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Topological K-theory and Bott Periodicity

Matthew Magill

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Examinator: Denis Gaidashev
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Department of Mathematics
Uppsala University

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Matt Magill
Supervisor: Thomas Kragh

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Abstract

Complex topological K-theory is defined to be the Grothendieck group of stable isomorphism classes of complex vector bundles. This construction is functorial and it is shown that the functor can be represented by homotopy classes of maps into a classifying space, for which we present an explicit model. Morse theory is then introduced and used to prove Bott periodicity. The consequences of Bott periodicity for K-theory are explored, leading to the conclusion that complex topological K-theory is a generalised cohomology theory. Finally, the K-theory of the n-torus and Lens space is computed.

Contents

Chapter 1

Introduction

In this thesis, the basic concepts of topological K-theory are introduced and studied using tools from Morse theory, homotopy theory and homological algebra. It will be shown that topological K-theory is a periodic, generalised cohomology theory in which the cocycles have a geometric interpretation as equivalence classes of vector bundles. The focus is on complex topological K-theory, however the results can be applied to the real case, with some changes. In particular, the calculations corresponding to Chapter 4 are significantly more involved in the real case.

The definition of K-theory, which we see in Chapter 2 has several well-known generalisations, which contribute to the appeal of K-theory as an object of study. One such example is equivariant K-theory, in which the classes of vector bundles are replaced with classes of vector bundles *with a group action*. This machinery is used in the proof of the Atiyah-Singer Index theorem, [1], a famous result from K-theory, which describes purely analytic data - the index of an elliptic operator - in terms of purely K-theoretic data. The methods used in this thesis, however, are not applicable to equivariant K-theory.

The outline of the thesis is as follows:

In Chapter 2 the category of principal bundles over a compact, Hausdorff space is introduced and the rigidity of this category is exploited to prove a classification theorem, Theorem 2.12; the presentation broadly follows [9]. In example 2.16 the classifying space of unitary principal bundles is computed and it is observed that the same space classifies GL_n principal bundles. By relating vector bundles and GL_n principal bundles, this agreement is understood to be a consequence of the fact that any vector bundle can be endowed with a Hermitian metric, Definition 2.21.

The set of isomorphism classes of vector bundles over a compact, Hausdorff space has a canonical semigroup operation, the direct sum. Groupifying this structure yields the K-group of the space. Although this procedure is quite simple, the theory obtained has deep connections throughout mathematics. A more algebraic viewpoint of this procedure can be taken, by observing that isomorphism classes of vector bundles over a compact base space are in bijective correspondence with isomorphism classes of finitely generated, projective $C(X)$-modules. From this perspective, it is natural that an interpretation of the procedure can be applied to the study of more general rings, see e.g. [2]; we do not investigate this aspect any further.

In Section 2.5 the classification theorem 2.12 is used to show the functor K is representable. In particular, we find that the product $\mathbf{B}\mathrm{U} \times \mathbb{Z}$ is a representative of the functor, K, where $\mathbf{B}\mathrm{U} = \mathrm{colim}_n\, Gr_n(\mathbb{C}^\infty)$.

We can probe the properties of the functor, K, by exploring the homotopy type of its representative. This is done using the tools of Morse theory. This framework allows one to deduce information about the homotopy of smooth spaces from the critical points of a generic function. By putting a Riemannian structure on the space, it is possible to apply the theory to the path space of a smooth manifold, via finite dimensional approximations. The basic results in this direction are presented in Chapter 3.

These results are then applied to the Lie group $U(n)$ in Chapter 4. Using elementary results on the geometry of Lie groups (collected in Appendix A) we are able to relate the homotopy type of the loop space of this group with Grassmannian spaces. Taking colimits, we will be able to deduce a homotopy equivalence of the K-representative, $\mathbf{B}\mathrm{U} \times \mathbb{Z}$, with its own second loop space. This is the phenomena known as Bott periodicity, which is a central result of this thesis. Chapters 3 and 4 heuristically follow [8]. There are some changes from the presentation therein, firstly to correct an error in Milnor's presentation, and secondly to reach our aim more directly.

The consequences of Bott periodicity on the functor K are explored in Chapter 5. We can immediately calculate the value of K on the spheres, but more general spaces require more tools. The most important tool to understand is the relationship between the K-ring of a closed subspace with the K-ring of the ambient space. To such a pair can be naturally associated a long exact sequence of K-rings, (5.11). This exact sequence is very useful in calculating examples, but it is also theoretically important, for it can be interpreted as satisfying one of the Eilenberg-Steenrod axioms. In fact, this sequence suggests a sequence of functors, K^*, generalising K and satisfying the Eilenberg-Steenrod axioms, with the exception of the dimension axiom. Therefore, K^* defines a generalised cohomology theory. This theory is periodic, with period two, as a consequence of Bott periodicity. In particular, the ring K^i is isomorphic to K^{i+2}, for all $i \in \mathbb{Z}$. By calculating some elementary examples, we see the effect that this periodicity has on computability.

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Chapter 2

Vector Bundles and K-Theory

In this chapter a functor from the category of compact spaces to the category of commutative rings is developed. This is accomplished by investigating certain equivalence classes of vector bundles, along with natural vector bundle operations, following the presentation in [4]. Viewing K-theory in this way makes many structural elements more intuitive, but obscures the periodic property that we wish to obtain. We aim to understand this property by viewing K-groups in a homotopy-theoretic context and therefore we begin by reviewing the notion of a principal bundle and its associated vector bundles, following [9]. The rigidity of principal bundles yields a classification theorem, which is inherited by associated vector bundles. The classification will eventually yield the main result of the chapter: that the functor is representable (see (2.18)). In later chapters, this will allow the use of tools from Morse theory to investigate the properties of these functors.

2.1 Principal Bundles

In this section, G will be a topological group with identity e.

A space, X, is called a left G-space if there exists a continuous map $G \times X \to X$, written $(g, x) \mapsto gx$, such that $ex = x$ and $g(hx) = (gh)x$, for all $x \in X$, $g, h \in G$. A right G-space is defined analogously. If we have two left G-spaces X, Y a map $\phi : X \to Y$ is called G-equivariant if $\phi(gx) = g\phi(x)$ for all $g \in G$, $x \in X$.

Definition 2.1. *Let X be a topological space with trivial G-action (i.e. $xg = x$ for all $x \in X$, $g \in G$). A* principal G-bundle *over X is a surjective G-equivariant map from a right G-space E to X, $\pi : E \to X$, with a local triviality condition. In particular, there exists an open cover $\{U_i\}$ of X and G-homeomorphisms $\pi^{-1}(U_i) \to U_i \times G$; the G-action on $U_i \times G$ is $(x, g)h := (x, gh)$.*

Any map $p : E \to X$ with a local triviality condition is called a local product and defines a *fibre bundle.*

It can be observed that a principal bundle has G-valued transition functions[1] i.e. $g_{ij} : U_i \cap U_j \to G$; in a more general fibre bundle with fibre F, it is only required that $g_{ij}(x) \in \mathrm{Homeo(F)}$. It is also important that the local triviality condition ensures the group action on a principal bundle is fibrewise-transitive and free. We will sometimes call a principal bundle a G-bundle.[2]

A morphism of G-bundles with base space X is a G-equivariant map between the total spaces, which commutes with the bundle surjections. An isomorphism is a bundle morphism that is also a homeomorphism. The collection of isomorphism classes of principal G-bundles over a fixed base space, X, is a category with the bundle morphisms as maps. Denote this category by $\mathcal{P}_G(X)$. The theory of principal bundles is very rigid, as the next proposition indicates.

[1] If (U_i, ϕ_i) and (U_j, ϕ_j) are trivialising neighbourhoods with $U_i \cap U_j \neq \varnothing$, then the transition function is defined by the composition $\phi_j \circ {\phi_i}^{-1} = \mathrm{Id}_{U_i \cap U_j} \times g_{ij}$

[2] Note that this terminology is sometimes used to mean a G-vector bundle in other contexts

Proposition 2.2 ([9], Proposition 2.1)**.** *Let E and F be principal G-bundles over X, and $\phi : E \to F$ a bundle morphism. Then ϕ is an isomorphism.*

Proof. We begin by assuming the bundles are both trivial $E = F = G \times X$. Then ϕ is of the form

$$\phi(x,g) = (x, f(x,g)) = (x, f(x,e)g)\,, \tag{2.1}$$

where $f(x,e) : X \to G$ is a continuous map. Set $f(x) \coloneqq f(x,e)$. The map $g : X \to G$ defined by $g(x) = (f(x))^{-1}$ induces a continuous inverse to ϕ.

More generally, let $\{U_i, \psi_i\}$ a cover of X trivialising E, and $\{U_i, \xi_i\}$ a cover of X trivialising F. There is no loss of generality in assuming the cover of X is the same in both cases, for if this is not the case, taking a common refinement and restricting the domain of the maps gives trivialisations with a common cover.

The above argument shows that $\phi|_{{\pi_E}^{-1}(U_i)} : {\pi_E}^{-1}(U_i) \to {\pi_F}^{-1}(U_i)$ is an isomorphism for all U_i. We can conclude that ϕ is bijective, and the map obtained by gluing together the local inverses is a continuous inverse to ϕ. Hence, ϕ is an isomorphism. □

Corollary 2.3 ([9], Proposition 2.2)**.** *A principal bundle is trivial if and only if it admits of a section.*

Proof. One implication is obvious. For the other, suppose $s : X \to E$ is a section of the principal bundle $\pi : E \to X$. As the group action on E is fibrewise-transitive and free we have that every element $e \in E$ can be written $e = s(\pi(e))g$ for unique $g \in G$. Let $E \to X \times G$ be given by $s(x)g \mapsto (x,g)$. This is a G-map lifting the identity on X, so it is a bundle morphism. By the above proposition, it is therefore an isomorphism. □

For a given $f : X \to Y$ and principal bundle $\pi : E \to Y$, define the *pullback bundle*, f^*E to be the pullback, in the categorical sense, of the maps f and π, i.e. f^*E is the limit of the diagram

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\;f\;} & Y \end{array} \tag{2.2}$$

which is (uniquely isomorphic to) the subspace $\{(x,\varepsilon) : f(x) = \pi(\varepsilon)\} \subset X \times E$, with projection $(x,e) \mapsto x$ and G-action $(x,\varepsilon)g = (x,\varepsilon g)$. It is not difficult to check that this is indeed a principal bundle. The following theorem is a crucial property of pullbacks and will be used many times below:

Theorem 2.4. *Let $f,g : X \to Y$ such that f is homotopic to g, and $P \to Y$ a principal G-bundle and X, Y are paracompact. Then the pullbacks f^*P, g^*P are isomorphic.*

This proof is taken from [4] Theorem 1.6, mildly modified for principal bundles. It is based on the following lemmas:

Lemma 2.5. *Let $I = [a,b]$, $I_1 = [a,c]$ and $I_2 = [c,b]$ for $a < c < b$. A principal bundle $P \to B \times I$ is trivial if and only if the restricted bundles $P_1 \to B \times I_1$ and $P_2 \to B \times I_2$ are trivial.*

Proof Of Lemma. We need only show the condition is sufficient.

Let $h_i : P_i \to B \times I_i \times G$ any two trivialisations. If they agree on the overlap $E|_{B\times\{c\}}$ then we are done, so it must be shown that they can be made to agree. Given $h_i(e) = (\pi(e), g_i(x,t))$ one can define a new trivialisation on P_2, $\tilde{h}_2(e) = (\pi(e), g_2(x,t)(g_2(x,c))^{-1} g_1(x,c))$ which is a trivialisation of P_2 that agrees with that of P_1 on the overlap. □

Lemma 2.6. *A principal bundle $P \to B \times I$, for a closed interval I, admits a trivialising cover of the form $\{U_i \times I\}$, where $\{U_i\}$ is a cover of B.*

Proof Of Lemma. This is a simple consequence of the last lemma.

Without loss of generality, let $I = [0,1]$. There exists a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ and a covering of B, $\{U_i\}$, such that $\pi^{-1}(U_i \times [t_j, t_{j+1}])$ is trivial. By the above lemma, we can extend that trivialisation over the next interval, $U_i \times [t_j, t_{j+2}]$, or $U_i \times [t_{j-1}, t_{j+1}]$. Finiteness of the partition means that in finite steps, one has found a trivialisation of $U_i \times I$. Hence $\{U_i \times I\}$ is a trivialising cover. □

Proof of Theorem 2.4. Let $F : X \times I \to Y$ a homotopy with $F_0 = f$, $F_1 = g$. Then $F^*P \to X \times I$ is a principal bundle and $F^*P|_{X\times 0} = f^*P$, $F^*P|_{X\times 1} = g^*P$. By the above lemmas X has a cover $\{U_i\}$ such that $F^*P|_{U_i \times I}$ is trivial. The theory of paracompact spaces tells us that there is a countable, locally finite open cover $\{V_j\}_{j\geq 1}$ such that each V_j is a disjoint union of open sets, each contained in some U_i, admitting of a subordinate partition of unity, $\{\xi_j\}$. In particular, $V_j \times I$ is a countable cover of $X \times I$, such that F^*P is trivial over each $V_j \times I$. Set $\Psi_i = \sum_{j=1}^{i} \xi_i$ and X_i the graph of Ψ_i, $X_i = \{(x, \Psi_i(x))\} \subset X \times I$, for $i \geq 1$; let $\Psi_0 = 0$ so that $X_0 = X \times \{0\}$. Denote the restriction of the bundle to these graphs $F^*P_i := F^*P|_{X_i}$.Thus, $X_0 = X \times 0$ and $F^*P_0 = f^*P$. Observe that for all $i \geq 1$ we have a natural homeomorphism $s_i : X_i \to X_{i-1}$, which is the identity above $X \smallsetminus V_i$. As F^*P is trivial over $V_j \times I$ for all j, the homeomorphism s_i can be lifted to a bundle isomorphism of $h_i : P_i \to P_{i-1}$, using the trivialisation over $V_i \times I$. Form the composition $h = h_0 \circ h_1 \circ \ldots : X \times \{1\} \to X \times \{0\}$. This is well defined because the cover V_j is locally finite, so in a neighbourhood of any point only finitely many of the ξ_i are non-zero, meaning only finitely many of the h_i are not the identity. As $X_1 = \lim_{i\to\infty} X_i$, it can be concluded that h is the sought after isomorphism. □

Observe that nowhere in the above did we use anything more specific than local triviality, so it can be concluded that in fact the statements hold for general fibre bundles.

Proposition 2.7. *If E is a bundle $E \to X \times I$, with X paracompact, then $E|_{X\times\{0\}}$ is isomorphic to $E|_{X\times\{1\}}$.*

The proof essentially repeats that of Theorem 2.4. It is important to observe that there is not a *canonical* isomorphism. Indeed, the isomorphism depends on the trivialisation of the vector bundle E. However, it can be checked that for a fixed trivialisation of, say $E|_{X\times\{0\}}$, the induced isomorphism is unique up to homotopy.

2.1.1 Associated Vector Bundles

In this section we generalise from principal G-bundles to bundles with arbitrary G-space fibres, and structure group G. The idea is that a bundle associated to a principal bundle is a locally trivial map with the same transition functions as the principal bundle, $g_{ij} : U_i \cap U_j \to G$, now acting on the fibre rather than G. This is formalised below. Associated bundles are key in constructing the classifying spaces in the next section, but they will also be important to us in that we regard vector bundles as associated GL_n-bundles.

