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Bootstrapping the Three-dimensional Ising Model

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Abstract

This thesis begins with the fundamentals of conformal field theory in three dimensions. The general properties of the conformal bootstrap are then reviewed. The three-dimensional Ising model is presented from the perspective of the renormalization group, after which the conformal field theory aspect at the critical point is discussed. Finally, the bootstrap programme is applied to the three-dimensional Ising model using numerical techniques, and the results analysed.

Layman's Summary

Matter most commonly appears as solid, liquid or gas. The state of a material depends on environmental factors, such as external temperature and pressure. Moreover, the properties of an object usually depend on the scale at which it is interacted with; the behaviour at a molecular level does not necessarily match the macroscopic properties.

The procedure of heating a block of ice such that it melts, the water boils and then evaporates, includes two *phase transitions*: one from solid to liquid and one from liquid to gas. Under certain conditions, a material undergoing a phase transition inhabits what is known as a *critical state*, at which point the physical properties of matter become scale invariant and the system appears the same regardless of the distance from which it is being observed. Scale invariance in turn allows the materials physical properties to be modelled by especially effective mathematical formulations which can be used to make experimentally verifiable predictions.

The aforementioned examples of phase transitions are not the only ones which appear in Nature, and the study of different critical states provides information about the phenomena in general. One remarkable critical state is the one which appears when a three-dimensional ferromagnetic material transitions between being non-magnetic and magnetic, which is explained by the *critical Ising model*. It is conjectured that the theory of a critical ferromagnet depends solely on real numbered parameters. Over the years, a wide variety of techniques which attempt to determine these parameters have been developed. This thesis is concerned with the study of a recently revived and promising approach to finding these numbers, known as the *conformal bootstrap programme*.

Populärvetenskaplig sammanfattning

Materia uppkommer vanligtvis i fast-, flytande- eller gasform. Ett materials tillstånd beror på externa faktorer, så som temperatur och tryck. Utöver detta beror ett objekts egenskaper på skalan som den interagerar på, det molekylära beteende behöver inte nödvändigtvis överensstämma med dess makroskopiska egenskaper.

Förloppet då ett block med is värms så att det smälter, vattnet börjar koka och sedan avdunstar, innehåller två *fasövergångar*: en övergång från fast till flytande och en från flytande till gas. Under särskilda förhållanden kommer ett material som genomgår en fasövergång att befinna sig i ett så kallat *kritiskt tillstånd*, då materians fysikaliska egenskaper blir skalinvarianta, det vill säga att systemets beteende är oberoende av avståndet som det observeras ifrån. Denna skalinvarians tillåter i sin tur att egenskaperna modelleras med särskilt effektiva matematiska formuleringar som sedan kan användas för att förutspå experimentellt verifierbara utfall.

Dessa fasövergångar är dock inte de enda som uppkommer i naturen, och studier av olika kritiska tillstånd förser oss med information om fenomenet i stort. Ett anmärkningsvärt kritiskt tillstånd är det som uppstår när en ferromagnet övergår mellan att vara icke-magnetiskt till magnetiskt, ett tillstånd som beskrivs av den *kritiska Ising-modellen*. Det är förmodat att teorin som beskriver en kritisk ferromagnet endast beror numeriska parametrar. Under lång tid har beräkningstekniker tagits fram för att försöka hitta dessa parametrar. Denna uppsats berör en speciellt lovande metod för detta ändamål, känd som *the conformal bootstrap programme*.

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Contents

1 Introduction

At a microscopic level, Nature behaves very differently compared to everyday life. From being deterministic at large scales, microscopic properties of matter become inherently statistical as distance scales decrease. This raises the number of degrees of freedom in a system, complicating calculations and invalidating frameworks which hold macroscopically.

There are several ways which these difficulties can be resolved, and the approach chosen depends on the system at hand. In high energy physics the typical way of dealing with the microscopic world is through the continuum description of quantum field theory (QFT), where the main objective is to calculate correlation functions which can be compared to experiment. For lower energy applications it may be sufficient to consider simplified statistical models of quantum interactions which capture the macroscopic observables without being construed by the details. However, there are circumstances where the two approaches overlap, at which point the tools of both frameworks can be combined and used for stronger calculations.

One system where such a correspondence exists is the Ising model, a simple statistical lattice spin model of ferromagnetism. Like a ferromagnet, the physical properties of the Ising model are dependent on temperature and the possible application of an external magnetic field. Most importantly, if no external magnetic field it present, at low temperatures the spins on the lattice align, giving rise to its own magnetic field. However, right before the magnetic field begins to appear, the Ising model goes into a so called critical state. At this point the statistical nature of the model changes, allowing a wide variety of mathematics to be used in calculations of physical quantities. Perhaps most importantly, while in a critical state the Ising model can be conjectured as a conformal field theory (CFT), a quantum field theory which exhibits invariance under the conformal algebra [1].

The Ising model has been successfully studied and exactly solved, using a variety of techniques, in one and two spacial dimensions; furthermore, in two dimensions the conformal invariance has been proven [2]. However, the perhaps most physically relevant three-dimensional manifestation has not bared as much fruit. Although there has been field-theoretical attempts at finding approximate values of thermodynamic quantities in three dimensions, the possible conformal invariance was long ignored [3]. This is mainly due to the finite nature of the conformal algebra in this dimension, which is not sufficient to by itself constrict the theory enough to be solved. However, it was suggested in [4] that the CFT aspect of the three-dimensional Ising model could be advantageously put to use in the *conformal bootstrap programme*.

The conformal bootstrap uses aspects of conformal field theory to constrict the so called CFT data, real numbers which allow for the calculation of all correlation functions of a theory. The method was first suggested by Polyakov in the year 1974 [5], and it was later shown that it could be used to solve so-called minimal models, a class in which the two-dimensional Ising

model belongs [2]. Unfortunately, it was long not understood how the technique should be implemented generally. However, the subject gained new life in 2008 when it was discovered that the method could be effectively treated numerically [6, 7].

The aim of this thesis is to study the numerical bootstrap approach to the three-dimensional Ising model, first done in [8, 9] and extended upon in [10]. The outline is as follows. Starting with the basics of conformal field theory, mainly following [11–13], section two contains a review of the essential aspects of conformal symmetry needed in the bootstrap approach. Section three presents the bootstrap formalism in general. The three-dimensional Ising model is studied, in accordance with [14], in section three; including aspects of the renormalization group, as well as conformal field theory. Section five is focused toward the specific application of the numerical bootstrap on the three-dimensional Ising model. Finally, the results are discussed in section six.

2 Conformal Field Theory

This section will present the notions of conformal symmetry, and by extension conformal field theory, which form the basis of future considerations. Before delving in to the subject, however, it is necessary to clear up some nomenclature. In conformal field theory, the term *field* is given to any coordinate dependent local quantity. For the sake of generality and clarity this terminology will be avoided. Instead, in the following such quantities will be referred to exclusively as *operators*, keeping the locality implicit.

2.1 Conformal Transformations

First and foremost it is natural to find the conformal transformations themselves. A conformal spacetime transformation is by definition a coordinate transformation $x \to x'$ which preserves the metric up to scale

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \tag{2.1}$$

where the scale factor $\Omega(x)$ is some smooth function. For the purpose of this thesis it will suffice to consider the case where $g_{\mu\nu}$ is a flat (p, q)-metric, up to some conformal transformation. It is immediately possible to conclude that the conformal group must contain translations and Lorentz rotations, as these transformations leave the metric unchanged. Furthermore, a global dilation yields a constant scale factor, hence dilations are also conformal.

To be more precise it is helpful to consider an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$. The tensorial transformation properties of the metric

$$g'_{\mu\nu}(x'^\mu) = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x^\mu), \tag{2.2}$$

yields the expression for the transformation, up to first order in ϵ^μ,

$$g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.3}$$

Comparing (2.3) with (2.1) and concluding that the right-hand side of (2.3) must be factorisable, constrains the additional infinitesimal term to be of the form

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}\,, \tag{2.4}$$

where $f(x)$ is a scalar function. The function can be found by contracting both sides of the above equation with the metric, which gives the conformal killing equation

$$f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho\,. \tag{2.5}$$

Further useful manipulations of (2.3) are possible, and will result in sets of equations for ϵ^μ which will return the form of the conformal spacetime transformations. Differentiating (2.3) yields

$$\partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu = g_{\mu\nu} \partial_\rho f. \tag{2.6}$$

Taking all permutations of the indices gives two more equations

$$\partial_\nu\partial_\mu\epsilon_\rho + \partial_\rho\partial_\nu\epsilon_\mu = g_{\mu\rho}\partial_\nu f, \tag{2.7}$$

$$\partial_\rho\partial_\mu\epsilon_\nu + \partial_\mu\partial_\nu\epsilon_\rho = g_{\rho\nu}\partial_\mu f. \tag{2.8}$$

Taking the linear combination (2.41) + (2.42) − (2.40) leaves the expression

$$2\partial_\mu\partial_\nu\epsilon_\rho = g_{\mu\rho}\partial_\nu f + g_{\nu\rho}\partial_\mu f - g_{\mu\nu}\partial_\rho f, \tag{2.9}$$

which after contraction with $g^{\mu\nu}$ and some simplification is equal to

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu f. \tag{2.10}$$

Acting with another derivative on the above equation returns

$$2\partial_\nu\partial^2\epsilon_\rho = (2-d)\partial_\mu\partial_\nu f, \tag{2.11}$$

or equivalently acting with ∂^2 on (2.3) gives, in a similar fashion to before,

$$2\partial_\nu\partial^2\epsilon_\mu = g_{\mu\nu}\partial^2 f. \tag{2.12}$$

Fusing the left-hand sides of the above equations gives

$$g_{\mu\nu}\partial^2 f = (2-d)\partial_\mu\partial_\nu f, \tag{2.13}$$

which after contraction with the metric and rearranging becomes

$$(d-1)\partial^2 f = 0. \tag{2.14}$$

It is evident that these equations have a dimensional dependence, and thus, so must conformal spacetime transformations. In fact, in one dimension (2.14) is trivially satisfied, meaning the notion of a one-dimensional conformal field theory does not exist. At $d = 2$ there is a special case, which will not be considered here. So, since the theory of interest for this thesis lives in three dimensions, the above equations can be used to derive the infinitesimal form of the conformal transformations. With $d = 3$, equation (2.13) becomes $\partial_\mu\partial_\nu f(x) = -g_{\mu\nu}\partial^2 f(x)$, while (2.14) is equal to $\partial^2 f(x) = 0$; meaning $f(x)$ must be linear in x, hence

$$f(x) = A + B_\mu x^\mu, \tag{2.15}$$

where A and B_μ are constants. Plugging this ansatz into (2.9) leaves constant terms proportional to the metric on the right-hand side, which implies that ϵ_μ is at most quadratic in the coordinates. The general form of ϵ_μ is thus

$$\epsilon_\mu = \alpha_\mu + \beta_{\mu\nu}x^\nu + \gamma_{\mu\nu\rho}x^\nu x^\rho, \tag{2.16}$$

where $\gamma_{\mu\nu\rho}$ is symmetric under the exchange of its second and third indices.

The constraints which were derived above each contain derivatives up to second order. Thus, the constant term in (2.16) is unconstrained, implying that it is an infinitesimal translation. The coordinate dependent terms are in turn constrained by separate equations and can be considered individually. The linear term will be constrained by equation (2.4), which after substitution it becomes

$$\beta_{\mu\nu} + \beta_{\nu\mu} = \frac{2}{d}\partial_\rho(\beta^\rho{}_\sigma x^\sigma)g_{\mu\nu}, \tag{2.17}$$

where the right-hand side is given by the conformal killing equation. Differentiating the right-hand side and simplifying yields

$$\beta_{\mu\nu} + \beta_{\nu\mu} = \frac{2}{d}\beta^\rho{}_\rho g_{\mu\nu}. \tag{2.18}$$

It is clear from the above equation that the symmetric part of $\beta_{\mu\nu}$ is proportional to the metric, while the anti-symmetric part is unconstrained. Hence the most general form of $\beta_{\mu\nu}$ is

$$\beta_{\mu\nu} = \lambda g_{\mu\nu} + m_{\mu\nu}, \tag{2.19}$$

where λ is a constant and $m_{\mu\nu}$ is anti-symmetric in its indices. Substituting this back into (2.16) it becomes manifest that the metric term corresponds to an infinitesimal scale transformation $x^\mu \to x^\mu + \lambda x^\mu$, while the anti-symmetric term represents an infinitesimal rotation $x^\mu \to x^\mu + m^\mu{}_\nu x^\nu$.

The quadratic term is constrained by (2.9). Substitution yields

$$2\partial_\nu\partial_\rho(\gamma_{\mu\alpha\beta}x^\alpha x^\beta) = \frac{2}{d}\left[g_{\mu\rho}\partial_\nu + g_{\mu\nu}\partial_\rho - g_{\nu\rho}\partial_\mu\right]\partial_\sigma\left(\gamma^\sigma{}_{\lambda\kappa}x^\lambda x^\kappa\right). \tag{2.20}$$

Differentiating the left-hand side gives $4\gamma_{\mu\nu\rho}$. Considering a single term on the right-hand side, differentiation gives

$$\begin{aligned}\partial_\nu\partial_\sigma\left(\gamma^\sigma{}_{\lambda\kappa}x^\lambda x^\kappa\right) &= \gamma^\sigma{}_{\lambda\kappa}\partial_\nu\left(\delta^\lambda{}_\sigma x^\kappa + x^\lambda\delta^\kappa{}_\sigma\right)\\ &= \gamma^\sigma{}_{\lambda\kappa}\left(\delta^\lambda{}_\sigma\delta^\kappa{}_\nu + \delta^\lambda{}_\nu\delta^\kappa{}_\sigma\right),\end{aligned} \tag{2.21}$$

so after simplification this becomes

$$\partial_\nu\partial_\sigma\left(\gamma^\sigma{}_{\lambda\kappa}x^\lambda x^\kappa\right) = 2\gamma^\sigma{}_{\sigma\nu}, \tag{2.22}$$

and similarly for the other terms. Finally, this amounts to

$$\gamma_{\mu\nu\rho} = g_{\mu\rho}b_\nu + g_{\mu\nu}b_\rho - g_{\nu\rho}b_\mu, \text{ where } b_\alpha = \frac{1}{d}\gamma^\sigma{}_{\sigma\alpha}. \tag{2.23}$$

Inserting this back into (2.16) forces the form of the special conformal transformation to be

$$x^\mu \to x^\mu + 2b_\rho x^\rho x^\mu - x_\sigma x^\sigma b^\mu. \tag{2.24}$$

The property which these transformations have in commons is that they preserve the angle between two crossing curves. Thus, an intuitive classification of conformal transformations can be made is this manner.

