

$\mathcal{N} = 2$ supersymmetric gauge theory on connected sums of $S^2 \times S^2$

Guido Festuccia,^a Jian Qiu,^{a,b} Jacob Winding^a and Maxim Zabzine^a

^a*Department of Physics and Astronomy, Uppsala University,
Box 516, SE-75120 Uppsala, Sweden*

^b*Mathematics Institute, Uppsala University,
Box 480, SE-75106 Uppsala, Sweden*

E-mail: guido.festuccia@physics.uu.se, jian.qiu@math.uu.se,
jacob.winding@physics.uu.se, Maxim.Zabzine@physics.uu.se

ABSTRACT: We construct 4D $\mathcal{N} = 2$ theories on an infinite family of 4D toric manifolds with the topology of connected sums of $S^2 \times S^2$. These theories are constructed through the dimensional reduction along a non-trivial $U(1)$ -fiber of 5D theories on toric Sasaki-Einstein manifolds. We discuss the conditions under which such reductions can be carried out and give a partial classification result of the resulting 4D manifolds. We calculate the partition functions of these 4D theories and they involve both instanton and anti-instanton contributions, thus generalizing Pestun's famous result on S^4 .

KEYWORDS: Supersymmetric gauge theory, Differential and Algebraic Geometry, Nonperturbative Effects

ARXIV EPRINT: [1611.04868](https://arxiv.org/abs/1611.04868)

Contents

1	Introduction	2
2	Conditions for reduction	4
2.1	Reduction of the spin bundle	6
2.2	Reduction of the Killing spinor on Sasaki-Einstein manifolds	7
2.3	Specialising to toric Sasaki-Einstein	9
3	A classification result	10
3.1	The geometry of the base	13
3.2	Intersection form and geometry of B	14
3.3	More examples not included in the classification	16
4	Reduction of $\mathcal{N} = 1$ SYM to 4D	17
4.1	Reduction of the action	17
4.2	The hyper-multiplets	20
4.3	Deformations of the action	21
4.3.1	Weyl rescaling	22
5	Partition function	23
5.1	Perturbative sector	23
5.2	Comparison with flat space results	25
5.3	Instanton sector	27
6	Summary	30
A	Details of the reduction conditions	31
B	Spinor conventions and bilinears	32
B.1	Spinor bilinears and some of their properties	33
C	The example of $Y^{p,q}$	33
D	The Weyl rescaled background	35
E	Cohomological variables	36
E.1	Weyl rescaled case	37

1 Introduction

Starting from the work [1] there has been huge activity on studying supersymmetric theories on curved manifolds and on the exact calculation of their partition functions using localization techniques. The original work [1] was devoted to $\mathcal{N} = 2$ gauge theory on S^4 , but since then there has been significant progress in diverse dimensions (from 2D to 7D) and on diverse backgrounds. For a recent overview of the field see [2]; localization computations in different dimensions are reviewed in [3] (for the 4D case see also [4]).

We have a precise classification of the geometries on which 4D $\mathcal{N} = 1$ theories can be placed preserving supersymmetry (see e.g. [5, 6]). The same is true for $\mathcal{N} = 2$ in 3D [5, 7] and $\mathcal{N} = (2, 2)$ theories in 2D [8]. Many localization calculations have been performed explicitly in lower dimension (2D and 3D) while in four dimensions applications of this technique to $\mathcal{N} = 1$ have concentrated on a limited set of geometries [9–12]. In the case of 4D $\mathcal{N} = 2$ theories the situation is even less satisfactory as we do not yet have a complete classification of the corresponding supersymmetric geometries. In particular, with a view towards applying localization techniques, we are interested in 4D manifolds that admit a toric action. It is interesting to notice that in 5D there exists a rich class of toric Sasaki-Einstein manifolds that admit $\mathcal{N} = 1$ theories. The goal of the present paper is to generate a rich class of toric 4D backgrounds which admit $\mathcal{N} = 2$ theories from dimensionally reducing these 5D examples. Essentially we will perform the reduction along non-trivial $U(1)$ fibration of the toric Sasaki-Einstein manifold in order to get a 4D supersymmetric theory. We also derive the exact 4D partition function for these theories. The manifolds we will consider have topological type $\#_k(S^2 \times S^2)$, and are a sub-class of the possible homeomorphism types of smooth simply connected spin 4-manifolds $(\pm M_{E_8})^{\#2m} \#(S^2 \times S^2)^{\#k}$.

Using the rigid supergravity approach [13] it is not easy to completely classify the geometries on which 4D $\mathcal{N} = 2$ theories can be placed preserving supersymmetry (see [14–18] for progress in this direction). The best studied cases are the round sphere [1] and the squashed sphere [16, 17]. The squashed sphere can be further generalized to local T^2 -bundle fibrations [17]. Equivariantly twisted theories on toric Kähler surfaces were also considered, with emphasis on $S^2 \times S^2$ [19] and CP^2 [20, 21]. The study of $\mathcal{N} = 2$ theories on $S^2 \times S^2$ was also started in [22].

The main result of this work is the explicit construction of $\mathcal{N} = 2$ SYM theories on an infinite family of 4D toric manifolds with the topology of connected sums $\#_k(S^2 \times S^2)$ via dimensional reduction from 5D. We would like to stress that our 4D examples are not generically Kähler and here by toric 4D manifolds we mean 4D manifold with smooth T^2 -action with the orbit space being convex polytope. We start by considering toric Sasaki-Einstein manifolds which admit a free $U(1)$ -action that preserves the Killing spinors, and we perform the reduction along this $U(1)$. We provide a partial classification of such toric Sasaki-Einstein manifolds. The resulting 4D theory has unusual properties originating from the fact that the $U(1)$ -fibre does not have a constant size with respect to the Sasaki-Einstein metric. As a result the 4D theory has a position dependent Yang-Mills coupling. If we add a θ -term to the 4D theory we can introduce the point dependent complex coupling τ ,

which takes value in the upper half plane

$$\tau(x) = \frac{4\pi i}{g_{\text{YM}}^2(x)} + \frac{\theta}{2\pi}, \tag{1.1}$$

where $g_{\text{YM}}(x)$ is the 4D dimensionless Yang-Mills coupling and its dependence from x comes from the Sasaki-Einstein metric in 5D, see section 4 for further explanation. The connected sum $\#_k(S^2 \times S^2)$ is a toric manifold with T^2 -action, and it has $(2 + 2k)$ -fixed points. The exact partition functions for these 4D theories is given by the classical term, one-loop term and the instanton term

$$Z = \int_t da e^{-\sum_{i=1}^{2k+2} \frac{4\pi^2 r^2}{\epsilon_1^i \epsilon_2^i g_{\text{YM}}^2(x_i)} \text{Tr}[a^2]} \cdot \frac{\det'_{\text{adj}} \Upsilon^C(ia|R^1, R^2)}{\det_{\underline{R}} \Upsilon^C(ia + im + \vec{\xi} \cdot \vec{R}/2|R^1, R^2)} Z_{\text{inst}}(a|\vec{R}), \tag{1.2}$$

where \vec{R} is related to the T^2 -action and Υ^C is a special function which gives the one-loop determinant. The above partition function corresponds to the $\mathcal{N} = 2$ vector multiplet coupled to a hypermultiplet in representation \underline{R} . The instanton contributions come from point-like instantons and anti-instantons which sit on the fixed points x_i ,

$$Z_{\text{inst}}(a|\vec{R}) = \prod_{i=1}^{k+1} Z_{\text{inst}}^{\mathbb{C}^2}(a, q_i|\epsilon_1^i, \epsilon_2^i) \times \prod_{i=k+2}^{2+2k} Z_{\text{inst}}^{\mathbb{C}^2}(a, \bar{q}_i|\epsilon_1^i, \epsilon_2^i), \tag{1.3}$$

where

$$q_i = q(x_i) = e^{2\pi i \tau(x_i)}. \tag{1.4}$$

Here $Z_{\text{inst}}^{\mathbb{C}^2}(a, q_i|\epsilon_1^i, \epsilon_2^i)$ is the Nekrasov partition function on \mathbb{C}^2 with equivariant parameters $\vec{\epsilon}_i$, that can be read off from the fixed points x_i . Note that the theories considered here are not the topologically twisted Donaldson-Witten theory, since we have a mixture of instanton- and anti-instanton-contributions. It is possible to specify further the toric geometry and find situations when the instanton and anti-instanton contributions pair together,

$$Z_{\text{inst}}(a|\vec{R}) = \prod_{i=1}^{k+1} |Z_{\text{inst}}^{\mathbb{C}^2}(a, q_i|\epsilon_1^i, \epsilon_2^i)|^2. \tag{1.5}$$

Thus our result generalizes Pestun's famous result on S^4 [1].

One may get nervous from the fact that τ depends on x . However this is not so exotic and it was discussed previously in [25, 26] in the context equivariant localization of gauge theories on \mathbb{R}^4 . Moreover the gauge theories with $\tau(x)$ can be obtained from the reduction of $(2, 0)$ 6D theory on elliptically fibered Kähler manifolds [23, 24]. Nevertheless we can deform 5D theory by performing a Weyl rescaling of our 5D manifold so that the length of S^1 -fiber is fixed to be a constant. Through a calculation using the rigid limit of minimal off-shell 5D supergravity, we check that this can be done without breaking supersymmetry. This deformation induces a Q-exact change of the action. After reducing to 4D using the rescaled background, we now find a theory with a constant Yang-Mills coupling, but where

the x -dependence is now shifted to a θ -term. It is important to stress that the partition function of the theory does not depend on $\tau(x)$ in general, but only on its values at the fixed points.

The paper is organised as follows: sections 2 and 3 are preparatory sections where we analyze the conditions under which the 5D $\mathcal{N} = 1$ theory on a non-trivial circle fibration can be reduced down to the 4D $\mathcal{N} = 2$ theory, while sections 4 and 5 contain the main result with the explicit construction of 4D $\mathcal{N} = 2$ theory and the calculation of its partition function. In section 2 we discuss in detail the criterion for pushing a bundle down an S^1 -fibre. In particular the parameters for the supersymmetry transformations are a pair of Killing spinors in 5D, and we seek conditions under which they can be reduced to 4D. This allows us to avoid dealing directly with the supersymmetry algebra in 4D. In section 3 we specialize to the case of toric Sasaki-Einstein manifolds and we present a simple classification of toric Sasaki-Einstein manifolds with a free $U(1)$ isometry preserving the holomorphic volume form (of the Calabi-Yau cone). The classification is not that of the regular toric SE manifolds and the resulting 4D geometry, which we study in sections 3.1 and 3.2, is more interesting. With this preparation in section 4 we reduce the action and the supersymmetry transformations of the 5D supersymmetric gauge theory on an Sasaki-Einstein manifold to 4D. We also discuss various features of the reduced $N = 2$ 4D theory and consider some of its supersymmetric deformations. In section 5 we discuss the partition function of the 4D theories, which can be obtained discarding non-zero Kaluza-Klein modes. We also consider the issue of assembling the instanton sector. Due to the misalignment of the aforementioned freely acting $U(1)$ and the Reeb vector field, one gets a mixture of instantons and anti-instantons. This is a main new feature of our theory that distinguishes it from the Donaldson-Witten theory. The paper contains appendices which complement the main text with some background and technical considerations.

2 Conditions for reduction

Performing dimensional reduction is straightforward if the 5D manifold is a trivial S^1 bundle over a 4D base manifold. If the S^1 bundle is non-trivial it is still possible to reduce. Locally this is Scherk-Schwarz reduction [27] but, since we are considering compact manifolds, we need to identify under which conditions there are no global obstructions. We will see that stating these conditions for differential forms is straightforward, but for spinors the issue is a more subtle. In general, the various fields that we wish to dimensionally reduce are sections of some vector bundles over our manifold. Hence, we will consider when bundles and sections of these bundles can be consistently pushed down from the 5D manifold to the 4D base. In the following we will state the relevant facts and give some examples. Proofs are presented in appendix A.

To set our notation, let $S^1 \rightarrow M \rightarrow B$ be a nontrivial circle fibration, and $E \rightarrow M$ be a vector bundle. We first give a criterion for being able to push the bundle E down to B . If E possesses a trivialization over patches of the form $[0, 2\pi] \times U_i$, with $\{U_i\}$ a cover of the base manifold B , such that the transition functions are independent of the circle direction, then E can be pushed down to B . We can reformulate this criterion as

follows: denote the coordinate of the circle fibre as α and let A be a connection of E , then if $P \exp i \int_0^{2\pi} d\alpha A_\alpha = id$, the bundle E can be pushed down. Moreover, when this is satisfied, sections of E such that $D_\alpha s = 0$ can be pushed down.

The push down is not unique but depends on the choice of connection. As an example, consider S^5 as the total space of the Hopf bundle $S^1 \rightarrow S^5 \xrightarrow{\pi} \mathbb{P}^2$. We want to push down the trivial bundle $S^5 \times \mathbb{C}$ to \mathbb{P}^2 . One way is to choose the zero connection and the pushdown is also the trivial bundle. Alternatively one may choose $A = n\kappa$, where κ is the contact 1-form of the contact structure of S^5 associated with the Hopf fibration structure. Concretely κ is described as $d\alpha + \pi^* \mathcal{A}$ where \mathcal{A} is the connection on \mathbb{P}^2 of the bundle $\mathcal{O}(1)$. The holonomy of κ is 2π , and the push down is $\mathcal{O}(n)$.

The above example shows that pushing down is not a canonical procedure. On the other hand, we know that there exists a canonical procedure to push down differential forms. Denote with $X = \partial_\alpha$ the vector field along the circle fiber. A 1-form ξ that satisfies $\iota_X \xi = 0$ and $L_X \xi = 0$ (where L_X is the Lie derivative along X and ι_X the contraction of a form with X) can be regarded as a 1-form on B canonically.

The following example clarifies this issue. Consider the subbundle $T_H^* M$ of 1-forms ξ with $\iota_X \xi = 0$, i.e. horizontal 1-forms. We want to push it down to B .

First without any loss of generality, we can choose a metric such that X is Killing and normalized to $\langle X, X \rangle = 1$. It follows that $J_{\mu\nu} = -\nabla_\mu X_\nu$ is anti-symmetric. From $2X^\rho \nabla_\rho X_\mu = 2X^\rho \nabla_\mu X_\rho = \partial_\mu \langle X, X \rangle = 0$ one has that J is horizontal with respect to X . The subbundle $T_H^* M$ possesses the connection

$$D_Y \xi = \nabla_Y \xi + gX \cdot \langle \nabla_Y X, \xi \rangle, \quad \xi \in T_H^* M, \quad Y \in TM,$$

where ∇ is the Levi-Civita connection. Hence the covariant derivative D_Y is written as

$$D_Y \xi_\mu = Y^\rho \partial_\rho \xi_\mu - Y^\sigma \Gamma_{\sigma\mu}^\rho \xi_\rho - X_\mu Y^\sigma J_{\sigma}^\rho \xi_\rho.$$

In particular setting $Y = X$

$$D_X \xi_\mu = X^\rho \partial_\rho \xi_\mu + (-J_\mu^\rho + \partial_\mu X^\rho) \xi_\rho = L_X \xi_\mu + J_\mu^\rho \xi_\rho.$$

Thus ξ can be pushed down if the right hand side vanishes

$$0 = D_X \xi = L_X \xi + J \xi. \tag{2.1}$$

This is not quite the usual condition $L_X \xi = 0$, but rather depends on the details of J . However one can write a different connection for $T_H^* M$ as

$$D_Y^{(n)} \xi = \nabla_Y \xi + gX \cdot \langle \nabla_Y X, \xi \rangle - n \langle Y, X \rangle J \xi, \tag{2.2}$$

which is valid since $\iota_X (J\xi) = 0$ from the horizontality of J . Choosing $n = 1$, we get the condition

$$0 = D_X^{(1)} \xi = L_X \xi$$

for pushing down ξ .