Definition 2.8. *Given a right G-space X and left G-space Y, we can define the* balanced product *to be $X \times_G Y = X \times Y/\sim$, where $(xg, y) \sim (x, gy)$. Equivalently, we have $(xg, g^{-1}y) \sim (x, y)$.*

In general, we can consider a left G-space as a right G-space (and *vice versa*) by defining right action $xg := g^{-1}x$. In this way, we can form the balanced product of two right spaces (or left spaces), where the equivalence relation above can be formulated as being the quotient of the product space $X \times Y$ by the right group action $(x, y)g := (xg, yg) = (xg, g^{-1}y)$. Two very useful examples involve the balanced product of any space with the one-point space, or with G itself:

Example 2.9. *Consider $\star$ the one-point space with trivial G-action, then the balanced product with any right G-space is $X \times_G \star = X/G$.*

Example 2.10. $X \times_G G = G \times_G X = X$.

These can both be checked directly from the definition.

Importantly, the balanced product is associative, up to unique homeomorphism. In particular, let X have a right G-action, Y have a left G-action and right H-action and Z a left H-action, for some groups G, H. Then the space $X \times_G Y$ has a right H-action, and $Y \times Z$ a left G-action, so $(X \times_G Y) \times_H Z$ and $X \times_G (Y \times_H Z)$ are well-defined. One can check they are naturally homeomorphic using some basic point-set topology.

Now suppose we have a principal G-bundle $\pi : P \to B$, and X some right G-space. Let $c : X \to \star$ the unique map to the one-point space. This is a G-map, and so it induces a map $\tilde{c} : P \times_G X \to P \times_G \star$, by $\tilde{c}([p,x]) = [p, cx] = [p, \star]$. By the above, $P \times_G \star = P/G$, and as P is a principal bundle $P/G \cong B$. We call $P \times_G X$ the fibre bundle associated to P, with fibre X and structure group G. The locally trivial structure on P induces a locally trivially structure on X as follows: for any cover of B over which P can be trivialised, say $\{(U_i, \phi_i\}$, we define $\tilde{\phi_i} : \tilde{c}^{-1}(U_i) \to U_i \times X$, by $\tilde{\phi_i}[p,x] = [\phi_i(p), x] \in (U_i \times G) \times_G X = U_i \times X$. This is well-defined, because $[p,x] \in \tilde{c}^{-1}(U_i)$ implies that $p \in \pi^{-1}(U_i)$.

It follows immediately that the fibre bundle associated to a principal bundle has the same transition functions as the principal bundle, now acting on the G-space X.

Example 2.11. *Of utmost importance for us will be the concept of vector bundles. We will work exclusively with complex vector bundles, though we could equally work over other fields at this stage. The space $\mathbb{C}^n$ is a $GL_n(\mathbb{C})$-space under the defining representation, i.e. $A \cdot v = Av$, $A \in GL_n(\mathbb{C})$, $v \in \mathbb{C}^n$ where the right hand side denotes the group action, and left hand side is matrix multiplication. A complex vector bundle of rank n is a fibre bundle with fibre $\mathbb{C}^n$, associated to a principal $GL_n(\mathbb{C})$-bundle. We will soon see that in fact complex vector bundle can equally be consider as an associate-$U(n)$ bundle.*

The fibres of an associated bundle $P \times_G \mathbb{C}^n$ have a canonical $\mathbb{C}$-linear structure on the fibres, induced by that on $\mathbb{C}^n$.

2.2 Classifying Spaces

A classifying space enables us to model the set of principal G-bundles as homotopy classes of maps. The goal of this section is to identify the classifying space of principal $U(n)$-bundles, which is achieved in Example 2.16

Proposition 2.12. *Let $P \to B$ a principal G-bundle, X a right G-space; there is a bijection $\mathrm{Hom}_G(P, X) \cong \Gamma(P \times_G X \to B)$, where Γ denotes the space of sections of the bundle.*

Proof. (This proof is taken from [9], Proposition 6.1) Let $\phi : \mathrm{Hom}_G(P,X) \to \Gamma(P \times_G X \to B)$ $\phi(f) \mapsto s_f$, where $s_f(b) = [p, f(p)]$, for any $p \in \pi^{-1}(b)$. This is well defined, because G acts freely and transitively on fibres, so if $\pi(p) = \pi(p')$, we know there is a unique g such that $p' = pg$. Then $[p', f(p')] = [pg, f(p)g] = [p, f(p)]$, where we used that f is a G-map. To see that ϕ is a bijection we use local triviality.

In the case P is trivial, $P \cong B \times G$, we have $\mathrm{Hom}_G(B \times G, X) = \mathrm{Hom}(B, X)$; similarly $\Gamma(B \times G \times_G X \to B) = \Gamma(B \times X \to B) = \mathrm{Hom}(B, X)$; $\phi = \mathrm{Id}$ in this case.

To extend to the general case, let $\{U_i\}$ a cover of B, trivialising P and therefore trivialising $P \times_G X$; let $U_{ij} := U_i \cap U_j$, P_i, P_{ij} the relevant restrictions. We have the commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_G(P,G) & \longrightarrow & \prod_i \mathrm{Hom}_G(P_i, X) & \rightrightarrows & \prod \mathrm{Hom}_G(P_{ij}, X) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \prod \phi_i} & & \downarrow{\scriptstyle \prod \phi_{ij}} \\ \Gamma(P \times_G X) & \longrightarrow & \prod \Gamma(P_i \times_G X) & \rightrightarrows & \prod \Gamma(P_{ij} \times_G X)\,; \end{array} \tag{2.3}$$

here the horizontal arrows are just restriction. As all P_i, P_{ij} are trivial, we know that the last two vertical maps are isomorphisms. Further, $\text{Hom}_G(P,G) \to \prod_i \text{Hom}_G(P_i, X)$ is the equalizer[3], and similarly for the second row. The universal property for equalizers then shows that there are unique morphisms $\text{Hom}_G(P,G) \underset{\phi}{\overset{\longleftarrow}{\longrightarrow}} \Gamma(P \times_G X)$ such that the diagram commutes. Uniqueness implies both that these maps are inverse to each other and hence bijections, so in particular, ϕ is a bijection. □

Theorem 2.13. *Let $\pi : P \to B$ a principal G-bundle with contractible total space P. Then for compact Hausdorff X, there is a bijective correspondence $[X,B] \to \mathcal{P}_G X$, given by $f \mapsto f^*P$; here $[X,B]$ denotes the set of homotopy classes of maps $X \to B$.*

Proof. **Surjectivity** Let $q : Q \to X$ be an arbitrary principal G-bundle and suppose there is some $f \in \text{Hom}_G(Q,P)$. Quotient out the G-action at both ends to obtain a map on the quotient spaces $\bar{f} : X \to B$, such that $\bar{f}q = \pi f$. By definition of the pullback, there must be a bundle morphism $Q \to \bar{f}^*P$, which must be an isomorphism by Proposition 2.2. Consequently, it suffices to show that $\text{Hom}_G(Q,P)$ is non-empty to conclude that Q is in the image of the map. Proposition 2.12 then implies that surjectivity can be deduced from the existence of a section of the fibre bundle $Q \times_G P \to X$, for arbitrary principal G-bundle, Q, which is proved in the following lemma.

Lemma 2.14. *Fibre bundles over compact, Hausdorff base space, with contractible fibre, admit a section.*

Proof Of Lemma. The lemma is clear for a trivial bundle, so we proceed by induction.

Let $\pi : E \to X$ the fibre bundle with contractible fibre, F, $\{U_i\}$ a cover of X and trivialisations $\phi_i : \pi^{-1}(U_i) \to U_i \times F$. As X is compact we can assume this cover is finite.

As the basis of induction, observe there is a section for U_1. Define $V_j = \bigcup_{i=1}^{j} U_i$ and let $E_j := E|_{V_j}$. Assuming there is a section $s : V_k \to E_k$ we will construct a section on V_{k+1} using local triviality and contractibility of the fibre.

As a first step, observe that the sets V_k and U_{k+1} form an open cover of V_{k+1}, so there is a partition of unity subordinate to this cover, say $\xi_k, \xi_{k+1} : V_{k+1} \to I$, with $\text{supp}(\xi_k) \subset V_k$, $\text{supp}(\xi_{k+1}) \subset U_{k+1}$. The section s induces $\tilde{s} : V_k \cap U_{k+1} \to F$ in the trivialisation given over U_{k+1}, and contractibility of F implies the existence of a homotopy $g : (V_k \cap U_{k+1}) \times I \to F$ such that $g_0 = f_0 \in F$ and $g_1 = \tilde{s}$. Define $s_{k+1} : V_{k+1} \to E_{k+1}$ to be

$$\sigma(x) = \begin{cases} s(x) & x \in V_k \smallsetminus (V_k \cap U_{k+1}) \\ \phi_{k+1}^{-1}(x, g(x, \xi_{k+1}(x))) & x \in V_k \cap U_{k+1} \\ \phi_{k+1}^{-1}(x, f_0) & x \in U_{k+1} \smallsetminus (V_k \cap U_{k+1}) \end{cases} . \tag{2.4}$$

Observe that the homotopy continuously interpolates over $V_k \cap U_{k+1}$ between the section s and the constant map f_0, hence the overall section is continuous. By induction, there exists a section. □

Injectivity Suppose we have $f_0^*P \cong f_1^*P$, for maps $f_0, f_1 : X \to B$. It needs to be shown that f_0 is homotopic to f_1. Let $q : Q \to X$, $Q \cong f_0^*P \cong f_1^*P$, and consider the bundle $Q \times I \to B \times I$. Set $R = (Q \times I)|_{X \times \partial I}$. The lifts of f_0, f_1 provide a morphism $R \to P$, which we want to extend to a morphism $Q \times I \to P$; by doing so, and passing to cosets, we obtain the required homotopy. Therefore, it suffices to show that the morphism $R \to P$ can be extended. By Proposition 2.12, this is equivalent to a section extension problem, which is solved in the following lemma.

[3] The equalizer of two maps $f, g : X \to Y$ is the limit of the diagram $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

Lemma 2.15. *If $E \to X \times I$ is a bundle with contractible fibres, and $s : X \times \partial I \to E|_{X\times\partial I}$ is any section, then s can be extended to a section of the whole base space $X \times I \to E$.*

Proof Of Lemma. In Lemma 2.6 we saw that a fibre bundle $Q \to X \times I$ admits of a trivialising cover of the form $\{U_i \times I\}_{i=1}^n$, where $\{U_i\}_{i=1}^n$ is an open cover of X. If Q is trivial, the lemma asserts that there is a homotopy between $s_0 : B \to F$ and $s_1 : B \to F$, which is true by contractibility of F.

The general case follows the same pattern as the previous lemma. In brief: by induction there exists a section over $V_k \times I$. A homotopy can be chosen between the maps $\sigma_0, \sigma_1 : U_{k+1} \to F$ induced by s_0, s_1 in the trivialisation of U_{k+1}. On the overlap, $(V_k \cap U_{k+1}) \times I$ we have to worry that the given section does not agree with the chosen homotopy. However, contractibility means the homotopies are themselves homotopic, so using the partitions of unity as above, we continuously pass from the section on V_k to a section on U_{k+1}, thereby defining a section on V_{k+1}. Thus, by induction, there exists an extension on the whole of $X \times I$. ☐

This completes the proof of injectivity and establishes the theorem.

☐

The base space of a contractible principal bundle is the classifying space of G and commonly denoted $\mathbf{B}G$. The total space of the bundle is often called the universal bundle and denoted EG.

Example 2.16. *The classifying space of $U(n)$ bundles will be fundamental in the rest of the chapter; essentially the same result for $O(n)$ can be obtained by replacing $\mathbb{C}$ with $\mathbb{R}$, whenever it appears in the following. This example is essentially an elaboration of Example 4.53 in [5].*

We shall show that the classifying space for $U(n)$ is the Grassmannian space of n-planes in $\mathbb{C}^\infty$, $Gr_n(\mathbb{C}^\infty)$. Here we define $\mathbb{C}^\infty = \operatorname{colim} \mathbb{C}^k$ as the colimit of the diagram

$$\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \hookrightarrow \cdots,$$

i.e. the set $\bigcup_{k=n}^\infty \mathbb{C}^k$, with the weak topology.

Let V_n be the space of orthonormal n-bases[4] in $\mathbb{C}^\infty$ and consider the natural projection map $p : V_n \to Gr_n(\mathbb{C}^\infty)$, mapping an n-basis to the n-plane spanned by it. More precisely, $Gr_n(\mathbb{C}^\infty)$, V_n are the colimits of the respective diagrams

$$V_n(\mathbb{C}^n) \hookrightarrow V_n(\mathbb{C}^{n+1}) \hookrightarrow \cdots$$
$$Gr_n(\mathbb{C}^n) \hookrightarrow Gr_n(\mathbb{C}^{n+1}) \hookrightarrow \cdots,$$

where the maps are induced by the inclusions $\mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$.

We topologize $V_n(\mathbb{C}^k)$ as a subspace of $(S^{2k-1})^{\times n}$ and induce a topology on $Gr_n(\mathbb{C}^k)$ by imposing that the projection p is a quotient map. In this way, we see that the limit surjection $p : V_n \to Gr_n(\mathbb{C}^\infty)$ is continuous. We wish to see that it is in fact a fibre bundle.

Let $P \in Gr_n(\mathbb{C}^\infty)$ some arbitrary plane, $k \in \mathbb{Z}$ big enough that $P \subset \mathbb{C}^k$ and $\pi_P : \mathbb{C}^k \to P$ be the orthogonal projection with respect to the standard hermitian metric; this induces the orthogonal projection $\pi_P : \mathbb{C}^\infty \to P$. Let $U \subset Gr_n(\mathbb{C}^\infty)$ the set containing all those n-planes, Q, such that $p_P|_Q$ is a surjection. In each finite case, this set is the quotient map image of $(S^{2k-1} \setminus S^{2k-1-2n})^{\times n} \cap V_n(\mathbb{C}^k)$ (where $k \geq n$ a priori, *and $S^{2k-1-2n}$ is the equatorial sphere orthogonal to the plane P) and hence an open neighbourhood of P in $Gr_n(\mathbb{C}^\infty)$. By continuously choosing an orthonormal basis of each plane in U we can obtain a trivialisation, $\psi_P : p^{-1}(U) \to U \times V_n(\mathbb{C}^n)$, via the coordinate vectors. We can make such a choice by first selecting an arbitrary orthonormal basis on P, denoted by $\Xi = (\xi_1, \ldots, \xi_n)$. This induces an n-basis on each $Q \in U$:*

$$(\pi_Q(\xi_1), \ldots, \pi_Q(\xi_n)),$$

[4] By an n-basis, I here mean a set of n linearly independent vectors. "Orthonormal" is with respect to the standard hermitian metric

so applying the Gram-Schimdt procedure yields an orthonormal basis of Q. This is a continuous procedure, and hence we can continuously identify $V_n(Q)$ with $V_n(\mathbb{C}^n)$. Using this, we can concretely write the trivialisation

$$\psi_P(v_1,\dots,v_n) = (p(v_1,\dots,v_n),(v_1^{\Xi},\dots,v_n^{\Xi})), \tag{2.5}$$

for v_i^{Ξ} the coordinate vector of $v_i \in Q$, corresponding to the choice of basis $GS(\pi_Q(\xi_1),\dots,\pi_Q(\xi_n))$.

Next, we observe that $V_n(\mathbb{C}^n)$ can be identified with $U(n)$ by identifying an n-basis, $\mathbf{v}$, with the unique element $g \in U(n)$ such that $\mathbf{e}g = \mathbf{v}$. Under this identification, it can be seen that the trivialisation (2.5) is a $U(n)$-map, and hence $V_n \to Gr_n(\mathbb{C}^\infty)$ is indeed a principal bundle.

