finishes the proof that (π_A, π_B) is a pullback of (φ, ψ). $\square$

For the existence of pushouts, suppose that an arbitrary diagram of the form

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \\
B &
\end{tikzcd}$$

is given and consider the so-called fiber sum (or fibered sum [5, p. 66]) of A and B (with respect to φ and ψ), which is the disjoint union

$$A \sqcup B = \{(a,0) \mid a \in A\} \cup \{(0,b) \mid b \in B\}$$

with the elements $(\varphi(y),0)$ and $(0,\psi(y))$ identified for every $y \in Y$. Formally, we may define an equivalence relation $\sim$ on $A \sqcup B$ as the equivalence relation generated by the conditions

$$(\varphi(y),0) \sim (0,\psi(y)), \quad \forall y \in Y$$

and define the fiber sum as the quotient set $(A \sqcup B)/{\sim}$ of all equivalence classes under $\sim$. As any quotient set, the fiber sum comes equipped with a canonical projection

$$\begin{aligned} \pi\colon A \sqcup B &\to (A \sqcup B)/{\sim} \\ x &\mapsto [x] \end{aligned}$$

where $[x]$ denotes the equivalence class under $\sim$ of x.

As morphisms for the square, consider the composition of the canonical inclusion $\iota_A\colon A \to A \sqcup B$ into the disjoint union and the projection π, and similarly for B:

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[d, "\pi\circ\iota_A"] \\ B \arrow[r, "\pi\circ\iota_B"'] & (A \sqcup B)/{\sim} \end{tikzcd}$$

It is immediate by the definition of $\sim$ that this square commutes: pick an arbitrary $y \in Y$ and traverse the two paths to the lower right corner. The upper path yields $[(\varphi(y),0)]$ and the lower path yields $[(0,\psi(y))]$. These representatives are equivalent by the generating condition $(\varphi(y),0) \sim (0,\psi(y))$, so the equivalence classes are equal.

It remains to show that any other completion of the square in a commutative way factors through the fiber sum. Consider therefore another commutative completion

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[ddr, bend left, "\alpha"] & \\ B \arrow[drr, bend right, "\beta"'] & & \\ & & X \end{tikzcd}$$

and look for a function $g\colon (A \sqcup B)/{\sim} \to X$ that makes the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[d, "\pi\circ\iota_A"] \arrow[ddr, bend left, "\alpha"] & \\ B \arrow[r, "\pi\circ\iota_B"'] \arrow[drr, bend right, "\beta"'] & (A \sqcup B)/{\sim} \arrow[dr, "g"] & \\ & & X \end{tikzcd}$$

The uniqueness of such a g follows by observing that g has to be defined by

$$\begin{aligned} g([(a,0)]) &= \alpha(a) \\ g([(0,b)]) &= \beta(b) \end{aligned}$$

in order for α and β to factor through g as necessary. To see that this g is well-defined and hence exists, the more concrete definition of $\sim$ as the transitive closure of the reflexive and symmetric closure of the generators from before is useful. Still more concretely, we may denote the reflexive and symmetric closure of the generators by G and express its transitive closure as

$$\mathrm{cl}_{\mathrm{trans}}\, G = \left\{ (x_1, x_n) \,\middle|\, \begin{array}{l} x_1, x_n \in A \sqcup B \\ (x_1, x_2), \ldots, (x_{n-1}, x_n) \in G \text{ for some } n \geq 2 \\ x_2, \ldots, x_{n-1} \in A \sqcup B \end{array} \right\}$$

For well-definedness of g, we need to verify that the right-hand sides in the definition of g are all equal for equivalent representatives.

Suppose that $(a, 0) \sim (a', 0)$. Then there are, by the concrete definition of the transitive closure, elements $x_1, \ldots, x_n \in G$ (for some $n \geq 1$) with $x_1 = (a, 0)$ and $x_n = (a', 0)$. By the definition of the generators, these x_i will be of the form

$$\begin{aligned} x_1 &= (a_1, 0) = (a, 0) \\ x_2 &= (0, b_2) \\ x_3 &= (a_3, 0) \\ &\vdots \\ x_{n-1} &= (0, b_{n-1}) \\ x_n &= (a_n, 0) = (a', 0) \end{aligned}$$

for some $a_i \in A$ and $b_i \in B$ where two consecutive elements a_i and b_{i+1} are images of the same element, some $y_i \in Y$, under φ and ψ, respectively, (and vice versa for consecutive elements b_j, a_{j+1}): $\varphi(y_i) = a_i$ and $\psi(y_i) = b_{i+1}$. Recalling that α and β complete the square in a commutative manner, we find that

$$\begin{aligned} \alpha(a) &= \alpha(a_1) \\ &= \alpha(\varphi(y_1)) \\ &= \beta(\psi(y_1)) \\ &= \beta(b_2) \\ &= \beta(\psi(y_2)) \\ &= \alpha(\varphi(y_2)) \\ &= \alpha(a_3) \\ &\vdots \\ &= \alpha(a_n) \\ &= \alpha(a') \end{aligned}$$

which shows that the right-hand sides in the definition of g are equal for equivalent representatives from the copy of A in $A \sqcup B$. By symmetry, this takes care of equivalent representatives from the copy of B as well. The very same idea as above also handles the final case with one representative from each copy, so g is well-defined and the fiber sum with the proposed functions is indeed a pushout.

Remark 2.49. [4, p. 62] Pullbacks and pushouts may be thought of as generalizing kernels and cokernels, respectively. More specifically, for any morphism f in a category with zero objects, consider the diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ 0 \arrow[r, "0"] & Y \end{tikzcd}$$

For any pullback completing the square as follows:

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "0"] & X \arrow[d, "f"] \\ 0 \arrow[r, "0"] & Y \end{tikzcd}$$

k is a kernel of f. Conversely, any kernel k of f completes a pullback square as above. This is so because the square commutes if and only if k precomposes with f to zero, and the factorization property of the pullback $(k, 0)$ via some unique g reduces to precisely the factorization property of the kernel k via this unique g seeing as no restrictions are imposed on g by 0.

Dually, the cokernels of f are precisely the morphisms c that complete the diagram to the left into a pushout square to the right:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "0"] & Y \\ 0 & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "0"] & Y \arrow[d, "c"] \\ 0 \arrow[r, "0"] & C \end{tikzcd}$$

Pullbacks and pushouts exhibit a handful of useful properties with regard to composition, which are given in the following proposition.

Proposition 2.50. *Let φ and ψ be morphisms with a common codomain:*

$$\begin{tikzcd} & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

1. *For any monomorphism m (postcomposable as in the diagram), the small square is a pullback square if and only if the large "square" is in the following diagram:*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \arrow[ddr, bend left, "m\circ\varphi"] & \\ B \arrow[r, "\psi"] \arrow[drr, bend right, "m\circ\psi"'] & Y \arrow[dr, "m"] & \\ & & Z \end{tikzcd}$$

which is to say that $(m \circ \varphi, m \circ \psi)$ admits the same pullbacks as (φ, ψ).

2. *For any isomorphism i (precomposable as in the diagram), the small square is a pullback square if and only if the large square is in the following diagram:*

$$\begin{tikzcd} W \arrow[dr, "i"] \arrow[drr, bend left, "\alpha\circ i"] \arrow[ddr, bend right, "\beta\circ i"'] & & \\ & X \arrow[r, "\alpha"'] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ & B \arrow[r, "\psi"] & Y \end{tikzcd}$$

which is to say that (α, β) is a pullback of (φ, ψ) if and only if $(\alpha \circ i, \beta \circ i)$ is a pullback of (φ, ψ).

3. *For any isomorphism i (precomposable as in the diagram), the small square is a pullback square if and only if the large square is in the following diagram:*

$$\begin{tikzcd} & A' \arrow[d, "i"] \\ X \arrow[ur, "i^{-1}\circ\alpha"] \arrow[r, "\alpha"'] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

which is to say that (α, β) is a pullback of (φ, ψ) if and only if $(i^{-1} \circ \alpha, \beta)$ is a pullback of $(\varphi \circ i, \psi)$.

Dually for pushouts, (1) precomposition by an epimorphism does not affect the pushouts, (2) postcomposing a pushout by an isomorphism yields another pushout, and (3) postcomposing one of the morphisms being pushed out by an isomorphism and precomposing the corresponding morphism in the pushout by the inverse isomorphism yields a pushout square.

Proof.

1. Because m is monic, the small square commutes if and only if the large square commutes. Furthermore, the factorization of any other commutative completion of either of the squares through α and β depends only on said morphisms and not on the morphisms φ and ψ (or $m \circ \varphi$ and $m \circ \psi$) being pulled back. Thus, α and β constitute a pullback of the small square if and only if they constitute a pullback of the large square.

2. Because i is epic, the small square commutes if and only if the large square commutes. Furthermore, if any other commutative completion (γ, δ) of the square factors through $(\alpha \circ i, \beta \circ i)$ via some unique χ, then said completion factors uniquely through (α, β) via $i \circ \chi$. Thus, if the large square is a pullback square, then so is the small square. The converse is immediate by writing (α, β) as $((\alpha \circ i) \circ i^{-1}, (\beta \circ i) \circ i^{-1})$.

3. Because the triangle at the top commutes, the small square commutes if and only if the large square commutes. Assume that the small square is a pullback square. Any commutative completion (γ', δ) of the large square gives rise to a commutative completion $(i \circ \gamma', \delta)$ of the small square and hence a factorization of $(i \circ \gamma', \delta)$ through (α, β) via a unique χ. This χ is the unique morphism via which the original completion (γ', δ) factors through $(i^{-1} \circ \alpha, \beta)$, which shows that the large square is a pullback square. In other words, if the small square is a pullback square, then so is the large square. The converse is immediate by writing (φ, ψ) as $(\varphi \circ i) \circ i^{-1}, \psi)$.

□

By Remark 2.49, the above properties of pullbacks and pushouts induce similar properties for kernels and cokernels.

Corollary 2.51. *Postcomposing a morphism by a monomorphism does not affect its kernels (as displayed to the left), precomposing a kernel by an isomorphism yields another kernel of the same morphism (as displayed in the middle), and precomposing a morphism by an isomorphism while postcomposing its kernel by the inverse isomorphism maintains the kernel–morphism relationship (as displayed to the right):*

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] & Y \arrow[d, "m"] \\ & & Z \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] & Y \\ K' \arrow[u, "i"] & & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] \arrow[d, bend right, "i"'] & Y \\ & X' \arrow[u, bend right, "i^{-1}"'] & \end{tikzcd}$$

The dual properties for cokernels are depicted below, where e is an epimorphism.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "c"] & C \\ W \arrow[u, "e"] & & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "c"] & C \arrow[d, "i"] \\ & & C' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "c"] \arrow[d, bend right, "i"'] & C \\ & X' \arrow[u, bend right, "i^{-1}"'] & \end{tikzcd}$$

2.5 Functors and subcategories

As for most algebraic structures, there is a notion of homomorphisms of the objects being studied. In order to avoid confusion with the morphisms *in* the categories, homomorphisms of categories go by the name of functors.

Definition 2.52. [4, p. 44] Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* (or *covariant functor*) F of $\mathcal{C}$ and $\mathcal{D}$, denoted by $F\colon \mathcal{C} \to \mathcal{D}$, consists of the following:

- for each object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$, and
- for each morphism $f\colon X \to Y$ in $\mathcal{C}$, a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$.

such that the composition of morphisms and identity morphisms are respected:

1. $F(g \circ f) = F(g) \circ F(f)$ for any composable morphisms f and g in $\mathcal{C}$
2. $F(1_X) = 1_{F(X)}$ for any object X in $\mathcal{C}$.

Some functors, notably the Hom functor of Example 2.56 with the second argument fixed, have a tendency to reverse morphisms. These require a revised version of the definition above.

Definition 2.53. [4, p. 46] Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *contravariant functor* F of $\mathcal{C}$ and $\mathcal{D}$, denoted by $F\colon \mathcal{C} \to \mathcal{D}$, consists of the following:

- for each object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$, and
- for each morphism $f\colon X \to Y$ in $\mathcal{C}$, a morphism $F(f)\colon F(X) \leftarrow F(Y)$ in $\mathcal{D}$.

such that the composition of morphisms and identity morphisms are respected:

1. $F(g \circ f) = F(f) \circ F(g)$ for any composable morphisms f and g in $\mathcal{C}$
2. $F(1_X) = 1_{F(X)}$ for any object X in $\mathcal{C}$.

Remark 2.54. One may understand a contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ as precisely a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ from the dual category [4, p. 46] or as precisely a covariant functor $F\colon \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ to the dual category. In other words, the following are equivalent for a mapping F of objects and morphisms in $\mathcal{C}$ to objects and morphisms in $\mathcal{D}$ to be: (1) a contravariant functor $\mathcal{C} \to \mathcal{D}$, (2) a covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, and (3) a covariant functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Intuitively, this is the case because the morphism-reversing nature of the contravariant functor is captured in the morphism-reversing nature of the opposite category.

More formally, consider a morphism $f\colon X \to Y$ in $\mathcal{C}$ (equivalently, a morphism $f\colon X \leftarrow Y$ in $\mathcal{C}^{\mathrm{op}}$) and the morphism $F(f)$. The implicit requirement on the domain and codomain of $F(f)$ in the definition of a covariant and contravariant functor is the same in all three cases, namely that $F(f)\colon F(X) \leftarrow F(Y)$ in $\mathcal{D}$ (equivalently, $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}^{\mathrm{op}}$). Similarly, the first functor axioms require in all three cases precisely that $F(g \circ_{\mathcal{C}} f) = F(f) \circ_{\mathcal{D}} F(g)$ (where the subscripts emphasize in which category the composition is done). This is seen by noting that the left-hand side may be written as $F(f \circ_{\mathcal{C}^{\mathrm{op}}} g)$ and the right-hand side may be written as $F(f) \circ_{\mathcal{D}^{\mathrm{op}}} F(g)$. The equivalence of the second functor axioms follows immediately from the fact that a category and its opposite have the same identity morphisms.

A beautiful example of a (covariant) functor is the one underlying the fundamental group in topology.

Example 2.55. [4, p. 45] [6, pp. 330–334] The map

$$(X, x) \mapsto \pi_1(X, x)$$
$$h \mapsto h_*$$

sending a pointed space (X, x) to its fundamental group $\pi_1(X, x)$ and a basepoint-preserving continuous map h between two pointed spaces (Y, y) and (Z, z) to the group homomorphism $h_* \colon \pi_1(Y, y) \to \pi_1(Z, z)$ induced by h defines a functor $\pi_1 \colon \mathrm{Top}_* \to \mathrm{Grp}$. That it is a functor amounts to saying that the morphism induced by a composition is the composition of induced morphisms:

$$(k \circ h)_* = k_* \circ h_*$$

and that the identity maps induce the identity group endomorphism:

$$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$$

both of which follow readily from the definition of the homomorphism induced by a continuous map.

Another, arguably more fundamental, example of functors is the Hom functors, which are defined for every category.

Example 2.56. For any category $\mathcal{C}$ and fixed object $A \in \mathcal{C}$, there is a covariant functor

$$\mathrm{Hom}_{\mathcal{C}}(A, -) \colon \mathcal{C} \to \mathrm{Sets}$$

and a contravariant functor

$$\mathrm{Hom}_{\mathcal{C}}(-, A) \colon \mathcal{C} \to \mathrm{Sets}$$

which are called Hom *functors*. They are defined in the apparent ways for the objects:

$$\mathrm{Hom}_{\mathcal{C}}(A, -)(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$$
$$\mathrm{Hom}_{\mathcal{C}}(-, A)(X) = \mathrm{Hom}_{\mathcal{C}}(X, A)$$

and in the only reasonable way for the morphisms:

$$\mathrm{Hom}_{\mathcal{C}}(A, -)(f) = f \circ -$$
$$\mathrm{Hom}_{\mathcal{C}}(-, A)(f) = - \circ f$$

for any morphism $f \colon X \to Y$, where $f \circ - \colon \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$, i.e., from $\mathrm{Hom}_{\mathcal{C}}(A, -)(X)$ to $\mathrm{Hom}_{\mathcal{C}}(A, -)(Y)$, is to be understood as the function in Sets mapping a morphism $h \colon A \to X$ to the morphism $f \circ h \colon A \to Y$ obtained by postcomposing by f illustrated as

$$\begin{tikzcd} A \arrow[r, "h"] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

and $- \circ f \colon \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$, i.e., from $\mathrm{Hom}_{\mathcal{C}}(-, A)(Y)$ to $\mathrm{Hom}_{\mathcal{C}}(-, A)(X)$, is to be understood as the function mapping $h \colon Y \to A$ to the morphism $h \circ f \colon X \to A$ obtained by precomposing by f illustrated as

$$\begin{tikzcd} Y \arrow[r, "h"] & A \\ X \arrow[u, "f"] & \end{tikzcd}$$

That the functors respect the composition is a straightforward consequence of the associativity of the composition: for any morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, we show that the functions $\mathrm{Hom}_{\mathcal{C}}(A,-)(g \circ f)$ and $\mathrm{Hom}_{\mathcal{C}}(A,-)(g) \circ \mathrm{Hom}_{\mathcal{C}}(A,-)(f)$ are equal by showing that they are equal for an arbitrary argument $h\colon A \to X$:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(A,-)(g \circ f)(h) &= \big((g \circ f) \circ -\big)(h) \\
&= (g \circ f) \circ h \\
&= g \circ (f \circ h) \\
&= (g \circ -)(f \circ h) \\
&= (g \circ -)\big((f \circ -)(h)\big) \\
&= \big((g \circ -) \circ (f \circ -)\big)(h) \\
&= \big(\mathrm{Hom}_{\mathcal{C}}(A,-)(g) \circ \mathrm{Hom}_{\mathcal{C}}(A,-)(f)\big)(h)
\end{aligned}$$

In a similar fashion, we have for the contravariant functor

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(-,A)(g \circ f)(h) &= \big(- \circ (g \circ f)\big)(h) \\
&= h \circ (g \circ f) \\
&= (h \circ g) \circ f \\
&= (- \circ f)(h \circ g) \\
&= (- \circ f)\big((- \circ g)(h)\big) \\
&= \big((- \circ f) \circ (- \circ g)\big)(h) \\
&= \big(\mathrm{Hom}_{\mathcal{C}}(-,A)(f) \circ \mathrm{Hom}_{\mathcal{C}}(-,A)(g)\big)(h)
\end{aligned}$$

where $h\colon Y \to A$ is an arbitrary morphism. That the functors respect the identity morphism is an immediate consequence of their definitions and the fact that the identity morphisms are identities with respect to the composition.

Remark 2.57. It should be noted that the Hom functors just defined may be thought of as induced by fixing one of the arguments of a more general functor (a so-called *bifunctor*) $\mathrm{Hom}_{\mathcal{C}}(-,-)$ or just $\mathrm{Hom}_{\mathcal{C}}$, which maps a pair of objects (X, Y) to $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and a pair of morphisms $f\colon X' \to X$ (to be thought of as $f\colon X \to X'$ in $\mathcal{C}^{\mathrm{op}}$, because of the contravariance in the first argument) and $h\colon Y \to Y'$ to the function between the Hom sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X', Y')$ defined by precomposition by f and postcomposition by h. In symbols, $(f, h) \mapsto (g \mapsto h \circ g \circ f)$. Diagrammatically, $\mathrm{Hom}_{\mathcal{C}}$ maps the pair of morphisms below to the left to the function between Hom sets (a morphism in Sets) that takes the morphism in the middle diagram to the composition from X' to Y' in the rightmost diagram:

$$\begin{tikzcd}
X & Y \arrow[d, "h"] \\
X' \arrow[u, "f"] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] & Y \arrow[d, "h"] \\
X' \arrow[u, "f"] & Y'
\end{tikzcd}$$

Formally, one may define the notion of a *product of categories* (not to be confused with the notion of a product of objects in a category introduced in Section 3.2). The pairs of objects and pairs of morphisms above are the objects and, respectively, morphisms of the product $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, and the bifunctor $\mathrm{Hom}_{\mathcal{C}}$ may be defined as a covariant functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sets}$. Bifunctors and products of categories will not be defined or explored in any more detail in this document. For more details, see [5, pp. 36–40].

The observation that every category has an identity functor, which simply fixes all the objects and morphisms, and that functors may be composed in a natural manner [4, p. 45] is captured in the following definition.

Definition 2.58. [5, p. 14] Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

The *identity functor* on $\mathcal{C}$, denoted by $1_{\mathcal{C}}$, is the covariant functor defined by

$$1_{\mathcal{C}}(X) = X$$
$$1_{\mathcal{C}}(f) = f$$

for any object X and morphism f in $\mathcal{C}$.

Given two covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$, their *composition* $G \circ F\colon \mathcal{C} \to \mathcal{E}$ is defined by

$$(G \circ F)(X) = G(F(X))$$
$$(G \circ F)(X \xrightarrow{f} Y) = G(F(X)) \xrightarrow{G(F(f))} G(F(Y))$$

for any object X and morphism f in $\mathcal{C}$.

Remark 2.59. [5, p. 14] Note that the composition of functors is associative (for the same reason that ordinary function composition is associative), i.e., that $H \circ (G \circ F) = (H \circ G) \circ F$ for any composable functors F, G, and H. Note also that the identity functors are identities with respect to the functor composition in the sense that $G \circ 1_{\mathcal{D}} = G$ and $1_{\mathcal{D}} \circ F = F$ for any functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. This hints at the possibility of gathering all categories as the objects of a category – the category of all categories – with all functors between the categories as morphisms. However, there are foundational difficulties to be wary of when collecting categories whose class of objects are proper (so-called *large* categories). If we instead restrict our attention to categories whose class of objects is a set (*small* categories), the foundationally sound category Cat of all small categories is obtained. This idea of a category of categories allows us to understand notions about categories and functors in a categorical sense. For instance, an isomorphism of (small) categories to be defined in Definition 2.60 may be understood as simply an isomorphism in Cat.

Definition 2.60. [5, p. 14] An *isomorphism* of categories is a covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ that has a (two-sided) inverse $G\colon \mathcal{D} \to \mathcal{C}$ with respect to functor composition. In other words, the functors should satisfy

$$G \circ F = 1_{\mathcal{C}}, \quad F \circ G = 1_{\mathcal{D}}$$

If there is an isomorphism from one category to another, the categories are said to be *isomorphic*.

Remark 2.61. As usual for associative operations with an identity, the inverse of a functor F is unique and may hence be denoted by F^{-1}.

Example 2.62. Recall Remark 1.7 where every $\mathbb{Z}$-module were shown to correspond very naturally to an abelian group and vice versa. We are now ready to put this correspondence into a formal context: the categories $\mathbb{Z}$-Mod and Ab are isomorphic. Specifically, the so-called *forgetful functor* $U\colon \mathbb{Z}\text{-Mod} \to \text{Ab}$ sending every $\mathbb{Z}$-module to its underlying abelian group and every $\mathbb{Z}$-module morphism to the additive function that it is is an isomorphism of categories. Its inverse is the functor from Ab to $\mathbb{Z}$-Mod sending every abelian group G to the unique $\mathbb{Z}$-module with underlying abelian group G and every morphism of abelian groups to itself as a morphism of the corresponding $\mathbb{Z}$-modules.

For most (if not all) algebraic structures, there is a notion of a substructure. Categories are no exception.

Definition 2.63. [5, p. 15] A *subcategory* $\mathcal{D}$ of a category $\mathcal{C}$ consists of a subclass $\operatorname{ob}\mathcal{D} \subseteq \operatorname{ob}\mathcal{C}$ of objects and for each pair of objects $X, Y \in \operatorname{ob}\mathcal{D}$ a subset $\mathcal{D}(X, Y) \subseteq \mathcal{C}(X, Y)$ of morphisms such that

(1) for every object $X \in \operatorname{ob}\mathcal{D}$, the identity morphism 1_X on X in $\mathcal{C}$ is in $\mathcal{D}(X, X)$ and

(2) the class of all morphisms in $\mathcal{D}$ is closed under the composition in $\mathcal{C}$.

The axioms ensure that $\mathcal{D}$ is a category in its own right with identity morphisms and law of composition inherited from $\mathcal{C}$.

Of particular importance among subcategories are the so-called full subcategories, which are subcategories for which the non-strict subset inequalities for the morphism sets are all equalities: $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$. A more abstract (but of course equivalent) definition of full subcategories, using general terminology of category theory, is presented in tbe next few paragraphs.

Remark 2.64. [5, p. 15] For any subcategory $\mathcal{D}$ of any category $\mathcal{C}$, there is a (covariant) functor from $\mathcal{D}$ to $\mathcal{C}$ called the *inclusion functor* that simply fixes every object and morphism of $\mathcal{D}$.

Definition 2.65. [4, p. 47] A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *full* if, for every pair of objects $X, Y \in \mathcal{C}$, the morphism set $\mathcal{C}(X,Y)$ is mapped surjectively by F onto $\mathcal{D}(F(X), F(Y))$.

Definition 2.66. [4, p. 43, 47] A subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *full* if the inclusion functor from $\mathcal{D}$ to $\mathcal{C}$ is full.

Remark 2.67. As Mac Lane notes ([5, p. 15]), full subcategories are determined entirely by their objects (and their supercategory). Moreover, *any* subclass of objects induces a full subcategory, because the fullness guarantees that the conditions on the morphisms that there is an identity morphism for every object and that the morphisms are closed under composition are satisfied.

Example 2.68. By the previous remark, full subcategories are plentiful. Some examples are

- the category $\operatorname{vec} K$ of all finite-dimensional (or equivalently finitely generated) vector spaces over a fixed field K (with all linear maps as morphisms and function composition as composition of morphisms) and the category of all even-dimensional vector spaces over K considered in Example 2.44 as subcategories of $\operatorname{Vec} K$,
- the categories R-mod of all finitely generated R-modules, R-Free of all free R-modules, and R-free of all finitely-generated free R-modules (or equivalently R-modules with finite bases) as subcategories of R-Mod
- the category Ab and the category of all finite groups as subcategories of Grp, and
- the category of all fields as a subcategory of Rings.

As an example of a subcategory that is not full, consider the category Rings of all rings with unity as a subcategory of the category Rngs of all rings with or without unity. For any two rings R and S with unity, the latter category may have a morphism $\varphi\colon R \to S$ that fails to respect the unity, which is to say that φ is not a morphism in the former category. Consider for instance the zero map from R to $S \neq \{0\}$. It is both additive and multiplicative but fails to map the unity of R to the unity $1_S \neq 0_S$ of S. This is to say that $\operatorname{Rings}(R,S) \subsetneq \operatorname{Rngs}(R,S)$ and that Rings is not a full subcategory of Rngs.

Example 2.69. Recall Remark 2.11 about viewing monoids as categories with a single object. In that context, a submonoid corresponds precisely to a subcategory (still containing the single object): the closedness under composition for the subcategory is equivalent to the closedness of the submonoid under the monoid operation. Similarly, the requirement on the subcategory to contain the same identity morphism as the supercategory is equivalent to the requirement on the submonoid to contain the same identity element as the supermonoid.

Isomorphisms of categories are for many purposes unnecessarily restrictive. The more general notion of equivalences of categories is more widely applicable while still strong enough for most ends. We will use equivalences in Sections 3 and 4 to draw conclusions about a category given information about an "equivalent" category.

