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U.U.D.M. Project Report 2016:49

2-Categories and Yoneda lemma

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Examensarbete i matematik, 15 hp
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Januari 2017

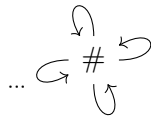
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January 3, 2017



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1 Introduction

Category theory is in some sense the theory of abstract functions. If one important general observation is that almost everything in mathematics can be realized as a set then another equally important observation might be that what we actually care about is how these mathematical objects relate to each other. Ideally we want to find deep, beautiful and surprising connections between different mathematical objects. In reality, and perhaps more so in the field of abstract algebra, such connections are often codified by structure preserving functions between objects of a certain type, for example group homomorphisms between groups. In a lot of cases these structure preserving functions can be composed in an associative way and we also generally have an identity function for every object. If this is the case we can form a category consisting of such objects and their associated structure preserving composable functions.

Category theory gives us a formal framework and language for investigating relations between objects. These abstract functions are called morphisms and much of the power of category theory can perhaps be summarized by saying that morphisms matter.

Category theory grew out of algebraic topology where it basically was needed in order to formalize what is meant by naturality. However, over time category theory has developed into its own self-contained diverse field of study and there is even real life applications of its theory in functional programming. It has also been suggested as a foundation of mathematics.

In this thesis we will review the some of the elementary concepts and results in category theory. We will define categories, consider some examples, define functors (functions between categories), natural transformations (functions between functors), adjoints, state and prove the classical Yoneda lemma.

In the last two sections we will define bicategories and in particular 2-categories, which are the simplest nontrivial higher categories, in which we not only have objects and morphisms between objects but also morphisms between morphisms. We will then define some 2-categorical analogues of functors and natural transformations and finally, we provide a 2-categorical version of the Yoneda lemma.

2 Categories, functors and natural transformations

We begin by defining categories, subcategories, functors and natural transformations between functors.

2.1 Definition and subcategories

Definition 1. A category \mathcal{C} consists of a class of objects a, b, c, \dots and a class of arrows f, g, h, \dots . To every arrow f we assign an object $a = \text{dom}(f)$ and an object $b = \text{cod}(f)$.

We also require that every object a has an identity arrow $\mathbb{1}_a: a \rightarrow a$ and that for arrows $f: a \rightarrow b, g: b \rightarrow c$ with $\text{cod}(f) = b = \text{dom}(g)$ we have composition arrow $g \circ f: a \rightarrow c$, which is associative i.e. for $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ the following equality always holds

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Finally, we have the unit law, which asserts that for all arrows $f: a \rightarrow b, g: b \rightarrow c$ composition with the identity arrow $\mathbb{1}_b$ gives

$$\mathbb{1}_b \circ f = f \text{ and } g \circ \mathbb{1}_b = g.$$

Arrows are also called morphisms and cod and dom should be understood as domain and codomain of a set theoretical function in the usual way, even though, as we shall see, in general morphisms need not be ordinary functions and objects might not be sets.

It is often emphasized that the most important thing in category theory are the morphisms and not the objects. This is not just semantics but true to the extent that we can dispense with objects altogether and only work with arrows. This is can be achieved by identifying each object with its identity arrow. The following definition makes this arrow-only construction precise.

Definition 2. A category \mathcal{C} consist of arrows, certain ordered pairs $\langle g, f \rangle$, called the composable pairs of arrows, and an operation assigning to each composable pair $\langle g, f \rangle$ an arrow gf , called their composite. We say that gf is defined for $\langle g, f \rangle$ is a composable pair.

We define an identity in \mathcal{C} to be an arrow u such that $fu = f$ whenever the composite $fu = f$ and $ug = g$ whenever ug is defined.

This data must obey the following three axioms

- For each arrow g in \mathcal{C} there exists identity arrows u and u' of \mathcal{C} such that $u'g$ and gu are defined.
- The composite $(kg)f$ is defined if and only if the composite $k(gf)$ is defined. When either is defined, they are equal and is written kgf .
- The triple composite kgf is defined whenever both composites kg and gf are defined.

It is easy to see that these two definitions of a category are equivalent.

Definition 3. For a category \mathcal{C} and $i, j \in \text{Obj}(\mathcal{C})$ we denote by; $\text{Hom}(i, j)$, $\text{Hom}_{\mathcal{C}}(i, j)$ or simply $\mathcal{C}(i, j)$, the set of all morphisms between objects i and j in \mathcal{C} .

This definition of homsets is actually somewhat sloppy since in general we have no guarantee that the collection of morphisms between any two objects is in fact a *set* (in the ZFC sense). A category where all homsets are sets is said to be (locally) *small*.

We prefer to ignore this and promptly assume that all our categories are small and leave foundations to the philosophers.

One remark regarding both of our definitions of a category; a category consists of objects, morphisms between objects and a rule of composition of morphisms. The right, and perhaps more categorical way to view composition of arrows in a category is to view it as a functions between homsets. This is a fair concept since we always assume that our categories are *small* and thus our homsets are sets and not proper classes.

Let \mathcal{C} be a category and let i, j, k, l be objects in \mathcal{C} then we can view composition \circ between morphisms as a well defined function:

$$\circ: \text{Hom}(j, k) \times \text{Hom}(i, j) \rightarrow \text{Hom}(i, k)$$

One of the main points of looking at composition this way it that we can express associativity as a commutative diagram:

$$\begin{array}{ccc}
 \text{Hom}(k, l) \times \text{Hom}(j, k) \times \text{Hom}(i, j) & \xrightarrow{(\text{id}, \circ)} & \text{Hom}(k, l) \times \text{Hom}(i, k) \\
 \downarrow (\circ, \text{id}) & & \downarrow \circ \\
 \text{Hom}(j, l) \times \text{Hom}(i, j) & \xrightarrow{\circ} & \text{Hom}(i, l)
 \end{array}$$

Explicitly, let $h \in \text{Hom}(k, l)$, $g \in \text{Hom}(j, k)$ and $f \in \text{Hom}(i, j)$ then

$$\begin{array}{ccc}
 (h, g, f) & \xrightarrow{\quad} & (h, g \circ f) \\
 \downarrow & & \downarrow \\
 (h \circ g, f) & \xrightarrow{\quad} & (h \circ g) \circ f = h \circ (g \circ f)
 \end{array}$$

We can also view identity morphisms in a similar way:

For every two objects i, j in \mathcal{C} there exists morphisms $\mathbb{1}_i \in \text{Hom}(i, i)$, $\mathbb{1}_j \in \text{Hom}(j, j)$

Such that

$$_ \circ \mathbb{1}_i : \text{Hom}(i, j) \rightarrow \text{Hom}(i, j)$$

$$\mathbb{1}_j \circ _ : \text{Hom}(j, i) \rightarrow \text{Hom}(j, i)$$

both are identity functions.

We will now provide a number of examples of categories.

2.2 The Category of Sets:

The most canonical example of a category is **Set** which has:

- Objects: Sets.
- Arrows: Functions between sets.
- Identity arrows: Identity functions.

It is easy to see that this obeys the category axioms. Composition of set theoretical functions are well defined and works as expected.

Every function has a domain and codomain and every set i has a identity function $\mathbb{1}_i : x \mapsto x$ for all x in i such that for $f : i \rightarrow j$, $f \circ \mathbb{1}_i = f$ and $g : j \rightarrow i$, $\mathbb{1}_i \circ g = g$. Composition of functions is of course associative.

2.3 The Category of Groups and categories of other algebraic structures

Our next example is slightly more interesting and is the category of groups, **Grp**.

- Objects: Groups $(G, \cdot_G, e_G), (H, \cdot_H, e_H), \dots$
- Arrows: Group homomorphisms, i.e. functions $\phi: G \rightarrow H$ such that $\phi(a \cdot_G b) = \phi(a) \cdot_H \phi(b)$, for all $a, b \in G$.
- Identity arrows: Identity functions.

Grp is a category since for any group (G, \cdot_G, e_G) , the identity function $\mathbb{1}_G: G \rightarrow G$ is a homomorphism. Composition of homomorphisms is a homomorphism, composition homomorphisms is associative and the unit law holds.

In order to verify this we actually have to prove all this but we will leave that as a straightforward exercise to the reader.

In fact, all algebraic structures; semigroups, monoids, abelian groups, rings, commutative rings, integral domains, fields, left R -modules, \mathbb{K} -vector spaces, unital associative \mathbb{K} -algebras, etc. all give rise to categories consisting of the class of all algebraic structures of a particular type as objects and as arrows; homomorphisms corresponding to that type, in other words functions preserving initial minimal algebraic structure.

This provides us with a great number of useful and easy examples. To name a few:

- **SGrp** - The category of semigroups.
- **Mon** - The category of monoids.
- **Ab** - The category of abelian groups.
- **Rng** - The category of all rings.
- **CRng** - The category of commutative rings.
- $R - \mathbf{Mod}$ - The category of left R -modules, for some ring R .
- **Vect**(\mathbb{K}) - The category of vector spaces over \mathbb{K} and \mathbb{K} -linear transformations for some field \mathbb{K}
- **Alg**(\mathbb{K}) The category of associative \mathbb{K} -algebras.

We will often write, for example $G \in \mathbf{Ab}$ even though we actually mean $G \in \text{Obj}(\mathbf{Ab})$, i.e. that G is an object in the category of abelian groups. However this should more often than not be completely clear from the context. This is sloppy but acceptable notation in the same way we often write "let G be a group" but we really mean "*let*(G, \cdot, e)*be a group*".

2.4 Monoids as a one-object category

Let (M, \cdot, e) be a monoid, that is M is a nonempty set and $\cdot: M \times M \rightarrow M$ is associative with the double sided identity e . It turns out that every monoid can be viewed as a category in the following way.

Given a monoid M we can form the category $M_{\#}$ with

- Objects: One formal object $\#$.
- Morphisms: The elements of M .

To verify that this indeed is a category we need to define composition and the identity morphisms.

We define composition in $M_\#$ to be the binary relation $\cdot : M \times M \rightarrow M$, so for every element $a \in M$ we have an arrow $a : \# \rightarrow \#$. So for $a, b, c \in M$ we have arrows $a \cdot b, b \cdot c$ are arrows in $M_\#$ such that;

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

since our binary operator is associative.

$M_\#$ has exactly one object, $\#$ so we need an identity arrow $\mathbb{1}_\# : \# \rightarrow \#$ such that composed with any other arrow we get that arrow back. We really only have one, very obvious, choice; $\mathbb{1}_\# := e$.

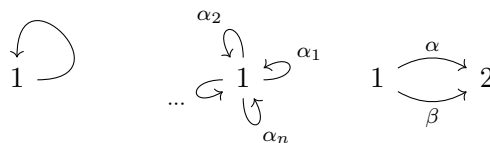
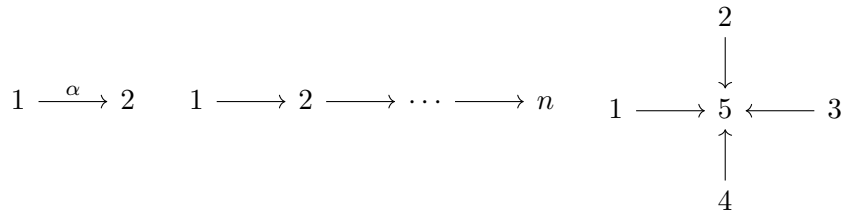
So, given a monoid M we can view it as a category $M_\#$ with elements as morphisms on one formal object. This raises an interesting perspective on the whole notion of categories. One way to think about categories is to view them as generalized monoids, in fact the category axioms are almost identical to the axioms constituting monoids, with the exception that we have multiple objects in a category which must play nicely with each other.

Note that a group G is a special monoid where every element is invertible. If we construct the one-object category G_X as above we get a category in which every morphism is invertible, in general such a category is called a *groupoid*, not to be confused with *groupoid* as in *magma*, the most primitive algebraic structure in which we only require closure of the binary operator.

2.5 Quivers and their path categories

A quiver $Q = (Q_0, Q_1)$ is defined to a directed graph consisting of a set of vertices Q_0 and a set of edges Q_1 between vertices in Q_0 . We allow loops, isolated points and multiple edges between vertices. In other words as the name suggests; a quiver is a box of arrows so we shouldn't be too surprised that quivers have a categorical interpretation.

Some examples by their visual representation as graphs:



Recall that, given some fixed field \mathbb{K} , any quiver $Q = (Q_0, Q_1)$ give rise to a (semigroup-)algebra, $Q(\mathbb{K})$, by taking all edges, including the trivial path ϵ_i for every vertex v_i to itself, and forming the path-semigroup of Q under concatenation (the product of any two non-composable is defined to be 0). $Q(\mathbb{K})$ is then the set of all formal sums over the path-semigroup with coefficients in \mathbb{K} . This makes \mathbb{K} an associative \mathbb{K} -algebra.

Similarly we can construct the path category \mathcal{Q}_Q for any quiver Q .

- Objects in \mathcal{Q}_Q are the vertices of Q , Q_0 .
- Morphisms are oriented paths.
- Identity morphisms are the trivial paths $\epsilon_i: v_i \rightarrow v_i$.
- Composition is concatenation of paths.

One subtle point is that the trivial paths are not actually a part of the design Q but something we impose on the quiver when constructing path algebras and path categories.

2.6 Posets

Let (X, \leq) be a poset (partial ordered set). We show that any poset can be understood as a category.

- Objects: Elements $x, y, z \in X$.
- Morphisms: There exists one, and only one, morphism between $x, y \in X$ if $x \leq y$.

To see that this is a category we need identities and well-behaved composition. This falls out quite naturally by the definition of posets.

For every $x \in X$ we have $x \leq x$ so there is a morphism $x \rightarrow x$ which we take to be the identity morphism.

Assume there exists morphisms $x \rightarrow y, y \rightarrow z$ then we can formally compose them in a clear way using transitivity of \leq :

$$(x \rightarrow y) \circ (y \rightarrow z) \implies x \leq y \implies x \leq y \implies x \rightarrow y$$

For associativity, assume $x \rightarrow y, y \rightarrow z, z \rightarrow w$ then we have

$$(x \rightarrow y) \circ [(y \rightarrow z) \circ (z \rightarrow w)] \implies (x \leq y) \circ (y \leq w) \implies x \leq w \implies x \rightarrow w$$

and

$$[(x \rightarrow y) \circ (y \rightarrow z)] \circ (z \rightarrow w) \implies (x \leq z) \circ (z \leq w) \implies x \leq w \implies x \rightarrow w$$

which show that associativity holds since the unique existence of a morphism is enough and hence every poset can be viewed as a category.

2.7 Categories of topological Spaces, regular surfaces and manifolds

Category theory was developed in the context of algebraic topology in the 1940s so it is appropriate to very briefly mention a few categories of the topological variety.

- **Top**

- Objects: Topological spaces.
- Morphisms: Continuous maps.

An example from elementary differential geometry is the category of regular surfaces.

- **reg**

- Objects: Regular surfaces (subsets of \mathbb{R}^3 with together with a parameterization from open set(s) in \mathbb{R}^2).
- Morphisms: Differential maps between regular surfaces.

These differentiable maps turns out to be composable and we always have the identity map. This **reg** is a special case of the more general category of manifolds:

- **Man^P**

- Objects: Manifolds of smoothness class C^P .
- Morphisms: p -times continuously differentiable maps.

2.8 The Category of Partitioned Binary Relations

A somewhat more complicated and exotic example of a category, from [7], is the category of Partitioned Binary Relations (PBRs) on finite sets.

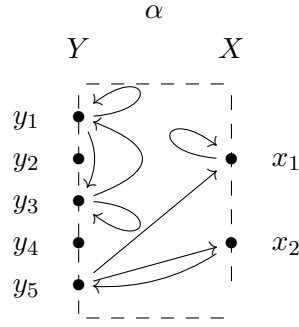
Let X and Y be finite sets. A *partitioned binary relation* (PBR) on (X, Y) is a binary relation α on the disjoint union of X and Y ($X \amalg Y$). X is called the domain of α and Y is called the codomain of α . $(a, b)\alpha$ is called an edge.

A PBR can be visualized as a directed rectangular graph. If α contains an edge $(a, b) \in (X \amalg Y)^2$ we denote this by $(a, b) \in \alpha$ and visualize it by an arrow from a to b on the graph.

So a PBR is basically a directed graph with $|X| + |Y|$ number of vertices (on fixed positions) and arrows between vertices that can go to any $x \in X \amalg Y$.

If we have the situation that $X \cap Y \neq \emptyset$ or $X = Y$, to distinguish between elements of the domain and the codomain, we write $a^{(d)}$ or $a^{(c)}$ for elements of $\text{Dom}(\alpha)$ and $\text{Cod}(\alpha)$.

Below we show an example of a partitioned binary relation α on $X = \{x_1, x_2\}$ to $Y = \{y_1, y_2, y_3, y_4, y_5\}$ in the form of a diagram:



If we have a PBR α on (X, Y) and another PBR β on (Y, Z) we can define a notion of composition of β and α so that $\beta \circ \alpha$ is a PBR on (X, Z) .

It will be convenient to start slightly more generally. Let $A = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k)$ be a sequence of composable partitioned binary relations.

We then define

- $X_i = \text{Dom}(\alpha_i)$ for $i = 1, 2, \dots, k$.
- $X_{k+1} = \text{Cod}(\alpha_k)$.
- $X_{\coprod} = \coprod_{i=1}^{k+1} X_i$.

Then a sequence of edges in a PBR;

$$(a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)$$

is said to be A -connected if

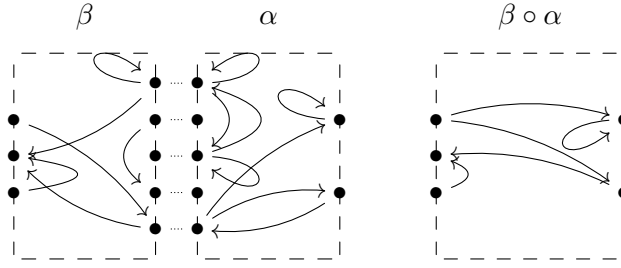
1. No two successive edges in the sequence are in the same PBR.
2. For every $i = 1, 2, \dots, m - 1$ we have $b_i = a_{i+1}$ (as elements of X_{\coprod}).

We can now finally define what it means to compose two PBRs:

Let α be a PBR on (X, Y) and β be a PBR on (Y, Z) . We define $\beta \circ \alpha$ to be a PBR on (X, Z) where for every $a, b \in X \coprod Z$ we have that $(a, b) \in \beta \circ \alpha$ if and only if there exists an (α, β) -connected sequence.

In other words, the composition PBR contains an edge (a, b) if and only if, when we put the graphs of β and α next to each other, we can draw a connect path beginning in a and ending in b , using existing "composable" edges in β and α .

We prove a visual example to illustrate how this composition works:



We claim that composition of partitioned binary relations is associative.

Proof. Let α be a PBR on (X, Y) , β be a PBR on (Y, Z) , and γ be a PBR on (Z, U) . Set $A = (\alpha, \beta, \gamma)$, $\xi = \beta \circ \alpha$ and $\zeta = \gamma \circ \beta$.

We will prove the claim by showing that

$$(a, b) \in \gamma \circ \xi \implies (a, b) \in \zeta \circ \alpha$$

and conversely that

$$(a, b) \in \zeta \circ \alpha \implies (a, b) \in \gamma \circ \xi.$$

Assume $(a, b) \in \gamma \circ \xi$ for some $(a, b) \in (X \amalg Z)^2$. Then there must exist some (ξ, γ) -connected path $S_1 : (a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$ connecting a to b .

From the definition of composition there must exist some some (α, β) -connected sequence connecting a_i to b_i in S_1 .

We can use this to define a new sequence S_2 by replacing every edge $(a_i, b_i) \in \xi$ with some (α, β) -connected sequence connecting a_i to b_i .

We must have that S_2 is A -connected.

Consider now all maximal consecutive subsequences in S_2 consisting exclusively of edges from β and γ .

By maximality it follows that any such subsequences is both preceded and followed by an edge from α , if any.

Since original sequence is A -connected it follows that any maximal subsequence is a (β, γ) -connected sequence connecting its first element to its last element.

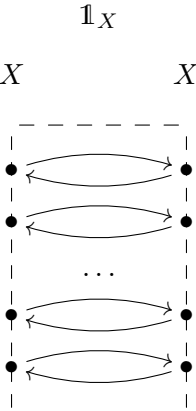
We can now construct one last sequence S_3 by replacing each such maximal (β, γ) -connected subsequence by the pair of elements which this subsequence connects.

This pair of elements gives an edge in ζ by definition. As a result, we obtain an (α, ζ) -connected sequence connecting a to b . Which implies that $(a, b) \in \zeta \circ \alpha$.

The converse follows in a similar manner.

□

Finally, we define identity PBRs. Let X be a finite set. The PBR $\mathbb{1}_X$ is the identity morphism for X with respect to composition, $\mathbb{1}_X \circ \alpha = \alpha$ for any PBR α on (Y, X) , and $\beta \circ \mathbb{1}_X = \beta$ for any PBR β on (X, Y) .



Thus, in conclusion we can define the category of partitioned binary relations in the following way:

- Objects: Finite sets.
- Morphisms: For finite sets X and Y $\text{Hom}(X, Y)$ is the set of all partitioned binary relations on (X, Y) .
- Composition of morphisms and identity morphisms as described above.

Since we have objects, morphisms, associative composition and identity morphisms this is indeed a category.

2.9 The Category of Propositional Proofs

Our next example comes from mathematical logic, [1] and is the category **Prf** of propositional proofs:

- Objects: Propositions.
- Morphisms: Formal conclusions (propositional proofs).

To formally show that this is a category requires some tedious work we will not do here but it basically boils down to the fact that proofs can be chained together, that is if B is a consequence of A and C is a consequence of B then C is a consequence of A . So we have composition. We also have the tautology proof-rule; any proposition is a consequence of itself, which serves as identity morphisms.

We believe that this informally stated example elegantly illustrates how categories pops up in the most unlikely places in mathematics.

2.10 Subcategories

When defining the category of groups **Grp** we briefly also mentioned **Ab**, the category of abelian groups. Of course every abelian group is a group so we have that every object and homset in **Ab** also lies in **Grp**. This kind of situation prompts us to define the notion of subcategories.