2.1.1 The Conformal Algebra

Using the above expressions for the infinitesimal coordinate transformations it is possible to determine the generators of the conformal algebra. A generator can be defined by the change in a local operator $\mathcal{O}(x)$ after an infinitesimal transformation evaluated at the same point, i.e.

$$\delta\mathcal{O}(x') = \mathcal{O}'(x') - \mathcal{O}(x') = -i\epsilon_a G_a \mathcal{O}(x'), \tag{2.25}$$

where G_a is the generator of some transformation. It is possible to determine G_a using the expressions for general infinitesimal transformations

$$x^\mu \to x'^\mu = x^\mu + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}, \tag{2.26}$$

and

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \mathcal{O}(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x), \tag{2.27}$$

where the function $\mathcal{F}$ relates the operator to its transformed form, i.e. $\mathcal{F}(\mathcal{O}(x)) = \mathcal{O}'(x')$. Using (2.26) to express x^μ in terms of x'^μ and the variation, $\mathcal{O}'(x')$ can be expressed as

$$\mathcal{O}'(x') = \mathcal{O}(x') - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu \mathcal{O}(x') + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x'), \tag{2.28}$$

where the derivative of the field appears due to the chain rule, and only terms up to first order in ϵ_a have been kept. Equation (2.25) now becomes

$$i\epsilon_a G_a \mathcal{O}(x') = \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu \mathcal{O}(x') - \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}(x'). \tag{2.29}$$

The effect which the generators have on operators will be considered separately, thus in the following $\delta\mathcal{F}/\delta\epsilon_a$ is taken to be zero. The expression of interest is then

$$i\epsilon_a G_a = \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu, \tag{2.30}$$

where the operators have been cancelled out, leaving only spacetime terms.

It is now a matter of finding the generator for each of the spacetime transformations derived in the previous section. Starting from translations,

$$x^\mu \to x^\mu + a^\mu, \tag{2.31}$$

the variation of the coordinate becomes $\delta x^\mu/\delta a_\nu = \delta^\mu{}_\nu$ and thus the generator is equal to

$$P_\mu = -i\partial_\mu. \tag{2.32}$$

Using the anti-symmetry of a Lorentzian rotation $x^\mu \to x^\mu + m_{\rho\nu} g^{\rho\mu} x^\nu$, it can be re-expressed as

$$x^\mu \to x^\mu + \frac{1}{2}\left(m_{\rho\nu} g^{\rho\mu} x^\nu - m_{\rho\nu} g^{\mu\nu} x^\rho\right), \tag{2.33}$$

and so, the variation is equal to

$$\frac{\delta x^\mu}{\delta m_{\rho\nu}} = \frac{1}{2}\left(g^{\mu\rho}x^\nu - g^{\mu\nu}x^\rho\right). \tag{2.34}$$

Plugging this into (2.30) gives

$$\frac{i}{2}m_{\rho\nu}L^{\rho\nu} = \frac{1}{2}m_{\rho\nu}\left(g^{\mu\rho}x^\nu - g^{\mu\nu}x^\rho\right)\partial_\mu, \tag{2.35}$$

where the factor one-half appears to compensate for over-counting due to contraction of indices. Using a fresh set of indices, the generator for Lorentz rotations is thus

$$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right). \tag{2.36}$$

The variation of the coordinate with respect to an infinitesimal scale transformation amounts to x^μ, yielding the generator of dilations

$$D = -ix^\mu\partial_\mu. \tag{2.37}$$

Studying (2.24) it becomes clear that the special conformal transformations are parametrised by b_μ. The coordinate variation will be

$$\frac{\delta x^\mu}{\delta b_\nu} = 2\delta^\nu{}_\rho x^\rho x^\mu - x^\sigma x_\sigma \delta^\mu{}_\nu. \tag{2.38}$$

After renaming some indices, the final generator of conformal transformations is

$$K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - x^\sigma x_\sigma\partial_\mu\right). \tag{2.39}$$

Using the above generators, some straightforward calculations gives the commutator relations of the conformal algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\left(\delta_{\nu\rho}L_{\mu\sigma} + \delta_{\mu\sigma}L_{\nu\rho} - \delta_{\mu\rho}L_{\nu\sigma} - \delta_{\nu\sigma}L_{\mu\rho}\right), \tag{2.40}$$

$$[P_\rho, L_{\mu\nu}] = i\left(\delta_{\rho\mu}P_\nu - \delta_{\rho\nu}P_\mu\right), \tag{2.41}$$

$$[K_\rho, L_{\mu\nu}] = i\left(\delta_{\rho\mu}K_\nu - \delta_{\rho\nu}K_\mu\right), \tag{2.42}$$

$$[K_\mu, P_\nu] = 2i\left(\delta_{\mu\nu}D - L_{\mu\nu}\right), \tag{2.43}$$

$$[D, P_\mu] = iP_\mu, \tag{2.44}$$

$$[D, K_\mu] = -iK_\mu. \tag{2.45}$$

The first and second commutators are manifestly the Poincaré algebra. However, the last two commutators are of most importance; namely, they suggest that P_μ and K_μ act interestingly in relation to D. What this means will become clear in the following.

2.1.2 Conformal Group Representations

The previous section was dedicated to spacetime transformations, ignoring any influence on the operators themselves. This section will concern itself with the latter aspect. The complete action of the generators of the conformal group must contain both a spacetime term and a term associated with the transformation of the operator, in accordance with (2.29). To this end, consider an operator which transforms in an irreducible representation of the the Lorentz group at the origin,

$$[G_a, \mathcal{O}(0)] = R_a \mathcal{O}(0) \tag{2.46}$$

where G_a is some generator of the conformal group, and R_a is the corresponding matrix representation which satisfies the analogous representational algebra of the conformal group. To be more general, it is natural to consider a transformation away from the origin. However, this will have a non-trivial effect on the generators. In order to take this effect into account, it is possible to translate the generator such that it is acting at some non-zero value of x. This yields

$$e^{ix^\rho P_\rho} G_a e^{-ix^\rho P_\rho} = G_a + ix^\rho [P_\rho, G_a] + \frac{i}{2!} x^\rho x^\sigma [P_\sigma, [P_\rho, G_a]] \tag{2.47}$$

where the right-hand side has been expanded using the Hausdorff formula for two operators A and B,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots \tag{2.48}$$

where higher order terms vanish for every conformal generator. The commutator terms in equation (2.47) will each yield one of the relations from the conformal algebra (2.40)-(2.45), thus the spacetime transformation aspect of the generator is automatically included. Plugging in the relevant terms one by one yields the expressions for generators acting away from the origin

$$e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu, \tag{2.49}$$

$$e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\sigma P_\sigma, \tag{2.50}$$

$$e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} = K_\mu + 2x_\mu D - 2x^\sigma L_{\mu\sigma} + 2x_\mu x^\alpha P_\alpha - x^\nu x_\nu P_\mu, \tag{2.51}$$

and P_μ is the same as (2.32). Denoting the value of R_a as $\mathcal{S}_{\mu\nu}$, κ_μ and $\tilde{\Delta}$, for the generators $L_{\mu\nu}$, K_μ and D, respectively, the action of the generators on a field will be

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x), \tag{2.52}$$

$$[L_{\mu\nu}, \mathcal{O}(x)] = \left[\mathcal{S}_{\mu\nu} + i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)\right] \mathcal{O}(x), \tag{2.53}$$

$$[D, \mathcal{O}(x)] = \left(\tilde{\Delta} - ix^\mu \partial_\mu\right) \mathcal{O}(x), \tag{2.54}$$

$$[K_\mu, \mathcal{O}(x)] = \left(\kappa_\mu + 2x_\mu \tilde{\Delta} - 2x^\nu \mathcal{S}_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\right) \mathcal{O}(x). \tag{2.55}$$

In order to find the eigenvalues of the conformal matrix representations it is necessary to make use of some representation theory. Schur's lemma states that any matrix which commutes with the generators of an irreducible representation must be proportional to the identity. In the current case of the operator belonging to the irreducible representation of the Lorentz group, the generator is the spin operators $\mathcal{S}_{\mu\nu}$. Analogously to the conformal algebra, $\mathcal{S}_{\mu\nu}$ commutes with $\tilde{\Delta}$. Moreover, since the Lorentz group is compact while the conformal group is not, the representation of dilations acting on $\mathcal{O}$ must be non-unitary. This leaves the value of $\tilde{\Delta}$ to be chosen as $-i\Delta$, where Δ is the *scaling dimension* of $\mathcal{O}$, defined by

$$\mathcal{O}'(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x), \tag{2.56}$$

where λ is the dilation parameter. The definition of the scaling dimension follows from the fact that the Jacobian of the *finite* scale transformation $x \to \lambda x$ is λ^d, and one thus requires the operator to transform in the above fashion to make the action invariant for some value of Δ.

It is important not to confuse the argument of non-unitarity of the conformal representation with the unitarity of a conformal field theory. Indeed, requiring a unitary theory will constrict the allowed values of scaling dimensions, which will be discussed in section 2.4.2

Finally, plugging the value of $\tilde{\Delta}$ into the representational form of (2.45) forces $\kappa_\mu = 0$. Hence dilations and special conformal transformations act as

$$[D, \mathcal{O}(x)] = -i\left(\Delta + x^\mu P_\mu\right)\mathcal{O}(x), \tag{2.57}$$

$$[K_\mu, \mathcal{O}(x)] = -\left(2ix_\mu\Delta + 2x^\nu\mathcal{S}_{\mu\nu} + 2ix_\mu x^\nu\partial_\nu - ix^2\partial_\mu\right)\mathcal{O}(x). \tag{2.58}$$

2.2 Scalar Correlators

Now that the conformal transformations have been established, it is time to study some of the consequences which arise in a theory which obeys their symmetry. It is advantageous to start by studying how correlation functions of local scalar operators are fixed by conformal symmetry. The results found in this section will at a later stage need to be modified slightly for non-zero spin-l operators, at which point any additional constraints will be discussed.

2.2.1 Two-Point Function

The simplest correlator is the two-point function, $\langle 0|\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)|0\rangle$. From rotational and translational invariance the correlator must be a function of the modulus of the distance between the points x_1 and x_2,

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = f(|x_1 - x_2|). \tag{2.59}$$

Scale invariance poses further constraints. Using equation (2.56) it is possible to conclude that the correlator transforms as

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \lambda^{\Delta_i+\Delta_j}\langle \mathcal{O}_i(\lambda x_1)\mathcal{O}_j(\lambda x_2)\rangle, \tag{2.60}$$

and from this it follows that

$$f(|x_1 - x_2|) = \lambda^{\Delta_i+\Delta_j} f(|\lambda x_1 - \lambda x_2|). \tag{2.61}$$

Demanding that the two-point function be symmetric under the above transformation gives the general form

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\lambda_{ij}}{|x_1 - x_2|^{\Delta_i+\Delta_j}}, \tag{2.62}$$

where λ_{ij} is a normalisation constant. At this point it is possible to draw conclusions about the scaling dimensions. In a physical theory operators are expected to obey cluster decomposition, meaning correlations between operators do not increase with the distance between the operators. Equation (2.62) shows that the desired behaviour demands that the scaling dimensions be positive or zero, else un-physical divergences are encountered at large coordinate separation.

Continuing, considering special conformal transformations yields additional constraints. In a similar fashion as before, the behaviour of the two point function under a finite transformation needs to be considered. This requires a generalisation of (2.56). The form which will be chosen is

$$\mathcal{O}(x) = \left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} \mathcal{O}(x'), \tag{2.63}$$

and must hold for any finite transformation $x \to x'$. The meaning of thus definition will be returned to in section 2.3. The finite form of a special conformal transformation is

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}, \tag{2.64}$$

which yields the factor in front of the transforming operator

$$\left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} = \frac{1}{(1 - 2b\cdot x + b^2x^2)^\Delta}. \tag{2.65}$$

Factorising the numerator and re-expressing the denominator of equation (2.64) as a square, the distance between two points will transform as

$$\left|x'_1 - x'_2\right| = \left|\frac{x_1}{1 - b\cdot x_1} - \frac{x_2}{1 - b\cdot x_2}\right|, \tag{2.66}$$

which after simplification is equal to

$$\left|x'_1 - x'_2\right| = \frac{|x_1 - x_2|}{\xi_1^{1/2}\xi_2^{1/2}}, \tag{2.67}$$

where $\xi_i = 1 - 2b\cdot x_i + b^2x_i^2$. Taking the above relations into consideration, the two-point function transforms as

$$\langle \mathcal{O}_i(x'_1)\mathcal{O}_j(x'_2)\rangle = \frac{\lambda_{ij}}{\xi_1^{\Delta_i}\xi_2^{\Delta_j}} \frac{(\xi_1\xi_2)^{(\Delta_i+\Delta_j)/2}}{|x_1 - x_2|^{\Delta_i+\Delta_j}}. \tag{2.68}$$

Invariance implies that a non-vanishing two-point function must contain only operators with the same value of the scaling dimension. Nevertheless, the operators can be independent. The final expression thus amounts to

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\lambda_{ij}\delta_{\Delta_i\Delta_j}}{|x_1 - x_2|^{\Delta_i+\Delta_j}}. \tag{2.69}$$

2.2.2 Three-Point Function

The appearance of the three-point function $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle$ is determined analogously. Reasoning about rotational and translational invariance will yield a similar result as earlier; however in this case the distance between all three points, $x_{mn} = x_m - x_n$, must be considered, yielding

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = f(|x_{mn}|). \tag{2.70}$$

Scale invariance implies the form

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^a|x_{13}|^b|x_{23}|^c}, \tag{2.71}$$

where $a + b + c = \Delta_i + \Delta_j + \Delta_k$; and λ_{ijk} is a real, non-trivial structure constant of the operator algebra which is fixed once a normalisation is chosen. Finally, applying a special conformal transformations produces the equation

$$\langle \mathcal{O}_i(x_1')\mathcal{O}_j(x_2')\mathcal{O}_k(x_3')\rangle = \frac{\lambda_{ijk}}{\xi_1^{\Delta_i}\xi_2^{\Delta_j}\xi_3^{\Delta_k}} \frac{\xi_1^{(a+b)/2}\xi_2^{(a+c)/2}\xi_3^{(b+c)/2}}{|x_{12}|^a|x_{13}|^b|x_{23}|^c}, \tag{2.72}$$

which means that $a+b = 2\Delta_i$, $a+c = 2\Delta_j$, and $b+c = 2\Delta_k$. Solving the system of equations for the exponents yields the relations

$$\begin{aligned} a &= \Delta_i + \Delta_j - \Delta_k, \\ b &= \Delta_i + \Delta_k - \Delta_j, \\ c &= \Delta_j + \Delta_k - \Delta_i, \end{aligned} \tag{2.73}$$

and thus the three-point function is equal to

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}} \tag{2.74}$$

This expression clearly holds independently of the scaling dimensions of the operators involved, and so a three-point function does not need to be between operators of the same scaling dimension.