Presented below is a pragmatic definition of equivalences. It may be compared to a more common and abstract definition in terms of natural isomorphisms (an important notion covered in, e.g., [4, 5] but not in this document), which may philosophically be more suitable as a definition but requires more setup. Depending on the foundation used for categories, the two definitions of an equivalence may or may not be equivalent; see [5, pp. 92–94] for the abstract definition and a proof that the two definitions are equivalent with the set-theoretical foundation involving no proper classes used by Mac Lane. Assuming the definition of a category given in this document, the pragmatic definition soon to be presented would seem to be strictly more general than its abstract counterpart. In any case, the definitions in question are closely related; all the results involving equivalences in this document hold regardless of which of the two definitions is used.

Dual in a sense to full functors are faithful functors.

Definition 2.70. [4, p. 47] A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *faithful* if, for every pair of objects $X, Y \in \mathcal{C}$, the morphism set $\mathcal{C}(X, Y)$ is mapped injectively by F into $\mathcal{D}(F(X), F(Y))$.

The notion of a functor that is surjective on objects up to isomorphism is given a name of its own.

Definition 2.71. [1, p. 410] A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *dense* if for every object $Y \in \mathcal{D}$, there is an object $X \in \mathcal{C}$ with $F(X)$ isomorphic to Y.

Definition 2.72. An *equivalence* of categories is a covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ that is full, faithful, and dense.

Remark 2.73. If there is an equivalence $F\colon \mathcal{C} \to \mathcal{D}$, one usually says that the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*. This terminology is appropriate when the abstract definition of an equivalence is used, because it may then be shown that equivalence of categories (as a "relation" between categories) has the properties of an equivalence relation [3, p. 590]. In particular, if there is an equivalence $F\colon \mathcal{C} \to \mathcal{D}$, then there is an equivalence $G\colon \mathcal{D} \to \mathcal{C}$, which is to say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if $\mathcal{D}$ and $\mathcal{C}$ are equivalent. This symmetry would, however, not seem to hold for arbitrary categories using the pragmatic definition of an equivalence (Definition 2.72), which makes the terminology inappropriate for us.

The following proposition says that equivalences of categories do indeed generalize isomorphisms of categories.

Proposition 2.74. *Every isomorphism of categories is an equivalence of categories.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an isomorphism of categories. The existence of a right inverse of F implies that F is full, and the existence of a left inverse of F implies that F is faithful. The existence of a right inverse of F also implies that F is surjective on objects and hence in particular dense. $\square$

Remark 2.75. Definitions 2.58, 2.60, 2.65 and 2.70 to 2.72 of composition of functors; isomorphism of categories; full, faithful, and dense functors; and equivalences of categories all extend very naturally to contravariant functors by viewing contravariant functors as covariant functors (see Remark 2.54).

One may observe that a covariant functor and a contravariant functor compose to a contravariant functor and that two contravariant functors compose to a covariant functor. This latter observation ensures that the definition of a contravariant isomorphism of categories makes sense (seeing as the identity functor is covariant). Note however that the notion of isomorphic categories should not be extended to allow for contravariant isomorphisms; a category and its dual are generally not to be thought of as isomorphic. Similarly, two categories are to be thought of as equivalent only if there is a *covariant* equivalence between the two.

3 Additive categories

In this section, we lay the foundation for the sections to come by defining additive categories – informally, categories where objects and morphisms may be added – and exploring their properties. This exploration culminates in a representation of morphisms as matrices and a characterization of pullbacks and pushouts in terms of kernels and cokernels.

3.1 Preadditive categories

In preparation for additive categories, we define the notion of a preadditive category, which is a category where the morphisms form abelian groups, each of whose operations is compatible with the composition in the category. Informally, it is a category where morphisms but not necessarily objects may be added (and even subtracted).

Definition 3.1. [5, p. 28] A *preadditive* category (also Ab-*category* or category *enriched over* Ab) is a category with the additional structure that the morphism sets are (additive) abelian groups such that the composition functions of the category are biadditive, i.e., for morphisms where the addition and composition are defined the following holds:

$$(f_1 + f_2) \circ g = (f_1 \circ g) + (f_2 \circ g)$$
$$f \circ (g_1 + g_2) = (f \circ g_1) + (f \circ g_2)$$

Remark 3.2. Every morphism set of a preadditive category is non-empty, seeing as groups are required to be non-empty by definition.

An immediate consequence of the biadditivity of the composition in a preadditive category $\mathcal{C}$ is that the zero element of a morphism group $\mathcal{C}(X, Y)$, which we may denote by $0_{\mathcal{C}(X,Y)}$, composes to a zero element.

Lemma 3.3. *Let $f \colon W \to X$ and $g \colon Y \to Z$ be morphisms of a preadditive category $\mathcal{C}$. Then,*

$$0_{\mathcal{C}(X,Y)} \circ f = 0_{\mathcal{C}(W,Y)}$$
$$g \circ 0_{\mathcal{C}(X,Y)} = 0_{\mathcal{C}(X,Z)}$$

Proof. In order to show that an element of an abelian group is the zero element, it suffices to establish that its sum with itself is itself. Using the biadditivity of the composition, we may show that this is the case for the left-hand sides above:

$$\begin{aligned}(0_{\mathcal{C}(X,Y)} \circ f) + (0_{\mathcal{C}(X,Y)} \circ f) &= (0_{\mathcal{C}(X,Y)} + 0_{\mathcal{C}(X,Y)}) \circ f \\ &= 0_{\mathcal{C}(X,Y)} \circ f\end{aligned}$$

and similarly

$$\begin{aligned}(g \circ 0_{\mathcal{C}(X,Y)}) + (g \circ 0_{\mathcal{C}(X,Y)}) &= g \circ (0_{\mathcal{C}(X,Y)} + 0_{\mathcal{C}(X,Y)}) \\ &= g \circ 0_{\mathcal{C}(X,Y)}\end{aligned}$$

□

In a preadditive category $\mathcal{C}$ with a zero object and hence zero morphisms, there would seem to be an evident risk of confusing the zero morphism 0_{XY} with the zero element $0_{\mathcal{C}(X,Y)}$ of the morphism group (for any objects X, Y). The following proposition shows that the morphisms coincide and hence that the confusion is of little matter in practice.

Proposition 3.4. *[4, p. 75] Let $\mathcal{C}$ be a preadditive category with a zero object. For any objects $X, Z \in \mathcal{C}$,*

$$0_{XZ} = 0_{\mathcal{C}(X,Z)}$$

Proof. Using the facts that zero morphisms compose into zero morphisms (Remark 2.27) and zero elements compose into zero elements of the morphism groups (Lemma 3.3), the equality is obtained immediately when considering the morphism $X \xrightarrow{0_{XY}} Y \xrightarrow{0_{\mathcal{C}(Y,Z)}} Z$ for arbitrary objects X, Y, Z: the composition is the zero morphism $X \xrightarrow{0_{XZ}} Z$ on one hand and the zero element $X \xrightarrow{0_{\mathcal{C}(X,Z)}} Z$ on the other hand, which is to say that $0_{XZ} = 0_{\mathcal{C}(X,Z)}$. □

Using the Lemma 3.3, we may give a characterization of zero objects in preadditive categories.

Proposition 3.5. *[5, p. 194] The following statements are equivalent for an object Y of a preadditive category $\mathcal{C}$.*

(1) Y is a zero object.

(2) Y is initial.

(3) Y is terminal.

(4) $\mathcal{C}(Y, Y)$ is the zero group.

(5) $0_{\mathcal{C}(Y,Y)} = 1_Y$.

Proof. The implications

$$\begin{aligned}(1) \implies (2) \implies (4) \implies (5)\\ (1) \implies (3) \implies (4) \implies (5)\end{aligned}$$

follow readily from the definition of everything involved: A zero object is by definition an initial and terminal object, initial and terminal objects by definition have a single endomorphism, and any two morphisms of the zero group are equal. Using Lemma 3.3, we may prove the implication (5) $\implies$ (1) as follows: Let X and Y be any objects of $\mathcal{C}$. Let further $f \colon X \to Y$ and $g \colon Y \to Z$ be arbitrary morphisms, which we know from Remark 3.2 exist. By the assumption, we find that

$$\begin{aligned} f &= 1_Y \circ f \\ &= 0_{\mathcal{C}(Y,Y)} \circ f \\ &= 0_{\mathcal{C}(X,Y)} \end{aligned}$$

and

$$\begin{aligned} g &= g \circ 1_Y \\ &= g \circ 0_{\mathcal{C}(Y,Y)} \\ &= 0_{\mathcal{C}(Y,Z)} \end{aligned}$$

We have shown that the arbitrary morphisms f and g starting and, respectively, ending in Y are the zero morphism of their respective morphism group, which is to say that the morphism sets $\mathcal{C}(X, Y)$ and $\mathcal{C}(Y, Z)$ consist of precisely one morphism each. Because X and Z were arbitrary, we may conclude that Y is a zero object. □

Example 3.6. We may by Proposition 3.5 and Remark 2.22, conclude that there is no preadditive structure that Sets can be endowed with to make a preadditive category, because it has initial objects that are not terminal and vice versa. Admittedly, a simpler way of coming to this conclusion would be to note that there are empty morphism sets in Sets, namely Sets$(X, \varnothing)$ for any non-empty set X, while recalling Remark 3.2 that the morphism sets are non-empty in preadditive categories.

Proposition 3.7. *In a preadditive category, the* Hom *functors may be viewed as functors into the category of all abelian groups rather than just sets. In other words, for any preadditive category $\mathcal{C}$ and fixed object $A \in \mathcal{C}$, we have* $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathrm{Ab}$ *and* $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to \mathrm{Ab}$.

Proof. The morphism sets in a preadditive category are abelian groups, so the mapping of the objects is sound. It remains to be shown that morphisms in $\mathcal{C}$ are mapped to morphisms in Ab, i.e., to additive functions. This follows readily from the definition of the Hom functors and the biadditivity of the composition in a preadditive category: for any morphism $f : X \to Y$, consider the function $\mathrm{Hom}_{\mathcal{C}}(A, -)(f) : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$ obtained by applying the covariant Hom functor to f and verify that it is additive. For any morphisms $g, h \in \mathcal{C}(A, X)$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(A, -)(f)(g + h) &= (f \circ -)(g + h) \\
&= f \circ (g + h) \\
&= (f \circ g) + (f \circ h) \\
&= (f \circ -)(g) + (f \circ -)(h) \\
&= \mathrm{Hom}_{\mathcal{C}}(A, -)(f)(g) + \mathrm{Hom}_{\mathcal{C}}(A, -)(f)(h)
\end{aligned}$$

where the third equality follows from the additivity in the right argument of composition. Similarly, we get for the contravariant Hom functor a function $\mathrm{Hom}_{\mathcal{C}}(-, A)f : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$, and for any morphisms $g, h \in \mathcal{C}(Y, A)$ we find that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(-, A)(f)(g + h) &= (- \circ f)(g + h) \\
&= (g + h) \circ f \\
&= (g \circ f) + (h \circ f) \\
&= (- \circ f)(g) + (- \circ f(h)) \\
&= \mathrm{Hom}_{\mathcal{C}}(-, A)(f)(g) + \mathrm{Hom}_{\mathcal{C}}(-, A)(f)(h)
\end{aligned}$$

where the third equality follows from the additivity in the left argument of composition. This shows that every morphism is mapped to a suitable morphism in Ab and finishes the proof. □

Example 3.8. [3, p. 600] As a first but not very typical example of a preadditive category, we consider any ring R (with unity). Recall from Remark 2.11 that monoids may be viewed as categories with a single object. We may thus consider the category $\mathcal{C}_R$ induced by the multiplicative monoid of the ring – in other words, the category with a single object ($*$, say) and the ring elements as morphisms with composition given by ring multiplication. Endow the only morphism set $\mathcal{C}_R(*, *) = R$ with the additive group structure of the ring. That is, addition of morphisms is done by adding them as ring elements.

Note now that this indeed defines a preadditive category: the only thing to verify is that the composition is biadditive, but this is exactly the requirement that multiplication distributes over addition in the ring, which is one of the ring axioms. One may also note the converse, that any preadditive category with a single object gives rise to a ring of morphisms with addition given by morphism addition and multiplication given by composition. As in Remark 2.11, the converse generalizes to categories with more than one object: in any preadditive category $\mathcal{C}$, every endomorphism set is a ring with the ring addition given by morphism addition and the ring multiplication given by morphism composition.

Two takeaways from this are that rings (with unity) may be thought of as precisely preadditive categories with a single object and that preadditive categories generalize rings in the same way that arbitrary categories generalize monoids.

3.2 Products and coproducts

Recall the universal property of the direct product and direct sum of R-modules from Proposition 1.29. This property turns out to be a useful way of defining the product (corresponding to the direct product of modules) and coproduct (corresponding to the direct sum of modules) in an arbitrary category (not necessarily preadditive).

Definition 3.9. [4, p. 54] A *product* of a family $\{X_i\}_{i\in I}$ of objects in a category $\mathcal{C}$ is a pair $(X, \{\pi_i\}_{i\in I})$, consisting of an object X and morphisms $\pi_i \colon X \to X_i$ (called *projection morphisms*), that satisfies the following universal property: every family of morphisms $\{f_i\}_{i\in I}$ with $f_i \colon W \to X_i$ for some object W factors through the product X via a unique morphism $f \colon W \to X$, denoted by

$$f = (f_i \mid i \in I)$$

in the sense that $f_i = \pi_i f$ for every $i \in I$.

The property is illustrated for $I = \{1, 2\}$ in the commutative diagram below.

$$\begin{tikzcd}
& W \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] \arrow[d, dotted, "f"] & \\
& X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

Dually, we define a *coproduct* of a family $\{X_i\}_{i\in I}$ as a pair $(X, \{\iota_i\}_{i\in I})$, consisting of an object X and morphisms $\iota_i \colon X_i \to X$ (called *inclusion morphisms*), that satisfies the dual universal property: every family of morphisms $\{g_i\}_{i\in I}$ with $g_i \colon X_i \to Y$ factors through the coproduct X via a unique morphism $g \colon X \to Y$, denoted by

$$g = \langle g_i \mid i \in I \rangle$$

in the sense that $g_i = g\iota_i$ for every $i \in I$.

The property is illustrated for $I = \{1, 2\}$ in the commutative diagram below.

$$\begin{tikzcd}
& Y & \\
& X \arrow[u, dotted, "g"] & \\
X_1 \arrow[uur, "g_1"] \arrow[ur, "\iota_1"'] & & X_2 \arrow[uul, "g_2"'] \arrow[ul, "\iota_2"]
\end{tikzcd}$$

Remark 3.10. A category need not have a product or coproduct for any two of its objects. Consider for instance the category $\mathcal{C}_G$ of Example 2.10 with a single object $*$ and morphism set $\mathcal{C}_G(*, *) = G$, for any non-trivial group G.

A product of the family with two instances of $*$ would have to be the only object $*$ of the category equipped with two group elements $g_1, g_2 \in G$ as projection morphisms. If $g_1 = g_2$, then a family of two distinct group elements a, b (which exist by the assumption that G is non-trivial) would fail to factor through the product, because the compositions of any factoring morphism h with each projection morphism would be equal. If $g_1 \neq g_2$, then a family of two equal group elements a, a would fail to factor through the product, because the compositions would in this case not be equal: $g_1 \circ h = g_1 h \neq g_2 h = g_2 \circ h$. This shows that there is no binary product of $*$ and $*$.

Similar reasoning with coproducts shows that there is no binary coproduct of $*$ and $*$ either.

Example 3.11. By Proposition 1.29, any family $\{M_i\}_{i\in I}$ of R-modules has a product and coproduct in R-Mod, namely the direct product $P = \prod_{i\in I} M_i$ equipped with the projection morphisms $\pi_i\colon P \to M_i$ of Definition 1.28 and, respectively, the direct sum $S = \bigoplus_{i\in I} M_i$ equipped with the inclusion morphisms $\iota_i\colon M_i \to S$ of Definition 1.28.

Example 3.12. [5, p. 63, 70] In Sets, the cartesian product $X \times Y$ of two sets X and Y together with the projection functions

$$\begin{aligned} \pi_X\colon X \times Y &\to X \\ (x, y) &\mapsto x \end{aligned} \qquad\qquad \begin{aligned} \pi_Y\colon X \times Y &\to Y \\ (x, y) &\mapsto y \end{aligned}$$

is a product, where the function (f_X, f_Y) induced by any two functions $f_X\colon Z \to X$ and $f_Y\colon Z \to Y$ is given by

$$(f_X, f_Y)(z) = (f_X(z), f_Y(z))$$

The disjoint union $X \sqcup Y = \{(x, 0) \mid x \in X\} \cup \{(0, y) \mid y \in Y\}$ together with the inclusions

$$\begin{aligned} \iota_X\colon X &\to X \sqcup Y \\ x &\mapsto (x, 0) \end{aligned} \qquad\qquad \begin{aligned} \iota_Y\colon Y &\to X \sqcup Y \\ y &\mapsto (0, y) \end{aligned}$$

is a coproduct in Sets, for the unique function defined on $X \sqcup Y$ through which a family $\{f_X, f_Y\}$ with $f_X\colon X \to Z$ and $f_Y\colon Y \to Z$ factors (via the inclusion functions) is given by

$$\begin{aligned} \langle f_X, f_Y\rangle(x, 0) &= f_X(x) \\ \langle f_X, f_Y\rangle(0, y) &= f_Y(y) \end{aligned}$$

Remark 3.13. The products of an empty collection of objects are precisely the terminal objects (of which there may be none, depending on the category), seeing as any morphism into a product vacuously factors through the empty collection of projections. The uniqueness condition is thus equivalent to there being exactly one morphism into the product, which is to say that the product is terminal. Similarly, the coproducts of an empty collection are precisely the initial objects. The situation is depicted below, for the product to the left and the coproduct to the right and with X denoting the product and, respectively, the coproduct.

$$\begin{tikzcd}
& W \arrow[d, "\exists!"'] & \\
& X \arrow[dl, dotted] \arrow[dr, dotted] & \\
{} & & {}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& Y & \\
& X \arrow[u, "\exists!"'] & \\
{} \arrow[ur, dotted] & & {} \arrow[ul, dotted]
\end{tikzcd}$$

Every category admits unary products and coproducts, for an object X has itself equipped with the identity morphism as both a product and coproduct (i.e., as a product and coproduct of the singleton family $\{X\}$). Given a morphism $f\colon W \to X$, the unique morphism through which f factors via the identity morphism (viewed as the projection) is f itself. In other words, $(f) = f$. Similarly, a morphism $g\colon X \to Y$ factors uniquely through itself via the identity morphism (viewed as the inclusion). The situation is shown below, for the product to the left and the coproduct to the right.

$$\begin{tikzcd}
W \arrow[d, "\exists!"'] & Y \\
X \arrow[d, "1_X"'] & X \arrow[u, "\exists!"'] \\
X & X \arrow[u, "1_X"']
\end{tikzcd}$$

If a family of objects has a product (coproduct), then the product (coproduct) is unique up to an isomorphism of sorts.

Proposition 3.14. *[4, p. 55] If $(X, \{\pi_i\}_{i\in I})$ and $(X', \{\pi'_i\}_{i\in I})$ are products of the family of objects $\{X_i\}_{i\in I}$, then there is a unique isomorphism $\xi\colon X' \to X$ such that $\pi'_i = \pi_i\xi$ for every $i \in I$.*

Dually for the coproduct, if $(X, \{\iota_i\}_{i\in I})$ and $(X', \{\iota'_i\}_{i\in I})$ are coproducts of the family of objects $\{X_i\}_{i\in I}$, then there is a unique isomorphism $\xi\colon X \to X'$ such that $\iota'_i = \xi\iota_i$ for every $i \in I$.

Proof. We prove the statement for the product and invoke duality to get the statement for the coproduct.

The situation that we strive for is depicted for $I = \{1, 2\}$ in the following commutative diagram with ξ being an isomorphism:

$$\begin{tikzcd}
 & X' \arrow[d, "\xi"] \arrow[ddl, "\pi'_1"'] \arrow[ddr, "\pi'_2"] & \\
 & X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

As the diagram suggests, the morphism ξ must (by the definition of the product) be the morphism $(\pi'_i \mid i \in I)$ induced by the projections from X'. The uniqueness of ξ in the proposition thus follows.

To prove that ξ is an isomorphism, consider the morphism $\xi' = (\pi_i \mid i \in I)\colon X \to X'$ induced by the projections from X, and show that ξ and ξ' are mutual inverses.

We need to prove that $\xi' \circ \xi = 1_{X'}$ and $\xi \circ \xi' = 1_X$. It suffices to verify the former equality, for then the latter must hold by symmetry.

Consider the compositions $\pi'_i \circ \xi' \circ \xi$ for every $i \in I$. By the universal property of X' and then that of X, the compositions reduce to $\pi'_i \circ \xi' \circ \xi = \pi_i \circ \xi = \pi'_i$. In general, π'_i is not a monomorphism so we cannot immediately conclude that $\xi' \circ \xi = 1_{X'}$. We can, however, apply the universal property of X' to itself with its projections to get a *unique* morphism f such that $\pi'_i = \pi'_i f$ for every $i \in I$, as depicted for $I = \{1, 2\}$ in the diagram below

$$\begin{tikzcd}
 & X' \arrow[d, dotted, "f"] \arrow[ddl, "\pi'_1"'] \arrow[ddr, "\pi'_2"] & \\
 & X' \arrow[dl, "\pi'_1"] \arrow[dr, "\pi'_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

It is plain that $1_{X'}$ is a morphism that satisfies the condition that $\pi'_i = \pi'_i 1_{X'}$ for every $i \in I$, and by the uniqueness in the universal property, $1_{X'}$ is the only such morphism. Thus we conclude that $\xi' \circ \xi = 1_{X'}$. By symmetry we get $\xi \circ \xi' = 1_X$ and conclude that ξ is an isomorphism. ☐

Remark 3.15. The uniqueness up to isomorphism of the product guaranteed by the proposition justifies the notation $\prod_{i\in I} X_i$ for the object X of any product of $\{X_i\}_{i\in I}$, and we may even sloppily speak of this object $X = \prod_{i\in I} X_i$ as the product, as long as it is immaterial which of the products is referred to.

Similarly, we may use the notation $\coprod_{i\in I} X_i$ for the object X of any coproduct of $\{X_i\}_{i\in I}$.

Proposition 3.16. *[4, p. 56] Let $\mathcal{C}$ be a category in which any two objects admit a product. Then, any non-empty finite family of objects in $\mathcal{C}$ admits a product.*

Dually, in a category where any two objects admit a coproduct, any non-empty finite collection of objects admit a coproduct.

Proof. As was noted in Remark 3.13 singleton families always admit a product and coproduct, which takes care of the case $n = 1$. For $n \geq 2$, prove the statement (for products, say) by induction over n.

Base ($n = 2$): This is the assumption of the proposition.

Induction step: Suppose that any family of k objects admits a product and consider a family $\{X_1, \dots, X_{k+1}\}$ of $k+1$ objects. The first k objects have a product $(\prod_{i=1}^{k} X_i, \{\pi_1, \dots, \pi_k\})$, where we may let X' denote the product object, by the induction hypothesis. Consider the product of this product X' by the last object X_{k+1}:

$$X = \left(\prod_{i=1}^{k} X_i\right) \Pi\, X_{k+1}$$

with projections $\pi_{X'} \colon X \to \prod_{i=1}^{k} X_i$ and $\pi_{k+1} \colon X \to X_{k+1}$

Claim. $(X, \{\pi_1\pi_{X'}, \dots, \pi_k\pi_{X'}, \pi_{k+1}\})$ *is a product of the family* $\{X_1, \dots, X_{k+1}\}$ *of* $k+1$ *objects.*

Proof. Let $\{f_1, \dots f_{k+1}\}$ be an arbitrary family of functions with $f_i \colon Z \to X_i$ (for every $i \in \{1, \dots, k+1\}$). We need to show that there is a unique morphism $f \colon Z \to X$ such that f_i factors through the ith projection for every i, i.e., a unique morphism f that gives rise to commutativity in the diagram

$$\begin{tikzcd}
 & & Z \arrow[d, dotted, "f"] \arrow[dddll, "f_1"'] \arrow[ddd, bend left, "f_k"] \arrow[ddr, bend left, "f_{k+1}"] & & \\
 & & X \arrow[dl, "\pi_{X'}"] \arrow[dr, "\pi_{k+1}"] & & \\
 & X' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_k"] & & X_{k+1} & \\
X_1 & \dots & X_k & &
\end{tikzcd}$$

Note that $f = ((f_1, \dots, f_k), f_{k+1})$ is one such morphism, seeing as $\pi_i \pi_{X'} f = \pi_i(f_1, \dots, f_k) = f_i$ for $1 \leq i \leq k$ and $\pi_{k+1} f = f_{k+1}$.

It is the unique such morphism, for if g is another morphism through which $f_1, \dots, f_{k+1}$ factor then $g = f$ as follows: the first k morphisms factor through both $\pi_{X'} f \colon Z \to X'$ and $\pi_{X'} g \colon Z \to X'$ in the sense that $\pi_i(\pi_{X'} f) = f_i = \pi_i(\pi_{X'} g)$ for $1 \leq i \leq k$, so $\pi_{X'} f = \pi_{X'} g$ by the uniqueness property for the product X'. But then the uniqueness property for the binary product of X' and X_{k+1} gives $f = g$. $\square$

This finishes the induction step; if every family of $k \geq 2$ objects admits a product, then every family of $k+1$ objects does so.

By the induction principle, we have shown that any non-empty finite collection of objects in a category with binary products admits a product. The corresponding statement for coproducts follows by duality. $\square$

With the above definitions taken care of, we are ready to define the important notion of an additive category.

Definition 3.17. [4, p. 75] An *additive* category is a preadditive category with a zero object in which every pair of objects admits a product.

Remark 3.18. The requirement that there exist a zero object may, by Remark 3.13, equivalently be viewed as a requirement that there are empty products. By this observation and Proposition 3.16, we may equivalently define an additive category as a preadditive category in which every finite collection of objects admits a product.

Later on (in Corollary 3.31), we shall see that "product" may be replaced by "coproduct" and even "biproduct" (Definition 3.26) in the definition of an additive category. The seeming asymmetry of requiring only products and not coproducts in an additive category is thus naught but a mirage, as is expanded upon in Remark 3.25.

As a first and foremost example of an additive category (formulated as a proposition), we consider the category of all R-modules over some fixed ring R.

Proposition 3.19. *The category* R-Mod *equipped with valuewise addition of morphisms is an additive category.*

Proof. As one might expect, the zero module $\{0\}$ is a zero object seeing as the only morphisms to and from $\{0\}$ are the zero morphisms. We noted in Example 3.11 that any two objects of R-Mod has a product, namely the direct product of the modules with the canonical projections. It remains for us to prove that R-Mod is enriched over Ab with valuewise addition of morphisms, i.e., that each morphism set forms an abelian group with said addition and that composition of morphisms is biadditive.

Claim. $\mathrm{Hom}_R(M, N)$ *is an abelian group with valuewise addition for any* R*-modules* M *and* N*.*

Proof. All the necessary properties for the morphisms are inherited from the module structure on N.