Definition 4. A subcategory \mathcal{B} of a category \mathcal{C} is a category such that

- $\text{Obj}(\mathcal{B})$ is a subclass of $\text{Obj}(\mathcal{C})$.
- For all $i, j \in \mathcal{B}$, $\mathcal{B}(i, j) \subset \mathcal{C}(i, j)$.
- For all $i \in \mathcal{B}$, $\mathbf{1}_i \in \mathcal{B}(i, i)$.
- For all $i, j, k \in \mathcal{B}$ and all $\alpha \in \mathcal{B}(i, j), \beta \in \mathcal{B}(j, k)$ we have $\beta \circ \alpha \in \mathcal{B}(i, k)$

We also have a special type of subcategories:

Definition 5. A subcategory \mathcal{B} of \mathcal{C} is called full when $\mathcal{B}(i, j) = \mathcal{C}(i, j)$ for all $i, j \in \mathcal{B}$

2.11 Subcategories of Set

Let us consider **Set** the category of all sets. We can form the category **FSet** with finite sets as objects and functions between finite sets as arrows. The reader can verify that this is a subcategory of **Set** but more interestingly is it a full subcategory of **Set** since for any two finite sets i, j the set of all functions from i to j is the same both in **FSet** and **Set**.

Three other subcategories of **Set** are **Inj**, **Sur** and **Bij**, which has the same objects as in **Set** but with respectively only injective, surjective and bijective functions between sets. This holds since the identity function is injective and surjective and composition of two injective functions is again injective and likewise for surjections. None of these subcategories are however full since

$$\mathbf{Inj}(i, j) = \mathbf{Set}(i, j) = \mathbf{Sur}(i, j)$$

does not hold in general.

A semi-interesting fact is that the subcategories **FSur**, **FInj** of **FSet** coincide.

2.12 Finite dimensional vector spaces

$\mathbf{Vect}_{\mathbb{K}}$ is the category of vector spaces over the field \mathbb{K} together with \mathbb{K} -linear transformations.

We can then form a full subcategory $\mathbf{vect}_{\mathbb{K}}$ consisting of all finite dimensional \mathbb{K} -vector spaces. To see that this is a full subcategory we take $V, W \in \mathbf{vect}_{\mathbb{K}}$ and note that there can't exist a $T \in \mathbf{Vect}_{\mathbb{K}}$ such that $T \notin \mathbf{vect}_{\mathbb{K}}$ so $\mathbf{vect}_{\mathbb{K}}$ must be full.

2.13 Isomorphic objects

Given some category \mathcal{C} we have the notion of isomorphisms between objects in \mathcal{C} , i.e. we have a categorical way of describing isomorphic objects.

Definition 6. *Two objects i, j in a category \mathcal{C} are said to be isomorphic, denoted $i \cong j$ if there exists some arrow $f \in \text{Hom}(i, j)$ and some arrow $g \in \text{Hom}(j, i)$ such that $g \circ f = \mathbf{1}_i$ and $f \circ g = \mathbf{1}_j$.*

The categorical isomorphism concept clearly corresponds to isomorphism in usual sense. For two easy examples consider isomorphisms between sets (bijections) and isomorphisms between groups (bijective group homomorphisms).

2.14 Functors

After defining what a category is, it is natural to want to compare categories to each other. The algebraic way to compare and understand mathematical objects is to define mappings which preserves algebraic structure i.e. if we have two groups, we can study them by constructing group homomorphisms between them. It turns out we can do the same thing with categories. A morphism between categories is called a functor and preserves categorical structure.

Definition 7. *For categories \mathcal{C} and \mathcal{D} , a functor $T: \mathcal{C} \rightarrow \mathcal{D}$ with domain \mathcal{C} and codomain \mathcal{D} consists of two functions,*

1. *The object function T which for every object i in \mathcal{C} assigns an object Ti in \mathcal{D} .*
2. *The arrow function T that assigns to every arrow $f: i \rightarrow j$ in \mathcal{C} an arrow $Tf: Ti \rightarrow Tj$ in \mathcal{D} such that*

$$\begin{aligned} T(\mathbf{1}_i) &= \mathbf{1}_{Ti}, \\ T(g \circ f) &= Tg \circ Tf, \end{aligned}$$

whenever $g \circ f$ is defined in \mathcal{C} .

Note that for categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and functors

$$T: \mathcal{C} \rightarrow \mathcal{D}, S: \mathcal{D} \rightarrow \mathcal{E}$$

then $S \circ T: \mathcal{C} \rightarrow \mathcal{E}$ is a functor. If we have two functors whose codomain and domain agree in the usual way, we can form a composition functor from them.

For every category \mathcal{C} we have the identity functor $id_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ which sends everything to itself, i.e. $id_{\mathcal{C}}(i) = i$ for all objects i and for a morphism $f: i \rightarrow j$, $id_{\mathcal{C}}(f) = f$.

Let \mathcal{B} be a subcategory of a category \mathcal{C} then we have the inclusion functor $I: \mathcal{B} \rightarrow \mathcal{C}$ which sends everything in \mathcal{B} to itself but seen from \mathcal{C} .

Consider the category of small groups **Grp** and group homomorphisms. We can define a forgetful functor to the category of sets

$$F: \mathbf{Grp} \rightarrow \mathbf{Set}$$

Such that for any group (G, \cdot, e) in \mathbf{Grp}

$$F[(G, \cdot, e)] := G \in \text{obj}(\mathbf{Set})$$

In other words we take a group and send it to its underlying set and completely forgets and disregards the binary operator in the group and thus also reducing any group homomorphism $\phi: (G, \cdot, e) \rightarrow (H, \cdot', e')$ to $\phi: G \rightarrow H$, a function between sets.

Similarly we can define a forgetful functor from the category of small rings \mathbf{Rng} to the category of abelian groups \mathbf{Ab} , where we send every ring $(R, +, \cdot, e)$ to its underlying abelian group $(R, +, e)$.

Definition 8. A functor $T: \mathcal{C} \rightarrow \mathcal{D}$ is said to be full if for every pair i, j of objects in \mathcal{C} and to every arrow $g: Ti \rightarrow Tj$ in \mathcal{D} there exists an arrow $f: i \rightarrow j$ of \mathcal{C} such that $Tf = g$

We can also have so called faithful functors:

Definition 9. A functor $T: \mathcal{C} \rightarrow \mathcal{D}$ is faithful (or an embedding) if when to every pair of objects i, j in \mathcal{C} with arrows $f, g: i \rightarrow j$ in \mathcal{C} $Tf = Tg \implies f = g$

Combining the two above definitions:

Definition 10. A functor is called fully faithful if it is both full and faithful.

Let \mathcal{B} be a subcategory of \mathcal{C} , then we have the canonical inclusion functor

$$i: \mathcal{B} \rightarrow \mathcal{C}$$

For every subcategory this inclusion functor is always faithful and if \mathcal{B} is a full subcategory then i is a full functor and hence fully faithful.

2.15 The Category of Categories

A curious consequence of the defining categories is they they pop up everywhere and we can use very elementary machinery to create new categories ad infinitum.

At this stage we basically have categories and functors between categories. It turns out that if we put these together we can form a new interesting category \mathbf{Cat} which has the following data:

- Objects: (Small) Categories.
- Morphisms: Functors.

It almost trivial to see that this is category. Every category has a identity functor, composable functors can be composed to get new functors and if we compose the identity functor with any other functor we get back that other functor.

We shall see more of this special and interesting category in a later part of this thesis.

2.16 Isomorphism between categories

All algebraic structures has the concept of isomorphism and categories are no exception.

Definition 11. *Two categories \mathcal{C} and \mathcal{D} are said to be isomorphic if there exists an functor $T: \mathcal{C} \rightarrow \mathcal{D}$ which is a bijection that respects both objects and morphisms. Or equivalently if there exists another functor $S: \mathcal{D} \rightarrow \mathcal{C}$ such that $S \circ T = Id_{\mathcal{C}}$, the identity functor of \mathcal{C} and $T \circ S = Id_{\mathcal{D}}$, the identity functor of \mathcal{D} .*

It is however quite rare to find isomorphic categories in wild. Also, if we have a fully faithful functor between two categories this does not imply an isomorphisms between categories since this functor might not be surjective, in the sense of isomorphism between categories. However one might argue that a fully faithful embedding is a weaker form of "sameness". In the next subsection we will study yet another form of sameness of categories.

2.17 Nontrivial example of isomorphic categories

Here we present one of those rare instances where two categories turn out to be isomorphic.

- We define \mathcal{C} to be the category with
 - Objects: $\underline{n} = \{1, \dots, n\}, n \in \mathbb{N} = \{1, 2, 3, \dots\}$.
 - Morphisms: $\text{Hom}(\underline{n}, \underline{k})$ is binary relations : $\underline{n} \rightarrow \underline{k}$. $\text{Hom}(\underline{n}, \underline{k}) \ni P \subset \underline{k} \times \underline{n}$.
 - Identity morphisms: $\mathbb{1}_{\underline{n}} = iRj \Leftrightarrow i = j$
- \mathcal{D} is the category with
 - Objects: $n \in \mathbb{N}$.
 - $\text{Hom}(n, k) = \text{Mat}_{k \times n}(\mathbb{B})$. Where $\mathbb{B} = \{0, 1\}$ is the boolean semiring with:

+	0	1	·	0	1
0	0	1	0	0	0
1	1	1	1	0	1
 - Identity morphisms: Identity $n \times n$ matrices, denoted E_n .
 - Composition: Matrix multiplication.

We claim that \mathcal{C} and \mathcal{D} are isomorphic categories.

Proof. We will show that $\mathcal{C} \cong \mathcal{D}$ by defining functors

$$F: \mathcal{C} \rightarrow \mathcal{D}, \quad G: \mathcal{D} \rightarrow \mathcal{C}$$

such that

$$G \circ F = Id_{\mathcal{C}} \text{ and } F \circ G = Id_{\mathcal{D}}$$

- First we define a functor from \mathcal{C} to \mathcal{D} and show that it is indeed functorial.

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$$\underline{n} \mapsto n$$

$$\text{Hom}(\underline{n}, \underline{k}) \ni P \mapsto M_p = (m_{i,j}) \in \text{Mat}_{k \times n}(\mathbb{B}),$$

$$m_{i,j} = \begin{cases} 1 & , (j, i) \in P \\ 0 & , (j, i) \notin P \end{cases}$$

To show that F is functorial is to show that it maps identity morphisms to identity morphisms and that it respect composition.

- For identity morphisms, $\mathbb{1}_{\underline{n}}$ we have:

$$F(\mathbb{1}_{\underline{n}}) = M_{\mathbb{1}_{\underline{n}}} = (m_{i,j})$$

Where $m_{i,j} = 1 \implies (j, i) \in \mathbb{1}_{\underline{n}} \implies j = i$ and $m_{i,j} = 0 \implies \notin \mathbb{1}_{\underline{n}} \implies j \neq i$. So $(m_{i,j})$ has nonzero entries on the diagonal which implies that:

$$F(\mathbb{1}_{\underline{n}}) = E_n = E_{F(\underline{n})}$$

- We also need to show that for $P_1 \in \text{Hom}(\underline{n}, \underline{k}), P_2 \in \text{Hom}(\underline{k}, \underline{m})$ the following equality holds:

$$F(P_2 \circ P_1) = F(P_2) \circ F(P_1)$$

So let $P_1 \in \text{Hom}(\underline{n}, \underline{k}), P_2 \in \text{Hom}(\underline{k}, \underline{m})$ so that $P_2 \circ P_1 \in \text{Hom}(\underline{n}, \underline{m})$

Then $F(P_2 \circ P_1) = M_{P_2 \circ P_1} = (m_{i,j}) \in \text{Mat}_{m \times n}(\mathbb{B})$ and

$F(P_2) = (a_{i,j}) = A \in \text{Mat}_{m \times k}, F(P_1) = (b_{i,j}) = B \in \text{Mat}_{k \times n}$ and let

$$F(P_2) \circ F(P_1) = AB = C = (c_{i,j})$$

We will prove that $C = F(P_2 \circ P_1)$ by showing that all entries in the two matrices are equal.

Every entry in C is obtained by matrix multiplication so we have the following expression:

$$c_{i,j} = \sum_{t=1}^k a_{i,t} b_{t,j}$$

Suppose that $m_{i,j} = 0$ in $F(P_2 \circ P_1)$ for some fixed i, j and assume towards contradiction that $c_{i,j} \neq 0$ Then we have that

$$c_{i,j} = \sum_{t=1}^k a_{i,t} b_{t,j} = a_{i,1} b_{1,j} + \dots + a_{i,k} b_{k,j} = 1$$

Since this sum is nonzero and since we have no additive inverses in \mathbb{B} there must exist some x such that $a_{i,x}b_{x,j} \neq 0$ which implies that

$$a_{i,x}b_{x,j} = 1$$

By the multiplication in $\mathbb{B} \implies a_{i,x} = b_{x,j} = 1$ So have that $(j, x) \in P_1$ and $(x, i) \in P_2 \implies (j, i) \in P_2 \circ P_1$. This is contradiction since this would imply that the corresponding entry $m_{i,j}$ in $F(P_2 \circ P_1)$ is 1 but $m_{i,j} = 0$ by our assumption. So we have no $a_{i,x}b_{x,j} \neq 0 \implies c_{i,j} = 0$

Now suppose that $m_{i,j} = 1$ in $F(P_2 \circ P_1)$ for some fixed i, j and assume towards yet another contradiction that for the same i, j ,

$$c_{i,j} = 0$$

which implies that $c_{i,j} = \sum_{t=1}^k a_{i,t}b_{t,j} = a_{i,1}b_{1,j} + \dots + a_{i,k}b_{k,j} = 0$. Again, the lack of additive inverses forces us to conclude that

$$a_{i,t}b_{t,j} = 0 \forall 1 \leq t \leq k$$

By looking at the multiplication table for \mathbb{B} we then have three cases

1. $a_{i,t} = b_{t,j} = 0 \implies (t, i) \notin P_2, (j, t) \notin P_1$,
2. $a_{i,t} = 1, b_{t,j} = 0 \implies (t, i) \notin P_2, (j, t) \in P_1$,
3. $a_{i,t} = 0, b_{t,j} = 1 \implies (t, i) \in P_2, (j, t) \notin P_1$.

Since every $t \leq k$ corresponds to one of these cases there is no t such that $(j, i) \in P_2 \circ P_1$ but this contradicts our assumption that $m_{i,j} = 1$ so we must have that $c_{i,j} = 1$. So the to matrices agrees in all entries hence as desired we have that

$$F(P_2 \circ P_1) = F(P_2) \circ F(P_2).$$

So F is functorial.

- We will now define our second functor

$$G: \mathcal{D} \rightarrow \mathcal{C}$$

$$n \mapsto \underline{n}$$

$$\text{Mat}_{m \times n}(\mathbb{B}) \ni M = (m_{i,j}) \mapsto \{(j, i) \in \underline{m} \times \underline{n}: m_{i,j} \neq 0\}$$

We need to show that G is functorial:

- Let $E = (e_{i,j}) = \mathbf{1}_n$ in \mathcal{D} . Then $G(\mathbf{1}_n) = G(E) = \{(j, i) \in \underline{n} \times \underline{n}: e_{i,j} \neq 0\} = \{(j, i) \in \underline{n} \times \underline{n}: i = j\} = \mathbf{1}_{\underline{n}} = \mathbf{1}_{G(n)}$

– Let $A \in \text{Mat}_{m \times k}(\mathbb{B})$ and $B \in \text{Mat}_{k \times n}(\mathbb{B})$. We need to show that

$$G(AB) = G(A) \circ G(B)$$

By definition $G(AB) = \{(j, i) \in \underline{m} \times \underline{n} : m_{i,j} \neq 0\}$. Let $(j, i) \in G(AB)$ then

$$\sum_{t=1}^k a_{i,t} b_{t,j} \neq 0 \implies \exists t \in \underline{k} \text{ such that } a_{i,t} b_{t,j} \neq 0 \implies a_{i,t} = b_{t,j} = 1.$$

Thus we have that $(t, i) \in G(A)$ and $(j, t) \in G(B)$ so $(j, i) \in G(A) \circ G(B)$ hence

$$G(AB) \subseteq G(A) \circ G(B)$$

Conversely, suppose that $(j, i) \in G(A) \circ G(B)$. Then there exists some $t \in \underline{k}$ such that $(j, t) \in G(B)$ and $(t, i) \in G(A) \implies a_{i,t} = b_{t,j} = 1$.

$$\text{If } AB = C = (c_{i,j}) \implies c_{i,j} \neq 0 \text{ since } a_{i,t} = b_{t,j} = 1 \implies a_{i,t} b_{t,j} = 1.$$

$$\text{If } c_{i,j} \neq 0 \implies (j, i) \in G(AB) \implies G(A) \circ G(B) \subseteq G(AB)$$

So we can conclude that

$$G(AB) = G(A) \circ G(B)$$

and hence G is functorial.

- Finally, we will show that F and G composed with each other gives identity functors and hence $\mathcal{C} \cong \mathcal{D}$.

– $F \circ G = Id_{\mathcal{D}}$ For objects;

$$F \circ G(n) = F(\underline{(n)}) = n$$

Let $M: n \rightarrow k$ be a morphism in \mathcal{D} , $M = (m_{i,j}) = M \in \text{Mat}_{k \times n}(\mathbb{B})$ then,

$$F \circ G(M) = F[\{(j, i) \in \underline{k} \times \underline{n} : m_{i,j} \neq 0\}] = F(P) = M_P = (m'_{i,j})$$

where

$$m'_{i,j} = \begin{cases} 1 & , (j, i) \in P \\ 0 & , (j, i) \notin P \end{cases}$$

So we have that $m'_{i,j} = m_{i,j}$ and thus $M_P = M$ and $F \circ G = Id_{\mathcal{D}}$.

– $G \circ F = Id_{\mathcal{C}}$

For objects it is clear; $G \circ F(\underline{n}) = G(n) = \underline{n}$.

Let $P \in \text{Hom}_{\mathcal{C}}(\underline{n}, \underline{k})$. Then $G \circ F(P) = G(M_P) = \{\{(j, i) \in \underline{k} \times \underline{n} : m_{i,j} \neq 0\}\} = \{\{(j, i) \in \underline{k} \times \underline{n} : (j, i) \in P\}\} = P$.

In other words, we have that

$$G \circ F = Id_{\mathcal{C}}.$$

Since we have that both $F \circ G = Id_{\mathcal{D}}$ and $G \circ F = Id_{\mathcal{C}}$ holds so \mathcal{C} and \mathcal{D} are isomorphic categories. □

2.18 The Yoneda Functor

One last but nevertheless hugely important example of a functor is the Yoneda functor. Let \mathcal{C} be a small category and i, j be objects in \mathcal{C} . Since \mathcal{C} is small, the set of all morphisms between i and j , $\text{Hom}(i, j)$ is a set. So we can define the functor

$$\begin{aligned} \text{Hom}(i, _): \mathcal{C} &\rightarrow \mathbf{Set} \\ j &\mapsto \text{Hom}(i, j) \end{aligned}$$

Let $f \in \text{Hom}(c, c')$ then $\text{Hom}(i, _)[f] = \text{Hom}(i, f)$ which works in the following way.

$$\begin{aligned} \text{Hom}(i, f): \text{Hom}(i, c) &\rightarrow \text{Hom}(i, c') \\ _ &\mapsto f \circ _ \end{aligned}$$

It is easy to check that the Yoneda functor is indeed functorial. We will see more of this later on.

2.19 Natural Transformations

It is often said that categories were defined in order to define functors and that functors were defined in order to define natural transformations.

Natural transformations are in essence morphisms between functors and shall play an important roll in later sections. We also provide some examples to illustrate this concept and why it is useful.

Definition 12. *Given two functors $S, T: \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\tau: S \rightarrow T$ is a function which assigns to each object i of \mathcal{C} an arrow $\tau_i = \tau i: Si \rightarrow Ti$ in \mathcal{D} such that for every morphism $f: i \rightarrow j$ in \mathcal{C} the following diagram commutes*

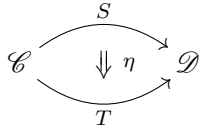
$$\begin{array}{ccc} i & Si & \xrightarrow{\tau_i} & Ti \\ \downarrow f & Sf \downarrow & & \downarrow Tf \\ j & Sj & \xrightarrow{\tau_j} & Tj \end{array}$$

In other words $Tf \circ \tau_i = \tau_j \circ Sf$.

Functors $S, T: \mathcal{C} \rightarrow \mathcal{D}$ are morphisms of categories:

$$\mathcal{C} \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \end{array} \mathcal{D}$$

Since natural transformations are morphisms of functors we sometimes write:



For a natural transformation $\eta: S \rightarrow T$. We will see more of this in some detail later on.

As we have seen, two objects in a category can be isomorphic but we can have an even stronger type sameness between objects by making use of functors and natural transformations.

Let $S, T: \mathcal{C} \rightarrow \mathcal{D}$ be two functors.

Definition 13. A natural transformation $\tau: S \rightarrow T$ is called a natural equivalence or natural isomorphism denoted $\tau: S \cong T$, if every τ_i is invertible in \mathcal{D} . I.e every $(\tau_i)^{-1}$ are the components of a natural isomorphism $\tau^{-1}: T \rightarrow S$.

2.20 Examples of natural transformations

- Fix a nonempty set S and consider the Yoneda functor

$$\text{Hom}(S, _): \mathbf{Set} \rightarrow \mathbf{Set}$$

Then we can take the evaluation function

$$\begin{aligned}
\tau_X: \text{Hom}(S, X) \times S &\rightarrow X \\
(f, s) &\mapsto f(s)
\end{aligned}$$

and by fixing some arbitrary element $s \in S$ we can define the function

$$\begin{aligned}
\tau^s: \text{Hom}(S, X) &\rightarrow X \\
f &\mapsto f(s)
\end{aligned}$$

We claim that τ^s is a natural transformation between $\text{Hom}(S, _)$ and the identity functor Id of \mathbf{Set}

Proof. To show that τ^s is natural transformation between the two functors in question, we must show that the following diagram commutes for every $f \in \text{Hom}(X, Y)$.

$$\begin{array}{ccc}
X & \text{Hom}(S, X) \xrightarrow{\tau_X^s} & X \\
\downarrow f & \begin{array}{c} f \circ _ \\ \downarrow \end{array} & \downarrow f \\
Y & \text{Hom}(S, Y) \xrightarrow{\tau_Y^s} & Y
\end{array}$$

To see that this is the case we take any $g \in \text{Hom}(S, X)$ and chase the diagram.

