Published for SISSA by Springer

Received: *January 10, 2017*
Revised: *February 10, 2017*
Accepted: *February 28, 2017*
Published: *March 8, 2017*

JHEP03(2017)042

All (4,1): Sigma models with $(4, q)$ off-shell supersymmetry

Chris Hull[a] **and Ulf Lindström**[a,b]

[a] *The Blackett Laboratory, Imperial College London, Prince Consort Road London SW7 @AZ, U.K.*

[b] *Department of Physics and Astronomy, Division of Theoretical Physics, Uppsala University, Box 516, SE-751 20 Uppsala, Sweden*

E-mail: c.hull@imperial.ac.uk, ulf.lindstrom@physics.uu.e

Abstract: Off-shell $(4, q)$ supermultiplets in 2-dimensions are constructed for $q = 1, 2, 4$. These are used to construct sigma models whose target spaces are hyperkähler with torsion. The off-shell supersymmetry implies the three complex structures are simultaneously integrable and allows us to construct actions using extended superspace and projective superspace, giving an explicit construction of the target space geometries.

Keywords: Extended Supersymmetry, Superspaces

ArXiv ePrint: 1611.09884

doi:10.1007/JHEP03(2017)042

Contents

1 Introduction

There is a rich interplay between supersymmetry and geometry in non-linear sigma models. Supersymetric sigma models have led to the discovery of a rich class of complex geometries. Our purpose here is to revisit this story, and we find that past results readily extend to the construction of some interesting new geometries with $(4,q)$ supersymmetry.

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The sigma model in 2 dimensions with $(1,1)$ supersymmetry has a target space with a metric g and closed 3-form H given locally in terms of a 2-form potential B, $H = dB$ [1]. The action can be written in $(1,1)$ superspace with coordinates (x^μ, θ^α) where $x^\mu = (x^{+\!\!+}, x^{=})$ are null coordinates, $x^{+\!\!+} = \tau + \sigma$, $x^{-} = \tau - \sigma$, and $\theta^\alpha = (\theta^+, \theta^-)$. We use spinor indices $+, -$ so that ψ^+ is a positive chirality 1-component Weyl spinor and ψ^- is a left-handed one, for any spinor ψ. If the target space coordinates are X^i, $i = 1, \ldots, n$, the map from the worldsheet superspace to the target space is given locally by scalar superfields $X^i(x, \theta)$ and the action is

$$S = \frac{1}{2} \int d^2x d^2\theta \, D_- X^i (g + B)_{ij}(X) D_+ X^j \,. \tag{1.1}$$

For particular geometries, the sigma model can have extended supersymmetry. The conditions for $(2,2)$ and (4,4) supersymmetry were found in [1] and the conditions for (2,0) supersymmetry were found in [2]. This was generalised to the case of (p,q) supersymmetry in [3] and the geometry was further studied in [4]. The $(1,1)$ theory will in fact have (p,q) supersymmetry (with $p, q = 1, 2$ or 4) if the target space has $p-1$ complex structures $J_{(+)}$ and $q-1$ complex structures $J_{(-)}$ satisfying

$$J^t_{(\pm)} g J_{(\pm)} = g \,, \quad (J_{(\pm)})^2 = -\mathbb{1} \,, \quad \nabla^{(\pm)} J_{(\pm)} = 0 \,, \tag{1.2}$$

where

$$\nabla^{(\pm)} := \left(\nabla^{(0)} \pm \frac{1}{2} g^{-1} H \right) \tag{1.3}$$

is the connection with torsion $\pm \frac{1}{2} g^{il} H_{ljk}$ added to the Levi-Civita connection $\nabla^{(0)}$. Then the extra supersymmetry transformations are given in terms of these complex structures by

$$\delta X^i = \epsilon^+_A \left(J^{(A)}_{(+)} \right)^i{}_j D_+ X^j + \epsilon^-_{\tilde{A}} \left(J^{(\tilde{A})}_{(-)} \right)^i{}_j D_- X^j \,, \tag{1.4}$$

where $A = 1, \ldots, p-1$ and $\tilde{A} = 1, \ldots, q-1$ label the complex structures.

Closure of the algebra requires that $J^{(A)}$ is an almost complex structure, $(J^{(A)})^2 = -\mathbb{1}$ and that it is integrable, i.e. the Nijenhuis tensor vanishes, $\mathcal{N}(J^{(A)}) = 0$, so that it is a complex structure. Similarly, the $J^{(\tilde{A})}$ are also complex structures. When $p > 1$ and/or $q > 1$, the commutator of supersymmetries $[\delta_{\epsilon_A}, \delta_{\epsilon_B}]$ gives a term with involving a tensor $\mathcal{N}(J^{(A)}, J^{(B)})$ constructed from the complex structures, known as the Nijenhuis concomitant, so that for closure it is necessary that this vanishes. For three anticommuting almost complex structures I, J, K satisfying the algebra of the quaternions it was shown in [5] that the vanishing of the Nijenhuis tensor of any two of the complex structures implies the vanishing of that of the third, and of all of the concomitants, so the integrability of the three complex structures $J^{(A)}$ is sufficient for closure, and in particular implies the vanishing of the Nijenhuis concomitant.

In what follows we shall be particularly interested in cases when there is a coordinate system (atlas) for which all the complex structures are constant in all coordinate patches, i.e. they are simultaneously integrable. Three anticommuting complex structures I, J, K are simultaneously integrable when a certain curvature formed from the three of them

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vanishes $R(I,J,K) = 0$ [4, 6, 7]. For two complex structures, $J^{(+)}$ and $J^{(-)}$ that commute, $[J^{(+)}, J^{(-)}] = 0$, it is instead the vanishing of the Magri-Morosi concomitant, $\mathcal{M}(J^{(+)}, J^{(-)})$ that signals simultaneous integrability. For details see [4].

If $H = 0$, then there are equal numbers of left and right handed supersymmetries, $p = q$, and the target space is Kähler for $(2,2)$ supersymmetry and hyperkähler for (4,4) supersymmetry. For the $(2,2)$ case, the supersymmetry algebra closes off-shell and the theory can be formulated in terms of chiral superfields, while for (4,4) supersymmetry, the supersymmetry algebra closes on-shell only, or after introducing an infinite number of auxiliary fields,[1] as the 3 complex structures are not simultaneously integrable.

For $H \neq 0$, there is a richer structure. For $(2,1)$ supersymmetry, the supersymmetry algebra closes off-shell and the theory can be formulated in terms of chiral superfields, while for $(2,2)$ supersymmetry the supersymmetry algebra closes off-shell only once suitable auxiliary fields are introduced. The theory can then be formulatd in terms of chiral superfields, twisted chiral superfields, and semi-chiral superfields [8]. For $(4,q)$ supersymmetry, the supersymmetry algebra only closes on-shell in general, but there are interesting cases in which the algebra closes off-shell, and the three complex structures $J^{(+)}$ are simultaneously integrable, i.e. there is a coordinate system where all of them are constant [4]. One example of this is the (4,4) supersymmetric model found in [1] that generalises that obtained from the dimensional reduction of $N = 2$ super-Yang-Mills theory in 4 dimensions. The aim of this paper is to investigate such cases with off-shell $(4,q)$ supersymmetry, with simultaneously integrable complex structures $J^{(+)}$. In such cases, there is an off-shell superfield formulation, and a superspace formulation of the action that gives a general local construction of the geometry in terms of certain potentials. In this paper, a number of new multiplets will be found and analysed. Actions for these multiplets will then be constructed using projective superspace. Projective superspace has a long history [9–15] paralleling and complementing that of harmonic superspace [16–19]. General superspaces of this type have been described in [20, 21]. For detailed reviews of projective $(4,4)$ superspace see [15] and the lectures [22].

The plan of the paper is as follows. In section 2 we define an off-shell $(4,1)$ multiplet that will play a key role in what follows. Its $(2,1)$ superspace formulation is given in section 3 and general $(4,1)$ sigma model actions written in $(2,1)$ superspace are introduced in section 4 and the geometric conditions for $(4,1)$ supersymmetry are studied. Related $(4,2)$ multiplets are discussed in section 5. General $(4,2)$ supersymmetric sigma models are studied in $(2,2)$ superspace in section 6 and the conditions for $(4,2)$ supersymmetry are analysed. The relationship to the $(4,4)$ hypermultiplet is discussed in section 7, while section 8 contains results on the $(4,1)$ superspace action. In section 9 we introduce $(4,q)$ projective superspace and use it to formulate multiplets and actions, giving explicit constructions of target space geometries.

2 (4,1) off-shell supermultiplets

In [1], a $(4,4)$ off-shell multiplet was found by dimensional reduction of $N = 2$ super Yang-Mills theory from 4 dimensions. Truncating this gives an off-shell $(4,1)$ supermultiplet that

[1] E.g. using projective or harmonic superspace.

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can be formulated as follows. We use $(4,1)$ superspace with coordinates $x^{\#}, x^{=}, \theta^+_a, \bar\theta^{+a}, \theta^-$ where the index $a = 1,2$ is an SU(2) index. Here θ^+_a are complex and θ^- is real. There are two right-handed complex spinorial covariant derivatives $\mathbb{D}^a_+$ and a real left-handed spinorial covariant derivative D_-, satisfying

$$
\begin{aligned}
&\{\mathbb{D}_{+a}, \bar{\mathbb{D}}^b_+\} = \ 2i\delta^b_a\partial_{\#}\,, \quad a,b, = 1,2.\\
&(D_-)^2 = i\partial_= \ .
\end{aligned}
\tag{2.1}
$$

The $(4,1)$ multiplet obtained by truncating the $(4,4)$ multiplet of [1] consists of a pair of $(4,1)$ superfields ϕ, χ satisfying the constraints

$$
\begin{aligned}
&\bar{\mathbb{D}}^1_+\phi = 0 = \mathbb{D}_{+2}\phi\,, \quad \bar{\mathbb{D}}^1_+\chi = 0 = \mathbb{D}_{+2}\chi\,,\\
&\bar{\mathbb{D}}^2_+\chi = -i\bar{\mathbb{D}}^1_+\bar\phi\,, \quad \bar{\mathbb{D}}^2_+\phi = i\bar{\mathbb{D}}^1_+\bar\chi\ .
\end{aligned}
\tag{2.2}
$$

The supersymmetry transformations can be put into the form (1.4) by expanding in $(1,1)$ superspace. The $(4,1)$ multiplet in (2.2) can be formulated in $(1,1)$ superspace by defining

$$
\phi\big|_{\theta^+_2=0,\theta^+_1=\bar\theta^+_1} = \tilde\phi, \qquad \chi\big|_{\theta^+_2=0,\theta^+_1=\bar\theta^+_1} = \tilde\chi\ . \tag{2.3}
$$

The constraints (2.2) then determine the terms in ϕ, χ of higher order in $\theta^+_2, \theta^+_1 - \bar\theta^+_1$ in terms of $\tilde\phi, \tilde\chi$ and give the supersymmetry transformations under the non-manifest supersymmetries. We define four real $(4,1)$ superspace spinor derivatives D_+ and $\check{D}^{(A)}_+$, $A = 1,2,3$ by

$$
\begin{aligned}
&\mathbb{D}_{+1} =: D_+ - i\check{D}^{(1)}_+\\
&\mathbb{D}_{+2} =: \check{D}^{(2)}_+ - \check{D}^{(3)}_+\ .
\end{aligned}
\tag{2.4}
$$

Then D_+ is the $(1,1)$ superspace spinor derivative and the three differential operators $\check{D}^{(A)}_+$, $A = 1,2,3$, determine the generators of nonmanifest supersymmetries $Q^{(A)}_+$ via the constraint (2.2)

$$
\check{D}^{(A)}_+\phi\Big|_{\theta^+_1=\bar\theta^+_1,\theta^+_2=0} = Q^{(A)}_+\tilde\phi\,, \tag{2.5}
$$

$$
\check{D}^{(A)}_+\chi\Big|_{\theta^+_1=\bar\theta^+_1,\theta^+_2=0} = Q^{(A)}_+\tilde\chi\ . \tag{2.6}
$$

This results in the following relation for the extended supersymmetries for d superfields,

$$
Q^{(A)}_+\begin{pmatrix}\tilde\phi\\ \tilde\chi\\ \bar{\tilde\phi}\\ \bar{\tilde\chi}\end{pmatrix} =: \mathbb{J}^{(A)}D_+\begin{pmatrix}\tilde\phi\\ \tilde\chi\\ \bar{\tilde\phi}\\ \bar{\tilde\chi}\end{pmatrix}, \tag{2.7}
$$

where the complex structures

$$
\mathbb{J}^{(A)} = \mathbb{I}^{(A)} \otimes \mathbb{1}_{d\times d} \tag{2.8}
$$

with

$$\mathbb{I}^{(1)} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}, \quad \mathbb{I}^{(2)} = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \mathbb{I}^{(3)} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{2.9}$$

are constant in this coordinate system and satisfy the quaternion algebra

$$\mathbb{J}^{(A)}\mathbb{J}^{(B)} = -\delta^{AB} + \epsilon^{ABC}\mathbb{J}^{(C)} . \tag{2.10}$$

Then this gives transformations for $\tilde{\phi}, \tilde{\chi}$ of the form (1.4).