2.2.3 Higher-Point Functions

For correlators involving four points and above, there is no way to constrain the correlators sufficiently enough to compute their exact form. At these levels it is possible to construct conformally invariant coordinate cross-ratios, on which the correlators can depend arbitrarily. For example, with four points the cross-ratios are

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{2.75}$$

which are manifestly invariant under conformal transformations seeing as all transformation factors will appear identically in the numerator and denominator.

For future purposes, the four-point $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4)\rangle$ function is of interest. For simplicity the following calculation will index the scaling dimensions by the location of the operators, for example $\Delta_1 = \Delta_i$. As before, the correlator must be a function of the distances x_{mn}; without making too many assumption this dependence can be taken to be

$$\prod_{m<n} |x_{mn}|^{a_{mn}}\,, \tag{2.76}$$

where a_{mn} is to be determined. Under scale transformations $x \to \lambda x$ this expression becomes

$$\prod_{m<n} \lambda^{a_{mn}} |x_{mn}|^{a_{mn}}\,. \tag{2.77}$$

In order for the function to be invariant, the factors of $\lambda^{a_{mn}}$ are required to cancel with the factors arising from the field transformation, i.e.

$$\sum_{m<n} a_{mn} \overset{!}{=} -\Delta, \tag{2.78}$$

where $\Delta = \sum_{q=1}^{4} \Delta_q$. Making an appropriate ansatz $a_{mn} = \alpha - \Delta_m - \Delta_n$, the equation becomes

$$6\alpha - 3\Delta = -\Delta, \tag{2.79}$$

which is satisfied by $\alpha = \Delta/3$. With this information, (2.76) becomes

$$\prod_{m<n} |x_{mn}|^{\Delta/3-\Delta_m-\Delta_n}\,. \tag{2.80}$$

Straight forward algebra using (2.67) gives that a special conformal transformation turns the coordinates into

$$\prod_{m=1}^{4} \xi_m^{\Delta_m} \prod_{m<n} |x_{mn}|^{\Delta/3-\Delta_m-\Delta_n}\,, \tag{2.81}$$

where the factors of $\xi_m^{\Delta_m}$ vanish when including the inevitable contributions from (2.65). Thus, the expression (2.80) is sufficiently symmetric. Because of the properties of the cross-rations,

the exact structure the the four-point function is ambiguous up to a function of u and v. Thus, it is possible to massage the form of (2.80). For future purposes the preferred form is

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4)\rangle = \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{ij}} \left(\frac{x_{14}}{x_{34}}\right)^{\Delta_{kl}} \frac{g(u,v)}{x_{12}^{\Delta_i+\Delta_j} x_{34}^{\Delta_k+\Delta_l}}, \tag{2.82}$$

where $\Delta_{mn} = \Delta_m - \Delta_n$. Thus, for the special case of four identical scalar operators, the correlator becomes

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}, \tag{2.83}$$

The choice of keeping x_{12} and x_{34} is arbitrary; any two values of x_{mn} are possible in the denominator. However, depending on the choice of spacetime dependence, the rest of the equation changes accordingly.

2.3 Primary and Descendant Operators

An important aspect of conformal symmetry concerns the classification of conformal operators [15]. To illustrate this, consider the generator P_μ acting on an operator at the origin. Consequently, this combination is acted upon by the dilation operator D. This introduces a subtlety which has not been important so far. In order to freely use the values of the conformal algebra (2.40)-(2.45), it is crucial distinguish between how differential operators and charges act on operators, as mentioned in [16]. In section 2.1.2 this distinction was implicit, but now it is important to define the relation

$$[G, \mathcal{O}] = \mathscr{D}\mathcal{O}. \tag{2.84}$$

Although seemingly trivial, this relation will have non-trivial effects on the form of repeated action of differential operators. In fact, the condition for the definition to hold enforces that repeated action reverses the order of operations, i.e.

$$[G_1, [G_2, \mathcal{O}]] = \mathscr{D}_2\mathscr{D}_1\mathcal{O}. \tag{2.85}$$

This is especially important when the the generators do not commute.

Remembering the above information, the action of P_μ followed by D on an operator at the origin is expressed as

$$[P_\mu, [D, \mathcal{O}(0)]]. \tag{2.86}$$

From the Jacobi identity, this can be written as

$$[P_\mu, [D, \mathcal{O}(0)]] = [D, [P_\mu, \mathcal{O}(0)]] - [[D, P_\mu], \mathcal{O}(0)]. \tag{2.87}$$

Using the value of the conformal algebra (2.44) and the value of the dilation operator (2.57), the above equation becomes

$$[P_\mu, [D, \mathcal{O}(0)]] = -i\,(\Delta + 1)\,[P_\mu, \mathcal{O}(0)]. \tag{2.88}$$

Knowing that the scaling dimension of $\mathcal{O}$ is Δ, the above equation says that acting with P_μ on said operator raises the value of its scaling dimension by one. Conversely, doing the same procedure for $[K_\mu, \mathcal{O}(0)]$ gives the relation

$$[K_\mu, [D, \mathcal{O}(0)]] = -i(\Delta - 1)[K_\mu, \mathcal{O}(0)]. \tag{2.89}$$

Thus, it has been shown that P_μ and K_μ act as raising and lowering operators for the scaling dimension, respectively.

However, this in not the full story. Since correlation functions constrain the scaling dimensions to be positive, the scaling dimensions of operators in a physical theory must be bounded from below. To this end, there must exist operators which are annihilated by K_μ,

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{2.90}$$

Such operators are called primary, and are the lowest weight representations of the conformal group. In fact, the transformation properties (2.63) and (2.56) are other defining aspect of primary operators. Thus, the fixed form of the correlation functions hold only for primaries.

Operators which are constructed by acting with P_μ on a primary operator are called descendants, i.e.

$$P_{\mu_1} P_{\mu_2} \dots P_{\mu_n} \mathcal{O}(0) \tag{2.91}$$

is a descendant operator with scaling dimension $\Delta + n$. Given the properties of the generator P_μ, descendants are n-th order derivatives of primary operators. Taken together, the primary and descendants operators form an irreducible representation of the conformal algebra. As will become clear shortly, the classification of operators as primaries and descendants is a crucial property of conformal field theory.

2.4 CFT in Radial Quantisation

Up until now, the treatment of conformal invariance has been general, without any reference to a quantisation. This section will introduce radial quantisation, and explore certain aspects of this choice of quantisation which prove useful in conformal field theory.

First of all, it is necessary to choose a signature for the metric. The quantisation at hand requires a Euclidean signature. This allows free foliation of spacetime, along any time direction which suits the properties of the theory. For a conformal field theory, it is handy to set the time direction to be radially outwards from the origin, hence foliating spacetime with S^{d-1} spheres. Moving between surfaces, i.e. moving in time, is done by acting with the generator of dilations, so D will play the role of the Hamiltonian.

2.4.1 State-Operator Correspondence

Each foliating sphere will have its own Hilbert space $\mathscr{H}$. The states living on the surfaces of the spheres are generated by operator insertions at some point within the spheres, the

corresponding states will then follow the same transformation rules as said operator. Since the Hamiltonian is the dilation generator, states will be classified by their scaling dimension, and will be denoted $|\Delta\rangle$. Naturally, the vacuum state $|0\rangle$ corresponds to no operator insertions, and is conformally invariant, as well as annihilated by all generators.

Inserting a local quantity, with scaling dimension Δ, at the origin creates a state on which dilations act as follows,

$$D\,|\Delta\rangle = [D, \mathcal{O}(0)]\,|0\rangle = -i\Delta\,|\Delta\rangle\,, \tag{2.92}$$

where the second equality is true from $D\,|0\rangle = 0$. The above result is reasonable, since a dilation will not perturb an operator insertion at the origin. Acting upon these states with P_μ or K_μ will raise or lower their scaling dimension, respectively, analogously to how operators were treated earlier. A state which is annihilated by K_μ is a primary state.

An operator insertion at non-zero x produces a different outcome. It can be written as

$$\mathcal{O}(x)\,|0\rangle = e^{ix\cdot P}\mathcal{O}(0)e^{-ix\cdot P}\,|0\rangle\,. \tag{2.93}$$

Using the fact that the vacuum is translation invariant this state becomes

$$e^{ix\cdot P}\mathcal{O}(0)e^{-ix\cdot P}\,|0\rangle = e^{ix\cdot P}\,|\Delta\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}(ix\cdot P)^n\,|\Delta\rangle\,. \tag{2.94}$$

Clearly, equation (2.94) states that any local operator insertion can be written as a linear combination of infinitely many descendants. This fact will be returned to shortly.

2.4.2 Unitarity Bounds

Further unitary restrictions on the value of the scaling dimension are set by requiring that the norm of all states be positive definite. Expressing the first level scalar descendant state as

$$|P_\mu\mathcal{O}\rangle\,, \tag{2.95}$$

the norm becomes

$$\langle\mathcal{O}P_\nu|P_\mu\mathcal{O}\rangle \geq 0. \tag{2.96}$$

Using that $P_\nu^\dagger = K_\nu$ in radial quantisation, the operators can be pulled out of the state, which gives

$$\langle\mathcal{O}P_\nu|P_\mu\mathcal{O}\rangle = \langle\Delta|K_\nu P_\mu|\Delta\rangle\,. \tag{2.97}$$

Since $K_\nu\,|\Delta\rangle = 0$, the above equation can be written as a commutator

$$\langle\mathcal{O}P_\nu|P_\mu\mathcal{O}\rangle = \langle\Delta|[K_\nu, P_\mu]|\Delta\rangle\,; \tag{2.98}$$

which, using the value of (2.43), is equal to

$$\langle\mathcal{O}P_\nu|P_\mu\mathcal{O}\rangle = 2i\,\langle\Delta|D|\Delta\rangle\,\delta_{\mu\nu}, \tag{2.99}$$

where the Lorentz term vanishes since the state is a scalar. Acting with the dilation operator thus leaves

$$\langle \mathcal{O}P_\nu | P_\mu \mathcal{O}\rangle = 2\Delta\delta_{\mu\nu}\,\langle\Delta|\Delta\rangle \geq 0, \tag{2.100}$$

and so it is once again confirmed that $\Delta \geq 0$.

A stronger constraint is set by the second level norm,

$$\langle \mathcal{O}P_\alpha P_\beta | P_\gamma P_\Delta \mathcal{O}\rangle \geq 0. \tag{2.101}$$

It is important to note that these states are constructed by repeated action of differential operators. At this point it is not crucial, as the action of P_μ is symmetric; however, it will have an effect shortly. Pulling out two of the operators yields

$$\langle \mathcal{O}P_\alpha P_\beta | P_\gamma P_\Delta \mathcal{O}\rangle = \langle \mathcal{O}P_\alpha | K_\beta P_\gamma | P_\Delta \mathcal{O}\rangle\,, \tag{2.102}$$

which can be written as

$$\langle \mathcal{O}P_\alpha P_\beta | P_\gamma P_\Delta \mathcal{O}\rangle = \langle \mathcal{O}P_\alpha | [K_\beta, P_\gamma] | P_\Delta \mathcal{O}\rangle + \langle \mathcal{O}P_\alpha | P_\gamma K_\beta | P_\Delta \mathcal{O}\rangle\,. \tag{2.103}$$

The first term on the right-hand side is equal to

$$\langle \mathcal{O}P_\alpha | [K_\beta, P_\gamma] | P_\Delta \mathcal{O}\rangle = 2i\,\langle \mathcal{O}P_\alpha | D | P_\Delta \mathcal{O}\rangle\,\delta_{\beta\gamma} - 2i\,\langle \mathcal{O}P_\alpha | L_{\beta\gamma} | P_\Delta \mathcal{O}\rangle\,. \tag{2.104}$$

Acting with D on the descendant state yields the value $-i(\Delta+1)$. The Lorentz term is trickier. Taking the $L_{\beta\gamma}$ inside the state gives $|L_{\mu\nu}P_\Delta\mathcal{O}\rangle$. Remembering the relation (2.85), this state is expressed in terms of commutators as

$$|L_{\mu\nu}P_\Delta\mathcal{O}\rangle = |[P_\Delta, [L_{\beta\gamma}, \mathcal{O}]]\rangle\,, \tag{2.105}$$

note that this is not zero, precisely because of the order of repeated action. Using the Jacobi identity gives

$$[P_\Delta, [L_{\beta\gamma}, \mathcal{O}]] = [L_{\beta\gamma}, [P_\Delta, \mathcal{O}]] - [[L_{\beta\gamma}, P_\Delta], \mathcal{O}], \tag{2.106}$$

where the first term on the right-hand side vanishes. Since what remains is a commutator acting on an operator, it can be taken out of the state without any issues, hence

$$|L_{\mu\nu}P_\Delta\mathcal{O}\rangle = -[L_{\beta\gamma}, P_\Delta]\,|\Delta\rangle\,. \tag{2.107}$$

Plugging this value into (2.104) leaves

$$\langle \mathcal{O}P_\alpha | [K_\beta, P_\gamma] | P_\Delta \mathcal{O}\rangle = 2(\Delta+1)\,\langle \mathcal{O}P_\alpha | P_\Delta \mathcal{O}\rangle\,\delta_{\beta\gamma} + 2i\,\langle \mathcal{O}P_\alpha | [L_{\beta\gamma}, P_\Delta] | \Delta\rangle\,. \tag{2.108}$$