It remains to show that V_n is contractible. The homotopy is constructed in two steps: the first uses the right-shift operator and the second is, in essence, a straight-line homotopy. To be precise, the linear operator $r : \mathbb{C}^\infty \to \mathbb{C}^\infty$, $r(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$ is injective, and repeating n-times gives an injection $r^n : \mathbb{C}^\infty \to 0^{\times n} \times \mathbb{C}^\infty$. Next, observe that the map $r_t := (1-t)\,\mathrm{Id} + tr^n$ is also injective for each $t \in [0,1]$. This implies that if we start with some $\mathbf{v} = (v_1,\dots,v_n) \in V_n$, then $\vec{r}_t(\mathbf{v}) := (r_t(v_1),\dots,r_t(v_n))$ is a linearly independent n-tuple for all t and hence the Gram-Schmidt procedure can be applied to obtain a homotopy $GS \circ \vec{r}_t : V_n \to V_n$. Observe that the Gram-Schmidt procedure is stationary on orthonormal bases, so that $GS \circ \vec{r}_0 = \mathrm{Id}$ and $GS \circ \vec{r}_1 = \vec{r}^n$. Thus, $\vec{r}_t$ induces a deformation retraction from V_n onto the proper subspace of n-bases with first n coordinates zero.

We can now use that $\{e_1,\dots,e_n,\vec{r}^n(\mathbf{v})\}$ is linearly independent and, further, that $\{(1-t)r^n(v_i) + te_i\}_{i=1}^n$ forms a linearly independent set for all $t \in [0,1]$. Therefore we can define $\vec{h}_t(\mathbf{v}) = GS((1-t)\vec{r}^n(\mathbf{v}) + t\mathbf{e})$, describing a homotopy between $\vec{r}^n$ and the constant map $\mathbf{e}$.

Putting these two homotopies together, we obtain a map

$$\vec{g}(\mathbf{v},t) = \begin{cases} \vec{r}_{2t}(\mathbf{v}) & t \in [0,\frac{1}{2}] \\ \vec{h}_{2t-1}(\mathbf{v}) & t \in [\frac{1}{2},1] \end{cases},$$

continuously depending on $\mathbf{v}$ and t, describing a homotopy from the identity on V_n to the constant point $\mathbf{e} = (e_1,\dots,e_n)$. This completes the proof that V_n is contractible, and we can finally conclude that $Gr_n(\mathbb{C}^\infty)$ is the classifying space of $U(n)$-bundles.

Looking carefully at this proof, it can be observed that if we replace V_n with the space W_n of *all* (not just orthonormal) n-bases, then the only substantive change in the above would be to omit the Gram-Schmidt process wherever it appears, implying the fibres of the resulting bundle are $W_n(\mathbb{C}^n) \cong GL_n(\mathbb{C})$. It follows that $\mathbf{B}GL_n = \mathbf{B}\mathrm{U}(n) = Gr_n(\mathbb{C}^\infty)$.

2.3 Vector Bundles

We earlier defined a complex vector bundle as a bundle with fibre $\mathbb{C}^n$, associated to a principal $GL_n(\mathbb{C})$-bundle, where $GL_n(\mathbb{C})$ acts in the defining representation (substituting "real" for "complex" everywhere leads to a real vector bundle). [5] It is important to remark that with this definition all fibres have the same dimension. We call the dimension of this fibre the rank of the vector bundle. Local triviality guarantees that the rank is locally constant.

A vector bundle morphism is a pair of maps (F, f), where f is a map between base spaces, F lifts f (i.e. $p_2F = fp_1$) and F is a linear map on fibres. An isomorphism of vector bundles over a fixed space is a continuous lift of the identity, which is a linear isomorphism on fibres. [6] For a given base space, X, we define the category $\mathrm{Vect}^n(X)$ with objects being isomorphism classes of vector bundles over X, and morphisms are vector bundle morphisms. Taking $\tilde{V}ect(X) = \bigcup_{n=0}^\infty \mathrm{Vect}^n(X)$, with all bundle morphisms, we get a category of constant rank vector bundles over X.

[5] This is not the common way to define vector bundles, but it can be checked that the category of isomorphism classes are equivalent, up to the issue of locally constant rank.

[6] In fact, an isomorphism is usually defined to be a fibrewise-linear homoeomorphism, lifting the identity, however the definition given here is in fact equivalent, see [4], Lemma 1.1.

In contrast to the category of principal bundles, these categories are not disconnected, i.e. there can be morphisms between non-isomorphic bundles. Therefore, this category is more complicated, however there is a bijection between the objects of $\mathrm{Vect}^n(X)$ and the objects of $\mathcal{P}_{GL_n(\mathbb{C})}X$, i.e. if two vector bundles are isomorphic $P \times_G \mathbb{C}^n \cong Q \times_G \mathbb{C}^n$, then the principal bundles are isomorphic, $P \cong Q$. This is not difficult to see, so further details are omitted.

It follows that there is a set bijection $\mathrm{Vect}^n(X) \cong [X, \mathbf{B}\mathrm{U}(n)]$, given by [7]

$$[f] \mapsto f^* V_n \mapsto f^* V_n \times_{U(n)} \mathbb{C}^n \cong f^*(V_n \times_{U(n)} \mathbb{C}^n) \,. \tag{2.6}$$

It can be recognised that $V_n \times_{U(n)} \mathbb{C}^n$ is isomorphic to the tautological bundle $EU(n) \to Gr_n(\mathbb{C}^\infty)$.[8] $EU(n)$ has the subspace topology, $EU(n) \subset Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty$, and the isomorphism is given by $\Psi : V_n \times_{U(n)} \mathbb{C}^n \to EU(n)$

$$\Psi[(v_1, \ldots, v_n), (z_1, \ldots, z_n)] = z_1 v_1 + \ldots + z_n v_n \,. \tag{2.7}$$

This continuous map lifts the identity $\mathrm{Id}_{Gr_n(\mathbb{C}^\infty)}$ and is a fibrewise-linear bijection, so is indeed a bundle isomorphism.

We also have the following analogue of 2.3 for vector bundles, which is a consequence of the bijection (2.6).

Corollary 2.17. *Define an n-frame on a vector bundle to be an n-tuple of sections that are fibrewise linearly independent. A vector bundle of rank n is trivial if and only if it admits an n-frame.*

Proof. It is clear that a trivial vector bundle of rank n has an n-frame, so we focus on sufficiency of the condition. We can mirror aspects of the proof of Corollary 2.3 to obtain the result.

In particular, let $\sigma_1, \ldots, \sigma_n$ form an n-frame of the rank n vector bundle $\pi : E \to X$. By definition, any element $v \in \pi^{-1}(x)$, for any $x \in X$, can be expressed as a unique linear combination of the basis $s_1(x), \ldots, s_n(x)$, say $v = \sum_i v_i s_i(x)$, with $v_i \in \mathbb{C}$. We can therefore define a map to the trivial bundle $\Phi : E \to X \times \mathbb{C}^n$ by $\Phi(v) = (\pi(v), (v_1, \ldots, v_n))$. Φ is a continuous lift of the identity, which is a linear isomorphism on the fibres and is therefore an isomorphism. □

Corollary 2.17 can also be understood as a direct consequence of Corollary 2.3, by observing that an n-frame induces a section of the associated principal bundle. Indeed, let $E \to X$ a vector bundle, and $s_1, \ldots, s_n : X \to E$ an n-frame. By the above, we know that $E \cong f^* EU(n)$, for some $f : X \to Gr_n(\mathbb{C}^\infty)$ so there is an induced n-frame on $f^* EU(n)$, denoted by the same names. The pullback comes equipped with a bundle morphism $\tilde{f} : f^* EU(n) \to EU(n)$, lifting f and which is a linear isomorphism on fibres, so there is an induced n-frame $\tilde{f}(s_i) : X \to EU(n)$. Applying the Gram-Schmidt process to the frame, we obtain a section $\tilde{\sigma} : X \to V_n$, satisfying $\pi \circ \tilde{\sigma} = f$.

Therefore, defining $\sigma : X \to f^* W_n = \{(x, (v)) \in X \times V_n \, : \, f(x) = \pi(v)\}$ by $\sigma(x) = (x, \tilde{\sigma}(x))$ there is an induced section on the principal bundle $f^* V_n$, which implies that it is trivial. Finally, using 2.6, we can conclude that $E \cong f^* V_n \times_{GL_n} \mathbb{C}^n \cong X \times \mathbb{C}^n$, which is the result.

Given a vector bundle $\pi : E \to X$ of rank n, a *subbundle* is a subspace $S \subset E$ such that:

- S intersects every fibre of E and the intersection is a vector subspace;
- the restriction $\pi|_S : S \to X$ is a vector bundle of rank $k \le n$.

A consequence of Corollary 2.17 is that a global k-frame exists in a bundle of rank $n \ge k$ if and only if there is a trivial rank k subbundle.

[7] The isomorphism $f^* V_n \times_{U(n)} \mathbb{C}^n \cong f^*(V_n \times_{U(n)} \mathbb{C}^n)$ is generally true of associated bundles, and can be checked by observing that the transition functions agree.

[8] Note that $EU(n)$ denotes the associated fibre bundle, which has elements in a fibre over a plane being vectors contained in the plane. It is not the universal $U(n)$ bundle

2.3.1 Operations on Vector Bundles

We can think of a vector bundle as a family of vector spaces continuously parametrised by the base space, suggesting that vector space operations, such as direct sum and tensor product, can be carried over to vector bundles. We turn now to these definitions.

Definition 2.18. *Given two bundles E, E' the product space is in fact a vector bundle $p \times p' : E \times E' \to B \times B'$. The fibre over a point (b, b') is $p^{-1}(b) \oplus p'^{-1}(b')$, and a trivialising cover is given by the products of a trivialising cover of E and E'.*

Definition 2.19. *Given two bundles E, E' over the same base space, B, we can take the pullback of the product bundle by the diagonal map $\Delta : B \to B \times B$ yielding the direct sum bundle $E \oplus E' \to B$.*

This definition agrees with what we expect of sum of bundles in that $\Delta^*(E \times E') = \{(b, e, e') : p(e) = p'(e') = b\}$, i.e. the fibres are exactly the direct sum $\pi^{-1}(b) = p^{-1}(b) \oplus p'^{-1}(b)$. The direct sum of vector bundles is, up to isomorphism, commutative $E \oplus E' \cong E' \oplus E$, and associative $E \oplus (E' \oplus E'') \cong (E \oplus E') \oplus E''$.

We define a tensor product of two bundles in a more direct, constructive manner.

Definition 2.20. *Given two bundles E, E' over the same base, consider the set $E \otimes E' := \coprod_b p^{-1}(b) \otimes p'^{-1}(b)$, which comes equipped with a surjection $\pi : E \otimes E' \to B$,. Choose trivialisations of E, E' over a common cover, $\{(U_i, \phi_i)\}$, $\{(U_i, \psi_i)\}$, and define maps $\phi_i \otimes \phi_i' : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^{nn'}$. Declaring these maps to be homeomorphisms defines a topology on $E \otimes E'$. With this topology, π is continuous, as can be checked.*

One point that was glossed over in the above construction is whether the resulting bundle depends on the chosen trivialisation. If this were the case, the tensor product would not be well-defined on isomorphism classes, which would be a problem. However, given two trivialisations of some vector bundle, which can be assumed to trivialise over the same cover, the composition: $\psi_i \circ \phi_i{}^{-1}(b, v) = (v, g(b)v)$, is a homeomorphism. It follows that the topology on $E \otimes E'$ is unaffected by the choice of trivialisation.

An important observation is that the direct sum and tensor product commutes with pulling back, i.e.

$$f^*(A \oplus B) \cong f^*A \oplus f^*B\,;$$
$$f^*(A \otimes B) \cong f^*A \otimes f^*B\,,$$

which can be checked directly from the definitions.

Furthermore, the tensor product is commutative, and distributes over the direct sum (up to isomorphism). These facts will be important in the following, and can be checked by simply writing out the definitions.

It will be of great importance later in the thesis to know that any bundle over a compact base space, E, is a subbundle of some trivial bundle. In fact, there exists a bundle E' such that $E \oplus E' \cong \Theta^n$, for some trivial bundle of rank n, Θ^n. This requires a compact base space. A tool we will use to prove this fact is a *bundle metric*; this concept also helps clarify the relation between $U(n)$ bundles and GL_n bundles.

Definition 2.21. *A metric bundle is a vector bundle, equipped with a continuously varying, fibrewise hermitian metric. That is, $\langle\cdot,\cdot\rangle : E \otimes E \to \mathbb{C}$, $v \otimes w \mapsto \langle v, w\rangle$, where $\langle\cdot,\cdot\rangle|_{fibre}$ is an Hermitian metric. $\langle\cdot,\cdot\rangle$ is called a bundle metric.*

Given a metric bundle with $F \subset E$ a subbundle, the orthogonal complement, $F^\perp \subset E$, is the subspace of E consisting of the orthogonal complements of the fibres of F. The restriction of π to $F^\perp$ can be checked to be locally trivial, and so $F^\perp$ is a vector bundle such that $F \oplus F^\perp = E$.

We will show that every bundle $E \to B$ can be endowed with a bundle metric, provided there exists a locally finite partition of unity subordinate to an open cover of B trivialising E. We will in fact restrict attention to compact base spaces, but the generalisation is straightforward.

On any trivial bundle, there is a bundle metric induced by the standard metric on $\mathbb{C}^n$. In the general case, suppose $\{(U_i, \phi_i)\}$ a trivialising cover of the bundle $p : E \to B$, and $\{\xi_i\}$ a partition of unity subordinate to $\{U_i\}$. Define a metric on each $E|_{U_i}$, say $\langle\cdot,\cdot\rangle_i$ and let $\langle\cdot,\cdot\rangle : E \otimes E \to \mathbb{C}$ given by $\langle\cdot,\cdot\rangle := \sum_i (\xi_i \circ p)\langle\cdot,\cdot\rangle_i$. This is hermitian, non-degenerate and positive-definite, and hence a metric.

Therefore, given any bundle E and subbundle $F \subset E$, there is a metric on E defining a subbundle $F^\perp$ such that $E \cong F \oplus F^\perp$. The next step is to see that the bundle $F^\perp$ is independent of the choice of metric. Indeed, given two metrics on some bundle E, $\langle\cdot,\cdot\rangle_{0,1}$ the convex combination $t\langle\cdot,\cdot\rangle_0 + (1-t)\langle\cdot,\cdot\rangle_1$ is a metric for any $t \in [0,1]$, i.e. the space of bundle metrics is convex. This combination defines a homotopy of metrics $g_t : E \otimes E \to \mathbb{C}$ such that $g_0 = \langle\cdot,\cdot\rangle_0$ and $g_1 = \langle\cdot,\cdot\rangle_1$. Consider the bundle $E \times I \to B \times I$, with bundle metric $\langle\cdot,\cdot\rangle(t) = g_t(\cdot,\cdot)$. Then, the subbundle $F \times I$ yields an orthogonal complement $\tilde{F}^\perp$, where $\tilde{F}^\perp|_0$ is the complement in E with respect to the $\langle\cdot,\cdot\rangle_0$ metric, and $\tilde{F}^\perp|_1$ is the complement with respect to the other given metric. Proposition 2.7 then shows that $\tilde{F}_0^\perp$ and $\tilde{F}_1^\perp$ are isomorphic.

As a consequence, for any subbundle of a trivial bundle, $E \subset \Theta^N$ there exists a vector bundle $E' \cong E^\perp$ such that $E \oplus E' \cong \Theta^N$. The next step is to show that any vector bundle on a compact space is in fact a subbundle of a trivial bundle, of rank say N. The key to this observation is that a subbundle inclusion $E \subset \Theta^N$ is equivalent to the existence of a fibrewise-linear and -injective map $g : E \to \mathbb{R}^N$ ([4], Proposition 1.4) . For, if we have an inclusion $E \hookrightarrow B \times \mathbb{R}^N$, then such a map is induced by projecting the inclusion onto $\mathbb{R}^N$. On the other hand, a fibrewise-linear and -injective g defines the required inclusion, via (p, g), where p is the bundle projection.