3 (2,1) superspace formulation

The general $(2,1)$ sigma model action can be written in $(2,1)$ superspace as [2, 23]

$$S = \int d^2x d^3\theta \left(k_\alpha D_- \varphi^\alpha + \bar{k}_{\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}} \right) . \tag{3.1}$$

The fields φ^α are $(2,1)$ chiral

$$\bar{\mathbb{D}}_+ \varphi^\alpha = 0 , \tag{3.2}$$

and $\bar{\varphi}^{\bar{\alpha}}$ are their complex conjugates $\bar{\varphi}^{\bar{\alpha}} = (\varphi^\alpha)^*$. The theory is defined locally by a 1-form potential $k_\alpha(\varphi, \bar{\varphi})$ with $\bar{k}_{\bar{\alpha}} = (k_\alpha)^*$, which is defined up to the addition of the gradient of a function $h(\varphi, \bar{\varphi})$ and a holomorphic 1-form $f_\alpha(\varphi)$,

$$k_\alpha(\varphi, \bar{\varphi}) \to k_\alpha(\varphi, \bar{\varphi}) + \partial_\alpha h(\varphi, \bar{\varphi}) + f_\alpha(\varphi) . \tag{3.3}$$

The metric g and B field for the model (3.1) are (in a particular gauge) [23]

$$\begin{aligned} g_{\alpha\bar{\beta}} &= i(\partial_\alpha \bar{k}_{\bar{\beta}} - \partial_{\bar{\beta}} k_\alpha) \\ B^{(2,0)}_{\alpha\beta} &= i(\partial_\alpha k_\beta - \partial_\beta k_\alpha) \\ B &= B^{(2,0)} + B^{(0,2)} \end{aligned} \tag{3.4}$$

as may be verified by reducing to the $(1,1)$ superspace formulation [2, 23].

The $(4,1)$ multiplet (2.2) can be expanded into $(2,1)$ superspace by writing[2]

$$\phi|_{\theta_2^+=0} = \tilde{\phi}, \qquad \chi|_{\theta_2^+=0} = \tilde{\chi} . \tag{3.5}$$

The constraints (2.2) then define the terms in ϕ, χ of higher order in θ_2 and give the supersymmetry transformations under the non-manifest supersymmetries. The $(4,1)$ derivative $\mathbb{D}_{+1}$ survives as the (2,1) derivative $\mathbb{D}_+$ while $\mathbb{D}_{+2}$ gives the generator Q of non-manifest supersymmetries, acting as:

$$\bar{Q}_+\tilde{\phi} = (\bar{\mathbb{D}}_+^2\phi)\Big|_{\theta^2=0}, \qquad \bar{Q}_+\tilde{\chi} = (\bar{\mathbb{D}}_+^2\chi)\Big|_{\theta^2=0}, \qquad \bar{Q}_+\bar{\tilde{\phi}} = 0 , \qquad \bar{Q}_+\bar{\tilde{\chi}} = 0 . \tag{3.6}$$

Complex conjugation then gives the action of the generator Q.

[2]We temporarily use the tilde notation for the $(2,1)$ components in this section, just as we did for the $(1,1)$ components in section 2.

The action for d $(4,1)$ multiplets must take the form (3.1) when written in $(2,1)$ superspace, with $(2,1)$ chiral superfields $\varphi^\alpha = (\tilde{\phi}^i, \tilde{\chi}^i)$ with $i = 1, \ldots, d$. We will henceforth drop the tildes on $\tilde{\phi}^i, \tilde{\chi}^i$. Then using the constraints (2.2), (3.6) gives the non-manifest supersymmetry transformations

$$\bar{Q}_+\phi = i\bar{\mathbb{D}}_+\bar{\chi}, \qquad \bar{Q}_+\chi = -i\bar{\mathbb{D}}_+\bar{\phi}\,, \qquad \bar{Q}_+\bar{\phi} = 0\,, \qquad \bar{Q}_+\bar{\chi} = 0\,. \tag{3.7}$$

The potential has components $k_\alpha = (k_{\phi^i}, k_{\chi^i})$ and the variation of the action (3.1) under the non-manifest supersymmetries generated by $\bar{Q}_+$ takes the form

$$\delta S = \int d^2x \mathbb{D}_+\bar{\mathbb{D}}_+ D_- \Delta \tag{3.8}$$

where

$$\begin{aligned}\Delta = \;& iD_-\phi^i \left(k_{\phi^i,\chi^j}\bar{\mathbb{D}}_+\bar{\phi}^j - k_{\phi^i,\phi^j}\bar{\mathbb{D}}_+\bar{\chi}^j\right)\\ & -iD_-\bar{\phi}^i\left[(\bar{k}_{\bar{\phi}^i,\phi^j} + k_{\chi^i\bar{\chi}^j})\bar{\mathbb{D}}_+\bar{\chi}^j - (\bar{k}_{\bar{\phi}^i,\chi^j} - k_{\chi^i,\bar{\phi}^j})\bar{\mathbb{D}}_+\bar{\phi}^j\right]\\ & -(\phi \leftrightarrow \chi)\,.\end{aligned} \tag{3.9}$$

The second line vanishes if k satisfies

$$\begin{aligned} k_{\phi^i,\bar{\phi}^j} + \bar{k}_{\bar{\chi}^i,\chi^j} &= 0\,,\\ k_{\phi^i,\bar{\chi}^j} - \bar{k}_{\bar{\chi}^i,\phi^j} &= 0\,,\end{aligned} \tag{3.10}$$

where the comma denotes a partial derivative, so that e.g. $k_{\phi^i,\bar{\phi}^j} = \partial k_{\phi^i}/\partial\bar{\phi}^j$. Then $\bar{\mathbb{D}}_+\Delta$ gives expressions that vanish after repeated use of (3.10) and their derivatives. Thus (3.10) implies that the variation (3.8) of the action under the extra supersymmetries vanishes.

We note that the vanishing of Δ and $\bar{\mathbb{D}}_+\Delta$ is sufficient for invariance, but not necessary. For invariance, it is only necessary that they reduce to terms that vanish when integrated, so that $\mathbb{D}_+\bar{\mathbb{D}}_+D_-\Delta$ is a total derivative with $\int d^2x\mathbb{D}_+\bar{\mathbb{D}}_+D_-\Delta = 0$ (up to a boundary term). This is essentially the condition that the variation of the action under the non-manifest supersymmetries can be cancelled by transformations of the form (3.3). The full necessary and sufficient conditions for supersymmetry will be given in the next section, from a geometric analysis. We will return to the $(4,1)$ superspace formulation of these actions in sections 4 and 8.

4 General (4,1) Sigma models

We now consider the general conditions for the $(2,1)$ superspace action (3.1) to be $(4,1)$ supersymmetric so that it is invariant under two further supersymmetries. Following [24] and [26], we make the ansatz

$$\begin{aligned}\delta\varphi^\alpha &= \bar{\epsilon}^+\bar{\mathbb{D}}_+ f^\alpha(\varphi,\bar{\varphi})\\ \delta\bar{\varphi}^{\bar{\alpha}} &= \epsilon^+\mathbb{D}_+\bar{f}^{\bar{\alpha}}(\varphi,\bar{\varphi})\end{aligned} \tag{4.1}$$

for the additional supersymmetries of the action (3.1). Up to central charge transformations, this is the most general ansatz compatible with the chirality properties [26].

Expanding in components and comparing with (1.4), we can read off the form of the complex structures. The manifest (2,1) supersymmetry involves the canonical complex structure

$$\mathbb{J}^{(1)} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix} , \tag{4.2}$$

while the transformation (4.1) yields second and third ones

$$\mathbb{J}^{(2)} = \begin{pmatrix} 0 & f^{\alpha}_{\bar\beta} \\ \bar f^{\bar\alpha}_{\beta} & 0 \end{pmatrix} , \quad \mathbb{J}^{(3)} = \begin{pmatrix} 0 & i f^{\alpha}_{\bar\beta} \\ -i\bar f^{\bar\alpha}_{\beta} & 0 \end{pmatrix} . \tag{4.3}$$

Here, the lower index on f denotes a derivative,

$$f^{\alpha}_{\bar\beta} := \frac{\partial f^{\alpha}}{\partial\bar\varphi^{\bar\beta}} . \tag{4.4}$$

From off-shell closure of the algebra,

$$[\delta_1,\delta_2]\varphi^{\alpha} = 2i\epsilon^{+}_{[2}\bar\epsilon^{+}_{1]}\partial_{+\!\!+}\varphi^{\alpha},$$

we deduce that (cf. [24])

$$f^{\alpha}_{\bar\beta}\bar f^{\bar\gamma}_{\ \alpha} = -\,\delta^{\bar\gamma}_{\ \bar\beta} , \qquad \bar f^{\bar\alpha}_{\ \beta} f^{\gamma}_{\ \bar\alpha} = -\,\delta^{\gamma}_{\ \beta} , \tag{4.5}$$

$$f^{\alpha}_{[\bar\alpha} f^{\beta}_{\ \bar\beta]\alpha} = 0 , \qquad \bar f^{\bar\alpha}_{[\alpha}\bar f^{\bar\beta}_{\ \beta]\bar\alpha} = 0 . \tag{4.6}$$

(See [24] for similar relations for $N = 2$ in $d = 4$.) Here $f^{\beta}_{\ \bar\beta\alpha} = \partial f^{\beta}/\partial\bar\varphi^{\bar\beta}\partial\varphi^{\alpha}$ etc. Then the matrices $\mathbb{J}^{(1)}, \mathbb{J}^{(2)}, \mathbb{J}^{(3)}$ satisfy the quaternion algebra and have vanishing Nijenhuis tensors

$$\mathcal{N}^{i}_{jk}(\mathbb{J}^{(A)}) = 0 , \tag{4.7}$$

so that they are each complex structures.

The remaining geometric constraints follow from invariance of the action. Varying the action we find

$$\delta S = \int d^2x d^3\theta\, \Delta \tag{4.8}$$

where

$$\Delta = \bar\epsilon^{+}\bar{\mathbb{D}}_{+} f^{\beta}\left(B_{\beta\alpha}D_{-}\varphi^{\alpha} + g_{\beta\bar\alpha}D_{-}\bar\varphi^{\bar\alpha}\right) . \tag{4.9}$$

Pushing in $\bar{\mathbb{D}}_{+}$ from the measure yields[3]

$$\bar{\mathbb{D}}_{+}\Delta = \bar\epsilon^{+}\bar{\mathbb{D}}_{+} f^{\beta}\bar{\mathbb{D}}_{+}\bar\varphi^{\bar\beta}\left(B_{\beta\alpha,\bar\beta}D_{-}\varphi^{\alpha} + g_{\beta\bar\alpha,\bar\beta}D_{-}\bar\varphi^{\bar\alpha}\right) - \bar\epsilon^{+}\bar{\mathbb{D}}_{+} f^{\beta}D_{-}\bar{\mathbb{D}}_{+}\bar\varphi^{\bar\alpha}g_{\beta\bar\alpha} . \tag{4.10}$$

[3]If we performed the full reduction to $(1,1)$ this would to parallel the calculation in [1] for $(2,2)$ supersymmetry.