As above S^5 can be used as an example. Then J is a complex structure transverse to the Hopf fibre and so $J^2 = -1$ on $T_H^*S^5$. This shows that for any integer n , the connection $D^{(n)}$ has holonomy $e^{2\pi i(n-1)}$ along the Hopf fibre, so that it is a valid choice of connection for pushing down $T_H^*S^5$ to \mathbb{P}^2 . For $n = 1$, the push down bundle is $T^*\mathbb{P}^2$, while for general n , it is the twisted $T^*\mathbb{P}^2 \otimes \mathcal{O}(n-1)$.

2.1 Reduction of the spin bundle

In this subsection we will consider the particular case of the spin bundle. According to the general discussion above, we need a spin connection with trivial holonomy and we will push down sections s satisfying $D_X s = 0$. Note that the push down bundle may be a spin bundle twisted by some line bundle or even a spin^c bundle.

To write down a spin connection, choose a vielbein $\{e^a \mid e^a \in \Gamma(TM), \langle e^a, e^b \rangle = \delta^{ab}\}$ and consider the Levi-Civita connection in this basis

$$\omega_Y^{ab} = \langle e^a, \nabla_Y e^b \rangle, \quad Y \in \text{vect}(M).$$

Then the spin connection is the lift $\mathfrak{so} \rightarrow \mathfrak{spin}$

$$D_Y = Y \cdot \partial + \frac{1}{4} \omega_Y^{ab} \Gamma^{ab}.$$

As the spin bundle is equipped with a spinor Lie-derivative L_Y^s for Y Killing [28, 29], a natural requirement for pushdown could be $L_X^s s = 0$. We will see that this condition can be made precise along the same lines as in the discussion about the cotangent bundle above.

We can pick $\{e^a\}$ to satisfy locally

$$L_X e^a = 0, \tag{2.3}$$

where X is along the S^1 fibre and normalised as always. We first show that when this is done, then $L_X^s = X \cdot \partial$, i.e. an ordinary derivative. The spinor Lie derivative along a Killing vector field is defined as

$$L_X^s = D_X + \frac{1}{4} (\nabla_m X_n) \Gamma^{mn} = X \cdot \partial + \frac{1}{4} \omega_X^{ab} \Gamma^{ab} + \frac{1}{4} (\nabla_\mu X_\nu) \Gamma^{\mu\nu},$$

where $\Gamma_\mu = \Gamma_a e_\mu^a$. Since $L_X e^a = 0$ one has $\nabla_X e_\mu^a = -e^{a\nu} \nabla_\mu X_\nu = (J e^a)_\mu$ so that $\omega_X^{ab} = \langle e^a, J e^b \rangle$. Thus

$$L_X^s = X \cdot \partial + \frac{1}{4} \langle e^a, J e^b \rangle \Gamma^{ab} - \frac{1}{4} J_{\mu\nu} \Gamma^{\mu\nu} = X \cdot \partial.$$

On the other hand, similarly to what we did in (2.2), one can modify the spin connection

$$D \rightarrow D^{(n)} = D - \frac{n}{4} g X \not{J},$$

so that $D_X^{(1)}$ will coincide with $L_X^s = \partial_\alpha$ when (2.3) holds.

In what follows we shall use $D^{(1)}$ for the connection and check its holonomy along the circle fibre. Note that $D_X^{(1)} = L_X^s = \partial_\alpha$ is a local expression and does not imply that the holonomy is 1. Indeed we have ignored the following global issue. Locally one adjusts

the trivialisation of TM to satisfy (2.3), but the adjustments may not be liftable to **spin**. A trivialisation of the spin bundle that it is independent of the S^1 -fibre might not exist. In particular, when the entire fibre does not lie in one patch, there could be a nontrivial transition function when going around the circle. An instance where this obstruction occurs is $S^{4k+1} \rightarrow \mathbb{P}^{2k}$. In such case, the reduction of the bundle cannot proceed straightforwardly, but one may instead push down the spin bundle into a spin^c bundle.

2.2 Reduction of the Killing spinor on Sasaki-Einstein manifolds

In this subsection we further specialize to the case where the 5D manifold is Sasaki-Einstein (SE). On any such manifold one can find Killing spinors

$$D_m \xi^1 = -\frac{i}{2} \Gamma_m \xi^1, \quad D_m \xi^2 = +\frac{i}{2} \Gamma_m \xi^2, \quad (2.4)$$

and we are interested in establishing under which conditions these Killing spinors can be pushed down to the base. We refer the reader to the appendix of [30] for a review of the Sasaki-Einstein geometry that we need (one may also consult [31] for a more comprehensive view).

Consider a SE manifold with metric $g_{\mu\nu}$. We will make use of the Reeb vector field R and the contact 1-form $\kappa = gR$ satisfying $\iota_R d\kappa = L_R d\kappa = 0$. We also need the complex structure J , acting on the plane transverse to R , which is related to R by $\nabla_Y R = JY$. For the Sasakian geometry, J induces a Kähler structure transverse to the Reeb, i.e. J satisfies the integrability condition

$$\langle Z, (\nabla_X J)Y \rangle = -\kappa(Z)\langle X, Y \rangle + \langle Z, X \rangle \kappa(Y), \quad (2.5)$$

where $\langle -, - \rangle$ is the inner product using the metric. We will use the same letter J for the complex structure as well as for the 2-form gJ . Finally the Sasaki-Einstein condition further implies that

$$R_{mn} = 4g_{mn}. \quad (2.6)$$

The Killing spinor equations (2.4) can be solved using the approach of [32]. Consider the rank 1 subbundle W_μ of the spin bundle W consisting of ψ satisfying

$$R\psi = -\psi, \quad \left(\mu JX - \frac{i}{2}(1+R)X \right) \psi = 0, \quad \forall X \in \Gamma(TM), \quad (2.7)$$

where $\mu = \pm 1$ and we have omitted Γ whenever Clifford multiplication is obvious. One then defines a connection for W_μ

$$\tilde{D}_X = D_X + \frac{i\mu}{2} X.$$

This is indeed a connection, i.e. it preserves W_μ , and furthermore it is flat when restricted to W_μ (more details can be found in [33]). If the SE manifold M is simply connected, there is a unique (up to a constant multiple) solution to

$$D_X \psi = -\frac{i\mu}{2} X \psi, \quad \mu = \pm 1.$$

Apart from (2.7), the solution satisfies

$$\not{J}\psi = -4i\mu\psi . \tag{2.8}$$

Since a section of the spin bundle can be reduced if $L_X^s s = D_X^{(1)} s = 0$, we now turn to compute the Lie derivative of a Killing spinor. The spinor Lie derivative along a Killing vector can be shown to satisfy the important properties

$$[L_X^s, Y \cdot \Gamma] = [X, Y] \cdot \Gamma, \quad [L_X^s, L_Y^s] = L_{[X, Y]}^s, \quad [L_X^s, D_Y] = D_{[X, Y]} . \tag{2.9}$$

Using these one sees that the Lie derivative of a Killing spinor ψ along a Killing vector X is also Killing. Using (2.7), (2.8) one can show

$$L_X^s \psi = \left(\frac{i\mu}{2} \langle X, R \rangle - \frac{i\mu}{8} \langle dX, J \rangle - \frac{1}{4} (\kappa \wedge L_X R) \cdot \Gamma \right) \psi . \tag{2.10}$$

In the formulae above we routinely identify vectors with their dual 1-form and vice versa.

For the next subsection we can assume that the Killing vector X commutes with R

$$L_X R = 0 .$$

In this case L_X^s preserves the rank 1 subbundle (2.7) so that $L_X^s \psi = i\mu f_X \psi$ for some constant f_X (the details are in appendix B of [33]¹). The last term in (2.10) is zero and hence

$$f_X = \frac{1}{2} \langle X, R \rangle - \frac{1}{8} \langle dX, J \rangle . \tag{2.11}$$

Knowing that f_X is a constant, this formula can be evaluated at a convenient point.

As we stressed above we also need to compute the holonomy of L_X^s . This is best done without resorting to local computation. To this end we will introduce a spinor representation using horizontal forms (see also section 2.6 of [34]).

Using the Reeb vector R , one can define the horizontal forms

$$\omega \in \Omega_H^\bullet(M) \quad \text{iff} \quad \iota_R \omega = 0 .$$

Using the transverse complex structure J one further decomposes $\Omega_H^p = \oplus_{i+j=p} \Omega_H^{i,j}$. Now one can define the so called canonical spin^c-structure. Consider

$$W_{\text{can}} = \bigoplus \Omega_H^{0,\bullet}(M) . \tag{2.12}$$

One has a representation of the Clifford algebra on W_{can} : let ψ be any section of W_{can} and χ a 1-form, define the Clifford action

$$\chi \cdot \psi = \begin{cases} \sqrt{2} \chi \wedge \psi & \chi \in \Omega_H^{0,1}(M) \\ \sqrt{2} \iota_{g^{-1}\chi} \psi & \chi \in \Omega_H^{1,0}(M) \\ (-1)^{\text{deg}+1} \psi & \chi = \kappa \end{cases} . \tag{2.13}$$

¹If one tries to check the calculation there, pay attention to the typo: the displayed equation before (87), $L_X^s = D_X - 1/8 \nabla_{[m} X_n] \Gamma^{mn}$ should be $L_X^s = D_X + 1/8 \nabla_{[m} X_n] \Gamma^{mn}$.

This in fact defines a priori a spin^c -structure whose characteristic line bundle (see chapter 5 in [35]) is the anti-canonical line bundle associated with the complex structure J . This latter line bundle is trivial on M for simply connected SE manifolds. Hence its square root is also a (trivial) line bundle, so that the spin^c is in fact spin .² With this concrete representation, the first condition in (2.7) says that ψ is in $\Omega_H^{0,2k}$ while the second tells whether its (0,0) or (0,2) depending on μ (as also does (2.8)).

We mentioned above the characteristic line bundle of a spin^c -structure, which in our case is generated by $\Omega_H^{0,2}$. For SE geometry this line bundle is trivialised by a nowhere vanishing section $\bar{\varrho}$ of $\Omega_H^{0,2}$. Thanks to the triviality, one can identify $W \simeq \oplus \Omega_H^{0,\bullet}$. However one needs to remember that this is a statement at the level of topology, while for covariant derivatives, spinor Lie derivatives etc., the isomorphism $W \simeq \oplus \Omega_H^{0,\bullet}$ has a non-trivial effect. This is especially important for reducing the spin bundle, which we turn to next.

Pick a Killing spinor ψ satisfying (2.9) with $\mu = 1$. Using this spinor one can write all other spinors by Clifford multiplying ψ with $\Omega_H^{0,\bullet}$

$$\xi = \eta \wedge \psi \in W, \quad \eta \in \Omega_H^{0,\bullet}.$$

Let now $X = \partial_\alpha$ be the vector field of the $U(1)$ -fibration. As we proved in section 2.1, if the vielbein on M is invariant under X then $D_X^{(1)} = L_X^s$, and so

$$D_X^{(1)}(\eta \wedge \psi) = L_X^s(\eta \wedge \psi) = (L_X \eta) \wedge \psi + \eta \wedge L_X^s \psi.$$

As X is induced from a circle action on M , the $L_X \eta$ term has the right period, so whether or not $D_X^{(1)}$ has trivial holonomy hangs on the last term $L_X^s \psi$. For our purposes $L_X^s \psi = if_X \psi$ for a constant f_X . Thus $f_X \in \mathbb{Z}$ ensures that the holonomy is trivial. When this condition fails, the reduction is not impossible, but rather one might need to adjust the spin connection.

2.3 Specialising to toric Sasaki-Einstein

In the toric setting M has isometry $U(1)^3$ generated by e_a , $a = 1, 2, 3$, and the Reeb vector is a constant combination of the three $U(1)$'s: $R = \sum_{a=1}^3 R^a e_a$. We also seek another combination $X = \sum_{a=1}^3 X^a e_a$, $X^a \in \mathbb{Z}$, so that X has closed orbits of period 2π and M is a regular foliation by the orbits. In other words M is a $U(1)$ -fibration over a 4D base B .

Let us investigate what requirement do we have on X^a so that f_X in (2.11) vanishes, that is $L_X^s \psi = 0$. Note that in the current setting $L_X R = 0$ trivially. Denoting with $\vec{X} = (X^1, X^2, X^3)$ the 3-vector parametrizing $X = X^a e_a$, we decompose (non-uniquely)

$$\vec{X} = \sum_{i=1}^n \lambda_i \vec{v}_i,$$

where n is the total number of faces of the moment map cone of M . In fact it is possible to choose $\lambda_i \in \mathbb{Z}$ since

$$\pi_1(M) = 0 \quad \Leftrightarrow \quad \text{span} \langle \vec{v}_1, \dots, \vec{v}_n \rangle = \mathbb{Z}^3.$$

²In general, SE manifolds with $H_1(M, \mathbb{Z})_{\text{tor}} = 0$, are spin (see theorem 7.5.27 in [36]).

Each \vec{v}_i represents a $U(1)$ that vanishes of degree 1 at face i , and so by a local computation

$$\langle dv_i, J \rangle = -2 ,$$

where we also use v_i to denote the vector field $\sum_a v_i^a e_a$. This shows $f_{v_i} = 1/2$ and $f_X = (1/2) \sum \lambda_i$. To formulate this quantity geometrically, we note that the SE condition implies that there exists a $\vec{\xi} \in \mathbb{Z}^3$ such that $\vec{\xi} \cdot \vec{v}_i = 1, \forall i$. Then

$$f_X = \frac{1}{2} \sum_{i=1}^n \lambda_i = \frac{1}{2} \vec{X} \cdot \vec{\xi} .$$

Hence the spin bundle is reducible to B if $\vec{X} \cdot \vec{\xi} = 2\mathbb{Z}$. Note that since such $\vec{\xi}$ must be primitive (its components have gcd 1), one may assume that $\vec{\xi} = [1, 0, 0]$.

The geometrical meaning of this condition is this: as the metric cone $C(M)$ over M is a Calabi-Yau, it has a holomorphic volume form Ω . From this one can construct a nowhere vanishing section $\varrho = \iota_R \Omega \in \Omega_H^{0,2}$. Then f_X is the charge of ϱ under X . For a geometry with $f_X = 0$ we can then simply declare that the spin bundle on M can be reduced to that of B .

3 A classification result

We first set up some nomenclature. The geometry of M is entirely encoded by a moment map cone $C_\mu(M) \subset \mathbb{R}^3$. Let $\vec{v}_i \in \mathbb{Z}^3, i = 1, \dots, m$ be the (primitive) inward pointing normals of the n faces of C_μ . Let $R = \sum_{a=1}^3 R^a e_a$ be the Reeb vector field, and assume that \vec{R} is within the dual cone C_μ^\vee , i.e.

$$\vec{R} = \sum_{i=1}^m \lambda_i \vec{v}_i, \quad \lambda_i > 0 . \tag{3.1}$$

With this assumption, the plane (where y^a are the coordinate of \mathbb{R}^3)

$$\left\{ \vec{y} \in \mathbb{R}^3 \mid \vec{R} \cdot \vec{y} = \frac{1}{2} \right\}$$

intersects C_μ at a convex polygon Δ_μ if C_μ is convex. Then the geometry of M is that of a $U(1)^3$ fibration over Δ_μ , except that at each faces of Δ_μ , a certain $U(1)$ becomes degenerate. More concretely if the normal associated with face i is \vec{v}_i , then the $U(1)$ given by $\sum_{a=1}^3 v_i^a e_a$ degenerates. In particular, at the intersection of faces, only one $U(1)$ remains non-degenerate and its orbit is a closed Reeb orbit. These are the only loci for closed Reeb orbits if \vec{R} is chosen generically.

We assume the following for C_μ (see [37])

1. Convexity (where $\vec{v}_{n+1} := \vec{v}_1$):

$$[\vec{v}_i, \vec{v}_{i+1}, \vec{v}_k] = (\vec{v}_i \times \vec{v}_{i+1}) \cdot \vec{v}_k > 0, \quad \forall k \neq i, i+1 . \tag{3.2}$$

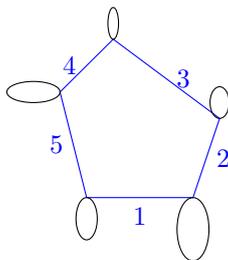


Figure 1. The polygon Δ_μ . The circles represent the closed Reeb orbits.