- $\tau_Y^s \circ (f \circ _)[g] = \tau_Y^s(f \circ g) = f \circ g(s)$
- $f \circ \tau_X^s(g) = f \circ g(s)$

The diagram commutes and hence $\tau_{_}^s$ is natural transformation between $\text{Hom}(S, _)$ and Id of **Set**.

□

- Let B and C be groups in the categorical sense as we saw earlier. Note that any functor between B and C is a group homomorphism.

We show here that there exists a natural transformation $\tau: S \rightarrow T$ of functors $S, T: B \rightarrow C$ if and only if S, T are conjugate. Recall that two group homomorphisms are conjugate if for all $g \in B$ there exists an $h \in C$ such that $Tg = h(Sg)h^{-1}$.

Proof. Assume there exists a natural transformation $\tau: S \rightarrow T$ then for all morphisms (group elements) $g: B \rightarrow B$ the following diagram commutes:

$$\begin{array}{ccc} B & SB & \xrightarrow{\tau_B} TB \\ \downarrow g & \downarrow Sg & \downarrow Tg \\ B & SB & \xrightarrow{\tau_B} TB \end{array}$$

This is equivalent to saying that the following equation holds

$$Tg \circ \tau_B = \tau_B \circ Sg$$

But composition on C is exactly group multiplication and all terms are group elements which implies that there exists an element τ_B^{-1} such that

$$Tg = \tau_B Sg \tau_B^{-1}$$

Which shows that if there exists a natural transformation between S and T then they are conjugate.

Conversely, assume that S and T are conjugate. Then for all $g \in B$ there exists an $h \in C$ such that $Tg = h(Sg)h^{-1}$. We want to show that there exists a natural transformation $\tau: S \rightarrow T$. We want

$$\begin{array}{ccc} B & SB & \xrightarrow{\tau_B} TB \\ \downarrow g & \downarrow Sg & \downarrow Tg \\ B & SB & \xrightarrow{\tau_B} TB \end{array}$$

to commute, so we have to explicitly define τ such that

$$Tg \circ \tau_B = \tau_B \circ Sg$$

We will use the fact that $Tg = h(Sg)h^{-1}$ which equivalent to $(Tg)h = h(Sg)$ which is very similar to $Tg \circ \tau_B = \tau_B \circ Sg$.

Define $\tau: S \rightarrow T$ as $\tau = h \in C$ such that $Tg = h(Sg)h^{-1}$ then the diagram commutes and $\tau: S \rightarrow T$ is a natural transformation, as desired.

□

- Fix a group H and define the functor

$$\begin{aligned} H \times _ : \mathbf{Grp} &\rightarrow \mathbf{Grp} \\ G &\mapsto H \times G \end{aligned}$$

Which has the morphisms functor

$$\mathrm{Hom}(G_1, G_2) \ni \phi \mapsto (\mathbf{1}_H, \phi) \in \mathrm{Hom}(H \times G_1, H \times G_2)$$

Then we have a natural transformation between any two such functors

$$\tau: H \times _ \rightarrow K \times _$$

by fixing some morphism $f: H \rightarrow K$ then we can define

$$\tau_G = (f, \mathbf{1}_G)$$

This is a natural transformation.

Proof. For every $\phi \in \mathrm{Hom}(G_1, G_2)$

$$\begin{array}{ccccc} G_1 & & H \times G_1 & \xrightarrow{\tau_{G_1}} & K \times G_1 \\ \downarrow \phi & & (\mathbf{1}_H, \phi) \downarrow & & \downarrow (\mathbf{1}_K, \phi) \\ G_2 & & H \times G_2 & \xrightarrow{\tau_{G_2}} & K \times G_2 \end{array}$$

Since

$$\tau_{G_2} \circ (\mathbf{1}_H, \phi) = (f, \mathbf{1}_{G_2}) \circ (\mathbf{1}_H, \phi) = (f \circ \mathbf{1}_H, \mathbf{1}_{G_2} \circ \phi) = (f, \phi) \text{ and}$$

$$(\mathbf{1}_K, \phi) \circ \tau_{G_1} = (\mathbf{1}_K, \phi) \circ (f, \mathbf{1}_{G_1}) = (\mathbf{1}_K \circ f, \phi \circ \mathbf{1}_{G_1}) = (f, \phi)$$

are equal, which follows by the category axioms, in particular the unit law. So τ is a natural transformation between $H \times _ \rightarrow K \times _$

□

2.21 $V \cong V^{**}$ is a natural isomorphism

We have the classic result in linear algebra that for any $V \in \mathbf{Vect}_K$ we have an injective map to the double dual space V^{**} of V and in particular if $V \in \mathbf{vect}_K$ (V is finite dimensional) then this map is also surjective so we have a isomorphism between V and V^{**} .

This result is often accompanied by a mumbling hand-wavy remark that this isomorphism is canonical in contrast to the isomorphism between V and V^* which requires an explicit choice of basis.

This is certainly not false but it is not the whole truth. We can make use of the machinery of category theory, in particular natural transformations between functors, to make this statement precise.

First we need some preliminaries

Let $V \in \text{obj}(\mathbf{Vect}_K)$ and recall that the *dual space* V^* of V is the set of all functionals

$$\Lambda: V \rightarrow K$$

We also denote the dual space $V^* = \mathcal{L}(V, K)$ We claim without proof that the dual space V^* of V is a K -vector space that it is (non-canonically) isomorphic to V .

Theorem 1. *The dual of the dual space V^{**} of a finite dimensional vector space V is (canonically) isomorphic to V .*

Proof. We will show that

$$\begin{aligned} \Psi: V &\rightarrow V^{**} \\ v &\mapsto \langle _, v \rangle \end{aligned}$$

is an isomorphism.

- First note that

$$\begin{aligned} \langle _, v \rangle: V^* &\rightarrow K \\ \Lambda &\mapsto \Lambda(v) \end{aligned}$$

is linear. Fix some arbitrary element $v \in V$ then

$$\begin{aligned} \langle \Lambda_1 + \Lambda_2, v \rangle &= [\Lambda_1 + \Lambda_2](v) = \Lambda_1(v) + \Lambda_2(v) = \langle \Lambda_1, v \rangle + \langle \Lambda_2, v \rangle, \forall \Lambda_1, \Lambda_2 \in V^* \\ k \langle \Lambda, v \rangle &= k(\Lambda(v)) = [k\Lambda](v) = \langle k\Lambda, v \rangle \forall k \in K \end{aligned}$$

So $\langle _, v \rangle$ is indeed linear.

- Ψ is injective. Let $v \in \text{Ker}(\Psi) \implies \Psi(v) = \langle _, v \rangle$ is the zero transformation for all $\Lambda \in V^* \implies \Lambda(v) = 0 \forall \Lambda \in V^*$ and we can only have one element in V which evaluates to zero on all functionals, namely $v = 0$. So Ψ has a trivial kernel and must be injective.

- Ψ is surjective, i.e. $\text{Im}(\Psi) = V^{**}$, let us assume that $\text{Dim}(V) = n < \infty$ then by the dimension formula

$\text{Dim}(V) = n = \text{Dim}(\text{Ker}(\Psi)) + \text{Dim}(\text{Im}(\Psi)) = 0 + \text{Dim}(\text{Im}(\Psi))$ since $\text{Ker}(\Psi) = \{0\}$ which implies that $\text{Dim}(\text{Im}(\Psi)) = n$ and we already know that a finite dimensional vector space and its corresponding dual has the same dimension so

$$n = \text{Dim}(V) = \text{Dim}(V^*) = \text{Dim}((V^*)^*) = \text{Dim}(V^{**}) = \text{Dim}(\text{Im}(\Psi)) = n$$

So Ψ is surjective.

Thus $V \cong V^{**}$ as desired. □

Instead of mumbling about how this isomorphism does not depend on any choice in V we will instead show that this is natural isomorphism in \mathbf{vect}_K in the categorical sense and thus making this result precise.

Theorem 2. *Let*

$$T: \mathbf{vect}_K \rightarrow \mathbf{vect}_K$$

be the double dual functor which sends

$$T: V \mapsto V^{**}$$

and for morphisms, i.e. linear maps, $f \in \mathcal{L}(V, W)$

$$T: f \mapsto f^{**}: V^{**} \rightarrow W^{**}$$

We claim that there exists a natural transformation $\tau: id \rightarrow T$, where id denotes the identity functor of \mathbf{Vect}_K

Proof. Technically we have one contravariant functor

$$S: \mathbf{vect}_K \rightarrow \mathbf{vect}_K$$

sending every vector space to its dual and every linear transformation to its dual map, such that the covariant double dual functor T can be described as $T = S \circ S$.

We will use the contravariant functor S when describing how T acts on morphisms.

Let $V, W \in \mathbf{vect}_K$ and $f \in \text{Hom}(V, W)$. Then $S(f) = f^*: W^* \rightarrow V^*$, such that for any $\lambda \in W^*$

$$f^*: \lambda \mapsto \lambda(f)$$

where $\lambda(f): V \rightarrow K, v \mapsto \lambda(f(v)) \in K$. Let us now consider the double dual functor $T: \mathbf{Vect}_K \rightarrow \mathbf{Vect}_K$. Since S is contravariant and $T = S \circ S$ we must have that T is covariant. For vector spaces, $T(V) = V^{**}$. How about morphisms?

$$T(f) = f^{**}: V^{**} \rightarrow W^{**}$$

For $\Lambda \in V^{**}$ we define

$$f^{**}(\Lambda) = \Lambda(f^*).$$

To see that this definition works, let $\xi \in W^*$ then $\Lambda(f^*)[\xi] = \Lambda(f^*(\xi)) \in K$ since $f^*(\xi) \in V^*$ and Λ maps functionals in V^* to elements in K .

We will now show that T is functorial.

- For $\mathbb{1}_V \in \text{Hom}(V, V)$:

$$\text{We have that } S(\mathbb{1}_V) = (\mathbb{1}_V)^*: \lambda \mapsto \lambda(\mathbb{1}_V) = \lambda \implies (\mathbb{1}_V)^* = \mathbb{1}_{V^*}$$

For our double dual functor we have that:

$$T(\mathbb{1}_V) = (\mathbb{1}_V)^{**}: \Lambda \mapsto \Lambda(\mathbb{1}_{V^*}) = \Lambda$$

Which implies that:

$$T(\mathbb{1}_V) = \mathbb{1}_{V^{**}}$$

- Suppose $f \in \text{Hom}(V, W)$ and $g \in \text{Hom}(W, U)$ then

$$T(g \circ f) = (g \circ f)^{**}: \Lambda \mapsto \Lambda((g \circ f)^*) = \Lambda(g^* \circ f^*) \text{ by assumption that the dual functor } S \text{ is factorial.}$$

On the other hand:

$$T(g) \circ T(f) = g^{**} \circ f^{**} = [g^{**}: \Xi \mapsto \Xi(g^*)] \circ [f^{**}: \Lambda \mapsto \Lambda(f^*)] = g^{**} \circ f^{**}: \Lambda \mapsto \Lambda(g^*(f^*)) = \Lambda(g^* \circ f^*) \text{ So we have that:}$$

$$T(g \circ f) = T(g) \circ T(f).$$

Which shows that T is functorial.

To show that τ is a natural transformation between id and T is an easy exercise:

$$\tau_V: V \rightarrow V^{**}$$

$$v \mapsto \langle _, v \rangle$$

If $\tau: \text{id} \rightarrow T$ is a natural transformation then for any $f \in \text{Hom}(V, W)$, the following diagram must commute:

$$\begin{array}{ccccc} V & \xrightarrow{\text{id}(V)} & T(V) & & V & \xrightarrow{\tau_V} & V^{**} \\ \downarrow f & \text{id}(f) \downarrow & \downarrow T(f) & & \downarrow f & & \downarrow f^{**} \\ W & \xrightarrow{\text{id}(W)} & T(W) & & W & \xrightarrow{\tau_W} & W^{**} \end{array}$$

Which is the case since

$$\begin{aligned}
T(f) \circ \tau_V[v] &= f^{**} \circ \tau_V[v] = f^{**}(\langle _, v \rangle) = \langle f^*(_), v \rangle = \langle f^*(_), _ \rangle [v] = \\
&f^*(\langle _, _ \rangle)[v] = \langle _, f(_) \rangle [v] = \langle _, f(v) \rangle = \tau_W(f(v)) = \tau_W \circ f[v].
\end{aligned}$$

We have shown that $\tau: \text{id} \rightarrow F$ is a natural transformation and thus $V \cong V^{**}$ is a natural transformation in the categorical sense and we are done. \square

2.22 Equivalence of Categories

Definition 14. A natural transformation $\tau: F \rightarrow G$ between functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ is called natural equivalence or natural isomorphism, denoted $\tau: F \cong G$ if with every component τc invertible in \mathcal{D} . The inverses $(\tau c)^{-1}$ in \mathcal{C} are the components of a natural isomorphism $\tau^{-1}: G \rightarrow F$.

We can use natural isomorphisms to define equivalence between categories:

Definition 15. Two categories \mathcal{C} and \mathcal{D} are said to be equivalent if for a pair of functors $F: \mathcal{C} \rightarrow \mathcal{D}, G: \mathcal{D} \rightarrow \mathcal{C}$ there exists natural isomorphisms $\tau: F \circ G \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta: \text{Id}_{\mathcal{C}} \rightarrow G \circ F$.

This concept allows us to compare categories alike but of different sizes and equivalence is often a better and more useful variant of "sameness" of categories, rather than strict isomorphisms.

2.23 Example of equivalence between categories

We show that \mathbf{vect}_K - the category of finite dimensional vector spaces and K -linear transformations and \mathbf{Matr}_K the category of natural numbers and rectangular matrices with entries in K , are equivalent.

Proof. We have the functors:

$$F: \mathbf{vect}_K \rightarrow \mathbf{Matr}_K$$

Which sends $V \in \mathbf{vect}_K$, with basis \underline{v} such that $|\underline{v}| = n$, to $n \in \text{Obj}(\mathbf{Matr}_K)$.

Let W be a finite dimensional K -vector space with basis \underline{w} , with $|\underline{w}| = m$, then for some some linear transformation $T \in \text{Hom}(V, W)$, F maps f to the matrix representation of f with respect to the bases \underline{v} and \underline{w} .

$$F(T) = [T]_{\underline{w}, \underline{v}} \in \text{Mat}_{n \times m}(K)$$

$$G: \mathbf{Matr}_K \rightarrow \mathbf{vect}_K$$

$$n \mapsto K^n$$

For some $A \in \text{Hom}(n, m)$, i.e. $A \in \mathbf{Mat}_{n \times m}(K)$, we can view A as a linear transformation between K^n and K^m so:

$$G(A) = A \in \text{Hom}(K^n, K^m)$$

We will now define our natural transformations:

$$\eta: F \circ G \rightarrow \text{Id}_{\mathbf{Matr}_K}$$

Note that for an object $n \in \mathbf{Matr}_K$

$$F \circ G(n) = F(K^n) = n = \text{Id}_{\mathbf{Matr}_K}(n)$$

and for some $A \in \text{Hom}(n, m)$

$$F \circ G(A) = F(A) = [A] = A.$$

So we end up with this diagram:

$$\begin{array}{ccc} n & & n \\ \downarrow A & & \downarrow A \\ m & & m \end{array} \quad \begin{array}{ccc} n & \xrightarrow{\eta_n} & n \\ \downarrow A & & \downarrow A \\ m & \xrightarrow{\eta_m} & m \end{array}$$

We want this diagram commute so we have no choice but to define the component function of $\eta: F \circ G \rightarrow \text{Id}_{\mathbf{Matr}_K}$ as the identity function.

We also need a natural transformation:

$$\tau: \text{Id}_{\mathbf{vect}_K} \rightarrow G \circ F$$

Let $V, W \in \mathbf{vect}_K$, with bases \underline{v} such that $|\underline{v}| = n$, \underline{w} with $|\underline{w}| = m$, respectively. Then for any $T \in \text{Hom}(V, W)$ we get the following diagram:

$$\begin{array}{ccc} V & & V \\ \downarrow T & & \downarrow T \\ W & & W \end{array} \quad \begin{array}{ccc} V & \xrightarrow{\tau_V} & G \circ F(V) \\ \downarrow T & & \downarrow G \circ F(T) \\ W & \xrightarrow{\tau_W} & G \circ F(W) \end{array}$$

Since $G \circ F(V) = G(n) = K^n$, $G \circ F(W) = G(m) = K^m$ and $G \circ F(T) = G([T]_{\underline{w}, \underline{v}}) = [T]_{\underline{w}, \underline{v}}$ we can rewrite the diagram as:

$$\begin{array}{ccc} V & & K^n \\ \downarrow T & & \downarrow [T]_{\underline{w}, \underline{v}} \\ W & & K^m \end{array} \quad \begin{array}{ccc} V & \xrightarrow{\tau_V} & K^n \\ \downarrow T & & \downarrow [T]_{\underline{w}, \underline{v}} \\ W & \xrightarrow{\tau_W} & K^m \end{array}$$

We can see that it is most appropriate to define the component function of $\tau: \text{Id}_{\mathbf{vect}_K} \rightarrow G \circ F$ as the canonical coordinate function associated to a finite dimensional vector space.

Since both η and τ are natural isomorphisms we have that the categories \mathbf{vect}_K and \mathbf{Matr}_K are equivalent. \square

2.24 Functor categories

This is a very important and perhaps our first nontrivial categorical construction which hints at the bottomless rabbit hole that is (higher) category theory.

Let \mathcal{C} and \mathcal{D} be any two categories. We can then form a new category called a functor category, denoted $\text{Hom}(\mathcal{C}, \mathcal{D})$ with the following data.

- Objects: Functors $F, G, H, \dots : \mathcal{C} \rightarrow \mathcal{D}$
- Morphisms: Natural transformations $\theta, \tau, \eta, \dots$ between functors between \mathcal{C} and \mathcal{D} .

We claim that $\text{Hom}(\mathcal{C}, \mathcal{D})$ is a category.

Proof. We need to back to the initial definition of category and check the following:

- Composition of natural transformations is a natural transformation.

Let $F, G, H \in \text{Hom}(\mathcal{C}, \mathcal{D})$ and let $\tau: F \rightarrow G, \theta: G \rightarrow H$ be natural transformations. Then for any $f \in \text{Hom}_{\mathcal{C}}(i, j)$ the following two diagrams commutes.

$$\begin{array}{ccccc} i & & Fi & \xrightarrow{\tau_i} & Gi & & Gi & \xrightarrow{\theta_i} & Hi \\ \downarrow f & & Ff \downarrow & & \downarrow Gf & & Gf \downarrow & & \downarrow Hf \\ j & & Fj & \xrightarrow{\tau_j} & Gj & & Gj & \xrightarrow{\theta_j} & Hj \end{array}$$

$$Gf \circ \tau_i = \tau_j \circ Ff, \quad Hf \circ \theta_i = \theta_j \circ Gf$$

We want to show that $\theta \circ \tau$ is a natural transformation so for $i \in \mathcal{C}$, we define:

$$(\theta \circ \tau)_i = \theta_i \circ \tau_i$$

We must show that the following diagram commutes.

$$\begin{array}{ccccc} i & & Fi & \xrightarrow{(\theta \circ \tau)_i} & Hi \\ \downarrow f & & Ff \downarrow & & \downarrow Hf \\ j & & Fj & \xrightarrow{(\theta \circ \tau)_j} & Hj \end{array}$$

Thus:

$$Hf \circ (\theta \circ \tau)_i = Hf \circ (\theta_i \circ \tau_i) = (Hf \circ \theta_i) \circ \tau_i = (\theta_j \circ Gf) \circ \tau_i = \theta_j \circ (Gf \circ \tau_i) = \theta_j \circ (\tau_j \circ Ff) = (\theta_j \circ \tau_j) \circ Ff = (\theta \circ \tau)_j \circ Ff$$

So the diagram commutes and we can conclude that that a composition of two natural transformations is a natural transformation.

- Every functor has an identity natural transformation to itself.

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor then we have natural transformation $\iota_F \in \text{Hom}(F, F)$ for any object $i \in \mathcal{C}$ we define $\iota_i = \mathbb{1}_{F(i)}$. To see that ι is a natural transformation, let $f \in \text{Hom}_{\mathcal{C}}(i, j)$ we get the following diagram:

$$\begin{array}{ccc} i & Fi & \xrightarrow{\iota_i} Fi \\ \downarrow f & Ff \downarrow & \downarrow Ff \\ j & Fj & \xrightarrow{\iota_j} Fj \end{array}$$

Which clearly commutes since

$$\begin{aligned} Ff \circ \iota_i &= Ff \circ \mathbb{1}_{Fi} = Ff \\ \iota_j \circ Ff &= \mathbb{1}_{Fj} \circ Ff = Ff \end{aligned}$$

It is also clear that for any natural transformation $\tau: F \rightarrow G$ we have $\tau \circ \iota_F = \tau$ and $\iota_G \circ \tau = \tau$

- Composition of natural transformations is associative.

