Phase transitions in long-range Ising models and an optimal condition for factors of g-measures

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Abstract

We weaken the assumption of summable variations in a paper by Verbitskiy [17] to a weaker condition, Berbee's condition, in order for a 1-block factor (a single site renormalisation) of the full shift space on finitely many symbols to have a g-measure with a continuous g-function. But we also prove by means of a counterexample, that this condition is (within constants) optimal. The counterexample is based on the second of our main results, where we prove that there is an inverse critical temperature in a one-sided long-range Ising model which is at most 8 times the critical inverse temperature for the (two-sided) Ising model with long-range interactions.

Contents

1 Introduction

The study of g-measures has a long history, with notable achievements in the 1930's in Romania and France under the name *chains with complete connections* and which refers to a generalization of Markov chains (and more generally, chains with finite memory) on a finite set to chains that have infinite memory. The g-measures are the not necessarily unique stationary distributions for such chains. We use the terminology of g-measures (with respect to a continuous transition probability function g) that was introduced by Keane in his 1972 paper [14] and used in other papers of ours related to this investigation ([11], [12], [13]).

Consider a left full shift map T on infinite strings of finitely many symbols, $X = S^{\mathbb{Z}_+}$, i.e., S is a finite set. Thus T acts on elements x of X, $x = (x_0, x_1, x_2, \ldots)$, in the following way (each x_i belongs to S):

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

A g-measure is a T-invariant Borel probability measure μ that is associated with a continuous function $g : X \to [0, 1]$ so that $g = d\mu/(d\mu \circ T)$ with $\sum_{y \in T^{-1}x} g(y) = 1$, for all $x \in X$.

Let us now define precisely what a factor of a measure means and, in particular, the precise form of our problem. Let $X = S_1^{\mathbb{Z}_+}$ and $Y = S_2^{\mathbb{Z}_+}$ and let $\pi : X \to Y$ be symbolic map in the sense that $\pi = \pi_0 \times \pi_1 \times \ldots$, i.e. if $y = \pi(x)$, then $y_i = \pi_i(x_i)$. (This explains the terms "single site renormalization" or the "1-block" factor.) Our problem is to find suitable conditions for a g-measure μ on X to be pushed down by π to a g-measure $\tilde{\mu}$ on Y (i.e., the 1-block factor $\tilde{\mu} = \mu \circ \pi^{-1}$ of μ is a g-measure).

In the literature, the best results are those by Verbitskiy [17], and Wang and Redig [15], respectively, where these authors assume summability of variations of the g-function, which means that

$$\sum_{n=1}^{\infty} \operatorname{var}_n g < \infty, \tag{1}$$

where

$$\operatorname{var}_n g = \sup_{x \sim_n y} |g(x) - g(y)|,$$

where $x \sim_n y$ if $x, y \in X$ coincide in the first n coordinates. In the ergodic theory literature one often imposes summability of the sequence $\operatorname{var}_n \log g$, but these conditions are equivalent so long as the g-function is regular, that is, if $g > 0$.

The results and open questions concerning factors of g-measures are closely related to sufficient conditions for uniqueness of a g-measure. Doeblin and Fortet [7] famously showed that uniqueness of a g-measure follows from condition (1). This condition was weakened by Berbee [3] to the condition

$$\sum_{n=1}^{\infty} e^{-r_1 - r_2 - \cdots r_n} = \infty, \tag{2}$$

where $r_n = \mathrm{var}_n \log g$. This includes the possibility of having $\mathrm{var}_n \log g = 1/n$, but not $\mathrm{var}_n \log g \geq 1/n^\alpha$ if $\alpha < 1$. In [11] we improved the situation for such sequences considerably, by only requiring square summability of the variations for uniqueness, that is

$$\sum_n (\mathrm{var}_n \log g)^2 < \infty \tag{3}$$

for uniqueness. In [13] we improved this result to obtain uniqueness (and the Bernoulli property) if we only assume $\mathrm{var}_n \log g = o(1/\sqrt{n})$, as $n \to \infty$.

Verbitskiy suggested in [17] that the class of g-measures satisfying (3) could be a natural class to consider for being closed under taking 1-block factors. We will present a counterexample to Verbitskiy's conjecture. The question remained if there is a broader natural class of g-functions to consider than those of summable variations to get a factored g-measure.

In fact we prove among other things the following results.

In **Corollary 3** of **Theorem 2** we prove that under the condition (2), the factor $\mu \circ \pi^{-1}$ of the unique g-measure is also a g-measure. This is an improvement of Verbitskiy's result in [17]. To prove Theorem 2 we combine Verbitskiy's methods with some estimates from Berbee's paper [4].

In **Theorem 4** we prove that there exists a g-function with $\mathrm{var}_n \log g = O(1/n)$, as $n \to \infty$, and a symbolic map $\pi : X \to Y$, such that the unique (because of condition (3)) g-measure on X has a 1-block factor $\mu \circ \pi^{-1}$ which is not a g-measure.

In view of the fact that $r_n = 1/n$ satisfies (2), we see that Berbee's condition is optimal within constants for the g-measure property to hold under taking 1-block factors.

Theorem 4 provides the counterexample to Verbitskiy's conjecture in [17] and is a construction of (a unique) g-measures when we have multiple Gibbs measures. The g-function is constructed from a general potential that admits two Gibbs measures, one of which dominates the other in that it gives a bigger value when integrating a strictly increasing function, and this gives a non-continuous induced g-function for a certain 1-block factor of the original g-measure. We now present a brief explanation and context for the construction.