Integrating D_- by parts and defining

$$\omega_{\bar{\beta}\bar{\alpha}} := f^{\beta}{}_{[\bar{\alpha}}g_{\bar{\beta}]\beta} = \frac{1}{2}\left(f^{\beta}{}_{\bar{\alpha}}g_{\beta\bar{\beta}} - f^{\beta}{}_{\bar{\beta}}g_{\beta\bar{\alpha}}\right) \tag{4.11}$$

we rewrite the last term as

$$-\bar{\epsilon}^{+}\left(\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\beta}}D_{-}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\alpha}}f^{\beta}{}_{(\bar{\beta}}g_{\beta\bar{\alpha})} + \frac{1}{2}D_{-}\omega_{\bar{\beta}\bar{\alpha}}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\alpha}}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\beta}}\right) - \frac{1}{2}D_{-}\left(\epsilon^{+}\omega_{\bar{\beta}\bar{\alpha}}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\alpha}}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\beta}}\right) \tag{4.12}$$

and drop the final term here as it is a total derivative.

Then the condition for supersymmetry is

$$\begin{aligned}&\bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}f^{\beta}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\beta}}\left(B_{\beta\alpha,\bar{\beta}}D_{-}\varphi^{\alpha} + g_{\beta\bar{\alpha},\bar{\beta}}D_{-}\bar{\varphi}^{\bar{\alpha}}\right)\\ &\quad -\bar{\epsilon}^{+}\left(\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\beta}}D_{-}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\alpha}}f^{\beta}{}_{(\bar{\beta}}g_{\beta\bar{\alpha})} + \frac{1}{2}D_{-}\omega_{\bar{\beta}\bar{\alpha}}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\alpha}}\bar{\mathbb{D}}_{+}\bar{\varphi}^{\bar{\beta}}\right) = 0\,.\end{aligned} \tag{4.13}$$

Then the independent terms in (4.13) give the equations[4]

$$f^{\beta}{}_{(\bar{\alpha}}g_{\bar{\gamma})\beta} = 0\,, \qquad \Rightarrow f^{\beta}{}_{\bar{\alpha}}g_{\bar{\gamma}\beta} = \omega_{\bar{\gamma}\bar{\alpha}}\,, \tag{4.14}$$

together with

$$\begin{aligned}\frac{1}{2}\omega_{\bar{\alpha}\bar{\gamma},\bar{\beta}} - g_{\beta\bar{\beta},[\bar{\alpha}}f^{\beta}{}_{\bar{\gamma}]} &= 0\,, \qquad \Rightarrow \nabla^{(+)}_{\bar{\beta}}\omega_{\bar{\alpha}\bar{\gamma}} = 0\,,\\ \frac{1}{2}\omega_{\bar{\alpha}\bar{\gamma},\beta} - B^{(2,0)}_{\sigma\beta,[\bar{\alpha}}f^{\sigma}{}_{\bar{\gamma}]} &= 0\,, \qquad \Rightarrow \nabla^{(+)}_{\beta}\omega_{\bar{\alpha}\bar{\gamma}} = 0\,,\end{aligned} \tag{4.15}$$

where we have used the geometric constraints on the connection and torsion that follow from the underlying $(2,1)$ geometry, as well as the definitions (3.4). Some of this structure is described in appendix A. The conditions (4.14) imply that the metric is hermitian with respect to the complex structures (4.3) while (4.15) implies that these complex structures are covariantly constant with respect to the connection with torsion $\Gamma^{(+)} = \Gamma^{(0)} + T$:

$$\nabla^{(+)}_{i}f^{\kappa}{}_{\bar{\lambda}} = 0\,, \tag{4.16}$$

where $\Gamma^{(0)}$ is the levi-Civita connection and the torsion is formed from the B field strength as $T = \frac{1}{2}g^{-1}H$. We note that this geometry is sometimes referred to as hyperkähler with torsion. Finally, the vanishing of the Nijenhuis tensor (4.7) in conjunction with the covariant constancy conditions in (4.15) leads to

$$H = d^{(A)}\omega^{(A)}\,, \tag{4.17}$$

for each A, where $\omega^{(A)}$ is the 2-form with components $\omega^{(A)}_{ij} = g_{ik}(\mathbb{J}^{(A)})^{k}{}_{j}$, and $d^{(A)}$ is the $i(\bar{\partial}-\partial)$ operator for the complex structure $\mathbb{J}^{(A)}$. This can also be derived from $\nabla^{(+)}J^{(A)} = 0$ and $\mathcal{N}_{jk}{}^{i}(J^{(A)}) = 0$.

[4] Pushing in additional $\mathbb{D}$s from the measure and/or partial integration of bosonic derivatives does not relate these terms or lead to any further simplifications.

The transformations (4.1) correspond to generalising the constraints (2.2) to

$$\begin{aligned} \bar{\mathbb{D}}^1_+ \varphi^\alpha &= 0\,, \\ \bar{\mathbb{D}}^2_+ \varphi^\alpha &= f^\alpha{}_{\bar\beta} \bar{\mathbb{D}}^1_+ \bar\varphi^{\bar\beta} \end{aligned} \tag{4.18}$$

in $(4,1)$ superspace. Note that the constraints (4.1) require the existence of a local product structure in addition to the structure required for $(4,1)$ geometry, as this is necessary to split the coordinates into two sets, $\varphi = (\phi, \chi)$. For $(4,2)$ or $(4,4)$ supersymmetry, the existence of this product structure follows from the conditions for extended supersymmetry.

For constant complex structures $f^\alpha{}_{\bar\beta}$, (4.1) implies that

$$\Gamma^{(+)}_{i\sigma\bar\kappa} f^\sigma{}_{\bar\lambda} + \Gamma^{(+)}_{i\bar\lambda\sigma} f^\sigma{}_{\bar\kappa} = 0\,, \tag{4.19}$$

where $i = (\beta, \bar\beta)$, we have lowered κ to $\bar\kappa$ and used the antisymmetry of the two-forms ω. This (non-covariant) condition can be rewritten using formulae from the appendix as

$$\begin{aligned} f^\sigma{}_{\bar\lambda} g_{\sigma\bar\kappa,\bar\beta} + 2 g_{\sigma\bar\beta,[\bar\lambda} f^\sigma{}_{\bar\kappa]} &= 0 \\ f^\sigma{}_{\bar\kappa} g_{\bar\lambda\sigma,\beta} + 2 f^\sigma{}_{[\bar\lambda} g_{\bar\kappa]\beta,\sigma} &= 0\,. \end{aligned} \tag{4.20}$$

For the constant complex structures (2.8), we have

$$f^\alpha{}_{\bar\beta} = i(\sigma_2)^\alpha{}_{\bar\beta} \tag{4.21}$$

and the hermiticity condition (4.14) becomes

$$\begin{aligned} \bar{k}_{\bar\phi^i,\phi^j} - k_{\phi^j,\bar\phi^i} - \bar{k}_{\bar\chi^j,\chi^i} + k_{\chi^i,\bar\chi^j} &= 0 \\ \bar{k}_{\bar\chi^{(i},\phi^{j)}} - k_{\phi^{(i},\bar\chi^{j)}} &= 0\,, \end{aligned} \tag{4.22}$$

while the covariant constancy conditions (4.15) or (4.20) become

$$\begin{aligned} \frac{1}{2}\left(k_{\phi^{[j},\bar\chi^{k]}} - \bar{k}_{\bar\chi^{[j},\phi^{k]}}\right)_{,\bar\beta} - \bar{k}_{\bar\beta,\phi^{[j}\bar\chi^{k]}} &= 0 \\ \frac{1}{2}\left(\bar{k}_{\bar\phi^k,\phi^j} + k_{\chi^k,\bar\chi^j} + \bar{k}_{\bar\chi^j,\chi^k} + k_{\phi^j,\bar\phi^k}\right)_{,\bar\beta} - \bar{k}_{\bar\beta,\phi^j\bar\phi^k} - \bar{k}_{\bar\beta,\chi^k\bar\chi^j} &= 0 \\ \frac{1}{2}\left(k_{\chi^{[j},\bar\phi^{k]}} - \bar{k}_{\bar\phi^{[j},\chi^{k]}}\right)_{,\bar\beta} - \bar{k}_{\bar\beta,\chi^{[j}\bar\phi^{k]}} &= 0\,. \end{aligned} \tag{4.23}$$

We note that if (3.10) are satisfied, then this implies that (4.22) and (4.23) are satisfied. The converse is not true, and (3.10) gives a special case of the general conditions (4.22) and (4.23). E.g. (3.10) requires that $k_{\phi^i,\bar\phi^j} + \bar{k}_{\bar\chi^i,\chi^j}$ vanishes whereas (4.22) only sets it equal to its hermitean conjugate.

5 (4,2) off-shell supermultiplets

Truncating the $(4,4)$ off-shell multiplet of [1] to $(4,2)$ superspace gives an off-shell $(4,2)$ supermultiplet that can be formulated as follows. We use $(4,2)$ superspace with coordinates

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$x^{\#}, x^{=}, \theta^{+a}, \bar{\theta}^{+}_{a}, \theta^{-}, \bar{\theta}^{-}$ where $a = 1, 2$ is an SU(2) index.[5] All fermionic coordinates are complex. There are two complex right-handed spinorial covariant derivatives $\mathbb{D}^{a}_{+}$ and a complex left-handed spinorial covariant derivative $\mathbb{D}_{-}$, satisfying

$$\begin{aligned}\{\mathbb{D}_{+a}, \bar{\mathbb{D}}^{b}_{+}\} &= 2i\delta^{b}_{a}\partial_{\#}\,, \quad a, b, = 1, 2,\\ \{\mathbb{D}_{-}, \bar{\mathbb{D}}_{-}\} &= 2i\partial_{=}\,,\end{aligned} \tag{5.1}$$

The $(4, 2)$ multiplet obtained from truncating the $(4, 4)$ multiplet of [1] consists of a pair of $(4, 2)$ superfields ϕ, χ satisfying the constraints

$$\begin{aligned}&\bar{\mathbb{D}}^{1}_{+}\phi = 0 = \mathbb{D}_{+2}\phi\,, && \bar{\mathbb{D}}^{1}_{+}\chi = 0 = \mathbb{D}_{+2}\chi\,,\\ &\bar{\mathbb{D}}^{2}_{+}\chi = -i\bar{\mathbb{D}}^{1}_{+}\bar{\phi}\,, && \bar{\mathbb{D}}^{2}_{+}\phi = i\bar{\mathbb{D}}^{1}_{+}\bar{\chi},\\ &\bar{\mathbb{D}}_{-}\phi = 0\,, && \mathbb{D}_{-}\chi = 0.\end{aligned} \tag{5.2}$$

An alternative truncation has the $\mathbb{D}_{-}$ constraints on the two fields switched. The two multiplets are related by interchanging $\theta_{-} \leftrightarrow \bar{\theta}_{-}$, so a theory written in terms of one multiplet is equivalent to one written in terms of the other. Indeed, we show in section 9.4 that their projective superspace formulations are isomorphic. However, just as for the $(2, 2)$ chiral and twisted chiral multiplets, one might suspect that there could be new non-trivial theories that have both kinds of supermultiplet. As far as we have been able to ascertain, this is not the case (as long as no further superfields are involved) as no supersymmetric interaction between the two kinds of multiplets seems possible.[6]

6 (4,2) supersymmetry in (2,2) superspace

In $(2, 2)$ superspace, chiral superfields φ satisfy

$$\bar{\mathbb{D}}_{\pm}\varphi = 0 \tag{6.1}$$

while twisted chiral superfields ψ satisfy

$$\bar{\mathbb{D}}_{+}\psi = 0, \qquad \mathbb{D}_{-}\psi = 0 \tag{6.2}$$

There are other possible $(2, 2)$ multiplets such as semichiral multiplets [11], but here we shall restrict ourselves to these two.