Let $F, G, H, K \in \text{Hom}(\mathcal{C}, \mathcal{D})$ and suppose that $\tau: F \rightarrow G, \theta: G \rightarrow H, \eta: H \rightarrow K$ are natural transformations. We want to prove that

$$\eta \circ (\theta \circ \tau) = (\eta \circ \theta) \circ \tau.$$

By definition, for all $i \in \mathcal{C}$, η_i, θ_i, τ_i are just morphisms in \mathcal{D} and we have already shown that composition of natural transformations is a natural transformations. Thus, consider the morphism $[\eta \circ (\theta \circ \tau)]_i$

$$[\eta \circ (\theta \circ \tau)]_i = \eta_i \circ (\theta \circ \tau)_i = \eta_i \circ (\theta_i \circ \tau_i).$$

Since these are just morphisms in \mathcal{D} it follows that

$$\eta_i \circ (\theta_i \circ \tau_i) = (\eta_i \circ \theta_i) \circ \tau_i = (\eta \circ \theta)_i \circ \tau_i = [(\eta \circ \theta) \circ \tau]_i$$

Which implies that

$$\eta \circ (\theta \circ \tau) = (\eta \circ \theta) \circ \tau$$

Since all points above holds, we have shown that $\text{Hom}(\mathcal{C}, \mathcal{D})$ is a category. □

3 Adjoints

This section contains a very short basic introduction to the concept of adjunctions; adjoint functors. We will define what it means for two functors to be adjoint, provide some examples and finally examine the beautiful and interesting connection between adjoints and equivalence of categories.

3.1 Motivation and definition

The concept of adjointness, due to Kan, formally provides an alternative description of properties of free objects and other universal constructions. Though informally adjointness is heavily inspired and motivated by adjointness of linear transformations and adjointness as it is defined in the theory of semigroups. We feel that adjoints are far less intimidating if one keep in mind the analogue of adjoint linear maps.

Let \mathcal{C} and \mathcal{D} be (small) categories.

Definition 16. An adjunction from \mathcal{C} to \mathcal{D} is a triple $\langle F, G, \phi \rangle: \mathcal{C} \rightarrow \mathcal{D}$. Where $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$ are functors and ϕ is a function which assigns to every pair of objects $i \in \mathcal{C}, j \in \mathcal{D}$ a bijection of sets

$$\phi = \phi_{i,j}: \mathcal{D}(Fi, j) \cong \mathcal{C}(i, Gj)$$

which is natural in both arguments i and j .

Let us deconstruct this definition to some extent. Since we assume that both our categories are small we have that $\mathcal{D}(Fi, j)$ and $\mathcal{C}(i, Gj)$ always are sets.

If we fix some $i \in \mathcal{C}$ we get two covariant functors:

$$\mathcal{D}(Fi, _), \mathcal{C}(i, Gj): \mathcal{D} \rightarrow \mathbf{Set}$$

If we instead fix some $j \in \mathcal{D}$ we get two contravariant functors:

$$\mathcal{D}(F(_), j), \mathcal{C}(_, Gj): \mathcal{D}^{\text{Op}} \rightarrow \mathbf{Set}$$

Contravariant meaning that it flips the direction of arrows, and \mathcal{D}^{Op} is the *opposite* category of \mathcal{D} , with the same objects as in \mathcal{D} but with all arrows reversed.

Just to make everything crystal clear we also review the required naturality of the arguments i and j which can be described by forcing the following two diagrams to commute, for all $f: j \rightarrow j'$ in \mathcal{D} and all $g: i' \rightarrow i$ in \mathcal{D}^{Op} :

$$\begin{array}{ccccc}
 j & \mathcal{D}(Fi, j) \xrightarrow{\phi} \mathcal{C}(i, Gj) & i' & \mathcal{D}(Fi, j) \xrightarrow{\phi} \mathcal{C}(i, Gj) \\
 \downarrow f & \downarrow f^* & \downarrow g & \downarrow (Fg)^* & \downarrow g^* \\
 j' & \mathcal{D}(Fi, j') \xrightarrow{\phi} \mathcal{C}(i, Gj') & i & \mathcal{D}(Fi', j) \xrightarrow{\phi} \mathcal{C}(i', Gj)
 \end{array}$$

Another equivalent way of looking at adjunctions is to define it as a bijection which assigns to each arrow $h: Fi \rightarrow j$ an arrow $\phi h: i \rightarrow Gj$ in such a way that the naturality condition

$$\phi(f \circ h) = Gf \circ \phi h, \phi(h \circ Fg) = \phi h \circ g$$

holds for all f and all arrows $f: j \rightarrow j'$ in \mathcal{D} and all $g: i' \rightarrow i$ in \mathcal{D}^{Op} . This is equivalent to require that ϕ^{-1} be natural.

If we have an adjunction then F is said to be the *left-adjoint* of G and G is called the *right-adjoint* of F .

Before digging into a well chosen example, why should we care about adjoints? Because they turn up everywhere in all branches of mathematics.

3.2 Free groups and free functors

This subsection contains some basic preliminaries before we we will show that the forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ has a left-adjoint.

Recall the definition of a free group.

Let S be any set. Assume that $F(S)$ is a group and $i: S \rightarrow F(S)$ is a function.

Definition 17. $F(S)$ is said to be free on S if for all groups G and all functions $f: S \rightarrow G$ there exists a unique group homomorphism $\phi: F(S) \rightarrow G$ such that the diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{i} & F(S) \\ f \downarrow & \swarrow \exists! \phi & \\ G & & \end{array}$$

That is, $\phi \circ i = f$.

A free group exists.

Proof. Let S be a nonempty set, we construct the free group $W(S)$ by assigning to every $s \in S$ another formal element $s^{-1} \in S^{-1}$. We call s^{-1} the inverse of s . The elements in $W(S)$ will be reduced finite words constructed from the alphabet $S \cup S^{-1}$. The binary operation is concatenation of words and the identity element is the empty word, ϵ .

It is easy to verify that

- Concatenation of two words is a word.
- Concatenation of a word with the empty word does not change the word.
- Every word has a unique inverse
- Concatenation of words is associative.

So $W(S)$ is a group.

$W(S)$ also satisfies the universal property in the definition of a free group.

Since S is nonempty we have the inclusion function $i : S \rightarrow W(S), s \mapsto s$. Let (G, \cdot, e) be any group and $f : S \rightarrow G$ a function. We then have that for any $w = s_1 i(s_2) \dots s_n \in W(S)$ since $i(s_j) = s_j$ seen as a one letter word, we have that

$$w = i(s_1) i(s_2) \dots i(s_n)$$

Assuming that w is reduced; $s_{j+1} \neq s_j^{-1}$ for all $j = 1, \dots, n-1$.

We can then define the map

$$\begin{aligned} \phi : W(S) &\rightarrow G \\ s_1 \dots s_n &\mapsto f(s_1) \cdot \dots \cdot f(s_n) \end{aligned}$$

ϕ is clearly a unique group homomorphism and the diagram

$$\begin{array}{ccc} S & \xrightarrow{i} & W(S) \\ f \downarrow & \swarrow \phi & \\ G & & \end{array}$$

commutes since for all $s \in S, \phi \circ i(s) = \phi(s)$ which by construction is equal to $f(s)$. □

We can now define a functor $G : \mathbf{Set} \rightarrow \mathbf{Grp}$ called the *free* functor, sending every set S to its associated free group $F(S)$.

For morphisms, let $f : S \rightarrow S'$ be a function between sets. $G(S) = F(S)$ and $G(S') = F(S')$ are free groups so we have the following two commutative diagrams.

$$\begin{array}{ccc} S & \xrightarrow{i} & F(S) \\ h \downarrow & \swarrow \exists! \phi & \\ G & & \end{array} \quad \begin{array}{ccc} S' & \xrightarrow{i'} & F(S') \\ h' \downarrow & \swarrow \exists! \phi' & \\ G' & & \end{array}$$

Using the universal property of $F(S)$ we can take G as $F(S')$ and h as $f : S \rightarrow S' \subset F(S')$ to obtain a unique group homomorphism $\phi : F(S) \rightarrow F(S')$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{i} & F(S) \\ f \downarrow & \swarrow \exists! \phi & \\ F(S') & & \end{array}$$

We define $G(f) = \phi$ as the unique group homomorphism between $F(S)$ and $F(S')$ extending f , described above.

To see that G is functorial:

- Consider the identity map $\mathbb{1}_S: S \rightarrow S$. If we let G act on this map we get the diagram

$$\begin{array}{ccc} S & \xrightarrow{i} & F(S) \\ \mathbb{1}_S \downarrow & \swarrow \exists! \phi = G(\mathbb{1}_S) & \\ F(S) & & \end{array}$$

Since for all $s \in S$ we have that $\phi \circ i(s) = \mathbb{1}_S(s) = s \implies \phi = \mathbb{1}_{F(S)} \implies G(\mathbb{1}_S) = \mathbb{1}_{F(S)}$.

- Let $f: S \rightarrow S', g: S' \rightarrow S''$ be maps between sets. We want to show that the functor G respects composition.

One one hand we have the commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{i} & F(S) \\ g \circ f \downarrow & \swarrow \exists! G(g \circ f) & \\ F(S'') & & \end{array}$$

On the other hand we have

$$\begin{array}{ccccc} & & S & \xrightarrow{i} & F(S) \\ & \swarrow f & \downarrow f & \swarrow \exists! G(f) & \\ S' & \xrightarrow{i'} & F(S') & & \\ \downarrow g & \swarrow \exists! G(g) & & & \\ F(S'') & & & & \end{array}$$

Which by the uniqueness in universal property of $F(S), F(S')$ and $F(S'')$ implies that:

$$\begin{array}{ccc} S & \xrightarrow{i} & F(S) \\ g \circ f \downarrow & \swarrow G(g) \circ G(f) = G(g \circ f) & \\ F(S'') & & \end{array}$$

3.3 The left-adjoint of the forgetful functor from Grp to Set

We have the forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ and in the previous subsection we defined the free functor $G: \mathbf{Set} \rightarrow \mathbf{Grp}$. We claim that G is a left-adjoint of U , in other words we have a bijection

$$\mathbf{Grp}(G(X), Y) \cong \mathbf{Set}(X, U(Y))$$

which is natural in both arguments $X \in \mathbf{Set}$ and $Y \in \mathbf{Grp}$.

Proof. We use the definition of a free group, in particular the universal property to define the function:

$$\Psi: \mathbf{Set}(X, U(Y)) \rightarrow \mathbf{Grp}(G(X), Y)$$

$$f \mapsto g$$

Where g is the map such that

$$\begin{array}{ccc} X & \xrightarrow{i} & U(G(X)) \\ f \downarrow & \swarrow U(g) & \\ U(Y) & & \end{array}$$

commutes.

By inspecting the above diagram we note that for any $g \in \mathbf{Grp}(G(X), Y)$, we have $U(g) \circ i \in \mathbf{Set}(X, U(Y))$. Where i is the inclusion $i: X \rightarrow U(G(X))$. So it is suitable to define the inverse function Φ in the following way:

$$\Phi: \mathbf{Grp}(G(X), Y) \rightarrow \mathbf{Set}(X, U(Y))$$

$$g \mapsto U(g) \circ i$$

Let $f \in \mathbf{Set}(X, U(Y))$ and $h \in \mathbf{Grp}(G(X), Y)$ then:

- $\Phi \circ \Psi(f) = \Phi(g)$. Where $g \in \mathbf{Grp}(G(X), Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i} & U(G(X)) \\ f \downarrow & \swarrow U(g) & \\ U(Y) & & \end{array}$$

$\Phi(g) = U(g) \circ i = f$, by the universal property of $G(X)$.

- $\Psi \circ \Phi(h) = \Psi(U(h) \circ i)$ which corresponds to the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{i} & U(G(X)) \\ U(g) \circ i \downarrow & \swarrow U(k) & \\ U(Y) & & \end{array}$$

Where $k \in \mathbf{Grp}(G(X), Y)$. We can extend the diagram:

$$\begin{array}{ccc}
X & \xrightarrow{i} & U(G(X)) \\
\downarrow i & & \searrow U(k) \\
U(G(X)) & & \\
\downarrow U(g) & & \\
U(Y) & &
\end{array}$$

The only possible way to make this commute is to set $k = h$.

$$\implies \Psi(U(h) \circ i) = h.$$

$$\Phi \circ \Psi(f) = f \text{ and } \Psi \circ \Phi(h) = h \implies \mathbf{Grp}(G(X), Y) \cong \mathbf{Set}(X, U(Y))$$

It remains to show that this isomorphism is natural in $X \in \mathbf{Set}$ and $Y \in \mathbf{Grp}$.

- Naturality in $Y \in \mathbf{Grp}$.

Fix some set $X \in \mathbf{Set}$ and define the natural transformation

$$\tau_{X, _} : \mathbf{Grp}(G(X), _) \rightarrow \mathbf{Set}(X, U(_))$$

By the function Φ ,

$$\begin{aligned}
\tau_{X, Y} : \mathbf{Grp}(G(X), Y) &\rightarrow \mathbf{Set}(X, U(Y)) \\
f &\mapsto \Phi(f) = U(f) \circ i
\end{aligned}$$

as defined earlier.

Now let $h: Y \rightarrow Z$ be any group homomorphism. To show naturality in Y is to show that the following diagram commutes:

$$\begin{array}{ccc}
Y & \mathbf{Grp}(G(X), Y) & \xrightarrow{\tau_{X, Y}} \mathbf{Set}(X, U(Y)) \\
h \downarrow & h \circ _ \downarrow & \downarrow U(h) \circ _ \\
Z & \mathbf{Grp}(G(X), Z) & \xrightarrow{\tau_{X, Z}} \mathbf{Set}(X, U(Z))
\end{array}$$

Let $f \in \mathbf{Grp}(G(X), Y)$ then:

$$\begin{aligned}
- U(h) \circ \tau_{X, Y}(f) &= U(h) \circ (U(f) \circ i) = U(h) \circ U(f) \circ i \\
- \tau_{X, Z} \circ (h \circ f) &= U(h \circ f) \circ i = U(h) \circ U(f) \circ i
\end{aligned}$$

The diagram commutes!

- Fix some group $Y \in \mathbf{Grp}$ and define the natural transformation

$$\theta_{_, Y} : \mathbf{Grp}(G(_), Y) \rightarrow \mathbf{Set}(_, U(Y))$$

as above.

Now, let X, X' be sets, $i': X' \rightarrow U(G(X')), i: X \rightarrow U(G(X))$ inclusions and finally let $h \in \text{Hom}(X, X')$.

We want the following diagram to commute for all $f \in \mathbf{Grp}(G(X'), Y)$:

$$\begin{array}{ccc}
 X & \mathbf{Grp}(G(X'), Y) & \xrightarrow{\theta_{X', Y}} \mathbf{Set}(X', U(Y)) \\
 h \downarrow & \downarrow \text{\scriptsize } _ \circ G(h) & \downarrow \text{\scriptsize } _ \circ h \\
 X' & \mathbf{Grp}(G(X), Y) & \xrightarrow{\theta_{X, Y}} \mathbf{Set}(X, U(Y))
 \end{array}$$

Let $f \in \mathbf{Grp}(G(X'), Y)$ by chasing the diagram we get:

- $(_ \circ h) \circ \theta_{X', Y}(f) = \theta_{X', Y}(f) \circ h = (U(f) \circ i') \circ h = U(f) \circ i' \circ h$
- $\theta_{X, Y} \circ (f \circ G(h)) = \theta_{X, Y}(f \circ G(h)) = U(f \circ G(h)) \circ i = U(f) \circ U(G(h)) \circ i$

We need to show that the equality $U(f) \circ i' \circ h = U(f) \circ U(G(h)) \circ i$ holds. However it suffices to show:

$$i' \circ h = U(G(h)) \circ i$$

Since $h \in \text{Hom}(X, X')$ and by the universal property of $G(X)$ and $G(X')$:

$$\begin{array}{ccc}
 X & \xrightarrow{i} & G(X) \\
 i' \circ h \downarrow & & \swarrow \exists! \phi \\
 & & G(X')
 \end{array}$$

There exists a unique homomorphism $\phi: G(X) \rightarrow G(X')$ such that $i \circ \phi = i' \circ h$ but we also have that $G(h)\phi$ which implies that:

$$i' \circ h = \phi \circ i = G(h) \circ i$$

So we must have that $i' \circ h = U(G(h)) \circ i$ which concludes the proof. □

One can view this result as demonstrating the existence of a *very* weak inverse to the forgetful functor.

3.4 Other examples

We have shown that the free functor is a left adjoint of the forgetful functor from the category of groups to the category of sets. This relation between a forgetful functor and a free functor is not unique to groups:

For the forgetful functors:

- $U: \mathbf{R-Mod} \rightarrow \mathbf{Set}$
- $U: \mathbf{Cat} \rightarrow \mathbf{Grph}$
- $U: \mathbf{Ab} \rightarrow \mathbf{Set}$
- $U: \mathbf{Mon} \rightarrow \mathbf{Set}$
- $U: \mathbf{Ab} \rightarrow \mathbf{Grp}$

We have corresponding free functors as left-adjoints.

In particular consider the forgetful functor $U: \mathbf{Fld} \rightarrow \mathbf{Dom}_m$ from the category of fields to the category of integral domains where the morphisms are restricted to monomorphisms; injective ring homomorphisms. Surprisingly we have a left-adjoint F to U .

$$F: \mathbf{Dom}_m \rightarrow \mathbf{Fld}$$

$$D \mapsto Q(D)$$

Where $Q(D)$ is the field of fractions of D , i.e. the generalization of how we construct the rational number field \mathbb{Q} from the integral domain \mathbb{Z} . $Q(D)$ is the smallest field containing D in the sense that every field of fractions K containing D comes with an injection $i: D \rightarrow K$ and for every $K \in \mathbf{Fld}$ such that $D \subset K$ there is a monomorphism:

$$f: Q(D) \rightarrow K$$

Which restricted to D , is the identity function. However, the inclusion $D \subset K$ can be replaced by any monomorphism $j: D \rightarrow K$, this universality shows why we need the restriction on the morphisms of \mathbf{Dom} to monomorphisms. Since for example considering all homomorphisms of \mathbf{Dom} this universality fails for $\mathbb{Z} \rightarrow \mathbb{Z}_p$ where p is a prime.

Another interesting example concerns posets and will be stated as a theorem without proof.

Theorem 3. *Let P and Q be two posets (viewed as categories) and $S: P \rightarrow Q^{Op}, T: Q^{Op} \rightarrow P$ two order-preserving functions (viewed functors). Then S is a left adjoint to T if and only if, for all $p \in P$ and $q \in Q$,*

$$Sp \geq q \in Q \Leftrightarrow p \leq Tq \in P$$

When this is the case, there is exactly one adjunction ϕ making S the left adjoint of T . For all p and q , $p \leq TSp$ and $STq \geq q$ so we also have that

$$Sp \geq LTLp \geq Sp, Tq \leq TsTq \leq Tq.$$

3.5 Connecting adjunction to equivalence of categories

Recall that functor $S: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if there exists a functor $T: \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms:

$$ST \cong \text{Id}: \mathcal{D} \rightarrow \mathcal{D}, \quad TS \cong \text{Id}: \mathcal{C} \rightarrow \mathcal{C}.$$

Our goal in this subsection is to connect the notion of equivalence of categories to the concept of adjunction.

Definition 18. *An adjoint equivalence of categories is an adjunction $\langle T, S, \tau, \theta \rangle: \mathcal{C} \rightarrow \mathcal{D}$ in which both $\tau: \text{Id} \rightarrow ST$ and $\theta: TS \rightarrow \text{Id}$ are natural isomorphisms.*

If $\langle T, S, \tau, \theta \rangle: \mathcal{C} \rightarrow \mathcal{D}$ is an adjoint equivalence, then we have that $\text{Id} \cong TS$ and $ST \cong \text{Id}$ which implies that τ^{-1} and θ^{-1} are also natural isomorphisms. Thus $\langle S, T, \theta^{-1}, \tau^{-1} \rangle: \mathcal{D} \rightarrow \mathcal{C}$ is an adjunction. This implies that for an adjoint equivalence $\langle T, S, \tau, \theta \rangle$ the functor $T: \mathcal{D} \rightarrow \mathcal{C}$ is the left adjoint of $S: \mathcal{C} \rightarrow \mathcal{D}$ with unit τ while at the same time T is the right adjoint of S with unit θ^{-1} .

We now have the machinery necessary to present and prove the main theorem of this subsection.

Theorem 4. *The following properties of a functor $S: \mathcal{D} \rightarrow \mathcal{C}$ are equivalent:*

1. S is an equivalence of categories.
2. S is part of an adjoint equivalence $\langle T, S, \tau, \theta \rangle: \mathcal{C} \rightarrow \mathcal{D}$.
3. S is fully faithful and all $i \in \mathcal{C}$ is isomorphic to Sj for some $j \in \mathcal{D}$.

Proof. We will proceed with a round-robin style proof, showing that:

$$2. \implies 1. \implies 3. \implies 2.$$

- 2. \implies 1.

Assume that $S: \mathcal{D} \rightarrow \mathcal{C}$ is part of an adjoint equivalence $\langle T, S, \tau, \theta \rangle: \mathcal{C} \rightarrow \mathcal{D}$.

Then by definition $\tau: \text{Id} \rightarrow ST, \theta: TS \rightarrow \text{Id}$ are natural isomorphisms which implies that S is an equivalence of categories.

- 1. \implies 3.

Now assume that $S: \mathcal{D} \rightarrow \mathcal{C}$ is an equivalence of categories. By definition this implies the existence of a functor $T: \mathcal{C} \rightarrow \mathcal{D}$ and natural isomorphisms $ST \cong \text{Id}$ and $TS \cong \text{Id}$.

$ST \cong \text{Id}: \mathcal{C} \rightarrow \mathcal{C} \implies$ for all $i \in \mathcal{C}$ we have $i \cong S(Ti)$. Set $j = Ti \in \mathcal{D}$.

Now we must show that S is fully faithful.

We have the natural isomorphism $\theta: TS \rightarrow \text{Id}$ which for all $f \in \mathcal{C}(j, j')$ gives us the following commutative square:

$$\begin{array}{ccc}
TSj & \xrightarrow{\theta_j} & j \\
TSf \downarrow & & \downarrow f \\
TSj' & \xrightarrow{\theta_{j'}} & j'
\end{array}$$

$$\implies f \circ \theta_j = \theta_{j'} \circ TSf \implies f = \theta_{j'} \circ TSf \circ \theta_j^{-1}.$$

Suppose there exists some $f' \in \mathcal{C}(j, j')$ such that $Sf = Sf'$.

Then we have that $f' = \theta_{j'} TSf' \circ \theta_j^{-1} = \theta_{j'} \circ TSf \circ \theta_j^{-1} = f$

So S is faithful.