The first term is proportional to the norm of a first level descendant, so its value is known from (2.100). Using (2.41) with a minus sign, the second term becomes

$$2i\,\langle \mathcal{O}P_\alpha | [L_{\beta\gamma}, P_\Delta] | \Delta\rangle = 2\,\langle \mathcal{O}P_\alpha | P_\gamma \Delta\rangle\,\delta_{\beta\Delta} - 2\,\langle \mathcal{O}P_\alpha | P_\beta \Delta\rangle\,\delta_{\gamma\Delta}, \tag{2.109}$$

which is again comprised of first level norms. Adding all the results above gives

$$\langle \mathcal{O}P_\alpha|[K_\beta, P_\gamma]|P_\Delta\mathcal{O}\rangle = 4\Delta\left((\Delta+1)\delta_{\beta\gamma}\delta_{\alpha\Delta} + \delta_{\alpha\gamma}\delta_{\beta\Delta} - \delta_{\alpha\beta}\delta_{\gamma\Delta}\right)\langle\Delta|\Delta\rangle\,. \tag{2.110}$$

This leaves the calculation of the second term in (2.103). In a similar fashion as before, the actions required to produce a commutator will yield a minus sign. However, since two commutators are needed, the minus signs will cancel, meaning

$$\langle \mathcal{O}P_\alpha|P_\gamma K_\beta|P_\Delta\mathcal{O}\rangle = \langle\Delta|[K_\alpha, P_\gamma][K_\beta, P_\Delta]|\Delta\rangle\,; \tag{2.111}$$

which simply gives

$$\langle \mathcal{O}P_\alpha|P_\gamma K_\beta|P_\Delta\mathcal{O}\rangle = 4\Delta^2\delta_{\alpha\gamma}\delta_{\beta\Delta}\langle\Delta|\Delta\rangle\,. \tag{2.112}$$

Adding (2.109) and (2.112) results in

$$\langle \mathcal{O}P_\alpha P_\beta|P_\gamma P_\Delta\mathcal{O}\rangle = 4\Delta\left((\Delta+1)\delta_{\alpha\Delta}\delta_{\beta\gamma} + (\Delta+1)\delta_{\alpha\gamma}\delta_{\beta\Delta} - \delta_{\alpha\beta}\delta_{\gamma\Delta}\right)\langle\Delta|\Delta\rangle\,. \tag{2.113}$$

Tracing with $\delta^{\alpha\beta}\delta^{\gamma\Delta}$ and simplifying yields the equation

$$\langle \mathcal{O}P_\alpha P_\beta|P_\gamma P_\Delta\mathcal{O}\rangle = \Delta(\Delta - (d-2)/2) \geq 0, \tag{2.114}$$

which imposes the new bounds

$$\Delta = 0 \quad \text{or} \quad \Delta \geq \frac{d-2}{2}, \tag{2.115}$$

where $\Delta = 0$ holds for the unit operator. Higher level states do not impose stronger bounds than what is found at the second level.

Naturally, not all operators in a theory are scalars. Although it will not be used directly, it is important to for future discussions to also state the unitarity bound for operators of non-zero $SO(d)$ spin-l. It is given by

$$\Delta \leq l + d - 2. \tag{2.116}$$

For further discussion of this, see [16].

2.5 Operator Product Expansion

Consider a state which consists of two primary scalar operator insertions, one at the origin and one at some non-zero value of x,

$$\mathcal{O}_i(x)\mathcal{O}_j(0)\,|0\rangle\,. \tag{2.117}$$

It was shown in section (2.4.1) that any state can be expressed as an infinite series of primaries and descendants. For a single operator insertion this expansion took the form of equation

(2.94). Following that example, a two particle state in the above configuration can be expanded into the general form

$$\mathcal{O}_i(x)\mathcal{O}_j(0)\left|0\right\rangle = \sum_k \mathscr{C}_{ijka}(x,P)\mathcal{O}_k^a(0)\left|0\right\rangle . \tag{2.118}$$

The sum in k runs over all primaries of the theory, including operators with non-zero $SO(d)$ spin-l, denoted with the index $a = \mu_1\mu_2\cdots\mu_l$; and the coefficient $\mathscr{C}_{ijka}(x,P)$ is a series of differential operators which produces an infinite number of descendants for each primary. The fact that the spin indices are contracted on the right-hand side means that the terms in the sum all transform as scalars. This is the conformally invariant *operator product expansion*, or OPE, introduced in the works [17–19]. As the name suggests, it enables a product of two operators to be expressed as a sum of all operators in the theory at the point of insertion of the second operator. The OPE is only convergent between neighbouring operators. In a radial quantisation language, this means that the OPE converges if it is possible to quantise around a point in such a way that the operators in the product are are the only operators living on some foliating sphere. Furthermore, the rate of convergence increases when the distance between the operator insertions decreases [20].

From the state-operator correspondence the OPE can be expressed as

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \mathscr{C}_{ijka}(x_{12},P)\mathcal{O}_k^a(x_2), \tag{2.119}$$

where the appearance of x_{12} in the differential operator is due to preservation of conformal invariance, and it is implied that the expansion occurs within a correlator of n operators such that $|x_{12}| \leq |x_{2n}|$. For example, taking the three-point function $\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle$ and performing the operator product expansion between operator one and two, gives

$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \sum_{k'} \mathscr{C}_{ijk'a}(x_{12},P)\left\langle\mathcal{O}_{k'}^a(x_2)\mathcal{O}_k(x_3)\right\rangle . \tag{2.120}$$

The fixed form of the two-and tree-point functions are known from (2.69) and (2.74), respectively. Choosing a basis such that $\lambda_{k'k}$ is unity, the above equation becomes

$$\frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}} = \mathscr{C}_{ijk}(x_{12},P)x_{23}^{-2\Delta_k}, \tag{2.121}$$

where the sum vanishes due to the factor $\delta_{k'k}$ in the two-point function, while only the scalar value of a yields a result. The above equation, along with conformal invariance, forces $\mathscr{C}_{ijk}(x_{12},P)$ to be an infinite series of derivatives with coefficients depending on Δ_n, with an overall proportionality to λ_{ijk}. Hence, using these facts, a more useful form of expressing the OPE for scalar operators is

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \lambda_{ijk}\mathscr{C}_a(x_{12},\partial_{x_2})\mathcal{O}_k^a(x_2), \tag{2.122}$$

where λ_{ijk} will from now on be referred to as the OPE coefficient, and the dependence of P_μ has been denoted as a derivative.

Additionally, the OPE between two identical scalar operators allows only even spin-l operators in the sum, or else λ_{ijk} is zero. This can be seen by considering a correlator between two scalars and a spin-l operators $\langle \mathcal{O}_i(x)\mathcal{O}_j(-x)\mathcal{O}^{\mu_1\mu_2\cdots\mu_l}(0)\rangle$, where the point configuration is reachable through conformal transformations applied to any three points.[1] Due to the spin-indices, this correlators must have a tensor structure. In fact, for the point configuration chosen, there is a unique form for this structure [21], namely

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(-x)\mathcal{O}^{\mu_1\mu_2\cdots\mu_l}(0)\rangle \propto x^{\mu_1}x^{\mu_2}\cdots x^{\mu_l}\lambda_{ijk}, \tag{2.123}$$

where the OPE has been taken between the scalars. Changing $x \to -x$ will yield a factor of $(-1)^l$ on the right-hand side, and switching the indices of the coefficient $i \leftrightarrow j$. If the scalar operators are identical, the correlator with exchanged coordinates should have the same form as (2.123), meaning l has to be even. On the other hand, if the scalar are not the same the sign flip of the coordinated imposes the relation $\lambda_{ijk} = (-1)^l\lambda_{jik}$.

Equation (2.120) illustrates a useful application of the operator product expansion; namely its ability to reduce an n-point function to an $(n-1)$-point function. This decomposing nature of the OPE is a property which plays a central role in the conformal bootstrap.

[1] Point configurations will be discussed in more detail in section 3.3.

3 The Conformal Bootstrap

The impressive ability of the operator product expansion to reduce n-point functions into $(n-1)$-point functions can be used recursively until only a two-or three-point function, as well as factors of λ_{ijk} and terms which depend on various scaling dimensions, remain. To this end, all correlation functions, and by extension any d-dimensional conformal field theory, can be determined by the OPE coefficients and the scaling dimensions. Taken together, the set of all λ_{ijk}'s and Δ's form the so called CFT data. Seeing as primaries and descendants of a unitary CFT cannot enjoy an arbitrary choice of scaling dimensions, there must exist some constraints on CFT data. Indeed, these restrictions are what form the conformal bootstrap programme, which has proven to be remarkably powerful in determining values of some CFT data. Beautifully, the conformal bootstrap works independently of a Lagrangian, meaning it can be used to study models which do not have such a description. This section will present the ideas and consequences of the conformal bootstrap in a general sense. The specific application on the three-dimensional Ising will be considered in section 5.

3.1 Crossing Symmetry

The conformal bootstrap is centred around a property of the four-point function known as *crossing symmetry*. The fundamental nature of crossing symmetry makes it a logical first topic of study. To this end, consider a four-point function with a point configuration which allows the following OPEs

$$\langle \overbrace{\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)}\overbrace{\mathcal{O}_k(x_3)\mathcal{O}_l(x_4)}\rangle = \sum_{m,m'} \lambda_{ijm}\lambda_{klm'}\mathscr{C}_a(x_{12},\partial_{x_2})\mathscr{C}_b(x_{34},\partial_{x_4})\,\langle \mathcal{O}^a_m(x_2)\mathcal{O}^b_{m'}(x_4)\rangle\,; \tag{3.1}$$

where the contraction channel will be referred to as (12)(34). The right hand side is completely fixed by conformal symmetry, with the correlator of two spin-l operators taking the form

$$\langle \mathcal{O}^a_m(x_i)\mathcal{O}^b_{m'}(x_j)\rangle = \delta_{mm'}\frac{I^{ab}(x_{ij})}{x_{ij}^{\Delta_m+\Delta_{m'}}}, \tag{3.2}$$

where $I^{ab}(x_{ij})$ is a tensor structure which is fixed by conformal symmetry. Without loss of generality, it is possible to configure the points in such a way that it is equally valid to do the OPE in the (14)(23) channel,

$$\langle \overbrace{\mathcal{O}_i(x_1)\overbrace{\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)}\mathcal{O}_l(x_4)}\rangle = \sum_{m,m'} \lambda_{kjm'}\lambda_{ilm}\mathscr{C}_a(x_{14},\partial_{x_4})\mathscr{C}_b(x_{23},\partial_{x_3})\,\langle \mathcal{O}^a_m(x_4)\mathcal{O}^b_{m'}(x_3)\rangle\,, \tag{3.3}$$

where the ordering of the indices in the coefficients is kept with respect to the unchanged labels. Since (3.1) and (3.3) are two different decompositions of the same correlator, the two

expressions should be equivalent. This symmetry can be expressed as

$$(12)(34) = (14)(23). \tag{3.4}$$

The contraction channels considered above are not necessarily the only possible choices. However, they are in agreement with conventions, and have proven to be the most useful. Considering crossing between other channels does not give more information, meaning the above symmetry is sufficient.

3.2 Conformal Blocks

The expressions for the OPE inside a four-point function may not be the most practical. Because of this, it is useful to develop a more compact notation. For this purpose it is necessary to consider, for example, the (12)(34) channel,

$$\langle \overbrace{\mathcal{O}_i(x_1)\mathcal{O}_j}(x_2)\overbrace{\mathcal{O}_k(x_3)\mathcal{O}_l}(x_4)\rangle = \sum_m \lambda_{ijm}\lambda_{klm}\mathscr{C}_a(x_{12},\partial_{x_2})\mathscr{C}_b(x_{34},\partial_{x_4})I^{ab}(x_{24})x_{24}^{-2\Delta_m}. \tag{3.5}$$

Remembering the expression for the four-point function derived from conformal invariance (2.82), the above contraction can be reduced to the more useful form

$$\langle \overbrace{\mathcal{O}_i(x_1)\mathcal{O}_j}(x_2)\overbrace{\mathcal{O}_k(x_3)\mathcal{O}_l}(x_4)\rangle = \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{ij}}\left(\frac{x_{14}}{x_{34}}\right)^{\Delta_{kl}}\frac{1}{x_{12}^{\Delta_i+\Delta_j}x_{34}^{\Delta_k+\Delta_l}}\sum_m \lambda_{ijm}\lambda_{klm}g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(u,v); \tag{3.6}$$

where

$$g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(u,v) = \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{ji}}\left(\frac{x_{14}}{x_{43}}\right)^{\Delta_{lk}}x_{12}^{\Delta_i+\Delta_j}x_{34}^{\Delta_k+\Delta_l}\mathscr{C}_a(x_{12},\partial_{x_2})\mathscr{C}_b(x_{34},\partial_{x_4})I^{ab}(x_{24})x_{24}^{-2\Delta_m}, \tag{3.7}$$

with $\Delta_{ij} = \Delta_i - \Delta_j$. The functions $g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(u,v)$ are called conformal blocks, and must be dependent on the cross ratios as a consequence of the relation to the four-point function; and the label l_m is associated with the spin-l of the m-th operator. Conformal blocks are valid up to factors of cross-ratios, meaning it is possible to their exact spacetime dependence can be tweaked, if necessary, by multiplying with factors of u and v. This way of dealing with correlation functions is known as conformal block decomposition.

The precise form of the conformal blocks will not be dwelt upon, and in fact they are not exactly known for theories in an odd number of spacetime dimensions. Thus, the applications which will be considered shortly all use numerical methods to approximate the conformal blocks to some order. Nevertheless, the conformal blocks depend upon the external operators in the four-point correlator. For a more involved treatment of conformal blocks, see [22–25]

3.3 Optimal Point Configuration

The above analytic discussion holds without reference to a specific point configuration. However, for numerical purposes there are coordinate values which are more beneficial than others. This

is because the convergence of the conformal blocks is dependent on the separation between the operators. In the channels used in the conformal bootstrap, the optimal configuration would be such that $u = v$. This is possible to achieve using conformally invariant maps from any set of four points as follows.

First, x_4 is placed at the origin. Space can then be inverted, sending $x_4 \to \infty$, where it will be unaffected by all the other operations. There are now three points left to manipulate. It is possible to translate the system such that $x_1 = 0$. Now, rotations of the system can place x_1 and x_3 on the same line; then a dilation can be used to fix x_3 at 1. This leaves the free coordinate x_2, it's position described by the complex coordinate z. With this configuration, the cross ratios become

$$u = |z|^2 , \quad \text{and} \quad v = |1 - z|^2 . \tag{3.8}$$

Thus, the crossing symmetric point is $z = \bar{z} = 1/2$, meaning $u = v = 1/4$, which places all points on the same line. Since a line can be mapped to a circle, the crossing symmetric configuration maps to the points which correspond to the vertices of a square such that the contractions in the OPE channels (12)(34) and (14)(23) are equally separated.