We exhibit two distinct eigen-measures of the adjoint L^* of the transfer operator L that acts on continuous functions f on spaces like $X = S^{\mathbb{Z}_+}$ as

$$Lf(x) = \sum_{y \in T^{-1}} e^{\phi(y)} f(y).$$

These correspond to probability measures ν that satisfy $L^*\nu = \lambda\nu$, where $\lambda > 0$ is the spectral radius of L, and these eigen-measures coincide with one-sided Gibbs measures. (This is relatively straightforward, but a clear proof is available in the recent [6].) We can also see that g-measures are special cases of such one-sided Gibbs measures, since for a given g-function g we can define a transfer operator L_g by

$$L_g f(x) = \sum_{y \in T^{-1}} g(y) f(y),$$

where we have imposed $\sum_{y \in T^{-1}x} g(y) = 1$ for all x. Thus the g-measures satisfy $L_g^*\mu = \mu$. In the construction of the counterexample we use that in general we do not have a unique Gibbs measure under the condition (3). In fact, it is known ([10],[1]) that there is a phase transition for the one dimensional two-sided sided Ising model with long-range interaction. We use this to obtain a one-sided potential ϕ for which $\mathrm{var}_n\,\phi = O(1/n)$, and such that there are multiple solutions ν to $L^*\nu = \lambda\nu$. This doesn't follow automatically, since we cannot use Sinai's famous "lemma" ([16]) that essentially says that we can study a two-sided Gibbs measure as a one-sided version if $\sum_n n\,\mathrm{var}_n\,\phi < \infty$.

In **Theorem 1** we prove that a "one-sided" version of the Ising model also has a phase transition at a critical inverse temperature at most 8 times the critical temperature of the original two-sided model. This implies that the critical level for obtaining multiple Gibbs measures are the same, i.e., when $\mathrm{var}_n\,\phi = O(1/n)$ (the variations are defined in an analogous way for two-sided systems). We conjecture that we have the same critical temperature for the one-sided Ising model. We use this one-sided long-range Ising model to construct a g-function for Theorem 4.

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2 Gibbs measures, g-measures and the long-range Ising model

2.1 Symbolic spaces

For a measurable space $(X, \mathcal{F})$, let $\mathcal{M}(X, \mathcal{F})$ denote the space of bounded measures. Let $\mathcal{C}(X)$ denote the space of continuous functions on a topological space X. In what follows S is a countable set and X a product set of the form $X = \prod_{i \in S} A_i$ where A_i are finite sets. (A "symbolic" space.) We assume X is equipped with the product topology and the corresponding Borel algebra $\mathcal{F} = \mathcal{B}(X)$. We say that X is homogeneous if the A_i are all equal, i.e. if $X = A^S$ for some fixed finite set A of symbols. If $S = \mathbb{Z}$ or $S = \mathbb{Z}_+$ we have the left-shift operator $T : X \to X$, by $(Tx)_n = x_{n+1}$.

For any subset F of S an element $x = (x_s) \in S$ in X can be represented as $x = x_F \times x_{\bar{F}}$, where $x_F \in X_F := \prod_{s \in F} A_s$. The sub sigma-algebra of $\mathcal{F}$ generated by x_F is $\mathcal{F}_F$. We write $x \sim_F y$ if $y_i = x_i$ for all $i \in F$. In the following, we use Λ to refer to a finite set $\Lambda \subset S$ and $\bar{\Lambda}$ mean the complement of Λ. We write $\Lambda_n \to S$ for taking limits with respect to an increasing sequence $\{\Lambda_n\}$ of finite sets such that, eventually, $F \subset \Lambda_n$ for any finite set F.

We will mostly work with S being the set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ of positive integers with finite sets Λ of the form $\Lambda_n = [0, n)$. In this case $x \sim_n y$ means $x \sim_{[0,n)} y$. We use (n) for the complement $\bar{\Lambda} = \overline{[0, n)} = [n, \infty)$ of $\Lambda = [0, n)$. We prefer to write $x_{(n)}$ for the tail sequence $(x_n, x_{n+1}, \dots)$.

For a measure $\mu \in \mathcal{M}(X)$ and a subset $F \subset S$, let $\mu_F = \mu \circ (x_F)^{-1} \in \mathcal{M}(X_F, \mathcal{F}_F)$ denote the marginal distribution of x_F.

2.1.1 Stochastic dominance

Here we will make some definitions that are used in the construction of our counterexample (Theorem 4). We assume that the symbolic space X is partially ordered: $x \leq y$ meaning that $x_i \leq y_i$ for all $i \in S$, where we assume the $x_i \in A_i$ are integers. This order induces a partial order $\preceq$ on the space $\mathcal{M}(X)$ of probability measures on X: For two probability measures $\mu, \mu' \in \mathcal{M}(X)$ on X, we say that μ' *stochastically dominates* μ if $\mu'(f) \geq \mu(f)$ for every increasing function $f : X \to \mathbb{R}$. We write the stochastic dominance relation $\mu \preceq \mu'$. Strict dominance, written $\mu \prec \mu'$, means that $\mu(f) < \mu'(f)$ for every strictly increasing increasing function f.

An equivalent formulation is that $\mu \preceq \mu'$ whenever we can define $x' \in X$ with distribution μ' and $x \in X$ with distribution μ on a common probability space, i.e., we *couple* μ and μ', in such manner that $\mathrm{P}(x \leq x') = 1$. Strict dominance means that the coupling allows $\mathrm{P}(x < x') = 1$.