The addition is well-defined in the sense that a sum of two morphisms is a morphism (i.e., is additive and homogeneous):

$$\begin{aligned}(f+g)(m+n) &= f(m+n) + g(m+n)\\ &= [f(m) + f(n)] + [g(m) + g(n)]\\ &= [f(m) + g(m)] + [f(n) + g(n)]\\ &= (f+g)(m) + (f+g)(n)\end{aligned}$$

and

$$\begin{aligned}(f+g)(rm) &= f(rm) + g(rm)\\ &= rf(m) + rg(m)\\ &= r[f(m) + g(m)]\\ &= r(f+g)(m)\end{aligned}$$

for any $r \in R$ and $m, n \in M$. The remaining properties are verified in a similar fashion using the fact that the zero morphism and valuewise negation of a morphism are both morphisms. ☐

Claim. *Composition of morphisms is biadditive in* R-Mod*.*

Proof. We need to prove that

$$(f_1 + f_2) \circ (g_1 + g_2) = (f_1 \circ g_1) + (f_1 \circ g_2) + (f_2 \circ g_1) + (f_2 \circ g_2)$$

holds for any suitable morphisms f_1, f_2, g_1, and g_2 in the R-Mod. We do so by proving equality for an arbitrary argument m in the domain of g_1 and g_2, keeping in mind that the morphisms are additive:

$$\begin{aligned}
[(f_1+f_2)\circ(g_1+g_2)](m) &= (f_1+f_2)[(g_1+g_2)(m)] \\
&= (f_1+f_2)[g_1(m)+g_2(m)] \\
&= f_1[g_1(m)+g_2(m)] + f_2[g_1(m)+g_2(m)] \\
&= f_1[g_1(m)] + f_1[g_2(m)] + f_2[g_1(m)] + f_2[g_2(m)] \\
&= (f_1\circ g_1)(m) + (f_1\circ g_2)(m) + (f_2\circ g_1)(m) + (f_2\circ g_2)(m)
\end{aligned}$$

□

The proposition follows from the claims and the above discussion about the existence of zero objects and products. □

The following proposition shows that equivalences of categories may be used to transfer the properties of an additive category to another category. Its proof, which is omitted, is a straightforward but ratber lengthy verification (requiring perseverance rather than ingenuity) that every axiom for $\mathcal{C}$ to be an additive category holds.

Proposition 3.20. *If $F\colon \mathcal{C} \to \mathcal{A}$ is an equivalence of categories, with $\mathcal{C}$ any category and $\mathcal{A}$ additive, then $\mathcal{C}$ with morphism addition inherited from $\mathcal{A}$ via F as follows is an additive category:*

$$f + g = F^{-1}(F(f) + F(g))$$

(Note that the right-hand side is well-defined seeing as F is bijective on each morphism group.) In other words, the addition is defined in the unique way to make F an additive functor.

Corollary 3.21. Ab *with valuewise addition of morphisms is an additive category.*

Proof. Recall the isomorphism Ab $\to$ $\mathbb{Z}$-Mod from Example 2.62. It is in particular an equivalence of categories and hence induces a morphism addition on Ab endowed with which Ab is an additive category. Concretely, the addition of morphisms in $\mathbb{Z}$-Mod is defined valuewise and the equivalence simply maps a group morphism into the "same" morphism of $\mathbb{Z}$-modules which have the same addition as the groups. Hence, the inherited morphism addition in Ab is also given by valuewise addition. □

The following proposition shows, in particular, that any finite collection of objects in an additive category admits a coproduct, which has many interesting consequences, one of which is mentioned in the remark following the proof.

Proposition 3.22. *[4, p. 76] [5, p. 194] Let $n \geq 0$, $X_1, X_2, \ldots, X_n$ be objects of some preadditive category $\mathcal{C}$, and $X = \prod_{j=1}^n X_j$ together with π_j (for $j = 1, \ldots, n$) be a product of the objects. Set (for $i = 1, \ldots, n$)*

$$\iota_i = (\delta_{ij})_{j=1}^n \colon X_i \to \prod_{j=1}^n X_j$$

where $\delta_{ij}\colon X_i \to X_j$ is the morphism

$$\delta_{ij} = \begin{cases} 1_{X_i} & \text{if } i = j \\ 0_{\mathcal{C}(X_i, X_j)} & \text{if } i \neq j \end{cases}$$

That is, ι_i is the morphism induced by a family consisting of a single identity morphism and the rest zero elements of the morphism groups. Then, $(\prod_{j=1}^n X_j, \{\iota_j\}_{j=1}^n)$ is a coproduct of $\{X_j\}_{j=1}^n$.

Before proving the proposition, we will state and prove a useful proposition to be understood in the above context.

Proposition 3.23. *[4, p. 76] [5, p. 195]* $\iota_1\pi_1 + \cdots + \iota_n\pi_n = 1_X$

Remark 3.24. For $n = 0$, the family of objects in Proposition 3.22 (and in similar statements to come) is understood to be the empty family. Similarly, the sum in the left-hand side of Proposition 3.23 (and in similar statements to come) is the empty sum, which is understood to be the zero element of the group of morphisms from X to X.

Proof of Proposition 3.23. For $n = 0$, note that X is the empty product, which is terminal (Remark 3.13). By Proposition 3.5, $0_{\mathcal{C}(X,X)} = 1_X$ and we get

$$\sum_{i=1}^{0} \iota_i\pi_i = 0_{\mathcal{C}(X,X)} = 1_X$$

For $n \geq 1$, consider the morphism $(\pi_1, \ldots, \pi_n)\colon X \to X$, i.e., the unique morphism through which π_j (for $j = 1, \ldots, n$) factors as $\pi_j = \pi_j(\pi_1, \ldots, \pi_n)$. On one hand, it is clearly the identity morphism 1_X for it has the required factorization property. On the other hand, we may show that it coincides with $\iota_1\pi_1 + \cdots + \iota_n\pi_n$ because it too has the required factorization property: use the definition of all the ι_i in terms of the identity morphism and zero elements and the fact that zero elements compose to zero elements to get

$$\begin{aligned}
\pi_j \circ (\iota_1\pi_1 + \cdots + \iota_n\pi_n) &= \pi_j\iota_1\pi_1 + \cdots + \pi_j\iota_n\pi_n \\
&= \pi_j\iota_1\pi_1 + \cdots + \pi_j\iota_j\pi_j + \cdots + \pi_j\iota_n\pi_n \\
&= 0_{\mathcal{C}(X_i,X_j)}\pi_1 + \cdots + 1_{X_j}\pi_j + \cdots + 0_{\mathcal{C}(X_n,X_j)}\pi_n \\
&= 0_{\mathcal{C}(X,X_j)} + \cdots + \pi_j + \cdots + 0_{\mathcal{C}(X,X_j)} \\
&= \pi_j
\end{aligned}$$

This shows that $\iota_1\pi_1 + \cdots + \iota_n\pi_n = 1_X$. □

Proof of Proposition 3.22. Let $f_i\colon X_i \to Z$ (for $i = 1, \ldots, n$) be a family of arbitrary morphisms into some object Z. We need to find a morphism $f\colon X \to Z$ through which all f_i factor via the proposed inclusion morphisms ι_i, and we need to prove that f is unique with this property.

Consider $f = f_1\pi_1 + \cdots + f_n\pi_n$. From the definition of the ι_i in terms of identity and zero elements, we see that f works: for any $j = 1, \ldots, n$,

$$\begin{aligned}
f\iota_j &= (f_1\pi_1 + \cdots + f_n\pi_n)\iota_j \\
&= f_1\pi_1\iota_j + \cdots + f_n\pi_n\iota_j \\
&= f_1\pi_1\iota_j + \ldots f_j\pi_j\iota_j + \cdots + f_n\pi_n\iota_j \\
&= f_1 0_{\mathcal{C}(X_j,X_1)} + \cdots + f_j 1_{X_j} + \cdots + f_n 0_{\mathcal{C}(X_j,X_n)} \\
&= 0_{\mathcal{C}(X_j,Z)} + \cdots + f_j + \cdots + 0_{\mathcal{C}(X_j,Z)} \\
&= f_j
\end{aligned}$$

To prove uniqueness of this f, let f' be any morphism with the same factorization property as f. By

Proposition 3.23, we may rewrite f and f' and use the fact that the f_i factor through them:

$$\begin{aligned}
f &= f1_X \\
&= f(\iota_1\pi_1 + \cdots + \iota_n\pi_n) \\
&= f\iota_1\pi_1 + \cdots + f\iota_n\pi_n \\
&= f_1\pi_1 + \cdots + f_n\pi_n \\
&= f'\iota_1\pi_1 + \cdots + f'\iota_n\pi_n \\
&= f'(\iota_1\pi_1 + \cdots + \iota_n\pi_n) \\
&= f'1_X \\
&= f'
\end{aligned}$$

Thus we have shown that $X = \prod_{j=1}^{n} X_j$ together with the ι_j (for $j = 1, \ldots, n$) is a coproduct of the objects $X_1, \ldots, X_n$. □

Remark 3.25. As a consequence of the proposition, one may equivalently define additive categories to also have coproducts for any finite collection of objects, which would make for a more symmetric but less practical definition. More precisely, the axioms required for a category to be additive are self-dual [4, p. 75]: the dual of an additive category is an additive category (where the morphism addition in the dual category is understood to be the very same as in the original category).

3.3 Biproducts

Using Proposition 3.22 and its dual, we may conclude that coproducts and products coincide as objects in an additive category and may hence consider the objects equipped with both projection and inclusion morphisms. This notion of a simultaneous product and coproduct turns out to be useful in the study of additive categories and is captured in the following definition.

Definition 3.26. [5, p. 194] Let $\mathcal{C}$ be a preadditive category and $X_1, \ldots, X_n$ (for $n \geq 0$) be a finite collection of objects in $\mathcal{C}$. A *biproduct* of $X_1, \ldots, X_n$ is an object X together with morphisms $\pi_1, \ldots, \pi_n$ (called *projection morphisms*) and $\iota_1, \ldots, \iota_n$ (called *inclusion morphisms*)

$$\begin{tikzcd}
& & X \arrow[dll, bend right, "\pi_1"'] \arrow[drr, bend left, "\pi_n"] & & \\
X_1 \arrow[urr, bend right, "\iota_1"'] & & \cdots & & X_n \arrow[ull, bend left, "\iota_n"]
\end{tikzcd}$$

satisfying for every $i, j \in \{1, \ldots, n\}$

$$\pi_j\iota_i = \delta_{ij} = \begin{cases} 1_{X_i} & \text{if } i = j \\ 0_{\mathcal{C}(X_i, X_j)} & \text{if } i \neq j \end{cases}$$

and

$$\iota_1\pi_1 + \cdots + \iota_n\pi_n = 1_X$$

Remark 3.27. The notion of a biproduct is essentially self-dual in the sense that if X together with $\pi_1, \ldots, \pi_n$ and $\iota_1, \ldots, \iota_n$ is a biproduct of $X_1, \ldots, X_n$ in some preadditive category $\mathcal{C}$, then X together with $\iota_1, \ldots, \iota_n$ and $\pi_1, \ldots, \pi_n$ is a biproduct of the same objects $X_1, \ldots, X_n$ in $\mathcal{C}^{\mathrm{op}}$ (with the same morphism addition).

Remark 3.28. In terms of biproducts, Proposition 3.23 in the setting of Proposition 3.22 may be interpreted as saying that any product (or, by duality, coproduct) in a preadditive category may be expanded to a biproduct.

Remark 3.29. [5, p. 195] One may note that the definition of the biproduct of a family of objects conveniently refers only to these objects and the biproduct object (the definition is "internal"), whereas the definition of the product and coproduct in terms of their universal properties refers to virtually every object of the category (the definition is "external").

By the very definition of the biproduct and in accordance with the previous remark, it is not obvious that every biproduct may be viewed as a simultaneous product and coproduct; the biproduct would seem to be more general than that. The following proposition shows, however, that biproducts indeed are simultaneous products and coproducts (with the obvious morphisms). Hence, finite products, finite coproducts, and biproducts coincide in preadditive categories in the sense that given one of them (e.g., a finite product) one may conjure up or forget about morphisms to get to the others (e.g., a finite coproduct and a biproduct).

Proposition 3.30. *[5, p. 194] Let X together with $\pi_1, \dots, \pi_n$ and $\iota_1, \dots, \iota_n$ be a biproduct of some objects $X_1, \dots, X_n$ in a preadditive category. Then,*

- $(X, \{\pi_1, \dots, \pi_n\})$ *is a product of* $X_1, \dots, X_n$ *and*
- $(X, \{\iota_1, \dots, \iota_n\})$ *is a coproduct of* $X_1, \dots, X_n$.

Proof. In order to prove that $(X, \{\iota_1, \dots, \iota_n\})$ is a coproduct, we may proceed exactly as in the proof of Proposition 3.22, where the identities $\pi_j \iota_i = \delta_{ij}$ and $\sum_{i=1}^n \iota_i \pi_i = 1_X$ were sufficient to prove the universal property of the coproduct (the former implied existence and the latter uniqueness of the factoring morphism).

It follows by the above and that biproducts are "essentially" self-dual (Remark 3.27) that $(X, \{\pi_1, \dots, \pi_n\})$ is a coproduct in the opposite category and hence a product in the original category. □

The following corollary to Propositions 3.22 and 3.30 summarizes the relationship between products, coproducts, and biproducts in a preadditive category.

Corollary 3.31. *[5, p. 194] [3, p. 601] For a collection $X_1, \dots, X_n$ of objects in a preadditive category, the following are equivalent:*

- $X_1, \dots, X_n$ *admit a product.*
- $X_1, \dots, X_n$ *admit a coproduct.*
- $X_1, \dots, X_n$ *admit a biproduct.*

Example 3.32. Recall Remark 1.27, which stated that the direct sum and the direct product of finitely many R-modules $M_1, \dots, M_n$ coincide. Interpreted categorically, we see that this is no coincidence: we know R-Mod to be additive (Proposition 3.19) and that the direct sum and direct product together with their inclusion and projection morphisms are a coproduct and, respectively, a product (Example 3.11). By the uniqueness of products and coproducts (Proposition 3.14) and the above propositions, we may conclude that the direct sum and the direct product would at the very least have to be isomorphic.

In fact, the direct sum and the direct product equipped with their canonical morphisms is a biproduct, which we may verify with ease. For any $i \in \{1, \dots, n\}$, consider the morphism $\pi_i \iota_i \colon M_i \to M_i$ and show that it is the identity morphism by noting that it fixes any element $m_i \in M_i$:

$$(\pi_i \iota_i)(m_i) = \pi_i((0, \dots, 0, m_i, 0, \dots, 0)) = m_i$$

Next, consider the morphism $\iota_1 \pi_1 + \dots + \iota_n \pi_n \colon \bigoplus_{i=1}^n M_i$ and show that it is the identity morphism by noting that it too fixes any element, $(m_1, \dots, m_n)$ say:

$$\begin{aligned}(\iota_1 \pi_1 + \dots + \iota_n \pi_n)((m_1, \dots, m_n)) &= (\iota_1 \pi_1)((m_1, \dots, m_n)) + \dots + (\iota_n \pi_n)((m_1, \dots, m_n)) \\ &= \iota_1(m_1) + \dots + \iota_n(m_n) \\ &= (m_1, 0, \dots, 0) + \dots + (0, \dots, 0, m_n) \\ &= (m_1, \dots, m_n)\end{aligned}$$

Proposition 3.33. *If* $(X, \{\pi_i\}_{i=1}^n, \{\iota_i\}_{i=1}^n)$ *and* $(X', \{\pi_i'\}_{i=1}^n, \{\iota_i'\}_{i=1}^n)$ *are biproducts of the same family of objects* $X_1, \ldots, X_n$*, then there is a unique isomorphism* $\varphi\colon X \to X'$ *that respects the projection and inclusion morphisms, i.e., with*

$$\pi_i = \pi_i' \circ \varphi \qquad \text{and} \qquad \iota_i' = \varphi \circ \iota_i$$

for every i*.*

Proof. For uniqueness, we may view the biproducts as products (or coproducts) via Proposition 3.30 and use Proposition 3.14 to conclude that there is precisely one isomorphism that respects the projection morphisms (or inclusion morphisms) and hence that there is at most one isomorphism that respects both projection and inclusion morphisms.

We may also show the uniqueness more directly by postcomposing φ with $1_{X'} = \sum \iota_i' \circ \pi_i'$. Using only the assumption that φ respects the projection morphisms, we obtain an expression for φ entirely in terms of the projection and inclusion morphisms:

$$\begin{aligned}
\varphi &= 1_{X'} \circ \varphi \\
&= (\sum_{i=1}^n \iota_i' \circ \pi_i') \circ \varphi \\
&= \sum_{i=1}^n (\iota_i' \circ \pi_i' \circ \varphi) \\
&= \sum_{i=1}^n (\iota_i' \circ \pi_i)
\end{aligned}$$

Alternatively, precompose φ with $1_X = \sum \iota_i \circ \pi_i$ and use only the assumption that φ respects the inclusion morphisms and arrive at the same expression:

$$\begin{aligned}
\varphi &= \varphi \circ 1_X \\
&= \varphi \circ (\sum_{i=1}^n \iota_i \circ \pi_i) \\
&= \sum_{i=1}^n (\varphi \circ \iota_i \circ \pi_i) \\
&= \sum_{i=1}^n (\iota_i' \circ \pi_i)
\end{aligned}$$

For existence, we may invoke Propositions 3.14 and 3.30 twice (once for the product and once for the coproduct). This yields two isomorphisms $\varphi, \varphi'\colon X \to X'$ with φ respecting the projection morphisms and φ' respecting the inclusion morphisms. By the chain of equalities above, $\varphi = \sum(\iota_i' \circ \pi_i) = \varphi'$. Hence the isomorphisms are the same and they respect both the projection morphisms and the inclusion morphisms.

Another more explicit approach for the existence is to show that $\varphi = \sum(\iota_i' \circ \pi_i)$ respects the projection and inclusion morphisms and to show that the morphism $\psi = \sum(\iota_i \circ \pi_i')$ is a two-sided inverse of φ.

In any case, we have shown existence and uniqueness of an isomorphism between the biproduct objects that respects the projection and inclusion morphisms, which proves the proposition. $\square$

Remark 3.34. Much like for products and coproducts, the uniqueness of the biproduct object up to isomorphism guaranteed by Proposition 3.33 justifies the notation $\bigoplus_{i=1}^n X_i$ for the object of any biproduct of $X_1, \ldots, X_n$. Somewhat sloppily, we will often use $\bigoplus_{i=1}^n X_i$ to refer to the entire biproduct (including the projection and inclusion morphisms).

The sum of two morphisms in an additive category may be expressed as a composition of morphisms by making clever use of biproducts. Hilton and Stammbach give a slightly more general statement in virtually the same setting [4, p. 76]. In Section 3.5, where we will see how morphisms may be represented by matrices, this proposition is generalized vastly and may be thought of as the special case $f + g = \begin{pmatrix} 1_Y & 1_Y \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$ for any morphisms f and g with codomain Y.

Proposition 3.35. *Let $f, g\colon X \to Y$ be morphisms in an additive category and consider a biproduct $Y \oplus Y$, which is a simultaneous product and coproduct. Let $(f, g)\colon X \to Y \oplus Y$ be the morphism induced by f and g into $Y \oplus Y$ as a product, and $\langle 1_Y, 1_Y\rangle\colon Y \oplus Y \to Y$ be the morphism induced by 1_Y from $Y \oplus Y$ as a coproduct, as shown in the following diagram:*

$$\begin{tikzcd}
& X \arrow[dl, bend right, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, bend left, "g"] & \\
Y \arrow[r, bend right, "\iota_1"'] \arrow[dr, bend right, "1_Y"'] & Y \oplus Y \arrow[l, bend right, "\pi_1"'] \arrow[r, bend left, "\pi_2"] \arrow[d, "{\langle 1_Y, 1_Y \rangle}"] & Y \arrow[l, bend left, "\iota_2"] \arrow[dl, bend left, "1_Y"] \\
& Y &
\end{tikzcd}$$

Then

$$f + g = \langle 1_Y, 1_Y\rangle \circ (f, g)$$

Proof. Insert the sum $\iota_1\pi_1 + \iota_2\pi_2$, which is $1_{Y \oplus Y}$ by the definition of the biproduct, into the composition and use the fact that (f, g) and $\langle 1_Y, 1_Y\rangle$ are induced by the product and coproduct to find that

$$\begin{aligned}
\langle 1_Y, 1_Y\rangle \circ (f, g) &= \langle 1_Y, 1_Y\rangle \circ 1_{Y \oplus Y} \circ (f, g) \\
&= \langle 1_Y, 1_Y\rangle \circ (\iota_1\pi_1 + \iota_2\pi_2) \circ (f, g) \\
&= (\langle 1_Y, 1_Y\rangle \circ \iota_1\pi_1 \circ (f, g)) + (\langle 1_Y, 1_Y\rangle \circ \iota_2\pi_2 \circ (f, g)) \\
&= (\langle 1_Y, 1_Y\rangle\iota_1 \circ \pi_1(f, g)) + (\langle 1_Y, 1_Y\rangle\iota_2 \circ \pi_2(f, g)) \\
&= (1_Y \circ f) + (1_Y \circ g) \\
&= f + g
\end{aligned}$$

□

Remark 3.36. [4, p. 76] Given that the biproduct $Y \oplus Y$ may be constructed from a product $Y \amalg Y$ as in Proposition 3.22 independently of the morphism addition in an additive category, Proposition 3.35 has the curious consequence that the morphism addition in an additive category is determined entirely by the underlying category. In other words, there is no freedom in the choice of morphism addition for a category with zero objects and binary products.

The following proposition lets us construct new additive categories from a given additive category by taking full subcategories and closing them under biproducts.

Proposition 3.37. *Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be a full subcategory of $\mathcal{A}$. If $\mathcal{B}$ has a zero object and is closed under binary biproducts (i.e., for any two objects $A, B \in \mathcal{B}$, every biproduct object $A \oplus B$ in $\mathcal{A}$ is also in $\mathcal{B}$), then $\mathcal{B}$ with morphism addition inherited from $\mathcal{A}$ is an additive category.*

Proof. Because $\mathcal{B}$ is taken to be a full subcategory, every morphism set in $\mathcal{B}$ is an abelian group with the addition from $\mathcal{A}$ (otherwise, the morphism set might not have been closed under the addition in $\mathcal{A}$). Moreover, it is immediate that the composition in $\mathcal{B}$ is biadditive, seeing as both the composition and the addition of morphisms are inherited from $\mathcal{A}$, where the composition is biadditive. Thus, $\mathcal{B}$ is at least a preadditive category with a zero object.

In order to show that $\mathcal{B}$ is additive, it suffices to show that any two objects $A, B \in \mathcal{B}$ admit a biproduct. By the assumption that $\mathcal{B}$ is closed under biproduct objects, there is an object X in $\mathcal{B}$ such that X together with some projection and inclusion morphisms is a biproduct in $\mathcal{A}$, like so:

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\pi_A"'] \arrow[dr, bend left, "\pi_B"] & \\
A \arrow[ur, bend right, "\iota_A"'] & & B \arrow[ul, bend left, "\iota_B"]
\end{tikzcd}$$

But the subcategory is full (and contains all the objects of the above diagram), so all the morphisms of the diagram are present in $\mathcal{B}$. Furthermore, the relevant morphism groups are the very same in $\mathcal{B}$ as in $\mathcal{A}$, which ensures that the morphisms satisfy the equations for the biproduct in $\mathcal{B}$. This shows that $\mathcal{B}$ has binary biproducts, which is to say that $\mathcal{B}$ is not just preadditive but additive. $\square$

Example 3.38. Consider the category R-Free of free R-modules for some fixed ring R. It is a full subcategory of the additive category R-Mod. In order to conclude that R-Free is also additive, it suffices to show that it is closed under biproducts. Let $M_1, \ldots, M_n$ (for $n \geq 0$) be a finite collection of free R-modules and recall from Example 3.32 that the direct sum $M = M_1 \oplus \cdots \oplus M_n$ is a biproduct of the modules. Thus we set out to prove that the direct sum M is free. Seeing as the summands are free, we may take bases $B_i \subseteq M_i$ (for $i = 1, \ldots, n$) for each of them. These bases may then be used to construct a basis of M by including each basis element into the appropriate "slot" in the direct sum, as per the following claim.

Claim. $B := \bigcup_{i=1}^{n} \{\iota_i(b) \mid b \in B_i\}$ *is a basis of* M.

Proof. To show that B generates M, consider an arbitrary element $(m_1, \ldots, m_n) \in M$ and write it as the sum

$$\begin{aligned}
(m_1, \ldots, m_n) &= (m_1, 0, \ldots, 0) + \cdots + (0, \ldots, 0, m_n) \\
&= \iota_1(m_1) + \cdots + \iota_n(m_n)
\end{aligned}$$

It is straightforward to show that each of these terms can be expressed as a linear combination of B, seeing as B_i generates M_i for every i. We may express every component m_i as a linear combination of the elements of the ith basis

$$m_i = \sum_{b \in B_i} r_b^{(i)} \cdot b$$

Thus we get

$$\begin{aligned}
\iota_1(m_1) + \cdots + \iota_n(m_n) &= \iota_1\left(\sum_{b \in B_1} r_b^{(1)} \cdot b\right) + \cdots + \iota_n\left(\sum_{b \in B_n} r_b^{(n)} \cdot b\right) \\
&= \sum_{b \in B_1} \left(r_b^{(1)} \cdot \iota_1(b)\right) + \cdots + \sum_{b \in B_n} \left(r_b^{(n)} \cdot \iota_n(b)\right)
\end{aligned}$$

where all the $\iota_i(b)$ are elements of B. This shows that B generates M.

For linear independence, consider an arbitrary linear combination evaluating to zero:

$$\sum_{c \in B} r_c \cdot c = 0$$

Group the terms by the direct summand that each basis element corresponds to in order to rewrite the linear combination as

$$\left(\sum_{b\in B_1} r_{\iota_1(b)}\cdot \iota_1(b)\right)+\cdots+\left(\sum_{b\in B_n} r_{\iota_n(b)}\cdot \iota_n(b)\right)=\iota_1\left(\sum_{b\in B_1} r_{\iota_1(b)}\cdot b\right)+\cdots+\iota_n\left(\sum_{b\in B_n} r_{\iota_n(b)}\cdot b\right)$$
$$=\left(\sum_{b\in B_1} r_{\iota_1(b)}\cdot b,\ldots,\sum_{b\in B_n} r_{\iota_n(b)}\cdot b\right)$$

The tuple in the right-hand side is assumed to be zero, which is to say that each of its components is zero. However, the components are linear combinations of the bases $B_1,\ldots,B_n$, which in particular are linearly independent. Thus all the coefficients $r_{\iota_i(b)}$ are zero, and these are precisely the coefficients r_c of the original linear combination, which must thus be the trivial linear combination. This shows that B is linearly independent. □

By the claim, a finite direct sum of free R-modules is free. That is, the category R-Free is closed under biproducts and is hence additive by Proposition 3.37.

To be pedantic, the proposition as stated requires *every* biproduct of free R-modules to be free (though a single biproduct suffices in the proof of the proposition), not just our canonical choice of biproduct: the finite direct sum with its projection and inclusion morphisms. But biproduct objects are unique up to isomorphism and R-Free is closed under isomorphism by Proposition 1.33, so every biproduct of free R-modules is indeed free.