3.4 The Bootstrap Equation

Crossing symmetry and conformal block decomposition will now be put to use. Two different scenarios of bootstrapping will be considered below, each treated separately.

3.4.1 Identical Scalar Correlator

A good starting point is the simplest case of a correlator of four identical scalars with scaling dimension $\Delta_{\mathcal{O}}$. Decomposing the four-point function in the (12)(34) channel yields

$$\langle \overbrace{\mathcal{O}(x_1)\mathcal{O}}(x_2)\overbrace{\mathcal{O}(x_3)\mathcal{O}}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_{\mathcal{O}}} x_{34}^{2\Delta_{\mathcal{O}}}} \sum_m \lambda^2_{\mathcal{O}\mathcal{O}m} g_{\Delta_m, l_m}(u, v), \tag{3.9}$$

where the superscript of the conformal block has been dropped for clarity. Since the OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}m}$ are real, their squares are positive. Taking the decomposition in the (14)(23) yields

$$\langle \overbrace{\mathcal{O}(x_1)\overbrace{\mathcal{O}(x_2)\mathcal{O}(x_3)}\mathcal{O}}(x_4)\rangle = \frac{1}{x_{14}^{2\Delta_{\mathcal{O}}} x_{23}^{2\Delta_{\mathcal{O}}}} \sum_m \lambda^2_{\mathcal{O}\mathcal{O}m} g_{\Delta_m, l_m}(v, u), \tag{3.10}$$

where the reversal of variable ordering in the conformal block argument is because $u \leftrightarrow v$ as $x_2 \leftrightarrow x_4$. Equating the above expressions and multiplying by $x_{14}^{2\Delta_{\mathcal{O}}} x_{23}^{2\Delta_{\mathcal{O}}}$, gives after some simplification

$$v^{\Delta_{\mathcal{O}}} \sum_m \lambda^2_{\mathcal{O}\mathcal{O}m} g_{\Delta_m, l_m}(u, v) = u^{\Delta_{\mathcal{O}}} \sum_m \lambda^2_{\mathcal{O}\mathcal{O}m} g_{\Delta_m, l_m}(v, u), \tag{3.11}$$

which is the bootstrap equation for identical scalars, and must hold for any valid conformal field theory. It is not satisfied term-by-term, which is manifest when considering the first order

of the expansion, i.e. the unit operator, on either side. Because of this intertwined nature, the bootstrap equation using identical external operators constrains the set of scaling dimensions of even spin-l operators, and the set of $\lambda_{\mathcal{O}\mathcal{O}m}$. Since the only information about the CFT contained within the conformal blocks is a variety of scaling dimensions and l's, which are determined by the symmetries of the theory, a solution is unique to the theory of the scalars $\mathcal{O}$. Although this form of the bootstrap equation is not strong enough to determine the full set of CFT data, it is a good first step.

The simple appearance of the bootstrap equation is deceiving, and is an artefact of the choice of notation. Indeed, the equation contains infinite sums of very non-trivial terms. Nevertheless, the form of equation (3.11) can easily be changed into the more practical sum rule

$$\sum_m \lambda^2_{\mathcal{O}\mathcal{O}m} F^{\Delta_\mathcal{O}}_{\Delta_m,l_m}(u,v) = 0, \tag{3.12}$$

with the definition of the convolved conformal block

$$F^{\Delta_\mathcal{O}}_{\Delta_m,l_m}(u,v) = v^{\Delta_\mathcal{O}} g_{\Delta_m,l_m}(u,v) - u^{\Delta_\mathcal{O}} g_{\Delta_m,l_m}(v,u). \tag{3.13}$$

When expressed as (3.12), one of the hurdles which needs to be overcome when solving the bootstrap equation becomes clear. Namely that an expression with positive coefficients should sum to zero. However, the positivity of $\lambda^2_{\mathcal{O}\mathcal{O}m}$ allows for an interpretation of each $F^{\Delta_\mathcal{O}}_{\Delta_m,l_m}(u,v)$ in the sum as a vector, in which case the validity of the sum rule is dependent on the orientation of all the vectors appearing in the sum. Geometrically speaking, if it is possible to find a plane such that all the vectors are on the same side, it is impossible for them to sum to zero for any values of $\lambda^2_{\mathcal{O}\mathcal{O}m}$. On the other hand, if such a plane cannot be found it is possible to satisfy (3.12); this does not, however, determine the exact values of the coefficients.

The case where a solution can be ruled out is expressed as

$$\begin{aligned} \alpha\left(F^{\Delta_\mathcal{O}}_{\Delta_m,l_m}\right) &\geq 0, \\ \alpha\left(F^{\Delta_\mathcal{O}}_{\Delta_0,l_0}\right) &= 1, \end{aligned} \tag{3.14}$$

where α is a linear functional acting on each convolved conformal block, yielding a real number; and $\mathcal{O}_0$ is the unit operator. The action of α can generally be expressed as derivatives around the crossing-symmetric point

$$\alpha\left(F^{\Delta_\mathcal{O}}_{\Delta_m,l_m}\right) = \sum_{i,j} a_{ij} \partial_z^i \partial_{\bar{z}}^j F^{\Delta_\mathcal{O}}_{\Delta_m,l_m}(z,\bar{z})\Big|_{z=\bar{z}=1/2}. \tag{3.15}$$

In this formulation, the constraints (3.14) become a set of linear inequalities which need to be solved for the coefficients a_{ij}.

3.4.2 Mixed Scalar Correlator

The generalisation of the bootstrap equation to correlators containing several distinct operators, $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4)\rangle$, is straightforward. The first instances of this approach can be found [26, 27], while this section is follows the notation of [10]. This requires using (3.5) in the (12)(34) channel, and the appropriate, analogous expression in the (14)(23) channel. Equating the two decompositions yields, after some algebra,

$$v^{\frac{\Delta_j+\Delta_k}{2}} \sum_m \lambda_{ijm}\lambda_{klm} g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(u,v) = u^{\frac{\Delta_i+\Delta_j}{2}} \sum_m \lambda_{kjm}\lambda_{ilm} g^{\Delta_{kj},\Delta_{il}}_{\Delta_m,l_m}(v,u); \tag{3.16}$$

where the factors of u and v appear from rearranging the pre-factors

$$\left(\frac{x_{24}}{x_{34}}\right)^{\Delta_{kj}} \left(\frac{x_{34}}{x_{13}}\right)^{\Delta_{il}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{ji}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{lk}} \frac{x_{12}^{\Delta_i+\Delta_j} x_{34}^{\Delta_k+\Delta_l}}{x_{23}^{\Delta_j+\Delta_k} x_{34}^{\Delta_i+\Delta_l}} = \frac{u^{\frac{\Delta_i+\Delta_j}{2}}}{v^{\frac{\Delta_j+\Delta_k}{2}}}. \tag{3.17}$$

This form of the bootstrap equation demonstrates the fact that finding the full set of CFT data requires bounds generated using all possible four-point functions. At this point there is a subtlety. Since the operators in the mixed correlator are not all identical, the OPE coefficients appearing in (5.8) are not necessarily related to each other. Therefore, their product can take any sign, and the possible ways of summing to zero increases. Thus, the reasoning which was valid in the case of a single correlator needs to be modified. Applying a linear functional on (5.8) yields

$$\sum_m \lambda_{ijm}\lambda_{klm} \left[\alpha^{ijkl}\left(v^{\frac{\Delta_j+\Delta_k}{2}} g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(u,v)\right) - \alpha^{kjil}\left(u^{\frac{\Delta_i+\Delta_j}{2}} g^{\Delta_{kj},\Delta_{il}}_{\Delta_m,l_m}(v,u)\right)\right] = 0, \tag{3.18}$$

where repeated indices are summed over, which allows for the factorisation. Here, the coefficients form vectors $\vec{\lambda}_m$; and the terms dependent on u and v can be expressed as elements in a matrix T_m. The above expression then becomes

$$\sum_m \vec{\lambda}_m^T T_m \vec{\lambda}_m = 0. \tag{3.19}$$

The general form of (3.14) is then

$$\begin{aligned} T_m &\succeq 0, \\ T_0 &\succ 0, \end{aligned} \tag{3.20}$$

where in this context $\succeq$ means positive semi-definite. Finding such a functional is no longer a linear problem, rather this task needs to be handled by *semi-definite programming*. This will be studied in further detail under simplifying circumstances in the context of the three-dimensional Ising model in section 5.3. Nevertheless, such problems are intrinsic to the conformal bootstrap, and case of identical scalar bootstrap is merely a scenario where the full procedure can be avoided. Finally, it is useful to define the functions

$$F^{ij,kl}_{\pm,\Delta_m,l_m}(u,v) = v^{\frac{\Delta_k+\Delta_j}{2}} g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(u,v) \pm u^{\frac{\Delta_k+\Delta_j}{2}} g^{\Delta_{ij},\Delta_{kl}}_{\Delta_m,l_m}(v,u), \tag{3.21}$$

in which case the general bootstrap equation takes a more familiar form

$$\sum_m \left(\lambda_{ijm}\lambda_{klm}F^{ij,kl}_{\mp,\Delta_m,l_m}(u,v) \pm \lambda_{kjm}\lambda_{ilm}F^{kj,il}_{\mp,\Delta_m,l_m}(v,u)\right) = 0. \tag{3.22}$$

4 Three-dimensional Ising Model

In order to understand the reasoning behind the application of the bootstrap programme to the three-dimensional Ising model, it is necessary to study the model in detail. This is the purpose of this section. The main content will be devoted to an analysis of the discreet, statistical Ising model using the renormalization group approach. This will naturally lead to a discussion about the Ising CFT.

4.1 Hamiltonian

One difficulty which arises in complicated physical systems is the construction of an effective Hamiltonian. In some cases, a Hamiltonian which takes into account all possible degrees of freedom would be too complicated to use in calculations using the exact expression for the partition function. In order to avoid issues and make a theory solvable, a simplified model, which nevertheless possesses the macroscopic properties of interest in some limit, is conceived and used for calculations of physically relevant quantities. This is the purpose of the Ising model, which serves as an effective description for several interesting phenomenon.[2]

The Ising model most commonly appears as a simple model for a uni-axial ferromagnet. In the three-dimensional case each node i of a cubic lattice is inhabited by a magnetic spin σ_i, which randomly takes the value $+1$ or -1, respectively corresponding to spin-up or spin-down. The spins also interact in nearest-neighbour pairs $\langle ij \rangle$. The most general form of the Ising model includes a term corresponding to application of a uniform, external magnetic field H, in which case the Hamiltonian takes the form

$$\mathcal{H}(\sigma) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - H \sum_{i=1}^{N} \sigma_i, \tag{4.1}$$

where N is the number of lattice sites. J is the exchange energy, which acts as a coupling between the sites i and j; if it has a positive value the energetically most favourable configuration is when all spins have aligned, yielding an interaction which is ferromagnetic by nature.

Using the above Hamiltonian it is possible to calculate the magnetisation, which is given by the expectation value of a single spin

$$M = \langle \sigma_i \rangle \,. \tag{4.2}$$

From the statistical formulation of the mean value this is expressed as

$$M = \frac{1}{NZ} \sum_{\sigma} \left(\sum_{i=1}^{N} \sigma_i \right) e^{-\beta \mathcal{H}(\sigma)}, \tag{4.3}$$

[2] This vague statement will be clarified shortly.

where the factor of $1/N$ compensates for the sum over i, and Z is the partition function defined by

$$Z = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)}, \tag{4.4}$$

with $\beta = 1/k_B T$. Remembering how to differentiate a logarithm, the above expression can be massaged into the form

$$M = \frac{1}{N} \frac{\partial F}{\partial H}, \tag{4.5}$$

where $F = \ln(Z)$ is the free energy.

Most of the future discussion will only be valid in the thermodynamic limit $N \to \infty$; and in the zero-field configuration $H \to 0$. In this limit, the interaction has a clear symmetry. Under a global $\mathbb{Z}_2$ transformation the individual spins are odd, i.e. they transform as $\sigma_i \to -\sigma_i$, while the Hamiltonian is even. Furthermore, this limit kills any boundary effects, which greatly simplifies the analysis.

4.2 Phase Transitions

Phase transition is the term given to the process when a physical system abruptly changes its macroscopic properties. Such transitions occur at one or more specific values of some external parameters. The specific point at which a phase transition occurs is called a critical point. There are two main types of phase transitions, aptly named discontinuous and continuous, alternatively first-or second-order, respectively. In the case of the former, the various possible states of matter are distinct from each other at the critical point. In such cases, the thermodynamic properties of the system will be discontinuous, taking on different but finite values for each state on either side of the critical point.

On the other hand, a continuous phase transition possesses its own *critical phase*. The effect of this is that the states on either side of the critical point are connected, yielding a continuous order parameter of the material. This does not guarantee, however, that all quantities are finite.

The ferromagnetic phase transition, which the Ising model undergoes, is continuous, the order parameter being the magnetisation. At some point the spins in the system align, giving rise to a magnetic field which spontaneously breaks the original $\mathbb{Z}_2$-symmetry of the Hamiltonian. The critical state of the Ising model lives at the Curie point, with critical temperature T_c and $H = 0$. It is the critical Ising model which provides an environment which is suitable for the calculations of interest here. Namely, it is the second-order phase transitions which are related to conformal invariance. Why this is the case will become clear in the following.

4.2.1 Critical Behaviour

Another fundamental trait of second-order phase transitions is the behaviour of the correlation length $\xi(T)$, the measure of the distance over which fluctuations of microscopic degrees of freedom are correlated. Continuing the example of the Ising model, away from the critical point the correlator of two spins decays exponentially as

$$\langle \sigma_i \sigma_j \rangle \propto e^{-r/\xi(T)}, \tag{4.6}$$

where $r = |i - j|$ is the separation in units of lattice spacing. For a system undergoing a continuous transition, the correlation length diverges as the critical point is approached, forcing the whole material to be in its critical state. Rather counter-intuitively, as $\xi(T) \to \infty$, the influence of microscopic variables on the macroscopic quantities diminishes. This is displayed by the fact that near the critical point some important functions take on the form of power-laws, becoming scale invariant. The characteristic feature of these functions are their *critical exponents*: pure numbers which depend only on global properties of a system, for example the symmetry of the Hamiltonian, or the dimension of space. As an illustration, the correlation length takes the form $\xi(T) \propto (T - T_c)^{-\nu}$, the critical exponent being ν. Once the values of the critical exponents have been determined, they can be used to make predictions for observables close to the critical point, solving the theory. The critical behaviour of the two-point function and other physical quantities of the Ising model will be discussed in tandem with the renormalization group.