- 2. Goodness:³ $\exists \vec{n}_i \in \mathbb{Z}^3$, such that $[\vec{n}_i, \vec{v}_i, \vec{v}_{i+1}] = 1, \forall i$.
- 3. Gorenstein: $\exists \vec{\xi} \in \mathbb{Z}^3$ such that $\vec{\xi} \cdot \vec{v}_i = 1, \forall i$, see [38].

The first condition is for compactness of M , the second for smoothness while the third guarantees the existence of a holomorphic volume form $\Omega \in \Omega^{3,0}(C(M))$, where $C(M)$ is the metric cone over of M . In other words, the Gorenstein condition is the Calabi-Yau condition for the cone over M .

One may assume without loss of generality that $\vec{\xi} = [1, 0, 0]$, and so we write

$$\vec{v}_i = \begin{bmatrix} 1 \\ x_i \\ y_i \end{bmatrix}. \tag{3.3}$$

Next let $\sum_{a=1}^3 X^a e_a$ represent the vector field X , we then have the correspondence

Proposition 3.1. *The 5D toric Sasaki-Einstein manifolds with a freely acting $U(1)$ that preserves Ω are in 1-1 correspondence (up to $SL(3, \mathbb{Z})$ transformation) with convex 2D-polygons whose vertices (x_i, y_i) are in \mathbb{Z}^2 , and furthermore the x -coordinate of neighbouring vertices must differ by ± 1 . This implies that the number of vertices is even $\mathfrak{m} = 2\mathfrak{n}$. If one requires $\pi_1 = 0$, then all the y_i 's should have greatest common divisor 1.*

Proof. That X acts freely means that at the intersection of face $i, i + 1$, one has

$$\det[\vec{X}, \vec{v}_i, \vec{v}_{i+1}] = \vec{X} \cdot (\vec{v}_i \times \vec{v}_{i+1}) = \pm 1, \tag{3.4}$$

so that not only the vector field X is nowhere zero, but its stability group is trivial for all points. This also ensures the smoothness of M since (3.4) implies goodness.

We focus on the case $f_X = \vec{\xi} \cdot \vec{X} = 0$, then with a further $SL(3, \mathbb{Z})$ transformation one can assume

$$\vec{X} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

³This condition was phrased in [37] as: $\mathbb{Z}^3 \cap \text{span}_{\mathbb{R}}\langle \vec{v}_i, \vec{v}_{i+1} \rangle = \text{span}_{\mathbb{Z}}\langle \vec{v}_i, \vec{v}_{i+1} \rangle$ for all i .

while preserving all the other assumptions we have made so far.⁴ With these assumptions (3.4) says

$$x_i - x_{i+1} = \pm 1, \tag{3.5}$$

and the convexity (3.2) says

$$\det \begin{bmatrix} x_i - x_k & x_{i+1} - x_i \\ y_i - y_k & y_{i+1} - y_i \end{bmatrix} > 0. \tag{3.6}$$

It is not difficult to see that the solution to (3.5), (3.6) are labelled by a convex polygon on the $x - y$ plane, for which the x -coordinates of successive vertices differ by ± 1 .

Finally for the toric manifolds considered $\pi_1 = \mathbb{Z}^3 / \text{span} \langle \vec{v}_1, \dots, \vec{v}_{2n} \rangle$, so from the explicit form of the \vec{v}_i 's, this is realised if $\text{gcd}(y_i) = 1$. \square

To fix the $\text{SL}(3, \mathbb{Z})$ redundancy, we enforce

1. the entire polygon lies to the right of y axis
2. vertex 1 and 2 are on $(0, 0)$ and $(1, 0)$
3. $\det[\vec{e}_1, \vec{e}_{n+1}] \geq 0$, and if $\det[\vec{e}_1, \vec{e}_{n+1}] = 0$, then $\det[\vec{e}_2, \vec{e}_{n+2}] \geq 0$ and so on,

where \vec{e}_i denotes the edge from vertex i to vertex $i + 1$.

Indeed using a cyclic permutation, one fixes the vertex with the smallest x -value as the 1st vertex, satisfying item 1 on the list above. The $\text{SL}(3, \mathbb{Z})$ redundancy now consists of lower triangular matrices only. With these, one can set $(x_1, y_1) = (0, 0)$, and a further transformation sets $(x_2, y_2) = (1, 0)$, satisfying item 2 of the list. If the resulting polygon does not satisfy item 3 we can act as follows. First flip the sign of all x_i, y_i (X is now $[0; 0; -1]$, but this does not affect anything). We can now repeat the steps above and make the polygon satisfy item 1, 2 and 3. This corresponds essentially to turning the polygon around so that the $(n + 1)$ th vertex (the right most one) becomes the first one. The first two pictures of figure 2 provide an explicit example of this flip.

Example 3.2 ($Y^{p,q}$ -spaces). Take a quadrangle with vertices placed at $[0, 0], [1, 0], [2, p - q], [1, p]$, with $p > q > 0$ and $\text{gcd}(p, q) = 1$, i.e. the normals are

$$[\vec{v}_1, \dots, \vec{v}_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & p - q & p \end{bmatrix}. \tag{3.7}$$

Note that the metric cone in this case can be obtained by a Kähler reduction of \mathbb{C}^4 with a $\text{U}(1)$ of weight $[-p, p + q, -p, p - q]$, cf. section 4 of [39]. From the explicit metric for $Y^{p,q}$ [40], that we write down in appendix C, the $\text{U}(1)$ fibration is obvious. In contrast, $L^{a,b,c}$ spaces [41] do not offer any free $\text{U}(1)$ and, if one writes down the normals, one sees that they do not fall into our classification.

⁴Keep in mind that if \vec{v}_i is transformed with $g \in \text{SL}(3, \mathbb{Z})$, then $\vec{\xi}$ is transformed with g^T .

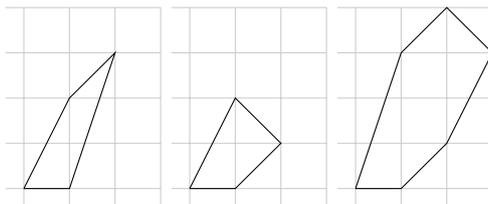


Figure 2. The first two are equivalent polygons representing the $Y^{2,1}$ space, and the last is a hexagon example

Figure 2 shows the normals of $Y^{2,1}$, as well as a hexagon example. For the hexagon, from the vertices we read off the normals

$$[\vec{v}_1, \dots, \vec{v}_6] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & 4 & 3 \end{bmatrix}. \tag{3.8}$$

Here is an octagon example

$$[\vec{v}_1, \dots, \vec{v}_8] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & 6 & 6 & 5 & 3 \end{bmatrix}.$$

Note that the polygons appearing here should not be confused with the polygons Δ_μ .

3.1 The geometry of the base

We fix the orientation of the 5-manifold by picking the volume form

$$\text{Vol}_M = \frac{1}{8} \kappa \wedge d\kappa \wedge d\kappa.$$

Since the vector field X is everywhere nonzero we fix the volume form of B as

$$\text{Vol}_B = \iota_X \text{Vol}_M. \tag{3.9}$$

At the intersection of two faces, there will be only one nondegenerate $U(1)$. Thus, R and X both being linear combinations of $U(1)$'s, must (anti)align at these loci. At the intersection of face i and $i + 1$, the three weights $\vec{R}, \vec{v}_i, \vec{v}_{i+1}$ always form a right-handed base. Indeed from the condition (3.1) one has

$$[\vec{R}, \vec{v}_i, \vec{v}_{i+1}] = \sum_{j=1}^n \lambda_j [\vec{v}_j, \vec{v}_i, \vec{v}_{i+1}] > 0.$$

The right hand side is greater than zero from (3.2). On the other hand $[\vec{X}, \vec{v}_i, \vec{v}_{i+1}] = \pm 1$, thus we conclude

$$\begin{aligned} [\vec{X}, \vec{v}_i, \vec{v}_{i+1}] &= +1, & R \text{ and } X \text{ parallel,} \\ [\vec{X}, \vec{v}_i, \vec{v}_{i+1}] &= -1, & R \text{ and } X \text{ anti-parallel,} \end{aligned}$$

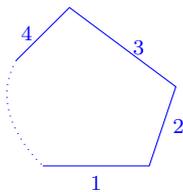


Figure 3. The momentum polytope of a 5D toric contact manifold.

at the locus corresponding to the intersection of face i and $i + 1$. Note that in the polygon picture of the normals, the $+1$ occurs for the sides of the polygon where the x -coordinate increase, and the -1 when it decreases (going around the polygon counter-clockwise). So they will occur the same number of times, which also is a way of seeing that $f_X = 0$.

Due to this misalignment of X with respect to R across the manifold, the orientation of B determined according to (3.9) does not always agree with that of $d\kappa \wedge d\kappa/8$. At a corner where $[\vec{X}, \vec{v}_i, \vec{v}_{i+1}] = -1$, the orientation of B is opposite to that of the transverse plane field of M . This will have important effect when we consider instantons, since the (anti-)self-duality condition depends on the choice of volume form, more about this in section 5.3.

3.2 Intersection form and geometry of B

To understand the geometry of the base manifold, we compute the pairing of $H_2(B, \mathbb{Z})$. Figure 3 represents the base of the moment map polytope of the 5D toric manifold. Taking a further quotient along X gives the base B . Note that in the classification above B is *not* toric Kähler, we are merely using the polytope for M to visualise the geometry of B .

The edges in figure 3 generate $H^2(M, \mathbb{Z})$; in fact each edge corresponds to a torus invariant 3D submanifold (some lens space). Taking the quotient along X , we get a generating set for $H^2(B, \mathbb{Z})$. There are relations among the generators. Denoting by $[x_i] \in H^2(B, \mathbb{Z})$ the generator associated with edge i , we have

$$\vec{r} \cdot \sum_{i=1}^{2n} \vec{v}_i [x_i] = 0, \quad \forall \vec{r} \in \mathbb{Z}^3, \text{ such that } \vec{r} \cdot \vec{X} = 0. \tag{3.10}$$

As we saw in the last section, one can assume that \vec{X} has been set to be $[0; 0; 1]$ and the normals have been put in the standard form

$$[\vec{v}_1, \dots, \vec{v}_n, \vec{v}_{n+1}, \vec{v}_{2n}] = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \dots & n & n-1 & \dots & 1 \\ 0 & 0 & * & * & * & * & * & * \end{bmatrix}, \tag{3.11}$$

then the relation (3.10) is simply

$$\sum_{i=1}^{2n} v_i^a [x_i] = 0, \quad a = 1, 2.$$

Looking at the first and second row of (3.11), we can take $[x_i]$, $i = 3, \dots, 2n$ as a free generating set of $H^2(B, \mathbb{Z})$.

The intersection form of $H^2(B, \mathbb{Z})$ can be computed as the intersection number of the $[x_i]$'s, which is

$$\begin{aligned} \langle [x_i], [x_{i+1}] \rangle &= \text{sgn}[\vec{X}, \vec{v}_i, \vec{v}_{i+1}] \\ \langle [x_i], [x_i] \rangle &= -\text{sgn}[\vec{X}, \vec{v}_{i-1}, \vec{v}_i] \text{sgn}[\vec{X}, \vec{v}_i, \vec{v}_{i+1}] [\vec{X}, \vec{v}_{i-1}, \vec{v}_{i+1}] \end{aligned} \quad (3.12)$$

and zero otherwise. Here the orientation we used for the pairing is that of (3.9).

Example 3.3. Take the $Y^{p,q}$ spaces as an example, the normals are in (3.7), and so the pairing matrix between $[x_3, x_4]$ is

$$\langle -, - \rangle_{Y^{p,q}} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

This pairing matrix is equivalent to the standard form

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (3.13)$$

that is, there is a matrix $g \in \text{SL}(2, \mathbb{Z})$ such that $\langle g-, g- \rangle = H$. Note that H is the intersection form of $S^2 \times S^2$.

Take now a hexagon example (3.8)

$$\langle -, - \rangle = \begin{bmatrix} -2 & 1 & & & & \\ 1 & 0 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & & \\ & & & -1 & 2 & \\ & & & & & \dots \end{bmatrix}$$

which is equivalent to $H \oplus H$, i.e. the intersection form of $\#_2(S^2 \times S^2)$.

By the notation $\#_k(S^2 \times S^2)$, we mean the connected sum of k copies of $S^2 \times S^2$. The connected sum of two manifolds joins them together near a chosen point of each, i.e. we delete a ball inside each manifold and glue together the resulting boundary spheres. Although the construction depends on the choice of balls, the result is unique up to diffeomorphism.

Proposition 3.4. *All manifolds appearing in the classification above are homeomorphic to $\#_k(S^2 \times S^2)$ with $k = n + 1$.*

Proof. We claim that all our intersection matrices are equivalent to a direct sum of terms H of (3.13). If this is so, then by a famous theorem of Freedman (theorem 1.5 [42]), there is a unique simply connected 4-manifold whose intersection form realizes the given quadratic form. This shows that the manifolds in question have to be $\#_k(S^2 \times S^2)$.

Next we prove the claim. We always assume that the normals are put in the standard form of equation (3.11); which makes the intersection form take the general form

$$\begin{bmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & 0 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ & & & & & \ddots & \end{bmatrix}.$$

It is easy to see that this pairing has even parity, i.e. $\langle x, x \rangle = \text{even}$ for any x .

Working over \mathbb{Q} and using elementary row and column operations, one can show that the pairing is equivalent to the diagonal matrix

$$\text{diag} \left[-2, -\frac{3}{2}, -\frac{4}{3}, \dots, -\frac{n-1}{n-2}, \frac{n-2}{n-1}, \frac{n-3}{n-2}, \dots, \frac{1}{2}, -\frac{1}{2}, 2 \right],$$

from which we see that its signature (the number of positive eigenvalues minus the number of negative eigenvalues) is zero. Moreover the determinant of the intersection form is $(-1)^{n-1}$ (since the matrices of the elementary row/column operations have determinant 1, one can compute the determinant using the above diagonal form) and hence it is invertible. Thus, our intersection form is of maximum rank, is even and of zero signature. It is easy to see that the same holds for the direct sum of factors of H , so by a theorem classifying the indefinite even quadratic forms (theorem 5.3 in chapter 2 of [43]), they are equivalent. \square

Remark 3.5. Note that the complex structure of the resulting 4-manifold is *not* inherited from the transverse complex structure of the 5-manifold, in contrast to the ones appearing below.

Remark 3.6. The manifolds $\#_k(S^2 \times S^2)$ are a sub-family of

$$(\pm M_{E_8})^{\#2m} \#(S^2 \times S^2)^{\#k} \tag{3.14}$$

where M_{E_8} is some 4-manifold with intersection form the Cartan matrix of E_8 . One has that any simply connected smooth 4-manifold has the *homeomorphism* type above (however the converse statement is an open problem). Indeed, the intersection form of a spin 4-fold must be indefinite, for by Donaldson’s theorem, a definite intersection form can be diagonalised to $+1$ or -1 and so not spin (since the intersection form of spin manifolds have even parity). Then the classification of the indefinite forms gives $\pm nE_8 \oplus kH$. Furthermore the number of copies of E_8 is even so that the intersection form has signature divisible by 16 according to Rohklin’s theorem. And if $m > 0$, one needs $k > 0$ so as not to have a definite form, leading to (3.14).

3.3 More examples not included in the classification

If one gives up the condition $L_X \Omega = f_X = 0$ or equivalently $\vec{\xi} \cdot \vec{X} = 0$, one can find some more sporadic cases. We do not consider these in this paper, leaving them for future study, but we make the following observation. Consider the condition

$$\vec{X} \cdot (\vec{v}_i \times \vec{v}_{i+1}) = \pm 1 .$$

The vector $\vec{w}_i \equiv \vec{v}_i \times \vec{v}_{i+1}$ is a generator of our cone; and thus it is also a normal vector of the dual cone. This means that we can think of the condition $\vec{X} \cdot \vec{w}_i = \pm 1 \forall i$ as a “generalized Gorenstein condition” for the dual cone. If we require to have strictly $\vec{X} \cdot \vec{w}_i = +1$, it is exactly the Gorenstein condition for the dual cone. Cones with this property, i.e. where both the cone and its dual are Gorenstein, are called *reflexive Gorenstein*. They are well studied [44, 45], since they are important and useful in the context of mirror symmetry: the cone and its dual give us a mirror pair of CY manifolds.