The general action for chiral and twisted chiral multiplets is given by [1]

$$S = \int d^{2}x d^{4}\theta\, K(\varphi, \bar{\varphi}, \psi, \bar{\psi}) \tag{6.3}$$

in terms of an unconstrained scalar potential $K(\varphi, \bar{\varphi}, \psi, \bar{\psi})$. Expanding in $(2, 1)$ superfields by writing

$$\varphi|_{\theta^{-}_{2}=0} = \tilde{\varphi}, \qquad \psi|_{\theta^{-}_{2}=0} = \tilde{\psi} \tag{6.4}$$

[5]There is a possible confusion between the SU(2) index 2 and a 2 indicating the square. This is resolved by noting that a bold face $\mathbb{D}_{\pm}$ never appears squared.

[6]Added in proof: the referee informs us that this is in agreement with the results of [25] derived using bi-harmonic superspace.

one finds the action (3.1) and the vector potentials are gradients of the scalar potential K [27],

$$k_\varphi = i\partial_\varphi K\,, \quad k_\psi = -i\partial_\psi K. \tag{6.5}$$

where the tildes and indices enumerating multiplets have been suppressed.

We now turn to the off-shell $(4,2)$ supermultiplet introduced in the last section. It contains a $(2,2)$ chiral superfield ϕ and a twisted chiral superfield χ with the transformation under the extra supersymmetries $Q, \bar{Q}$ given by

$$\bar{Q}_+\phi = i\bar{\mathbb{D}}_+\bar{\chi}, \qquad \bar{Q}_+\chi = -i\bar{\mathbb{D}}_+\bar{\phi}, \qquad \bar{Q}_+\bar{\phi} = 0, \qquad \bar{Q}_+\bar{\chi} = 0\,, \tag{6.6}$$

together with the complex conjugate expressions.

Consider a model with d multiplets ϕ^i, χ^i, so the action is

$$S = \int d^2x d^4\theta\, K(\phi^i, \chi^i, \bar{\phi}^i, \bar{\chi}^i)\ . \tag{6.7}$$

Then under the $\bar{Q}$ transformation

$$\delta S = \int d^2x \mathbb{D}_+\bar{\mathbb{D}}_+\mathbb{D}_-\bar{\mathbb{D}}_-\Delta \tag{6.8}$$

where

$$\Delta = \bar{Q}K = iK,_{\phi^i}\,\bar{\mathbb{D}}_+\bar{\chi}^i - iK,_{\chi^i}\,\bar{\mathbb{D}}_+\bar{\phi}^i\ . \tag{6.9}$$

Then acting with $\bar{\mathbb{D}}_+$ gives

$$\delta S = \int d^2x \mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_-(\bar{\mathbb{D}}_+\Delta) \tag{6.10}$$

where

$$\bar{\mathbb{D}}_+\Delta = \bar{\mathbb{D}}_+\bar{Q}K = \bar{\mathbb{D}}_+(iK,_{\phi^i}\,\bar{\mathbb{D}}_+\bar{\chi}^i - iK,_{\chi^i}\,\bar{\mathbb{D}}_+\bar{\phi}^i)\ . \tag{6.11}$$

This gives

$$\begin{aligned}\bar{\mathbb{D}}_+\Delta =\ & iK,_{\phi^i\bar{\phi}^j}\,\bar{\mathbb{D}}_+\bar{\phi}^j\bar{\mathbb{D}}_+\bar{\chi}^i + iK,_{\phi^i\bar{\chi}^j}\,\bar{\mathbb{D}}_+\bar{\chi}^j\bar{\mathbb{D}}_+\bar{\chi}^i \\ & -iK,_{\chi^i\bar{\phi}^j}\,\bar{\mathbb{D}}_+\bar{\phi}^j\bar{\mathbb{D}}_+\bar{\phi}^i - iK,_{\chi^i\bar{\chi}^j}\,\bar{\mathbb{D}}_+\bar{\chi}^j\bar{\mathbb{D}}_+\bar{\phi}^i \qquad (6.12)\\ =\ & i(K,_{\phi^i\bar{\phi}^j} + K,_{\chi^j\bar{\chi}^i})\bar{\mathbb{D}}_+\bar{\phi}^j\bar{\mathbb{D}}_+\bar{\chi}^i \\ & +iK,_{\phi^i\bar{\chi}^j}\,\bar{\mathbb{D}}_+\bar{\chi}^j\bar{\mathbb{D}}_+\bar{\chi}^i - iK,_{\chi^i\bar{\phi}^j}\,\bar{\mathbb{D}}_+\bar{\phi}^j\bar{\mathbb{D}}_+\bar{\phi}^i. \qquad (6.13)\end{aligned}$$

The first term vanishes if

$$K,_{\phi^i\bar{\phi}^j} + K,_{\chi^j\bar{\chi}^i} = 0\ . \tag{6.14}$$

This is a sufficient condition for full invariance, since using it one finds that the remaining terms vanish using $\bar{\mathbb{D}}_-$ or $\mathbb{D}_-$ from the remaining measure:

$$\begin{aligned}&\bar{\mathbb{D}}_-(K,_{\phi^i\bar{\chi}^j}\,\bar{\mathbb{D}}_+\bar{\chi}^j\bar{\mathbb{D}}_+\bar{\chi}^i) = 0\\ &\mathbb{D}_-(K,_{\chi^i\bar{\phi}^j}\,\bar{\mathbb{D}}_+\bar{\phi}^j\bar{\mathbb{D}}_+\bar{\phi}^i) = 0\ .\end{aligned} \tag{6.15}$$

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To find the necessary and sufficient conditions for $(4,2)$ supersymmetry, we start with the conditions for $(4,1)$ supersymmetry given by (4.22) and (4.23). For the sigma model to have $(4,2)$ supersymmetry requires in addition the condition (6.5) which here implies that the $(4,1)$ potential k is given by derivatives of a scalar potential K:

$$k_{\phi^i} = iK_{,\phi^i}\,, \quad k_{\chi^i} = -iK_{,\chi^i}\ . \tag{6.16}$$

Then the hermiticity condition (4.22) together with (6.16) gives precisely the condition (6.14), and then the remaining conditions (4.23) are all satisfied identically using (6.14) and (6.16), and give no further constraints. Thus (6.14) is the necessary and sufficient condition for a $(2,2)$ model to have $(4,2)$ supersymmetry.

In section 3, we considered $(4,1)$ models whose potentials satisfied the conditions (3.10). These models will have $(4,2)$ supersymmetry if (6.16) is satisfied, which implies (6.14) together with

$$K_{,\phi^i\bar{\chi}^j} = K_{,\phi^j\bar{\chi}^i}\ . \tag{6.17}$$

This gives a special class of $(4,2)$ models.

7 (4,4) supermultiplet and action

The $(4,4)$ off-shell multiplet of [1] is formulated in (4,4) superspace with two complex right-handed spinorial covariant derivatives $\mathbb{D}_{+a}$ and two complex left-handed spinorial covariant derivatives $\mathbb{D}_{-a}$, satisfying

$$\begin{aligned} \{\mathbb{D}_{+a}, \bar{\mathbb{D}}_+^b\} =&\ 2i\delta_a^b\partial_{+\!\!\!+}\,, \quad a, b, = 1, 2. \\ \{\mathbb{D}_{-a}, \bar{\mathbb{D}}_-^b\} =&\ 2i\delta_a^b\partial_{=}\,, \end{aligned} \tag{7.1}$$

The $(4,4)$ multiplet of [1] consists of a pair of superfields ϕ, χ satisfying the constraints

$$\begin{aligned} &\bar{\mathbb{D}}_+^1\phi = 0 = \mathbb{D}_{+2}\phi\,, && \bar{\mathbb{D}}_+^1\chi = 0 = \mathbb{D}_{+2}\chi\,,\,, && \mathbb{D}_{-a}\chi = 0 \\ &\bar{\mathbb{D}}_+^2\chi = -i\bar{\mathbb{D}}_+^1\bar{\phi}\,, && \bar{\mathbb{D}}_+^2\phi = i\bar{\mathbb{D}}_+^1\bar{\chi}, && \\ &\bar{\mathbb{D}}_-^2\chi = i\mathbb{D}_{-1}\phi\,, && \mathbb{D}_{-2}\phi = i\bar{\mathbb{D}}_-^1\chi\ . && \end{aligned} \tag{7.2}$$

As before, the action can be written in (2,2) superspace in terms of d (2,2) chiral multiplets ϕ^i and d twisted chiral multiplets χ^i, so the action is

$$S = \int d^2x d^4\theta\, K(\phi^i, \chi^i, \bar{\phi}^i, \bar{\chi}^i) \tag{7.3}$$

with the non-manifest supersymmetry transformations given by

$$\bar{Q}_+\phi = i\bar{\mathbb{D}}_+\bar{\chi}, \qquad \bar{Q}_+\chi = -i\bar{\mathbb{D}}_+\bar{\phi}, \qquad \bar{Q}_+\bar{\phi} = 0, \qquad \bar{Q}_+\bar{\chi} = 0\,, \tag{7.4}$$

and

$$Q_-\phi = i\bar{\mathbb{D}}_-\chi, \qquad Q_-\bar{\chi} = -i\bar{\mathbb{D}}_-\bar{\phi}, \qquad Q_-\bar{\phi} = 0, \qquad Q_-\chi = 0\,, \tag{7.5}$$

together with the complex conjugate expressions.

Then under the $\bar{Q}_+$ transformation

$$\delta S = \int d^2x \mathbb{D}_+ \bar{\mathbb{D}}_+ \mathbb{D}_- \bar{\mathbb{D}}_- \Delta \tag{7.6}$$

where

$$\Delta = \bar{Q}_+ K = iK,_{\phi^i} \bar{\mathbb{D}}_+ \bar{\chi}^i - iK,_{\chi^i} \bar{\mathbb{D}}_+ \bar{\phi}^i \tag{7.7}$$

and, as in the last section, the action is invariant if

$$K,_{\phi^i \bar{\phi}^j} + K,_{\chi^j \bar{\chi}^i} = 0 \ . \tag{7.8}$$

Under the Q_- transformation we obtain (7.6) but with

$$\Delta = Q_- K = iK,_{\phi^i} \bar{\mathbb{D}}_- \chi^i - iK,_{\bar{\chi}^i} \bar{\mathbb{D}}_- \bar{\phi}^i \tag{7.9}$$

Then a similar analysis to the above gives that the action is invariant under the Q_- transformation if

$$K,_{\phi^i \bar{\phi}^j} + K,_{\chi^i \bar{\chi}^j} = 0 \ . \tag{7.10}$$

Then the necessary and sufficient conditions for $(4,4)$ supersymmetry are (7.8) and (7.10).

Together, (7.8) and (7.10) imply

$$K,_{\phi^i \bar{\phi}^j} = K,_{\phi^j \bar{\phi}^i} \ . \tag{7.11}$$

We can then instead take the necessary and sufficient conditions for $(4,4)$ supersymmetry to be (7.11) and (7.10), which are precisely the conditions that were found in [1].

8 (4,1) superspace action

8.1 General

A superspace action for $\mathcal{N}$ supersymmetries in D dimensions involves integration over the $d = s\mathcal{N}$ fermi coordinates θ, where s is the dimension of the spinor representation in D dimensions (e.g. $s = 4$ in $D = 4$). This picks out the highest θ component from the superspace Lagrangian $\mathcal{L}$. Equivalence between Berezin integration and differentiation means that the integration may be written schematically as

$$\int d^D x d^d \theta \mathcal{L} = \int d^D x \frac{\partial^d \mathcal{L}}{\partial \theta^d} = \int d^D x D^d \mathcal{L}\Big|_{\theta=0}, \tag{8.1}$$

where the vertical bar denotes the θ-independent part of the expression and use has been made of the fact that the spinorial covariant derivatives D differ from the partial spinorial derivatives by θ terms involving a spacetime derivative, and total derivative terms are dropped from the spacetime integral. Since the product $DD \sim \partial$, with ∂ a space time derivative, it is clear that even if the Lagrangian $\mathcal{L}$ contains no derivatives, there is a limit to $d \leq 4$ in spacetime dimensions $D \geq 3$, for the action to be physical, i.e. for its bosonic part to be quadratic in space time derivatives. In $D = 2$ dimensions with (p, q) supersymmetry, $D_-D_- \sim \partial_=$ and $D_+D_+ \sim \partial_{\#}$ and a similar argument shows that $p \leq 2$ and $q \leq 2$ for the action to be physical.