Example 3.39. The category R-mod of all finitely generated R-modules is also a full subcategory of R-Mod that is closed under biproducts (finite direct sums and isomorphism). To see this, let $M=\bigoplus_{i=1}^n M_i$ be a direct sum of finitely generated R-modules M_i. Include for every i the generators of M_i into M using ι_i to get a finite set of generators C_i for the elements in M of the form $(0,\ldots,0,m_i,0,\ldots,0)$ with $m_i\in M_i$. Seeing as every element of M is of the form $(m_1,\ldots,m_n)=(m_1,0,\ldots,0)+\cdots+(0,\ldots,0,m_n)$, the union of these C_i (which is a finite union of finite sets) generates M. Thus, R-mod is closed under finite direct sums. For closedness under isomorphism, note that any isomorphism maps generating sets to generating sets, so that any module isomorphic to a finitely generated module is itself finitely generated. By Proposition 3.37, we conclude that R-mod is an additive category.

Example 3.40. An abundance of examples of additive categories may be obtained in the following way. Let M be any fixed R-module. Consider the singleton collection $\{M\}$ of this fixed module and close it simultaneously under direct sums, direct summands, and isomorphism. The full subcategory of R-Mod with the resulting class of objects is denoted by $\mathrm{Add}(M)$.

It is straightforward to verify that taking the closure with respect to isomorphism preserves closedness with respect to direct sums and that taking the closure with respect to direct summands preserves closedness with respect to both isomorphism and direct sums. Thus, the simultaneous closure of $\{M\}$ with respect to direct sums, direct summands, and isomorphism may be viewed as the closure of $\{M\}$ with respect to direct sums, isomorphism, and direct summands (in that order), which gives rise to the following concrete expression for the objects of $\mathrm{Add}(M)$:

$$\text{ob } \mathrm{Add}(M)=\left\{X\in R\text{-Mod}\;\middle|\;\exists Y\in R\text{-Mod}\colon X\oplus Y\cong\bigoplus_{i\in I} M^{(i)} \text{ for some set } I\right\}$$

Note that $\mathrm{Add}(M)$ has a zero object, seeing as the zero module is a direct summand of any module. Moreover, $\mathrm{Add}(M)$ is closed under direct sums (in particular binary direct sums) and isomorphism, so that any biproduct of two objects in $\mathrm{Add}(M)$ is also in $\mathrm{Add}(M)$. That is, $\mathrm{Add}(M)$ is a full subcategory of R-Mod with a zero object and is closed under binary biproducts. Proposition 3.37 has us conclude that $\mathrm{Add}(M)$ is an additive category.

3.4 Additive functors

Seeing as preadditive and additive categories are categories with additional structure, it makes sense to define "homomorphisms" of preadditive and additive categories as functors that respect this additional structure. It turns out that functors that respect the preadditive structure of morphism addition also respect zero objects and biproducts as seen in Propositions 3.43 and 3.44. Thus, the following definition, which a priori seems to be of a "homomorphism" of preadditive categories, is to be understood as of a "homomorphism" of additive categories (as hinted by the term "*additive* functor") when both categories are additive.

Definition 3.41. [4, p. 78] [5, p. 29, 197] An *additive* functor between two preadditive categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that the maps between the morphism sets are homomorphisms of abelian groups, i.e., the maps are additive:

$$F(f+g) = F(f) + F(g)$$

for any morphisms f and g of the same morphism set.

Remark 3.42. It is readily shown from the definitions of additive functors and composition of functors that the composition of two additive functors is again an additive functor.

Proposition 3.43. *If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between preadditive categories and $X \in \mathcal{C}$ is a zero object, then $F(X) \in \mathcal{D}$ is a zero object.*

Proof. Using the characterization of zero objects in Proposition 3.5 for X, we find that $0_{\mathcal{C}(X,X)} = 1_X$. On one hand, because F is additive, the left-hand side is mapped to $0_{\mathcal{D}(F(X),F(X))}$. On the other hand, the right-hand side is mapped to $1_{F(X)}$. In other words, $0_{\mathcal{D}(F(X),F(X))} = 1_{F(X)}$, which is precisely to say that $F(X)$ is a zero object (using the characterization again). $\square$

Proposition 3.44. *[4, p. 77] [5, p. 197] The following are equivalent for a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories:*

(1) F is additive.

(2) F respects binary products. In other words, if $(X \amalg Y, \{\pi_X, \pi_Y\})$ is a product of $X, Y \in \mathcal{A}$, then $(F(X \amalg Y), \{F(\pi_X), F(\pi_Y)\})$ is a product of $F(X), F(Y) \in \mathcal{B}$.

(3) F respects binary coproducts. In other words, if $(X \amalg Y, \{\iota_X, \iota_Y\})$ is a coproduct of $X, Y \in \mathcal{A}$, then $(F(X \amalg Y), \{F(\iota_X), F(\iota_Y)\})$ is a coproduct of $F(X), F(Y) \in \mathcal{B}$.

(4) F respects binary biproducts. In other words, if $(X \oplus Y, \{\pi_X, \pi_Y\}, \{\iota_X, \iota_Y\})$ is a biproduct of $X, Y \in \mathcal{A}$, then $(F(X \oplus Y), \{F(\pi_X), F(\pi_Y)\}, \{F(\iota_X), F(\iota_Y)\})$ is a biproduct of $F(X), F(Y) \in \mathcal{B}$.

Proof. The implication (1) $\implies$ (4) follows readily from the fact that every additive functor respects the ingredients of the biproduct definition: identity morphisms, zero elements of the morphism groups, and morphism composition and addition. Let us verify this by assuming that F is additive, applying it to a biproduct $(X \oplus Y, \{\pi_X, \pi_Y\}, \{\iota_X, \iota_Y\})$ in $\mathcal{A}$, and seeing that a biproduct in $\mathcal{B}$ is indeed obtained. The axioms for composing an inclusion morphism and a projection morphism that match hold (the case for Y is of course the same):

$$F(\pi_X) \circ F(\iota_X) = F(\pi_X \circ \iota_X) = F(1_X) = 1_{F(X)}$$

Similarly, mismatching morphisms compose to the zero element of the morphism group (again, the case with X and Y swapped is the same):

$$F(\pi_Y) \circ F(\iota_X) = F(\pi_Y \circ \iota_X) = F(0_{\mathcal{A}(X,Y)}) = 0_{\mathcal{B}(F(X),F(Y))}$$

Finally, the sum of compositions in the opposite order is the identity on the biproduct object:

$$
\begin{aligned}
F(\iota_X)F(\pi_X) + F(\iota_Y)F(\pi_Y) &= F(\iota_X\pi_X) + F(\iota_Y\pi_Y) \\
&= F(\iota_X\pi_X + \iota_Y\pi_Y) \\
&= F(1_{X \oplus Y}) \\
&= 1_{F(X \oplus Y)}
\end{aligned}
$$

Thus we have shown that an additive functor respects binary biproducts.

The implications (4) $\implies$ (2) and (4) $\implies$ (3) are immediate by Remark 3.28 and Proposition 3.30 stating that products and coproducts may be expanded to biproducts and that a biproduct may be viewed as a product and coproduct. Given a (binary) product or coproduct in $\mathcal{A}$, expand it to a biproduct, use the assumption that F respects biproducts to obtain a biproduct in $\mathcal{B}$, and view this biproduct as a product or coproduct in $\mathcal{B}$ (which will be of the required form, i.e., with the appropriate projection or inclusion morphisms) to conclude that F respects products and coproducts.

For the implication (2) $\implies$ (1), we will make use of the following lemma stating that a functor that respects binary products also respects zero elements in the morphism groups.

Lemma. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. If F respects binary products, then $F(0_{\mathcal{A}(X,Y)}) = 0_{\mathcal{B}(F(X),F(Y))}$ for any objects $X, Y \in \mathcal{A}$.*

Proof. Note that it suffices to show that F maps zero objects to zero objects, because then F respects zero morphisms which in additive categories are precisely zero elements of the morphism groups (by Proposition 3.4).

Let $0 \in \mathcal{A}$ be a zero object and note that $(0, \{1_0, 1_0\})$, i.e., the zero object itself equipped with its identity morphism as projection morphisms, is a binary product of itself. Both existence and uniqueness of the factoring morphism follows readily from the fact that 0 is terminal. The setting is shown in the following diagram, where the object at top is the product object and those at the bottom are the “factors”:

$$
\begin{tikzcd}
 & 0 \arrow[dl, "1_0"'] \arrow[dr, "1_0"] & \\
0 & & 0
\end{tikzcd}
$$

Applying F to the product yields a binary product in $\mathcal{B}$ by the assumption. Now consider any object $Z \in \mathcal{B}$ and any morphisms $f, g\colon Z \to F(0)$ (which exist, because $\mathcal{B}$ is additive and hence has non-empty morphism groups). By the definition of the product, there should be a morphism $h\colon Z \to F(0)$ through which f and g factor:

$$
\begin{tikzcd}
 & Z \arrow[d, dotted, "h"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
 & F(0) \arrow[dl, "F(1_0)"] \arrow[dr, "F(1_0)"'] & \\
F(0) & & F(0)
\end{tikzcd}
$$

Because the projection morphisms are equal, we find that

$$f = F(1_0) \circ h = g$$

so that there is exactly one morphism $Z \to F(0)$. That is to say that $F(0)$ is terminal and by Proposition 3.5 a zero object. We now have for any $X, Y \in \mathcal{A}$

$$\begin{aligned}
F(0_{\mathcal{A}(X,Y)}) &= F(0_{XY}) \\
&= F(X \xrightarrow{0} 0 \xrightarrow{0} Y) \\
&= F(X) \xrightarrow{F(0)} F(0) \xrightarrow{F(0)} F(Y) \\
&= F(X) \xrightarrow{F(0)} \hat{0} \xrightarrow{F(0)} F(Y) \\
&= 0_{F(X)F(Y)} \\
&= 0_{\mathcal{B}(F(X),F(Y))}
\end{aligned}$$

where $\hat{0}$ denotes a zero object in $\mathcal{B}$, which proves the lemma. $\square$

Suppose now that F respects binary products and prove that $F(f+g) = F(f) + F(g)$ for any morphisms $f, g\colon X \to Y$ and objects $X, Y \in \mathcal{A}$. The idea is to express the sums $f+g$ and $F(f)+F(g)$ as compositions via suitable biproducts using Proposition 3.35 and find, because F preserves products, that applying F to the former sum yields the latter sum.

Let $(Y \Pi Y, \{\pi_1, \pi_2\})$ be a product of Y with itself and equip it with inclusion morphisms as in Proposition 3.22, i.e., with $\iota_1 = (1_Y, 0_{\mathcal{A}(Y,Y)})$ and $\iota_2 = (0_{\mathcal{A}(Y,Y)}, 1_Y)$ induced by the product, to obtain a biproduct $(Y \Pi Y, \{\pi_1, \pi_2\}, \{\iota_1, \iota_2\})$. We may now express the sum $f+g$ as the composition $\langle 1_Y, 1_Y \rangle \circ (f,g)$ of the middle morphisms in

$$\begin{tikzcd}
& X \arrow[dl, bend right, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, bend left, "g"] & \\
Y \arrow[r, shift right, "\iota_1"'] \arrow[dr, bend right, "1_Y"'] & Y \Pi Y \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] \arrow[d, "{\langle 1_Y, 1_Y \rangle}"] & Y \arrow[l, shift left, "\iota_2"] \arrow[dl, bend left, "1_Y"] \\
& Y &
\end{tikzcd}$$

just as in Proposition 3.35. Now apply F to the diagram to obtain

$$\begin{tikzcd}
& F(X) \arrow[dl, bend right, "F(f)"'] \arrow[d, "{F((f,g))}"] \arrow[dr, bend left, "F(g)"] & \\
F(Y) \arrow[r, shift right, "F(\iota_1)"'] \arrow[dr, bend right, "1_{F(Y)}"'] & F(Y \Pi Y) \arrow[l, shift right, "F(\pi_1)"'] \arrow[r, shift left, "F(\pi_2)"] \arrow[d, "{F(\langle 1_Y, 1_Y \rangle)}"] & F(Y) \arrow[l, shift left, "F(\iota_2)"] \arrow[dl, bend left, "1_{F(Y)}"] \\
& F(Y) &
\end{tikzcd}$$

Note that $F(f+g)$ is the composition of the middle morphisms:

$$F(f+g) = F(\langle 1_Y, 1_Y \rangle \circ (f,g)) = F(\langle 1_Y, 1_Y \rangle) \circ F((f,g))$$

With some effort, we may also find $F(f)+F(g)$ as said composition. Note first that the middle row constitutes a biproduct diagram: the middle object $F(Y \Pi Y)$ together with $F(\pi_1)$ and $F(\pi_2)$ is certainly a product, seeing as F respects products. Furthermore, $F(\iota_1) = (1_Y, 0_{\mathcal{B}(F(Y),F(Y))})$ and $F(\iota_2) = (0_{\mathcal{B}(F(Y),F(Y))}, 1_Y)$ are the morphisms induced as in Proposition 3.22 by the product. To see this, it suffices by the uniqueness of the factoring morphism to note that $F(\iota_1)$ and $F(\iota_2)$ compose with the projection morphisms to the identity morphism and the zero element of the morphism group. For matching indices, this follows from the fact that functors respect identity morphisms, whereas mismatching indices require the lemma:

$$F(\pi_1)F(\iota_1) = F(\pi_1\iota_1) = F(1_Y) = 1_{F(Y)}$$

and

$$F(\pi_2)F(\iota_1) = F(\pi_2\iota_1) = F(0_{\mathcal{A}(Y,Y)}) = 0_{\mathcal{B}(F(Y),F(Y))}$$

(with virtually the same chain of equalities for the omitted compositions). By Proposition 3.23 the middle row is indeed a biproduct diagram.

Observe next that the vertical morphisms in the middle are those induced by the outer morphisms, again by noting that they compose properly with the projection and inclusion morphisms and invoking the uniqueness property for the factoring morphisms. We have

$$F(\pi_1) \circ F((f,g)) = F(\pi_1 \circ (f,g)) = F(f)$$

and

$$F(\pi_2) \circ F((f,g)) = F(\pi_2 \circ (f,g)) = F(g)$$

so that $F((f,g)) = (F(f), F(g))$ is the morphism induced by $F(f)$ and $F(g)$ with $F(Y \amalg Y)$ as a product. Similarly,

$$F(\langle 1_Y, 1_Y \rangle) \circ F(\iota_1) = F(\langle 1_Y, 1_Y \rangle \circ \iota_1) = F(1_Y) = 1_{F(Y)}$$

and

$$F(\langle 1_Y, 1_Y \rangle) \circ F(\iota_2) = F(\langle 1_Y, 1_Y \rangle \circ \iota_2) = F(1_Y) = 1_{F(Y)}$$

so that $F(\langle 1_Y, 1_Y \rangle) = \langle 1_{F(Y)}, 1_{F(Y)} \rangle$ is the morphism induced by $1_{F(Y)}$ and $1_{F(Y)}$ with $F(Y \amalg Y)$ as a coproduct. By Proposition 3.35, the composition of the vertical morphisms is thus the sum $F(f) + F(g)$:

$$\begin{aligned} F(f) + F(g) &= \langle 1_{F(Y)}, 1_{F(Y)} \rangle \circ (f,g) \\ &= F(\langle 1_Y, 1_Y \rangle) \circ F((f,g)) \end{aligned}$$

We conclude that $F(f+g) = F(f) + F(g)$ and, because f and g were arbitrary, that F is an additive functor.

Finally, we may reason by duality to prove (3) $\implies$ (1). Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ respects coproducts. Viewed as a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ between the opposite categories (recall Remark 2.54), F respects products and hence is additive by the implication (2) $\implies$ (1) showed previously. But the morphism addition in $\mathcal{A}^{\mathrm{op}}$ and $\mathcal{B}^{\mathrm{op}}$ is the very same as in $\mathcal{A}$ and $\mathcal{B}$, respectively, so that F viewed as a functor $\mathcal{A} \to \mathcal{B}$ is additive. □

The Hom functors for an additive category, which by Proposition 3.7 may be viewed as functors into the additive category Ab, are an important example of additive functors.

Proposition 3.45. *[5, p. 197] Let $\mathcal{C}$ be a preadditive category and $A \in \mathcal{C}$ be a fixed object. Then the* Hom *functors* $\mathrm{Hom}_{\mathcal{C}}(A, -)\colon \mathcal{C} \to \mathrm{Ab}$ *and* $\mathrm{Hom}_{\mathcal{C}}(-, A)\colon \mathcal{C} \to \mathrm{Ab}$ *are additive.*

Proof. Note first that the proposition makes sense, seeing as Ab is additive by Corollary 3.21. The proposition is a rather immediate consequence of the definition of the Hom functors in terms of composition and the biadditivity of the composition in an additive category: recall that the covariant Hom functor is defined by $\mathrm{Hom}_{\mathcal{C}}(A,-)(f) = f \circ -$. Additivity in the right argument of the composition ensures that this functor is additive. Similarly, the contravariant Hom functor is defined by $\mathrm{Hom}_{\mathcal{C}}(-,A)(f) = - \circ f$, and additivity in the left argument of the composition ensures that this functor is additive. This argument is made particularly clear if the composition is written as $- \circ -$. □

Example 3.46. By Proposition 3.44 and Proposition 3.45, the morphism set for two biproducts $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ of an additive category $\mathcal{A}$ decomposes as

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X_1 \oplus X_2, Y_1 \oplus Y_2) &= \mathrm{Hom}_{\mathcal{A}}(X_1 \oplus X_2, -)(Y_1 \oplus Y_2) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(X_1 \oplus X_2, -)(Y_1) \oplus \mathrm{Hom}_{\mathcal{A}}(X_1 \oplus X_2, -)(Y_2) \\ &= \mathrm{Hom}_{\mathcal{A}}(-, Y_1)(X_1 \oplus X_2) \oplus \mathrm{Hom}_{\mathcal{A}}(-, Y_2)(X_1 \oplus X_2) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(-, Y_1)(X_1) \oplus \mathrm{Hom}_{\mathcal{A}}(-, Y_1)(X_2) \oplus \mathrm{Hom}_{\mathcal{A}}(-, Y_2)(X_1) \oplus \mathrm{Hom}_{\mathcal{A}}(-, Y_2)(X_2) \\ &= \mathrm{Hom}_{\mathcal{A}}(X_1, Y_1) \oplus \mathrm{Hom}_{\mathcal{A}}(X_1, Y_2) \oplus \mathrm{Hom}_{\mathcal{A}}(X_2, Y_1) \oplus \mathrm{Hom}_{\mathcal{A}}(X_2, Y_2) \end{aligned}$$

3.5 Matrix notation

The reader may recall the decomposition of morphism sets in Example 3.46 from linear algebra, where the morphism sets are not just abelian groups but linear spaces (with valuewise addition and scalar multiplication). The morphisms in that setting, i.e., linear maps, may be viewed as matrices and the above decomposition then corresponds to dividing the matrices into blocks. Furthermore, matrix addition and multiplication corresponds to addition and composition of morphisms.

It turns out that the idea of viewing morphisms as matrices, with addition and composition of morphisms as matrix addition and multiplication, is fruitful not only in linear algebra (i.e., in the category $\mathrm{Vec}\,K$) but in *any* additive category. The details are found in the following proposition.

Proposition 3.47. *Let $X = \bigoplus_{j=1}^n X_j$ and $Y = \bigoplus_{i=1}^m Y_i$ be biproducts in an additive category $\mathcal{A}$. A morphism $\varphi\colon X \to Y$ induces a family of morphisms $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ with $\varphi_{ij}\colon X_j \to Y_i$ via $\varphi_{ij} = \pi_i \circ \varphi \circ \iota_j$. Conversely, any family of morphisms $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ with $\varphi_{ij}\colon X_j \to Y_i$ induces a unique map $\varphi\colon X \to Y$ via the universal properties of the biproducts viewed as a coproduct and a product. The setting is depicted in the following diagram:*

$$\begin{tikzcd}
\bigoplus_{j=1}^n X_j \arrow[r, "\varphi"] \arrow[dr, dotted, "\varphi_i"] & \bigoplus_{i=1}^m Y_i \arrow[d, "\pi_i"] \\
X_j \arrow[u, "\iota_j"] \arrow[r, "\varphi_{ij}"] & Y_i
\end{tikzcd}$$

For any morphism φ as above, arrange its corresponding morphisms into a matrix $[\varphi]$ as follows:

$$[\varphi] := \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}$$

1. *The correspondence is an isomorphism of the abelian groups $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathrm{Hom}_{\mathcal{A}}(X_j, Y_i)$. In particular, the matrix of a sum is obtained by matrix addition:*

$$[\varphi + \varphi'] = [\varphi] + [\varphi']$$

   where the addition of matrix entries in the right-hand side is just the morphism addition in the additive category.

2. *Let $W = \bigoplus_{k=1}^o W_k$ be a third biproduct. If $\psi\colon W \to X$ and $\varphi\colon X \to Y$, then*

$$(\varphi \circ \psi)_{ik} = \sum_{j=1}^n \varphi_{ij} \circ \psi_{jk}$$

   In other words, the matrix of the composition $\varphi \circ \psi$ is obtained by matrix multiplication

$$[\varphi \circ \psi] = [\varphi] \cdot [\psi]$$

   where multiplication of the entries is done by composition and addition is the addition on morphisms in the additive category.

Proof.

1. First verify that the given maps between the Hom space of sums and the sum of Hom spaces are mutual inverses and hence that they really define a one-to-one correspondence.

A morphism $\varphi\colon X \to Y$ between the biproducts induces a family $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ of morphisms where $\varphi_{ij} = \pi_i \circ \varphi \circ \iota_j$. We need to show that this family induces (by viewing the suitable biproducts as a coproduct and, respectively, a product) the morphism φ that we started with.

For every i, the family induces (by viewing the biproduct X as a coproduct) a unique morphism $\varphi_i\colon X \to Y_i$ (as depicted in the proposition) satisfying $\varphi_i \circ \iota_j = \varphi_{ij}$. But $\pi_i \circ \varphi$ is a morphism satisfying said condition, so $\varphi_i = \pi_i \circ \varphi$ by the uniqueness of such a morphism. Next, this family $\{\varphi_i\}_{i \in I}$ of "intermediate morphisms" induces (by viewing the biproduct Y as a product) a unique morphism $\varphi'\colon X \to Y$ satisfying $\pi_i \circ \varphi' = \varphi_i$. Recall that $\varphi_i = \pi_i \circ \varphi$ and conclude that $\varphi' = \varphi$ by the uniqueness.

Conversely, a morphism family $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ where $\varphi_{ij}\colon X_j \to Y_i$ (for every i and j) induces, by the above, a morphism φ satisfying $\varphi_{ij} = \pi_i \circ \varphi \circ \iota_j$, which in turn induces the family we started with.

Thus we have shown that the above defines a one-to-one correspondence between on one hand morphisms between the sums and on the other hand families of morphisms between the summands.

Next, we note that every such family of morphisms $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ may be viewed as an element of the biproduct $\bigoplus_{j=1}^n \bigoplus_{i=1}^m \operatorname{Hom}_{\mathcal{A}}(X_j, Y_i)$ of abelian groups, which we by Example 3.32 and Example 2.62 can take to be the familiar direct sum of $\mathbb{Z}$-modules, by identifying it with the tuple of tuples (or matrix if you will)

$$\left(\begin{pmatrix} \varphi_{11} \\ \varphi_{21} \\ \vdots \\ \varphi_{m1} \end{pmatrix} \quad \begin{pmatrix} \varphi_{12} \\ \varphi_{22} \\ \vdots \\ \varphi_{m2} \end{pmatrix} \quad \cdots \quad \begin{pmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \vdots \\ \varphi_{mn} \end{pmatrix} \right)$$

Finally, we verify that the correspondence is a morphism of abelian groups, i.e., that it is additive. Consider the sum $\varphi + \psi$ of two morphisms $\varphi, \psi\colon X \to Y$. The (i,j)th member of its corresponding family of morphisms is $\pi_i \circ (\varphi + \psi) \circ \iota_j$, which we may write as a sum

$$\begin{aligned} \pi_i \circ (\varphi + \psi) \circ \iota_j &= ((\pi_i \circ \varphi) + (\pi_i \circ \psi)) \circ \iota_j \\ &= (\pi_i \circ \varphi \circ \iota_j) + (\pi_i \circ \psi \circ \iota_j) \end{aligned}$$

But the right-hand side is the sum of the (i,j)th members of the families corresponding to φ and ψ. Seeing as the addition is defined componentwise in the direct sum (i.e., the biproduct), this shows that the correspondent of a sum is the sum of the correspondents; the bijection at hand is a morphism of abelian groups.

2. Consider the (generally non-commutative) diagram

$$\begin{tikzcd} \bigoplus_{k=1}^o W_k \arrow[r, "\psi"] & \bigoplus_{j=1}^n X_j \arrow[r, "\varphi"] \arrow[d, bend right, "\pi_j"'] & \bigoplus_{i=1}^m Y_i \arrow[d, "\pi_i"] \\ W_k \arrow[u, "\iota_k"] \arrow[r, "\psi_{jk}"] & X_j \arrow[u, bend right, "\iota_j"'] \arrow[r, "\varphi_{ij}"] & Y_i \end{tikzcd}$$

and note that $\sum_{j=1}^n \iota_j \pi_j = 1_X$ by the definition of the biproduct. We may thus write

$$\begin{aligned} (\varphi \circ \psi)_{ik} &= \pi_i \circ (\varphi \circ \psi) \circ \iota_k \\ &= \pi_i \circ \varphi \circ 1_X \circ \psi \circ \iota_k \\ &= \pi_i \circ \varphi \circ (\sum_{j=1}^n \iota_j \circ \pi_j) \circ \psi \circ \iota_k \end{aligned}$$

Because the composition is biadditive in additive categories, we may move out the sum and then use the definition of φ_{ij} and ψ_{jk} to obtain what we were looking for:

$$\begin{aligned}
\pi_i \circ \varphi \circ (\sum_{j=1}^{n} \iota_j \circ \pi_j) \circ \psi \circ \iota_k &= \sum_{j=1}^{n} \pi_i \circ \varphi \circ (\iota_j \circ \pi_j) \circ \psi \circ \iota_k \\
&= \sum_{j=1}^{n} (\pi_i \circ \varphi \circ \iota_j) \circ (\pi_j \circ \psi \circ \iota_k) \\
&= \sum_{j=1}^{n} \varphi_{ij} \circ \psi_{jk}
\end{aligned}$$

□

Remark 3.48. Proposition 3.47 might seem to deal with only very particular morphisms whose domain and codomain are biproducts. However, any object X may be viewed as a singleton biproduct of itself by equipping it with its identity morphism as the projection and inclusion morphism. In this sense, *any* morphism in an additive category may be thought of as a matrix.