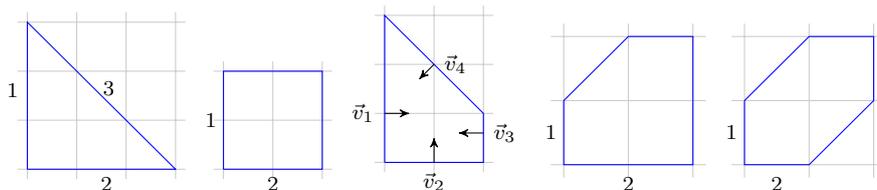


Figure 4. More examples that correspond to regular toric SE manifolds.

Reflexive Gorenstein cones are in one-to-one correspondence with reflexive polytopes (polytopes that contain exactly 1 interior lattice point). In 2D there are 16 such polytopes (up to $GL_2(\mathbb{Z})$ transformations). In figure 4 we have plotted some of these polytopes. In contrast to the previous examples, the base B is now a toric Kähler manifold. We also note that now X is always aligned with R at the loci of the closed Reeb orbits.

Depending on the details of the geometry, one may be able to push the spin bundle from M to a spin or spin^c bundle on B .

To give a bit more detail about these cases, let us work out some details of the examples in figure 4. The second and third one have normals given by

$$[\vec{v}_1, \dots, \vec{v}_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & k \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad k = -1, 0.$$

Denoting by $[i]$ the divisor of the i^{th} face of the moment polygons in the figure above, the canonical classes (and also the Kähler class) are

$$\begin{aligned} & 3[3], \\ & 2[3] + 2[4], \\ & 2[3] + 3[4], \\ & 2[3] + 2[4] + [5], \\ & [3] + 2[4] + 2[5] + [6], \end{aligned}$$

respectively. Here we have used relations among the divisors to eliminate $[1]$, $[2]$. Only in the second case is the canonical class divisible by 2 and one can reduce to a spin structure (the geometry is $S^2 \times S^2$ after all). For the rest, one gets spin^c structures.

4 Reduction of $\mathcal{N} = 1$ SYM to 4D

4.1 Reduction of the action

Starting from the works [46–48] the 5D $\mathcal{N} = 1$ supersymmetric Yang-Mills on a Sasaki-Einstein manifold has been constructed in [33, 49]. For $\mathcal{N} = 1$ vector multiplet the action has the following form (we refer the reader to appendix B where the conventions used here are spelled out)

$$S_{\text{vec}} = \frac{1}{(g_{\text{YM}}^{\text{5D}})^2} \int_M \text{Vol}_M \text{Tr} \left[\frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + 2\sigma t^{IJ} D_{IJ} - 10t^{IJ} t_{IJ} \sigma^2 + i\lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] - i t^{IJ} \lambda_I \lambda_J \right], \quad (4.1)$$

where the λ^I satisfy the symplectic Majorana condition

$$(\lambda^I)^* = \epsilon_{IJ} C \lambda^J ,$$

with C being the charge conjugation matrix. The supersymmetry transformations read

$$\begin{aligned} \delta A_m &= i\xi_I \Gamma_m \lambda^I , \\ \delta \sigma &= i\xi_I \lambda^I , \\ \delta \lambda_I &= -\frac{1}{2}(\Gamma^{mn} \xi_I) F_{mn} + (\Gamma^m \xi_I) D_m \sigma - \xi^J D_{JI} + 2t_I^J \xi_J \sigma , \\ \delta D_{IJ} &= -i\xi_I \Gamma^m D_m \lambda_J + [\sigma, \xi_I \lambda_J] + it_I^K \xi_K \lambda_J + (I \leftrightarrow J) . \end{aligned} \quad (4.2)$$

The spinor ξ_I is also symplectic Majorana and satisfies the Killing equation

$$\nabla_m \xi_I = t_I^J \Gamma_m \xi_J , \quad t_I^J = \frac{i}{2r} (\sigma_3)_I^J , \quad (\xi_I \xi_J) = -\frac{1}{2} \epsilon_{IJ} , \quad (4.3)$$

where $\sigma_3 = \text{diag}[1, -1]$. The supercharge squares to a translation along the Reeb vector $R^n = \xi^I \Gamma^n \xi_I$. In formula (4.3) r is a dimensionful parameter corresponding to the size of the manifold. Explicitly the metric on M is taken to be $r^2 ds_5^2$. In the limit $r \rightarrow \infty$ the theory approaches $\mathcal{N} = 1$ in flat space.

We assume now that M is a $U(1)$ bundle over a 4-manifold $S^1 \rightarrow M \xrightarrow{\pi} B$, satisfying the conditions described in the previous sections, and rewrite the action as a SYM theory over B . We set up some notation first. Let the $U(1)$ action be generated by ∂_α . The metric g_M on M is invariant along the fiber, that is ∂_α is a Killing vector. The metric can then be written in the following form,

$$r^2 ds_5^2 = r^2 \left(ds_4^2 + e^{2\phi} (d\alpha + b)^2 \right) , \quad (4.4)$$

where $r^2 ds_4^2$ is the metric on the 4D base B , α is the coordinate along the fiber, b is the connection one-form for the fibration and re^ϕ is the radius of the fiber. Because ∂_α is Killing both ϕ and b are constant along the flow it generates. In appendix C following [40], we present the metric of the $Y^{p,q}$ spaces, considered in example 3.3, in this form.

Generically e^ϕ is non-constant over the base, and the five dimensional volume form Vol_M is related to the volume form Vol_4 on the base by (note that this is not the same as Vol_B defined in (3.9))

$$\text{Vol}_M = r e^\phi d\alpha \wedge \text{Vol}_4 . \quad (4.5)$$

Finally let $\beta = r^{-1} e^{-2\phi} g_M \partial_\alpha = r(d\alpha + b)$. Since ∂_α is Killing, $d\beta$ is constant along its flow, that is $L_{\partial_\alpha} d\beta = 0$. Additionally $\iota_{\partial_\alpha} (d\beta) = 0$ and hence we can regard $d\beta$ as a 2-form on B .

In reducing we will take all the fields in the theory to be invariant under the (spinor) Lie-derivative along α . As explained in the previous sections they can then be regarded as fields on the base B . In particular we restrict the gauge bundle on M to be one pulled back from B , so that only gauge connections of the form $\pi^* A + \tilde{\varphi} \beta$ are considered. Here $\tilde{\varphi}$ is an adjoint scalar that is constant along the fiber. Note that $\tilde{\varphi} \beta$ is a globally defined adjoint valued 1-form and hence does not affect the topology type of the bundle.

This restriction on the fields is compatible with the supersymmetry transformations (4.2) as long as the spinor parameters ξ_I are constant along the fiber. Under this condition the reduction gives rise to a supersymmetric field theory on the base B .

The four dimensional supersymmetry variation parameter ξ_I satisfies

$$\begin{aligned}\nabla_\mu \xi_I &= -\frac{1}{4}e^\phi d\beta_{\mu\nu} \gamma^\nu \gamma_5 \xi_I + t_I^J \gamma_\mu \xi_J, \\ 0 &= \frac{1}{2}\partial_\mu \phi \gamma^\mu \xi_I - \frac{1}{8}e^\phi d\beta_{\mu\nu} \gamma^{\mu\nu} \gamma_5 \xi_I - t_I^J \xi_J,\end{aligned}\tag{4.6}$$

where we use γ_μ for the gamma matrices in four dimensions and we regard $d\beta$ as a form on B . The first equation above matches with the generalized Killing spinor equation stemming from the rigid limit of $\mathcal{N} = 2$ Poincaré supergravity [18]. The second equation is a constraint arising from the higher dimensional Killing spinor equation (4.3) along the fiber direction.

Plugging $\pi^*A + \tilde{\varphi}\beta$ into the curvature we obtain ($\langle \bullet, \bullet \rangle_M$ is the contraction using the 5D metric g_M , while $\langle \bullet, \bullet \rangle_B$ uses g_B)

$$\begin{aligned}F_5 &= F_4 + (D\tilde{\varphi}) \wedge \beta + \tilde{\varphi}d\beta, \\ \langle F_5, F_5 \rangle_M &= \langle F_4 + \tilde{\varphi}d\beta, F_4 + \tilde{\varphi}d\beta \rangle_B + 2e^{-2\phi} \langle D\tilde{\varphi}, D\tilde{\varphi} \rangle_B,\end{aligned}$$

Making use of (4.6) and setting $\varphi = e^{-\phi}\tilde{\varphi}$ the reduced supersymmetry transformations are given by

$$\begin{aligned}\delta A_\mu &= i\xi_I \gamma_\mu \lambda^I, \\ \delta \varphi &= i\xi_I \gamma_5 \lambda^I, \quad \delta \sigma = i\xi_I \lambda^I, \\ \delta \lambda_I &= -\frac{1}{4} \left(2\mathcal{F} + \varphi e^\phi d\beta \right) \xi_I + (\mathcal{D}\sigma + \gamma_5 \mathcal{D}\varphi) \xi_I - i[\varphi, \sigma] \gamma_5 \xi_I \\ &\quad - D_{IJ} \xi^J + 2(\sigma + \gamma_5 \varphi) t_I^J \xi_J, \\ \delta D_{IJ} &= -i\xi_I \mathcal{D}\lambda_J - [\varphi, \xi_I \gamma_5 \lambda_J] + [\sigma, \xi_I \lambda_J] + (I \leftrightarrow J).\end{aligned}\tag{4.7}$$

These are a specific instance of those arising from rigid $\mathcal{N} = 2$ supergravity [15–18]. The 5D supersymmetry transformations (4.2) provide a compact packaging of the 4D ones. Finally the 4D action reads (we suppressed subscripts $4, B$),

$$\begin{aligned}S_{\text{vec}}^{4D} &= \int_B \text{Vol}_4 \frac{2\pi r e^\phi}{(g_{\text{YM}}^{5D})^2} \text{Tr} \left[\frac{1}{2} \langle F + \tilde{\varphi}d\beta, F + \tilde{\varphi}d\beta \rangle + \langle D\varphi, D\varphi \rangle - \varphi^2 \nabla^2 \phi \right. \\ &\quad - \langle D\sigma, D\sigma \rangle + [\varphi, \sigma]^2 - \frac{1}{2} D_{IJ} D^{IJ} + 2\sigma t^{IJ} D_{IJ} - 10t^{IJ} t_{IJ} \sigma^2 + i\lambda_I \mathcal{D}\lambda^I \\ &\quad \left. + \frac{i}{8} e^\phi \lambda_I d\beta \gamma_5 \lambda^I + \frac{i}{2} \lambda_I \partial \phi \lambda^I - \lambda_I [\sigma - \gamma_5 \varphi, \lambda^I] - it^{IJ} \lambda_I \lambda_J \right],\end{aligned}\tag{4.8}$$

We see that after reduction the field theory defined by (4.8) has a position dependent YM coupling constant, the dependence coming from e^ϕ . This is expected since we know

that, when performing a Kaluza-Klein type reduction, the YM coupling picks up a factor of the radius of the S^1 fiber, which in our case is not of constant size. Nevertheless this theory is supersymmetric by construction. We define the 4D YM coupling in terms of the 5D as

$$\frac{1}{g_{\text{YM}}^2(x)} = \frac{2\pi r e^\phi}{(g_{\text{YM}}^{5\text{D}})^2}. \tag{4.9}$$

We want to point out that the action above reverts to the flat space SYM when $r \rightarrow \infty$. To see this one needs to remember that the geometric quantities such as the metric, β and t contain r explicitly, while derivatives of the conformal factor goes to zero since ϕ is slow varying across distances far smaller than r .

With the goal of reaching a more conventional theory, in section 4.3 we will study diverse deformations of (4.8). Along the way we will see that these deformations are Q -exact and hence do not affect supersymmetric observables.

With a view towards the discussion of instantons in section 5.3, we modify the 4D action by adding a θ -term

$$S_{\text{YM}} \rightarrow S_{\text{YM}} - \frac{i\theta}{8\pi^2} \text{Tr} \int F \wedge F,$$

where θ is constant. This term is supersymmetric by itself. It is now natural to define the position dependent complex coupling τ as

$$\tau(x) = \frac{4\pi i}{g_{\text{YM}}^2(x)} + \frac{\theta}{2\pi},$$

which takes values in the upper half complex plane. This is what will appear in the instanton partition function.

4.2 The hyper-multiplets

The 5D hyper-multiplet consists of an $SU(2)_R$ -doublet of complex scalars q_I^A , $I = 1, 2$ and an $SU(2)_R$ -singlet fermion ψ^A , with the reality conditions ($A = 1, 2, \dots, 2N$)

$$(q_I^A)^* = \Omega_{AB} \epsilon^{IJ} q_J^B, \quad (\psi^A)^* = \Omega_{AB} C \psi^B,$$

where Ω_{AB} is the invariant tensor of $USp(2N)$ and C is the charge conjugation matrix.

Suppressing the gauge group index, the on-shell supersymmetry variations are

$$\begin{aligned} \delta q_I &= -2i\xi_I \psi, \\ \delta \psi &= \Gamma^m \xi_I (D_m q^I) + i\sigma \xi_I q^I - 3t^{IJ} \xi_I q_J. \end{aligned} \tag{4.10}$$

The 5D supersymmetric action reads

$$\begin{aligned} S_{\text{hyp}} = \int_M \text{Vol}_M \left(\epsilon^{IJ} \Omega_{AB} D_m q_I^A D^m q_J^B - \epsilon^{IJ} q_I^A \sigma_{AC} \sigma_{CB} q_J^B + \frac{15}{2} \epsilon^{IJ} \Omega_{AB} t^2 q_I^A q_J^B \right. \\ \left. - 2i\Omega_{AB} \psi^A \not{D} \psi^B - 2\psi^A \sigma_{AB} \psi^B - 4\Omega_{AB} \psi^A \lambda_I q^{IB} - i q_I^A D_{AB}^{IJ} q_J^B \right). \end{aligned}$$

We refer the reader to [47] for more details on the hyper-multiplet. We do not explicitly present the reduction to 4D of this action and the supersymmetry transformation rules (4.10). These can be performed along the same lines as for the vector multiplet. It is important to note that the hypermultiplet action is Q -exact [46–48].

4.3 Deformations of the action

Here we will study supersymmetric deformations of the action (4.8) which give rise to a four dimensional theory with coupling constant g_{YM} which is position independent. To accomplish this it is convenient to go back to the five dimensional action (4.1) and rewrite it in terms of cohomological (twisted) variables that make the action of supersymmetry more transparent.

The cohomological complex for Yang-Mills theory on a Sasaki-Einstein manifold was introduced in [46] (see [50] for earlier work). Its bosonic variables comprise, besides the fields A_μ and σ , a two form $H_{\mu\nu}$ while the gauginos are embedded in a one form Ψ_μ and a two form $\chi_{\mu\nu}$. Appendix E includes a brief review of the definitions of these variables, their salient properties, and their transformation under supersymmetry.

In terms of twisted variables the supersymmetric action (4.1) can be written as the sum of a Q -closed contribution and various Q -exact terms:

$$S_{\text{YM}} = \frac{1}{(g_{\text{YM}}^{5\text{D}})^2} \left[CS_{3,2}(A + \sigma\kappa) + i\text{Tr} \int \kappa \wedge d\kappa \wedge \Psi \wedge \Psi \right] + QW_{\text{vec}}, \quad (4.11)$$

where

$$CS_{3,2}(A) = \text{Tr} \int \kappa \wedge F \wedge F, \\ W_{\text{vec}} = \frac{1}{(g_{\text{YM}}^{5\text{D}})^2} \text{Tr} \int \left[\Psi \wedge \star(-\iota_R F - d_A \sigma) - \frac{1}{2} \chi \wedge \star H + 2\chi \wedge \star F + \kappa \wedge d\kappa \wedge (\sigma\chi) \right]. \quad (4.12)$$

Here R is the Reeb vector and κ its dual one-form $\kappa = gR$. Note in particular that $\iota_R \kappa = 1$.