This bound on d or (p, q) can be circumvented by finding subspaces that are invariant under supersymmetry and integrating constrained Lagrangians over those. The prime example of such subspaces are the chiral and antichiral subspaces of $D = 4$, $\mathcal{N} = 1$ superspace, where the complex superfields ϕ obey the chirality condition $\bar{D}\phi = 0$, and a chiral Lagrangian is integrated with the chiral measure D^2, and an anti-chiral Lagrangian is integrated with the anti-chiral measure $\bar{D}^2$. The projective superspace construction described in section 9 below provides a systematic method of constructing such constrained superfields and the corresponding invariant subspaces, but we first describe the approach of [1].

8.2 The GHR approach

In [1] a general invariant action for an off-shell $(4, 4)$ multiplet was found. Here we adapt this to our $(4, 1)$ models.

In constructing an action for $(4, 1)$ multiplets we face the problem discussed above in section 8.1. The algebra involves four real or two complex positive chirality derivatives $\mathbb{D}_{+a}, \bar{\mathbb{D}}^a_+$, and so the full $(4, 1)$ superspace measure has too large a dimension. We then seek an invariant subspace and corresponding subintegration, similar to the chiral subspaces in $\mathcal{N} = 1, D = 4$ superspace. We use the procedure of [1] and define two linear combinations of positive chirality spinor derivatives:

$$\begin{aligned}\nabla_+ &= \beta\mathbb{D}_{+1} + i\alpha\mathbb{D}_{+2}\\ \Delta_+ &= \alpha\bar{\mathbb{D}}^1_+ + i\beta\bar{\mathbb{D}}^2_+\end{aligned} \tag{8.2}$$

for some choice of complex parameters α, β.

For a given choice of parameters α, β, the $(4, 1)$ superfields $\eta, \breve{\eta}$ given by

$$\eta := \alpha\phi + \beta\bar{\chi}\,, \quad \breve{\eta} := \beta\bar{\phi} - \alpha\chi \tag{8.3}$$

are annihilated by ∇_+ and Δ_+

$$\nabla_+\eta = \Delta_+\eta = 0\,, \quad \nabla_+\breve{\eta} = \Delta_+\breve{\eta} = 0\ .$$

Then for a Lagrangian constructed from these constrained superfields, a $(4, 1)$ supersymmetric action is given using the conjugate operators $\bar{\nabla}_+$ and $\bar{\Delta}_+$ to define the supermeasure. The action is then

$$i\int d^2x\bar{\nabla}_+\bar{\Delta}_+D_-L_- + \text{h.c.}\ . \tag{8.4}$$

where h.c. denotes hermitian conjugate, and we take

$$L_- := \lambda_i(\eta, \breve{\eta})D_-\eta^i + \tilde{\lambda}_i(\eta, \breve{\eta})D_-\breve{\eta}^i\,, \tag{8.5}$$

for a set of multiplets labelled by the index i, for some potentials $\lambda_i, \tilde{\lambda}_i$.

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A general action will be a linear superposition of actions of the form (8.4). We then allow the potentials $\lambda_i, \tilde{\lambda}_i$ to depend explicitly on α, β and integrate over all possible values of α, β. The $(4,1)$ supersymmetric action constructed from the constrained superfields in (8.3) is then

$$i \int d^2x \left[\int d\alpha d\beta \bar{\nabla}_+ \bar{\Delta}_+ D_- L_- \right] + \text{h.c.}$$
$$L_- := \lambda_i(\eta, \breve{\eta}; \alpha, \beta) D_- \eta + \tilde{\lambda}_i(\eta, \breve{\eta}; \alpha, \beta) D_- \breve{\eta} \,, \tag{8.6}$$

where the operators $\bar{\nabla}_+$ and $\bar{\Delta}_+$ define the supermeasure. The parameter integration must be specified as some contour integral.

In the special case when the action is a reduction of the $(4,4)$ action of [1] which has a scalar function L as its Lagrangian, one finds

$$-\tilde{\lambda}_i = \lambda_i = i\partial_{\eta^i + \breve{\eta}^i} L(\eta + \breve{\eta}) \ . \tag{8.7}$$

The measure in (8.4) can be rewritten in a form suitable for reduction to $(2,1)$ superspace using

$$\bar{\Delta}_+ = -\frac{\bar{\beta}}{\alpha} \nabla_+ + \frac{1}{\alpha}(|\alpha|^2 + |\beta|^2) \mathbb{D}_{+1} \ . \tag{8.8}$$

Since ∇_+ and Δ_+ annihilate the Lagrangian, the measure becomes

$$\bar{\nabla}_+ \bar{\Delta}_+ D_- \propto \mathbb{D}_{+1} \bar{\mathbb{D}}^1_+ D_- \ . \tag{8.9}$$

In the reduction we identify $\mathbb{D}_{+1} \to \mathbb{D}_+$ which gives the $(2,1)$ measure when the second θ^+ is set to zero

$$\mathbb{D}_{+1} \bar{\mathbb{D}}^1_+ D_- (\dots)| = \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- (\dots)| \ . \tag{8.10}$$

This gives rise to an expression for the potential k_α in terms of an integral of an expression constructed from the $\lambda_i, \tilde{\lambda}_i$; we will give similar forms explicitly in later sections. By construction, the potential k_α will necessarily satisfy the conditions (4.22), (4.23) for $(4,1)$ supersymmetry.

The form of $\eta, \breve{\eta}$ given in (8.3) implies that any function function $f(\eta, \breve{\eta})$ will automatically satisfy

$$\frac{\partial^2 f}{\partial \phi^i \partial \bar{\phi}^{\bar{k}}} + \frac{\partial^2 f}{\partial \chi^k \partial \bar{\chi}^{\bar{i}}} = 0 \ . \tag{8.11}$$

For the multiplet (8.3), this implies that the potential k_α constructed in this way will satisfy

$$k_{\alpha, \phi^k \bar{\phi}^j} + k_{\alpha, \chi^j \bar{\chi}^k} = 0 \,, \tag{8.12}$$

and its complex conjugate, in addition to the conditions (4.22), (4.23) for $(4,1)$ supersymmetry.

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Further, the potentials may be written

$$k_{\phi^i} = i\left(\int d\alpha d\beta \alpha\lambda_i - \int d\bar{\alpha}d\bar{\beta}\bar{\beta}\bar{\tilde{\lambda}}_i\right)$$
$$k_{\chi^i} = -i\left(\int d\alpha d\beta \alpha\tilde{\lambda}_i + \int d\bar{\alpha}d\bar{\beta}\bar{\beta}\bar{\lambda}_i\right) , \tag{8.13}$$

along with their complex conjugates. Using this form, it is easy to show that the potentials actually satisfy the stronger conditions (3.10). Thus the models constructed in this way constitute a subclass of the possible $(4, 1)$ models.

9 Projective superspace

The procedure from [1] used in the derivation of the action (8.6) was introduced to construct an action for a particular multiplet. It was later realised that there is a generalisation that works the other way: the superspace can be enlarged by an extra coordinate or coordinates in such a way that superfields and actions in this enlarged superspace automatically have extended supersymmetry. This is the Projective Superspace construction [9–12], a useful tool for finding new multiplets and constructing actions in various dimensions. We begin by making contact with the discussion in the previous section.

9.1 Relation of the GHR construction to projective superspace

In the previous section we summed over theories parameterised by complex variables (α, β). The overall scale is unimportant, so they can be viewed as homogeneous coordinates on $\mathbb{CP}^1$. It is useful to instead use an inhomogeneous coordinate

$$\zeta = i\alpha/\beta \tag{9.1}$$

in the region where $\beta \neq 0$, or $\zeta' = -i\beta/\alpha$ in the patch where $\alpha \neq 0$. Then the summation over theories corresponds to a contour integral on $\mathbb{CP}^1$, covered by two patches, one with inhomogeneous coordinate ζ and one with inhomogeneous coordinate ζ'. We now discuss Projective Superpace in more detail.

9.2 $(4, q)$ projective superspace defined

Projective superspace is defined to deal with the limitations outlined in section 8.1 and at the same time gives a constructive method for finding new multiplets. We shall be concerned with $(4, q)$ superspace for $q = 4, 2, 1$. In all these cases a full superspace measure has more spinorial derivatives than allowed and so we seek invariant subintegrations. Part of the construction is the same for all p, the difference is mainly in the form of the actions.

We start from the positive chirality part of the D algebra given in the first line of (2.1) or (5.1). A projective coordinate ζ on $\mathbb{CP}^1$ is used to construct the combinations[7]

$$\nabla_+ := \mathbb{D}_{+1} + \zeta\mathbb{D}_{+2} ,$$
$$\breve{\nabla}_+ := \bar{\mathbb{D}}^1_+ - \zeta^{-1}\bar{\mathbb{D}}^2_+ . \tag{9.2}$$

[7]The conventions have varied over time. The present choice are those of [14], up to an unimportant overall ζ factor multiplying $\breve{\nabla}$.

We introduce a conjugation acting on meromorphic functions of $f(\zeta)$ by

$$f(\zeta) \to \breve{f}(\zeta) \tag{9.3}$$

given by the composition of complex conjugation

$$f(\zeta) \to: f^*(\bar{\zeta}) \equiv (f(\zeta))^* \tag{9.4}$$

and the antipodal map

$$\zeta \to -\bar{\zeta}^{-1} \tag{9.5}$$

so that[8]

$$\breve{f}(\zeta) = f^*(-\zeta^{-1}) \ . \tag{9.6}$$

The derivatives (9.2) are related by the this conjugation. We shall be interested in projectively chiral superfields η that satisfy

$$\nabla_+\eta = 0\,, \quad \breve{\nabla}_+\eta = 0\,, \tag{9.7}$$

as well as being $(4,q)$ superfields. We assume that they have the ζ-expansion

$$\eta = \sum_{\mu=-m}^{n} \zeta^\mu \eta_\mu\,, \tag{9.8}$$

where η_μ is the expansion coefficient superfields for the μ'th power of ζ. The constraints (9.7) then lead to the following conditions on the fields η_μ:

$$\begin{aligned} \mathbb{D}_{+1}\eta_\mu + \mathbb{D}_{+2}\eta_{\mu-1} &= 0 \\ \bar{\mathbb{D}}^1_+\eta_\mu - \bar{\mathbb{D}}^2_+\eta_{\mu+1} &= 0 \ . \end{aligned} \tag{9.9}$$

Here $\eta_\mu = 0$ for $\mu < -m$ and $\mu > n$, so that the highest and lowest components are constrained

$$\begin{aligned} \mathbb{D}_{+1}\eta_{-m} &= 0 \\ \bar{\mathbb{D}}^1_+\eta_n &= 0 \ . \end{aligned} \tag{9.10}$$

To be able to write actions, two independent orthogonal derivatives are needed. The following pair can be used for the supermeasure for fields annihilated by the operators (9.2):

$$\begin{aligned} \Delta_+ &:= \mathbb{D}_{+1} - \zeta\mathbb{D}_{+2}\,, \\ \breve{\Delta}_+ &:= \bar{\mathbb{D}}^1_+ + \zeta^{-1}\bar{\mathbb{D}}^2_+ \ . \end{aligned} \tag{9.11}$$

The algebra obeyed by the ∇'s and Δ's is

$$\begin{aligned} &\{\nabla_+,\nabla_+\} = \{\breve{\nabla}_+,\breve{\nabla}_+\} = \{\Delta_+,\Delta_+\} = \{\breve{\Delta}_+,\breve{\Delta}_+\} = \{\nabla_+,\Delta_+\} = \{\breve{\nabla}_+,\breve{\Delta}_+\} = 0 \\ &\{\nabla_+,\breve{\Delta}_+\} = \{\breve{\nabla}_+,\Delta_+\} = 4i\partial_{+\!\!+} \ . \end{aligned} \tag{9.12}$$

[8]Projective superspace uses complex conjugation composed with the antipodal map on $\mathbb{CP}^1$ [10], as described here. It is the relevant conjugation in projective superspace, and in the literature it is often denoted by just a bar. A closely related conjugation in harmonic superspace was earlier introduced in [16].