Example 3.49. Let K be a field and consider the vector spaces K^m and K^n for some $m, n \in \mathbb{N}$ (i.e., up to isomorphism two arbitrary finite-dimensional vector spaces over K). They decompose as the direct sums (biproducts)

$$K^m = \bigoplus_{i=1}^{m} K^{(i)}, \qquad K^n = \bigoplus_{j=1}^{n} K^{(j)}$$

Proposition 3.47 states that any morphism $\varphi\colon K^n \to K^m$ may be expressed equivalently as the $m \times n$ matrix

$$[\varphi] = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}$$

of linear maps $\varphi_{ij} = \pi_i \circ \varphi \circ \iota_j \colon K^{(j)} \to K^{(i)}$, which is almost but not quite what we would expect; in linear algebra, the matrix entries are typically elements in the scalar field K rather than morphisms (linear maps) $K \to K$. Fortunately, one readily verifies that every morphism $\varphi\colon K \to K$ acts simply by scaling by some $k = \varphi(1) \in K$ and that the mapping of a morphism $\varphi\colon K \to K$ to the factor $\varphi(1) \in K$ by which it scales is a one-to-one correspondence that respects addition and multiplication (where "multiplication" of morphisms is understood to be composition) (formally, this correspondence is an isomorphism of unital algebras). In other words, we may think of the matrix of morphisms above as the matrix of scalars

$$\begin{pmatrix} \varphi_{11}(1) & \cdots & \varphi_{1n}(1) \\ \vdots & \ddots & \vdots \\ \varphi_{m1}(1) & \cdots & \varphi_{mn}(1) \end{pmatrix}$$

without running into trouble when adding or multiplying such matrices of scalars. Finally, one may verify that this is indeed the matrix of φ (in the standard bases) from linear algebra, for instance by observing that the jth column (for any j) is the image under φ of the jth standard basis vector, and hence that Proposition 3.47 gives a slightly different but in the end equivalent representation of linear maps as matrices compared to the one usually encountered in linear algebra.

In particular, we may evaluate the morphism by a matrix-vector multiplication, as is customary in linear algebra, of the matrix of morphisms and a vector $(k_1, \ldots, k_n) \in K^n$ with the convention that the product of a morphism and a scalar is the morphism applied to the scalar:

$$\begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} \varphi_{11}(k_1) + \cdots + \varphi_{1n}(k_n) \\ \vdots \\ \varphi_{m1}(k_1) + \cdots + \varphi_{mn}(k_n) \end{pmatrix}$$

Remark 3.50. The previous example generalizes from the setting of finite-dimensional vector spaces (vec K) to the setting of modules over a commutative ring R with the caveat that not all finitely generated R-modules are necessarily free (and hence might not decompose up to isomorphism as a direct sum of regular modules). In other words, the example generalizes nicely to the category R-free (which is additive by Examples 3.38 and 3.39) with R commutative.

Evaluating the morphism by a matrix-vector multiplication of sorts generalizes even further – to all of R-Mod with R any ring, where we might not be able to view the matrix of morphisms as a matrix of ring elements. To see that this is so, consider two finite direct sums of modules $M = \bigoplus_{j=1}^{n} M_j$ and $N = \bigoplus_{i=1}^{m} N_i$ and a morphism $\varphi\colon M \to N$ with corresponding matrix

$$[\varphi] = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}$$

An arbitrary element $x \in M$ may be written as $x = (x_1, \dots, x_n) = \iota_1(x_1) + \cdots + \iota_n(x_n)$. The ith element of the tuple $\varphi(x) \in N$ is thus

$$\begin{aligned} \pi_i(\varphi(x)) &= \pi_i(\varphi(\iota_1(x_1) + \cdots + \iota_n(x_n))) \\ &= \pi_i(\varphi(\iota_1(x_1)) + \cdots + \varphi(\iota_n(x_n))) \\ &= \pi_i(\varphi(\iota_1(x_1))) + \cdots + \pi_i(\varphi(\iota_n(x_n))) \\ &= (\pi_i \circ \varphi \circ \iota_1)(x_1) + \cdots + (\pi_i \circ \varphi \circ \iota_n)(x_n) \\ &= \varphi_{i1}(x_1) + \cdots + \varphi_{in}(x_n) \end{aligned}$$

which is precisely the ith element of the tuple obtained by matrix-vector multiplication.

Example 3.51. Consider an arbitrary biproduct $X = \bigoplus_{k=1}^{n} X_k$ in an additive category.

The matrix of the identity morphism on X is the identity matrix (with identity morphisms as ones):

$$[1_X] = \begin{pmatrix} 1_{X_1} & 0 & \cdots & 0 \\ 0 & 1_{X_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{X_n} \end{pmatrix}$$

seeing as the (i,j)th entry of the matrix is $\pi_i \circ 1_X \circ \iota_j = \pi_i \circ \iota_j = \delta_{ij}$ by the definition of the biproduct.

The matrix of the inclusion morphism ι_k of X_k in X is the column matrix with a one (identity morphism) in the kth row and the rest zeros:

$$[\iota_k] = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1_{X_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

seeing as the ith entry of the matrix is $\pi_i \circ \iota_k \circ \iota = \pi_i \circ \iota_k = \delta_{ik}$, where ι is the inclusion of X_k into the singleton biproduct of X_k, which can be taken as X_k itself with the identity morphism as both inclusion and projection.

The matrix of the projection morphism π_k of X on X_k is the row matrix with a one (identity morphism) in the kth column and the rest zeros:

$$[\pi_k] = \begin{pmatrix} 0 & \cdots & 0 & 1_{X_k} & 0 & \cdots & 0 \end{pmatrix}$$

seeing as the jth entry of the matrix is $\pi \circ \pi_k \circ \iota_j = \pi_k \circ \iota_j = \delta_{kj}$, where π is the projection of the singleton biproduct X_k onto X_k and may be taken as the identity morphism.

Let $Y = \bigoplus_{j=1}^m Y_j$ be another biproduct, and let $\mathcal{A}$ denote the category. The matrix of the zero morphism $0_{XY} = 0_{\mathcal{A}(X,Y)} \colon X \to Y$ is the zero matrix:

$$[0_{\mathcal{A}(X,Y)}] = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

This follows by noting that the zero element $0_{\mathcal{A}(X,Y)} \in \mathcal{A}(X,Y)$ of the morphism group corresponds to the zero element $0 \in \bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathcal{A}(X_j, Y_i)$, seeing as the correspondence is additive, and that this latter element (viewed as a matrix) must have all entries zero morphisms, by the componentwise addition of matrices.

Remark 3.52. In diagrams with biproducts, it is often convenient to specify morphisms to and from biproducts by their matrices rather than by their equivalent and cumbersome arithmetic expressions or by assigning them specific symbols, which is both less descriptive and more wasteful of unused symbols – a scarce resource at times. Henceforth, we will adopt this practice, sloppy as it may be, of identifying a morphism with its matrix in diagrams as well as in other contexts where deemed fit.

At this point, we may (fairly) painlessly prove that the biproduct exhibits the commutativity and associativity properties that one might expect it to have.

Proposition 3.53. *Let $X_1, \ldots, X_n$ be objects in an additive category. Let $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation of the indices and consider the corresponding permutation of the objects $X_{\sigma(1)}, \ldots, X_{\sigma(n)}$. Then there is a canonical isomorphism $\varphi \colon X_1 \oplus \cdots \oplus X_n \to X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(n)}$, whose matrix is the permutation matrix for σ, i.e., the matrix with $\varphi_{ij} := \delta_{j\sigma(i)} \colon X_j \to X_{\sigma(i)}$ as (i,j)th entry.*

Proof. Let $\psi \colon X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(n)} \to X_1 \oplus \cdots \oplus X_n$ be the morphism whose matrix is the permutation matrix of the inverse permutation σ^{-1}. By careful thinking, the (i,j)th entry of $[\psi]$ is $\delta_{\sigma(j)i} \colon X_{\sigma(j)} \to X_i$. In terms of rows and columns rather than individual elements of the matrices, the ith row of $[\varphi]$ has an identity morphism in column $\sigma(i)$ and zero morphisms elsewhere, and the ith *column* of $[\psi]$ has an identity morphism in *row* $\sigma(i)$ and zero morphisms elsewhere. It would thus seem that ψ is a right inverse of φ, which we may indeed verify by noting that the (i,j)th entry of the product $[\varphi] \cdot [\psi]$ is

$$\begin{aligned} \sum_{k=1}^n \varphi_{ik} \circ \psi_{kj} &= \sum_{k=1}^n \delta_{k\sigma(i)} \circ \delta_{\sigma(j)k} \\ &= \delta_{\sigma(i)\sigma(i)} \circ \delta_{\sigma(j)\sigma(i)} \\ &= 1_{X_{\sigma(i)}} \circ \delta_{\sigma(j)\sigma(i)} \\ &= \delta_{\sigma(j)\sigma(i)} \\ &= \begin{cases} 1_{X_{\sigma(i)}} & \text{if } \sigma(i) = \sigma(j) \\ 0_{X_{\sigma(j)} X_{\sigma(i)}} & \text{if } \sigma(i) \neq \sigma(j) \end{cases} \end{aligned}$$

Note that $\sigma(i) = \sigma(j)$ if and only if $i = j$ (because σ is a permutation). Thus, $[\varphi \circ \psi] = [\varphi] \cdot [\psi]$ is the identity matrix and $\varphi \circ \psi$ is the identity morphism on $X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(n)}$. For symmetry reasons, one would expect that φ is a right inverse of ψ, and this is indeed the case: the (i, j)th entry of the product $[\psi] \cdot [\varphi]$ is

$$\begin{aligned} \sum_{k=1}^{n} \psi_{ik} \circ \varphi_{kj} &= \sum_{k=1}^{n} \delta_{\sigma(k)i} \circ \delta_{j\sigma(k)} \\ &= \delta_{ii} \circ \delta_{ji} \\ &= 1_{X_i} \circ \delta_{ji} \\ &= \delta_{ji} \end{aligned}$$

The matrix $[\psi \circ \varphi] = [\psi] \cdot [\varphi]$ is thus the identity matrix, and $\psi \circ \varphi$ is the identity morphism on $X_1 \oplus \cdots \oplus X_n$. This shows that φ and ψ are mutually inverse isomorphisms; in particular, φ is an isomorphism. $\square$

Proposition 3.54. *Let $X_1, \ldots, X_k, \ldots, X_m, \ldots, X_n$ be objects of an additive category. Given any two biproducts $A := X_1 \oplus \cdots \oplus X_{k-1} \oplus (X_k \oplus \cdots \oplus X_m) \oplus X_{m+1} \oplus \cdots \oplus X_n$ and $B := X_1 \oplus \cdots \oplus X_n$, there are mutually inverse canonical isomorphisms φ and ψ between the biproducts, namely the identity matrices up to parenthesization:*

$$[\varphi] = \begin{pmatrix} 1 & \cdots & 0 & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \end{pmatrix} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \end{pmatrix} & 0 & \cdots & 1 \end{pmatrix} : A \to B$$

and

$$[\psi] = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : B \to A$$

Proof. Carrying out the matrix multiplication, one finds that

$$
\begin{aligned}
[\psi \circ \varphi] &= [\psi] \cdot [\varphi] \\
&= \begin{pmatrix}
1 & \cdots & 0 & \begin{pmatrix}0 & 0 & \cdots & 0 & 0\end{pmatrix} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \begin{pmatrix}0 & 0 & \cdots & 0 & 0\end{pmatrix} & 0 & \cdots & 0 \\
\begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix} & \cdots & \begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix} & \begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix} & \begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix} & \cdots & \begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix} \\
0 & \cdots & 0 & \begin{pmatrix}0 & 0 & \cdots & 0 & 0\end{pmatrix} & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \begin{pmatrix}0 & 0 & \cdots & 0 & 0\end{pmatrix} & 0 & \cdots & 1
\end{pmatrix}
\end{aligned}
$$

which by Example 3.51 is the identity matrix. Similarly for $\varphi \circ \psi$, one finds that

$$
\begin{aligned}
[\varphi \circ \psi] &= [\varphi] \cdot [\psi] \\
&= \begin{pmatrix}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\end{aligned}
$$

This shows that φ and ψ are mutual inverses. □

Remark 3.55. Propositions 3.53 and 3.54 in their general forms may appear intimidating. However, the special cases with two and, respectively, three objects are not only much more tangible but quite useful on their own.

For any two objects X, Y the canonical isomorphisms between $X \oplus Y$ and $Y \oplus X$ are

$$
\begin{tikzcd}
X \oplus Y \arrow[r, bend left, "\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)"] & Y \oplus X \arrow[l, bend left, "\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)"]
\end{tikzcd}
$$

Postcomposing a morphism (thought of as a matrix) into either of the biproducts with the corresponding isomorphism simply amounts to swapping the rows of the matrix. Similarly, precomposing a morphism from either of the biproducts with the corresponding isomorphism amounts to swapping the columns of the matrix.

For any three objects X, Y, Z, there are canonical isomorphisms

$$\begin{tikzcd}[column sep=huge]
(X \oplus Y) \oplus Z \arrow[r, shift left, "{\begin{pmatrix} (1\ 0) & 0 \\ (0\ 1) & 0 \\ (0\ 0) & 1 \end{pmatrix}}"] & X \oplus Y \oplus Z \arrow[l, shift left, "{\begin{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & 0 & 1 \end{pmatrix}}"] \arrow[r, shift right, swap, "{\begin{pmatrix} 1 & 0 & 0 \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}}"] & X \oplus (Y \oplus Z) \arrow[l, shift right, swap, "{\begin{pmatrix} 1 & (0\ 0) \\ 0 & (1\ 0) \\ 0 & (0\ 1) \end{pmatrix}}"]
\end{tikzcd}$$

It is straightforward to verify that composition with any of the above isomorphisms amounts to grouping or ungrouping rows or columns. More specifically, postcomposing a morphism into $X \oplus Y \oplus Z$ with either of the isomorphisms above amounts to grouping two rows into a single row, and precomposing a morphism from $X \oplus Y \oplus Z$ with either of the isomorphisms above amounts to grouping two columns into a single column. Postcomposing a morphism into $(X \oplus Y) \oplus Z$, say, with the isomorphism above amounts to ungrouping the first row into two rows, and precomposing a morphism from $(X \oplus Y) \oplus Z$ by the other isomorphism above amounts to ungrouping the first column into two columns.

Remark 3.56. Repeated use of Propositions 3.53 and 3.54 shows that any two "nested" biproducts of the same collection of objects, regardless of order and parenthesization, are isomorphic. Intuitively, this may be seen as follows: start with one of the biproduct expressions. Remove all parentheses using Proposition 3.54, starting with the outermost pairs. Rearrange the objects using Proposition 3.53. Add parentheses as needed using Proposition 3.54, starting with the innermost pairs, to obtain the other biproduct expression. The propositions guarantee that all the intermediate biproduct objects, including the last, are isomorphic to the original.

Remark 3.57. When passing to the dual category, the matrix of a morphism is transposed. To see that this is so, consider a morphism $\bigoplus_{j=1}^n X_j \xrightarrow{\varphi} \bigoplus_{i=1}^m Y_i$ in an additive category $\mathcal{A}$. In the opposite category $\mathcal{A}^{\mathrm{op}}$, we are looking at $\bigoplus_{j=1}^n X_j \xleftarrow{\varphi} \bigoplus_{i=1}^m Y_i$ where the roles of the inclusion and projection morphisms in the biproducts are swapped. The matrix of φ in the dual category, which we may denote by $[\varphi]^{\mathrm{op}}$, is thus an $n \times m$ matrix and its (j,i)th entry is $\iota_j \circ \varphi \circ \pi_i$. But this composition is in the original category $\pi_i \circ \varphi \circ \iota_j$, i.e., φ_{ij}. Thus, from the point of view of the original category, we have

$$[\varphi]^{\mathrm{op}} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{m1} \\ \vdots & \ddots & \vdots \\ \varphi_{1n} & \cdots & \varphi_{mn} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}^{\mathrm{tr}} = [\varphi]^{\mathrm{tr}}$$

where the superscripted "tr" denotes matrix transposition.

The following proposition and corollary provide sufficient conditions for a matrix of morphisms to correspond to an isomorphism.

Proposition 3.58. *Let φ be a morphism in an additive category. If $[\varphi]$ is a diagonal matrix with isomorphisms along the main diagonal, then φ is an isomorphism.*

Proof. Let $\varphi\colon \bigoplus_{i=1}^n X_i \to \bigoplus_{i=1}^n Y_i$ with matrix

$$[\varphi] = \begin{pmatrix} \varphi_1 & 0 & \cdots & 0 \\ 0 & \varphi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_n \end{pmatrix}$$

for some isomorphisms $\varphi_i\colon X_i \to Y_i$. Consider the matrix with the diagonal elements inverted (corresponding to the morphism $\psi\colon \bigoplus_{i=1}^n Y_i \to \bigoplus_{i=1}^n X_i$, say)

$$\begin{pmatrix} \varphi_1^{-1} & 0 & \cdots & 0 \\ 0 & \varphi_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_n^{-1} \end{pmatrix}$$

One readily verifies that the matrix products $[\psi] \cdot [\varphi]$ and $[\varphi] \cdot [\psi]$ are the identity matrices and hence that φ and ψ are mutually inverse morphisms. In particular, φ is an isomorphism. $\square$

A practical corollary of this proposition is that given an isomorphism in the form of a row matrix, we may precompose any of its components by an isomorphism and obtain (the matrix of) an isomorphism. Dually, any component of a column matrix of an isomorphism may be postcomposed by an isomorphism to yield an isomorphism.

Corollary 3.59. *If $\varphi = \begin{pmatrix} \varphi_1 & \cdots & \varphi_n \end{pmatrix}\colon A_1 \oplus \cdots \oplus A_n \to B$ and $\alpha\colon A_i' \to A_i$ are isomorphisms such that φ_i can be precomposed by α, then*

$$\begin{pmatrix} \varphi_1 & \cdots & \varphi_{i-1} & \varphi_i \circ \alpha & \varphi_{i+1} & \cdots & \varphi_n \end{pmatrix} : A_1 \oplus \cdots \oplus A_{i-1} \oplus A_i' \oplus A_{i+1} \oplus \cdots \oplus A_n \to B$$

is an isomorphism (for any i).

Dually, if $\psi\colon A \to B_1 \oplus \cdots \oplus B_n$ is an isomorphism whose matrix is a column matrix

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

and $\beta\colon B_i \to B_i'$ is an isomorphism such that ψ_i can be postcomposed by β, then

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{i-1} \\ \beta \circ \psi_i \\ \psi_{i+1} \\ \vdots \\ \psi_n \end{pmatrix} : A \to B_1 \oplus \cdots \oplus B_{i-1} \oplus B_i' \oplus B_{i+1} \oplus \cdots \oplus B_n$$

is an isomorphism (for any i).

Proof. For the first statement, note that the matrix

$$\begin{pmatrix} \varphi_1 & \cdots & \varphi_{i-1} & \varphi_i \circ \alpha & \varphi_{i+1} & \cdots & \varphi_n \end{pmatrix}$$

may be written as the product

$$\begin{pmatrix} \varphi_1 & \cdots & \varphi_n \end{pmatrix} \cdot \begin{pmatrix} 1_{A_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1_{A_2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{A_{i-1}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1_{A_{i+1}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1_{A_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1_{A_n} \end{pmatrix}$$

where the row matrix is an isomorphism by assumption and the diagonal matrix is an isomorphism by Proposition 3.58. Because a composition of isomorphisms is an isomorphism, we conclude that

$$\begin{pmatrix}\varphi_1 & \cdots & \varphi_{i-1} & \varphi_i \circ \alpha & \varphi_{i+1} & \cdots & \varphi_n\end{pmatrix}$$

is an isomorphism.

The second statement follows by duality. $\square$

3.6 Pullbacks and pushouts via matrices

Pullbacks and pushouts exhibit a connection with kernels and cokernels in additive categories.

Proposition 3.60. *Consider in an additive category a square*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"'] & Y \end{tikzcd}$$

and the following diagonal morphisms that it gives rise to:

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dd, "\beta"'] \arrow[dr, "\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)"'] & & A \arrow[dd, "\varphi"] \\ & A \oplus B \arrow[dr, "\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)"'] & \\ B \arrow[rr, "\psi"'] & & Y \end{tikzcd}$$

The square is

(1) a pullback square (i.e., (α, β) is a pullback of (φ, ψ)) if and only if $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ is a kernel of $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$, and

(2) a pushout square (i.e., (φ, ψ) is a pushout of (α, β)) if and only if $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$ is a cokernel of $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$.

Proof. The intuition for the statements is that

$$\begin{aligned} \varphi \circ \alpha = \psi \circ \beta &\iff \varphi \circ \alpha - \psi \circ \beta = 0 \\ &\iff \begin{pmatrix}\varphi & -\psi\end{pmatrix} \circ \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0 \end{aligned}$$

where the last equivalence follows from $(-\psi) \circ \beta = -(\psi \circ \beta)$, a fact that is an immediate consequence of the biadditivity of the morphism composition. In words, the equivalences say that the square commutes if and only if $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ precomposes with $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$ to zero (or equivalently, $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$ postcomposes with $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ to zero), which gives the crucial link between kernels and cokernels composing to zero and the commutativity of pullback and pushout squares.

Focusing on the first statement (about pullbacks and kernels), we see that the above is to say that (α, β) satisfies the first pullback condition if and only if $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ satisfies the first kernel condition.

For the second conditions, consider any pair of morphisms $\gamma\colon X' \to A, \delta\colon X' \to B$ (that may or may not complete the square with φ and ψ in a commutative manner, or equivalently, for which $\left(\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right)$ may or may not precompose with $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$ to zero) and note that they factor through α and β via some $\chi\colon X' \to X$ if and only if $\left(\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right)$ factors through $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ via the same χ:

$$\gamma = \alpha \circ \chi \text{ and } \delta = \beta \circ \chi \iff \begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\alpha \circ \chi\\ \beta \circ \chi\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \circ \chi$$

In particular, the above shows that the statements "every pair of morphisms (γ, δ) that completes the square commutatively factor through α and β via some unique χ" and "every morphism $\left(\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right)$ that precomposes with $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$ to zero factors through $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ via some unique χ" are equivalent, which is to say that (α, β) satisfies the second pullback condition if and only if $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ satisfies the second kernel condition.

Thus (α, β) is a pullback of (φ, ψ) if and only if $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ is a kernel of $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$, and we have proved the statement about pullbacks and kernels.

The statement about pushouts and cokernels follows by the dual of the first statement and the observation

$$\begin{pmatrix}\varphi & -\psi\end{pmatrix} \text{ is a cokernel of } \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \iff \begin{pmatrix}\varphi & \psi\end{pmatrix} \text{ is a cokernel of } \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$$

which follows readily from how negation behaves well with respect to composition. □

Remark 3.61. The placement of the negation sign in front of ψ in the matrices for the diagonal morphisms is arbitrary. As noted in the end of the proof, the negation could equally well be placed in front of β, and one readily shows in similar fashion that placing it in front of either α or φ instead yields an equivalent statement.

Example 3.62. We have seen in Example 2.42 that all morphisms in R-Mod (which is an additive category) have kernels and cokernels. In particular, all morphisms to and from binary biproducts have kernels and cokernels, so Proposition 3.60 has us conclude that R-Mod has all pullbacks and pushouts.

Moreover, an explicit kernel and cokernel for any R-module morphism was given in the same example, which invites us to determine explicitly a pullback and a pushout for a given pair of suitable morphisms (φ, ψ).

Starting with the pullback of

$$\begin{tikzcd} & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

we consider the module-theoretic kernel of $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)\colon A \oplus B \to Y$:

$$\begin{aligned}\ker\begin{pmatrix}\varphi & -\psi\end{pmatrix} &= \{(a, b) \in A \oplus B \mid \varphi(a) - \psi(b) = 0\}\\ &= \{(a, b) \in A \oplus B \mid \varphi(a) = \psi(b)\}\end{aligned}$$

which is the fiber product $A \times_Y B$ previously encountered in Example 2.48 when looking for a pullback in Sets. Let K denote this kernel. The canonical inclusion of K into $A \oplus B$ is then a category-theoretic kernel of $\left(\begin{smallmatrix}\varphi & -\psi\end{smallmatrix}\right)$. Its matrix is $\left(\begin{smallmatrix}\pi_A\\ \pi_B\end{smallmatrix}\right)$ where $\pi_A\colon K \to A$ denotes the restriction to $K \subseteq A \oplus B$ of the canonical projection of $A \oplus B$ onto A and similarly for $\pi_B\colon K \to B$. We thus conclude that (π_A, π_B) is a pullback of (φ, ψ) (depicted below), very much like what we had in Sets.

$$\begin{tikzcd} K \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

We may verify by module-theoretic means that this indeed is a pullback. The square commutes, because an element $(a, b) \in K$ is mapped to $\varphi(a)$ along the upper path and $\psi(b)$ along the lower path and $\varphi(a) = \psi(b)$ seeing as K is a fiber product. For any other commutative completion of the square

$$\begin{tikzcd}
X \arrow[drr, "\alpha"] \arrow[ddr, "\beta"'] & & \\
& & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"'] & Y
\end{tikzcd}$$

α and β induce the map $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)\colon X \to A \oplus B$, unique with the property that α and β factor through the canonical projections (π'_A and π'_B, say):

$$\begin{tikzcd}
X \arrow[dr, "\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, "\beta"'] & & \\
& A \oplus B \arrow[r, "\pi'_A"] \arrow[d, "\pi'_B"] & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"'] & Y
\end{tikzcd}$$

But in order for α and β to make the outer square commute, the image of $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ must consist of tuples of the form (a, b) with $\varphi(a) = \psi(b)$. This is to say that $\operatorname{im}\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right) \subseteq K$ and hence that we may replace $A \oplus B$ by K in the diagram above and restrict all the morphisms accordingly (letting f denote the restriction of $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ to K with respect to the codomain). In so doing, we see that α and β factor uniquely through π_A and π_B:

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, "\beta"'] & & \\
& K \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "\varphi"] \\
& B \arrow[r, "\psi"'] & Y
\end{tikzcd}$$

Finishing with the pushout of

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \\
B &
\end{tikzcd}$$

we consider the module-theoretic cokernel of $\left(\begin{smallmatrix}\varphi\\ -\psi\end{smallmatrix}\right)\colon Y \to A \oplus B$, i.e., the quotient module $(A \oplus B)/\operatorname{im}\left(\begin{smallmatrix}\varphi\\ -\psi\end{smallmatrix}\right)$. Let C denote this cokernel. The canonical projection π of $A \oplus B$ onto C is then a category-theoretic cokernel of $\left(\begin{smallmatrix}\varphi\\ -\psi\end{smallmatrix}\right)$. Its matrix is (by definition) $\left(\begin{smallmatrix}\pi\circ\iota_A\\ \pi\circ\iota_B\end{smallmatrix}\right)$ where $\iota_A\colon A \to A \oplus B$ denotes the canonical inclusion of A into $A \oplus B$ and similarly for $\iota_B\colon B \to A \oplus B$. We thus conclude that $(\pi \circ \iota_A, \pi \circ \iota_B)$ is a pushout of (φ, ψ):

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[d, "\pi\circ\iota_A"] \\
B \arrow[r, "\pi\circ\iota_B"] & C
\end{tikzcd}$$

Unlike the pullback, this is it at first sight quite different from the pushout in Sets. Upon closer inspection however, there is still a strong resemblance between the pushouts: the object is some sort of quotient of a coproduct and the morphisms are the canonical inclusions into the coproduct composed with some sort of projection.

We may verify by module-theoretic means that this really is a pushout. To see that the square commutes, pick any $y \in Y$ and traverse the two paths from Y to C. The upper path yields $[(\varphi(y), 0)]$ and the lower path yields $[(0, \psi(y))]$. Because $(\varphi(y), -\psi(y)) \in \operatorname{im}\left(\begin{smallmatrix}\varphi \\ -\psi\end{smallmatrix}\right)$ and $(\varphi(y), 0) = (0, \psi(y)) + (\varphi(y), -\psi(y))$, we conclude that the equivalence classes coincide and that the square commutes.