So, letting $p : E \to B$ a vector bundle over compact B, and $\{(U_i, \tilde{\phi})_i\}_{i=1}^m$ an arbitrary, finite trivialising cover of B, set ϕ_i the projection of $\tilde{\phi}_i$ onto the fibre. Let ξ_i a partition of unity subordinate to U_i[9] and define $g : E \to \mathbb{R}^{nm}$ by $g(e) = (\xi_1(p(e))\phi_1(e), \ldots, \xi_m(p(e))\phi_m(e))$. Observe that $\phi_i(e)$ need only be defined above the support of ξ_i, so this well defined. Direct calculation shows this to be fibrewise-linear and injective. Therefore, we have proved:

Proposition 2.22. *For any vector bundle over compact base space, $E \to B$, there exists some vector bundle $E' \to B$ such that $E \oplus E'$ is isomorphic to a trivial bundle of some finite rank, N.*

2.4 K-Group

We have seen that the set of vector bundles has two natural operations, the direct sum and the tensor product. A key strategy in algebraic topology has been to assign invariant algebraic structures, such as a group or ring, to a topological space. This is the key idea behind homotopy and homology groups. Given this historical background, it seems natural to attempt to use the algebraic structure on vector bundles to yield invariants of the base space. The existence of the classifying space $\mathbf{BU}(n)$ indicates that this will yield homotopy invariants of the base space. In this section, the natural operations on vector bundles are used to associate a ring to a compact, topological space in a functorial manner. From now on, it is not assumed that bundles have constant rank, and let $\mathrm{Vect}(X)$ the category of all vector bundles over X with this definition.

The direct sum is a commutative, associative operation on $\mathrm{Vect}(X)$ with identity the (unique) rank 0-bundle. Therefore, $\mathrm{Vect}(X)$ is a commutative monoid.[10] For such objects there exists a group satisfying a universal property, the groupification see e.g. [15].

Indeed, let $(M, +; 0)$ a commutative monoid; there exists an Abelian group G_M, and map $\iota : M \hookrightarrow G$ such that:

1. ι is a monoid morphism;

[9] We assume for notational simplicity that E has constant rank, say n

[10] Also known as an Abelian semigroup

2. For any group G and morphism $\jmath : M \to G$, there exists a unique group morphism $\phi : G_M \to G$ such that $\jmath = \phi \circ \iota$.

As is usual with objects satisfying a universal property, the group G_M is only defined up to unique isomorphism. It can be shown that such a G_M exists by explicit construction:

$$G_M := \{(m,n) \in M \times M\} / \sim , \tag{2.8}$$

where

$$(m,n) \sim (m',n') \iff m + n' + a = m' + n + a , \tag{2.9}$$

for some $a \in M$. It can be checked that $\sim$ is indeed an equivalence relation. Let $[m,n]$ denote the equivalence class represented by (m,n).

Endow the set G_M with the operation $[m,n] + [m',n'] = [m + m', n + n']$, which inherits commutativity and associativity from the monoid operation. The class $[m,m]$ acts as an identity, because $[a,b] + [m,m] = [a+m, b+m] = [a,b]$, and the class $[n,m]$ is an inverse to $[m,n]$, as $[m,n] + [n,m] = [m+n, m+n]$. Therefore G_M is indeed a group. The inclusion $\iota(m) \mapsto [m,0]$ is a monoid morphism satisfying the universal property. A more suggestive notion for elements of G_M is $[m,n] =: m - n$, and henceforth this symbolism is used.

Now apply this formalism to the semigroup $\mathrm{Vect}(X)$, so that for any compact X, we have the group[11] $K(X) = \{E - E' : E, E' \in \mathrm{Vect}(X)\}$. By definition, $E - E' = F - F'$ if $E \oplus F' \oplus A \cong E' \oplus F \oplus A$ for some vector bundle A. In particular, suppose that $E = F$ in $K(X)$, so that $E \oplus A \cong F \oplus A$ for some vector bundle A; let A' such that $A \oplus A' \cong \Theta^N$, which exists by compactness of X, then adding A' to both sides it follows that $E \oplus \Theta^N \cong F \oplus \Theta^N$. In that case, we say that E and F are *stably isomorphic*, and we write $E \cong_s F$. Thus, we can view $K(X)$ as generated by stable isomorphism classes of vector bundles and their (formal) inverses. A strength of this viewpoint is that stable isomorphism classes satisfy the cancellation property: $A \oplus B \cong_s A' \oplus B$ implies $A \cong_s A'$. This again follows from the existence of B' such that $B \oplus B'$ is trivial:

$$\begin{aligned} & A \oplus B \oplus B' \oplus \Theta^n \cong A' \oplus B \oplus B' \oplus \Theta^n \\ \Longrightarrow \quad & A \oplus \Theta^{n+N} \cong A' \oplus \Theta^{n+N} , \end{aligned}$$

so that $A \cong_s A'$. Thus, for stable isomorphism classes $[E] - [E'] = [F] - [F']$ in $K(X)$ implies $[E \oplus F'] \cong_s [E' \oplus F]$, which is conceptually simpler than (2.9). Of course, the only difference here is notational. In future, we denote elements of $K(X)$ simply by $E - E'$, meaning $[E] - [E']$ in an abuse of notation.

Using a similar trick to above, we can observe that if $E - E' \in K(X)$, and E'' is a bundle such that $E' + E'' \cong \Theta^N$, then $E - E' = E - (E' + E'') + E''$ in $K(X)$ and $E - (E' + E'') + E'' \cong E + E'' - \Theta^N$ and so $E - E' = E + E'' - \Theta^N$ in $K(X)$. It follows that $K(X) = \{E - \Theta^n : E \in \mathrm{Vect}(X)\}$. We call the elements of $K(X)$ virtual bundles, and the group itself will be called the K-group of X.

Recall that any map between compact topological spaces $f : X \to Y$ induces a monoid morphism $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$. Composing with the canonical injection $\iota : \mathrm{Vect}(X) \to K(X)$, we have a monoid morphism $\mathrm{Vect}(Y) \to K(X)$, so by the universal property for the K groups, we get an induced group morphism $K(f) : K(Y) \to K(X)$ - we usually denote this map simply by f^*. The fact that $(f \circ g)^* = g^* \circ f^*$ and $\mathrm{Id}_X^* = \mathrm{Id}_{K(X)}$ implies that K is a contravariant functor from the category of compact, Hausdorff spaces to Abelian groups.

In fact, $K(X)$ is a ring, as the tensor product of vector bundles induces a multiplication on virtual bundles

$$(E - E')(F - F') := (E \otimes F + E' \otimes F') - (E \otimes F' + E' \otimes F). \tag{2.10}$$

[11] Warning: in the literature, the letter K is frequently used to denote the full sequence of functors describing the cohomology theory that we describe in Chapter 5. K, as we describe it here, corresponds only to the degree zero group K^0.

To check this is well-defined assume that $E - E' = F - F' \in K(X)$, i.e. there is some n such that $E \oplus F' \oplus \Theta^n \cong E' \oplus F \oplus \Theta^n$ and suppose $D - D' \in K(X)$. The multiplication then gives

$$\begin{aligned}(D-D')(E-E') &= (D\otimes E + D'\otimes E') - (D'\otimes E + D\otimes E')\\ &= (D\otimes(E+F'+\Theta^n) + D'\otimes(E'+F+\Theta^n)) - (D'\otimes(E+F+\Theta^n) + D\otimes(E'+F'+\Theta^n))\\ &= D\otimes(F+E'+\Theta^n) + D'\otimes(F'+E+\Theta^n) - (D'\otimes(E+F+\Theta^n) + D\otimes(E'+F'+\Theta^n))\\ &= D\otimes F + D'\otimes F' - D\otimes F' - D'\otimes F\\ &= (D-D')(F-F'),\end{aligned}$$

implying it is indeed well-defined. This product inherits associativity and distributes over the addition operation, so that $K(X)$ is a ring, with unit given by $[\Theta^1]$.

It can be remarked that for any $x \in K(X)$, multiplying with the class Θ^n is the same as taking the direct sum $nx = x + \ldots + x$, for $n \in \mathbb{N}$. This is true for $n = 0, 1$ and induction yields the general case, for if it is true for $n = k$, then

$$\begin{aligned}\Theta^{k+1}A &= (\Theta^k + \Theta^1)A\\ &= \Theta^k A + A\\ &= \underbrace{A + \ldots A}_{k} + A,\end{aligned}$$

so it is true for $k+1$. Setting $\Theta^{-n} := -\Theta^n$ allows us to generalise to $n \in \mathbb{Z}$.

Finally, one can see that the morphism $K(f)$ induced from some map $f : X \to Y$ is in fact a ring morphism. Indeed, $f^*(E \otimes E') \cong f^*E \otimes f^*E'$ implies

$$\begin{aligned}f^*((E-E')(F-F')) &= f^*\left((E\otimes F + E'\otimes F') - (E\otimes F' + E'\otimes F)\right)\\ &= f^*E\otimes f^*F + f^*E'\otimes f^*F' - (f^*E\otimes f^*F' + f^*E'\otimes f^*F)\\ &= (f^*E - f^*E')(f^*F - f^*F').\end{aligned}$$

A particularly important map to consider is an inclusion $\iota : A \hookrightarrow X$, for $A \subset X$, closed. I call such an (A, X) a compact pair, following [10]. The inclusion induces a restriction morphism $\iota^* : K(X) \to K(A)$

$$E - \Theta^n(X) \mapsto E|_A - \Theta^n(A). \tag{2.11}$$

This will be important in Chapter 5; for now, we restrict our attention to the case $A = \{x_0\}$, for some $x_0 \in X$. Writing $K(x_0) := K(\{x_0\})$, observe that $K(x_0) = \{\Theta^n - \Theta^m : n, m \in \mathbb{Z}\} \cong \mathbb{Z}$, and the kernel of the restriction is $\ker \iota^* = \{E - \Theta^n : n \in \mathbb{N}, \mathrm{rank}(E|_{x_0}) = n\}$.

Suppose that $E - \Theta^n = E' - \Theta^{n'}$, where both $E - \Theta^n, E' - \Theta^{n'} \in \ker \iota^*$. By definition, this means that $E \oplus \Theta^{n+m} \cong E' \oplus \Theta^{n'+m}$ for some integer m and therefore we can identify $\ker \iota^*$ as the set of equivalence classes of vector bundles under the equivalence relation: $E \sim_s E'$ if and only if $E \oplus \Theta^n \cong E' \oplus \Theta^m$. In this sense, we can look at this kernel as being vector bundles with all trivial factors quotiented out, leaving only non-trivial factors. Denote the kernel by $\ker \iota^* =: \widetilde{K}(X)$ and call it the reduced K-group. Observe that as a kernel of a ring morphism, $\widetilde{K}(X)$ is an ideal in $K(X)$ and hence a commutative ring, though not usually with unit. The tensor product in $\widetilde{K}$ behaves as one would expect; let $E - n, E' - n' \in \ker(\iota^*)$:

$$\begin{aligned}[E]\otimes[E'] &= (E-n)(E'-n')\\ &= EE' + nn' - En' - nE'\\ &= [EE' + nn']\\ &= [E\otimes E']\end{aligned}$$

It is important to note that $\widetilde{K}$ depends on the chosen basepoint x_0, being the kernel of the inclusion $x_0 \hookrightarrow X$. Indeed, a representative $E - \Theta^n \in \ker(x_0 \hookrightarrow X)$ need not be in the kernel

$\ker(y_0 \hookrightarrow X)$, if x_0 and y_0 are in different connected components. Thus, identifying $\widetilde{K}(X)$ with the $\sim_s$ equivalence classes gives an invariant model of the reduced group, which will be helpful.

Given a basepoint preserving map, $f : (X, x_0) \to (Y, y_0)$, the pullback $K(f)$ maps the kernel of $y_0 \hookrightarrow Y$ into the kernel of $x_0 \hookrightarrow X$, so that there is a well-defined map $\widetilde{K}(f) = K(f)|_{\widetilde{K}(Y)}$. Functoriality of K thereby induces functoriality of $\widetilde{K}$ from the category of pointed compact spaces with basepoint preserving maps.

2.5 Homotopic formulation of the K-Groups

We earlier used the machinery of principal bundles to classify vector spaces of constant rank, via $\mathrm{Vect}^n(X) = [X, \mathbf{B}\mathrm{U}(n)]$. This will be used to find a representative for the functors K, $\widetilde{K}$.

It turns out that it is convenient to begin looking at the reduced K-group, initially assuming X connected. As the objects of $\widetilde{K}(X)$ can be identified as $\sim_s$ equivalence classes of vector bundles, define functions $\iota_{n,m} : \mathrm{Vect}^n(X) \to \mathrm{Vect}^{n+m}(X)$, for each $n, m \in \mathbb{N}$ by $\iota_{n,m}E := E \oplus \Theta^m$. Observe that $\iota_{n+m,k}\iota_{n,m} = \iota_{n,m+k}$, and so all the maps can be obtained by repeatedly composing $\iota_{n,n+1} =: \iota_n$, giving the diagram

$$\mathcal{D} : \quad \mathrm{Vect}^0(X) \xrightarrow{\iota_0} \mathrm{Vect}^1(X) \xrightarrow{\iota_1} \mathrm{Vect}^2(X) \xrightarrow{\iota_2} \dots \xrightarrow{\iota_{n-1}} \mathrm{Vect}^n(X) \xrightarrow{\iota_n} \dots$$

This is a directed system in the category of sets, so we can take the direct limit in the standard way (e.g. [15] p.42) yielding $\operatorname{colim}\mathcal{D} = \coprod_n Vect^n(X)/\sim$, where $E_i \sim E_j$ if there is some k such that $\iota_{i,k-i}E_i \cong \iota_{j,k-j}E_j$. This is precisely the set $\widetilde{K}(X)$, so rewriting the diagram $\mathcal{D}$ in terms of the homotopy description of $\mathrm{Vect}^n(X)$ will lead to the representative of $\widetilde{K}$. The first step in this program is to identify the analogues of the ι_n maps.

Specifically, we look for maps $\jmath_n : [X, \mathbf{B}\mathrm{U}(n)] \to [X, \mathbf{B}\mathrm{U}(n+1)]$, such that each of the squares

$$\begin{array}{ccc} [X, \mathbf{B}\mathrm{U}(n)] & \xrightarrow{\jmath_n} & [X, \mathbf{B}\mathrm{U}(n+1)] \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathrm{Vect}^n(X) & \xrightarrow{\iota_n} & \mathrm{Vect}^{n+1}(X) \end{array} \tag{2.12}$$

commute.