2.1.2 Potentials and one-point potentials

Let X be a symbolic space. A *potential* ϕ (or a Hamiltonian) is a limit of functions on X such that the difference $\phi(x) - \phi(y)$ is well-defined and finite whenever x and y coincides outside a finite set. For a potential ϕ and a finite set $\Lambda \subset S$ and an arbitrary but fixed mapping $\Lambda \to K(\Lambda) \in \mathbb{R}$, assigning a constant "ground potential" to each finite set Λ, we can define a function

$$\phi_\Lambda(x) = \max_y \{\phi(x) - \phi(y) + K(\Lambda) : y \sim_{\bar{\Lambda}} x\},$$

with $\phi_\emptyset(x) = 0$. We will assume that the potentials are *continuous*, which means that

$$\text{the functions } \phi_\Lambda(x) \text{ are all continuous on } X. \qquad (4)$$

Note that the sequence $\phi_\Lambda(x)$, $n \geq 0$, is tail adapted, i.e. $\phi_\Lambda(x)$ is $\mathcal{F}_{\bar{\Lambda}}$-measurable for all Λ.

For a given sequence $\Lambda_n \nearrow S$, $n \geq 0$, we can represent the potential $\phi(x)$ as the limit

$$\phi(x) = \lim_{\Lambda_n \nearrow S} \phi_{\Lambda_n}(x).$$

The limit may not exist, but the difference

$$\phi(x) - \phi(y) = \lim(\phi_{\Lambda_n}(x) - \phi_{\Lambda_n}(y))$$

should exist. We can also represent ϕ as an infinite sum

$$\phi(x) = \sum_n \phi_n(x),$$

where $\phi_n(x) = \phi_{\Lambda_n} - \phi_{\Lambda_{n-1}}$. The sequence $\phi_n(x) \in \mathcal{C}(X)$ is the *one-point potential* of ϕ. (It is the potential of the point $x_{\Lambda_n \setminus \Lambda_{n-1}}$.)

2.2 Gibbs distributions

A *Gibbs distribution* μ on X with potential ϕ is a probability distribution on X such that for any finite set Λ the conditional probability of x (or equivalently x_Λ) given $x_{\bar{\Lambda}}$ is proportional to $\exp(\phi_\Lambda(x))$. That is, a version of the conditional probability $\mu(\cdot|\mathcal{F}_{\bar{\Lambda}})$ satisfies

$$\mu(x \mid x_{\bar{\Lambda}}) = \rho_{\phi,\Lambda}(x \mid x_{\bar{\Lambda}}) := \frac{\exp\left(\phi_\Lambda(x)\right)}{Z_\Lambda(x_{\bar{\Lambda}})}, \tag{5}$$

where $Z_\Lambda = Z_\Lambda(\phi)$ denotes the local partition function

$$Z_\Lambda(x_{\bar{\Lambda}}) = Z_\Lambda(x_{\bar{\Lambda}}; \phi) = \sum_{y_\Lambda \in X_\Lambda} \exp\left(\phi_\Lambda(y_\Lambda \times x_{\bar{\Lambda}})\right).$$

For a given potential ϕ, we denote the set of corresponding Gibbs-distributions by $\mathcal{G}(\phi)$.

A probability measure μ is Λ-Gibbsian with respect to ϕ, $\mu \in \mathcal{G}_\Lambda(\phi)$, if for all $f \in \mathcal{C}(X)$

$$\mu(f) = \int f(x)\, \rho_{\phi,\Lambda}(x \mid x_{\bar{\Lambda}})\, d\mu_{\bar{\Lambda}}(x_{\bar{\Lambda}}).$$

Thus $\mathcal{G}(\phi) = \cap_\Lambda \mathcal{G}_\Lambda(\phi)$.

Alternatively, one can define $\mathcal{G}(\phi)$ as the set of weak limits in $\mathcal{M}(X)$ of finite support probability measures of the form given in (5) with respect to some filtration $\Lambda \nearrow S$. For a fixed $\xi \in X$ and a filtration Λ, we say that a limit

$$\mu = \lim_{\Lambda \nearrow S} \rho_{\phi,\Lambda}(x|\xi_{\bar{\Lambda}}) \tag{6}$$

corresponds to a Gibbs measure with *boundary condition* ξ.

2.2.1 The case $S = \mathbb{Z}_+$

In the case $S = \mathbb{Z}_+$, we use $\Lambda_n = [0, n)$ and the one-point potential sequence

$$\phi_n(x) = \phi_{[0,n+1)}(x) - \phi_{[0,n)}(x),$$

and the potential representation

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n(x).$$

If

$$\phi_n(x) = \phi_n(x_n, x_{n+1}, \dots) = \phi_0(T^n x)$$

then we say the potential sequence is *homogeneous*.

If, for the sequence of potentials (ϕ_n), we have that

$$\sum_{x_n \in A_n} \exp(\phi_n(x_n, x_{n+1}, \dots)) = 1, \quad \forall n \ \forall x$$

then (ϕ_n) is said to be *normalised*. Equivalently, we have that the local partition functions $Z_n(x) := Z_{[0,n)}(x) \equiv 1$ for all $n \geq 0$.

For a normalised and homogeneous sequence of one-point potentials $(\phi \circ T^n)$ the function $g(x) = e^{\phi_n(x)}$ is referred to as a *g-function* and the corresponding Gibbs measures are *g-measures*. A g-measure μ is always shift-invariant, i.e. $\mu = \mu \circ T^{-1}$.