Supersymmetry requires the overall coefficient of the term in square brackets in (4.11) to be constant. Reducing to four dimensions along $X = \partial_\alpha$ these terms give rise to a non constant θ term proportional to $\iota_X \kappa$

$$\text{Tr} \int_B \frac{2\pi r}{(g_{\text{YM}}^{5\text{D}})^2} (\iota_X \kappa) F \wedge F + \dots \quad (4.13)$$

Here the dots stand for several other terms, that are necessary to preserve supersymmetry and go away in the flat space limit $r \rightarrow \infty$. When considering the reduction of the complete action (4.11) this non constant θ term is cancelled by a contribution coming from the Q -exact terms.

Because the supercharge squares to a translation along R we can multiply each of the Q -exact terms by arbitrary functions, constant along R , preserving supersymmetry. Indeed such deformations are not just supersymmetric but Q -exact, thus they do not affect the value of supersymmetric observables. In the previous subsection we have considered reducing along a $U(1)$ fiber with length given by e^ϕ . This length is invariant along R hence we can multiply all the Q -exact terms by $e^{-\phi}$. Upon reduction we obtain a theory on the base B with constant g_{YM} . We must however pay attention to the fact that we could not rescale the supersymmetrized $CS_{3,2}$ term in the action. As a consequence the four dimensional theory will include the non-constant θ term (4.13) as it is no longer cancelled by the Q -exact terms.

4.3.1 Weyl rescaling

Performing a dimensional reduction along a $U(1)$ fiber of varying length leads to a four dimensional field theory with varying coupling constant. It is natural to consider if it is possible to write a supersymmetric theory on M with a deformed metric, so that the fiber is of constant length. For instance, this can be achieved via a Weyl rescaling of the five dimensional metric. Note that the factor e^ϕ in the metric (4.4) which controls the length of the fiber is constant along the Reeb vector, hence it is plausible that supersymmetry can be preserved under such a rescaling.

In order to address this question we can make use of the general framework for constructing supersymmetric field theories in curved space by taking a rigid limit of supergravity coupled to matter [13]. For $\mathcal{N} = 1$ field theories in five dimensions the appropriate rigid limit of supergravity has been studied in [51–54]. In order to preserve supersymmetry with the Weyl rescaled metric there must exist a solution to a Killing spinor equation generalizing (4.3)

$$D_m \xi_I - t_I^J \Gamma_m \xi_J - \mathcal{F}_{mn} \Gamma^n \xi_I - \frac{1}{2} \mathcal{V}^{pq} \Gamma_{mpq} \xi_I = 0. \quad (4.14)$$

Here D_m includes a background $SU(2)_R$ connection and \mathcal{F}_{mn} , \mathcal{V}^{pq} are background supergravity fields. For instance \mathcal{F}_{mn} is the graviphoton field strength.⁵

We also require that the supercharge continues to square to translations along the Reeb vector R and that the solution of the generalized Killing equation (4.14) is continuously connected to the original solution of (4.3) as the rescaling factor approaches unity. Under these conditions the rigid limit of supergravity will provide a deformation of the theory given by (4.1) that is supersymmetric on the Weyl rescaled manifold. This theory is given by $\mathcal{N} = 1$ SYM minimally coupled to the Weyl rescaled metric together with terms that vanish in the flat space limit and are required by supersymmetry.

The analysis of the Killing spinor equation (4.14), showing that the Weyl rescaling can be performed preserving supersymmetry, is presented in appendix D. The supersymmetric variations of the fields are deformed under rescaling, however the supercharge continues to square to translations along the Reeb vector R . As a consequence it is possible to define appropriate twisted variables giving rise to a cohomological complex of the same form as in the Sasaki-Einstein case (see appendix E). In particular the role of σ in the complex is now played by $\tilde{\sigma} = e^{-\phi} \sigma$. We can write an action in terms of the twisted variables as before:

$$S_{\text{YM}} = \frac{1}{(g_{\text{YM}}^{5\text{D}})^2} \left[-CS_{3,2}(A + \tilde{\sigma}\kappa) + i \int \kappa \wedge d\kappa \wedge \tilde{\Psi} \wedge \tilde{\Psi} + QW_{\text{vec}} \right]. \quad (4.15)$$

Here κ is the same one form as in the Sasaki-Einstein case. In particular $\iota_R \kappa = 1$. By taking the Q -exact terms of the same form as in (4.12) (rescaled by factors that are constant along R) all the leading terms in the action stemming from rigid supergravity can be matched to (4.15). It follows that, if the twisted variables are held fixed under Weyl rescaling, the theory (4.15) is a Q -exact deformation of the theory on the original Sasaki-Einstein manifold. Upon reduction the action (4.15) gives rise to $\mathcal{N} = 2$ SYM on

⁵An equation stemming from setting to zero the variation of the dilatino needs to be satisfied as well.

the Weyl rescaled base with constant coupling g_{YM} because the length of the fiber is now constant. As before however, there will be a position dependent θ term stemming from the reduction of $CS_{3,2}$.

5 Partition function

In this section, we use the results for the partition function of $\mathcal{N} = 1$ theories on toric Sasaki-Einstein manifolds [33, 49], to compute the partition function for the reduced 4D theory. This is done by discarding the contribution of non-zero KK modes along the $U(1)$ fibre. The answer has a similar structure to the partition function for $\mathcal{N} = 2$ on squashed S^4 [1, 16], in that it factorizes to a product of contributions corresponding to isolated points on the manifold. On S^4 , these points are the poles, while here they are the fixed points of the torus action on the 4D manifold. Just as for S^4 , half of them will support instantons while the other half support anti-instantons.

5.1 Perturbative sector

The results stated here are given in terms of a generic Reeb vector field, for which we do not have a Sasaki-Einstein metric. Because the partition function depends only on the Reeb, and the cohomological complex (see appendix E) has a straightforward generalisation for generic Reeb, our result below is still valid.

From the computation of [49], the perturbative contribution to the partition function of the 5D $\mathcal{N} = 1$ vector multiplet couple to hypermultiplet in representation \underline{R} reads

$$Z^{\text{pert}} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r^3}{(g_{\text{YM}}^5)^2} \varrho \text{Tr}[a^2]} \cdot \frac{\det'_{\text{adj}} S_3^C(ia|\vec{R})}{\det_{\underline{R}} S_3^C(ia + im + \vec{\xi} \cdot \vec{R}/2|\vec{R})}, \tag{5.1}$$

where $\varrho = \text{Vol}(M)/\text{Vol}(S^5)$, and S_3^C is the generalized triple sine associated to the cone C [55, 56], which is defined as

$$S_3^C(x|\vec{\omega}) = \prod_{\vec{m} \in C \cap \mathbb{Z}^3} (\vec{\omega} \cdot \vec{m} + x) \prod_{\vec{m} \in C^\circ \cap \mathbb{Z}^3} (\vec{\omega} \cdot \vec{m} - x). \tag{5.2}$$

Here C° is the interior of C , and C is the moment map cone of M . When the manifold is SE, there is a vector $\vec{\xi}$ such that $\vec{\xi} \cdot \vec{v}_i = 1 \forall i$, and so the product above can be written as

$$S_3^C(x|\vec{\omega}) = \prod_{\vec{m} \in C \cap \mathbb{Z}^3} (\vec{\omega} \cdot \vec{m} + x)(\vec{\omega} \cdot \vec{m} + \vec{\xi} \cdot \vec{\omega} - x). \tag{5.3}$$

The perturbative partition function for the reduced theory on B is obtained by keeping only the zero Kaluza-Klein modes along the S^1 fiber. Using the explicit description in section 3 of the cone C , the normals to its faces are

$$[\vec{v}_1, \dots, \vec{v}_n, \vec{v}_{n-1}, \dots, \vec{v}_{2n}] = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \dots & n & n-1 & \dots & 1 \\ 0 & 0 & * & * & * & * & * & * \end{bmatrix}.$$

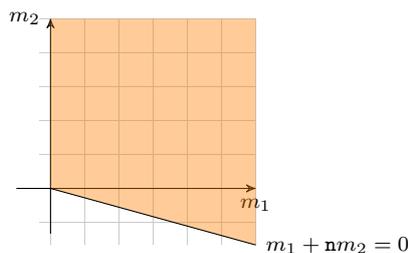


Figure 5. m_1, m_2 runs over the shaded region, which is the cone \tilde{C} .

Because $\vec{X} = [0; 0; 1]$, we keep only the modes $\vec{X} \cdot \vec{m} = m_3 = 0$. Geometrically, this is the intersection of the cone C with the plane with normal vector \vec{X} . Now the constraint $\vec{v}_i \cdot \vec{m} \geq 0, \forall i$ reads

$$m_1 + pm_2 \geq 0, \quad p = 0, \dots, n.$$

As a result, the region for m_1, m_2 is as in figure 5. This region is a 2D cone \tilde{C} . To describe the resulting perturbative partition function, we define the lower dimensional analogue of S_3^C :

$$\Upsilon^{\tilde{C}}(x|\vec{\omega}) = \prod_{\vec{m} \in \tilde{C} \cap \mathbb{Z}^2} (\vec{\omega} \cdot \vec{m} + x) \prod_{\vec{m} \in \tilde{C}^\circ \cap \mathbb{Z}^2} (\vec{\omega} \cdot \vec{m} - x). \quad (5.4)$$

This special function is the even-dimensional analogue of the multiple sine functions that appear in odd dimensions. It is a straightforward generalization of the perturbative answer that appeared in [1], see also [3].⁶

In terms of the above special function, the perturbative result can be written as

$$Z^{\text{pert}} = \int_t da e^{-\frac{8\pi^3 r^3}{(g_{\text{YM}}^{5D})^2} \text{Tr}[a^2]} \cdot \frac{\det'_{\text{adj}} \Upsilon^C(ia|R^1, R^2)}{\det_R \Upsilon^C(ia + im + \vec{\xi} \cdot \vec{R}/2|R^1, R^2)}. \quad (5.5)$$

Notice that it is the 5D YM coupling that appear in the classical action. In the next section, we will explain how the combination $\frac{g}{(g_{\text{YM}}^{5D})^2}$ actually is a natural 4D quantity that involves the position dependent 4D coupling evaluated at the torus fixed points.

Next, we will investigate the asymptotic behavior of (5.5) as we go the large radius limit where the local geometry approaches flat space and we can compare with well-known flat space results.

For computing the asymptotic behavior, we will use the approach of [33], for details we refer the reader to section 6 of that paper. The asymptotic behavior of the above matrix model is given by

$$Z^{\text{pert}} \sim \int_t da e^{-\frac{8\pi^3 r^3}{(g_{\text{YM}}^{5D})^2} \text{Tr}[a^2]} \cdot e^{\text{Tr}_{\text{adj}} V_v^{\text{asy}}(ia)} \cdot e^{\text{Tr}_R V_h^{\text{asy}}(ia+im)}, \quad (5.6)$$

⁶It is possible to write an expression for Υ^C as a factorized product over contributions from the torus fixed points. While we do not write it explicitly here, this factorization will be apparent in the next subsection.

where the functions $V_v^{\text{asy}}, V_h^{\text{asy}}$ give the asymptotic contributions of the vector and hypermultiplet respectively. They are given by

$$V_h^{\text{asy}}(x) = -\rho \left[6x^2 - \left(4x^2 + \frac{1}{3}(-\omega_1^2 + 2\mathbf{n}\omega_1\omega_2 + 2\omega_2^2) \right) \log|x| \right], \quad (5.7)$$

and

$$V_v^{\text{asy}}(x) = \rho \left[6x^2 - \left(4x^2 - \frac{2}{3}(\omega_1^2 + \mathbf{n}\omega_1\omega_2 - \omega_2^2) \right) \log|x| \right]. \quad (5.8)$$

Here ρ is given by

$$\rho = \frac{\mathbf{n}}{4R_2(\mathbf{n}R_1 - R_2)}. \quad (5.9)$$

We note that this only depends on the number of sides of our moment map cones, i.e. the number of fixed points, and is independent of the overall shape of the cone.

5.2 Comparison with flat space results

We consider the asymptotic contributions from the vector and hyper as given above in equations (5.7), (5.8). The two terms contribute to the effective action at the point $\sigma = ia$ on the Coulomb branch. We focus on the $a^2 \log(r|a|)$ terms and compare them to the 1-loop β -function of $\mathcal{N} = 2$ SYM in flat space. Putting together the classical action and the quantum generated effective action, focusing only on the log term we have

$$S_{\text{eff}} = -\frac{8r^3}{(g_{\text{YM}}^{\text{D}})^2} \text{Vol}_M \text{Tr}_f[a^2] - \frac{\mathbf{n}r^2}{R_2(R_2 - \mathbf{n}R_1)} \log(r|a|) \text{Tr}_{\text{adj}}[a^2] + \frac{\mathbf{n}N_f r^2}{R_2(R_2 - \mathbf{n}R_1)} \log(r|a|) \text{Tr}_R[a^2], \quad (5.10)$$

where in the log we have r^{-1} as the renormalisation scale at which g_{YM} is defined, and we have extracted the powers of r from the volume, so Vol_M here is just a number.

We now rewrite the above as a sum of contributions from the fixed points of $U(1)^2$ acting on B , which are also the loci of the closed Reeb orbits. To this end we rewrite the volume Vol_M as

$$\text{Vol}_M = \pi^3 \sum_i \frac{-[X, v_i, v_{i+1}]^2}{[R, v_i, v_{i+1}][R, v_i, X][R, v_{i+1}, X]}. \quad (5.11)$$

This formula is derived using localisation techniques on K-contact manifolds in [57]. The sum is over the corners of Δ_μ (see section 3 for notations). At each of these corners resides a closed Reeb orbit, and each contributes ($v_i = u, v_{i+1} = v$)

$$\frac{\pi^2}{2} \ell_O \cdot \frac{1}{\epsilon_1 \epsilon_2} \cdot (\iota_X \kappa)^2 = \frac{\pi^2}{2} \frac{2\pi}{[u, v, R]} \cdot \frac{-[R, u, v]^2}{[R, u, X][R, v, X]} \cdot \left(\frac{[X, u, v]}{[R, u, v]} \right)^2,$$

where ℓ_O is the length of the closed Reeb orbit O as measured by the contact form κ ; $\epsilon_{1,2}$ are the weights of X acting on the space transverse to O .⁷ It is an interesting exercise

⁷To match with [57] it is useful to note that $(\iota_X \kappa)^2$ is the 0-form component of the equivariantly completed form $(d\kappa)^2$.

to show that the sum in (5.11) actually is independent of X ; as of course the volume of M should be. Note also that for certain choices of R, X , summands of (5.11) may be ill-defined, yet the total sum still makes sense.