An element in $\mathbf{B}\mathrm{U}(n)$ can be identified as an n-plane in some $\mathbb{C}^k$, identified with its image under all the embeddings $\mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1} \hookrightarrow \dots$. This suggests the map $\mathbf{B}\mathrm{U}(n) \to \mathbf{B}\mathrm{U}(n+1)$, $P \mapsto \mathbb{C} \oplus P \subset \mathbb{C} \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$. To see this is well-defined, observe that it is induced by the inclusions $Gr_n(\mathbb{C}^k) \hookrightarrow Gr_{n+1}(\mathbb{C}^{k+1})$, given by $P \mapsto \mathbb{C} \oplus P \subset \mathbb{C} \oplus \mathbb{C}^k$. Indeed, we can compose these maps with the colimit morphisms $Gr_{n+1}(\mathbb{C}^{k+1}) \to Gr_{n+1}(\mathbb{C}^\infty)$, yielding $Gr_n(\mathbb{C}^k) \to \mathbf{B}\mathrm{U}(n+1)$. Then, because the following diagram commutes up to homotopy:

$$\begin{array}{ccccccccc} Gr_n(\mathbb{C}^n) & \hookrightarrow & Gr_n(\mathbb{C}^{n+1}) & \hookrightarrow & \dots & \hookrightarrow & Gr_n\mathbb{C}^{n+k} & \hookrightarrow & \dots \\ & \searrow & & \searrow & & & & \searrow & \\ & & Gr_{n+1}(\mathbb{C}^{n+1}) & \hookrightarrow & Gr_{n+1}(\mathbb{C}^{n+2}) & \hookrightarrow \dots \hookrightarrow & & Gr_{n+1}\mathbb{C}^{n+k+1} & \hookrightarrow \dots \end{array} \tag{2.13}$$

the universal property of the colimit $\mathbf{B}\mathrm{U}(n)$ induces a map

$$\chi_n : \mathbf{B}\mathrm{U}(n) \to \mathbf{B}\mathrm{U}(n+1)\,. \tag{2.14}$$

By construction, $\chi_n(P) = \mathbb{C} \oplus P$. Define $\jmath_n := \chi_n \circ - : [X, \mathbf{B}\mathrm{U}(n)] \to [X, \mathbf{B}\mathrm{U}(n+1)]$. It remains to show that with this definition, (2.13) commutes.

Suppose $\tilde{f} : X \to \mathbf{B}\mathrm{U}(n+1)$ is in the image of $\jmath_n$, then we can write $\tilde{f} = K \oplus f := (K \times f) \circ \Delta$, where $K : X \to \mathbf{B}\mathrm{U}(1)$ is constant and $\Delta : X \to X \times X$ is the diagonal map. In general, the composition $(f \times g) \circ \Delta =: f \oplus g$, where $f : X \to BU(n)$ and $g : X \to BU(m)$, defines a map

$f \oplus g : X \to \mathbf{B}\mathrm{U}(n+m)$. Comparing with the definition for the direct sum of vector bundles implies the isomorphism $(f \oplus g)^* EU(n+m) \cong f^* EU(n) \oplus g^* EU(m)$, and so in particular

$$\tilde{f}^* EU(n+1) \cong f^* EU(n) \oplus K^* EU(1) \cong f^* EU(n) \oplus \Theta^1 , \tag{2.15}$$

where it was used that the pullback of a constant map is trivial.

It follows that the square (2.12) commutes, yielding a bijective map between the colimit of the diagram $\mathcal{D}$ and the colimit of the diagram

$$\mathcal{D}' : \qquad \ldots \xrightarrow{\jmath_{n-1}} [X, \mathbf{B}\mathrm{U}(n)] \xrightarrow{\jmath_n} [X, \mathbf{B}\mathrm{U}(n+1)] \xrightarrow{\jmath_{n+1}} \ldots \tag{2.16}$$

Observing that $\operatorname{colim} \mathcal{D}' = [X, \operatorname{colim}_n \mathbf{B}\mathrm{U}(n)]$ we have obtained a bijection $\widetilde{K}(X) \cong [X, \mathbf{B}\mathrm{U}]$, where $\mathbf{B}\mathrm{U} \coloneqq \operatorname{colim}_n \mathbf{B}\mathrm{U}(n)$. It will later be important to know that $\mathbf{B}\mathrm{U}$ is path-connected; this is inherited from path-connectedness of $\mathbf{B}\mathrm{U}(n)$, for all n.

The identification of $\widetilde{K}(X)$ with the homotopy classes of maps $[X, \mathbf{B}\mathrm{U}]$ is *a priori* a bijection of sets, however the fact that $(f \oplus g)^* EU(n+m) \cong f^* EU(n) \oplus g^* EU(m)$ implies that the group operation on $\widetilde{K}(X)$ corresponds to the operation on $[X, \mathbf{B}\mathrm{U}]$, $(f, g) \mapsto f \oplus g$. It follows that the bijection $\widetilde{K}(X)$ is a group isomorphism.

What about the ring multiplication? For $f : X \to BU(n)$ and $g : X \to BU(m)$ define $f \otimes g : X \to BU(nm)$ by $(f \otimes g)(x) = f(x) \otimes g(x)$. To show that $f^* EU(n) \otimes g^* EU(m) \cong (f \otimes g)^* EU(nm)$ we can use the universal property of pullbacks to find a morphism $f^* EU(n) \otimes g^* EU(m) \to (f \otimes g)^* EU(nm)$ and then observing that this morphism is actually an isomorphism on fibres. This is sufficient to conclude that the induced morphism is a bundle isomorphism. I omit the details.

We can conclude that the ring $\widetilde{K}(X)$ is isomorphic to $[X, \mathbf{B}\mathrm{U}]$ with the described operations. Next, we aim to express the unreduced ring $K(X)$ in similar fashion, still assuming X is connected. By definition, $\widetilde{K}(X)$ is the kernel of the morphism induced by the point inclusion map, so there is a short exact sequence

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \longrightarrow K(x_0) \longrightarrow 0 ; \tag{2.17}$$

we have seen that $K(x_0) \cong \mathbb{Z}$ and, because this is a free $\mathbb{Z}$-module, the sequence splits. Indeed, the map $\pi : \Theta^n(x_0) - \Theta^m(x_0) \mapsto \Theta^n(X) - \Theta^m(X)$ is a ring morphism that is a section of ι^*, so implies a splitting (see e.g. [3], Proposition X.1.5). This implies $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$, and therefore

$$K(X) \cong [X, \mathbf{B}\mathrm{U}] \oplus \mathbb{Z} \cong [X, \mathbf{B}\mathrm{U} \times \mathbb{Z}] . \tag{2.18}$$

In fact, the above shows that it is generally true that $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$, independent of the connectedness of X. For disconnected spaces the definition of K implies that $K(X) = \bigoplus_i K(X_i)$, where $X = \coprod_i X_i$ and the X_i are connected. Therefore, in general $K(X) \cong \bigoplus [X_i, \mathbf{B}\mathrm{U} \times \mathbb{Z}] = [X, \mathbf{B}\mathrm{U} \times \mathbb{Z}]$. Finally, the reduced group for arbitrary compact spaces is the kernel of the restriction morphism $[X, \mathbf{B}\mathrm{U} \times \mathbb{Z}] \to [x_0, \mathbf{B}\mathrm{U} \times \mathbb{Z}]$, which is precisely the set of homotopy classes of maps sending x_0 into $\mathbf{B}\mathrm{U} \times \{0\}$. Given that $\mathbf{B}\mathrm{U}$ is path-connected, this set is precisely the basepoint preserving maps $[(X, x_0), (\mathbf{B}\mathrm{U} \times \mathbb{Z}, (y_0, 0))]$, for any $y_0 \in \mathbf{B}\mathrm{U}$. We write this set as $[X, \mathbf{B}\mathrm{U} \times \mathbb{Z}]_0$.

We have learnt that $K(X) \cong [X, \mathbf{B}\mathrm{U} \times \mathbb{Z}]$ and $\widetilde{K}(X) \cong [X, BU \times \mathbb{Z}]_0$, so that $\mathbf{B}\mathrm{U} \times \mathbb{Z}$ represents the functor, K. In the next chapter we will review the Morse theoretic tools that will be used to probe the homotopy type of these spaces. Remembering that we wish to identify a cohomology theory based on K, we will pay particular attention to loop spaces.

Chapter 3

Morse Theory of Path Spaces

Morse theory uses real-valued, smooth functions to probe the topology of their domain. The basic results of Morse theory concerning the relation between critical points of smooth functions and the topology of smooth finite-dimensional manifolds are stated without proof. Assuming these results, the main task will be to identify a smooth finite-dimensional manifold that models the path space of a manifold. The results of finite-dimensional Morse theory can then be applied to this non-finite case. The sweep of ideas follows that of Milnor, [8], however some alterations are made to avoid an error in that source. In particular, in [8], p.68 a variation is introduced and assumed to be continuous, which is not true in general. This construction can be avoided by immediately passing to the finite-dimensional model, thereby avoiding the calculus of variations altogether.

3.1 Introductory Morse Theory

The idea of Morse theory is to probe the topology of smooth manifolds using smooth, real-valued functions. Therefore, in the following two chapters we will work in the category of smooth manifolds with smooth functions; unless specified otherwise, "manifold" will mean "finite-dimensional smooth manifold" and all maps and functions will be infinitely differentiable. Denote the set of smooth mappings $M \to N$ by $C^\infty(M,N)$ and real-valued smooth functions $C^\infty(M)$.

Definition 3.1. *Let M, N manifolds and $f : M \to N$ a smooth mapping. A point $x \in M$ is a* critical point *of f if the map $f_* : T_xM \to T_{f(x)}N$ is not surjective. A point $y \in N$ is a* critical value *of f if there exists $x \in f^{-1}(y)$ such that x is a critical point. If $y \in N$ is not a critical value, it is said to be a* regular value*.*

In particular, a point $x \in M$ is a critical point of $f \in C^\infty(M)$ if and only if the tangent map $f_* : T_xM \to T_{f(x)}\mathbb{R}$ is the zero map; in local coordinates $(x^1, \ldots, x^m)$ this means that $\frac{\partial f}{\partial x^1} = \frac{\partial f}{\partial x^2} = \ldots = \frac{\partial f}{\partial x^m} = 0$.

Regular values are well-behaved in that the preimage of a regular value is a smooth submanifold ([11], p.21). In fact, topologically speaking, not much happens when passing regular values, however the critical values contain topological information about the domain manifolds. We begin the study of Morse theory listing some results which make this precise.

Definition 3.2. *For $f \in C^\infty(M)$ and $x \in M$ a critical point, there is a symmetric bilinear form, denoted $f_{**} : T_xM \times T_xM \to \mathbb{R}$ and called the* Hessian*, of f. For $X, Y \in T_xM$, we can extend to vector fields $\tilde{X}, \tilde{Y}$, and then define $f_{**}(X,Y) = X(\tilde{Y}(f)) = Y(\tilde{X}(f))$. To see that the second equality actually holds, observe that $[\tilde{X}, \tilde{Y}]_x(f) = 0$, because x is a critical point. This observation shows that $f_{**}(X,Y)$ is independent of the extensions of X and Y, and that it is symmetric.*

In local coordinates, the Hessian has the matrix form $f_{**} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij}$. Being a symmetric, bilinear form, the Hessian has real spectrum. The *nullity* of a critical point is the dimension of the nullspace of the Hessian. Similarly, the *index* of the critical point is the maximal dimension of a negative-definite subspace of $T_x M$.

A critical point is called non-degenerate if the Hessian at that point is non-degenerate. A function is called a *Morse function* if all its critical points are non-degenerate.

One can show that nondegenerate critical points are isolated. The idea is that there exist local coordinates around a non-degenerate critical point, x_0 in which the function f has the form

$$f(x) = f(x_0) - x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2 \tag{3.1}$$

where λ is the index of the critical point. See [8], Lemma 2.2 for proof.

It is an important result that Morse functions are generic, in that any function can be perturbed an arbitrarily small amount to yield a Morse function. This will not be made precise here, but the following lemma is an important corollary.

Theorem 3.3. *On any manifold M, there eixsts a smooth function $f \in C^\infty(M)$ with no degenerate critical points, and for which the sublevel set $M^a = \{x \in M : f(x) \le a\}$ is compact.*

The proof is in [8], Theorem 6.6 and Corollary 6.7.

The next two theorems serve to indicate the topological structure that can be inferred from a Morse function.

Theorem 3.4. *Let $f \in C^\infty(M)$ and suppose the interval $[a,b] \subset \mathbb{R}$ is such that:*

1. *$f^{-1}[a,b]$ is compact;*
2. *$f^{-1}[a,b]$ contains no critical points;*

Then M^a is diffeomorphic to M^b, and in fact is a deformation retraction of M^b.

Sketch of Proof. We give the idea of the proof; for more details, see [8], Theorem 3.1.

A gradient vector field, ∇f, can be defined on M using an auxiliary Riemann metric:

$$\langle X, \nabla f \rangle := X(f)\,.$$

This vector field is orthogonal to the level sets $f^{-1}(c)$ and vanishes precisely at the critical points. By multiplying with a properly normalised bump function, we can obtain a compactly supported vector field that is non-vanishing on $f^{-1}[a,b]$. This induces a one-parameter family of diffeomorphisms $\phi_t : M \to M$, which are the identity outside a neighbourhood of $f^{-1}[a,b]$. The effect of the diffeomorphisms ϕ_t (when ∇f is properly normalised) is, in essence, to pull the sublevel set M^a to the sublevel set M^{a+t}; the proper choice of t provides the diffeomorphism in the theorem, and by varying t we can find the deformation retraction. □

This shows that the topology on M is, in a precise sense, constant away from the critical points of a Morse function M, so it seems inevitable that something *does* happen at the critical points. The next theorem shows that not only is this true, but we can deduce what happens based on the index of the critical point.

Theorem 3.5. *Let $f \in C^\infty(M)$ and $p \in M$ a non-degenerate critical point of index λ, $f(x) = c$. Suppose there exists $\epsilon > 0$ such that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact, and contains no critical point other than x. For all such ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

I here indicate only why this is plausible - [8], Theorem 3.2 gives the detailed proof.

Sketch of Proof. Recalling that there are coordinates $(x^1, \ldots, x^n)$ in which f looks like (3.1) in neighbourhood of the critical point, the preimage of the interval $[c-\epsilon, c+\epsilon]$ looks locally like a generalised saddle, see Figure 3.1, with λ decreasing directions. As indicated in the picture, we can think of the λ-cell as bridging between the $(c-\epsilon)$-hyperbolae, and the $(c+\epsilon)$-hyperbolae retract onto the complex in the manner indicated by the arrows.

□

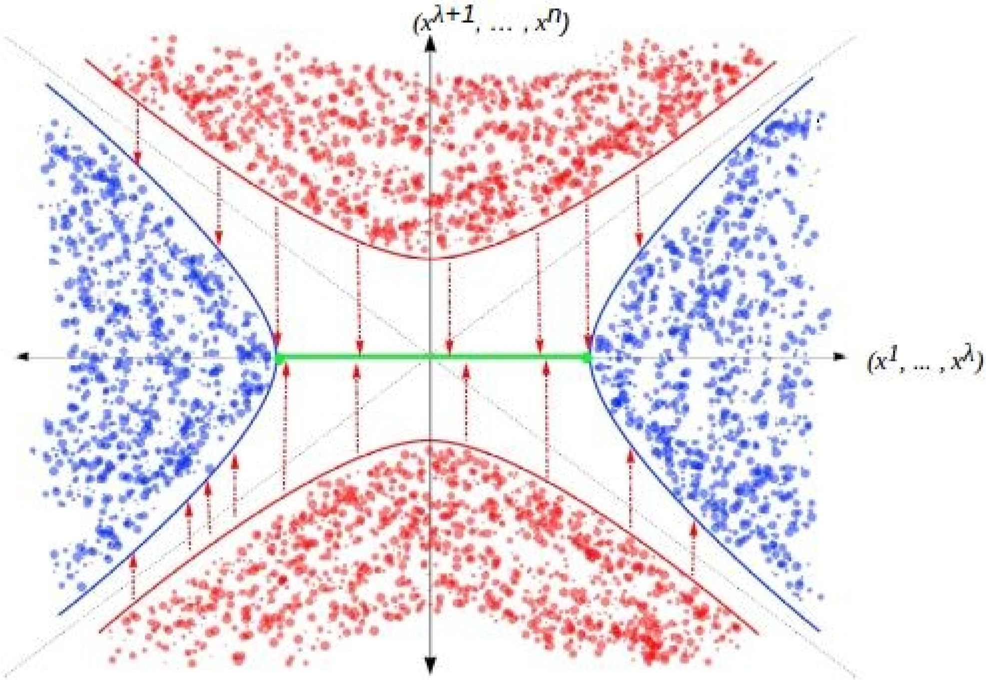

Figure 3.1: The blue hyperbolae represent the preimage $f^{-1}(c-\epsilon)$, bounding the sublevel set depicted by blue-fill; the red hyperbolae represent $f^{-1}(c+\epsilon)$ bounding the set $f^{-1}[c+\epsilon, \infty)$ in red-fill; the thick green line depicts the λ-cell attached to $f^{-1}(c-\epsilon)$. The dotted red arrows illustrate the retraction of $f^{-1}(c+\epsilon)$ to $f^{-1}(c-\epsilon) \coprod_{\partial D^\lambda} D^\lambda$.