2.3 The long-range Ising model

A relevant example of a potential and a Gibbs measure is the *long-range (ferromagnetic) Ising model* on

$$U = \{-1, 1\}^S,$$

where we compare the two-sided case $S = \mathbb{Z}$ with the one-sided case $S = \mathbb{Z}_+$. Let $S^{(2)}$ denote the set of unordered pairs ij of elements in S. We refer to $S^{(2)}$ as the complete graph on S and its elements ij as edges. We usually exclude loops, i.e. the edges of the form ii for $i \in S$.

The long-range Ising model is defined by the potential $\varphi(u)$, $u = (u_i) \in U$, given by

$$\varphi(u) = \varphi(\alpha, \beta)(u) := \beta \sum_{ij \in S^{(2)}} \frac{u_i u_j}{|i-j|^\alpha}, \quad \alpha > 1, \ \beta \geq 0. \tag{7}$$

The potential $\varphi(\alpha, \beta)$ is not well-defined for $\alpha \leq 1$.

For $S = \mathbb{Z}_+$, we obtain the potential φ from the homogeneous sequence of potentials $(\varphi_0(T^n u))$ where (with $\alpha = 2$)

$$\varphi_0(u) = K + \beta \sum_{j=1}^{\infty} \frac{u_0 u_j}{j^2}, \tag{8}$$

where we can choose K to be arbitrary.

Let $\mathcal{I} = \mathcal{I}(\beta, \alpha, S)$ denote the set $\mathcal{G}(\varphi(\alpha, \beta))$ of Gibbs measures for the Ising-potential above. We use ν_+ and ν_- in $\mathcal{I}$ to denote the Gibbs-measures obtained as limits by taking the constant sequences $\xi = \overline{+1}$ and $\xi = \overline{-1}$ as boundary conditions, respectively. In the two-sided case, it is well-known (see [1]) that for α in the range $(1, 2]$ there is a *critical inverse temperature* $\beta_c = \beta_c(\alpha)$. This means that for $\beta > \beta_c$ we have the strict stochastic dominance relation $\nu_+ \succ \nu_-$. Moreover, $\nu_+ = \nu_-$ whenever $\beta < \beta_c$ so that the Gibbs measure is unique. We also have that $\nu_+ \succ \nu_-$, when $\beta = \beta_c$ but we do not need this result. For $\alpha > 2$, we have uniqueness of Gibbs measures.

We use this result to prove that there is a critical inverse temperature in the one-sided case as well. We need to establish the existence of a critical temperature in order to construct a counterexample to Verbitskiy's conjecture later.

Theorem 1. *For the one sided case, i.e. considering $\mathcal{I} = \mathcal{I}(\alpha, \beta, \mathbb{Z}_+)$, $1 < \alpha \leq 2$ and $\beta \geq 0$, there is a similar critical inverse temperature β_c^+. Moreover, $\beta_c^+(\alpha) \leq 8\beta_c(\alpha)$.*

The proof is postponed until Section 4.

It should be remarked that the factor 8 in the bound on β_c^+ is an artifact of the proof. We conjecture that β_c^+ is indeed equal to the two-sided β_c for the relevant values of α.

3 An optimal condition for factors of g-measures

3.1 Uniqueness of Gibbs measures and Berbee's condition

3.1.1 Transfer operator

The study of Gibbs measures in [4] is based on the analysis of the generalised "transfer operator" $L = (L_n)$ and its dual $L^* = (L_n^*)$: For a given sequence of potentials (ϕ_n), let $\mathcal{M}_{(n)} = \mathcal{M}(X_{(n)}, \mathcal{F}_{(n)})$. We define the transfer operator $L = (L_n)$ as the system of maps

$$\mathcal{C}(X_{(0)}) \xrightarrow{L_0} \mathcal{C}(X_{(1)}) \xrightarrow{L_1} \cdots \mathcal{C}(X_{(n)}) \xrightarrow{L_n} \cdots$$

where $L_n : \mathcal{C}(X_{(n)}) \to \mathcal{C}(X_{(n+1)})$ is given by

$$(Lf)(x_{(n+1)}) = \sum_{x_n \in A_n} \phi_{n-1}(x_n, x_{(n+1)}) f(x_n, x_{(n+1)}).$$

Dually, we obtain the system L^* of maps between measures

$$\mathcal{M}_{(0)} \xleftarrow{L_0^*} \mathcal{M}_{(1)} \xleftarrow{L_1^*} \cdots \mathcal{M}_{(n)} \xleftarrow{L_n^*} \cdots$$

where $L_{n+1}^* : \mathcal{M}_{(n+1)} \to \mathcal{M}_{(n)}$ is given by "multiplication by $\exp(\phi_n(x))$".

A Gibbs measure μ on X corresponds to a projective limit for the system above: Recall that $\mu_{(n)}$ denotes the restriction of μ to $\mathcal{F}_{(n)}$. If

$$\mu_n := \frac{1}{Z_n}\mu_{(n)}, \quad n \geq 0$$

then it is readily checked that the sequence $(\mu_0, \mu_1, \dots) \in \prod_n \mathcal{M}_{(n)}$ satisfies

$$\mu_n = L^*\mu_{n+1}.$$

We write $\mu = \mu_0 = (L^*)^n\mu_n$. It is also clear that if L^* is defined as multiplication by e^{ϕ_n} then any such sequence $(\mu_0, \mu_1, \dots)$ where $\mu_n = L^*\mu_{n+1}$ gives the Gibbs measure $\mu_0 \in \mathcal{G}(\sum_n \phi_n)$.