For any $h: Sj \rightarrow Sj'$ take $f = \theta_{j'} \circ Th \circ \theta_j^{-1}$.

$$\begin{array}{ccc}
TSj & \xrightarrow{\theta_j} & j \\
Th \downarrow & & \downarrow f = \theta_{j'} Th \circ \theta_j^{-1} \\
TSj' & \xrightarrow{\theta_{j'}} & j'
\end{array}$$

$$\implies f \circ \theta_j = (\theta_{j'} Th \circ \theta_j^{-1}) \circ \theta_j = \theta_{j'} \circ Th.$$

Since S is faithful this implies that $Sf = h \implies S$ full and thus fully faithful.

- 3. \implies 2.

To show the final implication suppose that S is fully faithful and all $i \in \mathcal{C}$ is isomorphic to Sj for some $j \in \mathcal{D}$.

We must construct an adjoint equivalence of which S is a part of.

To define a left adjoint T of S , note that for all $i \in \mathcal{C}$ we can choose some $j_0 = T_0(i) \in \mathcal{D}$ and an isomorphism θ_i :

$$\begin{array}{ccc}
\theta_i: i & \xrightarrow{\cong} & S(T_0i) & & T_0i \\
& \searrow f & \downarrow Sg & & \downarrow g \\
& & S(j) & & j
\end{array}$$

For all arrows $f: i \rightarrow Sj$ the composite $f \circ \theta_i^{-1}$ has the form Sg for some unique g which follows from the assumption that S is fully faithful. This implies that $f = Sg \circ \theta_i$ for some unique g . Then we can construct $T: \mathcal{C} \rightarrow \mathcal{D}$ in exactly one way such that $\theta: \text{Id} \rightarrow ST$ is natural. So T is a left adjoint of S , with unit isomorphism θ .

For the counit $\tau: S\theta_j \circ \tau_{Sj} = 1_{Sj} \implies S\theta_j = \tau_{Sj}^{-1}$ and since S is fully faithful and the counit τ is also invertible and we have an adjoint equivalence, as desired.

□

4 The Yoneda Lemma

We have now reached the first real highlight of this thesis; the Yoneda Lemma, a classic categorical representation result formulated, but not proved by, Nobuo Yoneda in a paper from 1954. The Yoneda Lemma is a beautiful result which can be viewed from many different angles with interesting corollaries corollaries.

It may be one of the more powerful theorems in basic category theory and interestingly enough we do not need much in terms of technical machinery to prove it.

4.1 Formulation and proof

We begin by reviewing two concepts we have seen earlier.

Let \mathcal{C} be a category and fix an object $i \in \mathcal{C}$. Recall the covariant Yoneda functor:

$$\mathcal{C}(i, _): \mathcal{C} \rightarrow \mathbf{Set}$$

This functor sends objects $j \in \mathcal{C}$ to the homset $\mathcal{C}(i, j)$ and sends morphisms $f \in \mathcal{C}(k, l)$ to $f \circ _$, post-composition by f .

We also have the contravariant Yoneda functor:

$$\mathcal{C}(_, i): \mathcal{C}^{\text{Op}} \rightarrow \mathbf{Set}$$

Which works in a similar way as the covariant Yoneda functor but sends morphisms to precomposition.

Also, we should recall the notion of a functor category. Let \mathcal{C} and \mathcal{D} be categories then we can form the functor category denoted $[\mathcal{C}, \mathcal{D}]$, $\text{Nat}(\mathcal{C}, \mathcal{D})$ or $\text{Mor}(\mathcal{C}, \mathcal{D})$ consisting of as objects; functors from \mathcal{C} to \mathcal{D} and as morphisms; natural transformations between such functors. Thus, if we have two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ then $\text{Nat}(\mathcal{C}, \mathcal{D})(F, G)$ is the set of all natural transformations η :

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow \eta \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

We are now ready to state and prove the Yoneda Lemma:

Theorem 5. The Yoneda Lemma. *Let \mathcal{C} be a locally small category. Then*

$$\text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F) \cong Fi$$

natural in $i \in \mathcal{C}$ and $F \in \text{Nat}(\mathcal{C}, \mathbf{Set})$.

Proof. Let us fix some $i \in \mathcal{C}$ and $F \in \text{Nat}(\mathcal{C}, \mathbf{Set})$. To prove the Yoneda Lemma we have to find a bijection between $\text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F)$ and Fi and show that it is natural in both arguments.

For $\eta \in \text{Nat}(\mathcal{C}, \mathbf{Set})$, $\eta: \mathcal{C}(i, _) \rightarrow F$ is a natural transformation, we claim that the function:

$$\begin{aligned} \Psi: \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F) &\rightarrow Fi \\ \eta &\mapsto \eta_i(\mathbf{1}_i) \end{aligned}$$

Gives us a bijection. To show this we need to prove that Ψ is both injective and surjective.

To show that Ψ is injective. Suppose that $\Psi(\eta) = \Psi(\tau)$ for two morphisms $\eta, \tau \in \text{Nat}(\mathcal{C}, \mathbf{Set})$. Since η, τ are natural transformations, we have, for any $\alpha \in \mathcal{C}(i, j)$, the following two commutative squares:

$$\begin{array}{ccc} \begin{array}{ccc} i & \mathcal{C}(i, i) & \xrightarrow{\eta_i} & Fi \\ \alpha \downarrow & \alpha \circ _ \downarrow & & \downarrow F\alpha \\ j & \mathcal{C}(i, j) & \xrightarrow{\eta_j} & Fj \end{array} & & \begin{array}{ccc} \mathcal{C}(i, i) & \xrightarrow{\tau_i} & Fi \\ \alpha \circ _ \downarrow & & \downarrow F\alpha \\ \mathcal{C}(i, j) & \xrightarrow{\tau_j} & Fj \end{array} \end{array}$$

We now make a choice and take $\mathbf{1}_i \in \mathcal{C}(i, i)$ and chase the diagrams:

$$\implies \begin{cases} F\alpha \circ \eta_i(\mathbf{1}_i) = \eta_j \circ (\alpha \circ \mathbf{1}_i) = \eta_j(\alpha) \\ F\alpha \circ \tau_i(\mathbf{1}_i) = \tau_j \circ (\alpha \circ \mathbf{1}_i) = \tau_j(\alpha) \end{cases}$$

By assumption $\Psi(\eta) = \Psi(\tau)$ so we get the equality:

$$\eta_j(\alpha) = F\alpha \circ \eta_i(\mathbf{1}_i) = F\alpha \circ \Psi(\eta) = F\alpha \circ \Psi(\tau) = F\alpha \circ \tau_i(\mathbf{1}_i) = \tau_j(\alpha)$$

$$\eta_j(\alpha) = \tau_j(\alpha) \implies \eta = \tau.$$

Since we have equality of the components functions of η and τ for an arbitrary $\alpha \in \mathcal{C}(i, j)$, η and τ must be the same natural transformation and hence equal. So $\Psi(\eta) = \Psi(\tau) \implies \eta = \tau$. Ψ is injective.

To show that Ψ is surjective; $\text{Im}(\Psi) = Fi$, we must for every $x \in Fi$ construct an explicit natural transformation $\xi^x \in \text{Nat}(\mathcal{C}, \mathbf{Set})$ such that $\Psi(\xi^x) = \xi_i^x(\mathbf{1}_i) = x \in Fi$.

Again, by choosing $\mathbf{1}_i \in \mathcal{C}(i, i)$ and some arbitrary $\alpha \in \mathcal{C}(i, j)$ and inspecting the diagram:

$$\begin{array}{ccc}
i & \mathcal{C}(i, i) & \xrightarrow{\xi_i^x} Fi \\
\alpha \downarrow & \alpha \circ _ \downarrow & \downarrow F\alpha \\
j & \mathcal{C}(i, j) & \xrightarrow{\xi_j^x} Fj
\end{array}$$

We see that it is suitable to define the component function of natural transformation as $\xi_j^x(\alpha) := F\alpha[x]$, the evaluation of $F\alpha \in \text{Hom}(Fi, Fj)$ at $x \in Fi$. Then:

$$\Psi(\xi^x) = \xi_i^x(\mathbb{1}_i) = F(\mathbb{1}_i)[x] = \mathbb{1}_{Fi}(x) = x$$

We also have to check that ξ^x indeed is a natural transformation. To show this we need to show that for all $\alpha \in \mathcal{C}(j, k)$ the square commutes:

$$\begin{array}{ccc}
j & \mathcal{C}(i, j) & \xrightarrow{\xi_j^x} Fj \\
\alpha \downarrow & \alpha \circ _ \downarrow & \downarrow F\alpha \\
k & \mathcal{C}(i, k) & \xrightarrow{\xi_k^x} Fk
\end{array}$$

Choose some arbitrary $\beta \in \mathcal{C}(i, j)$, if the diagram commutes we have:

$$\begin{array}{ccc}
\beta & \xrightarrow{\xi_j^x} & \xi_j^x(\beta) \\
\alpha \circ _ \downarrow & & \downarrow F\alpha \circ _ \\
\alpha \circ \beta & \xrightarrow{\xi_k^x} & \xi_k(\alpha \circ \beta) = F\alpha \circ \xi_j^x(\beta)
\end{array}$$

We need to show that $\xi_k(\alpha \circ \beta) = F\alpha \circ \xi_j^x(\beta)$,

$$F(\alpha \circ \beta)[x] = \xi_k^x(\alpha \circ \beta) = F\alpha \circ \xi_j^x(\beta) = F\alpha \circ F(\beta)[x] = F(\alpha \circ \beta)[x].$$

The diagram commutes which implies that ξ^x is a natural transformation and hence Ψ is surjective. Since Ψ is both surjective and injective we have shown that it is a bijection.

Now we have to show that this bijection is natural in both argument. Let Ψ_i^F denote the function giving the bijection $\text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F) \cong Fi$.

First, note that if $\alpha \in \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F)$ then, given some arrow $f: \mathcal{C}(i, j)$, we can get a natural transformation in $\text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(j, _), F)$ by $\alpha \mapsto \alpha \circ [_ \circ f] \in \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(j, _))$

Where $\alpha \circ [_ \circ f]$ for some $a \in \mathcal{C}$ has the component function:

$$\begin{aligned}
(\alpha \circ [_ \circ f])_a &: \mathcal{C}(j, a) \rightarrow F(a) \\
_ &\mapsto \alpha_a(_ \circ f)
\end{aligned}$$

To see that $\alpha \circ [_ \circ f]$ really is a natural transformation let $f \in \mathcal{C}(a, b)$. We show that

$$\begin{array}{ccc}
a & \mathcal{C}(j, a) & \xrightarrow{(\alpha \circ [_ \circ f])_a} Fa \\
g \downarrow & g \circ _ \downarrow & \downarrow Fg \\
b & \mathcal{C}(j, b) & \xrightarrow{(\alpha \circ [_ \circ f])_b} Fb
\end{array}$$

commute. Let $\phi \in \mathcal{C}(j, a)$ then we have

$$Fg \circ (\alpha \circ [_ \circ f])_a(\phi) = Fg \circ \alpha_a(\phi \circ f)$$

and

$$(\alpha \circ [_ \circ f])_b(g \circ \phi) = \alpha_b((g \circ \phi) \circ f) = \alpha_b((\phi \circ f) \circ g).$$

Since α is a natural transformation,

$$\begin{array}{ccc}
a & \mathcal{C}(j, a) & \xrightarrow{\alpha_a} Fa \\
g \downarrow & g \circ _ \downarrow & \downarrow Fg \\
b & \mathcal{C}(j, b) & \xrightarrow{\alpha_b} Fb
\end{array}$$

commutes, in other words $Fg \circ \alpha_a = \alpha_b \circ (g \circ _)$ and since $\phi \circ f \in \mathcal{C}(j, a)$, we have that:

$$Fg \circ \alpha_a(\phi \circ f) = \alpha_b(g \circ (\phi \circ f))$$

So $Fg \circ (\alpha \circ [_ \circ f])_a = (\alpha \circ [_ \circ f])_b(g \circ _)$ which shows that $\alpha \circ [_ \circ f] \in \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(j, _), F)$.

Now, to show naturality in i , we must prove that for some fixed functor $F \in \text{Nat}(\mathcal{C}, \mathbf{Set})$,

$$\Psi_{[_]}^F: \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}([_], _), F) \rightarrow F([_])$$

is a natural transformation. That is, for every $f: i \rightarrow j \in \mathcal{C}$ the square

$$\begin{array}{ccc}
i & \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F) & \xrightarrow{\Psi_i^F} Fi \\
f \downarrow & (_) \circ [_ \circ f] \downarrow & \downarrow Ff \\
j & \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(j, _), F) & \xrightarrow{\Psi_j^F} Fj
\end{array}$$

must commute; $\Psi_j^F \circ ((_) \circ [_ \circ f]) = Ff \circ \Psi_i^F$. To show this let $\tau \in \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F)$ by chasing the diagram:

$$Ff \circ \Psi_i^F(\tau) = Ff \circ (\tau_i(\mathbf{1}_i)) = Ff(\tau_i(\mathbf{1}_i))$$

On the other hand:

$$\Psi_j^F \circ ((_) \circ [_ \circ f])[\tau] = \Psi_j^F \circ (\tau \circ [_ \circ f]) = \Psi_j^F(\tau \circ [_ \circ f]) = (\tau \circ [_ \circ f])_j[\mathbf{1}_j] = \tau_j(\mathbf{1}_j \circ f) = \tau_j(f),$$

but since τ is a natural transformation we have that

$$Ff \circ \tau_i = \tau_j \circ (f \circ _).$$

Take $\mathbf{1}_i \in \mathcal{C}(i, i)$ in the above equation to obtain:

$$Ff \circ \tau_i(\mathbf{1}_i) = \tau_j \circ (f \circ \mathbf{1}_i) = \tau_j(f) \implies \Psi_j^F \circ ((_) \circ [_ \circ f]) = Ff \circ \Psi_i^F.$$

So $\Psi_{[_]}^F$ is a natural transformation and we have established naturality in i .

Naturality in $F \in \text{Nat}(\mathcal{C}, \mathbf{Set})$ states that for some fixed $i \in \mathcal{C}$

$$\Psi_i^{[_]}: \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), [_]) \rightarrow [_](i)$$

is a natural transformation. This means that for every natural transformation:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathbf{Set} \\ & \Downarrow \tau & \\ \mathcal{C} & \xrightarrow{G} & \mathbf{Set} \end{array}$$

the following square must commute:

$$\begin{array}{ccc} \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F) & \xrightarrow{\Psi_i^F} & Fi \\ \tau \circ _ \downarrow & & \downarrow \tau_i \\ \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), G) & \xrightarrow{\Psi_i^G} & Gi \end{array}$$

$$\implies \Psi_i^G \circ (\tau \circ _) = \tau_i \circ \Psi_i^F.$$

Let $\theta \in \text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F)$ then

$$\Psi_i^G \circ (\tau \circ _)[\theta] = \Psi_i^G \circ (\tau \circ \theta) = \Psi_i^G(\tau \circ \theta) = (\tau \circ \theta)_i[\mathbf{1}_i] = \tau_i \circ \theta_i(\mathbf{1}_i)$$

and

$$\tau_i \circ \Psi_i^F[\theta] = \tau_i \circ \theta_i(\mathbf{1}_i)$$

So the diagram commutes and we have established naturality in both F and i which completes the proof. □

We also have a contravariant form of the Yoneda Lemma:

Theorem 6. The Yoneda Lemma. *Let \mathcal{C} be a locally small category. Then*

$$\text{Nat}(\mathcal{C}^{Op}, \mathbf{Set})(\mathcal{C}(_, i), F) \cong F(i)$$

naturally in $i \in \mathcal{C}$ and $F \in \text{Nat}(\mathcal{C}, \mathbf{Set})$.

The proof is very similar.

4.2 The Yoneda Embedding

Yoneda gives us a natural bijection $\text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), F) \cong F(i)$. As a first nice corollary note that if we for some $j \in \mathcal{C}$ let F be another homfunctor;

$$\mathcal{C}(j, _): \mathcal{C} \rightarrow \mathbf{Set}$$

Then according to Yoneda Lemma we have that: $\text{Nat}(\mathcal{C}, \mathbf{Set})(\mathcal{C}(i, _), \mathcal{C}(j, _)) \cong \mathcal{C}(j, i)$. Which shows that natural transformations between homfunctors correspond exactly to reversed arrows. In fact, we can use this to define a special functor from the dual category \mathcal{C}^{Op} to the functor category $\text{Nat}(\mathcal{C}, \mathbf{Set})$.

Theorem 7. The Yoneda Embedding. *There exists a functor*

$$Y: \mathcal{C}^{Op} \rightarrow \text{Nat}(\mathcal{C}, \mathbf{Set})$$

which is fully faithful and injective on objects.

Proof. By Yoneda Lemma, Y maps objects i to homfunctors $\mathcal{C}(i, _)$ and a reversed morphism $f \in \mathcal{C}(j, i)$ is mapped to the natural transformation $_ \circ f: \mathcal{C}(i, _) \rightarrow \mathcal{C}(j, _)$. It is clear that Y is functorial.

To see that Y is faithful let $f, g \in \mathcal{C}^{Op}(j, i)$ and suppose that $Y(f) = Y(g)$ which implies that:

$$_ \circ g = _ \circ f: \mathcal{C}(i, _) \rightarrow \mathcal{C}(j, _)$$

Since these are two equal natural transformations. Choose the morphism $\mathbf{1}_i \in \mathcal{C}(i, i)$ to get:

$$\begin{cases} (\mathbf{1}_i \circ _) \circ f = \mathbf{1}_i \circ (_ \circ f) \\ (\mathbf{1}_i \circ _) \circ g = \mathbf{1}_i \circ (_ \circ g) \end{cases}$$

Choose $\mathbf{1}_i \in \mathcal{C}(i, i)$ again to get $\mathbf{1}_i \circ (\mathbf{1}_i \circ f) = \mathbf{1}_i \circ (\mathbf{1}_i \circ g) \implies f = g$. So Y is faithful.

Y is full as well since for all $i, j \in \mathcal{C}^{Op}$ and all natural transformations

$$\eta: \mathcal{C}(i, _) \rightarrow \mathcal{C}(i, _)$$

are exactly on the form $_ \circ f$ for some $f \in \mathcal{C}^{Op}(j, i)$.

Finally, we show that Y is injective on objects. Suppose that $Y(i) = Y(j)$ for $i, j \in \mathcal{C}^{Op}$, which implies that:

$$\mathcal{C}(i, _) = \mathcal{C}(j, _): \mathcal{C} \rightarrow \mathbf{Set}$$

Then $i \mapsto \mathcal{C}(i, i) = \mathcal{C}(j, i)$ and since $\mathbb{1}_i \in \mathcal{C}(i, i) \implies \mathbb{1}_i \in \mathcal{C}(j, i) \implies j = i$ and we are done. □

What does this mean? The dual of locally small category \mathcal{C} is embedded fully faithfully in the functor category. This means that the dual of a category essentially is a full subcategory of $\mathbf{Nat}(\mathcal{C}, \mathbf{Set})$.

We also have the dual Yoneda embedding giving a fully faithful functor from \mathcal{C} to $\mathbf{Nat}(\mathcal{C}, \mathbf{Set})$.

This is a very useful result in practice, say that we have want to study a very complicated category then it might be easier to just apply the Yoneda functor on this category and study natural transformations between homsets instead.

4.3 Cayley's Theorem

Cayley's theorem is a famous theory in group theory stating that every group G is isomorphic to a subgroup of a symmetric group S_n , for some n . We will in this section state and prove this theorem in the setting of ordinary basic group theory and then view Cayley's theorem

Definition 19. A left group action of a group G on a set X is a mapping $G \times X \rightarrow X, (g, x) \mapsto g \cdot x$ such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G, x \in X$. We say that G acts on the left on X .

Note that in a left group action of a group G on a set X , the action $\sigma_g : x \mapsto g \cdot x$ is a permutation of X and the map $g \mapsto \sigma_g$ is a homomorphism from G into S_G .

Proof. If $\sigma_g : x \mapsto g \cdot x$ is a permutation of X then the map is a bijection from X to X . Suppose that $\sigma_g(x) = \sigma_g(y)$ for some $x, y \in X$. Then:

$$g \cdot x = g \cdot y \implies g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (g \cdot y) \implies g^{-1}g \cdot x = g^{-1}g \cdot y \implies x = y$$

From the definition of a left group action, $g^{-1} \cdot x \in X$ so for every $x \in X$ we have:

$$\sigma_g(g^{-1} \cdot x) = g \cdot (g^{-1} \cdot x) = (gg^{-1}) \cdot x = 1 \cdot x = xi.$$

So $\sigma_g : x \mapsto g \cdot x$ is a bijection and $\sigma_g \circ \sigma_h(x) = \sigma_g(h \cdot x) = g \cdot (h \cdot x) = gh \cdot x = \sigma_{gh}$ shows that it is a group homomorphism as well. □

Theorem 8. Cayley's Theorem. Every group G is isomorphic to a subgroup of the symmetric group S_G .

Proof. Let G act on itself by left multiplication. We must show that $\phi : g \mapsto \sigma_g$ is a *injective* homomorphism from G into S_G .

Since we already know that ϕ is a homomorphism it suffices to show that it is injective.

Let $g, h \in G$ and suppose that $\phi(g) = \phi(h) \implies \sigma_g(x) = \sigma_h(x)$ for all $x \in G$. We act on G by left multiplication so we have that $hx = gx \implies hxx^{-1} = gxx^{-1} \implies h = g$.

□

Now let us think about this from a categorical point of view. Let $G \in \mathbf{Grp}$ and consider the category (groupoid) $\#_G$ with one formal object $\#$ and the elements of G as arrows in $\#_G$. By Yoneda Lemma we have the natural bijection:

$$[\#_G, \mathbf{Set}](\mathrm{Hom}(\#, _), \mathrm{Hom}(\#, _)) \cong \mathrm{Hom}(\#, \#)$$

Note that $\mathrm{Hom}(\#, \#)$ is the set of all morphisms from $\#$ to itself in the category $\#_G$, so $\mathrm{Hom}(\#, \#) = G$, seen as a set. This means that we can identify each element of the group G with a natural transformation $\eta : \mathrm{Hom}(\#, _) \rightarrow \mathrm{Hom}(\#, _)$, $\eta \in \mathrm{Mor}([\#_G, \mathbf{Set}])$. What can we say about these natural transformations?