Theorem 3.5 will play an important role in what follows. The next theorem takes more of a background role, but as it is incredibly useful, I state it here.

Theorem 3.6. *If $f \in C^\infty(M)$ is Morse, and each M^a is compact, then M has the homotopy type of a CW-complex, with a cell of dimension λ for each critical point of index λ.*

In the light of 3.3, we have the following corollary.

Corollary 3.7. *Every smooth manifold has the homotopy type of a CW-complex.*

The above results rely on the fact that the real-valued function is Morse, in particular that the critical points are non-degenerate. We can actually relax the condition and still obtain meaningful results. In particular, the next two lemmas show that if the set of minimal values form a submanifold, and the index of all other critical points is bounded from below, then we can deduce the low-dimensional homotopy groups of the full manifold from the critical submanifold.

Lemma 3.8 ([8] Lemma 22.4)**.** *Let $K \subset \mathbb{R}^n$ compact, and U an open neighbourhood of K. Let $f : U \to \mathbb{R}$ smooth, such that every critical point of f has index at least λ; then for small enough $\epsilon > 0$, any $g : U \to \mathbb{R}$ satisfying*

$$|\partial_i f(x) - \partial_i g(x)| \leq \epsilon \qquad \forall x \in K,\ i = 1, \ldots, n\,;$$
$$|\partial_i \partial_j f(x) - \partial_i \partial_j g(x)| \leq \epsilon \qquad \forall x \in K\, i, j = 1, \ldots, n\,;$$

has the property that every critical point of g has index greater than or equal λ.

The proof is purely technical, and so omitted. The main application of this theorem is the following lemma.

Lemma 3.9 ([8] Lemma 22.5)**.** *Let M a smooth manifold, $f : M \to \mathbb{R}$ a smooth function which attains its minimum at 0, and such that the sublevel sets $f^{-1}[0,a]$ are compact. If $f^{-1}(0) =: M^0$ is a topological manifold, and every critical point of f in $M \setminus M^0$ has index at least λ, then the relative homotopy groups $\pi_i(M, M^0) = 0$, $\forall i = 1, \dots, \lambda - 1$.*

The idea is that there exists a small perturbation of the function which is Morse in a neighbourhood of the minimal submanifold and, away from a smaller neighbourhood of the minimal locus, only adds critical points of index at least λ.

Proof. We start with an arbitrary map $h : (I^i, \partial I^i) \to (M, M^0)$, with $0 \leq i < \lambda$, and want to show that there is a homotopy within (M, M^0) to a map $h' : (I^i, \partial I^i) \to (M^0, M^0)$.

The assumption that M^0 is a compact manifold implies that it is a Euclidean neighbourhood retract ([5], Corollary A.9) and hence, there is an open neighbourhood of M^0, $U \subset M$, such that M^0 is a deformation retraction of U in M.

Now, as the hypercube I^i is compact, the set $f(h(I^i))$ achieves a maximum, which we denote by c. Similarly, the sets $M^a \setminus U$ are compact, so that $f(M^a \setminus U)$ achieves a minimum, which (for a big enough) is equal to the minimum of $f(M \setminus U)$, call this number 3δ.

Using Lemma 3.9 and the discussion preceding Theorem 3.3 we can deduce the existence of a map, $g : M^{c+2\delta} \to \mathbb{R}$, such that:

1. The sublevel sets of g are compact;
2. All critical points of g are non-degenerate;
3. For all $x \in M^{c+2\delta}$, $|f(x) - g(x)| < \delta$;
4. Every critical point of g in the compact set $f^{-1}[\delta, c + 2\delta]$ has index at least λ.

In particular, $g^{-1}[2\delta, c + \delta] \subset f^{-1}[\delta, c + 2\delta]$, and we conclude from Theorem 3.5 that $g^{-1}(-\infty, c + \delta]$ has the homotopy type of $g^{-1}(-\infty, 2\delta]$ with cells of dimension at least λ attached. Given that $h(I^i) \subset M^c \subset g^{-1}(-\infty, c + \delta]$, we can can think of h having codomain $h : (I^i, \partial I^i) \to (g^{-1}(-\infty, c + \delta], M^0)$ and the given cell structure indicates that h is homotopic in $(g^{-1}(-\infty, c + \delta], M^0)$ to some $h' : (I^i, \partial I^i) \to (g^{-1}(-\infty, 2\delta], M^0)$, using that $i \leq \lambda_0 - 1$. Observe that $g^{-1}(-\infty, 2\delta] \subset U$, and therefore composing with the retraction yields the requisite homotopy. In particular, $\pi_i(M, M^0) = 0$, for any $0 \leq i < \lambda$. □

In light of the long exact sequence of homotopy groups, we can conclude that the maps induced by the inclusion, $\iota_* : \pi_i(M^0) \to \pi_i(M)$, are isomorphisms for all $1 \leq i \leq \lambda - 2$, and a surjection for $i = \lambda - 1$.

3.2 Morse Theory on the Path Space

Next, the application of Morse theory to the study of loop spaces is presented. As these spaces are not generally endowed with a smooth structure, we begin with an approximation of the loop space that is amenable to Morse theory and aim to see that the approximation is valid. For a specific choice of real-valued function, we are able to identify the critical points and find a geometric expression for the index.

A smooth manifold M is endowed with a choice of Riemannian structure[1], so that the tools of Riemannian geometry can be exploited. For $x, y \in M$, define $\Omega(M; p, q)$ the space of piecewise smooth paths $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. When not likely to cause confusion, I denote this space by ΩM, or even Ω. A path, γ is "piecewise smooth" if γ is continuous and there

[1] this is the real analogue of the hermitian structures considered in Chapter 2, applied to the tangent bundle. A Riemannian structure always exists, following similar reasoning as in the complex case.

is a finite partition of the unit interval $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $\gamma|_{[t_i,t_{i+1}]}$ is smooth, for all $i = 0\ldots,n-1$.

A real-valued function can be defined $E : \Omega(M;p,q) \to \mathbb{R}$, with the aid of the metric, g:

$$E(\gamma) = \int_0^1 g(\dot\gamma(t),\dot\gamma(t))\,dt\,. \tag{3.2}$$

E is called the energy functional and will be instrumental in the definition of a finite-dimensional approximation of the path space. We shall also study its critical points and their index.

3.2.1 Path Space Topology

Given that we have a Riemannian manifold, there is an induced distance function on the manifold, given by

$$\Lambda(p,q) = \inf\{L(\gamma) : \gamma \in \Omega(M;p,q)\}\,, \tag{3.3}$$

where $L(\gamma) = \int_0^1 \sqrt{g(\dot\gamma(t),\dot\gamma(t)}\,dt$ is the length of the path. The distance function induces a metric on $\Omega(M;p,q)$ by

$$\tilde d(\gamma,\gamma') = \max_{t\in[0,1]} \Lambda(\gamma(t),\gamma'(t))\,. \tag{3.4}$$

Following [8] we modify this metric as follows:

$$d(\gamma,\gamma') = \tilde d(\gamma,\gamma') + \sqrt{\int_0^1 (\dot\gamma(t) - \dot\gamma'(t))^2\,dt}\,; \tag{3.5}$$

it is straightforward to check that this is positive-definite, symmetric and satisfies the triangle inequality, so is a metric. The modification ensures that the energy functional is continuous with respect to the induced topology, as can be checked using the Minkowski inequality. Endow Ω with the topology induced by d.

The path space commonly employed by topologists is the space of continuous paths in M between p and q with the compact-open topology, which we denote by $\Omega^*(M;p,q)$ (or Ω^*, Ω^*M, when no confusion is risked). Obviously, $\Omega \subset \Omega^*$ as sets. Further, we know that for M a metric space, Y compact, the space of continuous maps $Y \to M$, M^Y, with the compact-open topology agrees with the topology induced by the analogue of (3.4), (see e.g. [5], Proposition A.13). Due to the inequality $\tilde d \le d$, it can be concluded that the inclusion $\Omega \hookrightarrow \Omega^*$ is continuous. As a matter of fact, the two spaces are homotopy equivalent:

Proposition 3.10. *The inclusion $\iota : (\Omega, d) \hookrightarrow \Omega^*$ is a homotopy equivalence.*

The proof is technical and not particularly illuminating, so is omitted, see [8], Theorem 17.1.

In particular, we have the following corollary:

Corollary 3.11. *For any smooth manifold, M, with $p,q \in M$, the piecewise smooth path space, $\Omega(M;p,q)$, has the homotopy type of a CW-complex*

Proof. In [7] it is shown that Ω^* has the homotopy type of a CW-complex, Corollary 2. So, Proposition 3.10 implies the result. □

Proposition 3.10 is encouraging for it indicates that as far as homotopy is concerned, the space of piecewise smooth curves is a faithful model of the ordinary path space. We now approximate Ω by finite-dimensional subspaces endowed with a natural smooth structure, which are well-behaved with respect to the energy functional. The procedure is motivated by the tools of Morse theory, in particular the sublevel sets. In the following, we fix M to be a geodesically complete manifold,[2] and set arbitrary $p,q \in M$, with Ω to be understood as $\Omega(M;p,q)$.

[2] Geodesic completeness means all geodesics can be extended indefinitely

For $c \in \mathbb{R}$, let $\Omega^c := E^{-1}[0,c]$ and $\operatorname{Int}\Omega^c := E^{-1}[0,c)$. We assume c is chosen so that $\Omega^c \neq \emptyset$ and construct a finite-dimensional approximation to this space.

For any partition of the unit interval $0 = t_0 < t_1 \ldots < t_n = 1$, set $P = \{t_1, \ldots, t_{n-1}\}$ and let $\Omega(P)$ denote the space of piecewise smooth curves on M, between p and q, such that the restriction to the subintervals $[t_i, t_{i+1}]$ are geodesic; that is $\gamma \in \Omega(P)$ implies $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma|_{[t_i,t_{i+1}]}$ is geodesic for all $i = 0, \ldots, n-1$. Set $\Omega^c(P) = \Omega^c \cap \Omega(P)$ and $\operatorname{Int}\Omega^c(P) = \operatorname{Int}\Omega^c \cap \Omega(P)$.

Proposition 3.12 ([8], Lemma 16.1)**.** *For M a complete Riemannian manifold, $c \in \mathbb{R}$ such that $\Omega^c \neq \emptyset$, and P a sufficiently fine partition of the unit interval, the set $\operatorname{Int}\Omega^c(P)$ can be canonically endowed with a smooth manifold structure.*

This statement follows from some technical considerations of complete Riemannian manifolds and so is only sketched here - more details are found in [8].

Sketch of Proof. We can define a compact set $S = \{x \in M : \Lambda(p,x) \leq \sqrt{c}\}$, and observe that for any $\sigma \in \Omega^c$, $\sigma(I) \subset S$. By completeness, there exists $\epsilon > 0$ such that, within S, any two points separated by less than ϵ are joined by a unique geodesic contained within the ϵ-ball, which is minimal, and depends smoothly on the endpoints. It can then be shown that for any sufficiently fine partition of I, depending on ϵ and c so that for any $\gamma \in \operatorname{Int}\Omega^c$, $\Lambda(\gamma(t_i), \gamma(t_{i+1})) < \epsilon$ and hence the component $\gamma|_{[t_i,t_{i+1}]}$ is determined by its endpoints. In particular, any $\gamma \in \operatorname{Int}\Omega^c$ is determined by the $(n-1)$-tuple $(\gamma(t_1), \ldots, \gamma(t_{n-1})) \in M^{\times(n-1)}$. One can check that this identification is a homeomorphism to an open subset of the product, and can hence induce a smooth structure on $\operatorname{Int}\Omega^c(P)$. □

It will be important to observe that as $\operatorname{Int}\Omega^c(P)$ can be identified with an *open* subset of a product of M, then the tangent space above any curve, ω can be canonically identified with a direct sum of the tangent space of M, i.e. $T_\omega\Omega(P) \cong \bigoplus_{t_i \in P} T_{\omega(t_i)}M$. On the other hand, we can observe that for any $a < c$, the subset $\Omega^a(P) \subset \operatorname{Int}\Omega^c(P)$ is compact. For this, it is sufficient to observe that $\Omega^a(P)$ is diffeomorphic to the $(n-1)$-tuples $(x_1, \ldots, x_{n-1}) \in S^{\times(n-1)}$ such that $\sum_{i=0}^{n-1} \frac{\Lambda(x_i,x_{i+1})^2}{t_i - t_{i+1}} \leq a$. Recalling that S is compact, it follows that $\Omega^a(P)$ is compact.

The next theorem will show that for sufficiently fine partitions, the space $\operatorname{Int}\Omega^c(P)$ is homotopy equivalent to the space $\operatorname{Int}\Omega^c$, and we can therefore probe the homotopy of $\operatorname{Int}\Omega^c$ by applying Morse theory to the smooth manifold $\operatorname{Int}\Omega^c(P)$.

Theorem 3.13. *There is a deformation retraction of the space $\operatorname{Int}\Omega^c$ to $\operatorname{Int}\Omega^c(P)$, with the property that the restriction to Ω^a, for $a < c$, is a retraction to $\Omega^a(P)$.*

Proof. The construction is taken from the proof of Theorem 16.2 in [8]. For convenience, set $L := \operatorname{Int}\Omega^c(P)$, and $L^a := \Omega^a(P)$ for arbitrary $a < c$.

We can define $r_1 : \operatorname{Int}\Omega^c \to L$, given by $r_1(\gamma)|_{[t_i,t_{i+1}]}=$ the unique geodesic between $\gamma(t_i)$ and $\gamma(t_{i+1})$. (One can check that this is indeed well-defined, using the explicit definition of the partition P, which I have omitted (see [8])); as $L \subset \operatorname{Int}\Omega^c$, the map r_1 induces $r_1 : \operatorname{Int}\Omega^c \to \operatorname{Int}\Omega^c$. Define a homotopy $r : \operatorname{Int}\Omega^c \times I \to \operatorname{Int}\Omega^c$ by:

$$r(\gamma, s)(t) = \begin{cases} r_1(\gamma)(t) & t \in [0, t_i] \\ \mu(t) & t \in [t_i, s] \\ \gamma(t) & t \in [s, 1] \end{cases} \quad \text{whenever } s \in [t_i, t_{i+1}]; \tag{3.6}$$

here $\mu : [t_i, s] \to M$ is the unique minimal geodesic between $\gamma(t_i)$ and $\gamma(s)$. It can be observed that $E(r(\gamma, s)) \leq E(\gamma) < c$ and that $r(\gamma, s)$ is piecewise smooth for all s, so the map is well-defined. It can be checked that r is continuous, and because $r(\cdot, 0) = \mathrm{Id}$ and $r(\cdot, 1) = r_1$, this is a homotopy between $r_0 = \mathrm{Id}$ and the map r_1, such that $r|_L = \mathrm{Id}_L$. Hence, this describes the required deformation retraction.