4.2.2 Universality

Power-law dependence and critical exponents do not only simplify complex phenomena, the behaviour also carries deep information about critical states of matter in general. Namely, it turns out that critical exponents are not uniquely valued for a specific phase transition. In fact, several distinct systems, meaning systems with different order parameters, can have the same values for their critical exponents. Seeing as the critical exponents determine the behaviour of a critical system, this means that overlapping systems are effectively the same at the critical point. By extension, the critical systems are governed by a similar Hamiltonians. This is known as *universality*, and systems which have the same critical exponents are categorised into the same universality class. For example, it is known that the Ising model shares a critical point with the liquid-gas phase transition. The spins σ_i which appear in the Hamiltonian (4.1) are in the case of a fluid system re-expressions of site occupation numbers, $\sigma_i = 2n_i - 1$, where $n_i \in \{0, 1\}$. Thus, close to criticality the zero-field Ising model's $\mathbb{Z}_2$ symmetry appears also in a fluid.

Universality is not limited to statistical physics. The link between critical phenomenon and conformal symmetry suggests that universality should emerge also in conformal field theories. Indeed, further indications of this will appear in the next section.

4.3 The Renormalization Group

At this point, with all the above information in mind, it is appropriate to delve into a formal approach to critical phenomena by donning the spectacles of the renormalization group. The main goal of this approach is to systematically eliminate the influence of microscopic degrees of freedom of statistical systems without disrupting the observable quantities. There is no single way of achieving this, each model must be considered in uniquely. The specific considerations below mostly concern lattice spin systems, and will be focused towards the Ising universality class.

4.3.1 Block-spin Renormalization

When choosing a renormalization group scheme it is favourable to pick one which makes use of the properties of the theory near criticality. The most effective way of dealing with the Ising model is by introducing block spins, σ'_I. This is done by grouping the spins into blocks with sides of length b, in terms of an odd number lattice spacings, each block containing b^d spins. σ'_I shares a value of $+1$ or -1 with the majority of the constituent spins.[3] This defines a map to a new, rescaled Hamiltonian

$$\mathcal{H}'(\sigma') = -J' \sum_{\langle IJ \rangle} \sigma'_I \sigma'_J - H' \sum_I \sigma'_I, \tag{4.7}$$

where the factor J' absorbs the interactions of the boundary spins within the block, meaning it scales as $J' \sim b^{d-1} J$.

Seeing that the statistical nature of the system, not least the magnetisation, is characterised by the partition function, it is appropriate to consider its behaviour under block spin renormalization. The partition function for the original system is given by

$$Z = \sum_\sigma e^{-\mathcal{H}(\sigma)}, \tag{4.8}$$

where the factor of β has been absorbed into the Hamiltonian, yielding the reduced coupling $K = \beta J$, and magnetic field $h = \beta H$. More formally, the block Hamiltonian can be defined by patching together areas of the original model, taking into account all possible spin configurations which agree with the value of the blocks in the renormalized model. This can be expressed as

$$e^{-\mathcal{H}'(\sigma')} = \sum_\sigma \prod_{\sigma'_I} f(\sigma'_I; \sigma_i) e^{-\mathcal{H}(\sigma)}. \tag{4.9}$$

where for each σ'_I the function $f(\sigma'_I; \sigma_i)$ is defined as

$$f(\sigma'_I; \sigma_i) = \begin{cases} 1 & \text{if} \quad \sigma'_I \sum_i \sigma_i > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

[3] The majority rule is the reason for choosing to build a block spin using an odd number of spins, since this eliminates the possibility of having equal number of spin-up and spin-down.

The purpose of $f(\sigma'_I, \sigma_i)$ is to pick out only the spin configurations which give rise to the value of each block σ'_I. In order for a spin configuration to contribute, their sum must have the same sign as the block in which they are contained; this constricts the values of the σ_i's, forcing only agreeing configurations to appear in the sum over σ in (4.9).

Since the function $f(\sigma'_I; \sigma_i)$ must contribute with a value of 1 for each block it has the identity

$$\sum_{\sigma'_I} f(\sigma'_I; \sigma_i) = 1. \tag{4.11}$$

Using this fact on equation (4.9) gives the block partition function

$$Z' = \sum_{\sigma'} e^{-\mathcal{H}'(\sigma')} = \sum_{\sigma} e^{-\mathcal{H}(\sigma)} = Z, \tag{4.12}$$

meaning that the statistical properties of the system are invariant under the renormalization group transformation. This should not come as a surprise since the block transformation is merely a re-formulation of the fundamental lattice system, nevertheless it is a comforting realisation and validates the approach.

4.3.2 Renormalization Group Flow

In practice the renormalization process does not end after a single iteration. This has several consequences. Primarily, the blocking has an effect on the perceived order of the system. If the starting temperature of the system is low, the spins will be mostly in alignment; thus, when constructing blocks, the order will increase since the majority of the spins inside the blocks will tend to be pointing in the same overall direction. After several transformations, the final block spin system will be completely ordered, ending up at a *fixed point* corresponding to an effective temperature of zero. On the other hand, if the initial temperature is high, the spins will in be a more disordered distribution. In this case repeated blocking will drive the system toward a fixed point corresponding to a paramagnetic state, with $T \to \infty$.

The fact that a system moves toward some state through renormalization iterations is called renormalization group flow. In the case of the Ising model, the two fixed points at either end of the temperature spectrum are attractive, meaning there must exists some intermediate temperature at which repeated blocking does not alter the effective order of the system. Indeed, this is the critical temperature. If $T > T_c$ the system flows to the high temperature phase, while if $T < T_c$ the system ends up in the $T = 0$ phase. The two endpoints are stable, while the critical point is unstable.

Remembering that the reduced coupling contains a factor of T, it can be used to described the flow. This requires further considerations. After each repeated blocking, the couplings from the previous block spins become absorbed into the interactions of the higher level blocks. Essentially, this generates a set of couplings between spins which span further than the original

nearest neighbours. It is thus necessary to consider not only a single coupling K, but rather a set of all possible couplings $\{K\}$. Then the action of renormalization can be expressed as

$$\{K'\} = \mathscr{R}(\{K\}), \tag{4.13}$$

which means that the renormalization group flow takes place in a multi-dimensional space of parameters. The above equation is known as the renormalization group equation, and will be considered in more detail in the next section.

In terms of the couplings, the Ising critical point $\{K^*\}$ is a surface which satisfies $\{K^*\} = \mathscr{R}(\{K^*\})$; the other fixed points are at $\{K\} \to \infty$ and $\{K\} = 0$ for $T = 0$ and $T \to \infty$, respectively. The criticality of $\{K^*\}$ can be argued as follows. In the limit $\{K\} \to \{K^*\}$ it will take an infinite number of block transformation, n, to leave the vicinity of the critical point. For each step the correlation length will decrease by a factor of b. The correlation length at the n-th iteration can be expressed in terms of the correlation length near the critical point as $\xi(\{K^{'\cdots'}\}) = b^{-n}\xi(\{K\})$. At some finite temperature away from the critical value, the correlation length $\xi^{'\cdots'}$ should be of order one lattice spacing. Reversing these arguments gives $\xi = b^n\xi^{'\cdots'}$, which diverges since $n \to \infty$, indicating a critical point.

4.3.3 Renormalization Group Equation

The renormalization group equation (4.13), and by extension its flow, can be studied in very general terms. Assuming it is differentiable at the critical point, the function $\mathscr{R}$ can be Taylor expanded around the points K_i^* in coupling space. Expanding only to first order yields the equation for each coupling

$$K_i' - K_i^* \approx \sum_j \left(K_j - K_j^*\right) \frac{\partial K_i'}{\partial K_j}\bigg|_{K_j = K_j^*}, \tag{4.14}$$

where the index j denotes the couplings in the non-renormalized system; the critical coupling on the left-hand side appears from the first term in the expansion, $K_i^* = \mathscr{R}(K_j^*)$; and, the term K_i' on the right-hand side is the result of $\mathscr{R}(K_j)$. The derivatives form a matrix T_{ij} which satisfies

$$\sum_j e_i^k T_{ij} = \lambda^k e_j^k, \tag{4.15}$$

with eigenvalues and eigenvectors λ^k and e_i^k, respectively. Using the above quantities it is possible to construct scaling variables u_i, defined as

$$u_i = \sum_j e_j^i (K_j - K_j^*). \tag{4.16}$$

Combining (4.14) and (4.15) it is manifest that the scaling variables transform as

$$u_i' = \lambda^i u_i. \tag{4.17}$$

The matrix eigenvalues can be defined by $\lambda_i = b^{y_i}$, where y_i is called a renormalization group eigenvalue. The value of y_i will govern the transformation properties near the critical point of the corresponding scaling variable, which can be seen by considering equation (4.17) in the form $u_i' = b^{y_i} u_i$. If $y_i > 0$ the exponent of λ^i will grow for each iteration, meaning u_i moves away from its fixed point value, in which case u_i is called a relevant variable. If $y_i < 0$ the value of the pre-factor grows in the denominator, sending u_i to its critical point value, zero, as the transformations are repeated. Variables which flow toward the critical point are called irrelevant. Finally, in the marginal case when $y_i = 0$ the linear form of the renormalization equation is not sufficient to determine the behaviour of the scaling variable. This prompts the distinction *exactly marginal* scaling variables which stay at the same value as the theory flows, and variables which exhibit corrections at higher orders.

The space of irrelevant scaling variables defines a hypersurface which flows toward the critical fixed point. Thus, when trying to produce a critical system these variables can be ignored, meaning only the relevant scaling variables need to be tuned. This is accomplished by adjusting the dependencies of the couplings, each parameter corresponding to a relevant variable. Reaching the critical point of the Ising model requires specific values of the temperature and the external magnetic field, meaning that the theory must have two relevant scaling variables.[4] The like the spins, the magnetic scaling variable will be odd under $\mathbb{Z}_2$-transformation, and will be denoted u_σ. Conversely, the thermal variable will be $\mathbb{Z}_2$-even, like the energy, and denoted u_ε. The respective eigenvalues of the two scaling variables are y_σ and y_ε.

Using the fact that the value of the scaling variables must tend to zero as the critical point is approached, it is possible to find approximate expressions for u_σ and u_ε. Defining the reduced temperature $t = (T - T_c)/T_c$ and remembering that the critical point occurs as $h \to 0$, the variables can be expressed, up to first order, as

$$u_\sigma = h/h_0 \quad \text{and} \quad u_\varepsilon = t/t_0, \tag{4.18}$$

with scaling constants h_0 and t_0.

The remainder of this thesis will be focused on the study of the relevant quantities and their CFT counterparts. Nevertheless, the irrelevant variables are still significant, as they contribute in small amounts as correction to several important expressions. So, in order to predict precise measurements, calculations require the inclusions of irrelevant scaling variables up to some desired order.

4.3.4 Critical Exponents

Most of thermodynamic quantities of interest in the Ising model depend on the free energy. Thus, the first task at hand is to study its transformation properties. Defining the free energy

[4] This is also true for the liquid-gas critical point, the two relevant scaling variables corresponding to pressure and temperature.

per site as

$$f(\{K\}) = -N^{-1} \ln Z \tag{4.19}$$

it is possible to write the partition function as

$$Z = e^{-Nf(\{K\})}. \tag{4.20}$$

The partition function is invariant under block-spin renormalization; in terms of the free energy this constraint becomes

$$e^{-Nf(\{K\})} = e^{-N'f(\{K'\})}, \tag{4.21}$$

where $N' = b^{-d}N$ in general. Clearly, this gives the free energy scaling law

$$f(\{K\}) = b^{-d} f(\{K'\}). \tag{4.22}$$

In the neighbourhood where the linearised renormalization group equation (4.14) holds, the correspondence between the scaling variables and the couplings means free energy can be expressed in terms of the scaling variables. Focusing only on the relevant variables, the n-th transformation of the free energy looks like

$$f(u_\varepsilon, u_\sigma) = b^{-dn} f(b^{ny_\varepsilon} u_\varepsilon, b^{ny_\sigma} u_\sigma). \tag{4.23}$$

This expression is only valid as long as the values of the scaling variables are small. Since their values grow for each iteration, it is important to limit n. This is achieved by defining a value of u_ε at which the iterations end. The stopping point is defined by setting $|b^{ny_\varepsilon} u_\varepsilon| = u_{\varepsilon 0}$, where the absolute value takes care of the two cases $t > 0$ and $t < 0$. Plugging this into (4.23) gives the more definite expression

$$f(u_\varepsilon, u_\sigma) = |u_\varepsilon / u_{\varepsilon 0}|^{d/y_\varepsilon} \, f(\pm u_{\varepsilon 0}, u_\sigma |u_\varepsilon / u_{\varepsilon 0}|^{y_\varepsilon / y_\sigma}). \tag{4.24}$$

Absorbing the value of $u_{\varepsilon 0}$ into t_0, the free energy of a renormalized theory becomes

$$f(u_\varepsilon, u_\sigma) = |t/t_0|^{d/y_\varepsilon} \, \Psi \left(\frac{h/h_0}{|t/t_0|^{y_\sigma / y_\varepsilon}} \right), \tag{4.25}$$

where the function Ψ is the scaling function, and its explicit dependency on $u_{\varepsilon 0}$ has been dropped.

Using this form of the free energy it is straightforward to calculate its derivatives. Most importantly, the spontaneous magnetisation at $h = 0$ is given by

$$\left. \frac{\partial f}{\partial h} \right|_{h=0} \propto (-t)^{(d - y_\sigma)/y_\varepsilon}, \tag{4.26}$$

where the choice of sign of t corresponds to its value at $T = 0$. This expression defines the critical exponent

$$\beta = \frac{d - y_\sigma}{y_\varepsilon}. \tag{4.27}$$

The arguments in deriving the magnetisation have been very general. Furthermore, the final expression is very simple; the only non-general aspects being the eigenvalues y_i, which have been classified by the symmetries of the theory at hand. Since any theory which possesses the same symmetries as the Ising model will have the same classifications of the eigenvalues, and thus the same values of the critical exponents, the above expression is a clear indication of universality. For example, as mentioned in section 4.2.2, the Ising model shares a critical point with the liquid-gas system. Indeed, the critical exponent for the order parameter of the liquid-gas system is defined identically, namely $\rho_L - \rho_G \propto (-t)^\beta$, where the eigenvalues y_i have been classified accordingly.