9.3 (4,1) projective superspace

For the $(4,1)$ theories the algebra is (2.1). The $(2,1)$ content of (9.9) is then obtained as discussed previously in section 2, by identifying the $(2,1)$ derivative as $\mathbb{D}_+ = \mathbb{D}_{+1}$ and the generator of the non-manifest extra supersymmetries[9] as $\mathbb{Q}_+ = \mathbb{D}_{+2}$. Most of the relations in (9.9) will just give the $\mathbb{Q}_+$ action of the second supersymmetry on the ζ coefficients fields η_μ. Only the first and last fields in the ζ-expansion in (9.9) will be constrained[10]

$$\begin{aligned} \mathbb{D}_+\eta_{-m} &= 0 \\ \bar{\mathbb{D}}_+\eta_n &= 0\,. \end{aligned} \tag{9.13}$$

The rest of the fields η_μ are unconstrained, with the conditions (9.9) giving relations between η_μ and $\eta_{\mu\pm1}$. A $(4,1)$ Lagrangian is

$$i\oint_C \frac{d\zeta}{2\pi i\zeta}\Delta_+\check{\Delta}_+D_-\left(\lambda_\alpha(\eta,\check{\eta};\zeta)D_-\eta^\alpha + \check{\lambda}_\alpha(\eta,\check{\eta};\zeta)D_-\check{\eta}^\alpha\right)\,. \tag{9.14}$$

The potentials $\lambda, \check{\lambda}$ can depend explicitly on ζ, and we perform a contour integration over a suitable contour C. In many examples, C will be a small contour encircling the origin. Since it follows from (9.2) and (9.11) that Δ anticommutes with D_-, and that

$$\Delta_+ = 2\mathbb{D}_+ - \nabla_+\,, \tag{9.15}$$

and since further ∇ annihilates η , we may make the following replacement in reducing a Lagrangian to $(2,1)$ superspace:

$$i\oint_C \frac{d\zeta}{2\pi i\zeta}\Delta_+\check{\Delta}_+D_-\mathcal{L}_-(\eta,\check{\eta}) \to i\oint_C \frac{d\zeta}{2\pi i\zeta}\mathbb{D}_+\bar{\mathbb{D}}_+D_-\mathcal{L}_-(\eta,\check{\eta})\,. \tag{9.16}$$

The relation of λ_α to k_α in (3.1) depends on the form of η, as illustrated in the examples below.

After the reduction, (9.16) gives a $(4,1)$ supersymmetric action written in $(2,1)$ superspace with the non-manifest supersymmetry ensured by the construction. For the multiplet (8.3), this will lead to constraints on $\mathcal{L}_-$ of the type (8.11). As before, these lead to a potential k satisfying (8.12) in addition to the conditions (4.22), (4.23) for $(4,1)$ supersymmetry. Thus for this multiplet, the models constructed in projective superspace represent a subclass of the general $(4,1)$ models.

9.3.1 Examples

If we consider η's with $m=0, n=1$, and denote $\eta_0 = \bar{\phi}$, $\eta_1 = \chi$, we have

$$\begin{aligned} \eta^i &= \bar{\phi}^i + \zeta\chi^i \\ \check{\eta}^i &= \phi^i - \zeta^{-1}\bar{\chi}^i\,, \end{aligned} \tag{9.17}$$

[9] See the comments following (2.4).

[10] We suppress the tildes that we previously used to denote $(2,1)$ superfields.

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with $i = 1 \dots d$ for d fields η^i. From (9.9) we find that the coefficients obey

$$\begin{aligned} &\bar{\mathbb{D}}_+\phi^i = 0\,, && \bar{\mathbb{D}}_+\chi^i = 0\,, && \mathbb{Q}_+\phi^i = 0\,, && \mathbb{Q}_+\chi^i = 0 \\ &\bar{\mathbb{Q}}_+\phi^i = -\bar{\mathbb{D}}_+\bar{\chi}^i\,, && \bar{\mathbb{Q}}_+\chi^i = \bar{\mathbb{D}}_+\bar{\phi}^i\,. \end{aligned} \tag{9.18}$$

For each i, this is (2.2) with $i\mathbb{D}_{+2} = \mathbb{Q}_+$. From (9.14), the $(2,1)$ Lagrangian is

$$i \oint_C \frac{d\zeta}{2\pi i\zeta}\mathbb{D}_+\bar{\mathbb{D}}_+ D_- \left(\lambda_{\eta^i}(\eta,\breve{\eta})D_-\eta^i + \breve{\lambda}_{\breve{\eta}^i}(\eta,\breve{\eta})D_-\breve{\eta}^i\right)\,, \tag{9.19}$$

In this case, the relation of λ_i to k_i in (3.1) is given by[11]

$$\begin{aligned} k_{\phi^i} &= \oint_C \frac{d\zeta}{2\pi i\zeta}\breve{\lambda}_i\,, & \bar{k}_{\bar{\phi}^i} &= \oint_C \frac{d\zeta}{2\pi i\zeta}\lambda_i \\ k_{\chi^i} &= \oint_C \frac{d\zeta}{2\pi i\zeta}\zeta\lambda_i\,, & \bar{k}_{\bar{\chi}^i} &= -\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1}\breve{\lambda}_i\,. \end{aligned} \tag{9.20}$$

By construction, these potentials satisfy (8.12) as well as (4.22) and (4.23). In fact, again, a direct calculation using (9.20) shows they satisfy the stronger condition (3.10). As a result, the Lagrangian (9.14) is not the most general one with $(4,1)$ supersymmetry.

To see that the vector potentials in (9.20) satisfy (3.10), we form their derivatives, using (9.17),

$$\begin{aligned} k_{\phi^i,\bar{\phi}^j} &= \oint_C \frac{d\zeta}{2\pi i\zeta}\breve{\lambda}_{i,\eta^j}\,, & \bar{k}_{\bar{\chi}^i,\chi^j} &= -\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1}\breve{\lambda}_{i,\eta^j}\zeta. \\ k_{\phi^i,\bar{\chi}^j} &= -\oint_C \frac{d\zeta}{2\pi i\zeta}\breve{\lambda}_{i,\eta^j}\zeta^{-1}\,, & \bar{k}_{\bar{\chi}^i,\phi^j} &= -\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1}\breve{\lambda}_{i,\eta^j}\,. \end{aligned} \tag{9.21}$$

They clearly satisfy (3.10).

Consider now the example of a quadratic Lagrangian for d multiplets η^i given by

$$\mathcal{L}_- = i\oint_C \frac{d\zeta}{2\pi i\zeta}\left(\eta^i D_-\breve{\eta}^i - \breve{\eta}^i D_-\eta^i\right)\,, \tag{9.22}$$

where the contour C is a small circle around the origin. Using (9.17) and performing the ζ integration results in the following action

$$-\int d^2x\mathbb{D}_+\bar{\mathbb{D}}_+ D_-\left(\bar{\phi}^i D_-\phi^i + \bar{\chi}^i D_-\chi^i - \phi^i D_-\bar{\phi}^i - \chi^i D_-\bar{\chi}^i\right) \tag{9.23}$$

with

$$k_{\phi^i} = i\bar{\phi}^i\,, \quad k_{\chi^i} = i\bar{\chi}^i\,, \tag{9.24}$$

in agreement with (9.20).

[11]Note that the ζ measure is invariant under conjugation.

A more interesting example arises if we take a general real function $L(\eta+\breve{\eta})$ and set $\lambda = -\breve{\lambda} = iL$. The vector potentials may be immediately read off from (9.20) using these expressions:

$$\begin{aligned} k_\phi &= \oint_C \frac{d\zeta}{2\pi i\zeta}\breve{\lambda} = -iL_0\,, & \bar{k}_{\bar{\phi}} &= \oint_C \frac{d\zeta}{2\pi i\zeta}\lambda = iL_0 \\ k_\chi &= \oint_C \frac{d\zeta}{2\pi i\zeta}\zeta\lambda = iL_{-1}\,, & \bar{k}_{\bar{\chi}} &= -\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1}\breve{\lambda} = iL_1\,, \end{aligned} \tag{9.25}$$

where L_μ are the coefficients in a expansion of L in powers of ζ. This will lead to a metric g and B field given by the ζ^1, ζ^0 and ζ^{-1} components of the derivative of L according to

$$E = g + B = \begin{pmatrix} 0 & L_0' & L_{-1}' & 0 \\ L_0' & 0 & 0 & -L_1' \\ -L_{-1}' & 0 & 0 & L_0' \\ 0 & L_1' & L_0' & 0 \end{pmatrix}\,, \tag{9.26}$$

with prime denoting derivative with respect to the argument and rows and columns ordered as $(\phi, \bar{\phi}, \chi, \bar{\chi})$. As an example, a function

$$L = \frac{1}{3!}(\eta+\breve{\eta})^3 \tag{9.27}$$

gives

$$\begin{aligned} g_{\phi\bar{\phi}} &= g_{\chi\bar{\chi}} = (\phi+\bar{\phi})^2 - 2\chi\bar{\chi} \\ B_{\phi\chi} &= -2(\phi+\bar{\phi})\bar{\chi}\,, \\ B_{\bar{\phi}\bar{\chi}} &= -2(\phi+\bar{\phi})\chi\,. \end{aligned} \tag{9.28}$$

An example involving unconstrained fields arises when $m \neq 0$. We then consider

$$\eta = \sum_{-m}^{n} \zeta^\mu \eta_\mu \tag{9.29}$$

where the top coefficient $\eta_n \equiv \chi$ and the bottom component $\eta_{-m} \equiv \bar{\phi}$ give chiral fields ϕ, χ in the $(2,1)$ reduction, while the rest of the fields η^i_μ for $-m < \mu < n$ are unconstrained. The $(4,1)$ transformations that follow from the constraints are

$$\begin{aligned} &\mathbb{D}_+\eta_{-m} = \mathbb{D}_+\bar{\phi} = 0\,, & &\bar{\mathbb{D}}_+\eta_n = \bar{\mathbb{D}}_+\chi = 0 & & \\ &\bar{\mathbb{Q}}_+\eta_{\mu+1} = \bar{\mathbb{D}}_+\eta_\mu\,, & &\mathbb{Q}_+\eta_{\mu-1} = -\mathbb{D}_+\eta_\mu\,, & &\mu = n-1, \dots, -m+1 \\ &\bar{\mathbb{Q}}_+\eta_{-m} = \bar{\mathbb{Q}}_+\bar{\phi} = 0\,, & &\mathbb{Q}_+\eta_n = \mathbb{Q}_+\chi = 0\,. & & \end{aligned} \tag{9.30}$$

This last example goes beyond the models described by the action (3.1), and introduces new unconstrained superfields. In particular, consider the following η with $m = 1 = n$:

$$\begin{aligned} \eta &= \zeta^{-1}\bar{\phi} + X + \zeta\chi \\ \breve{\eta} &= -\zeta\phi + \bar{X} - \zeta^{-1}\bar{\chi}\,, \end{aligned} \tag{9.31}$$

The fields satisfy

$$\begin{aligned} \mathbb{D}_+\bar{\phi} &= 0\,, & \bar{\mathbb{D}}_+\chi &= 0\,, \\ \bar{\mathbb{Q}}_+X &= \bar{\mathbb{D}}_+\bar{\phi}\,, & \mathbb{Q}_+X &= -\,\mathbb{D}_+\chi\,, \\ \mathbb{Q}_+\bar{\phi} &= -\,\mathbb{D}_+X\,, & \bar{\mathbb{Q}}_+\chi &= \bar{\mathbb{D}}_+X \\ \bar{\mathbb{Q}}_+\bar{\phi} &= 0\,, & \mathbb{Q}_+\chi &= 0\,, \end{aligned} \tag{9.32}$$

which leaves X unconstrained. A quadratic Lagrangian is

$$\mathcal{L}_- = i\oint_C \frac{d\zeta}{2\pi i\zeta}\left(\eta D_-\breve{\eta} - \breve{\eta}D_-\eta\right)\ . \tag{9.33}$$

Performing the ζ integration results in the $(2,1)$ action taking the form

$$S = -\int d^2x\mathbb{D}_+\bar{\mathbb{D}}_+D_-\left(\bar{\phi}D_-\phi - \phi D_-\bar{\phi} + XD_-\bar{X} - \bar{X}D_-X - \bar{\chi}D_-\chi + \chi D_-\bar{\chi}\right). \tag{9.34}$$

The superfields ϕ, χ are chiral and satisfy the standard free field equations

$$\mathbb{D}_+D_-\phi = 0, \qquad \mathbb{D}_+D_-\chi = 0\ . \tag{9.35}$$

However, note that their kinetic terms in the action have opposite sign. The superfield X is unconstrained and its field equation is $D_-X = 0$, which implies $\partial_= X = 0$. The components $X|, \mathbb{D}_+X|, \bar{\mathbb{D}}_+X|, \mathbb{D}_+\bar{\mathbb{D}}_+X|$ are all right-moving, i.e. are independent of $x^=$, while the remaining components $D_-X|, \mathbb{D}_+D_-X|, \bar{\mathbb{D}}_+D_-X|, \mathbb{D}_+\bar{\mathbb{D}}_+D_-X|$ are all set to zero by the field equations.