Let $\eta : \mathrm{Hom}(\#, _) \rightarrow \mathrm{Hom}(\#, _)$ be such a natural transformation. We claim that η is a bijection from G to itself.

Proof. Since $\eta : \mathrm{Hom}(\#, _) \rightarrow \mathrm{Hom}(\#, _)$ is a natural transformation and since we only have one object $\# \in \#_G$, we only have one component function $\eta_{\#} : \mathrm{Hom}(\#, \#) \rightarrow \mathrm{Hom}(\#, \#)$ so every η . We will write η but mean the component function $\eta_{\#}$. For simplicity we define $\mathrm{Hom}(\#) := \mathrm{Hom}(\#, \#)$.

By definition we have that for every morphism $g : \# \rightarrow \#$ the following diagram commutes:

$$\begin{array}{ccc} \# & \mathrm{Hom}(\#) & \xrightarrow{\eta} \mathrm{Hom}(\#) \\ g \downarrow & g \circ _ \downarrow & \downarrow g \circ _ \\ \# & \mathrm{Hom}(\#) & \xrightarrow{\eta} \mathrm{Hom}(\#) \end{array}$$

Which is equivalent to requiring that $\eta(g \circ h) = g \circ \eta(h)$ holds for all $h \in \mathrm{Hom}(\#)$

It is clear that η is a function from G to G .

Suppose that $\eta(x) = \eta(y)$ for some $x, y \in G$ then $\eta(x) = \eta(xe) = x\eta(e)$ and $\eta(y) = \eta(ye) = y\eta(e)$ then $x\eta(e) = y\eta(e) \implies x\eta(e)\eta(e)^{-1} = y\eta(e)\eta(e)^{-1} \implies x = y$. So $\eta : G \rightarrow G$ is injective.

Assume towards a contradiction that $\mathrm{Im}(\eta) \neq G$. Then there exists some $y \in G$ such that $\eta(x) \neq y$ for all $x \in G$. Since η is a natural transformation we have that for all $g, h \in G$:

$$\eta(gh) = g\eta(h)$$

Take $g := y\eta(h)^{-1}$ then $\eta(gh) = g\eta(h) = (y\eta(h)^{-1})\eta(h) = y(\eta(h)^{-1}\eta(h)) = ye = y$.

So there is in fact an element $x := y\eta(h)^{-1} \in G$ such that $\eta(x) = y$. This is a contradiction so we must have that $\text{Im}(\eta) = G$. η is indeed a bijection from G to itself. \square

We have shown that a group G is bijective to a set of bijections G to G which is precisely Cayley's Theorem. In, particular if $[\#_G, \mathbf{Set}](\text{Hom}(\#, _), \text{Hom}(\#, _))$ happens to be the set of *all* bijections on G then G is a symmetric group.

Another nearly identical approach is to consider one of the Yoneda embeddings of the category $\#_G$ and then meditate on the natural transformations.

So Cayley's Theorem follows from Yoneda's Lemma as a special case. This is not just a simple implication but we can in fact view Yoneda's Lemma as a vast generalization of Cayley's Theorem in terms of representations, of groups in the case of Cayley and categories for Yoneda.

5 Bicategories

A peculiar and interesting property inherent in category theory is that once we have defined the notion of a category they pop up everywhere. One nontrivial example of this might be the functor category we saw in the last section, consisting of functors and natural transformations.

What if we have a category in which every homset happens to be a category?

This begs the question if can we have define a notion of higher dimensional category theory which not only has objects and morphisms between objects but also morphisms between morphisms, morphisms between morphisms between morphisms... together with appropriate composition and identities. It turns out that the answer is yes, we can define a notion of higher dimensional categories and there exists such mathematical objects. The complexity increases rapidly with n , for n -categories so we shall restrict our attention to 2-categories, and the more general bicategories, i.e. categories with objects, morphisms between objects and morphisms between morphisms.

The motivation for such a generalization mainly comes from **Cat** which turns out to be the most canonical example of a 2-category.

The theory of 2-categories is nontrivial and certainly more complicated than ordinary category theory but easy and accessible enough so 2-categories is a good place to start.

Much of the material in this section comes from a paper by Leinster [3].

5.1 Definition and examples

We begin in the abstract by defining bicategory which is a *lax* 2-category.

Definition 20. A bicategory \mathcal{B} consists of the following data subject to the following axioms:

Data

- A collection of objects $\text{Obj}(\mathcal{B})$ (0-cells) A, B, \dots
- Categories $\mathcal{B}(A, B)$ with objects f, g, \dots (1-cells) and morphisms (2-cells) α, β, \dots
- Functors

$$\begin{aligned} \mathcal{C}_{ABC}: \mathcal{B}(B, C) \times \mathcal{B}(A, B) &\rightarrow \mathcal{B}(A, C) \\ (g, f) &\mapsto g \circ f = gf \\ (\beta, \alpha) &\mapsto \beta * \alpha \end{aligned}$$

and $\mathbb{1}_A: \mathbf{1} \rightarrow \mathcal{B}(A, A)$ (thus $\mathbb{1}_A$ is a 1-cell $A \rightarrow A$).

- Natural isomorphisms

$$\begin{array}{ccc} \mathcal{B}(C, D) \times \mathcal{B}(B, C) \times \mathcal{B}(A, B) & \xrightarrow{\mathbb{1} \times \mathcal{C}_{ABC}} & \mathcal{B}(C, D) \times \mathcal{B}(A, C) \\ \mathcal{C}_{BCD} \times \mathbb{1} \downarrow & & \downarrow \mathcal{C}_{ACD} \\ \mathcal{B}(B, D) \times \mathcal{B}(A, B) & \xrightarrow{\mathcal{C}_{ABD}} & \mathcal{B}(A, D) \end{array}$$

$$\begin{array}{ccc}
\mathcal{B}(A, B) \times \mathbf{1} & & \mathbf{1} \times \mathcal{B}(A, B) \\
\downarrow 1 \times \mathbb{1}_A & \searrow \sim & \downarrow \times \mathbb{1}_B \times \mathbf{1} \\
\mathcal{B}(A, B) \times \mathcal{B}(A, A) & \xrightarrow{c_{AAB}} & \mathcal{B}(A, B) & \mathcal{B}(B, B) \times \mathcal{B}(A, A) & \xrightarrow{c_{ABB}} & \mathcal{B}(A, B)
\end{array}$$

thus 2-cells

$$a_{hgf}: (hg)f \rightarrow h(gf)$$

$$r_f: f \circ \mathbb{1}_a \rightarrow f$$

$$l_f: \mathbb{1}_B \circ f \rightarrow f.$$

Axioms

We require the following two diagram to commute.

$$\begin{array}{ccccc}
& & ((kh)g)f & \xrightarrow{a*1} & (k(hg))f \\
& \swarrow a & & & \searrow a \\
(kh)(gf) & & & & k((hg)f) \\
& \searrow a & & & \swarrow 1*a \\
& & k(h(gf)) & & \\
& & & & \\
& & (gI)F & \xrightarrow{a} & g(I f) \\
& & \swarrow r*1 & & \swarrow 1*l \\
& & gf & &
\end{array}$$

This definition is fairly long and abstract compared to the material presented earlier but it looks more complicated than it actually is.

Informally we can say that a bicategory consists of 0-cells (objects), 1-cells (arrows/morphisms between objects) and 2-cells (arrows/morphisms between 1-cells) together with a notion of well behaved composition of 1-cells and 2-cells, respectively, such that associativity of composition and unit laws holds up to *isomorphism*, for 1-cells.

Note that equality is a trivial isomorphisms which brings us to strict bicategories which we will look more into in the next section. For the time being let us restrict our attention to proper weak bicategories.

5.2 Bicategory of bimodules

We will here define a proper bicategory in the sense that that it is not strict with respect to associativity of composition of 1-cells.

Before defining our bicategory we have to define what we mean by a bimodule.

Definition 21. *Let R, S be two rings. A bimodule is an abelian group M such that:*

1. A is a left R -module,
2. A is a right S -module,
3. $(ra)s = r(as)$ for all $r \in R, s \in S$ and $a \in A$.

We sat that if M is a bimodule, then $M \in R\text{-Mod-}S$.

It is easy to see that a ring R itself and any two-sided ideal in R provides examples of R - R -bimodules.

If M is a left R -module then $M \in R\text{-Mod-}\mathbb{Z}$.

A bimodule homomorphism between two bimodules is just a module homomorphism respecting both the right and the left action.

We can now define our proper bicategory \mathcal{B} consisting of:

- 0-cells are rings R, S, T, \dots ,
- 1-cells are bimodules $M \in R\text{-Mod-}S, N \in S\text{-Mod-}T \dots$,
- 2-cells are bimodule homomorphisms $M \rightarrow N$.

Composition of 2-cells is just ordinary function composition of bimodule homomorphisms. However, we have to define composition of 1-cells (i.e. bimodules).

Consider the bimodules $M \in R\text{-Mod-}S$ and $N \in S\text{-Mod-}T$, then we can construct a new bimodule via the tensor product $M \otimes_S N \in R\text{-Mod-}T$. So we define composition of 1-cells to be the tensor product.

But if $M \in R\text{-Mod-}S, N \in S\text{-Mod-}T$ and $N' \in T\text{-Mod-}T'$ then we have the well known canonical isomorphism:

$$M \otimes_S (N \otimes_T N') \cong (M \otimes_S N) \otimes_T N'$$

Which shows that composition of 1-cells in general only holds up to isomorphism. It is also clear that $\mathcal{B}(R, S)$ is a category and the reader might want to check that the axioms holds.

5.3 Opposite bicategories

Given a bicategory \mathcal{B} , we can form the opposite bicategory \mathcal{B}^{Op} by revering 1-cells but leaving 2-cells as they were. So if we have a 2-cell α in \mathcal{B} :

$$\begin{array}{ccc}
& f & \\
A & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & B \\
& g &
\end{array}$$

Then \mathcal{B}^{Op} has the 2-cell:

$$\begin{array}{ccc}
& f & \\
B & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & A \\
& g &
\end{array}$$

Note that this is not canonical, we made a choice in this definition, in fact we have three different opposite bicategories:

- only reverse 1-cells (the case we defined above),
- only reverse 2-cells,
- reverse both 1-cells and 2-cells.

As we shall see towards the end of this thesis, our choice in the definition turns out to have an interesting and useful application.

5.4 Functors between bicategories

Given two bicategories \mathcal{B} and \mathcal{B}' we can of course have morphisms between them.

Ideally, we would like to call these morphisms bifunctors but sadly a bifunctor is already well established taken to be a functor from a product category to some other category

$$F: \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D}.$$

Definition 22. A morphism (2-functor) $F := (F, \phi): \mathcal{B} \rightarrow \mathcal{B}'$ of bicategories \mathcal{B} and \mathcal{B}' consists of the following data:

- Function $F: \text{Obj}(\mathcal{B}) \rightarrow \text{Obj}(\mathcal{B}')$
- Functors $F_{AB}: \mathcal{B}(A, B) \rightarrow \mathcal{B}'(FA, FB)$
- Natural transformations

$$\begin{array}{ccc}
\mathcal{B}(B, C) \times \mathcal{B}(A, B) & \xrightarrow{c} & \mathcal{B}(A, C) \\
\downarrow a' & & \downarrow F_{AC} \\
\mathcal{B}'(FB, FC) \times \mathcal{B}'(FA, FB) & \xrightarrow{c'} & \mathcal{B}'(FA, FC)
\end{array}$$

$$\begin{array}{ccc}
1 & \xrightarrow{I_A} & \mathcal{B}(A, C) \\
id \downarrow & & \downarrow F_{AA} \\
1 & \xrightarrow{I'_{FA}} & \mathcal{B}'(FA, FA)
\end{array}$$

Thus 2-cells $\phi_{gf}: Fg \circ Ff \rightarrow F(g \circ f)$ and $\phi_A: I'_{FA} \rightarrow FI_A$.

Axioms. The Following diagrams commute:

$$\begin{array}{ccccc}
(Fh \circ Fg) \circ Ff & \xrightarrow{\phi * 1} & F(h \circ g) \circ Ff & \xrightarrow{\phi} & F((h \circ g) \circ f) \\
a' \downarrow & & & & \downarrow F_a \\
Fh \circ (Fg \circ Ff) & \xrightarrow{1 * \phi} & Fh \circ F(g \circ f) & \xrightarrow{\phi} & F(h \circ (g \circ f))
\end{array}$$

$$\begin{array}{ccc}
Ff \circ I'_{FA} \xrightarrow{1 * \phi} Ff \circ FI_A \xrightarrow{\phi} F(f \circ I_A) & I'_{FB} \circ Ff \xrightarrow{\phi * 1} FI_B \circ Ff \xrightarrow{\phi} F(I_B \circ f \circ) \\
r' \downarrow & \downarrow Fr & l' \downarrow & \downarrow Fl \\
Ff \xrightarrow{id} Ff & & Ff \xrightarrow{id} Ff &
\end{array}$$

If ϕ_{ABC} and ϕ_A are natural isomorphisms such that $Fg \circ Ff \cong F(g \circ f)$ and $FI \cong I'$ then F is called a morphism (2-functor). In the special case when ϕ_{ABC} and ϕ_A are all identities so that $Fg \circ Ff = F(g \circ f)$ and $FI = I'$ then F is said to be a strict 2-functor.

We will see examples of 2-functors in the section concerning 2-categories.

5.5 Transformations

Given two 2-functors $G, F: \mathcal{B} \rightarrow \mathcal{B}'$ of bicategories \mathcal{B} and \mathcal{B}' we can have transformations between them, i.e. bi/2-(natural transformations) for functors between bicategories.

Definition 23. A transformation

$$\begin{array}{ccc}
& F & \\
\mathcal{B} & \begin{array}{c} \curvearrowright \\ \downarrow \sigma \\ \curvearrowleft \end{array} & \mathcal{B}' \\
& G &
\end{array}$$

where $F := (F, \phi)$ and $G := (G, \psi)$ are morphisms (2-functors), consists of the following data and axioms. Note that $h_*: \mathcal{B}(C, D) \rightarrow \mathcal{B}(C, E)$ here denotes the functor induces by a 1-cell $h: D \rightarrow E$ in \mathcal{B} and similarly for the contravariant $h^*: \mathcal{B}(E, C) \rightarrow \mathcal{B}(D, C)$.

Data

- 1-cells $\sigma_A: FA \rightarrow GA$

- *Natural transformations*

$$\begin{array}{ccc}
\mathcal{B}(A, B) & \xrightarrow{F_{AB}} & \mathcal{B}'(FA, FB) \\
G_{AB} \downarrow & & \downarrow (\sigma_B)_* \\
\mathcal{B}'(GA, GB) & \xrightarrow{(\sigma_A)^*} & \mathcal{B}'(FA, GB)
\end{array}$$

Thus 2-cells $\sigma_f: Gf \circ \sigma_A \rightarrow \sigma_B \circ Ff$

Axioms

The following diagrams commute:

$$\begin{array}{ccccccccccc}
(Gg \circ Gf) \circ \sigma_A & \xrightarrow{a'} & Gg \circ (Gf \circ \sigma_A) & \xrightarrow{1 * \sigma_f} & Gg \circ (\sigma_B \circ Ff) & \xrightarrow{a'^{-1}} & (Gg \circ \sigma_B) \circ Ff & \xrightarrow{\sigma_g * 1} & (\sigma_C \circ Fg) \circ Ff & \xrightarrow{a'} & \sigma_C \circ (Fg \circ Ff) \\
\psi * 1 \downarrow & & & & & & & & & & \downarrow 1 * \phi \\
G(g \circ f) \circ \sigma_A & \xrightarrow{\sigma_{gf}} & & & & & & & & & \sigma_C \circ F(g \circ f)
\end{array}$$

$$\begin{array}{ccc}
I'_{GA} \circ \sigma_A & \xrightarrow{l'} & \sigma_A \xrightarrow{r'^{-1}} \sigma_A \circ I'_{FA} \\
\psi * 1 \downarrow & & \downarrow 1 * \phi \\
GI_A \circ \sigma_A & \xrightarrow{(\sigma_{I_A})} & \sigma_A \circ FI_A
\end{array}$$

If σ_{AB} are all natural isomorphisms then σ is called a strong transformation. If σ_{AB} are all identities then σ is called a strict transformation.

5.6 Modifications

Given two transformations between 2-functors we also have morphisms between transformations, called modifications.

Definition 24. Given transformations $\sigma, \sigma': F \Rightarrow G: \mathcal{B} \rightarrow \mathcal{B}'$, a modification $\Gamma: \sigma \Rrightarrow \sigma'$ consists of the following data and axioms:

Data

• 2-cells $FA \begin{array}{c} \xrightarrow{\sigma_A} \\ \Downarrow \Gamma_A \\ \xrightarrow{\sigma'_A} \end{array} GA$

Axioms

The following diagram commutes:

$$\begin{array}{ccc}
Gf \circ \sigma_A & \xrightarrow{1*\Gamma_A} & Gf \circ \sigma'_A \\
\sigma_f \downarrow & & \downarrow \sigma'_f \\
\sigma_B \circ Ff & \xrightarrow{\Gamma_B*1} & \sigma'_B \circ Ff
\end{array}$$

We have no variants of modifications between transformations but we will see another, fully equivalent, easier definition of this concept in the next section.

6 2-Categories

We will now restrict our study to strict bicategories which we call 2-categories.

6.1 Definitions

Definition 25. A 2-category is a strict bicategory, in the sense that associativity and the unit law for 1-cells holds with equality, not only up to isomorphism.

We can also define 2-categories directly without any reference to bicategories:

Definition 26. A 2-category \mathcal{C} consists of

- Objects (0-cells) i, j, k, \dots
- For every two objects i, j we have a category $\mathcal{C}(i, j)$
 - Objects in $\mathcal{C}(i, j)$ are denoted f, g, h, \dots and are call 1-cells.
 - Morphisms in $\mathcal{C}(i, j)$ are called 2-cells $\alpha, \beta, \gamma \dots$ and composition of such morphism are called vertical composition, denoted by \bullet .
- For every category $\mathcal{C}(i, i)$ there exists a 1-cell $\mathbb{1}_i$ which is an identity morphism with respect to 1-cells.
- For every two categories $\mathcal{C}(i, j), \mathcal{C}(j, k)$ there exists a bifunctor

$$\text{Comp: } \mathcal{C}(j, k) \times \mathcal{C}(i, j) \rightarrow \mathcal{C}(i, k)$$

Which is strictly unital and associative with respect to 1-cells.

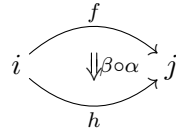
This is of course exactly identical to viewing a 2-category as a special case of a bicategory but it is somewhat easier to work with 2-categories independent of bicategories, therefor we will redefine some bicategorical concepts in the context of 2-categories. However first we will unpack and review the definition.

Let \mathcal{C} be a 2-category. Then we have 0-cells (i, j, k, \dots) , 1-cells (f, g, h, \dots) and 2-cells $(\alpha, \beta, \gamma, \dots)$ together with three types of strict composition. So a typical structural unit in \mathcal{C} looks something like:

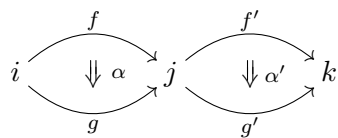
$$\begin{array}{ccc}
 & f & \\
 i & \curvearrowright & j \\
 & \Downarrow \alpha & \\
 & g &
 \end{array}$$

Since every $\mathcal{C}(i, j)$ is a category we have vertical composition of 2-cells in $\mathcal{C}(i, j)$, so for 2-cells

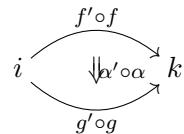
$$\begin{array}{c}
 f \\
 \curvearrowright \\
 i \quad \Downarrow \alpha \quad j \\
 \curvearrowleft \\
 g
 \end{array}
 \text{ and }
 \begin{array}{c}
 g \\
 \curvearrowright \\
 i \quad \Downarrow \beta \quad j \\
 \curvearrowleft \\
 h
 \end{array}$$
 we have a 2-cell:



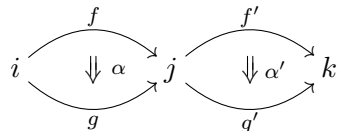
By the bifunctoriality of our composition functor we also have horizontal composition of 1-cells and 2-cells, meaning that if we have:



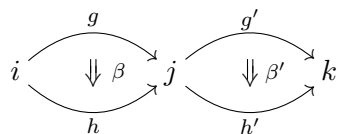
We have the composite:



Again by the bifunctoriality of comp we have that given



and



We have that the following equation holds:

$$(\beta' \bullet \alpha') \circ (\beta \bullet \alpha) = (\alpha' \circ \alpha) \bullet (\beta' \circ \beta): f' \circ f \Rightarrow h' \circ h: i \rightarrow k$$

This equation is sometimes called the interchange law.

We also have that horizontal composition of any two vertical identity 2-cells is again a vertical identity 2-cell.

6.2 The Category of Categories

If **Set**, the category of sets and set theoretical functions, is the canonical example of a (1-)category then **Cat**, the category of categories, functors and natural transformations, is the canonical example of a 2-category.

We already know that **Cat** is a category so we will only provide horizontal and vertical composition of 2-cells (natural transformations). The reader might want to check the details, perhaps in particular that everything holds strictly.

Let \mathcal{C} and \mathcal{D} be categories and $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$ be functors. If $\tau: F \Rightarrow G, \tau': G \Rightarrow H$ are natural transformations we can define the composite natural transformation $\tau' \bullet \tau: F \Rightarrow H$, for any object $i \in \mathcal{C}$ we define the component morphism $(\tau' \bullet \tau)_i := \tau'_i \circ \tau_i$. To see that $\tau' \bullet \tau$ is a natural transformation, let $f: i \rightarrow j$ be a morphism in \mathcal{C} , then we have following diagram:

$$\begin{array}{ccccc} i & & F(i) & \xrightarrow{\tau_i} & G(i) & \xrightarrow{\tau'_i} & H(i) \\ \downarrow f & & \downarrow F(f) & & \downarrow G(f) & & \downarrow H(f) \\ j & & F(j) & \xrightarrow{\tau_j} & G(j) & \xrightarrow{\tau'_j} & H(j) \end{array}$$

Since the two squares commutes, the whole thing commutes so $\tau' \bullet \tau$ is a natural transformation.