3.1.2 Berbee's condition

The relation between smoothness of the transfer operator and uniqueness of Gibbs measures is a central object of study. We measure the smoothness of the sequence of potentials (ϕ_n) with uniform variations r_k, $k \geq 1$, defined as

$$r_k = \sup_n \operatorname{var}_k \phi_n(x_{(n)}). \tag{9}$$

It is shown in [4] that the Gibbs measure $\mu \in \mathcal{G}(\sum_n \phi_n)$ is unique whenever

$$\sum_{n=1}^{\infty} e^{-r_1-\cdots-r_n} = \infty. \tag{10}$$

We refer to this as "Berbee's condition".

For two bounded measures $\nu, \tilde{\nu}$ on a symbolic space X, we define

$$\rho_k(\nu, \tilde{\nu}) = \inf_{C \in \mathcal{F}_k} \frac{\tilde{\nu}(C)}{\nu(C)}.$$

Let $P = (P_{ij})_{i=0,j=0}^{\infty,\infty}$ be the Markov matrix given by $P_{00} = 1$ and $P_{0j} = 0$ and for $i > 0$

$$P_{ij} = \begin{cases} 0 & j < i-1 \\ e^{-r_j} & j = i-1 \\ e^{-r_{j+1}} - e^{-r_j} & j \geq i \end{cases},$$

and where $r_j = r_j((\phi_n))$ are the variations from (9). Let $X = B_0 \times B_1 \times \dots$ and $X' = B_1 \times B_2 \times \dots$ and consider a transfer operator $L : \mathcal{C}(X) \to \mathcal{C}(X')$ given by

$$Lf(x') = \sum_{x_0} f(x_0, x')e^{\phi(x_0,x')}.$$

A crucial estimate in [4] is that

$$\rho_n(L^*\nu, L^*\tilde{\nu}) \geq P\rho_{(n+1)}(\nu, \tilde{\nu}), \tag{11}$$

whenever $\sup_n \mathrm{var}_k\, \phi_n(x) \leq r_k$.

Let $N \geq 1$. A probability measure $\nu \in \mathcal{M}(X)$ is $[0, N)$-Gibbsian if it is extended by L from the restriction $\nu_{(N)}$ on $X_{(N)}$ (the "boundary condition"). That is, if we have

$$\nu = (L^*)^N \nu_N = L_0^* L_1^* \dots L_{N-1}^* \nu_{(N)}.$$

Berbee shows with a clever induction argument that for any pair of $[0, N)$-Gibbsian measures $\nu, \tilde{\nu}$ it we have that

$$\rho_k(\nu, \tilde{\nu}) \geq \mathrm{P}(Z_N = 0 | Z_0 = k)^2, \tag{12}$$

where Z_t, $t = 0, 1, \dots$ denotes a Markov chain on state-space $[0, \infty)$ with transition matrix P. Note that Z_t is absorbing at state 0 and that Berbee's condition (10) is equivalent to stating that Z_t is recurrent and hence that

$$\lim_{N \to \infty} \mathrm{P}(Z_N = 0 | Z_0 = k)^2 = 1.$$

The uniqueness of the Gibbs measure then follows.

3.2 One-factors and Gibbsianity

Let $X = \prod_{n=1}^{\infty} A_n$ and $Y = \prod_{n=1}^{\infty} \tilde{A}_n$ be two symbolic spaces as defined above. For our purposes a *symbolic map* is a map $\pi : X \to Y$ obtained from a sequence of surjective coordinate-wise maps $\{\pi_i : A_i \to \tilde{A}_i\}$ so that $\pi(x)_i = \pi_i(x_i)$.

Theorem 2. *Assume that μ is Gibbs measure on X with respect to the potential $\sum_n \phi_n$ and that the sequence (ϕ_n) of one-point potentials satisfies Berbee's condition* (10)*. For any symbolic map $\pi : X \to Y$ it holds that*

- *(i) the conditional probability measure $\mu(x|y) \in \mathcal{M}(X)$ is a continuous function of $y = \pi(x) \in Y$.*

- *(ii) If, in addition, the potential sequence (ϕ_n) is normalised, then the distribution of y, $\tilde{\mu} = \mu \circ \pi^{-1}$ is given by the normalised potential sequence* $(\log \tilde{p}_n)$ *where*

$$\tilde{p}_n(y_{(n)}) = \int \sum_{x_n \in \pi_n^{-1}(y_n)} e^{\phi_n(x_n, x_{(n+1)})}\, d\mu(x_{(n+1)} \mid y_{(n+1)}) \tag{13}$$

Proof of Theorem 2. The argument is in many ways similar to that of Verbitskiy in [17]: We note that $\mu(x|y)$ is a Gibbs measure on the non-homogeneous symbolic space

$$\pi^{-1}(y) = \pi_0^{-1}(y_0) \times \pi_1^{-1}(y_1) \times \cdots,$$

with respect to (ϕ_n) restricted to $\pi^{-1}(y)$: For any $x \in \pi^{-1}(y)$, the probability $\mu(x \mid y, x_{(n)})$ differ from $\mu(x \mid x_{(n)})$ by the factor $1/\mu(\pi^{-1}(y) \mid x_{(n)}) > 0$ and hence it is proportional to the product $\exp(\sum_{k=0}^{n-1} \phi_k(x_{(k)})$, since this holds for $\mu(x \mid x_{(n)})$ by assumption.