For any other commutative completion of the square

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[ddr, "\alpha"] & \\
B \arrow[drr, "\beta"'] & & \\
& & X
\end{tikzcd}$$

α and β induce the map $\left(\begin{smallmatrix}\alpha & \beta\end{smallmatrix}\right) \colon A \oplus B \to X$:

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[d, "\iota_A"] \arrow[ddr, bend left, "\alpha"] & \\
B \arrow[r, "\iota_B"] \arrow[drr, "\beta"'] & A \oplus B \arrow[dr, "(\alpha\ \beta)"] & \\
& & X
\end{tikzcd}$$

Note that in order for the outer square to commute, we must for every $y \in Y$ have $\alpha(\varphi(y)) - \beta(\psi(y)) = 0$, which is to say that $\left(\begin{smallmatrix}\alpha & \beta\end{smallmatrix}\right)\begin{pmatrix}\varphi(y) \\ -\psi(y)\end{pmatrix} = 0$. In other words, the module-theoretic kernel of $\left(\begin{smallmatrix}\alpha & \beta\end{smallmatrix}\right)$ contains the module-theoretic image of $\left(\begin{smallmatrix}\varphi \\ -\psi\end{smallmatrix}\right)$. By Proposition 1.23 $\left(\begin{smallmatrix}\alpha & \beta\end{smallmatrix}\right)$ thus factors through the canonical projection π of $A \oplus B$ onto $(A \oplus B)/\operatorname{im}\left(\begin{smallmatrix}\varphi \\ -\psi\end{smallmatrix}\right) = C$ via the unique $g \colon C \to X$ given by $g([(a, b)]) = \alpha(a) + \beta(b)$:

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[d, "\psi"] & A \arrow[d, "\iota_A"] \arrow[ddrr, bend left, "\alpha"] & & \\
B \arrow[r, "\iota_B"] \arrow[ddrr, "\beta"'] & A \oplus B \arrow[dr, "\pi"] & & \\
& & C \arrow[dr, "g"] & \\
& & & X
\end{tikzcd}$$

This g is the unique morphism via which the original completion of the square factors through the pushout square.

4 Abelian categories

The notion of an additive category is too general to do homological algebra effectively [4, p. 78]. In this section, we will therefore strengthen the notion of an additive category to that of an abelian category, which turns out to be an adequate setting in which to define so-called short exact sequences – a cornerstone of homological algebra – and indeed practise homological algebra in general.

4.1 Basics

An abelian category may be viewed as an additive category that behaves well with respect to kernels and cokernels.

Definition 4.1. [3, p. 602] An *abelian* category is an additive category where

(A1) every morphism admits a kernel and a cokernel, and

(A2) every monomorphism is a kernel and every epimorphism is a cokernel.

Remark 4.2. Note that the axioms are trivially self-dual, i.e., that the dual of an abelian category is abelian, which allows for reasoning by duality with abelian categories.

Remark 4.3. Recall that every kernel is always a monomorphism (Proposition 2.37) and every cokernel is always an epimorphism (Proposition 2.40). The second axiom for the abelian category may thus be understood as stating that monomorphisms and kernels coincide and epimorphisms and cokernels coincide.

The primary example of an abelian category for us to consider is that of R-Mod [5, p. 199][3, p. 602].

Example 4.4. For a fixed ring R, the additive category R-Mod (recall Proposition 3.19) is an abelian category. In Example 2.42, we noted that any morphism in R-Mod admits a kernel and a cokernel, which leaves to verify only that every monomorphism is a kernel and every epimorphism is a cokernel.

Let $f\colon M \to N$ be a monomorphism in R-Mod, i.e., an R-module monomorphism. Guided by Example 2.42, consider the cokernel $\pi\colon N/\operatorname{im} f$ of f and verify that f is a kernel of π. We know that the inclusion morphism $\iota\colon \operatorname{im} f \to N$ of the kernel (in the module sense) of π, which is $\operatorname{im} f$, is a kernel of π (in the categorical sense), so by Corollary 2.51 it suffices to find an isomorphism g making the following diagram commute:

$$\begin{tikzcd}
N \arrow[rr, "\pi"] & & N/\operatorname{im} f \\
& M \arrow[ul, "f"'] \arrow[ur, "0"] \arrow[d, dotted, "g"] & \\
& \operatorname{im} f \arrow[uul, "\iota"] \arrow[uur, "0"'] &
\end{tikzcd}$$

If $g\colon M \to \operatorname{im} f$ is taken to be the restriction (with respect to the codomain) of f from N to $\operatorname{im} f$, the diagram commutes. This g is by the assumption injective and by construction surjective and hence an isomorphism, which shows that f is a kernel of π.

For the dual axiom, let $f\colon M \to N$ be an epimorphism in R-Mod, i.e., an R-module epimorphism. Consider the inclusion morphism $\iota\colon \ker f \to M$ of the kernel of f in the module sense and verify that f is a cokernel of ι. We know that the quotient morphism $\pi\colon M \to M/\operatorname{im}\iota = M/\ker f$ is a cokernel of ι (in a categorical sense), so by Corollary 2.51 it suffices to find an isomorphism g making the following diagram commute:

$$\begin{tikzcd}
M \arrow[dr, "f"] \arrow[ddr, "\pi"'] & & \ker f \arrow[ll, hook', "\iota"'] \arrow[dl, "0"'] \arrow[ddl, "0"] \\
& N & \\
& M/\ker f \arrow[u, dotted, "g"] &
\end{tikzcd}$$

By the assumption that f is an epimorphism, we have $N = \operatorname{im} f$ and we are really looking for an isomorphism $g\colon M/\ker f \to \operatorname{im} f$ making the following diagram commute:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\pi"] & N \arrow[d, equal] \\
M/\ker f \arrow[r, dotted, "g"] & \operatorname{im} f
\end{tikzcd}$$

This is a perfect fit for the first isomorphism theorem of modules (Proposition 1.24), which guarantees the existence of such an isomorphism g. We conclude that f is a cokernel of ι.

Thus we have shown that R-Mod with the usual additive structure is an abelian category.

As the term *abelian* category would suggest, the category of all abelian groups is abelian. To show this, we will first show the analog of Proposition 3.20 for abelian categories: that equivalences of categories transfer the properties of an abelian category to another category.

Proposition 4.5. *If $F\colon \mathcal{C} \to \mathcal{A}$ is an equivalence of categories, with $\mathcal{C}$ any category and $\mathcal{A}$ abelian, then $\mathcal{C}$ with morphism addition inherited from $\mathcal{A}$ via F as follows is an abelian category:*

$$f + g = F^{-1}(F(f) + F(g))$$

In other words, the addition is defined in the unique way to make F an additive functor.

Proof. By Proposition 3.20, we know that $\mathcal{C}$ is additive. Using Remark 2.54 twice, we may view $F\colon \mathcal{C} \to \mathcal{A}$ as a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$, which is readily seen to also be an equivalence. Because the notion of an abelian category is self-dual, $\mathcal{A}^{\mathrm{op}}$ is abelian and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ is an equivalence into an abelian category. Thus we may reason by duality and prove only the parts of the axioms for abelian categories about kernels (invoking duality for the cokernels).

For (A1), let $f\colon X \to Y$ be any morphism in $\mathcal{C}$ and consider the morphism $F(f)\colon F(X) \to F(Y)$. It has a kernel $k'\colon K' \to F(X)$ by (A1) for $\mathcal{A}$. By picking the kernel more carefully, we may find a kernel of the form $F(k)$ for some k in $\mathcal{C}$. This is so because F is dense, so $K' \cong F(K)$ for some $K \in \mathcal{C}$, and precomposing $k'\colon K' \to F(X)$ by an isomorphism $\varphi\colon F(K) \to K'$ yields another kernel, $k' \circ \varphi\colon F(K) \to F(X)$, of $F(f)$ (by Corollary 2.51).

Because F is full, this $k' \circ \varphi$ has a preimage $k\colon K \to X$ in $\mathcal{C}$. This k is readily shown to be a kernel of f.

In preparation for (A2), note first that equivalences preserve monomorphisms (for any categories, not necessarily abelian):

Lemma. *If $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, then*

$$f \text{ is monic} \iff F(f) \text{ is monic}$$

for any morphism f in $\mathcal{C}$.

Proof. The implication from right to left is straightforward and uses only that F is faithful: if $F(f)$ is monic and $f \circ g = f \circ h$, then $F(f \circ g) = F(f \circ h)$, which is to say that $F(f) \circ F(g) = F(f) \circ F(h)$. By the monicness of $F(f)$, we have $F(g) = F(h)$ and the faithfulness of F implies $g = h$, so f is monic.

For the implication from left to right, suppose that $f\colon X \to Y$ in $\mathcal{C}$ is monic and that $F(f) \circ g' = F(f) \circ h'$ for some $g', h'\colon W' \to F(X)$ in $\mathcal{D}$. Because F is dense, there is an object $W \in \mathcal{C}$ with $F(W) \cong W'$ and hence an isomorphism $\varphi\colon F(W) \to W'$. Consider the morphisms $g' \circ \varphi, h' \circ \varphi\colon F(W) \to F(X)$. By the fullness of F, they have preimages $g, h\colon W \to X$, which allows us to utilize the monicness of f as follows:

$$\begin{aligned} F(f) \circ g' = F(f) \circ h' &\implies F(f) \circ g' \circ \varphi = F(f) \circ h' \circ \varphi \\ &\implies F(f) \circ F(g) = F(f) \circ F(h) \\ &\implies F(f \circ g) = F(f \circ h) \\ &\implies f \circ g = f \circ h \\ &\implies g = h \\ &\implies g' \circ \varphi = F(g) = F(h) = h' \circ \varphi \end{aligned}$$

Now because φ is an isomorphism, we may precompose the above by φ^{-1} to obtain $g' = h'$, which shows that $F(f)$ is monic. □

For (A2), let $m\colon M \to X$ be a monomorphism in $\mathcal{C}$. By the lemma, $F(m)$ is monic and hence is a kernel of some $f'\colon F(X) \to Y'$ by (A2) in $\mathcal{A}$. By picking the morphism more carefully, we may find a morphism of the form $F(f)$ with kernel $F(m)$ for some f in $\mathcal{C}$, in much the same way as in the proof of $(A1)$: by the denseness of F, there is a $Y \in \mathcal{C}$ with $Y' \cong F(Y)$ and hence an isomorphism $\varphi\colon Y' \to F(Y)$. Postcomposing f' by φ yields another morphism $\varphi \circ f'\colon F(X) \to F(Y)$ with $F(m)$ as kernel (by Corollary 2.51).

Because F is full, this morphism $\varphi \circ f'$ has a preimage $f\colon X \to Y$ in $\mathcal{C}$. This f is readily shown to have m as a kernel. □

Corollary 4.6. Ab *with valuewise morphism addition is an abelian category.*

Proof. Recall Corollary 3.21 where Ab was noted as being an additive category with valuewise morphism addition inherited from $\mathbb{Z}$-Mod via an equivalence from Ab to $\mathbb{Z}$-Mod. In light of Example 4.4 and Proposition 4.5, we may immediately conclude that Ab with valuewise addition of morphisms is not just additive but abelian. □

That not all additive categories are abelian is made apparent by the following example.

Example 4.7. [4, p. 78] Consider the category $\mathbb{Z}$-Free of all free $\mathbb{Z}$-modules (up to isomorphism the category of all free abelian groups), which we saw is additive in Example 3.38. Consider next the morphism $L_2 = \mathbb{Z} \xrightarrow{2} \mathbb{Z}$, i.e., the morphism defined by left-multiplication by 2 in the regular $\mathbb{Z}$-module: $n \mapsto 2 \cdot n = 2n$. That this really is a morphism may be taken to be a consequence of the following more general lemma, whose proof is omitted but follows readily from the module axioms:

Lemma. *Let R be a ring and M be an R-module. Left-multiplication $L_r\colon M \to M$ defined by $m \mapsto r \cdot m$ is an R-module morphism for any element r of the multiplicative center of R.*

It is immediate that L_2 is injective and hence a monomorphism. In order for $\mathbb{Z}$-Free to be an abelian category, L_2 must be a kernel of some morphism. Assume towards a contradiction that there is a morphism $f\colon \mathbb{Z} \to M$ (for some free $\mathbb{Z}$-module M) such that L_2 is a kernel of f.

We may without loss of generality take M to be a direct sum $\bigoplus_{i \in I} \mathbb{Z}^{(i)}$ (for some set I), as made evident by the following reasoning: there is by Proposition 1.33 a $\mathbb{Z}$-module isomorphism $g\colon M \to \bigoplus_{i \in I} \mathbb{Z}^{(i)}$. This g is an isomorphism (and in particular a monomorphism) also in the categorical sense seeing as $\mathbb{Z}$-Free is a full subcategory of $\mathbb{Z}$-Mod, where we know the latter category to have precisely the $\mathbb{Z}$-module isomorphisms as isomorphisms. Postcomposing a morphism by a monomorphism yields a morphism with the very same kernels, so L_2 is also a kernel of the composition $\mathbb{Z} \xrightarrow{f} M \xrightarrow{g} \bigoplus_{i\in I} \mathbb{Z}^{(i)}$.

Note next that $\mathbb{Z}$-Free has the very same zero objects as $\mathbb{Z}$-Mod, namely the singleton modules, seeing as these are free on the empty basis. Hence, the zero morphisms in the module sense are precisely the zero morphisms in the categorical sense in $\mathbb{Z}$-Free.

Consider now what it means for L_2 to precompose with $f\colon \mathbb{Z} \to \bigoplus_{i\in I} \mathbb{Z}^{(i)}$ to zero: for every $n \in \mathbb{Z}$, $(f \circ L_2)(n) = f(2 \cdot n) = 2 \cdot f(n) = f(n) + f(n)$ is zero. But the order of any nonzero element in $\bigoplus_{i\in I} \mathbb{Z}^{(i)}$ as an abelian group is infinite, so $f(n) = 0$. In other words, f is the zero morphism.

Any kernel of a zero morphism is an isomorphism, by the uniqueness of the kernel up to isomorphism and the ready observation that the identity morphism is one of its kernels. Thus $L_2\colon \mathbb{Z} \to \mathbb{Z}$ must be an isomorphism, which contradicts its blatant lack of surjectivity. We conclude that L_2, despite being monic, is not a kernel of any morphism and hence that the additive category $\mathbb{Z}$-Free is not an abelian category.

Now we set out to prove an important proposition that characterizes isomorphisms in abelian categories as precisely the morphisms that are both monic and epic. The reader may recall Remark 2.19, where this statement was noted to be false in an arbitrary category. We shall first require two lemmas.

Lemma 4.8. *[3, p. 602] Let μ be a monomorphism and σ be an epimorphism in an abelian category. Then μ is a kernel of σ if and only if σ is a cokernel of μ.*

Proof. Assume that $\mu\colon K \to X$ is a kernel of $\sigma\colon X \to C$ and show that σ is a cokernel of μ. It is clear by the assumption that σ and μ compose to zero, so it suffices to show that any other morphism $d\colon X \to D$ with $d \circ \mu = 0$ factors through σ as $d = g \circ \sigma$ via some unique $g\colon C \to D$. This uniqueness is automatic, seeing as σ is an epimorphism. We may thus depict the setting as the following commutative diagram:

$$\begin{tikzcd}
& D & \\
X \arrow[ur, "d"] \arrow[rr, "\sigma"] & & C \arrow[ul, dotted, "\exists g?"'] \\
& K \arrow[ul, "\mu"'] &
\end{tikzcd}$$

By the axioms of the abelian category, σ is a cokernel of some morphism $a\colon A \to X$. If a were to compose with d to 0, the factorization property of the cokernel σ would guarantee the (unique) existence of a g as required. Thus it suffices to show that $d \circ a = 0$.

σ is a cokernel of a, so σ postcomposes with a to zero. Equivalently, a precomposes with σ to zero, which by the factorization property of the kernel μ of σ induces a (unique) morphism $h\colon A \to K$ with $a = \mu \circ h$:

$$\begin{tikzcd}
& & D & \\
& X \arrow[ur, "d"] \arrow[rr, "\sigma"] & & C \arrow[ul, dotted, "\exists g?"'] \\
A \arrow[ur, "a"] \arrow[rr, "h"] & & K \arrow[ul, "\mu"'] &
\end{tikzcd}$$

By the assumption that $d \circ \mu = 0$, we find that $d \circ a = d \circ \mu \circ h = 0 \circ h = 0$. Thus d gives rise via the cokernel property of σ to a (unique) morphism g as required.

This shows that if μ is the kernel of σ then σ is the cokernel μ. The implication in the opposite direction follows immediately by duality. $\square$

Remark 4.9. In particular, this lemma shows that (in an abelian category) a monomorphism is a kernel of any of its cokernels and not just any morphism (as guaranteed by the first axiom for abelian categories). Similarly, an epimorphism is a cokernel of any of its kernels. Some authors ([4, p. 78]) require this seemingly stronger property as an axiom for abelian categories.

The next lemma states that, in any additive category (or just preadditive with a zero object) monomorphisms are characterized as precisely the morphisms with a zero kernel and dually that epimorphisms are characterized as morphisms with a zero cokernel. The reader may compare this to Proposition 1.19 and similar results from the study of groups and vector spaces, for instance.

Lemma 4.10. *[3, p. 603] [4, p. 78] Let f be a morphism in an additive category. Then the following are equivalent:*

(1) f is a monomorphism,

(2) f has a zero morphism as kernel, and

(3) f has a zero object as kernel object.

Dually, the following are also equivalent:

(1) f is an epimorphism,

(2) f has a zero morphism as cokernel, and

(3) f has a zero object as cokernel object.

Proof. Let us first show the equivalence of (2) and (3). It is clear that (3) $\implies$ (2), seeing as the only morphism from a zero object is the zero morphism. Conversely, if a zero morphism $k = 0_{KX}$ (let f be a morphism from X to Y) is a kernel of f, then it must be terminal and hence a zero object. More precisely, consider the zero morphism $l = 0_{LX}$ from any object L and how the factorization property of the kernel k states that l factors as $l = k \circ g$, i.e., $0_{LX} = 0_{KX} \circ g$ for some *unique* g. But this factorization holds for *any* $g\colon L \to K$ (and L is an arbitrary object), so K must be initial and hence a zero object. The setting is shown in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& K \arrow[ul, "0"'] & \\
& L \arrow[u, dotted, "\exists ! g"'] \arrow[uul, "0"] &
\end{tikzcd}$$

It remains to show that (1) is equivalent to either of (2) and (3).

The implication (1) $\implies$ (2) is readily shown to hold in any category with a zero object. Suppose that $f\colon X \to Y$ is a monomorphism, consider the morphism $0_{0X}\colon 0 \to X$, and show that it is a kernel of f. Note that 0_{0X} composes with any morphism into a zero morphism, in particular with f, so only the factorization property of the kernel remains to be shown. It is a straightforward consequence of f being a monomorphism that only zero morphisms precompose with f to zero, i.e., that $f \circ g = 0$ if and only if $g = 0$. Thus the factorization property needs to hold only for zero morphisms as shown below (for any object L):

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & 0 \arrow[ul, "0"'] & \\ & L \arrow[uul, "0"] \arrow[u, dotted, "\exists! g"'] & \end{tikzcd}$$

Seeing as the zero object is terminal, there is precisely one morphism from L to 0, namely the zero morphism, which does make the triangle commute. Hence the factorization property holds and 0_{0X} is a kernel of f.

The implication (2) $\implies$ (1) is shown using the morphism addition (subtraction) in the additive category. Suppose that f has a zero morphism as kernel. Observe that this implies that only zero morphisms precompose with f to zero, i.e., that $f \circ h = 0 \implies h = 0$, seeing as the only way for the following diagram to commute is for l to be zero:

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & K \arrow[ul, "0"'] & \\ & L \arrow[uul, "l"] \arrow[u, "g"'] & \end{tikzcd}$$

Now consider any two morphisms a and b with $f \circ a = f \circ b$. By the above, we have:

$$\begin{aligned} f \circ a = f \circ b &\implies (f \circ a) - (f \circ b) = 0 \\ &\implies f \circ (a - b) = 0 \\ &\implies (a - b) = 0 \\ &\implies a = b \end{aligned}$$

which is to say that f is a monomorphism. $\square$

Remark 4.11. The proof of the implication (2) $\implies$ (1) and reasoning by duality suggests the following practical characterizations of monomorphisms and epimorphisms in any additive category:

- A morphism f is monic if and only if only zero morphisms precompose with f to zero.
- A morphism f is epic if and only if only zero morphisms postcompose with f to zero.

Equipped with the above lemmas, we are now ready to take on the proposition.

Proposition 4.12. *[3, p. 603] [4, p. 79] If f is a morphism in an abelian category, then*

$$f \text{ is an isomorphism} \iff f \text{ is a monomorphism and an epimorphism}$$

Proof. The implication from left to right holds in any category and was shown in Remark 2.19.

For the implication from right to left, let $f\colon X \to Y$ be a monic epimorphism. Being an epimorphism, f has a zero cokernel (to a zero object) by Lemma 4.10 and, being a monomorphism, f is a kernel of this cokernel by Lemma 4.8. We may depict the setting as

$$\begin{tikzcd} & 0 & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "0"'] \end{tikzcd}$$

Now consider the identity morphism 1_Y for Y. It does, like any other morphism (with codomain Y), compose with the cokernel 0 to a zero morphism. Because f is a kernel of this cokernel, this induces a (unique) morphism $g\colon Y \to X$ such that $1_Y = f \circ g$, i.e., a right inverse of f. We may depict the situation as the following commutative diagram

$$\begin{tikzcd}
 & 0 & \\
X \arrow[rr, "f"] & & Y \arrow[ul, "0"'] \\
 & & Y \arrow[u, "1"'] \arrow[ull, "g"]
\end{tikzcd}$$

In order to show that this g is also a left inverse of f, we may observe (as is done in [3, p. 603]) that a right inverse β of any monomorphism α (in any category) is also a left inverse:

$$\alpha \circ 1 = \alpha = 1 \circ \alpha = (\alpha \circ \beta) \circ \alpha = \alpha \circ (\beta \circ \alpha) \implies 1 = \beta \circ \alpha$$

Seeing as f is a monomorphism, we conclude that g is a left-sided inverse. Alternatively, we may dually also find some right-sided inverse h of f and then observe that a left-sided inverse α and a right-sided inverse γ of a morphism β must coincide (in any category) and hence be a two-sided inverse:

$$\alpha = \alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma = \gamma$$

Here, we would find that $g = h$ is a two-sided inverse of f.

In any case, we find that f is an isomorphism, which is what we set out to prove. $\square$

4.2 The first isomorphism theorem

Definition 4.13. [3, p. 603] In any category $\mathcal{C}$ with a zero object, an *image* of a morphism $f\colon X \to Y$ is a kernel (B, κ) of a cokernel (C, c) of f. Dually, a *coimage* of f is a cokernel (A, γ) of a kernel (K, k) of f.

We will call the objects A and B a *coimage object* and an *coimage object*, respectively, and the morphisms γ and κ a *coimage morphism* and an *image morphism*. Depending on the context, we may refer to any of them as just a "coimage" or "image", just like for kernels and cokernels.

These notions are depicted below:

$$\begin{tikzcd}
 & X \arrow[rrr, "f"] \arrow[dr, "\gamma"] & & & Y \arrow[dr, "c"] & \\
K \arrow[ur, "k"] & & A & B \arrow[ur, "\kappa"] & & C
\end{tikzcd}$$

Remark 4.14. Note that the first axiom of the abelian category ensures that every morphism of an abelian category admits an image and a coimage. We shall see that any two image objects are isomorphic and that any two coimage objects are isomorphic, i.e., that the image and coimage objects are unique up to isomorphism. Perhaps more surprisingly, any image and coimage object of a morphism in an abelian category are isomorphic to each other, which we will see in Corollary 4.19.

Remark 4.15. By the definition of the image and coimage as well as Proposition 2.37 and its dual result, images are monic and coimages are epic.

Much like kernels and cokernels, images and coimages are both unique up to an isomorphism of sorts.

Proposition 4.16. *[3, p. 603] If $\kappa\colon K \to Y$ and $\lambda\colon L \to Y$ are images of a morphism $f\colon X \to Y$, then there is a unique isomorphism $g\colon L \to K$ with $\lambda = \kappa \circ g$.*

Dually, if $\gamma\colon X \to C$ and $\delta\colon X \to D$ are coimages of a morphism $f\colon X \to Y$, then there is a unique isomorphism $g\colon C \to D$ with $\delta = g \circ \gamma$.

Proof. Let f be a morphism with images κ and λ. Then, by the definition of the image, there are cokernels c and d of f such that κ is a kernel of c and λ is a kernel of d:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[dr, "c"] \arrow[ddr, "d"'] & \\
& K \arrow[ur, "\kappa"] & & C \\
& L \arrow[uur, "\lambda"] & & D
\end{tikzcd}$$

By the uniqueness of kernels (see Proposition 2.38), it suffices to show that λ is a kernel of not only d but also c. First note that the dual proposition (Proposition 2.41) lets us factor c through d as $c = h \circ d$ for some isomorphism h:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[dr, "c"] \arrow[ddr, "d"'] & \\
& K \arrow[ur, "\kappa"] & & C \\
& L \arrow[uur, "\lambda"] & & D \arrow[u, "h"']
\end{tikzcd}$$

Thus $c \circ \lambda = h \circ d \circ \lambda = 0$, which shows the first condition for λ being a kernel of c, that λ precomposes with c to 0.

For the second condition, bethink the general statement that the "zeroness" of a morphism is invariant under composition of an isomorphism (to the left or to the right). In other words, for a morphism a and isomorphism b where the composition $b \circ a$ is defined, $a = 0 \iff b \circ a = 0$. The implication to the right is trivial (by Remark 2.27) and the converse follows by noting that b^{-1} is an isomorphism.

In the case at hand, we find that a morphism l precomposes with c to 0 if and only if it precomposes with d to 0, seeing as $(c \circ l) = 0 \iff h \circ (c \circ l)$. The second kernel property for λ as a kernel of d may then be used to ascertain that any morphism l precomposing with c to 0 factors uniquely through λ.

Thus λ is a kernel of c, just like κ, and the uniqueness of the kernel gives rise to the unique morphism $g\colon L \to K$ making the left triangle commute in the diagram below, which finishes the proof of the uniqueness of the image.

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[dr, "c"] \arrow[ddr, "d"'] & \\
& K \arrow[ur, "\kappa"] & & C \\
& L \arrow[u, "g"] \arrow[uur, "\lambda"] & & D \arrow[u, "h"']
\end{tikzcd}$$

The uniqueness statement for coimages follows by duality. ∎

Example 4.17. [3, p. 603] Consider a morphism $f\colon M \to N$ in the category R-Mod of all R-modules (for some fixed ring R). We have seen in Example 2.42 that the canonical inclusion $\iota\colon \ker f \to M$ of the kernel $\ker f$ (in the module sense) into the domain M is a kernel (in the category sense) and that the canonical projection $\pi\colon N \to N/\operatorname{im} f$ of the codomain N into the cokernel $\operatorname{cok} f = N/\operatorname{im} f$ (in the module sense) is a cokernel (in the category sense). Observe further that the module-theoretical kernel of π is $\operatorname{im} f$ and that the module-theoretical cokernel of ι is $M/\operatorname{im}\iota = M/\ker f$. Hence, the canonical inclusion of the module-theoretical image $\operatorname{im} f$ in the codomain N is a category-theoretical image of f with corresponding image object $\operatorname{im} f$. Also, the canonical projection of M on $M/\ker f$ is a coimage of f with corresponding coimage object $M/\ker f$. Comparing with Definition 1.17, one may note that the category-theoretical definition of the image and coimage agrees with the module-theoretical definition.