9.4 (4,2) projective superspace

For $(4,2)$ superspace the derivative algebra is (5.1):

$$\begin{aligned} \{\mathbb{D}_{+a}, \bar{\mathbb{D}}^b_+\} &= 2i\delta^b_a\partial_{+\!\!+}\,, \quad a, b, = 1, 2. \\ \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} &= 2i\partial_=\ . \end{aligned} \tag{9.36}$$

As before, we introduce projectively chiral superfields η, now in $(4,2)$ superspace, that satisfy

$$\nabla_+\eta = 0\,, \quad \breve{\nabla}_+\eta = 0\,, \tag{9.37}$$

which have the ζ-expansion

$$\eta = \sum_{-m}^{n}\zeta^\mu\eta_\mu\ . \tag{9.38}$$

In addition, we impose chirality constraints, to obtain irreducible multiplets

$$\mathbb{D}_-\eta = 0\,, \quad \Rightarrow \quad \bar{\mathbb{D}}_-\breve{\eta} = 0\ . \tag{9.39}$$

Then the top coefficient $\eta_n \equiv \chi$ and the bottom component $\eta_{-m} \equiv \bar{\phi}$ give fields ϕ, χ in the $(2,2)$ reduction, where ϕ is chiral; and χ is twisted chiral.

An invariant action is

$$\oint_C \frac{d\zeta}{2\pi i\zeta}\Delta_+\bar{\Delta}_+\mathbb{D}_-\bar{\mathbb{D}}_-L(\eta,\breve{\eta}:\zeta) = \oint_C \frac{d\zeta}{2\pi i\zeta}\mathbb{D}_+\bar{\mathbb{D}}_+\mathbb{D}_-\bar{\mathbb{D}}_-L(\eta,\breve{\eta}:\zeta)$$
$$=: \mathbb{D}_+\bar{\mathbb{D}}_+\mathbb{D}_-\bar{\mathbb{D}}_-K| \tag{9.40}$$

where L and its ζ integral K are real potentials. Note that terms of the form $f(\eta)+\breve{f}(\breve{\eta})$ integrate to zero in the action and thus shifts $L \to L + f(\eta) + \breve{f}(\breve{\eta})$ constitute "Kähler gauge transformations". The non-manifest supersymmetry transformations are

$$\bar{\mathbb{Q}}_+\eta = \zeta\bar{\mathbb{D}}_+\eta\ . \tag{9.41}$$

The reduction to $(2,2)$ superspace (9.40) gives a potential K which automatically satisfies

$$\frac{\partial^2 K}{\partial\phi^i\partial\bar{\phi}^{\bar{k}}} + \frac{\partial^2 K}{\partial\chi^k\partial\bar{\chi}^{\bar{i}}} = 0\ . \tag{9.42}$$

This is precisely the condition (6.14) for $(4,2)$ supersymmetry, so in this case projective superspace gives the most general $(4,2)$ supersymmetric model.

Note that a variant multiplet $\tilde{\eta}$ arises if we replace (9.39) by

$$\bar{\mathbb{D}}_-\hat{\eta} = 0\,, \quad \Rightarrow \quad \mathbb{D}_-\breve{\hat{\eta}} = 0 \tag{9.43}$$

which corresponds to $\theta_1^- \leftrightarrow \bar{\theta}^{1-}$. However, it is easy to see that $\hat{\eta}(\bar{\phi},\chi)$ is equivalent to $\breve{\eta}(-\bar{\phi},\chi)$ for $\eta = \bar{\phi} + \zeta\chi$.

9.4.1 Example

A simple example of a $(4,2)$ multiplet is

$$\eta = \bar{\phi} + \zeta\chi\ . \tag{9.44}$$

The projective chirality constraints result in $(2,2)$ superfields ϕ, χ with ϕ chiral and χ twisted chiral. They also yield the transformations

$$\bar{\mathbb{Q}}_+\bar{\phi} = 0\,, \quad \bar{\mathbb{Q}}_+\bar{\chi} = 0\,, \quad \bar{\mathbb{Q}}_+\chi = \bar{\mathbb{D}}_+\bar{\phi}\,, \quad \bar{\mathbb{Q}}_+\phi = -\bar{\mathbb{D}}_+\bar{\chi} \tag{9.45}$$

corresponding to the truncation of the $(4,4)$ multiplet. The formulae in (9.45) are those in (5.2) $\mathbb{Q}_+ \to i\mathbb{D}_{+2}$. We obtain the scalar potential

$$K = i\oint_C \frac{d\zeta}{2\pi i\zeta}L\ . \tag{9.46}$$

From the action (9.40), reducing to $(2,1)$ and setting $\lambda_{\eta^i} \to -iL_{,\eta^i}$ we find

$$k_{\phi^i} = i\oint_C \frac{d\zeta}{2\pi i\zeta}L_{,\breve{\eta}^i}\,, \qquad \bar{k}_{\bar{\phi}^i} = -i\oint_C \frac{d\zeta}{2\pi i\zeta}L_{,\eta^i}$$
$$k_{\chi^i} = -i\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta L_{,\eta^i}\,, \qquad \bar{k}_{\bar{\chi}^i} = -i\oint_C \frac{d\zeta}{2\pi i\zeta}\zeta^{-1}L_{,\breve{\eta}^i}\ . \tag{9.47}$$

9.4.2 Flat space

In particular, considering a quadratic function of d multiplets

$$L = \eta^i \breve{\eta}^i \,, \tag{9.48}$$

with the contour C a small circle around the origin, gives the following $(2,1)$ action

$$\int d^2x \oint_C \frac{d\zeta}{2\pi i \zeta} \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- \left(\breve{\eta}^i D_- \eta^i - \eta^i D_- \breve{\eta}^i \right) \,.$$

Performing the ζ integration reproduces the component action (9.23), but with the fields now being chiral and twisted chiral. The target space geometry is $2d$ dimensional, flat with zero B-field.

9.4.3 Curved space

We can construct a more interesting model using the propeller contour Γ in figure 1; a similar construction was used in [9]. We use the the $(4,2)$ multiplet η in (9.44) and consider the Lagrangian given by the following integral over the contour Γ:

$$-\oint_\Gamma \frac{d\zeta}{2\pi i \zeta} (\eta + \breve{\eta}) ln(\eta\breve{\eta}) \,. \tag{9.49}$$

Regarding $\eta\breve{\eta}$ as a function of ζ, it has two zeroes

$$\zeta_1 = -\frac{\bar{\phi}}{\chi} =: -\frac{1}{r} e^{i\theta} \,, \qquad \zeta_2 = \frac{\bar{\chi}}{\phi} =: r e^{i\theta} \,, \tag{9.50}$$

and these are branch points of $ln(\eta\breve{\eta})$. We take one branch cut to go from ζ_1 to $-\infty$ on the real axis, and the other to go from ζ_2 to $+\infty$ on the real axis. For any $f(\zeta)$, the integral

$$\frac{1}{2\pi i} \int_\Gamma d\zeta f(\zeta) ln(\eta\breve{\eta})$$

gives the definite integral

$$\int_{\zeta_1}^{\zeta_2} d\zeta f(\zeta)$$

along the straight line between ζ_1 and ζ_2.[12]

For $f(\zeta) = \frac{1}{\zeta}(\eta + \breve{\eta})$, the resulting $(2,2)$ scalar potential is then

$$K = -(\phi + \bar{\phi}) \left(\frac{\chi\bar{\chi}}{\phi\bar{\phi}} + ln\,(\frac{\chi\bar{\chi}}{\phi\bar{\phi}}) \right) \,, \tag{9.51}$$

[12] This can be seen as follows. The real and imaginary axes divide the ζ-plane into four quadrants. Choose a branch where the integral is $\int d\zeta f(\zeta) ln(\eta\breve{\eta})$ along the part of the curve below the negative real axis, i.e. in the bottom left quadrant. Above the negative and below the positive real axes (i.e. in the upper left and lower right quadrants) we then have $\int d\zeta f(\zeta)(ln(\eta\breve{\eta}) + 2\pi i)$ and above the positive real axis (i.e. in the upper right quadrant), changing sheet in the opposite direction, it is $\int d\zeta f(\zeta) ln(\eta\breve{\eta})$ again. Combining the integrals and paying attention to the directions of integration the net result is $\int_{\zeta_1}^{\zeta_2} d\zeta f(\zeta)$.

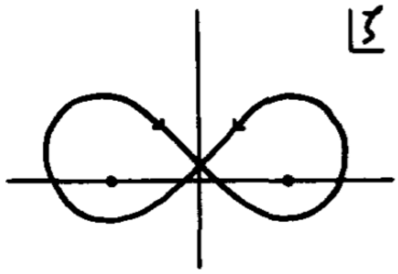

Figure 1. A propeller contour encircling two singularities of $\ln(\bullet)$. The zeros are depicted as lying on the real axis, but in our example they lie on a line tilted to an angle θ with the real axis, see (9.50).

(up to Kähler gauge transformations), which is indeed invariant under the additional supersymmetry in (9.45). The geometry has a conformally flat metric g and is given by

$$
\begin{aligned}
g_{a\bar{b}} &= \begin{pmatrix} g_{\phi\bar{\phi}} & 0 \\ 0 & g_{\chi\bar{\chi}} \end{pmatrix} = \frac{(\phi+\bar{\phi})}{\phi\bar{\phi}} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \\
H &= \phi^{-2} d\chi \wedge d\bar{\chi} \wedge d\phi - \bar{\phi}^{-2} d\chi \wedge d\bar{\chi} \wedge d\bar{\phi} \\
R &= \frac{3}{2} \frac{\phi\bar{\phi}}{(\phi+\bar{\phi})^3}
\end{aligned} \tag{9.52}
$$

where $H = dB$ and R is the curvature scalar (see, e.g., [28]). Note the vector field $\partial/\partial\chi$ generates an isometry.