On the other hand (note $[X, v_i, v_{i+1}] = \pm 1$)

$$\begin{aligned}
 & \sum_i \frac{-[R, v_i, v_{i+1}]^2}{[R, v_i, X][R, v_{i+1}, X]} \cdot \left(\frac{[X, v_i, v_{i+1}]}{[R, v_i, v_{i+1}]} \right)^2 \\
 &= \sum_i \frac{-1}{[R, v_i, X][R, v_{i+1}, X]} \\
 &= \frac{-1}{(-R_2)(R_1 - R_2)} + \frac{-1}{(R_1 - R_2)(2R_1 - R_2)} + \cdots + \frac{-1}{((\mathbf{n} - 1)R_1 - R_2)(\mathbf{n}R_1 - R_2)} \\
 & \quad + \frac{-1}{(\mathbf{n}R_1 - R_2)((\mathbf{n} - 1)R_1 - R_2)} + \cdots + \frac{-1}{(R_1 - R_2)(-R_2)} \\
 &= \frac{2\mathbf{n}}{R_2(\mathbf{n}R_1 - R_2)} = 8\rho,
 \end{aligned}$$

where we recognize ρ defined in (5.9). Using these two results, we can write both ρ and Vol_M as a sum over the corners of Δ_μ . Starting from (5.10) the i^{th} corner contribution is,

$$\begin{aligned}
 S_{\text{eff}}|_{i^{\text{th}} \text{ corner}} & \stackrel{\log}{\sim} \left(-\frac{4\pi^2}{(g_{\text{YM}}^{5\text{D}})^2} \frac{2\pi r}{[R, v_i, v_{i+1}]} - \frac{1}{2} \log(r|a|) \frac{c_{\text{adj}}}{c_f} + \frac{N_f}{2} \log(r|a|) \frac{c_R}{c_f} \right) \\
 & \quad \times \frac{r^2}{[R, v_i, X][v_{i+1}, R, X]} \text{Tr}_f[a^2],
 \end{aligned}$$

where c_R is the Casimir in the representation R , i.e. $\text{Tr}_R[t^a t^b] = c_R \delta^{ab}$. To interpret this formula, we note that the local geometry close to a corner is that of $S^1 \times \mathbb{C}^2$ where the radius of S^1 is

$$\frac{r}{[R, v_i, v_{i+1}]} = r e^{\phi(x_i)}.$$

So we can recognize the 4D position dependent coupling constant,

$$\frac{2\pi r e^{\phi(x_i)}}{(g_{\text{YM}}^{5\text{D}})^2} = \frac{1}{g_{\text{YM}}^2(x_i)}, \tag{5.12}$$

as defined in (4.9). Furthermore, $[R, v_i, X]$, $[v_{i+1}, R, X]$ are proportional to the weights of X acting on the space transverse to S^1 , i.e. \mathbb{C}^2 . It follows that

$$\frac{1}{[R, v_i, X][v_{i+1}, R, X]} = \text{Vol}_{\mathbb{C}_{\epsilon_1, \epsilon_2}},$$

i.e. the volume of \mathbb{C}^2 computed equivariantly. Putting these together

$$S_{\text{eff}}|_{i^{\text{th}} \text{ corner}} \stackrel{\log}{\sim} \left(-\frac{4\pi^2}{g_{\text{YM}}^2(x_i)} - \frac{1}{2} \log(r|a|) \frac{c_{\text{adj}}}{c_f} + \frac{N_f}{2} \log(r|a|) \frac{c_R}{c_f} \right) \text{Vol}_{\mathbb{C}_{\epsilon_1, \epsilon_2}} \text{Tr}_f[r^2 a^2].$$

The quantity in the brace gives the well-known 1-loop running coupling for $\mathcal{N} = 2$ theories.

The formula (5.11) for Vol_M can be used to write the classical action as

$$S_{\text{cl}}(ia) = \sum_i \frac{4\pi^2 r^2}{g_{\text{YM}}^2(x_i)} \frac{1}{\epsilon_1^i \epsilon_2^i} \text{Tr}[a^2].$$

This matches known results on the squashed S^4 [1, 16].

5.3 Instanton sector

For the instanton sector we proceed with the same strategy as for the perturbative sector. We restrict the 5D results to the zero KK mode along X . The instanton sector for the 5D theory is computed by gluing together flat space results

$$Z_{\text{inst}}^{\mathbb{C}^2 \times S^1}(a|\beta, \epsilon_1, \epsilon_2), \tag{5.13}$$

one copy for each closed Reeb orbit. Here β is the radius of the Reeb orbit, and ϵ_1, ϵ_2 are the equivariant rotation parameters, which are determined by the local geometry [49]. The role of instanton counting parameter is played by $q = \exp[-16\pi^3 \frac{\beta}{(g_{\text{5D}}^{\text{YM}})^2}]$. The argument leading to this result is that the point like instantons propagating along closed Reeb orbits are the only solution invariant under the torus action. A rigorous proof is not available at the moment, though in 4D and \mathbb{P}^2 some tests have confirmed this expectation [21].

In 5D, the instanton equation reads

$$F_5 = - *_{\text{5}} \kappa \wedge F_5, \tag{5.14}$$

which is called the contact instanton equation. After reducing to 4D, it turns into some PDE's, whose expression at a general point is not illuminating. They can however be analyzed at the fixed-points of the torus action, which correspond to the corners of the polygon Δ_{μ} .⁸

Take the hexagon example (3.8). In figure 6 we have marked with \pm whether X aligns or anti-aligns with R , (e.g. at the corner 61, since $[\vec{v}_6, \vec{v}_1, \vec{X}] = -1$ we get anti-alignment.) At any toric fixed point where two of the torus actions degenerate, we can decompose the vector X as its part along R and the rest:

$$X = (\iota_X \kappa)_R + X^{\perp},$$

where X^{\perp} is a locally degenerate vector with zero norm at the fixed point. Using that the Reeb is normalized, we have that at the fixed point,

$$\langle X, X \rangle = (\iota_X \kappa)^2 \langle R, R \rangle = (\iota_X \kappa)^2,$$

on the other hand from the metric (4.4) we have $\langle X, X \rangle = e^{2\phi}$ everywhere (here we set $r = 1$). Dropping the zero norm part we conclude that at the fixed point

$$X = (\iota_X \kappa)_R = \pm e^{\phi} R. \tag{5.15}$$

With this observation, we can reduce the 5D instanton equation at a fixed point as follows. The horizontal part of the 5D instanton equation (5.14) reads

$$F_4 + \tilde{\varphi} d\beta = - *_{\text{5}} \kappa \wedge (F_4 + \tilde{\varphi} d\beta) = - \iota_R *_{\text{5}} (F_4 + \tilde{\varphi} d\beta) = \pm e^{-\phi} \iota_X *_{\text{5}} (F_4 + \tilde{\varphi} d\beta) = \pm *_{\text{4}} (F_4 + \tilde{\varphi} d\beta),$$

⁸We emphasize that some of the 4-manifolds B are not toric Kähler, we are merely using the moment polygon of M to visualize the geometry of B .

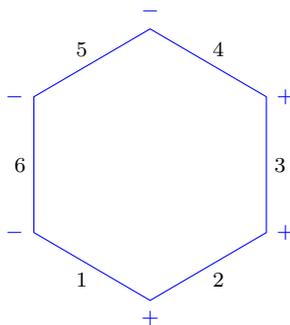


Figure 6. The momentum polygon Δ_μ whose normals are (3.8).

where \star_4 in the final step uses the metric of the 4D base. The factor $e^{-\phi}$ precisely makes up the difference between the 5D and 4D volume form. The 5D contact instanton equations at the fixed-points thus reduce to the deformed 4D instanton equations

$$F = \pm \star F - \tilde{\varphi}(d\beta \mp \star d\beta) , \tag{5.16}$$

which we also can write as

$$F^\pm = -\tilde{\varphi}d\beta^\pm .$$

At each fixed point, keeping only the zero KK mode of the 5D answer (5.13), we get

$$\prod_i Z_{\text{inst}}^{\mathbb{C}^2}(a, q_i | \epsilon_1^i, \epsilon_2^i) , \tag{5.17}$$

where $q_i = \exp[-\frac{8\pi^2}{g_{\text{YM}}^2(x_i)}]$. This is valid before turning on a θ -term. With $\theta \neq 0$ in 4D, we need to distinguish which of these contributions will arise from point-like instantons versus anti-instantons. This can be seen from the local behavior of the reduced contact instanton equation (5.16), namely we will have an instanton wherever X and R align, and an anti-instanton when they anti-align.

Thus the counting parameter for an instanton at fixed point i will be

$$e^{2\pi i \tau(x_i)} = q_i .$$

Similarly the counting parameter for an anti-instanton will be $\bar{q}_j = \exp(-2\pi i \bar{\tau}(x_j))$.

Using the form of the normals given in (3.11), the total instanton partition function will be a product of $2n$ factors, each of which depends holomorphically on either q or \bar{q} ,

$$Z_{\text{inst}}^B(a|\vec{R}) = \prod_{i=1}^n Z_{\text{inst}}^{\mathbb{C}^2}(a, q_i | \epsilon_1^i, \epsilon_2^i) \times \prod_{i=n+1}^{2n} Z_{\text{inst}}^{\mathbb{C}^2}(a, \bar{q}_i | \epsilon_1^i, \epsilon_2^i) .$$

The fact that the first n fixed points support instantons and the rest anti-instantons follow from the form of the normals. Hence for the geometries we are considering, by construction we will always have an equal number of instanton and anti-instanton contributions.

We can further write down the equivariant parameters at fixed-point i . For $i = 1, \dots, n$ these are

$$\epsilon_1^i = (i - 1)R_1 - R_2 , \quad \epsilon_2^i = iR_1 - R_2 ,$$

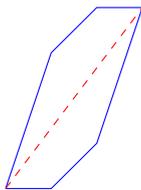


Figure 7. An example of the polygons that exhibit a reflection symmetry about the dashed red line. For the corresponding manifold the local geometry at the fixed points corresponding to parallel sides will be the same, and as a consequence the instanton partition function can be written as something explicitly real.

and for $i = n + 1, \dots, 2n$ they are

$$\epsilon_1^i = (2n + 1 - i)R_1 - R_2, \quad \epsilon_2^i = (2n - i)R_1 - R_2 .$$

Thus for a given $i \leq n$, fixed points i and $2n + 1 - i$ have the same equivariant parameters, and they will support instantons and anti-instantons respectively. It is thus tempting to try and combine the two corresponding instanton partition functions into something of the form $|Z_{\text{inst}}(q_i)|^2$, however this cannot be done in general because of the position dependent τ . In other words, generically we do not have $\tau(x_i) = \tau(x_{2n+1-i})$, since $\tau(x_i)$ unlike $\epsilon_{1,2}^i$ depends on the shape of the polygon. However, for all n , there exists a sub-class of polytopes, that allow a choice of Reeb for which $\tau(x_i) = \tau(x_{2n+1-i})$. In particular, inside this class are polytopes that have \mathbb{Z}_2 symmetry about their diagonal axis, see figure 7. In these cases, after appropriately selecting R , the instanton partition function takes the form

$$Z_{\text{inst}}^B(a|\vec{R}) = \prod_{i=1}^n |Z_{\text{inst}}^{\mathbb{C}^2}(a, q_i|\epsilon_1^i, \epsilon_2^i)|^2 ,$$

which very closely mimics the answer on S^4 found in [1].

For concreteness we will work out one example in some detail. Consider again the $S^2 \times S^2$ coming from the reduction of a $Y^{p,q}$ space.

Example 5.1 ($S^2 \times S^2$, from reduction of $Y^{p,q}$). For these spaces with topology $S^2 \times S^2$ (we emphasize that their metric is not canonical), we have 4 fixed-points, corresponding to the ways of choosing one pole from each of the S^2 's. Using the normals given in (3.7), and the prescriptions explained in section 5.2 for computing the local data, we find the following parameters

	1	2	3	4
$[v_i, v_{i+1}, X]$	+1	+1	-1	-1
$\beta^{-1} = e^{-\phi(x_i)}$	R_3	$R_3 + (p - q)(R_1 - R_2)$	$-R_3 + pR_1 + q(R_1 - R_2)$	$-R_3 + pR_2$
ϵ_1	$-R_2$	$R_1 - R_2$	$2R_1 - R_2$	$R_1 - R_2$
ϵ_2	$R_1 - R_2$	$2R_1 - R_2$	$R_1 - R_2$	$-R_2$

The first line of the table tells us if X and R align or anti-align at the fixed-point, and the second line tells us the inverse radius of the Reeb orbit. The local complex coupling $\tau(x_i)$

at fixed-point i given by

$$\tau(x_i) = \frac{4\pi i \beta_i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi} .$$

We see that the expression for β_i depends on R_3 and on the shape of the polygon, i.e. on the integers p, q , whereas the equivariant parameters $\epsilon_{1,2}$ do not. Thus the instanton partition function for these spaces takes the form

$$\begin{aligned} & Z_{\text{inst}}^{\mathbb{C}^2}(a|q_1, -R_2, R_1 - R_2) Z_{\text{inst}}^{\mathbb{C}^2}(a|q_2, R_1 - R_2, 2R_1 - R_2) \\ & \times Z_{\text{inst}}^{\mathbb{C}^2}(a|\bar{q}_3, 2R_1 - R_2, R_1 - R_2) Z_{\text{inst}}^{\mathbb{C}^2}(a|\bar{q}_4, R_1 - R_2, -R_2) . \end{aligned}$$

6 Summary

In this paper we have constructed $\mathcal{N} = 2$ 4D gauge theories on a wide class of toric manifolds with the topology of connected sums of $S^2 \times S^2$. The construction comes from the reduction of toric Sasaki-Einstein manifolds along a free $U(1)$ chosen in such a way that it preserves 5D Killing spinors. We can reduce to the 4D geometrical data from the 5D toric Sasaki-Einstein geometry. However at the moment we are missing a description of the 4D geometry in intrinsic 4D terms. It would be important to further study this 4D geometry and see if our 4D toric manifolds are part of a bigger class of toric manifolds which allow $\mathcal{N} = 2$ theories. Another important issue is that the resulting 4D theories have a point dependent coupling constant. We think that this feature of the 4D theory should be taken seriously, and one should study the supergravity origin of this when placing the supersymmetric theory on curved manifolds.

We also calculated the exact partition function for these 4D theories, by reducing the 5D answer. The toric manifolds we consider have an even number of fixed points, half of which corresponds to instanton contribution to the partition function and half to anti-instanton contributions. Although S^4 is formally outside of our analysis, the formal expression for the partition function coincides with Pestun's result (as well as for the squashed S^4). We conjecture that the $\mathcal{N} = 2$ partition function on any toric simply connected 4D manifold will have the same structure that we have found here. It would be curious to see if our results has any AGT-like explanation coming from the reduction of 6D theories, especially taking into account the fact that the coupling τ is point dependent.

In this paper we concentrate on the reduction of 5D supersymmetric gauge theory to 4D supersymmetric gauge theory and our framework requires that the underlying four manifold is spin. The present analysis can be extended to a wider class of theories and manifolds for instance by formulating it in cohomological terms. Namely we can start from 5D cohomological theory defined in [46] and reduce down to 4D cohomological theory. In this case we should expect a similar effect when in 4D we glue together the contributions of instantons for some of the fixed points and contributions of anti-instantons for the other fixed points. We plan to explore these more general 4D theories elsewhere.

Acknowledgments

We thank first of all Dario Martelli who shared with us his insight that inspired this project. We also thank Marco Chiodaroli, Marco Golla, Thomas Kragh, Yiwen Pan and Luigi Tiziano for helpful discussions, and finally Mark Hamilton, who pointed out an imprecision in our statement about 4-manifolds. The work of MZ is supported in part by Vetenskapsrådet under grant #2014-5517, by the STINT grant, and by the grant “Geometry and Physics” from the Knut and Alice Wallenberg foundation. The work of GF is supported by the ERC STG grant 639220.

A Details of the reduction conditions

Here, we give the proofs of statements in section 2. Remember that $S^1 \rightarrow M \rightarrow B$ is our nontrivial circle bundle and that $E \rightarrow M$ is the bundle we wish to push down.

Proposition A.1. *We use α as the coordinate of the circle fibre and we let A be a connection of E , then if $P \exp i \int_0^{2\pi} d\alpha A_\alpha = id$, the bundle E can be pushed down.*

Proof. The bundle M can be trivialised as $S^1 \times U_i$, where $\{U_i\}$ is a cover of B . Then choose a cover of M of the form $V_{is} = (a_{is}, b_{is}) \times U_i$, where (a_{is}, b_{is}) , $s = 1, \dots, n_i$ covers an interval of the circle fibre. Assume that the cover is chosen fine enough so that E is trivialised over $\{V_{is}\}$, and let $g_{is,jt}$ be the transition function of E . We first show that the transition function can be made independent of α .