The inequality $E(r(\gamma, s)) \leq E(\gamma)$ implies that r restricts to a retraction $\Omega^a \to L^a$, for all $a < c$. □

The energy functional on the path space induces a function $\operatorname{Int}\Omega^c(P) \to \mathbb{R}$, which by a mild abuse of notation is also labelled E. Observe that this function is smooth because the distance between two points smoothly depends on the points. Indeed $E(\omega) = \sum_{i=1}^n \frac{\Lambda(\omega_{i-1},\omega_i)^2}{t_i - t_{i-1}}$, for any $\omega \in \operatorname{Int}\Omega^c(P)$, where $\omega_i := \omega(t_i)$ and distance smoothly depends on the endpoints. Therefore, given any fine enough partition, P, the differential, critical points and Hessian can be calculated.

Proposition 3.14. *Let $E : \operatorname{Int}\Omega^c(P) \to \mathbb{R}$ the energy function. Its differential at the path ω is given by*

$$E_*(X) = -2\sum_{i=1}^n g(X_i, \Delta_i V)\,, \tag{3.7}$$

where $V(t) = \dot\omega(t)$ and $\Delta_i V = \lim_{t\to t_i^+} V(t) - \lim_{t\to t_i^-} V(t) =: V_i^+ - V_i^-$, and we use the decomposition $T_\omega\Omega^c(P) \cong T_{\omega_1}M \oplus \ldots \oplus T_{\omega_{n-1}}M$ to write $X = (X_1, \ldots, X_{n-1})$.

Proof. By linearity, it suffices to consider tangent vectors of the form $X = (0,\ldots,0,X_i,0\ldots,0)$, which shall be denoted by X_i.

To compute $E_*(X_i)$, we need to compute the variation of the energy when we infinitesimally shift the point ω_i in the direction of X_i, i.e.

$$E_*(X_i) = \frac{d}{ds}\left(\frac{\Lambda(\omega_{i-1},\omega_i(s))^2}{t_i - t_{i-1}} + \frac{\Lambda(\omega_i(s),\omega_{i+1})^2}{t_{i+1} - t_i}\right)_{s=0} \tag{3.8}$$

Let $\tilde V_{i-1} \in T_{\omega_{i-1}}M$ such that $\exp_{\omega_{i-1}}\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\tilde V_{i-1}\right) = \omega(t)$, for all $t \in [t_{i-1}, t_i]$. By taking the derivative at $t = t_{i-1}$ we can conclude that $\tilde V_{i-1} = (t_i - t_{i-1})V_{i-1}^+$. Define $\tilde X = (\exp_{\omega_{i-1},*})^{-1}_{\tilde V_{i-1}}(X_i) \in T_{\tilde V_{i-1}}T_{\omega_{i-1}}M$ and observe that

$$\gamma_s(t) = \exp_{\omega_{i-1}}\left(\frac{t - t_{i-1}}{t_i - t_{i-1}}(\tilde V_{i-1} + s\tilde X)\right), \quad t \in [t_{i-1}, t_i] \tag{3.9}$$

forms a family of geodesics starting at ω_{i-1} such that $\gamma_0(t) = \omega(t)$ and $\partial_s\gamma_s(t_i)|_{s=0} = X_i$. Therefore, the first term of (3.8) can be computed:

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\Lambda(\omega_{i-1},\omega_i(s))^2}{t_i - t_{i-1}}\right)_{s=0} &= (t_i - t_{i-1})\frac{d}{ds}g\left(\frac{\tilde V_{i-1} + s\tilde X}{t_i - t_{i-1}}, \frac{\tilde V_{i-1} + s\tilde X}{t_i - t_{i-1}}\right)_{s=0} \\
&= \frac{2}{(t_i - t_{i-1})}g(\tilde V_{i-1}, \tilde X) \\
&= \frac{2}{(t_i - t_{i-1})}g((\exp_{\omega_{i-1},*})_{\tilde V_{i-1}}\tilde V_{i-1}, X) = 2g(V_i^-, X)\,.
\end{aligned}$$

The last line used that $g((\exp_*)_V V, (\exp_*)_V X = g(V, X)$ (see e.g. [6], Corollary 5.2.3), and the definition of $\tilde V_{i-1}$.

Performing a similar calculation for the second term of (3.8) yields

$$\frac{d}{ds}\left(\frac{\Lambda(\omega_i(s),\omega_{i+1})^2}{t_{i+1} - t_i}\right)_{s=0} = -2g(V_i^+, X)\,, \tag{3.10}$$

where the minus sign comes from the fact that the geodesics we introduce get traced in the opposite direction to ω.

Therefore, $E_*(X_i) = -2g(X_i, V_i^+ - V_i^-) = -2g(X, \Delta_i V)$. □

Corollary 3.15. *The critical points of the energy functional are the geodesics.*

Proof. Suppose ω is such that $E_* = 0$. By Proposition 3.14, this means

$$\sum_i g(X_i, \Delta_i V) = 0 \tag{3.11}$$

for all X. By choosing vectors of the form X_i, as above, we find that $g(\Delta_i V, \cdot) = 0$ for all i. Non-degeneracy of g implies that $\Delta_i V = 0$ for all i, i.e. the tangent vector to ω is continuous, so by local uniqueness of geodesics, we can conclude that ω is everywhere a geodesic. □

It will be convenient to observe that the tangent space to $\operatorname{Int}\Omega^c(P)$ over a geodesic can be identified with the space of Jacobi fields over the curve. Under this identificiation, the vector $(X_1, \ldots, X_{n-1})$ maps to the unique vector field over the curve, J such that $J(p) = 0$, $J(\omega_1) = X_1$, ..., $J(\omega_{n-1}) = X_{n-1}$, $J(q) = 0$ and $J([t_i, t_{i+1}])$ is Jacobi.[3] This is natural from the point of view that the tangent space at a point can be considered as the space of derivatives of curves at that point. In the smooth manifold $\operatorname{Int}\Omega^c(P)$ a curve is specified by a tuple $(\omega_1(s), \omega_2(s), \ldots, \omega_{n-1}(s))$, corresponding in the path space to a map $\alpha(t,s)$ such that $\alpha(t_i, s) = \omega_i(s)$ and $\alpha([t_i, t_{i+1}], s)$ is a geodesic segment. It is known that the derivative $\partial_s \alpha(t)_{s=0}$ is indeed Jacobi (see Proposition A.7).

We now look at the Hessian of the energy functional over a geodesic.

Proposition 3.16. *Let $\gamma \in \Omega^c$ a geodesic. The Hessian $E_{**} : T_\gamma \operatorname{Int}\Omega^c(P) \otimes T_\gamma \operatorname{Int}\Omega^c(P) \to \mathbb{R}$ is given by:*

$$E_{**}(X, Y) = -2 \sum_{i \in P} g(\Delta_i D_t X, Y)\,, \tag{3.12}$$

where we identify the vector X with its Jacobi field, and $\Delta_i D_t X = \lim_{t \to t_i^+} D_t X(t) - \lim_{t \to t_i^-} D_t X(t)$ is the discontinuity in its derivative.

Proof. By definition $E_{**}(X, Y) = X(Y(E))_\gamma$, where we extend Y to a vector field in a neighbourhood of γ. We have seen that this is in fact symmetric, and independent of the choice of vector field extension. Using the same notation as for the calculation of the differential, it suffices to evaluate $E_{**}(X_i, Y_j)$, by bilinearity.

The manifold structure of $\operatorname{Int}\Omega^c(P)$ means we can identify a neighbourhood of γ with a product of open neighbourhoods of each γ_i, $\prod_{i \in P} U_i$. A neighbourhood of γ_j can be identified with its tangent space $T_{\gamma_j} M$ using the exponential, so we can use this to define an extension of Y_j:

$$Y_j(Z) = Z + Y_j\,. \tag{3.13}$$

More precisely, we have $Y_j(x) = (\exp_{\gamma_i,*})_{(\exp_{\gamma_i}^{-1}(x))}(Y_j)$ whenever $x \in U_j$.

With this extension, the expression for the differential obtained in Proposition 3.14 yields $X_i(Y_j(E)) = -2 X_i \left(\sum_k g(Y_j + Z, (\Delta_Z)_k V)|_{Z=0}\right)$, where $(\Delta_Z)_k V$ is the discontinuity in the velocity of the piecewise-geodesic path from γ_{k-1} to $\exp_{\gamma_k}(Z)$ and from $\exp_{\gamma_k}(Z)$ to γ_{k+1}. Observe that for $k \neq j-1, j, j+1$, we must have $(\Delta_Z)_k V = 0$, and so we can conclude that $X_i(Y_j(E)) = 0$ unless $i \in \{j-1, j, j+1\}$. We can evaluate:

$$\begin{aligned} X_i g(Y_j + Z, \Delta_Z V)|_{Z=0} &= g(D_{X_i}(Y_j + Z), \Delta_0 V) + g(Y_j, D_{X_i} \Delta_Z V)_{Z=0} \\ &= g(Y_j, D_{X_i} V_Z^+ - D_{X_i} V_Z^-)\,, \end{aligned}$$

where it was used that $\Delta_0 V = 0$. Therefore, the main calculation remaining is $D_{X_i} V_Z^+ - D_{X_i} V_Z^-$.

Case 1: i = j

As $i = j$, I will drop the subscripts on X_i, Y_i.

Using similar calculations as in the computation of the differential (here, I have suppressed the correct normalising factors), we have

$$\begin{aligned} D_X V_Z^- &= D_X \partial_t \exp_{\gamma_{i-1}}(t(V + \tilde{Z}))_{t=t_i} \\ &= D_t X(\exp_{\gamma_{i-1}}(t(V + \tilde{Z})))_{t=t_i} \\ &= D_t \frac{d}{ds} \exp_{\gamma_{i-1}}(t(V + s\tilde{X}))|_{s=0, t=t_i} \\ &= D_t (\exp_{\gamma_{i-1},*})_{tV}(t\tilde{X})_{t=t_i}\,. \end{aligned}$$

Here, $\tilde{Z} = (\exp_{\gamma_{i-1},*})_V^{-1}(Z)$ and similar for $\tilde{X}$. It can be checked that (up to the normalisation factors I omitted) $J = (\exp_{\gamma_{i-1},*})_{tV}(t\tilde{X})$ is precisely the Jacobi field over $\gamma([t_{i-1}, t_i])$ such that $J(\gamma_{i-1}) = 0$ and $J(\gamma_i) = X_i$ (see e.g. [6], Corollary 5.2.2).

[3] This specifies the field uniquely, because the Jacobi field over any geodesic between non-conjugate points is uniquely determined by its endpoints, see Proposition A.9.

Repeating the calculation over the interval $[t_i, t_{i+1}]$ yields $D_X V_0^+ = D_t X_i^+$. Putting these together yields the result.

Case 2: $\mathbf{i = j \pm 1}$

I will focus on $i = j - 1$, the $j + 1$ case is essentially the same. As in case 1, the remaining calculation is $D_{X_{j-1}} V_Z^+ - D_{X_{j-1}} V_Z^-$. This case is actually simpler, as only $D_X V_Z^-$ is non-zero. Indeed,

$$D_{X_{j-1}} V_Z^+ = D_t \frac{d}{ds} \exp_{\gamma_i}(tV_i)|_{s=0,t=t_i} = 0.$$

On the other hand

$$D_X V_Z^- = D_t \frac{d}{ds} \exp_{\gamma_{i-1}(s)}(t\tilde{V}_{i-1}(s)). \tag{3.14}$$

We can recognise the curve as varying the starting point of a geodesic, whilst keeping the end point fixed. To calculate this we view it from the other end, i.e. with fixed start point, and varying endpoint. From this viewpoint the calculation is essentially a repetition of $D_X V_Z^+$ in Case 1, so we conclude:

$$X_{j-1} g(Y_j + Z, \Delta_Z V)|_{Z=0} = g(Y_j, \Delta_j D_t X)$$

□

It should be emphasised that, although the expression $g(X, \Delta_i D_t Y)$ does not look symmetric in X and Y, it in fact is by symmetry of E_{**}. Hence, $E_{**}(X,Y) = -2\sum_i g(X, \Delta_i D_t Y) = -2\sum_i g(\Delta_i D_t X, Y)$.

Corollary 3.17. *The nullspace of E_{**} at γ can be identified with the Jacobi fields over γ.*

Proof. The proof is very similar to that of Corollary 3.15.

Suppose that $X \in \text{Null}(E_{**})$ so that $E_{**}(X, \cdot) = 0$. By Proposition 3.16 it can be deduced that

$$g(\Delta_i D_t X, \cdot) = 0 \tag{3.15}$$

and therefore $\Delta_i D_t X = 0$, i.e. $D_t X$ is continuous. A Jacobi field is uniquely specified by its inital value and initial derivative, so continuity of $D_t X$ implies that the Jacobi segments of X piece together to a Jacobi field over γ. □

Before calculating the index of a geodesic, we will want to be sure that it does not depend on the particular partition chosen. A key ingredient in this will be the following.

Proposition 3.18. *E_{**} is positive semi-definite over minimal geodesics.*

Proof. The key idea is that for a minimal geodesic $L(\gamma)^2 = E(\gamma)$, and therefore, for any other piecewise-geodesic curve ω:

$$E(\gamma) = L(\gamma)^2 \le L(\omega)^2 \le E(\omega), \tag{3.16}$$

which implies that E is minimized on minimal geodesics. From standard calculus, we can conclude that E_{**} is positive-semi definite. □

Proposition 3.19. *Let P, P' any two partitions of the unit interval, which are sufficiently fine for Proposition 3.12 to hold. Then the index of a geodesic with respect to the energy functional is the same for $\text{Int}\,\Omega^c(P)$ and $\text{Int}\,\Omega^c(P')$.*

Proof. Firstly, observe that a geodesic, γ, is in $\Omega(P)$ for any partition P, so the proposition makes sense. For convenience, let $\lambda(P)$ denote the index of γ, with respect to E, in the space $\text{Int}\,\Omega^c(P)$. It will be shown that adding a point to the partition P conserves the index, implying that $\lambda(P) = \lambda(\bar{P})$ for all refinements $\bar{P}$. This suffices for the general result, since applied to the common refinement $P \cup P'$ it yields $\lambda(P) = \lambda(P \cup P') = \lambda(P')$.

Let $\tau \in (0,1) \setminus P$, and set $P' = P \cup \{\tau\}$. For concreteness, let $t_i < \tau < t_{i+1}$. There is a canonical splitting $T_\gamma \Omega(P') = T_\gamma \Omega(P) \oplus T_{\gamma(\tau)} M$, where we identify $T_\gamma\Omega(P)$ as piecewise-Jacobi fields that are smooth away from P, and $T_{\gamma(\tau)}M$ can be identified with the piecewise Jacobi fields that are zero away from $\gamma(t_i, t_{i+1})$. This splitting is orthogonal with respect to the bilinear form E_{**}, due to the explicit formula of the Hessian, (3.12). If it can be shown that E_{**} is positive-definite when restricted to this subspace, then the result follows. By definition, $\gamma([t_i, t_{i+1}])$ is minimal, so the result follows from Proposition 3.18. □

Corollary 3.20. *Let γ a geodesic such that $E(\gamma) < c$. For any $c' \in \mathbb{R}$ such that $c' \geq c$, the index of γ in Ω^c is equal to the index of γ in $\Omega^{c'}$.*

Proof. By Proposition 3.19 the index is independent of the partition, so choose a partition, P which is sufficiently fine to endow a smooth manifold structure on $\operatorname{Int}\Omega^{c'}(P)$. As $\operatorname{Int}\Omega^c(P) \subset \operatorname{Int}\Omega^{c'}(P)$ is an open subset, the result follows. □

We can now compute the index of a geodesic.

Theorem 3.21 (Index Theorem). *The index of a geodesic, γ, with respect to the energy functional has index, λ, equal to the number of points $\gamma(t_0)$, $t_0 \in (0,1)$, such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, counted with multiplicity.*

Recall that two points, $t_0 \neq t_1$ are called *conjugate* if there exists a non-zero Jacobi field, J, over γ such that $J(t_0) = 0 = J(t_1)$. The multiplicity of the conjugate pair is the dimension of the space of such Jacobi fields.