In order to prove continuity of the map $y \mapsto \mu(x|y)$, we use the explicit mixing rate in (12). If y and y' are two different element in Y such that $y \sim_N y'$ then $\mu(x|y)$ and $\mu(x|y')$ are $[0, N)$-Gibbs measures on the space

$$\pi_0^{-1}(y_1) \times \cdots \times \pi_{N-1}^{-1}(y_{N-1}) \times A_N \times A_{N+1} \times \cdots$$

with respect to (ϕ_n). Hence, by (12), we have

$$|\log \mu([x]_k|y) - \log \mu([x]_k|y')| \leq |\log \mathrm{P}(Z_N = 0|Z_0 = k)|$$

which tends to zero as $N \to \infty$ by Berbee's condition.

To show (ii), we note that $(\tilde{\phi}_n)$ is clearly normalised, since

$$\sum_{y_n \in \tilde{A}_n} \tilde{p}_n(y_n, y_{(n+1)}) = \int \underbrace{\left(\sum_{x_n} p_n(x_n, x_{(n+1)}) \right)}_{=1} d\mu(x_{(n+1)}|y_{(n+1)}) = 1,$$

where $p_n(x) = e^{\phi_n(x)}$ and $\tilde{p}_n(x) = e^{\tilde{\phi}_n(x)}$. That $\tilde{p}_n$ is continuous follows from the continuity of $y \mapsto \mu(\cdot|y_{(n+1)})$.

That the distribution of y, i.e. $\tilde{\mu} = \mu \circ \pi^{-1}$, is a Gibbs measure with the normalised potential $(\log \tilde{p}_n)$ follows if $\tilde{\mu}(y_n|y_{(n+1)}) = \tilde{p}_n(y_n, y_{(n+1)})$. But this is immediate from the definition of $\tilde{p}$, since

$$\tilde{\mu}(y_n|y_{(n+1)}) = \int \sum_{x_n \in \pi_n^{-1}(y_n)} p_n(x_n, x_{(n+1)})\, d\mu(x_{(n+1)}|y_{(n+1)}) = \tilde{p}_n(y_n, y_{(n+1)}).$$

□

A symbolic map $\pi : X \to Y$ between two homogeneous spaces $X = A^S$ and $Y = \tilde{A}^S$ is homogeneous if it has the form $y_i = \pi(x_i)$ where $\pi : A \to \tilde{A}$ is a fixed surjective map between finite sets. From the explicit form (13) of the induced potential sequence, it is clear that homogeneity is preserved if the symbolic map π is homogeneous. We obtain, as a corollary, the result by Verbitskiy in [17] under weaker assumptions.

Corollary 3. *Assume that μ is a g-measure where the g-function satisfies Berbee's condition. If $\pi : X \to Y$ is a homogeneous factor then $\tilde{\mu} = \mu \circ \pi^{-1}$ is a g-measure.*

It follows that the class of g-measures satisfying Berbee's conditions is closed under taking 1-block factors.

3.3 The counterexample to a conjecture by Verbitskiy

In [17] (p. 328), Verbitskiy argues that it would be natural to conjecture that for any homogeneous symbolic map π and any g-measure μ with respect to a g-function having square summable variations the measure $\mu \circ \pi^{-1}$ is a g-measure. The condition of square summability variations, i.e. $\sum_n (\mathrm{var}_n\, g)^2 < \infty$, is a condition that in [11] was used to prove uniqueness of the g-measure. Verbitskiy speculates that the condition of square summability could be closed under taking 1-block factors, perhaps by adapting arguments from Fan and Pollicott [9].

We show that this is not the case using the following counterexample, where we have a square summable g-function. In fact, the g-function satisfies $\mathrm{var}_n\, g = O(1/n)$ and the sequence of variations is thus only a factor away from satisfying Berbee's condition. The construction also connects our investigation with the principle proposed by van Enter *et al.* in [8] and discussed in [17] that non-Gibbsianity of factors is linked to the presence of a "hidden phase transition".

Theorem 4. *There exists a g-function g with $\mathrm{var}_n\, g = O(1/n)$ and with unique g-measure μ, and a symbolic map $\pi : X \to Y$, so that the corresponding 1-block factor $\mu \circ \pi^{-1}$ is not a g-measure.*

Proof. Let $(\varphi(\beta)_n)$ be the homogeneous potential of the one-sided long-range Ising model on $U = \{-1, +1\}^{\mathbb{Z}_+}$ defined in (8). We can choose the constant K in (8), so that

$$q(\pm 1, u) < 1/2, \quad \text{for all } u \in U, \tag{14}$$

where $q(u) = \exp(\phi_0(u))$. Let $\nu_N^{\pm} \in \mathcal{G}_{[0,n)}(\varphi)$ denote the $[0, N)$-Gibbsian measure on U obtained from the boundary condition $\xi = \overline{\pm 1}$. By construction $\nu_N^{\pm} \to \nu^{\pm}$ and we choose $\beta > \beta_c^+$ from Theorem 1, so that $\nu^+ \succ \nu^-$.