On a patch V_{is} , we denote the connection as A^{is} and so on the intersection $V_{is} \cap V_{jt}$ the connections are related as

$$A^{is} = g_{jt,is}^{-1} dg_{jt,is} + g_{jt,is}^{-1} A^{jt} g_{jt,is} .$$

We first adjust locally the trivialisation by multiplying with a Wilson line

$$h_{is}(\alpha, x) = P \exp \int_{a_{is}}^{\alpha} d\alpha (-A_\alpha^{is}) ,$$

where (α, x) are the fibre and base coordinates. The Wilson lines h_{is} satisfies

$$\partial_\alpha h_{is}(\alpha, x) = -A_\alpha^{is} h_{is}(\alpha, x) .$$

Then the new transition function becomes $\tilde{g}_{is,jt} = h_{is}^{-1} g_{is,jt} h_{jt}$. Let us look at

$$\begin{aligned} \partial_\alpha \tilde{g}_{is,jt} &= \partial_\alpha (h_{is}^{-1} g_{is,jt} h_{jt}) = h_{is}^{-1} A_\alpha^{is} g_{is,jt} + h_{is}^{-1} (\partial_\alpha g_{is,jt}) h_{jt} - h_{is}^{-1} g_{is,jt} A_\alpha^{jt} h_{jt} \\ &= h_{is}^{-1} g_{is,jt} \left(g_{is,jt}^{-1} A_\alpha^{is} g_{is,jt} - A_\alpha^{jt} + g_{is,jt}^{-1} \partial_\alpha g_{is,jt} \right) h_{jt} = 0 . \end{aligned}$$

So the new transition function is independent of the α direction, and the new connection satisfies $\tilde{A}_\alpha^{is} = 0$ by construction.

Now we assume that such adjustment has been made and all transition functions are α independent and $A_\alpha = 0$. Now for each fixed i , we readjust the trivialisation on $V_{i2}, V_{i3}, \dots, V_{in_i}$ by multiplying by

$$g_{i1,i2}, g_{i2,i3} g_{i1,i2}, g_{i3,i4} g_{i2,i3} g_{i1,i2}, \dots$$

This way we can make the transition functions $g_{is,i(s+1)}$ identity except possibly when one goes a full circle, i.e. $g_{in_i,i1}$ (nonetheless it is still α -independent). But the quantity $g_{in_i,i1}$ can be computed as the holonomy along the circle fibre of the original connection A . If this last holonomy is trivial then $g_{in_i,i1} = 1$, which means that $V_{i1}, V_{i2}, \dots, V_{in_i}$ for fixed i can be pieced together and become one single open set that covers the whole circle, i.e. $V_i = [0, 2\pi] \times U_i$. The transition is by construction α -independent, and so one can push down the bundle E to B \square

Proposition A.2. *Suppose that the criterion in proposition A.1 is satisfied, then the sections of E satisfying $D_\alpha s = 0$ can be pushed down.*

Proof. Repeat the adjustment as in proposition A.1 to make the transition functions independent of α , and we continue to use the notation there. Let s be a section, if on each patch V_i one has $\partial_\alpha s = 0$, then clearly s can be regarded as a section of the pushdown bundle on B .

Now undo the adjustments of trivialisation then $\partial_\alpha s = 0$ reverts to $D_\alpha s = 0$. \square

B Spinor conventions and bilinears

We follow the convention for spinors of [47]. Let $\{e^a\}$ be a set of vielbein which reduces the structure group of M^5 to $SO(5)$. The gamma matrices satisfy the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab},$$

and the charge conjugation relation

$$C^{-1}(\Gamma^a)^T C = \Gamma^a, \quad C^T = -C, \quad C^* = C.$$

Denote by $\Gamma_m = \Gamma^a e_m^a$, which satisfy $\{\Gamma_m, \Gamma_n\} = 2g_{mn}$.

The spinor bi-linears are formed using C ,

$$\psi^T C \chi \xrightarrow{\text{abbreviate}} \psi \chi, \tag{B.1}$$

throughout the paper, the bi-linears are abbreviated as $(\psi \chi)$, following [47].

Denote by

$$\Gamma^{a_1 \dots a_n} = \frac{1}{n!} \Gamma^{[a_1} \dots \Gamma^{a_n]}$$

and similarly for their curved space counterpart. We use Dirac's slash notation

$$\not{M} = M \cdot \Gamma = M_{i_1 \dots i_p} \Gamma^{i_1 \dots i_p}, \quad M \in \Omega^p(M),$$

we will even drop the slash whenever confusion is unlikely.

The $SU(2)$ R-symmetry index are raised and lowered from the left

$$\xi^I = \epsilon^{IJ} \xi_J, \quad \xi_I = \epsilon_{IJ} \xi^J, \quad \epsilon^{IK} \epsilon_{KJ} = \delta^I_J, \quad \epsilon^{12} = -\epsilon_{12} = 1.$$

In 5D, one cannot impose the Majorana condition on a spinor, but we can instead impose the symplectic Majorana condition, which for a pair of spinors ξ_I reads

$$\overline{\xi_I^\alpha} = C_{\alpha\beta} \epsilon^{IJ} \xi_J^\beta. \tag{B.2}$$

B.1 Spinor bilinears and some of their properties

Given a symplectic Majorana spinor ξ_I we can construct the following spinor bilinears out of it:

$$s = -\xi^I \xi_I > 0, \quad R^m = \xi^I \Gamma^m \xi_I, \quad \Theta_{mn}^{IJ} = \xi^I \Gamma_{mn} \xi^J. \quad (\text{B.3})$$

These will satisfy various relations, and we use the following when solving the Killing spinor equation:

$$R_m R^m = s^2, \quad R^m \Theta_{mn}^{IJ} = 0, \quad s \Theta_{mn}^{IJ} = \frac{1}{2} \epsilon_{mnpqr} R^p \Theta^{IJqr}. \quad (\text{B.4})$$

In particular when the spinor solves the Killing spinor equation on a SE manifold, we have $s = 1$ and the vector R is the Reeb of our contact structure. For this case, when we choose $t_I^J = \frac{i}{2} (\sigma_3)_I^J$ and denote the contact one-form as $\kappa = g(R)$, we have

$$d\kappa_{mn} = 2i(\sigma_3)_{IJ} \Theta_{mn}^{IJ}. \quad (\text{B.5})$$

C The example of $Y^{p,q}$

To make our procedure a bit more concrete, let us give some details on the example of $Y^{p,q}$, where we can write an explicit metric. We essentially take all the relevant information from Gauntlett, Martelli, Sparks and Waldram [40], where the metric on $Y^{p,q}$ is given as

$$ds^2 = \frac{1-y}{6} (d\theta + \sin^2 \theta d\phi^2) + \frac{dy^2}{w(y)q(y)} + \frac{q(y)}{9} [d\psi - \cos \theta d\phi]^2 + w(y) \left[d\alpha + \frac{a-2y+y^2}{6(a-y^2)} [d\psi - \cos \theta d\phi] \right]^2 \quad (\text{C.1})$$

where

$$w(y) = \frac{2(a-y^2)}{1-y},$$

$$q(y) = \frac{a-3y^2+2y^3}{a-y^2}.$$

Here the coordinates (θ, ϕ, y, ψ) describe the 4D base and α describes the S^1 fiber. The coordinates run over the following ranges

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad y_1 \leq y \leq y_2, \quad 0 \leq \psi \leq 2\pi, \quad 0 < \alpha < 2\pi l,$$

and the constant a is chosen in the range $0 < a < 1$, for which case the equation $q(y) = 0$ has one negative and two positive roots. We choose y_1 to be the negative root and y_2 to be the smallest positive root. This makes sure that the base manifold described by (θ, ϕ, y, ψ) has the topology of $S^2 \times S^2$, and that $w(y) > 0$ everywhere so that α describe a no-where degenerating S^1 fiber. More precisely, as explained in detail in [40], to get the proper SE manifold $Y^{p,q}$, we need to pick a such that $y_2 - y_1 = \frac{3q}{2p}$; which they show that you can always do for any coprime $p > q$. This also fixes the constant l which determine the range of α .

The metric above makes very explicit the S^1 fibration structure, and in our construction we dimensionally reduce along the α direction; which we emphasize is not the Reeb. The canonical Reeb vector in these coordinates is given by

$$R = 3 \frac{\partial}{\partial \psi} - \frac{1}{2} \frac{\partial}{\partial \alpha}, \tag{C.2}$$

which has constant unit norm. On the other hand we see that the radius of the S^1 fiber is given by $\sqrt{w(y)}$, which clearly is not constant over the base manifold.

One can also see that the 4D base manifold has the topology of $S^2 \times S^2$ by looking at the metric. First we can observe that for fixed y the first term in (C.1) describes the S^2 covered by coordinates (θ, ϕ) . We also see that y is running over an interval, and the last term of the first line describes a circle parametrized by ψ that degenerate at the ends of the interval (since $q(y_i) = 0$); which gives the second S^2 . From this, we know that the base has the structure of an S^2 bundle over S^2 , but it is not immediately clear that its the trivial $S^2 \times S^2$ rather than some non-trivial fibration. However this is shown in [40], and we won't repeat the argument here. So the 4D base is $S^2 \times S^2$, equipped with a non-standard metric given by the first line of (C.1).

Next, let us briefly explain how this is connected to the toric description in terms of a moment map cone and its normals, a story first told in [58]. As explained in section 3 the toric picture of the manifold is that of a T^3 fibration over a polygon where some S^1 fibers degenerate as we go to the edges. At the vertices, an entire T^2 degenerate and we have a local geometry of $\mathbb{C}^2 \times S^1$. For $Y^{p,q}$ we know that the base polygon has 4 edges (as seen in figure 2). Each edge corresponds to a particular pole of one of the S^2 , where the rotation of that S^2 has a fix point and thus degenerates. From the metric we see that the edges are given by $\{\theta = 0\}$, $\{\theta = \pi\}$, $\{y = y_1\}$ and $\{y = y_2\}$. We can then find vectors that generate rotations, i.e. combinations of $\partial_\phi, \partial_\psi$ and ∂_α , that are such that their norm vanishes at each of these poles. Doing this, we find

$$v_1 = \partial_\phi + \partial_\psi, \quad v_2 = \partial_\psi + \frac{p-q}{2l} \partial_\alpha, \quad v_3 = -\partial_\phi + \partial_\psi, \quad v_4 = \partial_\psi - \frac{p+q}{2l} \partial_\alpha, \tag{C.3}$$

which degenerate at $\theta = 0, y = y_1, \theta = \pi$ and $y = y_2$ respectively. Observe that we here rescale ∂_α by $1/l$ so that it has a normal period. In this computation, we use properties of the roots y_1, y_2 that relates them to p, q :

$$\frac{y_1 - 1}{3ly_1} = p + q, \quad \frac{1 - y_2}{3ly_2} = p - q.$$

These vectors v_1, \dots, v_4 are precisely the inward normals of the moment map cone, but to relate them to the ones given in section 3, we need to make a change of basis. Instead of using the basis $\partial_\phi, \partial_\psi, \partial_\alpha$, we should use a basis of vectors whose orbits all close. The orbits of $l^{-1} \partial_\alpha$ closes everywhere since it is a proper fibration, but that is not true of the orbits of ∂_ϕ and ∂_ψ . One suitable basis is instead given by

$$e_1 = \partial_\phi + \partial_\psi, \quad e_2 = -\partial_\phi + \frac{p-q}{2l} \partial_\alpha, \quad e_3 = -\frac{1}{l} \partial_\alpha,$$

where of course this choice is far from unique: any $SL_3(\mathbb{Z})$ transformation of this give us an equally good basis. In this basis, the vectors v_1, \dots, v_4 has the components

$$v_1 = [1, 0, 0], \quad v_2 = [1, 1, 0], \quad v_3 = [1, 2, p - q], \quad v_4 = [1, 1, p],$$

and we recognize the normals of the moment map cone as given in example 3.2. So we have seen explicitly how the toric description and the explicit metric is related to one-another.

D The Weyl rescaled background

As discussed in section 4.3, after performing a Weyl transformation where the metric is rescaled $g \rightarrow \tilde{g} = e^{-2\phi}g$, we wish to show that we can still have rigid supersymmetry on this new background. The scale factor ϕ is invariant along both the U(1) fiber, and along the Reeb, i.e. $L_X\phi = L_R\phi = \iota_R d\phi = 0$.

To do this we use the minimal off-shell 5D supergravity [59], and focus on the Killing spinor equation coming from requiring the supersymmetry variation of the gravitino to vanish:

$$D_m \xi_I - t_I^J \Gamma_m \xi_J - \mathcal{F}_{mn} \Gamma^n \xi_I - \frac{1}{2} \mathcal{V}^{pq} \Gamma_{mpq} \xi_I = 0, \quad (\text{D.1})$$

where D_m includes the coupling to the background $SU(2)_R$ gauge field A_{mI}^J , as $D_m \xi_I = \nabla_m \xi_I - A_{mI}^J \xi_J$. Here, $\mathcal{F} = d\mathcal{A}$ is the field strength of the graviphoton, \mathcal{V} is a 2-form background field and t_I^J is background $SU(2)_R$ triplet scalar. We will use a $\tilde{}$ to denote new quantities after the rescaling, while non-tilded variables denotes ‘old’ quantities. The idea now is that we can solve this equation by turning on these various background fields so that the new Killing spinor is a rescaling of the old one. In particular, we require that the new spinor $\tilde{\xi}_I$ is such that the bilinear giving us the Reeb vector is unchanged, i.e.

$$R^m = \xi^I \Gamma^m \xi_I = \tilde{\xi}^I \tilde{\Gamma}^m \tilde{\xi}_I,$$

and since $\tilde{\Gamma}^m = \Gamma_a \tilde{e}_a^m$ scales like the inverse vierbein, i.e. with e^ϕ , this fixes

$$\tilde{\xi}_I = e^{-\phi/2} \xi_I.$$

Next, we can compute how the spin connection changes because of the rescaling and then use our old solution to get rid of the derivative of the spinor from the equation. The spin connection changes as

$$\tilde{\omega}_m^{ab} = \omega_m^{ab} + (\partial^n \phi)(e_n^a e_m^b - e_n^b e_m^a), \quad (\text{D.2})$$

and using this as well as our old equation (2.4), our Killing spinor equation becomes

$$\begin{aligned} -\frac{1}{2}(\partial_m \phi)\tilde{\xi}_I - \frac{1}{2}(\partial^n \phi)\Gamma_{mn}\tilde{\xi}_I + A_{mI}^J \tilde{\xi}_J \\ + (t_I^J \Gamma_m - \tilde{t}_I^J \tilde{\Gamma}_m)\tilde{\xi}_J - \mathcal{F}_{mn} \tilde{\Gamma}^n \tilde{\xi}_I - \frac{1}{2} \mathcal{V}^{mp} \tilde{\Gamma}_{mnp} \tilde{\xi}_I = 0. \end{aligned} \quad (\text{D.3})$$

This is now an algebraic equation for $\mathcal{F}, \mathcal{V}, A$ and \tilde{t}_I^J , and it is a straightforward but somewhat tedious exercise to solve it. When solving, it is helpful to note that a symplectic

Majorana spinor χ_I is completely determined by the contractions $\xi^I \Gamma_m \chi_I$ and $\chi_{(J} \xi_{I)}$. So performing these contractions of equation (D.3), we get a set of equations that only involves spinor bilinears, and using the properties we know about the various bilinears, see appendix B.1, we find the following solution for our various background fields:

$$\begin{aligned} \mathcal{F} &= d\mathcal{A}, \quad \mathcal{A} = -\frac{1}{2}(e^{-\phi} - e^{-\phi_p})\kappa, \\ \mathcal{V} &= \frac{1}{2}e^{-\phi}d\phi \wedge \kappa, \\ \tilde{t}_I^J &= -\frac{i}{2}(e^\phi - 2e^{2\phi-\phi_p})(\sigma_3)_I^J, \\ A_I^J &= -i(1 - e^{\phi-\phi_p})(\sigma_3)_I^J \kappa. \end{aligned} \tag{D.4}$$

Here, ϕ_p is a free constant of our solution; and κ is the old contact 1-form. If we choose ϕ_p to be the scale factor ϕ evaluated at some point p on our manifold, then when ϕ is a constant scaling, we note that the background fields \mathcal{F}, \mathcal{V} and A all vanish, and the \tilde{t}_I^J becomes a simple scaling of the old t_I^J field. This shows that our solution is smoothly connected to the old SE solution.

From supergravity, we also get a second equation that we need to solve, the dilatino equation. This involves one further background scalar field C , which one can solve for directly in terms of the other background fields. Through a tedious computation, one can then check that our solution also solves this equation so that we indeed have a valid rigid supersymmetric background.