9.4.4 Semichiral superfields

When we want to generalise the constructions along the lines of the second $(4,1)$ example (9.29) above, we run into an interesting problem. When $n \leq 1$, and η is a series as in (9.29) with the additional condition $D_-\eta = 0$, its reduction to $(2,2)$ superspace contains right semichiral fields rather than unconstrained superfields, e.g.,

$$
\eta = \zeta^{-1}\eta_{-1} + \eta_0 + \zeta\eta_1 = \zeta^{-1}\bar{\phi} + \bar{r} + \zeta\chi\,, \tag{9.53}
$$

the constraints imply that ϕ is chiral, χ twisted chiral and r right semichiral: $\bar{\mathbb{D}}_- r = 0$. However, to construct a sigma model with a non-degenerate kinetic term, one needs an equal number of left and right semichiral superfields. Here by necessity we get right semichiral superfields but no left semichiral superfields. Such a model typically contains right-moving multiplets [11]. The $(4,2)$ projective superfields are thus restrictive when it comes to constructing sigma models. To construct a sigma model with non-degenerate kinetic term, the multiplets considered here would need to be combined with other multiplets.

9.5 (4,4) projective superspace

This case is well documented in the literature [9]–[15], and we make no claim of completeness for the following brief presentation. The construction is always off-shell, typically involving auxiliary fields (sometimes an infinite number). The application to our present type of multiplets, notably the $(4,4)$ twisted multiplet, requires use of the doubly projective superspace based on $\mathbb{CP}^1 \otimes \mathbb{CP}^1$. The two coordinates on these are labeled ζ_L and ζ_R, respectively. The linear combinations of the four $(4,4)$ derivatives are

$$\begin{aligned} \nabla_+ &:= \mathbb{D}_{+1} + \zeta_L \mathbb{D}_{+2}\,, \qquad & \nabla_- &:= \mathbb{D}_{-1} + \zeta_R \mathbb{D}_{-2} \\ \Delta_+ &= \mathbb{D}_{+1} - \zeta_L \mathbb{D}_{+2}\,, \qquad & \Delta_- &= \mathbb{D}_{-1} - \zeta_R \mathbb{D}_{-2}\,, \end{aligned} \tag{9.54}$$

and their conjugates. Now the anticommutation relations are (9.12) for positive chirality derivatives ∇_+, Δ_+ with similar relations for the negative chirality ones ∇_-, Δ_-. A projectively chiral superfield η satisfies

$$\nabla_\pm \eta = 0\ . \tag{9.55}$$

We consider the real multiplet

$$\eta = \bar{\phi} + \zeta_L \chi + \zeta_R \bar{\chi} - \zeta_L \zeta_R \phi\,, \tag{9.56}$$

where the components and transformations are those of the $(4,4)$ twisted multiplet and the reality condition is

$$\eta = -\zeta_L^{-1} \zeta_R^{-1} \breve{\eta}\ . \tag{9.57}$$

A $(4,4)$ Lagrangian is

$$\oint_{C_L} \frac{d\zeta_L}{2\pi i \zeta_L} \oint_{C_R} \frac{d\zeta_R}{2\pi i \zeta_R} \Delta_+ \Delta_- \breve{\Delta}_+ \breve{\Delta}_- L(\eta)\,, \tag{9.58}$$

where C_L and C_R are some suitable contours. By construction, this will be invariant under the full $(4,4)$ supersymmetry. In fact, $L = L(\eta^i)$ ensures that the potential K satisfies the general $(4,4)$ conditions (7.10) and (7.11), where the indices now refer to a set of η^is.

Other multiplets involving semichirals and auxiliaries may be constructed as in [11] and [12].

Finally, we mention that other extended superspaces, such as Harmonic Superspace [16]–[19], have also been used to describe off-shell $(4,4)$ multiplets and actions. The construction closest to what we describe in this section uses bi-harmonic superspace, as described in, e.g., [29].

10 Conclusion

In this paper we introduce new $(4,1)$ and $(4,2)$ multiplets and construct actions for them using new projective superspaces and their progenitors in the GHR formalism. We find the conditions for additional supersymmetries as conditions on the geometric objects: the vector or scalar potentials for the metric and B-field. Our multiplets and actions display off-shell supersymmetry and simultaneously integrable complex structures.

The general conditions for a $(2,1)$ model to have $(4,1)$ symmetry are given in (4.22) and (4.23). The conditons for a $(2,2)$ model to have $(4,2)$ symmetry are (6.14), and the conditons for a $(2,2)$ model to have $(4,4)$ symmetry are the well known relations (7.11) and (7.10). We also consider a stronger condition (3.10) that is sufficient but not necessary for a $(2,1)$ model to have $(4,1)$ symmetry.

Actions for the $(4,1)$ multiplet (2.2) as well as for $(4,2)$ multiplets are constructed both using the GHR approach and novel $(4,1)$ and $(4,2)$ projective superspaces.

We briefly reviewed the $(4,4)$ models. General $(4,4)$ models were formulated in $(4,4)$ superspace using the GHR approach in [1] later using projective superspace actions. In both approaches, the scalar potential satisfies certain conditions by construction. These full conditions for $(4,4)$ supersymmetry arise when we combine the conditions for $(4,2)$ with the conditions for $(2,4)$, supersymmetry.

Examining the $(4,p)$ supersymmetric actions constructed in $(4,p)$ superspace using both the projective superspace and GHR constructions, we find that they give the most general $(4,p)$ supersymmetric sigma models for both the $(4,2)$ and $(4,4)$ cases, but for the $(4,1)$ case we obtain only the special class of models for which the constraint (2.2) is satisfied. This can be viewed as follows. The $(4,1)$ actions we have constructed are based on superfields that depend on additional parameters apart from the worldsheet superspace coordinates. The additional parameters enter in such a way that the second derivative conditions (8.11) are satisfied. In addition to this, the form of the actions leads to vector potentials that satisfy (3.10). Together these conditions are stronger than the general conditions (4.22) and (4.23) for extra supersymmetry of a $(2,1)$ action. This is in contrast to the $(4,2)$ case where the conditions derived for the scalar potential that depends on extra parameters satisfies the general condition (6.14) for $(4,2)$ supersymmetry. At present we do not fully understand this discrepancy, but perhaps there is a more general construction which gives a manifest formulation of the general $(4,1)$ case.

Acknowledgments

During the final stage of writing this article, we were informed of interesting results by Naveen Prabhakar and Martin Roček on $(0,4)$ and $(0,2)$ multiplets and projective superspace that partly overlap, but mostly complement our presentation. We hope to return to the topic in a joint publication. Discussions with both these authors, as well as the stimulating atmosphere at the 2016 Simons workshop on geometry and physics, are gratefully acknowledged. UL gratefully acknowledges the hospitality of the theory group at Imperial College, London, as well as partial financial support by the Swedish Research Council through VR grant 621-2013-4245. This work was supported by the EPSRC programme grant "New Geometric Structures from String Theory", EP/K034456/1.

A Connections

In the $(2,1)$ formulation one complex structure, $\mathbb{J}^{(1)}$ has its canonical form and is preserved by a connection with torsion. The form of $\mathbb{J}^{(1)}$ follows from the reduction of the $(2,1)$ constraint

$$\bar{\mathbb{D}}_+\varphi^\alpha = 0 \tag{A.1}$$

to $(1,1)$ as in (2.4):

$$\mathbb{D}_+ =: D_+ - i\check{D}_+ ,$$

which implies that

$$\check{D}_+ \begin{pmatrix} \varphi^\alpha \\ \bar{\varphi}^{\bar{\alpha}} \end{pmatrix}\Bigg| = Q_+ \begin{pmatrix} \varphi^\alpha \\ \bar{\varphi}^{\bar{\alpha}} \end{pmatrix} = \mathbb{J}^{(1)} \begin{pmatrix} \varphi^\alpha \\ \bar{\varphi}^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix} D_+ \begin{pmatrix} \varphi^\alpha \\ \bar{\varphi}^{\bar{\alpha}} \end{pmatrix} . \tag{A.2}$$

Invariance of the action implies

$$\nabla^{(+)}\mathbb{J}^{(1)} = 0 . \tag{A.3}$$

For the torsion-free case, $\nabla^{(0)}\mathbb{J}^{(1)} = 0$ implies that the Levi-Civita connection has no mixed "holonomy" components, i.e. $\Gamma_{i\;\bar{\beta}}^{\;\alpha} = 0$. SImilarly, for the connection $\Gamma^{(+)}$ with torsion

$$T_{ij}{}^k = \frac{1}{2} g^{kl} H_{ijl} \tag{A.4}$$

Eq. (A.3) implies the connection $\Gamma^{(+)}$ has no mixed "holonomy" components, so that

$$\Gamma^{(+)\alpha}_{i\;\bar{\alpha}} = \Gamma^{(0)\alpha}_{i\;\bar{\alpha}} + T_{i\bar{\alpha}}{}^{\alpha} = 0 , \quad \Gamma^{(+)\bar{\alpha}}_{i\;\alpha} = \Gamma^{(0)\bar{\alpha}}_{i\;\alpha} + T_{i\alpha}{}^{\bar{\alpha}} = 0 . \tag{A.5}$$

In addition, the hermiticity condition

$$\mathbb{J}^{(1)t} g \mathbb{J}^{(1)} = g , \tag{A.6}$$

implies that the metric has only mixed components, $g_{\alpha\beta} = 0$, and this determines the Levi Civita connections:

$$\begin{aligned}
&\Gamma^{(0)}_{\alpha\beta\gamma} = 0 , && \Gamma^{(0)}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = ;0 \\
&\Gamma^{(0)}_{\alpha\beta\bar{\gamma}} = g_{\bar{\gamma}(\alpha,\beta)} , && \Gamma^{(0)}_{\bar{\alpha}\bar{\beta}\gamma} = g_{\gamma(\bar{\alpha},\bar{\beta})} \\
&\Gamma^{(0)}_{\alpha\bar{\beta}\gamma} = \Gamma^{(0)}_{\bar{\beta}\alpha\gamma} = g_{\bar{\beta}[\gamma,\alpha]} , && \Gamma^{(0)}_{\alpha\bar{\beta}\bar{\gamma}} = \Gamma^{(0)}_{\bar{\beta}\alpha\bar{\gamma}} = -g_{\alpha[\bar{\beta},\bar{\gamma}]} .
\end{aligned} \tag{A.7}$$

where for any connection we define

$$\Gamma_{ijk} = g_{kl}\Gamma^l_{ij} \tag{A.8}$$

The formulae (3.4) for the metric and B-field imply that

$$\begin{aligned}
g_{\bar{\beta}[\gamma,\alpha]} &= i(\bar{k}_{\bar{\beta},[\gamma} - k_{[\gamma,\bar{\beta}})_{,a]} = -ik_{[\gamma,\alpha]\bar{\beta}} = \frac{1}{2}B^{(2,0)}_{\gamma\alpha\bar{\beta}} = \frac{1}{2}H^{(2,1)}_{\gamma\alpha\bar{\beta}} = T_{\gamma\alpha\bar{\beta}} , \\
-g_{\alpha[\bar{\beta},\bar{\gamma}]} &= \frac{1}{2}B^{(0,2)}_{\bar{\beta}\bar{\gamma},\alpha} = \frac{1}{2}H^{(1,2)}_{\bar{\beta}\bar{\gamma}\alpha} = T_{\bar{\beta}\bar{\gamma}\alpha} .
\end{aligned} \tag{A.9}$$

Combining (A.7) and (A.9) we see that the relations in (A.5) are satisfied.

The connection with torsion is then given by

$$\Gamma^{(+)}_{\alpha\beta\bar{\gamma}} = \Gamma^{(0)}_{\alpha\beta\bar{\gamma}} + T_{\alpha\beta\bar{\gamma}} = g_{\bar{\gamma}(\alpha,\beta)} + g_{\bar{\gamma}[\alpha,\beta]} = g_{\bar{\gamma}\alpha,\beta} \tag{A.10}$$

and

$$\Gamma^{(+)}_{\alpha\bar{\beta}\gamma} = \Gamma^{(0)}_{\alpha\bar{\beta}\gamma} + T_{\alpha\bar{\beta}\gamma} = 2g_{\bar{\beta}[\gamma,\alpha]} \tag{A.11}$$

together with

$$\Gamma^{(+)}_{\alpha\beta\gamma} = \Gamma^{(+)}_{\alpha\bar{\beta}\bar{\gamma}} = 0 \tag{A.12}$$

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