The first attempts at solving the three-dimensional Ising model were concerned with finding the critical exponents using various techniques stemming from the renormalization group. These attempts were relatively successful, and approximate values of the exponents have been found in many ways. Indeed, some of the results gained from the bootstrap equation can be translated into critical exponents, making it the most effective tool yet.

4.3.5 Correlation Function

The last task at hand is to compute the general form of the correlation function using the renormalization group. In order to not have to focus on specific variables, like σ_i, some new quantities are needed. The transformation properties of u_i make them a good candidate to build a basis around. Since the scaling variables are linear combinations of couplings, they should each pair with interaction terms I_a in the Hamiltonian. Using this, it is possible to define the local scaling operators ϕ_i, such that

$$\sum_i u_i \phi_i = \sum_a (K_a - K_a^*) I_a. \tag{4.28}$$

In this basis the two-point function is to be taken between the operators, and it is given by

$$\Gamma(r) = \langle \phi_i(r_1)\phi_i(r_2)\rangle - \langle \phi_i(r_1)\rangle \langle \phi_i(r_2)\rangle , \tag{4.29}$$

where $r = |r_1 - r_2|$ in units of lattice spacing. For the purpose of simplifying the calculation, it is possible to subtract the left hand side of (4.28) from the Hamiltonian in the partition function,

$$\mathcal{H} \to \mathcal{H} - \sum_{r_j} u_i(r_j)\phi_i(r_j), \tag{4.30}$$

where $r_1, r_2 \in \{r_j\}$, in which case the correlator can be expressed as follows

$$\Gamma(r) = \left. \frac{\partial^2 \ln Z}{\partial u_i(r_1)\partial u_i(r_2)} \right|_{u_i=0}. \tag{4.31}$$

For example, the correlator of two spins corresponds to the special case of adding a term acting as a local magnetic field, with $\phi_i(r_j) = \sigma_i(r_j)$ and $u_i(r_j) = h(r_j)$. From the properties

of the scaling variables the extended Hamiltonian transforms as expected, with the primed scaling variables given by $u_i'(r_j') = b^{y_i} u_i(r_j)$, where $r_j' = r_j/b$. The invariance of the partition function in turn gives

$$\frac{\partial^2 \ln Z'}{\partial u_i'(r_1')\partial u_i'(r_2')} = \frac{\partial^2 \ln Z}{\partial u_i'(r_1')\partial u_i'(r_2')}, \tag{4.32}$$

and both sides must be evaluated at $u_i' = 0$. Here, the left-hand side is simply the correlation function between block spins, given by $\Gamma(r/b)$.

To obtain a useful transformation relation, the left-hand side requires some massaging. Perturbing one of the blocked scaling variables $u_i'(r_1') \to u_i'(r_1') + \delta u_i'(r_1')$ affects all of the variables $u_i(r_j)$ contained within the block. By the established relationships this gives the scaling $\delta u_i(r_j) = b^{-y_i}\delta u_i'(r_1)$. Effectively, each perturbation generates correlations between the u_i's in block number one, and u_i's block number two. Since two variables are renormalized, the perturbation must give rise to b^{2d} number of two point functions. However, if the separation between the blocks is significantly larger than b, the individual correlation functions will be very close in value and can thus be approximated by a product of all correlators. By this reasoning the left hand side of (4.32) is equal to $b^{2(d-y_i)}\Gamma(r)$. Therefore, the transformation becomes, after some rearranging,

$$\Gamma(r) = b^{-2(d-y_i)}\Gamma(r/b). \tag{4.33}$$

The blocking procedure can be iterated at the critical point, stopping at a constant value of r/b^n which is still much greater than the fundamental lattice spacing. Solving this for b^n and plugging it into the above equation yields a scaling function on the right-hand side. Finally, then, the two-point function takes the form

$$\Gamma(r) \propto |r_1 - r_2|^{-2\Delta_i} \tag{4.34}$$

where $\Delta_i = d - y_i$.

4.4 Three-dimensional Ising CFT

The expression (4.34) is reminiscent of the two-point function derived in section 2.2.1 using conformal symmetry. Considering that the correlations of a theory are experimentally measurable, the likeness of the two-point functions suggests that the critical point of a statistical system can be described by a conformal field theory where the quantities ϕ_i are analogous to primary operators $\mathcal{O}_i$. This would mean that scale invariance, which the critical point possesses, alone implies conformal invariance. Whether this holds in general is not yet fully understood [28–31]. Furthermore, the appearance of y_i in what can be considered the scaling dimension brings universality into the domain of quantum field theory.

The statements above need to be justified for each theory at hand. In the case of the three-dimensional Ising model, conformal invariance of the critical point is to be considered a

promising conjecture. In order to make this claim, a continuum limit, where the lattice spacing $a \to 0$, needs to be taken. This does not violate any of the assumptions and approximations made when deriving the form of $\Gamma(r)$. In this limit the scaling function, which gives the proportionately of (4.34), becomes unity, yielding an equality.

The two relevant quantities ϕ_σ and ϕ_ε will each have a corresponding primary scalar operator in the CFT, denoted σ and ε, for the $\mathbb{Z}_2$-odd spin and $\mathbb{Z}_2$-even energy-density operators, respectively. Since y_i will determine the value of the dimension of the continuum space operators, σ and ε are the two lowest dimensional operators in the CFT description. Naturally, the Ising CFT will also contain operators corresponding to irrelevant scaling operators, having scaling dimension exceeding the spacetime dimension. Such 'irrelevant' operators do in general not have intuitive interpretations, seeing as they do not need to be tuned in order to reach the CFT; rather, in the following they will only be considered in terms of operators which appear in the OPE of the relevant operators. In fact, since the OPE must satisfy the same overall symmetries as the product of operators, the OPE for $\sigma \times \sigma$ and $\varepsilon \times \varepsilon$ will contain only $\mathbb{Z}_2$-even operators, with even spin-l, while $\sigma \times \varepsilon$ only includes $\mathbb{Z}_2$-odd operators of any spin.

Finally, a CFT must contain a stress-energy tensor, $T^{\mu\nu}$. For the Ising model it is classified as a $\mathbb{Z}_2$-even marginal operator. A further study of this operator is outside of the scope of this thesis, however.

5 Bounds for the Three-dimensional Ising Model

It is time to put all the above information to use. This section will present the bootstrapping programme specifically applied to the three-dimensional Ising model. The discussion will focus only on finding bounds for the scaling dimensions of σ and ε, disregarding the numerical values of the OPE coefficients, or specifically discussing scaling dimensions of other operators in the theory. This is a straightforward approach to the semi-definite programming problem discussed in section 3.4, in which properties of conformal blocks are used to determine if a solution to the bootstrap equation is possible for any arbitrary set of λ's. Although this does not provide the full set of data needed to solve the CFT, it still yields important information concerning physical properties of the Ising model. This is due to the connection between scaling dimensions and the renormalization group eigenvalues y_i, and in turn the critical exponents. This means that precise bounds on the Δ_i's can be used to determine values of experimentally verifiable quantities, for example the heat capacity, which are of phenomenological interest. In this sense, the RG theory at the critical point can be considered as solved when the operator dimensions are found. Furthermore, the exact solution of the two-dimensional Ising model suggests that the three-dimensional solution is extremal, meaning that determining the spectrum uniquely fixes the allowed values of the coefficients in the OPE. To this end, finding the values of Δ_σ and Δ_ε also provides information about the whole theory.

5.1 Numerical Approach

The results which will be presented in the next two section have been computed using numerical techniques specifically developed for the conformal bootstrap [32, 33]. The method for finding values of the lowest scalar $\mathbb{Z}_2$-even and odd operator dimensions will be, briefly and qualitative, presented below. Because of the nature of the approach taken here, the results are inherently approximate. It is therefore important to also consider the limitations which are a part of the procedure. For more detailed accounts of the methods discussed, see the papers of the programs and references therein.

5.1.1 Method

The results presented below regard the allowed values of the scaling dimensions of Δ_σ and Δ_ε, determined using both single-and multiple correlator methods, i.e. the scaling dimensions of the operators which are included in the original correlators which form the equation. Determining which values of Δ_σ and Δ_ε are allowed is a two step process.

In this work, the first step is handled by the Python wrapper `PyCFTBoot` [33]. The objective of this interface is to produce a file containing all of the information needed to study the bootstrap equation using a computer. To this end, given some assumed values of the scaling dimensions which want to be found, `PyCFTBoot` produces convolved conformal blocks out of

rational approximations of conformal blocks as sums over poles in the scaling dimensions of the included operators. Thereafter, it applies the derivatives associated with α, as defined in (3.15), up to some chosen order.

When all this is completed, the information carried by the semi-definite program is packaged as a polynomial matrix program and passed to the solver `SDPB` [32]. It uses a variety of techniques to determine whether or not the matrix of convolved blocks satisfies the constraints 3.20. If it does, the assumed values of the scaling dimensions are ruled out. If the constraints are not satisfied, the input values are kept as *possible* values. This procedure is iterated, scanning over a set of scaling dimensions of the operators in the correlator, until a satisfactory result is found. The values of the allowed scaling dimensions are then plotted.

5.1.2 Shortcomings

As with most numerical work, the largest source of uncertainty and error is a consequence of truncation. In the case of the numerically studying the conformal bootstrap, this appears at several levels.

The most obvious case of truncation appears in the rational approximations of the OPE. Since a computer cannot handle the infinite number of terms in the OPE, the expansion must be stopped at some stage. The truncating parameters for the OPE are l, and the number of poles in the approximation. Naturally, the larger the amount of terms which are included in the expansion, the more values need to be considered, and the more constricted the data becomes.

The same principal is true for the action of α. The number of derivatives will affect the precision of the functional. The more derivatives, the stricter the bounds get. However, the number of derivatives does not automatically yield stronger constraints. In order to meet a optimal result, an increase in derivatives should be followed by a larger expansion.

Furthermore, restrictions need to be set on the resolution of the scaling dimensions of the operators appearing in the OPE. When constructing the convolved blocks in `PyCFTBoot` it is necessary to assign a reasonable interval of allowed values of the scaling dimension for each of the operators in the OPE, except for the input values of the operators of interest. Although the other operators in the theory will have discreet scaling dimensions, they can take values from a continuous set of unknown numbers. Encoding this is not feasible. Thus, the interval of allowed scaling dimensions is discretized for each operator. This naturally involves some choice of precision which, although not as significant as the above issues, nevertheless has an effect.

5.2 Single Correlator

The simplest way of bootstrapping the lowest dimensional operators of the three-dimensional Ising model is, perhaps not surprisingly, by using crossing symmetry of a single correlator of

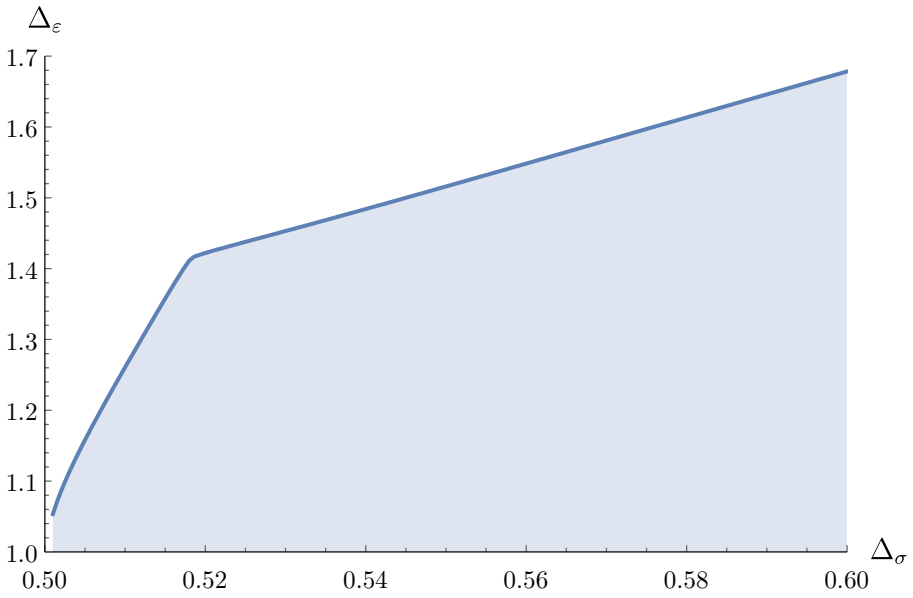

Figure 1: Curve fitted to allowed values of scaling dimension for the lowest laying $\mathbb{Z}_2$-odd and $\mathbb{Z}_2$-even operators of the three-dimensional Ising model, σ and ε, respectively. Bounds for Δ_σ and Δ_ε found assuming a unitary Ising CFT and crossing-symmetry of the four-point function $\langle\sigma\sigma\sigma\sigma\rangle$. Lower bound corresponds to the unitarity bound for three-dimensional scalars, while the upper bound is chosen arbitrarily.

identical scalars. However, the choice of which operator to include in the correlators is not arbitrary. Because the OPE of σ or ϵ with itself contains only $\mathbb{Z}_2$-even operators, the only way that Δ_σ can appear in the bootstrap equation of identical Ising operators is via the exponent of u or v in equation (3.11); hence, σ needs to be included in the correlator. Using crossing symmetry of the four-point function $\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$ yields the bootstrap equation, expressed as the sum rule,

$$\sum_{\mathcal{O}\in\sigma\times\sigma} \lambda_{\sigma\sigma\mathcal{O}} F^{\Delta_\sigma}_{\Delta_\mathcal{O},l_\mathcal{O}}(u,v) = 0, \tag{5.1}$$

where the sum runs over over all operators in the $\sigma\times\sigma$ OPE, including ε; and the convolved conformal block given by

$$F^{\Delta_\sigma}_{\Delta_\mathcal{O},l_\mathcal{O}}(u,v) = v^{\Delta_\sigma} g_{\Delta_\mathcal{O},l_\mathcal{O}}(u,v) - u^{\Delta_\sigma} g_{\Delta_\mathcal{O},l_\mathcal{O}}(v,u). \tag{5.2}$$

The above form is readily suitable for numerical study by scanning over values of $(\Delta_\sigma,\Delta_\varepsilon)$. Starting near the unitarity bound (2.115), `SDPB` checks if a solution to (5.1) is feasible for each point, using the constraints (3.14) and further assuming unitary values of the remaining $\Delta_\mathcal{O}$.