For horizontal composition, let α and β be natural transformation:

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{D} \\ \begin{array}{c} \curvearrowright \\ F \\ \downarrow \alpha \\ G \\ \curvearrowleft \end{array} & & \begin{array}{c} \curvearrowright \\ H \\ \downarrow \beta \\ H' \\ \curvearrowleft \end{array} \\ \mathcal{C} & & \mathcal{E} \end{array}$$

We want to define the horizontal composite natural transformation $\beta \circ \alpha: H \circ F \Rightarrow H' \circ G$.

Since $\alpha: F \Rightarrow G$ is a natural transformation, then the following diagram commutes for any morphism $f: i \rightarrow j$ in \mathcal{C} :

$$\begin{array}{ccccc} i & & F(i) & \xrightarrow{\alpha_i} & G(i) \\ \downarrow f & & \downarrow F(f) & & \downarrow G(f) \\ j & & F(j) & \xrightarrow{\alpha_j} & G(j) \end{array}$$

Since $\beta: H \Rightarrow H'$ is a natural transformation and α_x is a morphism in \mathcal{D} for every object x in \mathcal{C} we have the following commutative square:

$$\begin{array}{ccc} HF(i) & \xrightarrow{\beta F i} & H'F(i) \\ H\alpha_i \downarrow & & \downarrow H'\alpha_i \\ HG i & \xrightarrow{\beta G i} & H'G i \end{array}$$

So we define the component morphism of $\beta \circ \alpha$ to be the diagonal of this square.

$$(\beta \circ \alpha)_i := \beta G i \circ H \alpha_i = H' \alpha_i \circ \beta \circ F i$$

To see that this is natural transformation, the following diagram commutes

$$\begin{array}{ccccc} i & HF(i) & \xrightarrow{H\alpha_i} & HG(i) & \xrightarrow{\beta G(i)} & H'G(i) \\ \downarrow f & HFf \downarrow & & \downarrow HG(f) & & \downarrow H'Gf \\ j & HF(j) & \xrightarrow{H\alpha_j} & HG(j) & \xrightarrow{\beta Gj} & H'Gj \end{array}$$

6.3 Another example

Let (S, \cdot, e, \leq) be a monoid with a partial order such that, for all $s, t \in M$

$$s \geq t \implies rs \leq rt \text{ and } sr \leq tr, \forall r \in S.$$

We will now define the 2-category \mathcal{C}_S consisting of

- one formal 0-cell $\#$,
- one category $\mathcal{C}_S(\#, \#)$ with
 - objects (1-cells) $s, t, \dots \in S$
 - $\text{Hom}(s, t) = \begin{cases} \emptyset & \text{if } s \not\leq t \\ m_{s,t} & \text{if } s \leq t \end{cases}$

2-cells $m_{s,t}$ should be understood to be formal morphism.

To see that this a 2-category let $s, t, s', t' \in S$ such that $s \leq t$ and $s' \leq t'$, i.e. we have 2-cells:

$$\begin{array}{ccc} \# & \begin{array}{c} \xrightarrow{s} \\ \Downarrow m_{s,t} \\ \xrightarrow{t} \end{array} & \# \\ \# & \begin{array}{c} \xrightarrow{s'} \\ \Downarrow m_{s',t'} \\ \xrightarrow{t'} \end{array} & \# \end{array}$$

Since $s \leq t$ and $s' \leq t'$ we have that $rs \leq rt$ and $sr \leq tr$, $\forall r \in S$. So we have that $s's \leq s't$ and $s't \leq t't$ by transitivity it follows that $s't \leq t't$ so there exists a 2-cell $m_{s',s,t't}$ which implies that horizontal composition is well defined, $m_{s',t'} \circ m_{s,t} := m_{s',s,t't}$.

Since $s \leq s$ for every $s \in S$ and in particular we have an identity 2-cell $m_{e,e}$ where e is the identity element in S . $s \leq t \implies m_{e,e} \circ m_{s,t} = m_{es,et} = m_{s,t}$.

Associativity of composition of arrows in the horizontal category of 2-cells follows from associativity of S .

6.4 2-functors

Functors between 2-categories are morphisms between (strict) bicategories which we defined in the previous section. However it is better and easier to consider 2-functors in a purely 2-categorical setting.

A 2-functor of 2-categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is a triple sending 0-cells to 0-cells, 1-cells to 1-cells and 2-cells to 2-cells such that all three types of composition and all identities are preserved.

This means that a typical structural unit

$$\begin{array}{ccc}
 & f & \\
 i & \curvearrowright & j \\
 & \Downarrow \alpha & \\
 & g &
 \end{array}$$

in \mathcal{C} is sent via F to a structural unit in \mathcal{C}

$$\begin{array}{ccc}
 & F(f) & \\
 F(i) & \curvearrowright & F(j) \\
 & \Downarrow F\alpha & \\
 & F(g) &
 \end{array}$$

A 2-functor must also respect composition of 1-cells, identity 1-cells, vertical identity 2-cells, vertical composition of 2-cells and horizontal composition of 2-cells. Note that we define a 2-functor to here always be a strict 2-functor.

6.5 The Yoneda 2-functor

We will now provide an example of a 2-functor. Let \mathcal{C} be a 2-category and i a 0-cell in \mathcal{C} we can then define a 2-functor $\mathcal{C}(i, _): \mathcal{C} \rightarrow \mathbf{Cat}$.

- Any 0-cell j is mapped to the vertical category $\mathcal{C}(i, j)$
- 1-cells $f: j \rightarrow k$ is mapped to the functor

$$\begin{aligned}
 f \circ _ &: \mathcal{C}(i, j) \rightarrow \mathcal{C}(i, k) \\
 g &\mapsto f \circ g \\
 (\gamma: g \Rightarrow g') &\mapsto \mathbb{1}_f \circ \gamma
 \end{aligned}$$

- A 2-cell $\alpha: f \Rightarrow g$ in \mathcal{C} is mapped to horizontal post-composition:

$$(\alpha: f \Rightarrow g) \mapsto \alpha \circ _$$

The 2-functoriality of the Yoneda functor follows directly from the 2-categorical structure of \mathcal{C} , in particular by the fact that any homset is a category and that horizontal and vertical composition of 2-cells are well behaved (by definition), so post-composition gives well defined functoriality.

6.6 Transformations between 2-functors

As with bicategories, we have both strong and weak transformations (2-natural transformations) between 2-functors but we will only work with strong such transformations so we take transformation to mean strong transformation.

Definition 27. Let \mathcal{C} and \mathcal{D} be 2-categories and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two 2-functors. A transformation $\theta: F \Rightarrow G$ induces, for every 0-cell i in \mathcal{C} , a 1-cell $\theta_i: F(i) \rightarrow G(i)$ in \mathcal{D} such that, for every for every 2-cell $\alpha: f \Rightarrow g: i \rightarrow j$ in \mathcal{C} , the following diagram commutes in the strictest possible sense:

$$\begin{array}{ccc}
 F(i) & \begin{array}{c} \xrightarrow{Ff} \\ \Downarrow F\alpha \\ \xrightarrow{Fg} \end{array} & F(j) \\
 \theta_i \downarrow & & \downarrow \theta_j \\
 G(i) & \begin{array}{c} \xrightarrow{Gf} \\ \Downarrow G\alpha \\ \xrightarrow{Gg} \end{array} & G(j)
 \end{array}$$

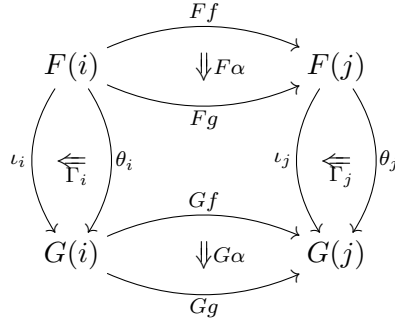
We can, of course, have weak transformations between (strict) 2-functors as well, but we will restrict our study mostly to strong transformations.

6.7 Modifications of transformations

Modifications, morphisms between transformations, can only be defined in one way, independent of the strictness of the functors and transformations involved, we have no variants.

For completeness we provide the definition in the context of 2-categories.

Definition 28. Let \mathcal{C} and \mathcal{D} be 2-categories and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two 2-functors and $\theta, \iota: F \Rightarrow G$ be two transformations. A modification $\Gamma: \theta \Rrightarrow \iota$ induces, for every 0-cell i in \mathcal{C} , a 2-cell $\Gamma_i: \theta_i \Rightarrow \iota_i$ in \mathcal{D} such that, for every for every 2-cell $\alpha: f \Rightarrow g: i \rightarrow j$ in \mathcal{C} , the following diagram commutes:



It is hard to find elementary examples of modifications, but we shall see some in the proof of the 2-categorical Yoneda lemma.

6.8 Mor(F,G) - Higher dimensional morphism categories

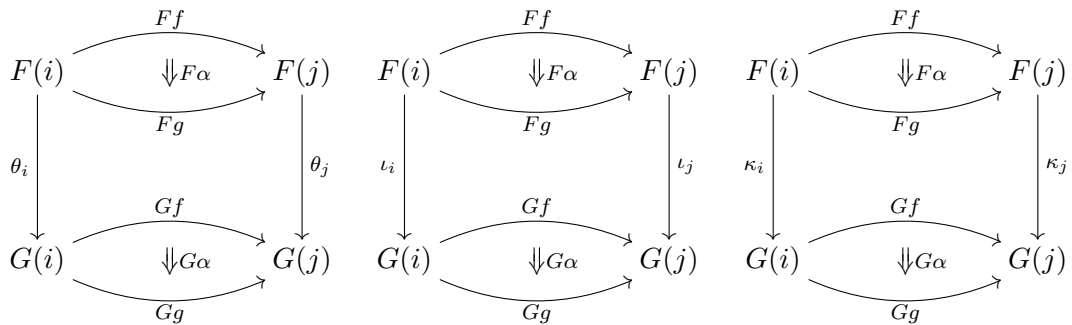
Recall that given two categories \mathcal{A} and \mathcal{B} we can form the functor category $\text{Func}(\mathcal{A}, \mathcal{B})$ with objects as functors between \mathcal{A} and \mathcal{B} and as morphisms natural transformations between such functors.

Similarly we can form construct higher dimensional analogue of the functor category $\text{Func}(\mathcal{A}, \mathcal{B})$. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two strict 2-functors between 2-categories. By $\text{Mor}(F, G)$ we denote the category with:

- Objects: Strong transformations between F and G .
- Morphisms: Modifications of such transformations.

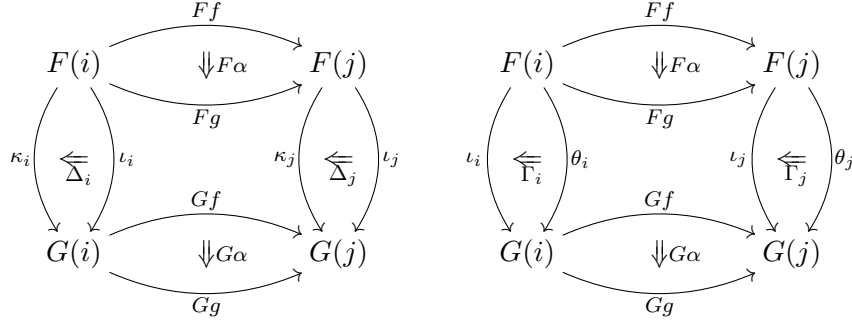
We claim that $\text{Mor}(F, G)$ is a category:

Proof. Let $\theta, \iota, \kappa: F \Rightarrow G$ be transformations. By definition, for every 2-cell $\alpha: f \Rightarrow g: i \rightarrow j$ in \mathcal{C} , the following three diagrams commutes:



Let $\Gamma: \theta \Rightarrow \iota$ and $\Delta: \iota \Rightarrow \kappa$ be modifications then for $\alpha: f \Rightarrow g$ in \mathcal{C} as above we have the

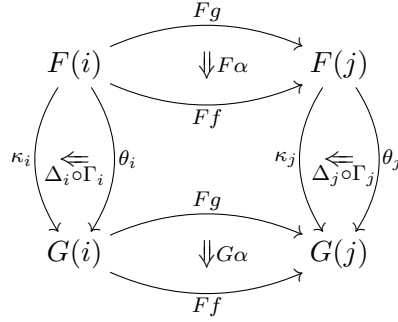
following two commutative diagrams:



The composite $\Delta \circ \Gamma: \theta \Rightarrow \kappa$ is defined in the obvious way by defining:

$$(\Delta \circ \Gamma)_k := \Delta_k \circ \Gamma_k: \theta_k \Rightarrow \kappa_k \text{ for } k \in \mathcal{C}.$$

Since the diagram



clearly commutes we have that $\Delta \circ \Gamma: \theta \Rightarrow \kappa$ is a modification so composition of modifications is a modification.

For every transformation $\theta: F \Rightarrow G$ we have an identity modification $\mathbb{1}_\theta: \theta \Rightarrow \theta$ with for $j \in \mathcal{C}$, $(\mathbb{1}_\theta)_k$ is defined to be the vertical identity $\mathbb{1}_{\theta_k}$ of θ_k . To see that this is an identity let $\Gamma: \theta \rightarrow \iota$ and $\Delta: \kappa \rightarrow \theta$ be modifications then by vertical composition in \mathcal{D} :

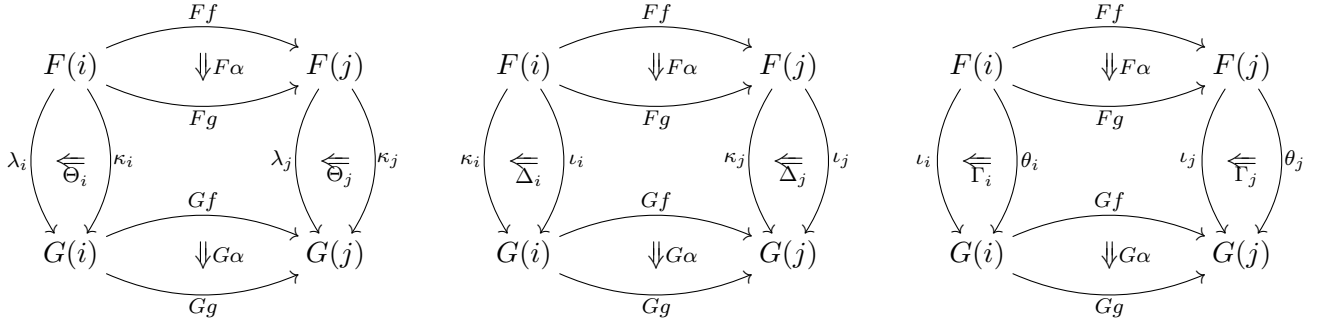
- $\Gamma \circ \mathbb{1}_\theta = \Gamma$ since for any $k \in \mathcal{C}$, $(\Gamma \circ \mathbb{1}_\theta)_k = \Gamma_k \circ (\mathbb{1}_\theta)_k = \Gamma_k \circ \mathbb{1}_{\theta_k} = \Gamma_k \implies \Gamma \circ \mathbb{1}_\theta = \Gamma$.
- Similarly $\mathbb{1}_\theta \circ \Delta = \Delta$ since $\mathbb{1}_{\theta_k} \circ \Delta_k = \Delta_k$.

Finally, we show that composition of modifications is associative.

Let $\Gamma: \theta \Rightarrow \iota$, $\Delta: \iota \Rightarrow \kappa$ and $\Theta: \kappa \Rightarrow \lambda$ be modifications. We want to show that

$$\Theta \circ (\Delta \circ \Gamma) = (\Theta \circ \Delta) \circ \Gamma.$$

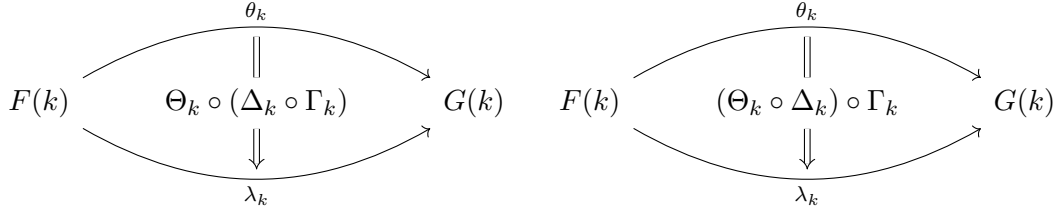
By definition we have:



To show that $\Theta \circ (\Delta \circ \Gamma) = (\Theta \circ \Delta) \circ \Gamma$ is to show that $[\Theta \circ (\Delta \circ \Gamma)]_k = [(\Theta \circ \Delta) \circ \Gamma]_k$ for all objects k in \mathcal{C} . Evaluating these we get:

- $[\Theta \circ (\Delta \circ \Gamma)]_k = \Theta_k \circ (\Delta \circ \Gamma)_k = \Theta_k \circ (\Delta_k \circ \Gamma_k)$
- $[(\Theta \circ \Delta) \circ \Gamma]_k = (\Theta \circ \Delta)_k \circ \Gamma_k = (\Theta_k \circ \Delta_k) \circ \Gamma_k$

The equality $\Theta_k \circ (\Delta_k \circ \Gamma_k) = (\Theta_k \circ \Delta_k) \circ \Gamma_k$ holds since all terms are arrows in the vertical category of 2-cells $\mathbf{T}(F(k), G(k))$ in \mathcal{D} .



This implies that $\Gamma \circ (\Delta \circ \Theta) = (\Gamma \circ \Delta) \circ \Theta$ so we have established associativity of composition of modifications thus we have proved that $\text{Mor}(F, G)$ is a category. \square

Note that we can form a 2-category **2-Cat** consisting of 2-categories, 2-functors and transformations but the result we just proved shows that **2-Cat** is in fact a 3-category.

6.9 Yoneda Lemma for 2-categories

Recall Yoneda lemma for 1-categories. Given a locally small category \mathcal{C} , a functor $F: \mathcal{C} \rightarrow \mathbf{Set}$ and an object $i \in \mathcal{C}$, we have a natural bijection of sets:

$$\text{Func}(\mathcal{C}(i, _), F) \cong F(i)$$

Let \mathcal{C} be a 2-category, $F: \mathcal{C} \rightarrow \mathbf{Cat}$ a 2-functor and i a 0-cell in \mathcal{C} .

Theorem 9. *$\text{Mor}(\mathcal{C}(i, _), F) \cong F(i)$ is an isomorphism of categories.*

Proof. To prove this we have to define two functors and show that they are each others inverses.

Since F is a functor from the 2-category \mathcal{C} to \mathbf{Cat} it follows that $F(i)$ is a category we have to define a functor:

$$\Psi: \text{Mor}(\mathcal{C}(i, _), F) \rightarrow F(i)$$

For a transformation $\theta: \mathcal{C}(i, _) \Rightarrow F$ Yoneda Lemma directly gives us a way to map θ to a object in $F(i)$, namely the evaluation of the arrow $\theta_i: \mathcal{C}(i, i) \rightarrow F(i)$ at $\mathbb{1}_i$. To see that this is an object in $F(i)$ note that θ_i is a 1-cells in \mathbf{Cat} i.e. a functor and since $\mathbb{1}_i$ is an object in the category $\mathcal{C}(i, i)$ we must have that $\theta_i(\mathbb{1}_i)$ is an object in $F(i)$. We define:

$$\Psi(\theta) := \theta_i(\mathbb{1}_i)$$

Let $\Gamma: \theta \Rightarrow \iota$ be a modification. We have to find a suitable morphism between $\theta_i(\mathbb{1}_i)$ and $\iota_i(\mathbb{1}_i)$ in $F(i)$. We define

Since $\theta_i, \iota_i: \mathcal{C}(i, i) \rightarrow F(i)$ are functors, the 2-cell $\Gamma_i: \theta_i \Rightarrow \iota_i$ induced by Γ must be a natural transformation. Thus if we evaluate Γ_i at $\mathbb{1}_i$, here denoted $\Gamma_{i, \mathbb{1}_i}$, we get the desired morphism between $\theta_i(\mathbb{1}_i)$ and $\iota_i(\mathbb{1}_i)$ in $F(i)$. We define

$$\Psi(\Gamma) := \Gamma_{i, \mathbb{1}_i}$$

It is clear that Ψ is well defined. Now we show that it is functorial.

- Let $\mathbb{1}_\theta: \theta \Rightarrow \theta$ be the identity modification of a transformation $\theta: \mathcal{C}(i, _) \Rightarrow F$. Then $\Psi(\mathbb{1}_\theta) = (\mathbb{1}_\theta)_{i, \mathbb{1}_i} = \mathbb{1}_{\theta_i(\mathbb{1}_i)} = \mathbb{1}_{\Psi(\theta)}: \theta_i(\mathbb{1}_i) \rightarrow \theta_i(\mathbb{1}_i)$.
- To show that Ψ respects composition let $\Gamma: \theta \Rightarrow \iota, \Delta: \iota \Rightarrow \kappa$ be modifications. Then

$$\Psi(\Delta \circ \Gamma) = (\Delta \circ \Gamma)_{i, \mathbb{1}_i} = (\Delta_i \circ \Gamma_i)[\mathbb{1}_i] = \Delta_{i, \mathbb{1}_i} \circ \Gamma_{i, \mathbb{1}_i} = \Psi(\Delta) \circ \Psi(\Gamma).$$

So $\Psi: \text{Mor}(\mathcal{C}(i, _), F) \rightarrow F(i)$ is functorial.