E Cohomological variables

The 5D supersymmetry given of our vector multiplet looks like

$$QA_m = i\xi^I \Gamma_m \lambda_I, \tag{E.1}$$

$$Q\sigma = i\xi^I \lambda_I, \tag{E.2}$$

$$Q\lambda_I = -\frac{1}{2}F_{mn}\Gamma^{mn}\xi_I + (D_m\sigma)\Gamma^m\xi_I + D_I^J\xi_J + 2\sigma\left(t_I^J\xi_J + \frac{1}{2}\mathcal{F}_{mn}\Gamma^{mn}\xi_I\right), \tag{E.3}$$

$$QD_{IJ} = -i(\xi_I\Gamma^m D_m\lambda_J) + [\sigma, \xi_I\lambda_J] + it_I^K\xi_K\lambda_J - \frac{i}{2}\mathcal{V}_{mn}(\xi_I\Gamma^{mn}\lambda_J) + (I \leftrightarrow J), \tag{E.4}$$

where the full set of background supergravity fields are included. In the SE case, the only non-zero background field is t_I^J , the others (\mathcal{F}, \mathcal{V}) vanish. We can make the structure of the supersymmetry clearer by switching to cohomological variables, following for example [48]. This change of variables is given by

$$\begin{aligned} \Psi_m &= \xi_I \Gamma_m \lambda^I, \quad \chi_{mn} = \xi_I \Gamma_{mn} \lambda^I + (\kappa_m \Psi_n - \kappa_n \Psi_m), \\ H &= Q\chi = 2F_H^+ + \Theta^{IJ}(D_{IJ} + 2t_{IJ}\sigma), \end{aligned}$$

where in the last line we have used our particular SE background to only keep t_{IJ} . Here Ψ is a fermionic one-form, and χ, H are horizontal, transversally self-dual 2-forms. F_H^+

denotes the self-dual part of the horizontal part of F ; and we see that χ and H essentially becomes the auxiliary fields. In these variables, the supersymmetry variation takes the form of the cohomological complex [46, 50],

$$QA = i\Psi, \tag{E.5}$$

$$Q\Psi = -\iota_R F + d_A(\sigma), \tag{E.6}$$

$$Q\sigma = -i\iota_R \Psi, \tag{E.7}$$

$$Q\chi = H, \tag{E.8}$$

$$QH = -iL_R^A \chi - [\sigma, \chi]. \tag{E.9}$$

Written in these variables it is clear that $Q^2 = -iL_R^A + G_\sigma$ where G_σ denotes a gauge transformation with parameter σ , and L_R^A is the gauge covariant Lie derivative along the Reeb, $L_R^A = L_R + G_{\iota_R A}$.

E.1 Weyl rescaled case

In the Weyl rescaled background as described in appendix D, we can perform the same change of variables. Now, the Reeb vector that appears in our supersymmetry is no longer normalized, so we have to insert its norm in the appropriate places in our change of variables. And since the background fields appear in the variation of χ , the definition of H will also change. So for our new background we make the change of variables

$$\begin{aligned} \tilde{\Psi}_m &= \tilde{\xi}_I \tilde{\Gamma}_m \lambda^I, & \tilde{\chi}_{mn} &= \tilde{\xi}_I \tilde{\Gamma}_{mn} \lambda^I + e^{-\phi}(\kappa_m \tilde{\Psi}_n - \kappa_n \tilde{\Psi}_m), \\ \tilde{H} &= Q\tilde{\chi} = 2e^{-\phi} F_H^+ + 2\sigma e^{-\phi}(e^{-\phi} - e^{\phi_p}) d\kappa^+ + (D_{IJ} + 2\sigma t_{IJ}) \tilde{\Theta}^{IJ}, \end{aligned}$$

where we have used the specific form of our background. Computing the supersymmetry variations of our new cohomological variables, we find that it is natural to make the field redefinition

$$\tilde{\sigma} = e^{-\phi} \sigma, \tag{E.10}$$

because in terms of this field, the new complex takes the form

$$QA = i\tilde{\Psi}, \tag{E.11}$$

$$Q\tilde{\Psi} = -\iota_R F + d_A(\tilde{\sigma}), \tag{E.12}$$

$$Q\tilde{\sigma} = -i\iota_R \tilde{\Psi}, \tag{E.13}$$

$$Q\tilde{\chi} = \tilde{H}, \tag{E.14}$$

$$Q\tilde{H} = -iL_R^A \tilde{\chi} - [\tilde{\sigma}, \tilde{\chi}], \tag{E.15}$$

which has exactly the same form as the complex before the rescaling. In the computation, various cancellations between the background fields take place, and we are left with the above result. This is to be expected, since the parameters of the square of the supersymmetry depends on the two spinor bilinears $\tilde{\xi}^I \tilde{\Gamma}^m \tilde{\xi}_I = \tilde{R}^m = R^m$ and $-\sigma \tilde{\xi}^I \tilde{\xi}_I = e^{-\phi} \sigma = \tilde{\sigma}$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [[arXiv:0712.2824](#)] [[INSPIRE](#)].
- [2] V. Pestun et al., *Localization techniques in quantum field theories*, [arXiv:1608.02952](#) [[INSPIRE](#)].
- [3] V. Pestun and M. Zabzine, *Introduction to localization in quantum field theory*, [arXiv:1608.02953](#) [[INSPIRE](#)].
- [4] K. Hosomichi, *$N = 2$ SUSY gauge theories on S^4* , [arXiv:1608.02962](#) [[INSPIRE](#)].
- [5] C. Klare, A. Tomasiello and A. Zaffaroni, *Supersymmetry on curved spaces and holography*, *JHEP* **08** (2012) 061 [[arXiv:1205.1062](#)] [[INSPIRE](#)].
- [6] T.T. Dumitrescu, G. Festuccia and N. Seiberg, *Exploring curved superspace*, *JHEP* **08** (2012) 141 [[arXiv:1205.1115](#)] [[INSPIRE](#)].
- [7] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *Supersymmetric field theories on three-manifolds*, *JHEP* **05** (2013) 017 [[arXiv:1212.3388](#)] [[INSPIRE](#)].
- [8] C. Closset and S. Cremonesi, *Comments on $\mathcal{N} = (2, 2)$ supersymmetry on two-manifolds*, *JHEP* **07** (2014) 075 [[arXiv:1404.2636](#)] [[INSPIRE](#)].
- [9] F. Benini, T. Nishioka and M. Yamazaki, *4D index to 3D index and 2D TQFT*, *Phys. Rev. D* **86** (2012) 065015 [[arXiv:1109.0283](#)] [[INSPIRE](#)].
- [10] B. Assel, D. Cassani and D. Martelli, *Localization on Hopf surfaces*, *JHEP* **08** (2014) 123 [[arXiv:1405.5144](#)] [[INSPIRE](#)].
- [11] T. Nishioka and I. Yaakov, *Generalized indices for $\mathcal{N} = 1$ theories in four-dimensions*, *JHEP* **12** (2014) 150 [[arXiv:1407.8520](#)] [[INSPIRE](#)].
- [12] C. Closset and I. Shamir, *The $\mathcal{N} = 1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization*, *JHEP* **03** (2014) 040 [[arXiv:1311.2430](#)] [[INSPIRE](#)].
- [13] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [[arXiv:1105.0689](#)] [[INSPIRE](#)].
- [14] R.K. Gupta and S. Murthy, *All solutions of the localization equations for $N = 2$ quantum black hole entropy*, *JHEP* **02** (2013) 141 [[arXiv:1208.6221](#)] [[INSPIRE](#)].
- [15] C. Klare and A. Zaffaroni, *Extended supersymmetry on curved spaces*, *JHEP* **10** (2013) 218 [[arXiv:1308.1102](#)] [[INSPIRE](#)].
- [16] N. Hama and K. Hosomichi, *Seiberg-Witten theories on ellipsoids*, *JHEP* **09** (2012) 033 [[arXiv:1206.6359](#)] [[INSPIRE](#)].
- [17] V. Pestun, *Localization for $\mathcal{N} = 2$ Supersymmetric Gauge Theories in Four Dimensions*, in *New dualities of supersymmetric gauge theories*, J. Teschner ed., Springer, Germany (2016), [arXiv:1412.7134](#) [[arXiv:EPRINT+arXiv:1412.7134](#)].
- [18] D. Butter, G. Inverso and I. Lodato, *Rigid 4D $\mathcal{N} = 2$ supersymmetric backgrounds and actions*, *JHEP* **09** (2015) 088 [[arXiv:1505.03500](#)] [[INSPIRE](#)].
- [19] A. Bawane, G. Bonelli, M. Ronzani and A. Tanzini, *$\mathcal{N} = 2$ supersymmetric gauge theories on $S^2 \times S^2$ and Liouville gravity*, *JHEP* **07** (2015) 054 [[arXiv:1411.2762](#)] [[INSPIRE](#)].
- [20] D. Rodriguez-Gomez and J. Schmude, *Partition functions for equivariantly twisted $\mathcal{N} = 2$ gauge theories on toric Kähler manifolds*, *JHEP* **05** (2015) 111 [[arXiv:1412.4407](#)] [[INSPIRE](#)].

- [21] M. Bershtein, G. Bonelli, M. Ronzani and A. Tanzini, *Exact results for $\mathcal{N} = 2$ supersymmetric gauge theories on compact toric manifolds and equivariant Donaldson invariants*, *JHEP* **07** (2016) 023 [[arXiv:1509.00267](#)] [[INSPIRE](#)].
- [22] M. Sinamuli, *On $\mathcal{N} = 2$ supersymmetric gauge theories on $S^2 \times S^2$* , *JHEP* **05** (2016) 062 [[arXiv:1411.4918](#)] [[INSPIRE](#)].
- [23] L. Martucci, *Topological duality twist and brane instantons in F-theory*, *JHEP* **06** (2014) 180 [[arXiv:1403.2530](#)] [[INSPIRE](#)].
- [24] B. Assel and S. Schäfer-Nameki, *Six-dimensional origin of $\mathcal{N} = 4$ SYM with duality defects*, *JHEP* **12** (2016) 058 [[arXiv:1610.03663](#)] [[INSPIRE](#)].
- [25] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [[hep-th/0206161](#)] [[INSPIRE](#)].
- [26] A.S. Losev, A. Marshakov and N.A. Nekrasov, *Small instantons, little strings and free fermions*, [hep-th/0302191](#) [[INSPIRE](#)].
- [27] J. Scherk and J.H. Schwarz, *How to get masses from extra dimensions*, *Nucl. Phys.* **B 153** (1979) 61 [[INSPIRE](#)].
- [28] Y. Kosmann, *Dérivées de lie des spineurs*, *Ann. Matem. Pura Appl.* **91** (1971) 317.
- [29] J.M. Figueroa-O'Farrill, *On the supersymmetries of Anti-de Sitter vacua*, *Class. Quant. Grav.* **16** (1999) 2043 [[hep-th/9902066](#)] [[INSPIRE](#)].
- [30] J. Qiu and M. Zabzine, *Review of localization for 5d supersymmetric gauge theories*, [arXiv:1608.02966](#) [[INSPIRE](#)].
- [31] J. Sparks, *Sasaki-Einstein manifolds*, *Surveys Diff. Geom.* **16** (2011) 265 [[arXiv:1004.2461](#)] [[INSPIRE](#)].
- [32] T. Friedrich and I. Kath, *Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator*, *J. Diff. Geom.* **29** (1989) 376.
- [33] J. Qiu and M. Zabzine, *5D super Yang-Mills on $Y^{p,q}$ Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **333** (2015) 861 [[arXiv:1307.3149](#)] [[INSPIRE](#)].
- [34] D. Martelli, J. Sparks and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, *Commun. Math. Phys.* **280** (2008) 611 [[hep-th/0603021](#)] [[INSPIRE](#)].
- [35] D. Salamon, *Spin geometry and Seiberg-Witten invariants*, (1996).
- [36] C.P. Boyer and K. Galicki, *Sasakian geometry*, Oxford University Press, Oxford, U.K. (2008).
- [37] E. Lerman, *Contact toric manifolds*, [math/0107201](#) [[INSPIRE](#)].
- [38] D. Martelli, J. Sparks and S.-T. Yau, *The geometric dual of α -maximisation for Toric Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **268** (2006) 39 [[hep-th/0503183](#)] [[INSPIRE](#)].
- [39] C.P. Boyer, *Completely integrable contact hamiltonian systems and toric contact structures on $S^2 \times S^3$* , *SIGMA* **7** (2011) 058 [[arXiv:1101.5587](#)] [[INSPIRE](#)].
- [40] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$* , *Adv. Theor. Math. Phys.* **8** (2004) 711 [[hep-th/0403002](#)] [[INSPIRE](#)].
- [41] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, *Phys. Rev. Lett.* **95** (2005) 071101 [[hep-th/0504225](#)] [[INSPIRE](#)].

- [42] M.H. Freedman, *The topology of four-dimensional manifolds*, *J. Diff. Geom.* **17** (1982) 357.
- [43] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Springer, Berlin Germany (1973).
- [44] H. Skarke, *How to classify reflexive gorenstein cones*, in *Strings, gauge fields, and the geometry behind: The legacy of Maximilian Kreuzer*, A. Rebhane et al. eds., World Scientific, Singapore (2012), [arXiv:1204.1181](#) [INSPIRE].
- [45] D.A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs volume 68, American Mathematical Society, U.S.A. (1999).
- [46] J. Källén and M. Zabzine, *Twisted supersymmetric 5D Yang-Mills theory and contact geometry*, *JHEP* **05** (2012) 125 [[arXiv:1202.1956](#)] [INSPIRE].
- [47] K. Hosomichi, R.-K. Seong and S. Terashima, *Supersymmetric gauge theories on the five-sphere*, *Nucl. Phys. B* **865** (2012) 376 [[arXiv:1203.0371](#)] [INSPIRE].
- [48] J. Källén, J. Qiu and M. Zabzine, *The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere*, *JHEP* **08** (2012) 157 [[arXiv:1206.6008](#)] [INSPIRE].
- [49] J. Qiu, L. Tizzano, J. Winding and M. Zabzine, *Gluing Nekrasov partition functions*, *Commun. Math. Phys.* **337** (2015) 785 [[arXiv:1403.2945](#)] [INSPIRE].
- [50] L. Baulieu, A. Losev and N. Nekrasov, *Chern-Simons and twisted supersymmetry in various dimensions*, *Nucl. Phys. B* **522** (1998) 82 [[hep-th/9707174](#)] [INSPIRE].
- [51] Y. Pan, *Rigid supersymmetry on 5-dimensional Riemannian manifolds and contact geometry*, *JHEP* **05** (2014) 041 [[arXiv:1308.1567](#)] [INSPIRE].
- [52] Y. Imamura and H. Matsuno, *Supersymmetric backgrounds from 5d $N = 1$ supergravity*, *JHEP* **07** (2014) 055 [[arXiv:1404.0210](#)] [INSPIRE].
- [53] Y. Pan and J. Schmude, *On rigid supersymmetry and notions of holomorphy in five dimensions*, *JHEP* **11** (2015) 041 [[arXiv:1504.00321](#)] [INSPIRE].
- [54] A. Pini, D. Rodriguez-Gomez and J. Schmude, *Rigid supersymmetry from conformal supergravity in five dimensions*, *JHEP* **09** (2015) 118 [[arXiv:1504.04340](#)] [INSPIRE].
- [55] L. Tizzano and J. Winding, *Multiple sine, multiple elliptic gamma functions and rational cones*, [arXiv:1502.05996](#) [INSPIRE].
- [56] J. Winding, *Multiple elliptic gamma functions associated to cones*, [arXiv:1609.02384](#) [INSPIRE].
- [57] O. Goetsches, H. Nozawa and D. Toeben, *Localization of Chern-Simons type invariants of Riemannian foliations*, [arXiv:1508.07973](#) [INSPIRE].
- [58] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, *Commun. Math. Phys.* **262** (2006) 51 [[hep-th/0411238](#)] [INSPIRE].
- [59] M. Zucker, *Minimal off-shell supergravity in five-dimensions*, *Nucl. Phys. B* **570** (2000) 267 [[hep-th/9907082](#)] [INSPIRE].