Consider the homogenous space $X = A^{\mathbb{Z}_+}$ on four symbols

$$A = \{+1, -1, +\tilde{1}, -\tilde{1}\}$$

and the symbolic map $\alpha : X \to U$ defined by $\alpha(+1) = \alpha(+\tilde{1}) = +1$ and $\alpha(-1) = \alpha(-\tilde{1}) = -1$. We define a g-function $g(x)$ from $q(u)$ by setting

$$g(x) = \begin{cases} q(\alpha(x)), & \text{if } x_0 = \pm 1, \\ \frac{1}{2} - q(\alpha(x)), & \text{if } x_0 = \pm \tilde{1}. \end{cases}$$

It is obvious that g is a g-function.

A simple estimate shows that the log-variations of q in (8) satisfy

$$\mathrm{var}_n \log q \le \beta \frac{2}{n},$$

and thus the variations of the g-function g satisfies $\mathrm{var}_n \log g = O(1/n)$. By [11] (or [13]), we have a unique g-measure μ on X.

Let Y be the symbolic space $Y = B^{\mathbb{Z}_+}$ on three symbols $B = \{0, +\tilde{1}, -\tilde{1}\}$ and consider the the shift-invariant factor $\pi : X \to Y$ defined by $\pi(\pm 1) = 0$ and $\pi(\pm\tilde{1}) = \pm\tilde{1}$. Let $\tilde{\mu} = \mu \circ \pi^{-1}$ be the distribution of $y \in Y$ and let

$$\tilde{g}(y) = \tilde{\mu}(y_0 | y_{(1)}) = \mu\left(\pi(x_0) = y_0 \mid \pi(x_{(1)}) = y_{(1)}\right).$$

We claim that the induced g-function $\tilde{g}(y)$ for the factor on Y is discontinuous at $y = \overline{0} \in Y$, where $\overline{0} = (0, 0, 0, \dots)$.

Let $\overline{0}_N^{\pm} \in Y$ be defined by $y_i = -1$ for $i = 0, 1, \dots, N-1$ and $y_i = \pm\tilde{1}$ for $i \geq N$. Since $\alpha(\pi^{-1}(\overline{0}_N^{+})) = \{+1, -1\}^{[0,N)} \times \overline{+1}$, it follows that

$$\tilde{g}(+\tilde{1}, \overline{0}_N^{\pm}) = \mu(y_0 = +\tilde{1} \mid y_{(1)} = \overline{0}_N^{\pm}) = \frac{1}{2} - \int q(+1, u)\, d\nu_N^{\pm}(u)$$

which tends to

$$\frac{1}{2} - \int q(+1, u)\, d\nu^{\pm}(u)$$

as $N \to \infty$.

Thus, since $u \mapsto q(+1, u)$ is a strictly increasing function on U, we obtain that

$$\lim_{N\to\infty} \tilde{g}(+\tilde{1}, \overline{0}_N^{+}) \neq \lim_{N\to\infty} \tilde{g}(+\tilde{1}, \overline{0}_N^{-}),$$

which shows the discontinuity of $\tilde{g}$ since $\overline{0}_N^{\pm} \to \overline{0}$ as $N \to \infty$. □

4 The one-sided long-range Ising model

In order to prove Theorem 1, we work with the random cluster model (as in Aizenman *et al* [1]) instead of working with the Ising model directly.

4.1 The random cluster model

A random cluster model on S is a certain type $\mathcal{R}(p, q, S)$ of distribution of a random subgraph t of $S^{(2)}$. We consider $t = (t_{ij})$ as an element of $T(S) := \{0, 1\}^{S^{(2)}}$ and obtain $\mathcal{R} = \mathcal{R}(p, q, S)$ as the class of Gibbs measures to the potential on $T(S)$ given by

$$\log q \cdot c(t) + \sum_{ij} \log(1 - p_{ij})(1 - t_{ij}) + \log p_{ij} t_{ij}.$$

Here $c(t)$ denotes the number of connected components (clusters) in the graph t, which readily can be defined as a potential, although it is not necessarily continuous. The random cluster model has two parameters: The edge-probability $p : S^{(2)} \to [0, 1]$, $ij \mapsto p_{ij}$, and the parameter q which is a number $q \geq 1$. Note that if S is finite, the distribution of the random graph $t \sim \phi \in \mathcal{R}(p,q,S)$ is a probability proportional to

$$q^{c(t)} \cdot \prod_{t_{ij}=1} p_{ij} \cdot \prod_{t_{ij}=0} (1-p_{ij}).$$

If $q = 1$ then we obtain the standard Bernoulli random graph distribution on S.

We obtain the (free boundary) random cluster distribution $\phi = \phi(p,q,S) \in \mathcal{R}(p,q,S)$ as the limit having fixed boundary $\xi = \overline{0}$ with respect to the sequence $\Lambda^{(2)} \nearrow S^{(2)}$ for $\Lambda \nearrow S$. (The so-called wired distribution ϕ^w is obtained by taking $\xi = \overline{1}$ and $\Lambda_n^{(2)} = \overline{(\bar{\Lambda})^{(2)}} \nearrow S^{(2)}$.) By the monotonicity in p, see below, one can deduce that the free boundary limit ϕ is well defined. In all cases of interest in this paper, ϕ is actually the unique random cluster distribution of type $\mathcal{R}(p,q,S)$.