We now need a functor in the other direction. For objects a, b and $\gamma \in F(i)(a, b)$, we define:

$$\Phi: F(i) \rightarrow \text{Mor}(\mathcal{C}(i, _), F).$$

$$\begin{aligned}
a &\mapsto \xi^a: \mathcal{C}(i, _) \Rightarrow F \\
\gamma &\mapsto \Lambda^\gamma: \xi^a \Rrightarrow \xi^b
\end{aligned}$$

An object a is mapped to the transformation $\xi^a: \mathcal{C}(i, _) \Rightarrow F$ which, for every object x in \mathcal{C} , induces the component functor:

$$\begin{aligned}
\xi_x^a: \mathcal{C}(i, x) &\rightarrow F(x) \\
- &\mapsto F()[a]
\end{aligned}$$

This also comes from (the inverse of) Yoneda Lemma but this is not only a natural transformation we claim that ξ^a is a transformation, that is, for every 2-cell $\alpha: f \Rightarrow g \in \mathcal{C}(j, k)$ where j, k are arbitrary objects in \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}(i, j) & \begin{array}{c} \xrightarrow{f \circ _} \\ \Downarrow \alpha \circ _ \\ \xrightarrow{g \circ _} \end{array} & \mathcal{C}(i, k) \\
\downarrow \xi_j^a & & \downarrow \xi_k^a \\
F(j) & \begin{array}{c} \xrightarrow{Ff} \\ \Downarrow F\alpha \\ \xrightarrow{Fg} \end{array} & F(k)
\end{array}$$

Which holds since for any $h \in \mathcal{C}(i, j)$ we have:

- $\xi_k^a \circ (f \circ h) = \xi_k^a(f \circ h) = F(f \circ h)[a] = F(f) \circ F(h)[a] = F(f) \circ \xi_j^a(h)$,
- $\xi_k^a \circ (g \circ h) = F(g) \circ \xi_j^a(h)$ similarly, and
- $\xi_k^a \circ (\alpha \circ h) = F(\alpha \circ h)[a] = F(\alpha) \circ F(h)[a] = F(\alpha) \circ \xi_j^a(h)$.

Which shows that ξ^a is a transformation.

Now let $\gamma: a \rightarrow b$ be a morphism in $F(i)$. We defined $\Phi(\gamma)$ to be Λ^γ with $\Lambda_x^\gamma := F(_)[\gamma]$ for every $x \in \mathcal{C}$. To see that this works: Note that if we take some $h \in \mathcal{C}(i, x)$ then $Fh: F(i) \rightarrow F(x)$ is a functor (since since F is a functor from \mathcal{C} to \mathbf{Cat}) then $F(h)[\gamma]: F(h)[a] \rightarrow F(h)[b]$ is a morphism in $F(x)$.

$$\begin{array}{ccc}
& \xrightarrow{\xi_x^a = F(_)[a]} & \\
\mathcal{C}(i, x) & \begin{array}{c} \Downarrow \Lambda_x^\gamma := F(_)[\gamma] \\ \Downarrow \end{array} & F(x) \\
& \xrightarrow{\xi_x^b = F(_)[b]} &
\end{array}$$

We also have to show that $\Lambda^\gamma: \xi^a \Rrightarrow \xi^b$ is in fact a modification. By definition this means that for all 2-cells $\alpha: f \Rightarrow g \in \mathcal{C}(j, k)$ the diagram

$$\begin{array}{ccc}
& \mathcal{C}(i, j) & \xrightarrow{f \circ _} & \mathcal{C}(i, k) \\
& \downarrow \alpha & \Downarrow \alpha \circ _ & \downarrow \\
& \mathcal{C}(i, j) & \xrightarrow{g \circ _} & \mathcal{C}(i, k) \\
& \downarrow \alpha & \Downarrow \alpha \circ _ & \downarrow \\
& \mathcal{C}(i, j) & \xrightarrow{F(f)} & \mathcal{C}(i, k) \\
& \downarrow \alpha & \Downarrow F(\alpha) & \downarrow \\
& \mathcal{C}(i, j) & \xrightarrow{F(g)} & \mathcal{C}(i, k)
\end{array}$$

commutes. In other words, given some $h \in \mathcal{C}(i, j)$, we have to show that

$$F(\alpha) \circ \Lambda_j^\gamma[h] = \Lambda_k^\gamma \circ (\alpha \circ h).$$

- $\Lambda_k^\gamma \circ (\alpha \circ h) = F(_)[\gamma] \circ (\alpha \circ h) = F(\alpha \circ h)[\gamma] = F(\alpha) \circ F(h)[\gamma]$ and,
- $F(\alpha) \circ \Lambda_j^\gamma[h] = F(\alpha) \circ F(h)[\gamma]$.

The diagram commutes which implies that $\Lambda^\gamma: \xi^a \Rightarrow \xi_b$ is a modification. So Φ is well defined on both objects and morphisms.

Next we prove that Φ is functorial.

First let $a \in F(i)$ be some object and consider its identity morphism $\mathbb{1}_a \in F(i)(a, a)$. We want to show that $\Phi(\mathbb{1}_a) = \mathbb{1}_{\Phi(a)}: \Phi(a) \Rightarrow \Phi(a)$.

We have that $\Phi(\mathbb{1}_a) = \Lambda_j^{\mathbb{1}_a}: \xi^a \Rightarrow \xi^a$

$$\begin{array}{ccc}
& \xi_j^a = F(_)[a] & \\
& \downarrow & \\
\mathcal{C}(i, j) & \xrightarrow{\Lambda_j^{\mathbb{1}_a} = F(_)[\mathbb{1}_a]} & F(j) \\
& \downarrow & \\
& \xi_j^a = F(_)[b] &
\end{array}$$

By the functoriality of F and the fact that, for every 1-cell $h \in \mathcal{C}(i, j)$, we have a functor $F(h): F(i) \rightarrow F(j)$, it follows that:

$$\Lambda_j^{\mathbb{1}_a}[h] = F(h)[\mathbb{1}_a] = \mathbb{1}_{F(h)[a]} = \mathbb{1}_{\xi_j^a[h]} \implies \Phi(\mathbb{1}_a) = \mathbb{1}_{\xi^a} = \mathbb{1}_{\Phi(a)}.$$

Now let $\gamma: a \rightarrow b, \delta: b \rightarrow c$ be morphisms in $F(i)$, we want to show that $\Phi(\delta \circ \gamma) = \Phi(\delta) \circ \Phi(\gamma)$.

$$\Phi(\delta \circ \gamma) = \Lambda^{\delta \circ \gamma}: \xi^a \Rightarrow \xi_c$$

Again, let $h \in \mathcal{C}(i, j)$ be an arbitrary morphism. Then

$$\Phi(\delta \circ \gamma)[h] = \Lambda^{\delta \circ \gamma}[h] = F(h)[\delta \circ \gamma]: \xi_j^a \Rightarrow \xi_j^c.$$

Yet again, by functoriality of $F(h)$:

$$F(h)[\delta \circ \gamma] = F(h)[\delta] \circ F(h)[\gamma] = (F(_)[\delta] \circ F(_)[\gamma])[h] = (\Lambda^\delta \circ \Lambda^\gamma)[h].$$

Which shows that $\Phi(\delta \circ \gamma) = \Phi(\delta) \circ \Phi(\gamma)$ and hence $\Phi: F(i) \rightarrow \text{Mor}(\mathcal{C}(i, _), F)$ is functorial.

We now have two functors:

$$\Psi: \text{Mor}(\mathcal{C}(i, _), F) \rightarrow F(i)$$

$$\Phi: F(i) \rightarrow \text{Mor}(\mathcal{C}(i, _), F).$$

Our goal is to show that $\text{Mor}(\mathcal{C}(i, _), F) \cong F(i)$ which means that we have to show that composition of our functors gives us identity functors.

$$\begin{array}{ccc} & \Psi & \\ & \curvearrowright & \\ \Phi \circ \Psi = \text{id}_{\text{Mor}(\mathcal{C}(i, _), F)} & \text{Mor}(\mathcal{C}(i, _), F) & F(i) \\ & \curvearrowleft & \\ & \Phi & \\ & \Psi \circ \Phi = \text{id}_{F(i)} & \end{array}$$

We begin with the easy direction, showing that

$$\Psi \circ \Phi = \text{Id}_{F(i)}.$$

- Let a be an object in $F(i)$ then

$$\Psi \circ \Phi(a) = \Psi(\xi^a) = \xi_i^a[\mathbb{1}_a] = F(\mathbb{1}_i)[a] = \text{Id}_{F(i)}(a) = a.$$

- Let $\gamma: a \rightarrow b$ be a morphism in $F(i)$. Applying the composite functor $\Psi \circ \Phi$ to γ , we get:

$$\Psi \circ \Phi(\gamma) = \Psi(\Lambda^\gamma) = \Lambda_{i, \mathbb{1}_i}^\gamma = \Lambda_i^\gamma[\mathbb{1}_i] = F(\mathbb{1}_i)[\gamma] = \text{Id}_{F(i)}(\gamma) = \gamma.$$

This proves that $\Psi \circ \Phi = \text{Id}_{F(i)}$.

The other direction $\text{id}_{\text{Mor}(\mathcal{C}(i, _), F)} = \Phi \circ \Psi: \text{Mor}(\mathcal{C}(i, _), F) \rightarrow \text{Mor}(\mathcal{C}(i, _), F)$ is slightly more involved.

Let θ be an object in $\text{Mor}(\mathcal{C}(i, _), F)$ i.e. a transformation $\theta: \mathcal{C}(i, _) \Rightarrow F$. Then we have

$$\Phi \circ \Psi(\theta) = \Phi(\theta_i(\mathbb{1}_i)) = \xi^{\theta_i(\mathbb{1}_i)}.$$

We know for sure that θ and $\xi^{\theta_i(\mathbb{1}_i)}$ are transformations, meaning that for every 2-cell $\alpha: f \Rightarrow g$

in \mathcal{C} we have two commutative diagrams:

$$\begin{array}{ccc}
\begin{array}{ccc}
f & & \\
\Downarrow \alpha & & \\
g & &
\end{array} &
\begin{array}{ccc}
\mathcal{C}(i, j) & \xrightarrow{f \circ _} & \mathcal{C}(i, k) \\
\downarrow \theta_j & \Downarrow \alpha \circ _ & \downarrow \theta_k \\
F(j) & \xrightarrow{Ff} & F(k) \\
\downarrow Fg & \Downarrow F\alpha & \\
F(j) & \xrightarrow{Fg} & F(k)
\end{array} &
\begin{array}{ccc}
\mathcal{C}(i, j) & \xrightarrow{f \circ _} & \mathcal{C}(i, k) \\
\downarrow \xi_j^{\theta_i(\mathbb{1}_i)} & \Downarrow \alpha \circ _ & \downarrow \xi_k^{\theta_i(\mathbb{1}_i)} \\
F(j) & \xrightarrow{Ff} & F(k) \\
\downarrow Fg & \Downarrow F\alpha & \\
F(j) & \xrightarrow{Fg} & F(k)
\end{array}
\end{array}$$

We have to show that $\theta = \xi^{\theta_i(\mathbb{1}_i)}$ which is equivalent to showing that $\theta_j = \xi_j^{\theta_i(\mathbb{1}_i)}$, for all $j \in \mathcal{C}$.

If we choose some $h \in \mathcal{C}(i, j)$, we have, by definition, the following two equalities:

$$\begin{cases}
\theta_k \circ (\alpha \circ h) = F(\alpha) \circ \theta_j[h]: \theta_k \circ (f \circ h) = F(f) \circ \theta_j \Rightarrow \theta_k \circ (g \circ h) = F(g) \circ \theta_j[h] \\
\xi_k^{\theta_i(\mathbb{1}_i)} \circ (\alpha \circ h) = F(\alpha) \circ \xi_j^{\theta_i(\mathbb{1}_i)}[h]: \xi_k^{\theta_i(\mathbb{1}_i)} \circ (f \circ h) = F(f) \circ \xi_j^{\theta_i(\mathbb{1}_i)}[h] \Rightarrow \xi_k^{\theta_i(\mathbb{1}_i)} \circ (g \circ h) = F(g) \circ \xi_j^{\theta_i(\mathbb{1}_i)}[h]
\end{cases}$$

First we show that the equality $F(\alpha) \circ \theta_j[h] = F(\alpha) \circ \xi_j^{\theta_i(\mathbb{1}_i)}[h]$ holds. Since θ is a transformation, it is also a natural transformation in the usual 1-categorical sense. Thus, for the same $h: i \rightarrow j$ in \mathcal{C} , as above, we have that

$$\begin{array}{ccc}
i & \mathcal{C}(i, i) & \xrightarrow{\theta_i} & F(i) \\
\downarrow h & h \circ _ & & \downarrow F(h) \\
j & \mathcal{C}(i, j) & \xrightarrow{\theta_j} & F(j)
\end{array}$$

commutes, i.e. $\theta_j \circ (h \circ _) = Fh \circ \theta_i$. In particular, take $\mathbb{1}_i \in \mathcal{C}(i, i)$ and we get that

$$\theta_j \circ (h \circ \mathbb{1}_i) = \theta_j \circ h = \theta_j(h) = F(h) \circ \theta_i(\mathbb{1}_i).$$

This implies that $F(\alpha) \circ \theta_j(h) = F(\alpha) \circ F(h) \circ \theta_i(\mathbb{1}_i) = F(\alpha) \circ \xi_j^{\theta_i(\mathbb{1}_i)}[h]$.

Similarly we can show that $\theta_k \circ (g \circ h) = \xi_k^{\theta_i(\mathbb{1}_i)} \circ (g \circ h)$.

The equality of 2-cells follows since we have:

$$\theta_k \circ (\alpha \circ h) = F(\alpha) \circ \theta_j[h]$$

and

$$\xi_k^{\theta_i(\mathbb{1}_i)} \circ (\alpha \circ h) = F(\alpha) \circ \xi_j^{\theta_i(\mathbb{1}_i)}[h].$$

Thus we have have that $\Phi \circ \Psi(\theta) = \theta$.

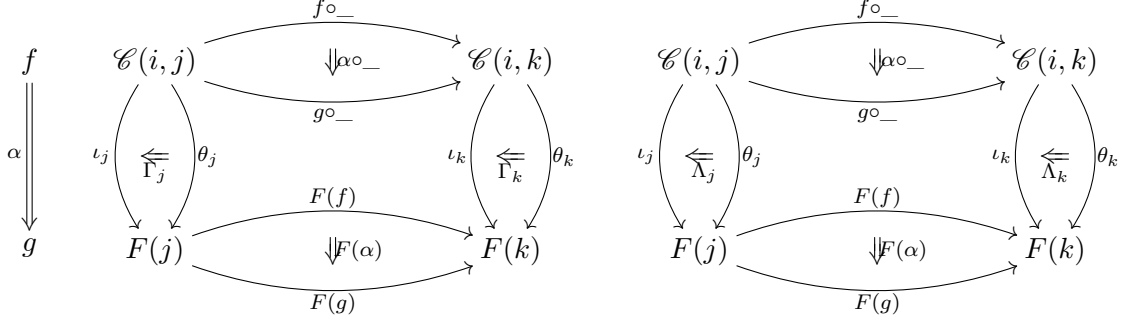
Finally, for any modification $\Gamma: \theta \Rrightarrow \iota$ in $\text{Mor}(\mathcal{C}(i, _), F)$, we have to prove that $\Phi \circ \Psi(\Gamma) = \Gamma$.

We have

$$\Phi \circ \Psi(\Gamma) = \Phi(\Gamma_{i, \mathbb{1}_i}) = \Lambda^{\Gamma_{i, \mathbb{1}_i}}.$$

To show $\Lambda^{\Gamma_{i, \mathbb{1}_i}} = \Gamma$, we have to show that $\Lambda_j^{\Gamma_{i, \mathbb{1}_i}} = \Gamma_j$, for all $j \in \mathcal{C}$.

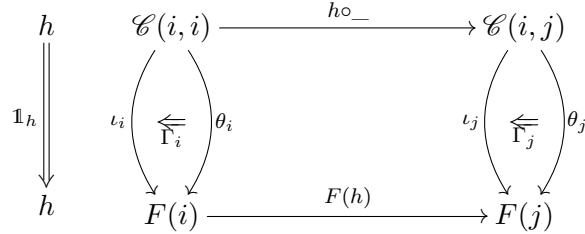
By definition, we have that a typical structural unit $\alpha: f \Rightarrow g$ gives rise to the two commutative diagrams:



In other words, given some $h \in \mathcal{C}(i, j)$, we have the following two equalities of 2-cells:

$$\begin{cases} \Gamma_k \circ (\alpha \circ h) = F(\alpha) \circ \Gamma_j[h] \\ \Lambda_k^{\Gamma_{i, \mathbb{1}_i}} \circ (\alpha \circ h) = F(\alpha) \circ \Lambda_j^{\Gamma_{i, \mathbb{1}_i}}[h] \end{cases}$$

So, it is sufficient to prove that $F(\alpha) \circ \Gamma_j[h] = F(\alpha) \circ \Lambda_j^{\Gamma_{i, \mathbb{1}_i}}[h]$. To show this, we use the fact that Γ is a modification and consider the same h as before together with its vertical identity 2-cell $\mathbb{1}_h: h \Rightarrow h$ to get the commutative diagram:



We can now make yet another choice and take $\mathbb{1}_i \in \mathcal{C}(i, i)$ together with its vertical identity to get

$$\Gamma_j \circ h = F(h) \circ \Gamma_{i, \mathbb{1}_i}.$$

We want to show that $F(\alpha) \circ \Gamma_j[h] = F(\alpha) \circ \Lambda_j^{\Gamma_{i, \mathbb{1}_i}}[h]$. Using the above equality, we can evaluate the right hand side:

$$F(\alpha) \circ \Lambda_j^{\Gamma_{i, \mathbb{1}_i}}[h] = F(\alpha) \circ (F(h)[\Gamma_{i, \mathbb{1}_i}]) = F(\alpha) \circ (F(h) \circ \Gamma_{i, \mathbb{1}_i}) = F(\alpha) \circ (\Gamma_j \circ h) = F(\alpha) \circ \Gamma_j[h]$$

Similarly for any 2-cell in $\mathcal{C}(i, j)$.

So, we have shown that $\Lambda_j^{\Gamma_i, \mathbb{1}_i} = \Gamma_j$ which implies that $\Gamma = \Lambda^{\Gamma_i, \mathbb{1}_i}$ and hence $\Phi \circ \Psi(\Gamma) = \Gamma$ and $\Phi \circ \Psi = \text{id}_{\text{Mor}(\mathcal{C}(i, _), F)}$. Thus we have an isomorphism between categories $\text{Mor}(\mathcal{C}(i, _), F) \cong F(i)$ and we are done.

Moreover, if we consider i and F in $\Psi: \text{Mor}(\mathcal{C}(i, _), F) \rightarrow F(i)$ to be parameters, then by construction, Ψ is 2-natural in these arguments.

□

6.10 2-Categorical Yoneda Embedding

We have just proved that $\text{Mor}(\mathcal{C}(i, _), F) \cong F(i)$. By defining F to be another (2-categorical) homfunctor, $F := \mathcal{C}(j, _) \rightarrow \mathbf{Cat}$. we get that

$$\text{Mor}(\mathcal{C}(i, _), \mathcal{C}(j, _)) \cong \mathcal{C}(j, i).$$

Recall that we defined the opposite bicategory of a bicategory to be the bicategory of reversed 1-cells. Since every 2-category is a bicategory \mathcal{C}^{Op} have reversed 1-cells.

We have that the vertical category $\mathcal{C}(j, i)$ of \mathcal{C}^{Op} is isomorphic to the category $\text{Mor}(\mathcal{C}(i, _), \mathcal{C}(j, _))$ meaning that reversed arrows and their 2-cells correspond exactly to transformations between 2-categorical homfunctors and modifications between such homfunctors. We can use this to construct a 2-categorical Yoneda embedding:

$$\begin{aligned} Y: \mathcal{C}^{\text{Op}} &\rightarrow \text{Mor}(\mathcal{C}, \mathbf{Cat}) \\ j &\mapsto \mathcal{C}(j, _) \\ (f: j \rightarrow k) &\mapsto _ \circ f \\ (\alpha: f \rightarrow g) &\mapsto _ \circ \alpha \end{aligned}$$

As a side note, $\text{Mor}(\mathcal{C}(i, _), F)$ is a full subcategory of $\text{MOR}(\mathcal{C}(i, _), F)$ which we define to be the category of weak transformations and modifications.

We claim, without proof, that $\text{MOR}(\mathcal{C}(i, _), F) \simeq F(i)$ is an equivalence of categories, giving rise to the following interesting diagram:

$$\begin{array}{ccc} \text{Mor}(\mathcal{C}(i, _), F) & & \\ \downarrow & \searrow \cong & \\ \text{MOR}(\mathcal{C}(i, _), F) & \xrightarrow{\simeq} & F(i) \end{array}$$

We also have an even more general situation if we consider bicategories.

6.11 Bicategorical Coherence

We proved Yoneda lemma for 2-categories in this section, this is slightly easier than proving the more general statement for bicategories but it turns out that every bicategory is biequivalent to a strict 2-category, meaning that it is in some sense enough to consider the strictest possible case. In this subsection we will give a very rough sketch of this result.

Let \mathcal{B} and \mathcal{B}' be bicategories. A biequivalence from \mathcal{B} to \mathcal{B}' consists of a pair of bicategorical functors $F: \mathcal{B} \rightarrow \mathcal{B}'$, $G: \mathcal{B}' \rightarrow \mathcal{B}$ together with an equivalence $1 \rightarrow G \circ F$ and an equivalence $F \circ G \rightarrow 1$ in the two bicategories consisting of bicategorical functors, strong transformations and modifications.

Another way to define a biequivalence $F: \mathcal{B} \rightarrow \mathcal{B}'$ is to require F to be a local equivalence and surjective -up-to-equivalence of objects.

To show that every bicategory is biequivalent to a strict 2-category we take the covariant bicategorical Yoneda (embedding) map Y of a bicategory \mathcal{B} and define \mathcal{B}' to be the full image of Y and define $Y': \mathcal{B} \rightarrow \mathcal{B}'$ to be the restriction of Y to \mathcal{B}' . Then Y' is a homomorphism, surjective on 0-cells and locally an equivalence. Since \mathcal{B}' is a 2-category the result follows.

Apparently, $n = 2$ is the highest n for which this holds true, i.e. we have some tricategory not being triequivalent to some strict 3-category.

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