Plotting the upper limit of allowed values $(\Delta_\sigma,\Delta_\varepsilon)$ results in the blue shaded region in figure 1. The bound is clearly not very strong. The large amount of possible values of Δ_σ and Δ_ε is an artefact of the treatment of the individual intervals for each $\Delta_\mathcal{O}$ used when

approximating the conformal blocks, as well as the limited number of operators available in the OPE. Indeed, assuming gaps in the $\Delta_{\mathcal{O}}$ spectrum, in accordance with the fact that ε is the only relevant $\mathbb{Z}_2$-even operator, carves out the lower portions of the plot (ref). This possibility will not be considered at this instant, but will be returned to in the next section. Nevertheless, the most eye-catching property of the bound is the presence of a kink. The first change in behaviour happens at $(\Delta_\sigma, \Delta_\varepsilon) = (0.5181, 1.4123)$, where the value of Δ_σ has been determined by eye with a vertical and horizontal resolution of 10^{-4}. The kink happens to be in the region of values approximated by other techniques, e.g. Monte-Carlo simulations [34]. This suggests that the appearance of a kink says something about the spectrum. Indeed, if one considers tightening a region from above, keeping some arbitrary point fixed, the resulting boundary would look something similar to this, the kink appearing at the stationary point. Taking this analogy further, the kink suggests that the three-dimensional Ising model lives on the boundary of allowed CFT data. This suspicion is strengthened when considering that the analytically derived values of the scaling dimensions of two-dimensional Ising model sit exactly at the boundary of a kink [7]. If this still holds in three dimensions, finding a precise value of Δ_σ and Δ_ε fixes the spectrum of CFT data.

5.3 Mixed Correlators

Clearly, the bound achieved using a single correlator is promising. It is, however, not optimal. Most notably, it does not take into consideration the full spectrum of $\mathbb{Z}_2$-odd operators when restricting the values of Δ_σ and Δ_ε, nor do the conformal blocks contain operators of odd spin-l. This part of the operator spectrum can be reached by considering the bootstrap of four-point functions containing both σ and ε, since the OPE of $\sigma \times \epsilon$ contains only $\mathbb{Z}_2$-odd operators. The simplest way of doing this is by considering four-point functions with an equal number of σ and ε. In order to avoid the situation of only having access to the odd spectrum, it is necessary to consider crossing-symmetry of a system of correlators, including those with identical operators. At this level there are four forms of the four-point function which produce independent constraints,

$$\langle\sigma\sigma\sigma\sigma\rangle\,, \quad \langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle\,, \quad \langle\sigma\varepsilon\sigma\varepsilon\rangle \quad \text{and} \quad \langle\sigma\sigma\varepsilon\varepsilon\rangle\,. \tag{5.3}$$

Their respective bootstrap equations are

$$0 = \sum_{\mathcal{O}\,\text{even}} \lambda^2_{\sigma\sigma\mathcal{O}} F^{\sigma\sigma,\sigma\sigma}_{-,\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v), \tag{5.4}$$

$$0 = \sum_{\mathcal{O}\,\text{even}} \lambda^2_{\varepsilon\varepsilon\mathcal{O}} F^{\varepsilon\varepsilon,\varepsilon\varepsilon}_{-,\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v), \tag{5.5}$$

$$0 = \sum_{\mathcal{O}\,\text{odd}} \lambda^2_{\sigma\varepsilon\mathcal{O}} F^{\sigma\varepsilon,\sigma\varepsilon}_{-,\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v), \tag{5.6}$$

$$0 = \sum_{\mathcal{O}\,\text{even}} \lambda_{\sigma\sigma\mathcal{O}}\lambda_{\varepsilon\varepsilon\mathcal{O}} F^{\sigma\sigma,\varepsilon\varepsilon}_{\mp,\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v) \pm \sum_{\mathcal{O}\,\text{odd}} (-1)^l \lambda^2_{\sigma\varepsilon\mathcal{O}} F^{\varepsilon\sigma,\sigma\varepsilon}_{\mp,\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v), \tag{5.7}$$

which contains five separate convolved conformal blocks. The factor of $(-1)^l$ appears due to $\lambda_{\varepsilon\sigma\mathcal{O}} = (-1)^l \lambda_{\sigma\varepsilon\mathcal{O}}$, where l is the spin of $\mathcal{O}$. The system of equations can be combined into single vector equation such that the action of $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$ yields

$$\sum_{\mathcal{O}\,\text{even}} (\lambda_{\sigma\sigma\mathcal{O}}, \lambda_{\varepsilon\varepsilon\mathcal{O}})\, \vec{\alpha}\cdot\vec{V}_{+,\Delta_\mathcal{O},l_\mathcal{O}} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\varepsilon\varepsilon\mathcal{O}} \end{pmatrix} + \sum_{\mathcal{O}\,\text{odd}} \lambda^2_{\sigma\varepsilon\mathcal{O}}\, \vec{\alpha}\cdot\vec{V}_{-,\Delta_\mathcal{O},l_\mathcal{O}} = 0, \tag{5.8}$$

with

$$\vec{\alpha}\cdot\vec{V}_{+,\Delta_\mathcal{O},l_\mathcal{O}} = \begin{pmatrix} \alpha^1[F^{\sigma\sigma,\sigma\sigma}_{-,\Delta_\mathcal{O},l_\mathcal{O}}] & \frac{1}{2}\alpha^4[F^{\sigma\sigma,\varepsilon\varepsilon}_{-,\Delta_\mathcal{O},l_\mathcal{O}}] + \frac{1}{2}\alpha^5[F^{\sigma\sigma,\varepsilon\varepsilon}_{-,\Delta_\mathcal{O},l_\mathcal{O}}] \\ \frac{1}{2}\alpha^4[F^{\sigma\sigma,\varepsilon\varepsilon}_{-,\Delta_\mathcal{O},l_\mathcal{O}}] + \frac{1}{2}\alpha^5[F^{\sigma\sigma,\varepsilon\varepsilon}_{-,\Delta_\mathcal{O},l_\mathcal{O}}] & \alpha^2[F^{\sigma\sigma,\sigma\sigma}_{-,\Delta_\mathcal{O},l_\mathcal{O}}] \end{pmatrix}, \tag{5.9}$$

and

$$\vec{\alpha}\cdot\vec{V}_{-,\Delta_\mathcal{O},l_\mathcal{O}} = \alpha^3 F^{\sigma\epsilon,\sigma\epsilon}_{-,\Delta_\mathcal{O},l_\mathcal{O}} + (-1)^l \alpha^4 F^{\epsilon\sigma,\sigma\epsilon}_{-,\Delta_\mathcal{O},l_\mathcal{O}} - (-1)^l \alpha^5 F^{\epsilon\sigma,\sigma\epsilon}_{+,\Delta_\mathcal{O},l_\mathcal{O}}. \tag{5.10}$$

The numbered α^i could in principle be distinct, for example containing derivatives of different orders. Such a case will not be considered here, however.

Evidently, the the dual form of the bootstrap equation of multiple correlators contains a matrix term (5.9) and a scalar term (5.10). The unit operator will only appear in the $\mathbb{Z}_2$-even OPE, thus an invalid assumption will be signified by

$$(1,1)\, \vec{\alpha}\cdot\vec{V}_{+,0,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0, \tag{5.11}$$

$$\vec{\alpha}\cdot\vec{V}_{+,\Delta_\mathcal{O},l_\mathcal{O}} \succeq 0 \tag{5.12}$$

$$\vec{\alpha}\cdot\vec{V}_{-,\Delta_\mathcal{O},l_\mathcal{O}} \geq 0 \tag{5.13}$$

where the second constraint should hold for $\mathbb{Z}_2$-even operators with even spin-l, and the third inequality concerns $\mathbb{Z}_2$-odd operators of any spin.

Although this treatment of the bootstrap equation gives a greater access to constraints, from both of the $\mathbb{Z}_2$ sectors, it does not automatically strengthen the numerical bounds produced by a single correlator. In fact, without further input, the above system yields the same plot as in figure 1. This is not surprising, seeing that the $\mathbb{Z}_2$-odd spectrum from a numerical point of view could potentially be the same as the even spectrum, not giving any new information. The additional input required is the statement that σ and ε are the only relevant scalars of the theory, and thus $\Delta_\mathcal{O} > 3$. These constraint can be imposed separately, however constraining both sections of the spectrum yields the most stringent bound; a comparison can be found in figure 3. The region of allowed scaling dimensions using this set-up is marked by the orange island centred at the kink in figure 2.

The process of eliminating values of the scaling dimensions is more demanding in the multiple bootstrap case, and thus the number of terms in the various expansions had to be lowered to be able to run the calculations locally. The dashed upper single correlator bound

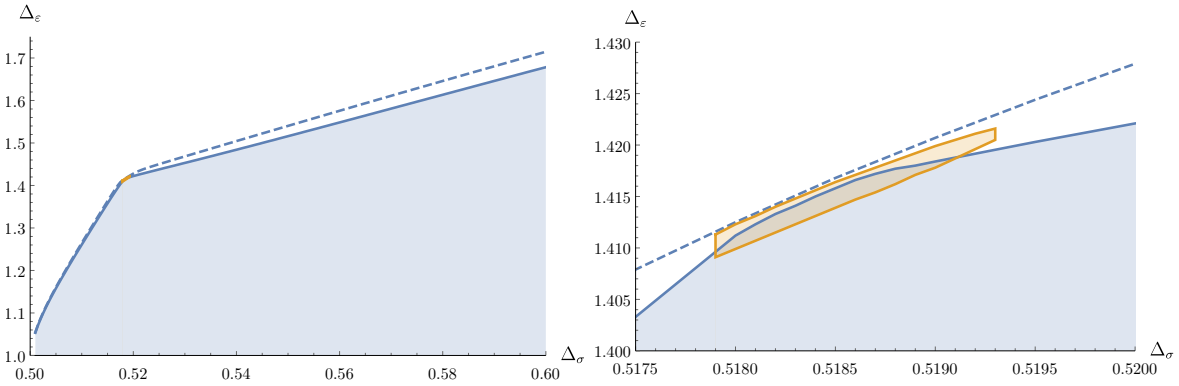

Figure 2: **Left:** Small orange region signifies values of Δ_σ and Δ_ε which may satisfy the bootstrap equation of multiple scalars (5.8). Single-correlator bounds of same order (*dashed line*) and higher order, corresponding to figure 1 (*solid line*) are included as comparisons. **Right:** Zoomed view of allowed region for system of correlators.

in these figures has the same input values as the multiple correlator, clearly illustrating the gain as a result of using a system of bootstrap equations, as well as the fact that the kink is not as pronounced at lower order expansions. Nevertheless, since the system of multiple correlators includes the correlator considered in the previous section, the values above the lower, solid blue bound should not hold at higher order. Thus, it is not unreasonable to look at the intersection between the allowed regions in the plot to get a narrower region of allowed scaling dimensions. Furthermore, the fact that the upper boundary is not affected as strongly as the lower bound regions again signals that the three-dimensional Ising model saturates the boundary.

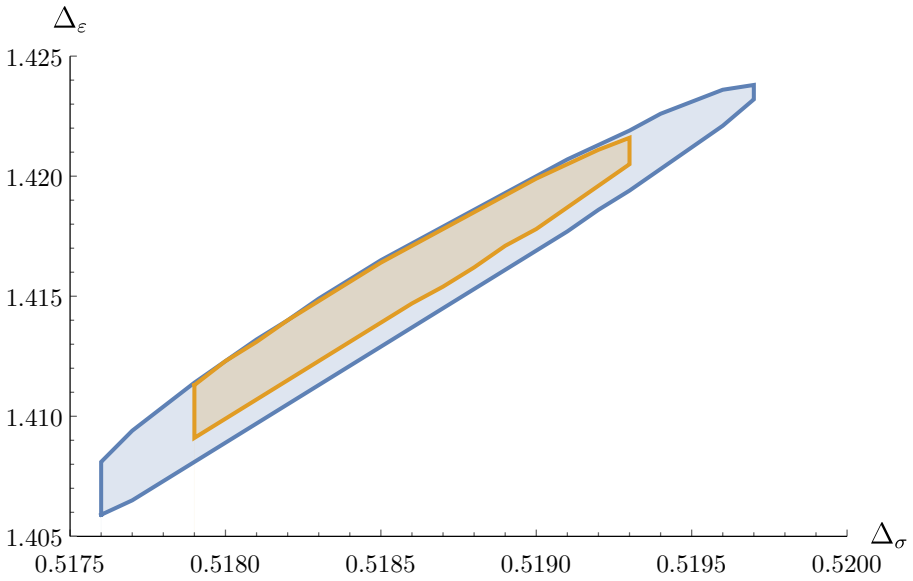

Figure 3: Mixed-correlator bounds on scaling dimensions of the two lowest dimensional operators; assuming only one relevant $\mathbb{Z}_2$-odd scalar (*blue region*) and two relevant scalars, σ and ε (*orange region*).

6 Discussion

Conformal invariance clearly provides powerful tools which can be used to better understand a theory. The bounds produced by the numerical bootstrap programme and presented in the previous section exhibit very interesting properties. It is important to remark, however, that the results have been severely limited by computational power, and should be considered as illustrations of a general behaviour. Nevertheless, these bounds do not tell the whole CFT story. Indeed, the approach taken here does not carry any information about the OPE coefficients, and only the scaling dimensions of the two lowest valued operators have been considered explicitly. Scaling dimensions of other Ising operators are treated in [8–10, 35]; moreover, [35] finds some values of OPE coefficients and also contains the strongest bounds for Δ_σ and Δ_ε achieved so far.

This thesis, as well as other work focused on the Ising model, has only considered the bootstrap constraints up to mixed correlators of σ and ε, while the full set of constraints is infinite. Arguably, in order to solve the whole three-dimensional Ising CFT, the full set of constraints needs to be tapped further. If infinite computational power is needed in order to achieve single valued bounds, or if the results stabilise at an earlier point, is up for speculation.

The application of the conformal bootstrap discussed here has clearly shed light on some

deep questions, but the impact of the programme is not limited to current topic. Analytical approaches to the bootstrap in higher dimensions than two have been considered in several instances, for example [36, 37]. Furthermore, the study of superconformal field theories through the conformal bootstrap has also proven to be useful, e.g. in the works [38–40]. Further developments are currently at the forefront of CFT research, and may be of use both phenomenologically and purely theoretically. There is still much work to be done, but the future is undoubtedly encouraging.

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