Next up is a generalization of the first isomorphism theorem for modules (Proposition 1.24) to the setting of any abelian category. Its corollary (Corollary 4.19) should be compared to Corollary 1.25 and Remark 1.18.

Proposition 4.18 (The first isomorphism theorem for abelian categories)**.** *[3, p. 603] Let $\alpha\colon X \to Y$ be a morphism in an abelian category and $\pi\colon X \to A$ and $\iota\colon B \to Y$ be a coimage and an image of f. Then there is a unique morphism $\theta\colon A \to B$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \\ A \arrow[r, dotted, "\theta"'] & B \arrow[u, "\iota"'] \end{tikzcd}$$

Moreover, θ is an isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \arrow[r, "c"] & C \\ & A & B \arrow[u, "\iota"'] & \end{tikzcd}$$

where k is a kernel of α and c is a cokernel of α. Because c is a cokernel of α, we have $c \circ \alpha = 0$. In other words, α is a morphism that precomposes with c to 0. Now ι is a kernel of c, so α factors through ι uniquely via some $g\colon X \to B$:

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] \arrow[dr, dotted, "g"] & Y \arrow[r, "c"] & C \\ & A & B \arrow[u, "\iota"'] & \end{tikzcd}$$

Consider next the composition $g \circ k$. The composition $\alpha \circ k$ is zero, seeing as k is a kernel of α. By the definition of g, this is to say that $\iota \circ g \circ k = 0$. Note that ι, being an image and hence a kernel, is monic, and conclude that $g \circ k = 0$. In other words, g postcomposes with k to zero and so must factor through the cokernel π of k via some unique morphism $\theta\colon A \to B$. We thus get the commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] \arrow[dr, dotted, "g"] & Y \arrow[r, "c"] & C \\ & A \arrow[r, dotted, "\theta"'] & B \arrow[u, "\iota"'] & \end{tikzcd}$$

where, in particular, the square commutes. This shows the existence of a morphism $\theta\colon A \to B$ making the square commute. Uniqueness follows immediately from the coimage π being epic and the image ι being monic: $\iota \circ \theta$ is the unique morphism postcomposing with π to α, and θ is the unique morphism precomposing with ι to this morphism $\iota \circ \theta$. Hence θ is unique with $\iota \circ \theta \circ \pi = (\iota \circ \theta) \circ \pi = \alpha$.

It remains to show that θ is an isomorphism. By Proposition 4.12, it suffices to show that θ is monic and epic, and by duality, it suffices to show that θ is epic, say. Following the proof of [3, p. 603], let $e\colon B \to E$ be an arbitrary morphism postcomposing with θ to a zero morphism, i.e., with $e \circ \theta = 0$, depicted in the diagram below, and show that $e = 0$. Then any cokernel of θ must be a zero morphism, which by Lemma 4.10 is to say that θ is an epimorphism.

$$\begin{tikzcd}
K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \arrow[r, "c"] & C \\
& A \arrow[r, "\theta"'] & B \arrow[u, "\iota"'] \arrow[r, "e"'] & E
\end{tikzcd}$$

Note finally that if the kernel, l say, of e is right-invertible, then $e = 0$, because

$$e \circ l = 0 \implies e = e \circ l \circ l^{-1} = 0 \circ l^{-1} = 0$$

In summary, we have set up the following chain of implications and need to prove its very first statement:

$$\begin{aligned}
l \text{ is right-invertible} &\implies e = 0 \\
&\implies \text{any cokernel of } \theta \text{ is a zero morphism} \\
&\implies \theta \text{ is an epimorphism} \\
&\implies \theta \text{ is a monic epimorphism} \\
&\implies \theta \text{ is an isomorphism}
\end{aligned}$$

Consider the cokernel $d\colon Y \to D$ of the composition $\iota \circ l$, as shown in the diagram.

$$\begin{tikzcd}
& & D & \\
K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \arrow[r, "c"] \arrow[u, "d"] & C \\
& A \arrow[r, "\theta"'] & B \arrow[u, "\iota"'] \arrow[r, "e"'] & E \\
& & L \arrow[u, "l"'] &
\end{tikzcd}$$

$\iota \circ l$ is a monomorphism, seeing as it is a composition of kernels, which are monic. Hence, by Lemma 4.8, $\iota \circ l$ is a kernel of d. By showing that ι alone precomposes with d to 0, the kernel property of $\iota \circ l$ would have ι factor via $\iota \circ l$ as $\iota = \iota \circ l \circ l'$ for some $l'\colon B \to L$, as shown below.

$$\begin{tikzcd}
& & D & \\
K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \arrow[r, "c"] \arrow[u, "d"] & C \\
& A \arrow[r, "\theta"'] & B \arrow[u, "\iota"'] \arrow[r, "e"'] \arrow[d, bend left, dotted, "l'"] & E \\
& & L \arrow[u, bend left, "l"] &
\end{tikzcd}$$

Because ι is monic, we would find that $1 = l \circ l'$, i.e., that l' is a right inverse of l. Thus we set out to prove that $d \circ \iota = 0$ and we do so by factoring d via c.

By first factoring θ (which precomposes with e to 0) as $l \circ h$ for some $h \colon A \to L$ using the fact that l is a kernel of e

$$\begin{tikzcd}
& & D & \\
K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \arrow[u, "d"] \arrow[r, "c"] & C \\
& A \arrow[r, "\theta"'] \arrow[dr, dotted, "h"'] & B \arrow[u, "\iota"'] \arrow[r, "e"'] & E \\
& & L \arrow[u, "l"'] &
\end{tikzcd}$$

it is apparent that d postcomposes with α to 0:

$$\begin{aligned}
d \circ \alpha &= d \circ (\iota \circ \theta \circ \pi) \\
&= d \circ \iota \circ (l \circ h) \circ \pi \\
&= (d \circ \iota \circ l) \circ h \circ \pi \\
&= 0 \circ h \circ \pi \\
&= 0
\end{aligned}$$

c is a cokernel of α, so d factors as $b \circ c$ for some $b \colon C \to D$, as shown below.

$$\begin{tikzcd}
& & D & \\
K \arrow[r, "k"] & X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Y \arrow[u, "d"] \arrow[r, "c"] & C \arrow[ul, dotted, "b"'] \\
& A \arrow[r, "\theta"'] \arrow[dr, "h"'] & B \arrow[u, "\iota"'] \arrow[r, "e"'] & E \\
& & L \arrow[u, "l"'] &
\end{tikzcd}$$

Now note that $d \circ \iota = b \circ c \circ \iota = 0$, where the last equality follows from c being a cokernel of ι and hence postcomposing with ι to 0. Thus we have shown

$$\begin{aligned}
d \text{ factors via } c &\implies d \circ \iota = 0 \\
&\implies l \text{ is right-invertible} \\
&\quad \vdots \\
&\implies \theta \text{ is an isomorphism}
\end{aligned}$$

and the proof is finished. $\square$

Corollary 4.19. *Any image object and coimage object of a morphism are isomorphic.*

Remark 4.20. One may interpret the isomorphism theorem as stating that any morphism factors as a composition of a coimage and an image, seeing as $\theta \circ \pi$ may be taken as the coimage, say [3, p. 603]. Some authors ([4, p. 78]) explicitly include among the axioms of the abelian category the similar statement that any morphism factors as a composition of an epimorphism and a monomorphism.

4.3 Exact sequences

Now we turn to a notion of paramount importance in homological algebra: the notion of a short exact sequence. In the next section, we will explore a kind of categories revolving around short exact sequences that generalizes abelian categories.

Definition 4.21. [4, p. 79] In an abelian category, a *short exact sequence* is a diagram

$$X \xrightarrow{\mu} Y \xrightarrow{\sigma} Z$$

where μ is a kernel of σ and σ is a cokernel of μ.

The following proposition shows that the above notion of short exact sequences is compatible with the notion defined in Example 1.22.

Proposition 4.22. *A short exact sequence in* R-Mod *(in the sense of Definition 4.21) is precisely a pair of morphisms f and g that fit into a short exact sequence of R-module morphisms in the sense of Example 1.22:*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

In other words, f is a monomorphism, g is an epimorphism, and $\operatorname{im} f = \ker g$.

Proof. A sufficient condition (by Lemma 4.8) for the diagram $L \xrightarrow{f} M \xrightarrow{g} N$ to be a short exact sequence (in the sense of Definition 4.21) is to show that g is a cokernel of f. This can be proved by showing that g differs from the canonical projection $\pi\colon M \to M/\operatorname{im} f$, which we know to be a cokernel of f, by a composition with an isomorphism.

By the first isomorphism theorem for modules (Proposition 1.24), there is an isomorphism θ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\pi"'] & N \\ M/\ker g \arrow[r, "\theta"'] & \operatorname{im} g \arrow[u, "\iota"'] \end{tikzcd}$$

But by assumption, g is surjective and $\ker g = \operatorname{im} f$, so the diagram is really of the form

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\pi"'] & N \arrow[d, equal] \\ M/\operatorname{im} f \arrow[r, "\theta"'] & N \end{tikzcd}$$

which is to say that π is a categorical cokernel of f and that $g = \theta \circ \pi$ is the composition of a cokernel of f and an isomorphism. Thus g is a cokernel of f and the diagram is a short exact sequence.

For the converse, let

$$X \xrightarrow{\mu} Y \xrightarrow{\sigma} Z$$

be a short exact sequence. Kernels are monic and cokernels are epic. Moreover, $\sigma \circ \mu = 0$, which is to say that $\operatorname{im} \mu \subseteq \ker \sigma$. It thus remains to show only that the inclusion is an equality: $\operatorname{im} \mu = \ker \sigma$. Consider the canonical inclusion ι of $\ker \sigma$ into Y:

$$\begin{tikzcd} X \arrow[r, "\mu"] & Y \arrow[r, "\sigma"] & Z \\ & \ker \sigma \arrow[u, "\iota"'] & \end{tikzcd}$$

Note that ι precomposes with σ to zero. By the assumption that μ is a kernel of σ, ι should therefore factor through μ:

$$\begin{tikzcd} X \arrow[r, "\mu"] & Y \arrow[r, "\sigma"] & Z \\ & \ker\sigma \arrow[u, "\iota"'] \arrow[ul, dotted, "g"] & \end{tikzcd}$$

Note now that

$$\begin{aligned} \ker\sigma &= \operatorname{im}\iota \\ &= \operatorname{im}(\mu \circ g) \\ &\subseteq \operatorname{im}\mu \end{aligned}$$

which lets us conclude that $\operatorname{im}\mu = \ker\sigma$. $\square$

5 Exact categories

The aim of this section is to explore an axiomatization of short exact sequences (previously defined in abelian categories) in additive categories, which gives rise to the notion of an exact category. This allows for homological algebra to be done in settings more general than that of abelian categories.

Most of the contents of this section are from Bühler's exposition of exact categories [2].

5.1 Basics

Definition 5.1. [2, pp. 4–5] Let $\mathcal{A}$ be an additive category. A *kernel–cokernel pair* in $\mathcal{A}$ is a pair of morphisms

$$X \xrightarrow{i} E \xrightarrow{p} Y$$

where i is a cokernel of p and p is a cokernel of i.

With respect to a class $\mathcal{E}$ of kernel–cokernel pairs, a morphism is said to be an *admissible monic* if there is a kernel–cokernel pair in $\mathcal{E}$ whose kernel is f. Dually, f is said to be an *admissible epic* if it is the cokernel of any pair in $\mathcal{E}$.

An *exact structure* on $\mathcal{A}$ is a class $\mathcal{E}$ that is closed under isomorphism, which is to say that if $(i, p) \in \mathcal{E}$ and there are vertical isomorphisms making the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & E \arrow[r, "p"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "i'"] & E' \arrow[r, "p'"] & Y' \end{tikzcd}$$

commute then $(i', p') \in \mathcal{E}$, and satisfies the following axioms:

(E0) For any object $X \in \mathcal{A}$, the identity morphism 1_X is an admissible monic.

(E0)$^{\mathrm{op}}$ For any object $X \in \mathcal{A}$, the identity morphism 1_X is an admissible epic.

(E1) The class of admissible monics is closed under composition; the composition of two admissible monics is an admissible monic.

(E1)$^{\mathrm{op}}$ The class of admissible epics is closed under composition; the composition of two admissible epics is an admissible epic.

(E2) The class of admissible monics is closed under pushout in the sense that the every admissible monic admits a pushout along any morphism and that this yields an admissible monic.

(E2)$^{\mathrm{op}}$ The class of admissible epics is closed under pullback in the sense that every admissible epic admits a pullback along any morphism and that this yields an admissible epic.

Diagrammatically (where arrows with a tail denote admissible monics and double-headed arrows denote admissible epics), (E2) says that for any diagram to the left below, there should be a pushout square depicted to the right:

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \\ X' & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, tail] & Y' \end{tikzcd}$$

Similarly, (E2)$^{\mathrm{op}}$ says that for any diagram to the left, the pullback square to the right should exist:

$$\begin{tikzcd} & Y' \arrow[d] \\ X \arrow[r, two heads] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, two heads] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

Finally, an *exact* category is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is an exact structure on $\mathcal{A}$. The elements of $\mathcal{E}$ are called *short exact sequences*.

Remark 5.2. [2, p. 5] Let $\mathcal{A}$ be an additive category. Note that a pair of morphisms (f, g) is a kernel–cokernel pair in the opposite category $\mathcal{A}^{\mathrm{op}}$ if and only if (g, f) is a kernel–cokernel pair in the original category $\mathcal{A}$.

This gives rise to a notion of a dual exact structure as follows: let $\mathcal{E}$ be an exact structure on $\mathcal{A}$. The class $\mathcal{E}^{\mathrm{op}}$ of pairs (p, i) for every $(i, p) \in \mathcal{E}$ is then an exact structure on the opposite category $\mathcal{A}^{\mathrm{op}}$; the class is readily shown to be closed under isomorphisms and every explicit axiom for $\mathcal{E}^{\mathrm{op}}$ follows from the dual axiom for $\mathcal{E}$.

Thus, any exact category $(\mathcal{A}, \mathcal{E})$ has a dual category $(\mathcal{A}, \mathcal{E})^{\mathrm{op}} := (\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$ that is exact, and the dual of the dual is the original category. This allows for reasoning by duality with exact categories.

Remark 5.3. If i is an admissible monic with respect to any class $\mathcal{E}$ of kernel–cokernel pairs that is closed under isomorphism, then $(i, p) \in \mathcal{E}$ for *any* cokernel p of i. To see this, let q be a cokernel of i with $(i, q) \in \mathcal{E}$. Because cokernels are unique up to isomorphism, any cokernel p of i is of the form $p = \varphi \circ q$ for some isomorphism φ. Thus we may form the following commutative diagram, where the vertical morphisms constitute an isomorphism between (i, q) and (i, p):

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "q"] \arrow[d, equal] & Y \arrow[d, "\varphi"] \\ X' \arrow[r, "i"] & E' \arrow[r, "p"] & Y' \end{tikzcd}$$

Because $\mathcal{E}$ is closed under isomorphism, $(i, p) \in \mathcal{E}$.

Dually, *any* kernel of an admissible epic is an admissible monic like above.

Proposition 5.4. *Let $\mathcal{E}$ be any class of kernel–cokernel pairs satisfying the implicit axiom of being closed under isomorphism of sequences. Then, the classes of admissible monics and admissible epics with respect to $\mathcal{E}$ are both closed under composition by isomorphisms.*

Proof. Let i be an admissible monic and φ be a precomposable isomorphism. These fit into the following commutative diagram with $(i, p) \in \mathcal{E}$ in the first row and vertical isomorphisms constituting an isomorphism of sequences:

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "\varphi^{-1}"] & E \arrow[r, two heads, "p"] \arrow[d, equal] & Y \arrow[d, equal] \\ X' \arrow[r, "i \circ \varphi"] & E \arrow[r, "p"] & Y \end{tikzcd}$$

Because $\mathcal{E}$ is closed under isomorphism of sequences, $i \circ \varphi$ in the second row is an admissible monic. For a postcomposable isomorphism ψ, consider the diagram

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, equal] & E \arrow[r, two heads, "p"] \arrow[d, "\psi"] & Y \arrow[d, equal] \\ X \arrow[r, "\psi \circ i"] & E' \arrow[r, "p \circ \psi^{-1}"] & Y \end{tikzcd}$$

and conclude in the same way that $\psi \circ i$ is an admissible monic. The statement for admissible epics follows by duality. $\square$

Remark 5.5. At first, it might seem that the truthfulness of axiom (E2) (and $(\text{E2})^{\text{op}}$) for a given class of kernel–cokernel pairs $\mathcal{E}$ depends on the choice of pushout, i.e., that an admissible monic might have two pushouts along some arbitrary morphism and that one of the pushouts contains an admissible monic while the other does not. Fortunately, this is not the case assuming that $\mathcal{E}$ is closed under isomorphism of sequences: either all the pushouts contain an admissible monic or none of them do. To see this, note that two pushouts of the same morphisms differ by an isomorphism (by the uniqueness up to isomorphism given in Proposition 2.47) and use Proposition 5.4.

Proposition 5.6. *The following statement is equivalent to (E0) for a class of kernel–cokernel pairs in an additive category $\mathcal{A}$ assuming only the implicit axiom that the class is closed under isomorphism of sequences:*

(E0') For any object $X \in \mathcal{A}$, the zero morphism $X \xrightarrow{0} 0$ is an admissible epic.

Dually, $(E0)^{op}$ is equivalent to

$(E0')^{op}$ For any object $X \in \mathcal{A}$, the zero morphism $0 \xrightarrow{0} X$ is an admissible monic.

In particular, (E0') and $(E0')^{op}$ are true in every exact category.

Proof. For the implication $(E0) \implies (E0')$, we have for any $X \in \mathcal{A}$ that 1_X is an admissible monic. Note that 0_{X0} is a cokernel of 1_X and hence that 0_{X0} is an admissible epic. For the converse, note similarly that 1_X is a kernel of 0_{X0} and is hence an admissible monic for any object $X \in \mathcal{A}$.

The second equivalence follows by duality. $\square$

As a first example of an exact category, we consider an abelian category together with all its short exact sequences in the sense of Definition 4.21, i.e., with the class of all kernel–cokernel pairs. Bühler gives the more general statement that any quasi-abelian category (a notion that is not covered in this document) with all its kernel–cokernel pairs is an exact category [2, p. 16]. Lemma 5.8 may be interpreted as stating that every abelian category is quasi-abelian. Moreover, Bühler states that the class of all kernel–cokernel pairs in the most general case of an additive category is not always an exact structure [2, p. 2].

Example 5.7. Let $\mathcal{A}$ be an abelian category and $\mathcal{E}_{\max}$ be the class of all kernel–cokernel pairs. In other words, $\mathcal{E}_{\max}$ is the class of all short exact sequences as defined in abelian categories (see Definition 4.21). Then $(\mathcal{A}, \mathcal{E}_{\max})$ is an exact category.

To show this, we verify the axioms one by one (including the implicit axiom that $\mathcal{E}_{\max}$ is closed under isomorphism). For the implicit axiom, consider a commutative diagram

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[d, "f"] & E \arrow[r, two heads, "p"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[r, "i'"] & E' \arrow[r, "p'"] & Y'
\end{tikzcd}$$

where (i, p) is a kernel–cokernel pair and the vertical morphisms are isomorphisms and show that (i', p') is a kernel–cokernel pair. Seeing as i' is monic and p' is epic, it suffices by Lemma 4.8 to show that i' is a kernel of p'. Express i' and p' as $i' = g \circ i \circ f^{-1}$ and $p' = h \circ p \circ g^{-1}$

$$\begin{tikzcd}
X \arrow[r, tail, "i"] & E \arrow[r, two heads, "p"] \arrow[d, bend right, "g"'] & Y \arrow[d, "h"] \\
X' \arrow[u, "f^{-1}"] \arrow[r, "i'"] & E' \arrow[r, "p'"] \arrow[u, bend right, "g^{-1}"'] & Y'
\end{tikzcd}$$

Then note that $i \circ f^{-1}$ is a kernel of $h \circ p$ (because f and h are isomorphisms), and realize (for instance by recalling Corollary 2.51) that the kernel–morphism relation is stable under composition by an isomorphism in the midst of things, by which the author means that

$$i \circ f^{-1} \text{ is a kernel of } h \circ p \iff g \circ i \circ f^{-1} = i' \text{ is a kernel of } h \circ p \circ g^{-1} = p'$$

Thus $(i', p') \in \mathcal{E}_{\max}$ and the class is closed under isomorphism.

For the explicit axioms, note first that an admissible monic is precisely a monomorphism: an admissible monic is a kernel and hence a monomorphism. Conversely, in an abelian category, every monomorphism forms a kernel–cokernel pair with any of its cokernels (which exist).

The first and second pair of axioms are immediate consequences of this observation: every identity morphism is an isomorphism, hence in particular a monic epimorphism, and hence an admissible monic and epic. A composition of admissible monics is monic and hence also an admissible monic (and similarly for admissible epics).

The remaining axiom pair (E2) and (E2)$^{\mathrm{op}}$ requires a little more work to verify. Their statements are also interesting in their own right and are therefore neatly arranged in the following lemma.

Lemma 5.8. *[5, p. 203] Consider a square in an abelian category*

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "g_B"] & Y \end{tikzcd}$$

If the square is a pushout square and f_A is a monomorphism, then g_B is a monomorphism. Dually, if the square is a pullback square and g_B is an epimorphism, then f_A is an epimorphism.

Proof. Let f_B and g_A denote the unnamed morphisms:

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, "f_B"] & A \arrow[d, "g_A"] \\ B \arrow[r, "g_B"] & Y \end{tikzcd}$$

Suppose that the square is a pushout, i.e., that $\begin{pmatrix} g_A & g_B \end{pmatrix}$ is the cokernel of $\begin{pmatrix} f_A \\ -f_B \end{pmatrix}$. To show that g_B is a monomorphism, enlarge and modify the diagram (at the expense of commutativity) by inserting the biproduct $A \oplus B$:

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, "f_B"'] \arrow[dr, "\begin{pmatrix} f_A \\ -f_B \end{pmatrix}"] & A \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \\ B \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[drr, "g_B"'] & A \oplus B \arrow[dr, "\begin{pmatrix} g_A & g_B \end{pmatrix}"] & \\ & & Y \end{tikzcd}$$

Then add any morphism $z\colon Z \to B$ that precomposes with g_B to zero:

$$\begin{tikzcd}
 & X \arrow[r, "f_A"] \arrow[d, "f_B"'] \arrow[dr, "\left(\begin{smallmatrix} f_A \\ -f_B \end{smallmatrix}\right)"] & A \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \\
Z \arrow[r, "z"] & B \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[drr, "g_B"'] & A \oplus B \arrow[dr, "\left(\begin{smallmatrix} g_A & g_B \end{smallmatrix}\right)"] & \\
 & & & Y
\end{tikzcd}$$

Note that $g_B \circ z = \left(\begin{smallmatrix} g_A & g_B \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \circ z$, so $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \circ z = \left(\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right)$ precomposes with $\left(\begin{smallmatrix} g_A & g_B \end{smallmatrix}\right)$ to zero. But $\begin{pmatrix} f_A \\ -f_B \end{pmatrix}$ is the kernel of $\left(\begin{smallmatrix} g_A & g_B \end{smallmatrix}\right)$, so $\left(\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right)$ factors through $\begin{pmatrix} f_A \\ -f_B \end{pmatrix}$ via some $h\colon Z \to X$:

$$\begin{tikzcd}
 & X \arrow[r, "f_A"] \arrow[d, "f_B"] \arrow[dr, "\left(\begin{smallmatrix} f_A \\ -f_B \end{smallmatrix}\right)"] & A \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \\
Z \arrow[ur, "h"] \arrow[r, "z"] & B \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[drr, "g_B"'] & A \oplus B \arrow[dr, "\left(\begin{smallmatrix} g_A & g_B \end{smallmatrix}\right)"] & \\
 & & & Y
\end{tikzcd}$$

f_A being monic now gives

$$\begin{aligned}
\begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} f_A \\ -f_B \end{pmatrix} \circ h = \begin{pmatrix} f_A \circ h \\ (-f_B) \circ h \end{pmatrix} &\implies f_A \circ h = 0 \\
&\implies h = 0 \\
&\implies 0 = \begin{pmatrix} f_A \\ -f_B \end{pmatrix} \circ h = \begin{pmatrix} 0 \\ z \end{pmatrix} \\
&\implies z = 0
\end{aligned}$$

Thus we have shown that only zero morphisms precompose with g_B to zero, which (because the category is additive) is precisely to say that g_B is monic.

The statement about pullbacks and epimorphisms follows by duality. □

A class of short exact sequences that is particularly important is that of the split short exact sequences. A priori, it is not obvious that they are exact, but we will soon see that they are and, moreover, that they give rise to a minimal exact structure on any additive category.

Definition 5.9. A "short sequence" $A \xrightarrow{f} E \xrightarrow{g} B$ in an additive category $\mathcal{A}$ is said to be *split short exact* if it is isomorphic to the sequence $A \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} A \oplus B \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)} B$, i.e., if there are vertical isomorphisms making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

The outer isomorphisms α and β may without loss of generality be taken as the identity morphisms for split short exact sequences.

Proposition 5.10. *A short sequence* $A \xrightarrow{f} E \xrightarrow{g} B$ *is split short exact if and only if there is an isomorphism* γ' *making the following diagram commute:*

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma'"] & B \arrow[d, equal] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd} \tag{1}$$

Proof. The veracity of the "if" claim is immediate, seeing as the identity morphisms are isomorphisms and the diagram is assumed to commute.