The following three stochastic dominance relations for random cluster models are well-known (see [1]). Firstly, the random cluster distribution $\phi(p,q,S)$ increases with p, i.e.

$$p \leq p' \implies \phi(p,q,S) \preceq \phi(p',q,S). \tag{15}$$

It decreases in q, so that

$$q \leq q' \implies \phi(p,q,S) \succeq \phi(p,q',S). \tag{16}$$

Finally, we can compare a random cluster distribution with the corresponding Bernoulli distribution. That is,

$$\phi(p,q,S) \succeq \phi\left(\frac{p}{p+(1-p)q}, 1, S\right), \tag{17}$$

where

$$\left(\frac{p}{p+(1-p)q}\right)_{ij} = \frac{p_{ij}}{p_{ij}+(1-p_{ij})q}.$$

4.1.1 The random cluster model and the Ising model coupled

The long range Ising model $\mathcal{I} = \mathcal{I}(\beta,\alpha,S)$, $S = \mathbb{Z}$ or $S = \mathbb{Z}_+$, can be constructed from the random cluster model. In fact, we may couple $\nu^+, \nu^- \in \mathcal{I}$ using a random cluster distribution $\phi(\rho, 2, S)$ where the edge-probability $\rho = \rho(\beta,\alpha)$ is given by

$$\rho_{ij} = 1 - \exp\left(-\frac{\beta}{|i-j|^\alpha}\right). \tag{18}$$

That is, the probability of non-occurrence of the edge ij is $\exp(-\beta/|i-j|^\alpha)$.

We can ([1]) construct a spin-configuration $u^{\pm} \in U$ distributed according to $\nu^{\pm}$ as follows: Choose a graph $t \sim \phi$ on vertex-set S according to the distribution $\phi = \phi(\rho, 2, S)$ and assign each *infinite* cluster in X the fixed spin-value ± 1. For each finite cluster in t a spin-value in $\{-1, +1\}$ is chosen independently and uniformly at random. Then the spin-configuration $u^{\pm} = u^{\pm}(t)$ is defined by setting u_i equal to the spin of the cluster containing i. The spin-configuration u^- is thus equal to u^+ except that for i belonging to the infinite cluster the spin u_i is changed from $+1$ to -1. It is hence clear that

$$\nu^+ \succ \nu^- \quad \text{precisely if } \phi(A_\infty) = 1$$

where A_∞ is the event

$$A_\infty = \text{“}t \text{ has an infinite cluster”}.$$

Note that A_∞ is a tail event and that satisfies a zero-one law, i.e. the probability $\phi(A_\infty)$ is either zero or one.

4.2 Proof of Theorem 1

Let α be fixed where $1 < \alpha \leq 2$. Let $\phi(\beta, q, S)$ denote the random cluster distribution $\phi(\rho, q)$, with $\rho = \rho(\alpha, \beta, S)$ as in (18) above. Note that ρ is increasing as a function of β. As stated above, it is known (see [1]) that if t has distribution $\phi = \phi(\beta, 2, \mathbb{Z})$ then $\phi(A_\infty) = 1$, precisely when $\beta \in [\beta_c(\alpha), \infty)$.

We assume $\beta \geq \beta_c$ and we shall prove that for $\phi = \phi(8\beta, 2, \mathbb{Z}_+)$, we have $\phi(A_\infty) = 1$. Since A_∞ is an increasing event, it is enough to establish a stochastic dominance $\phi(8\beta, 2, \mathbb{Z}_+) \succeq \tilde{\phi}$ for some distribution $\tilde{\phi}$ on $T(\mathbb{Z}_+)$, where $\tilde{\phi}(A_\infty) = 1$.

Let $F : T(\mathbb{Z}) \to T(\mathbb{Z}_+)$ be defined by

$$F(t)_{ab} = 1 - \prod_{|i|=a, |j|=b} (1 - t_{ij}), \quad t \in T(\mathbb{Z}). \tag{19}$$

Then F corresponds to the graph homomorphism (loops “silently removed”) induced by the vertex map $i \mapsto |i|$. Thus for any pair i, j of vertices connected by a path in t, the images $|i|$ and $|j|$ under F remain connected in $F(t)$. It is therefore clear that if t has an infinite cluster then $F(t)$ has an infinite cluster. Hence, if we construct $\tilde{\phi}$ as the push-forward $\tilde{\phi} = \phi(\beta, 1, \mathbb{Z}) \circ F^{-1}$ of the long-range Bernoulli random graph on $\mathbb{Z}$ then $\tilde{\phi}(A_\infty) = 1$ whenever $\phi(\beta, 1, \mathbb{Z})(A_\infty) = 1$. Since $\phi(\beta, 1, \mathbb{Z}) \succeq \phi(\beta, 2, \mathbb{Z})$ and since $\beta \geq \beta_c$, we can deduce that $\tilde{\phi}(A_\infty) = 1$. By independence, it follows from (19) that $\tilde{\phi}$ is a Bernoulli random graph distribution $\tilde{\phi} = \phi(\gamma, 1)$ with edge-probability γ

$$\gamma_{ij} = 1 - \prod_{|i'|=i,|j'|=j} \exp - \frac{\beta}{|i'-j'|^\alpha} = 1 - \exp\left\{-\beta(2|i-j|^{-\alpha} + 2|i+j|^{-\alpha})\right\}, \tag{20}$$

for $i, j \geq 0$. From (17) and (15), it follows that

$$\phi' \succeq \phi(4\beta, 1, \mathbb{Z}_+)$$

and we are done since (20) implies that

$$\gamma_{ij} \geq= 1 - \exp\left(-4\frac{\beta}{|i-j|^\alpha}\right)$$

and thus

$$\phi(4\beta, 1, \mathbb{Z}_+) \succeq \tilde{\phi}.$$

This concludes the proof. □

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