For the "only if" claim, let $A \xrightarrow{f} E \xrightarrow{g} B$ be a split short exact sequence. By the definition, there are vertical isomorphisms making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

and we are looking for an isomorphism γ' making the diagram Eq. (1) with equalities commute. Precompose the outer isomorphisms by their respective inverse to obtain equalities (identity morphisms) along the sides from top to bottom:

$$\begin{tikzcd}
A \arrow[d, "\alpha^{-1}"] & & B \arrow[d, "\beta^{-1}"] \\
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

Then fill out the rest of the row just like the middle row

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha^{-1}"] & A \oplus B \arrow[r, "(0\ 1)"] & B \arrow[d, "\beta^{-1}"] \\
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

and note that the canonical inclusion and projection behave well enough for us to push around the α^{-1} and β^{-1} and complete simultaneously both squares at the top in a commutative manner with the isomorphism $\gamma'' := \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ (recall Proposition 3.58 that a diagonal matrix of isomorphisms is an isomorphism):

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha^{-1}"] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma''"] & B \arrow[d, "\beta^{-1}"] \\
A \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

The outer square in this diagram is of the required form with equalities along the sides. Explicitly, for $\gamma = (\gamma_1\ \gamma_2)$, the middle isomorphism γ' is $\gamma \circ \gamma'' = (\gamma_1 \circ \alpha^{-1}\ \gamma_2 \circ \beta^{-1})$:

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma'"] & B \arrow[d, equal] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

$\square$

Remark 5.11. The notion of being a split short exact sequence is self-dual in the sense that a sequence (f, g) is split short exact in $\mathcal{A}$ if and only if the sequence (g, f) is split short exact in $\mathcal{A}^{\text{op}}$. To see this, construct the diagram

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, "\gamma"] & B \arrow[d, equal] \\
A \arrow[r, "f"] & E \arrow[r, "g"] & B
\end{tikzcd}$$

for (f, g) in $\mathcal{A}$ as per Proposition 5.10. Then consider the same diagram in the dual category (with the matrices adjusted accordingly)

$$\begin{tikzcd}
A \arrow[d, equal] & A \oplus B \arrow[l, "(1\ 0)"'] & B \arrow[l, "\binom{0}{1}"'] \arrow[d, equal] \\
A & E \arrow[l, "f"] \arrow[u, "\gamma"'] & B \arrow[l, "g"]
\end{tikzcd}$$

which is almost of the required form. Swap A and B in the biproduct by adding another isomorphic sequence to the diagram with the canonical isomorphism of $A \oplus B$ and $B \oplus A$ in the middle:

$$\begin{tikzcd}
A \arrow[d, equal] & B \oplus A \arrow[l, "(0\ 1)"'] & B \arrow[l, "\binom{1}{0}"'] \arrow[d, equal] \\
A \arrow[d, equal] & A \oplus B \arrow[l, "(1\ 0)"] \arrow[u, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"'] & B \arrow[l, "\binom{0}{1}"] \arrow[d, equal] \\
A & E \arrow[l, "f"] \arrow[u, "\gamma"'] & B \arrow[l, "g"]
\end{tikzcd}$$

Thus the top and bottom rows are isomorphic, which is to say that (g, f) is split short exact in the opposite category $\mathcal{A}^{\text{op}}$. Repeat the procedure for (g, f) in $\mathcal{A}^{\text{op}}$ for the converse implication.

The term "split short *exact* sequence" is justified by the following proposition.

Proposition 5.12. *[2, p. 6] In any exact category $(\mathcal{A}, \mathcal{E})$, every split short exact sequence is exact.*

Proof. Because $\mathcal{E}$ is closed under isomorphism, it suffices to show that

$$A \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} A \oplus B \xrightarrow{\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)} B$$

is exact. Note first that this is a kernel–cokernel pair: their composition is zero, and any morphism that precomposes with $\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)$ to zero is of the form $\left(\begin{smallmatrix}f\\0\end{smallmatrix}\right)$ for some $f\colon X \to A$ and thus factors uniquely through $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ via this f as $\left(\begin{smallmatrix}f\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \circ f$. Thus $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is a kernel of $\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)$. For the cokernel part, one may reason similarly or note that the kernel part generalizes to any split short exact sequence and reason by duality.

To show that the sequence is exact, it suffices to show that $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is an admissible monic. To this end, consider the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\ B \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] & A \oplus B \end{tikzcd}$$

which is a pushout square, because the biproduct $A \oplus B$ is a coproduct. Moreover, all the morphisms in the diagram are admissible monics: Proposition 5.6 implies that the morphisms being pushed out (the top and left morphisms) are admissible monics and (E2) then implies that the morphisms of the pushout (the bottom and right morphisms) are admissible monics. In particular, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is an admissible monic, and the statement of the proposition follows. ☐

The following example, which Bühler mentions briefly in his introduction and list of examples of exact categories [2, p. 2, 50], shows that the class $\mathcal{E}_{\min}$ of all split short exact sequences in an additive category $\mathcal{A}$ is an exact structure on $\mathcal{A}$. By Proposition 5.12, this is the smallest exact structure on $\mathcal{A}$, in the sense that it is contained in every other exact structure on $\mathcal{A}$.

Example 5.13. For any additive category $\mathcal{A}$, the class $\mathcal{E}_{\min}$ of all split short exact sequences in $\mathcal{A}$ is an exact structure on $\mathcal{A}$.

Let us verify that this is indeed the case. The implicit requirement that $\mathcal{E}_{\min}$ be closed under isomorphism holds by the very definition of split short exact sequences.

It suffices to verify only that (E0), (E1), and (E2) hold, because the opposite axioms follow by duality. For (E0), consider for any object $X \in \mathcal{A}$ the commutative diagram

$$\begin{tikzcd} X \arrow[r, tail, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, equal] & X \oplus 0 \arrow[r, two heads, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)"] & 0 \arrow[d, equal] \\ X \arrow[r, "1"] & X \arrow[r, "0"] & 0 \end{tikzcd}$$

The vertical morphism is a projection morphism from a biproduct, which is to say that it has a right inverse, namely the corresponding inclusion morphism $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\colon X \to X \oplus 0$. The right inverse is in this case also a left inverse:

$$\begin{aligned} \begin{pmatrix} 1_X \\ 0_{X0} \end{pmatrix} \circ \begin{pmatrix} 1_X & 0_{0X} \end{pmatrix} &= \begin{pmatrix} 1_X & 0_{0X} \\ 0_{X0} & 0_{00} \end{pmatrix} \\ &= \begin{pmatrix} 1_X & 0_{0X} \\ 0_{X0} & 1_0 \end{pmatrix} \end{aligned}$$

This shows that the vertical morphism in the diagram is an isomorphism and hence that $X \xrightarrow{1} X \xrightarrow{0} 0$ is a split short exact sequence. In particular 1_X is an admissible monic.

For (E1), consider an arbitrary split short exact sequence (the bottom row)

$$\begin{tikzcd}
A \arrow[r, tail, "\binom{1}{0}"] \arrow[d, equal] & A \oplus B \arrow[r, two heads, "(\,0\ 1\,)"] \arrow[d, "\varphi = (\,\varphi_1\ \varphi_2\,)"] & B \arrow[d, equal] \\
A \arrow[r, tail, "a"] & E \arrow[r, two heads, "b"] & B
\end{tikzcd}$$

and another split short exact sequence whose first morphism composes with a, i.e., of the form

$$\begin{tikzcd}
E \arrow[r, tail, "\binom{1}{0}"] \arrow[d, equal] & E \oplus C \arrow[r, two heads, "(\,0\ 1\,)"] \arrow[d, "\psi = (\,\psi_1\ \psi_2\,)"] & C \arrow[d, equal] \\
E \arrow[r, tail, "e"] & F \arrow[r, two heads, "c"] & C
\end{tikzcd}$$

where the vertical morphisms are isomorphisms, and find a split short exact sequence with the composition $e \circ a$ as the first morphism, i.e., of the form

$$\begin{tikzcd}
A \arrow[r, tail, "\binom{1}{0}"] \arrow[d, equal] & A \oplus D \arrow[r, two heads, "(\,0\ 1\,)"] \arrow[d, "\chi = (\,\chi_1\ \chi_2\,)"] & D \arrow[d, equal] \\
A \arrow[r, tail, "e \circ a"] & F \arrow[r, two heads, "d"] & D
\end{tikzcd}$$

for some d and isomorphism χ (all diagrams are assumed to commute). Note that

$$F \cong E \oplus C \cong (A \oplus B) \oplus C \cong A \oplus (B \oplus C)$$

must necessarily hold, so take D to be $B \oplus C$. Note further that, given an isomorphism χ, we may take d simply to be $(\,0\ 1\,) \circ \chi^{-1}$ in order to make the right square commute. Thus, we are looking for an isomorphism $\chi\colon A \oplus D = A \oplus (B \oplus C) \to F$ that makes the first square commute. But the first square commuting is precisely to say that χ is of the form $(\,e \circ a\ \chi_2\,)$ for any χ_2. Similarly, the first squares of the previous two diagrams commuting is precisely to say that $\varphi_1 = a$ and $\psi_1 = e$.

Consider the isomorphism $(\,\psi_1 \circ \varphi\ \psi_2\,) = (\,e \circ \varphi\ \psi_2\,) = (\,(\,e \circ \varphi_1\ e \circ \varphi_2\,)\ \psi_2\,)\colon (A \oplus B) \oplus C \to F$ (which really is an isomorphism by Corollary 3.59). It induces an isomorphism $(\,e \circ \varphi_1\ e \circ \varphi_2\ \psi_2\,) = (\,e \circ a\ e \circ \varphi_2\ \psi_2\,)\colon A \oplus B \oplus C \to F$ (via Proposition 3.54), which in turn induces an isomorphism $(\,e \circ a\ (\,e \circ \varphi_2\ \psi_2\,)\,)\colon A \oplus (B \oplus C) \to F$. This latter isomorphism is of the required form, and we conclude that (E1) holds.

For (E2), consider first the case that the admissible monic being pushed out is of the nice form $A \xrightarrow{\binom{1}{0}} A \oplus B$. Let $f\colon A \to X$ be any morphism with the same domain. The following is then a pushout square:

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "f"] & A \oplus B \arrow[d, "\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}"] \\
X \arrow[r, "\binom{1}{0}"] & X \oplus B
\end{tikzcd}$$

The commutativity of the square is immediate. Moreover. any other pair (φ, ψ), where we may write

$\varphi = \left(\, \varphi_1 \; \varphi_2 \,\right)$, completing the square commutatively factors uniquely as follows:

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "f"'] & A \oplus B \arrow[d, "\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[ddr, bend left, "(\, \varphi_1 \; \varphi_2 \,)"] & \\
X \arrow[r, "\binom{1}{0}"] \arrow[drr, bend right, "\psi"'] & X \oplus B \arrow[dr, "(\, \psi \; \varphi_2 \,)"'] & \\
& & D
\end{tikzcd}$$

It is immediate that the bottom triangle commutes, and the commutativity of the right-hand triangle follows from $\psi \circ f = \varphi_1$ (which holds by the commutativity of the outer square). Uniqueness follows by noting that the first component of the factoring morphism must be ψ in order for the bottom triangle to commute and the second component must be φ_2 in order for the right-hand triangle to commute. In other words, the factoring morphism must be $(\, \psi \; \varphi_2 \,)$. Note that the pushout $\binom{1}{0} \colon X \to X \oplus B$ of $\binom{1}{0} \colon A \to A \oplus B$ along f is an admissible monic.

Consider now an arbitrary first morphism $g \colon A \to E$ of a split short exact sequence, for which there by Proposition 5.10 is an object B and an isomorphism φ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & A \oplus B \arrow[d, "\varphi"] \\
A \arrow[r, "g"] & E
\end{tikzcd}$$

In other words, g is $\binom{1}{0}$ postcomposed by an isomorphism. By Proposition 2.50, pushouts of g along f not only exist but they coincide with the pushouts of $\binom{1}{0} \colon A \to A \oplus B$ along f. The latter pushouts are admissible monics, and so we conclude that (E2) holds.

5.2 Extension-closed subcategories

So far, we have seen two examples of exact categories (Examples 5.7 and 5.13). In this subsection, we will see how one may construct new exact categories from a given one by considering appropriate subcategories.

Definition 5.14. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. An *extension* of a pair of objects $A, B \in \mathcal{A}$ is an object $E \in \mathcal{A}$ for which there is a short exact sequence of the form $A \rightarrowtail E \twoheadrightarrow B$.

An *extension-closed subcategory* is a subcategory $\mathcal{B}$ of $(\mathcal{A}, \mathcal{E})$ that is closed under extension. Explicitly, for any short exact sequence $A \rightarrowtail E \twoheadrightarrow B$ in $(\mathcal{A}, \mathcal{E})$ with endpoints A, B in $\mathcal{B}$, the extension E is also in $\mathcal{B}$.

It turns out that full extension-closed subcategories with zero objects are themselves exact when equipped with all the short exact sequences from the supercategory with objects in the subcategory. In order to prove this, we will require two lemmas.

The first lemma can be understood as stating that admissible monics that are part of a pushout square have the same cokernel objects and their cokernel morphisms are related in a simple way. Bühler gives the lemma as part of a list of equivalent statement [2, p. 7].

Lemma 5.15. *If*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\ A' \arrow[r, tail, "i'"] & B' \end{tikzcd}$$

is a pushout square, then it can be expanded to a commutative diagram with exact rows:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

Proof. Consider any cokernel p of i and the zero morphism $0\colon A' \to C$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] \arrow[rr, bend right, "0"'] & B' & C \end{tikzcd}$$

The outer square commutes and hence induces, by the pushout property, a unique morphism $p'\colon B' \to C$ making the diagram commute, which explicitly is to say that $p = p' \circ f'$ and $0 = p' \circ i'$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] \arrow[rr, bend right, "0"'] & B' \arrow[r, "p'"] & C \end{tikzcd}$$

It suffices to show that this p' is a cokernel of i'. We saw that it postcomposes with i' to 0, so it remains to show that any other morphism $d\colon B' \to D$ that postcomposes with i' to 0 factors uniquely through p'. If $d \circ i' = 0$, then in particular $d \circ i' \circ f = 0$, which by the commutativity of the pushout square is equivalent to $d \circ f' \circ i = 0$. In other words, $d \circ f'$ postcomposes with i to 0. Seeing as p is a cokernel of i, this induces a unique morphism $g\colon C \to D$ with $d \circ f' = g \circ p$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, "p'"] \arrow[dr, "d"'] & C \arrow[d, "g"] \\ & & D \end{tikzcd}$$

The uniqueness of g in the cokernel property for p establishes the uniqueness of the factorization for p': for any $g'\colon C \to D$ with $d = g' \circ p'$, precomposition by f' gives $d \circ f' = g' \circ p' \circ f' = g' \circ p$, which implies $g' = g$.

For existence, the commutativity of the triangle ($d = g \circ p'$) still remains to be shown. This would be immediate if f' were epic, but that is not generally the case. Instead, we may use the epicness of sorts provided by the uniqueness in the pushout factorization as follows. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] & \\
A' \arrow[r, tail, "i'"] & B' \arrow[dr, "d"] & \\
& & D
\end{tikzcd}$$

and note that the diagram commutes (the compositions along the paths from A to D both yield 0). The pushout property of the original pushout square now asserts that there is a unique morphism $B \to D$ via which $d \circ f'$ and $d \circ i'$ factor through f' and i'. On one hand, said unique morphism is obviously d. On the other hand, we may note that $g \circ p'$ works equally well: $(g \circ p') \circ i' = g \circ 0_{AC} = 0_{AD} = d \circ i'$ and $(g \circ p') \circ f' = g \circ p = d \circ f'$ by the definition of g. We thus conclude that $d = g \circ p'$ and that p' is a cokernel of i'.

We have shown that the commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C
\end{tikzcd}$$

has exact rows, which finishes the proof. $\square$

Remark 5.16. Note that the cokernel p of i was chosen arbitrarily in the proof of Lemma 5.15. That is to say that we may expand any diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \\
A' \arrow[r, tail, "i'"] & B' &
\end{tikzcd}$$

where the square is a pushout square and the first row is exact into a diagram with exact rows as in the lemma

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C
\end{tikzcd}$$

The second lemma may be viewed as a generalization to exact categories of the second isomorphism theorem (of modules say), which states that there is a canonical isomorphism from C/B to the quotient of quotients $(C/A)/(B/A)$ for nested submodules $A \subseteq B \subseteq C$ (using Grillet's numbering, this is the *first* isomorphism theorem [3, p. 323]).

Bühler states the lemma with the additional conclusion that the upper right square is bicartesian (i.e., both a pullback square and a pushout square) [2, p. 12], which we will not require.

Lemma 5.17 (The second isomorphism theorem for exact categories)**.** *Consider the commutative diagram*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \\
& Z \arrow[r, equal] & Z
\end{tikzcd}$$

where the first and second rows as well as the middle column are short exact. Then there is a unique third column that is short exact and makes the extended diagram commute:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d, tail] \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads] \\
& Z \arrow[r, equal] & Z
\end{tikzcd}$$

Proof. For the morphism $X \to Y$, use the fact that $B \to X$ is a cokernel of $A \to B$ and that $B \to C \to Y$ postcomposes with $A \to B$ to zero, seeing as the upper left square commutes. This induces a morphism $X \to Y$ that makes the upper right square commute. Furthermore, said square is a pushout square. To see this, consider another commutative completion of the square

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d] \arrow[ddr] & \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[drr] & Y & \\
& & & D
\end{tikzcd}$$

The upper path $B \to X \to D$ postcomposes with $A \to B$ to zero, so $A \to B \to C \to D$ and hence $A \to C \to D$ are zero by commutativity. But $C \to Y$ is a cokernel of $A \to C$, so there is a unique morphism $Y \to D$ making the bottom triangle commute:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d] \arrow[ddr] & \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[drr] & Y \arrow[dr] & \\
& & & D
\end{tikzcd}$$

Because $B \to X$ is epic, this factoring morphism makes the upper triangle commute as well. Explicitly, using the appropriate commutativity properties,

$$\begin{aligned}
B \to X \to Y \to D &= B \to C \to Y \to D \\
&= B \to C \to D \\
&= B \to X \to D
\end{aligned}$$

Epicness of $B \to X$ now gives that $X \to Y \to D$ and $X \to D$ are equal, which finishes the proof that the square is a pushout square.

Thus we have the diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d, tail] \\ A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \\ & Z \arrow[r, equal] & Z \end{tikzcd}$$

and may apply Lemma 5.15 and Remark 5.16 to find a morphism $Y \to Z$ such that the diagram commutes and the rightmost column is short exact:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d, tail] \\ A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads] \\ & Z \arrow[r, equal] & Z \end{tikzcd}$$

This shows the existence part of the lemma. The uniqueness part follows immediately from the epicness of $B \to X$ and $C \to Y$. $\square$

Now we are ready to state and prove the main proposition of this subsection. Bühler gives a variant of the proposition with a more lenient notion of extension-closed subcategories but with the slightly stronger assumption that the subcategory is additive [2, p. 40].

Proposition 5.18. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let $\mathcal{B}$ be a full extension-closed subcategory of $(\mathcal{A}, \mathcal{E})$ with a zero object and $\mathcal{F}$ be the subclass of all short exact sequences* $A \rightarrowtail E \twoheadrightarrow B$ *in $\mathcal{E}$ with $A, E, B \in \mathcal{B}$. Then, $(\mathcal{B}, \mathcal{F})$ is an exact category.*

Proof. Note first that $\mathcal{B}$ is closed under binary biproducts: for any $A, B \in \mathcal{B}$, the split short exact sequence

$$A \xrightarrow{\binom{1}{0}} A \oplus B \xrightarrow{(0\ 1)} B$$

is short exact (by Proposition 5.12) in the supercategory. Because $\mathcal{B}$ is extension-closed, it contains the biproduct $A \oplus B$.

Thus $\mathcal{B}$ is a full subcategory of an additive category that is closed under biproducts and has a zero object. Proposition 3.37 lets us conclude that $\mathcal{B}$ is additive (with morphism addition inherited from $\mathcal{A}$).

Note next that $\mathcal{F}$ consists of kernel–cokernel pairs with respect to $\mathcal{B}$ and not just $\mathcal{A}$. By fullness, the morphisms of such a pair compose to the zero morphism in $\mathcal{B}$ and the factoring morphisms for the kernel and cokernel property in $\mathcal{B}$ are simply the factoring morphisms in $\mathcal{A}$.

Closedness of $\mathcal{F}$ under isomorphism is straightforward to show: let

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d] & E \arrow[r, two heads, "p"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "i'"] & E' \arrow[r, "p'"] & B' \end{tikzcd}$$

be a commutative diagram in $\mathcal{B}$ with $(i,p) \in \mathcal{F}$ and vertical morphisms isomorphisms. Viewed in $\mathcal{A}$, the top row is short exact and the vertical morphisms are still isomorphisms (isomorphisms of a subcategory are always isomorphisms in the supercategory). By the implicit axiom for $\mathcal{E}$, the bottom row is short exact, and seeing as its objects are all in $\mathcal{B}$, we find that $(i',p') \in \mathcal{F}$,

We may reduce the work needed to show that $\mathcal{F}$ is an exact structure on $\mathcal{B}$ by observing that $\mathcal{B}^{\mathrm{op}}$ is a full extension-closed subcategory of $(\mathcal{A},\mathcal{E})^{\mathrm{op}}$ with a zero object and that $\mathcal{F}^{\mathrm{op}}$ is the subclass of $\mathcal{E}^{\mathrm{op}}$ of all sequences with objects in $\mathcal{B}^{\mathrm{op}}$. These observations are all immediate. Thus, it suffices to show (E0), (E1), and (E2) for $(\mathcal{B},\mathcal{F})$; the opposite axioms follow by duality.

(E0) trivially holds, seeing as $X \xrightarrow{1} X \xrightarrow{0} 0$ is short exact in $(\mathcal{A},\mathcal{E})$ with objects in $\mathcal{B}$ and hence short exact in $(\mathcal{B},\mathcal{F})$.

For (E1), it is immediate (by (E1) in $(\mathcal{A},\mathcal{E})$) that a composition $i' \circ i$ of admissible monics in $\mathcal{B}$ is an admissible monic in $\mathcal{A}$, but said composition might a priori have all its cokernel objects outside of $\mathcal{B}$ and hence not be an admissible monic in $\mathcal{B}$. The idea is to use the second of the two lemmas (Lemma 5.17) to find a short exact sequence in $\mathcal{A}$ with a cokernel object of $i' \circ i$ as the extension and objects in $\mathcal{B}$ as endpoints and conclude by extension-closedness that the sequence is short exact in $\mathcal{B}$ as well.

Let $i\colon A \to E$ and $i'\colon E \to F$ be two composable admissible monics in $\mathcal{B}$ and let $p\colon E \to B$ and $p'\colon F \to D$ be any of their cokernels in $\mathcal{B}$. By (E1) for $(\mathcal{A},\mathcal{E})$, the composition $i' \circ i$ is an admissible monic in $\mathcal{A}$. Thus we may consider a cokernel $p''\colon F \to C$ of $i' \circ i$ in $\mathcal{A}$ and form the commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & E \arrow[r, two heads, "p"] \arrow[d, tail, "i'"] & B \\
A \arrow[r, tail, "i'\circ i"] & F \arrow[r, two heads, "p''"] \arrow[d, two heads, "p'"] & C \\
 & D \arrow[r, equal] & D
\end{tikzcd}$$

where all the objects except possibly C are in $\mathcal{B}$. Lemma 5.17 now gives a short exact sequence (in $\mathcal{A}$) as the third column:

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, equal] & E \arrow[r, two heads, "p"] \arrow[d, tail, "i'"] & B \arrow[d, tail] \\
A \arrow[r, tail, "i'\circ i"] & F \arrow[r, two heads, "p''"] \arrow[d, two heads, "p'"] & C \arrow[d, two heads] \\
 & D \arrow[r, equal] & D
\end{tikzcd}$$

Note now that $B, D \in \mathcal{B}$ and hence that $C \in \mathcal{B}$ by extension-closedness. We may thus view the above diagram in $\mathcal{B}$ and conclude in particular that the composition $i' \circ i$ is an admissible monic in $\mathcal{B}$.

For (E2), we will use extension-closedness and the first of the two lemmas (Lemma 5.15). Consider the diagram in $\mathcal{B}$

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & E \\
A' &
\end{tikzcd}$$

where i is an admissible monic with respect to $\mathcal{F}$ and f is any morphism. Add a cokernel p of i to the diagram:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & E \arrow[r, two heads, "p"] & B \\ A' & & \end{tikzcd}$$

By (E2) for $(\mathcal{A}, \mathcal{E})$, we may extend the diagram in $(\mathcal{A}, \mathcal{E})$ to include a pushout square

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & E \arrow[r, two heads, "p"] \arrow[d, "f'"] & B \\ A' \arrow[r, tail, "i'"] & E' & \end{tikzcd}$$

with i' an admissible monic (with respect to $\mathcal{E}$), and by Lemma 5.15 and Remark 5.16, we may extend the diagram further to

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & E \arrow[r, two heads, "p"] \arrow[d, "f'"] & B \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & E' \arrow[r, two heads, "p'"] & B \end{tikzcd}$$

with the second row exact in the supercategory.

Seeing as $A', B \in \mathcal{B}$ and $\mathcal{B}$ is extension-closed, $E \in \mathcal{B}$. Furthermore, because $\mathcal{B}$ is a full subcategory, all the morphisms in the diagram are also in $\mathcal{B}$ and so the diagram may be viewed as a diagram in $\mathcal{B}$ with exact rows. The fullness also implies that the left square is a pushout square not only in $\mathcal{A}$ but also in $\mathcal{B}$ (in much the same way as for kernels and cokernels, as noted previously in the proof). Thus we have shown in particular that the pushout of an admissible monic i along an arbitrary morphism f exists in $\mathcal{B}$ and that it moreover yields an admissible monic i'. $\square$

The following example, which presupposes some familiarity with homological algebra for R-modules, shows how Proposition 5.18 may be applied to R-Mod to find exact categories that are subcategories of R-Mod.

Example 5.19. Consider R-Mod (for some fixed ring R). For every R-module M, there is an additive covariant functor $\mathrm{Ext}^1_R(M, -)\colon R\text{-Mod} \to \mathrm{Ab}$ with the property that for every short exact sequence $X \longrightarrow E \longrightarrow Y$ of R-modules there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, X) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, Y)$$

$$\longrightarrow \mathrm{Ext}^1_R(M, X) \longrightarrow \mathrm{Ext}^1_R(M, E) \longrightarrow \mathrm{Ext}^1_R(M, Y)$$

of abelian groups. For more details, see [3, pp. 487–491].

Consider now the full subcategory $\mathcal{A}$ of R-Mod whose objects are the modules N for which $\mathrm{Ext}^1_R(M, N) = 0$. It has zero objects, seeing as additive functors map zero objects to zero objects (by Proposition 3.43) and hence $\mathrm{Ext}^1_R(M, 0) = 0$. Furthermore, we may show that it is extension-closed: suppose that $X, Y \in \mathcal{A}$ and that $X \longrightarrow E \longrightarrow Y$ is a short exact sequence in R-Mod. Then there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, X) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, Y)$$

$$\longrightarrow \mathrm{Ext}^1_R(M, X) \longrightarrow \mathrm{Ext}^1_R(M, E) \longrightarrow \mathrm{Ext}^1_R(M, Y)$$

By the assumption that $X, Y \in \mathcal{A}$, the sequence is of the form

$$0 \longrightarrow \mathrm{Hom}_R(M, X) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, Y)$$

$$\longrightarrow 0 \longrightarrow \mathrm{Ext}^1_R(M, E) \longrightarrow 0$$

so that the exactness in $\mathrm{Ext}^1_R(M, E)$ have us conclude that $\mathrm{Ext}^1_R(M, E) = 0$ and hence $E \in \mathcal{A}$.

Thus, $\mathcal{A}$ is an extension-closed subcategory of R-Mod with zero objects. By Proposition 5.18, $\mathcal{A}$ with short exact sequences precisely those that are short exact in R-Mod is an exact category.

References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[2] Theo Bühler. Exact categories. *Expo. Math.*, 28(1):1–69, 2010.

[3] Pierre Antoine Grillet. *Abstract algebra*, volume 242 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2007.

[4] Peter John Hilton and Urs Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, first edition, 1971.

[5] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[6] James Raymond Munkres. *Topology*. Prentice Hall, New Jersey, second edition, 2000.

Index