Proof. It is useful in this proof to identify the tangent space at γ with the piecewise-Jacobi fields over γ. Fix $c \in \mathbb{R}$ such that $E(\gamma) < c$, so we can focus on $\Omega^c(P)$.

Let $\tau \in [0,1]$, and define $\gamma_\tau := \gamma|_{[0,\tau]}$. As γ_τ is a geodesic between $\gamma(0)$ and $\gamma(\tau)$ it is a geodesic in $\Omega(M;\gamma(0),\gamma(\tau))$ and hence a critical point of the energy functional $E^\tau = \int_0^\tau g(\dot\gamma,\dot\gamma)\,dt < c$. Let $\lambda(\tau)$ denote the index of γ_τ. It is possible to choose the partition, P, so that $t_i < \tau < t_{i+1}$, set $P_\tau = \{0, t_1, \ldots, t_i, \tau\}$ define $T\Omega_\tau(P_\tau)$ similarly to $T_\gamma\Omega(P)$.

The first thing to observe is that λ is monotone increasing: $\lambda(\tau) \leq \lambda(\tau')$ whenever $\tau < \tau'$. Indeed, suppose that $V_\tau \subset T\Omega^c_\tau(P_\tau)$ is a negative definite subspace such that $\dim V_\tau = \lambda(\tau)$. Let $V_{\tau'}$ be the vector fields on $\gamma_{\tau'}$ such that $X \in V_{\tau'}$ implies $X|_{[0,\tau]} \in V_\tau$ and $X_{[\tau,\tau']} = 0$. Observe that $X \in V_\tau$ means $X_\tau = 0$, so the vector field is continuous. As $\dim V_{\tau'} = \dim V_\tau$ and $E^{\tau'}_{**}$ is negative-definite on $V_{\tau'}$, $\lambda(\tau) \leq \lambda(\tau')$.

Next, observe that λ is continuous from the left, i.e $\lambda(\tau - \delta) = \lambda(\tau)$ for sufficiently small $\delta > 0$. To see this, suppose i is such that $t_i < \tau < t_{i+1}$, and P_τ as above. Then, we can use the canonical isomorphism $T\Omega^c_\tau(P_\tau) \cong \bigoplus_{j=1}^i T_{\gamma(t_j)}M =: \Sigma_i$. By construction, E^τ_{**} continuously depends on τ so $E_{**} : (t_i, t_{i+1}] \times \Sigma_i \times \Sigma_i \to \mathbb{R}$ is continuous. Therefore, if V_τ is a negative-definite subspace of Σ_i, with respect to E^τ_{**}, then so too is it a negative definite subspace of Σ_i for E^t_{**} with t in a small neighbourhood of τ, $N \subset (t_i, t_{i+1}]$. It follows that $\lambda(\tau') \geq \lambda(\tau)$ for all $\tau' \in N$. But, monotonicity implies $\lambda(\tau - \delta) = \lambda(\tau)$ for small enough δ.

Thirdly, suppose that E^τ_{**} has nullity n. Then for small enough $\delta > 0$ it is claimed that $\lambda(\tau+\delta) = \lambda(\tau) + n$. Indeed, the inequality $\lambda(\tau + \delta) \leq \lambda(\tau) + n$ can be seen by similar reasoning to the above. Indeed, using the same notation, we have $\dim \Sigma_i = i \dim M = im$, and so Σ_i has a subspace, say Π, which is positive-definite with respect to L_τ and of dimension $im - \lambda(\tau) - n$. Again, Π must be positive-definite with respect to $L_{\tau'}$ for τ' in a small neighbourhood of τ; thus, $\lambda(\tau') \leq \lambda(\tau) + n$.

It remains only to show the inequality $\lambda(\tau + \delta) \geq \lambda(\tau) + n$. This is the most technical part of the proof.

Let $V \subset T\Omega_\tau(P_\tau)$ a negative-definite subspace, with basis of vector fields v_j, $j = 1, \ldots, \lambda(\tau)$. These can be extended by zero, to yield a negative-definite subspace $\tilde V \subset T\Omega(P_{\tau+\delta})$; we call the extended vector fields by the same name.

Now, we know that the nullspace of E^{τ}_{**} corresponds to a vector space of Jacobi fields over γ_τ, vanishing at the endpoints, say with a basis $J_1, \ldots, J_n$. It is claimed that the vector fields $D_t J_i(\tau)$ form a linearly independent set. Indeed, if α_i is such that $\sum_i \alpha_i D_t J_i(\tau) = 0$, the Jacobi field $J = \sum_i \alpha_i J_i$ satisfies

$$J(\tau) = 0\,, \tag{3.17}$$

$$D_t J(\tau) = 0\,. \tag{3.18}$$

This specifies a Jacobi field going backwards along γ, i.e. a Jacobi field along the curve $\bar{\gamma}_\tau(t) = \gamma_\tau(\tau - t)$, which is equivalent to a Jacobi field along γ. But a Jacobi field is uniquely specified by its initial points (final, in this case), which implies that $J = 0$. However, by construction the J_i are linearly independent and so $J = \sum \alpha_i J_i = 0$ only if $\alpha_i = 0$, for all $i = 1, \ldots, n$.

Of course, given any linearly independent set in an inner product space, $D_1, \ldots, D_k$, there exists a linearly independent $X_1, \ldots, X_k$ such that $g(D_i, X_j) = \delta_{ij}$, so applying this to $\{D_t J_i(\tau)\}$ gives us a specific linearly independent set in $T_{\gamma(\tau)}M$, $\tilde{X}_1, \ldots, \tilde{X}_n$ such that $g(D_t J_i, \tilde{X}_j) = \delta_{ij}$. Then we can choose n piecewise Jacobi fields on $\gamma_{\tau+\delta}$, $X_1, \ldots, X_n \in T\Omega_\tau(P_{\tau+\delta})$, satisfying:

1. $X_i(\tau) = X_i$;
2. $X_i(\tau + \delta) = 0$;
3. Each X_i is smooth except possibly at τ.

Extend the Jacobi fields J_i by zero to $\gamma_{\tau+\delta}$, implying that $\Delta_\tau J_i = -D_t J_i$. Directly applying the formula for the Hessian (3.12) yields:

$$E^{\tau+\delta}_{**}(J_i, v_j) = 0$$

$$E^{\tau+\delta}_{**}(J_i, X_j) = 2\delta_{ij}\,.$$

Therefore, consider the space spanned by $v_1, \ldots, v_{\lambda(\tau)}, Y_1 = c^{-1}\, J_1 - cX_1, \ldots, Y_n = c^{-1}\, J_n - cX_n$. With this basis, $E^{\tau+\delta}_{**}$ has the matrix:

$$E^{\tau+\delta}_{**} = \begin{pmatrix} (E^{\tau+\delta}_{**}(v_i, v_j))_{i,j} & c(E^{\tau+\delta}_{**}(v_i, X_j))_{i,j} \\ c(E^{\tau+\delta}_{**}(X_i, v_j))_{i,j} & -4I + c^2(E^{\tau+\delta}_{**}(X_i, X_j))_{i,j} \end{pmatrix}. \tag{3.19}$$

Therefore, choosing c small enough ensures that this matrix is negative-definite, so in particular $\lambda(\tau + \delta) \geq \lambda(\tau) + n$.

Finally, we observe that for τ sufficiently small, $\lambda(\tau) = 0$. Indeed, if $\tau \in [0, t_1]$ then γ_τ is a minimal geodesic and hence Proposition 3.18 implies $\lambda(\tau) = 0$.

Therefore the index can be deduced by counting, with multiplicity, points conjugate to 0 and this completes the proof of the index theorem. □

We now have a convenient geometric expression for the index of the energy functional. Its key application will be in the following theorem, which is essentially a corollary of Lemma 3.9 and Proposition 3.10.

Theorem 3.22 (Minimal Geodesic Index, [8], Theorem 22.1)**.** *Let M a complete Riemannian manifold, p and q distinct points such that the space of minimal geodesics between them is a topological manifold and every non-minimal geodesic has index greater than or equal to some λ_0, then the relative homotopy groups $\pi_i(\Omega_{min}, \Omega)$ are zero, for all $0 \leq i < \lambda_0$. In particular, the inclusion $\Omega_{min} \hookrightarrow \Omega$ induces an isomorphism on the i-th homotopy groups, for all $0 \leq i < \lambda_0$.*

Proof of Theorem 3.22. Let $\sqrt{l} = \Lambda(p, q)$, so that Ω^l is the manifold of minimal geodesics. For arbitrary $c \in \mathbb{R}$, we know that $\operatorname{Int}\Omega^c$ has a smooth manifold $\operatorname{Int}\Omega^c(P)$ as a deformation retract, so that the relative homotopy groups are isomorphic, $\pi_i(\operatorname{Int}\Omega^c, \Omega^l) \cong \pi_i(\operatorname{Int}\Omega^c(P), \Omega^l)$. If we can show that the latter group is trivial for arbitrary $c > l$, then we are done. We have $E' : \operatorname{Int}\Omega^c(P) \to \mathbb{R}$ a smooth function *almost* satisfying the hypotheses of Lemma 3.9, with the problem being that $E(\operatorname{Int}\Omega^c(P)) \subset [l, c)$, as opposed to $[0, \infty)$. Being as that these intervals are diffeomorphic this poses no challenge, and so Lemma 3.9 completes the proof. $\square$

Chapter 4

Bott Periodicity

In this chapter the Morse theoretic technology developed in Chapter 3 is used to calculate the homotopy type of the stable unitary group $U = \operatorname{colim} U(n)$ and its classifying space $\mathbf{B}\mathrm{U}$. In particular, it will be shown that the homotopy groups of U have periodicity two, i.e. $\pi_i(U) \cong \pi_{i+2}(U)$ for all $i \geq 0$, and similarly, the space $\mathbf{B}\mathrm{U} \times \mathbb{Z}$ has periodicity two. The key to the calculation is to realise that the space of minimal geodesics in $U(n)$, between the identity and its negative, is a smooth manifold. Therefore we can use Theorems 3.21 and 3.22 to calculate the low-dimensional homotopy groups of $\Omega U(n)$ in terms of the corresponding homotopy groups of the minimal geodesic manifold. This chapter largely follows the ideas from [8], section 23; where material is from elsewhere, the source has been indicated.

To begin with, we focus on $U(n)$ for arbitrary n, and look at the piecewise smooth path-space from the identity to its negative, $\Omega(U(n); I, -I)$. The first step is to exhibit a left- and right-invariant Riemannian metric, which will yield several tools to explore the space of minimal geodesics with respect to this metric. It suffices to give an inner product on the tangent at the identity, which we can then extend to a metric on the whole space using the parallellization induced by left-invariant vector fields. This will automatically be left-invariant, so it will only remain to check right-invariance.

The tangent space at the identity, $T_I U(n)$ is the algebra of $n \times n$ antihermitian matrices with the anticommutator as Lie bracket. For arbitrary $X, Y \in T_I U(n)$ we can define [1]

$$\langle X, Y \rangle := Tr(AB^{\dagger}) \tag{4.1}$$

which is bilinear, positive-definite and symmetric so defines an inner product. We need to check the induced Riemannian metric is right invariant, which is done using the adjoint action of the group on its Lie algebra.

Any Lie group acts on its Lie algebra via $Ad_S : \mathfrak{g} \to \mathfrak{g}$, $Ad_S X = (L_S R_S{}^{-1})_* X$, for arbitrary $S \in G$, L_S, R_S are the left (resp. right) action of G on itself and $\mathfrak{g}$ is the algebra of left invariant vector fields, which we identify with the tangent space at the identity. As $U(n)$ is a matrix group, we can use the matrix exponential and deduce

$$Ad_S X = \frac{d}{dt}(L_S R_S{}^{-1} \exp(tX))|_{t=0} = SXS^{-1} \tag{4.2}$$

so in particular, $Ad_S X = SXS^{-1}$ for any $X \in \mathfrak{g}$, $S \in G$.

We can see that the inner product 4.1 is invariant under the adjoint action:

$$\langle Ad_S X, Ad_S Y \rangle = Tr(SXS^{-1}\, SY^{\dagger}S^{-1}) = \langle X, Y \rangle\,. \tag{4.3}$$

Given that the Riemannian metric is left-invariant, it can be concluded that the hermitian metric is right-invariant:

$$\langle R_{S*}X, R_{S*}Y \rangle = \langle L_{S^{-1}}{}_* R_{S*}X, L_{S^{-1}}{}_* R_{S*}Y \rangle = \langle Ad_{S^{-1}}X, Ad_{S^{-1}}Y \rangle = \langle X, Y \rangle\,. \tag{4.4}$$

[1] the dagger indicates Hermitian conjugation

In Theorem A.10, it is observed that the geodesics beginning at the identity are precisely the one-parameter subgroups. For matrix groups such as $U(n)$ the one-parameter subgroups are given by the matrix exponential applied to the tangent space at the identity, so that the matrix exponential coincides with the geodesic exponential.

Therefore, the geodesics starting at the identity of $U(n)$, equipped with the metric induced by (4.1), are given by $\gamma(t) = \exp(tX)$, for any $n \times n$, antihermitian matrix, X, where

$$\exp(tX) = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} . \tag{4.5}$$

Elementary linear algebra can tell us that antihermitian matrices are diagonalisable by the adjoint action of some unitary matrix; closure of the algebra under the adjoint action means the diagonal entries must be purely imaginary.

In particular, for $X \in T_I U(n)$ such that $\exp X = -I$ we can use that, for all $T \in U(n)$,

$$\exp(Ad_T X) = T(\exp X)T^{-1} = -I,$$

so that we can find all the geodesics between I and $-I$ by conjugating certain diagonal matrices. In particular, if $X \in T_I U(n)$ is diagonal,

$$X = \begin{pmatrix} ix_1 & & & \\ & ix_2 & & \\ & & \ddots & \\ & & & ix_n \end{pmatrix} \tag{4.6}$$

satisfying

$$\exp X = -I . \tag{4.7}$$

we conclude that $\exp(ix_j) = -1$, for all $j = 1, \ldots, n$ meaning $x_j = p_j \pi$, for p_j an odd integer.

The length of the geodesic described by X, γ_X, is given by

$$L(\gamma_X) = \int_0^1 \sqrt{\left\langle \frac{d\gamma_X(t)}{dt}, \frac{d\gamma_X(t)}{dt} \right\rangle} dt = \sqrt{\langle X, X \rangle} \tag{4.8}$$

by definition of the geodesic. It follows that $L(\gamma_X) = \pi\sqrt{\sum p_i^2}$, and so the minimal geodesics correspond to matrices with eigenvalues $\pm i\pi$. A matrix X (corresponding to a minimal geodesic, but not necessarily diagonalised) can be identified with a linear operator $\mathbb{C}^n \to \mathbb{C}^n$, which is uniquely specified by its eigenbasis and eigenvalues. Therefore, a minimal geodesic between I and $-I$ is specified by the space of eigenvectors with eigenvalue $i\pi$ (or, equally good, with eigenvalue $-i\pi$) and this gives an identification between the space of minimal geodesics and the space of k-planes in $\mathbb{C}^n$, for all $k = 0, \ldots, n$, $\Omega_{min} = \coprod_{k=0}^{n} Gr_k(\mathbb{C}^n)$. Note that the topology of Ω_{min} as a subspace of $\operatorname{Int}\Omega^a$ (for any $a > \pi\sqrt{n}$) coincides with the topology induced by the tangent space, which follows from the definition of the topology on $\operatorname{Int}\Omega^a$ and the fact that exp is a local diffeomorphism.

In particular, we can apply the minimal geodesic theorem, Theorem 3.22 to this situation. In this case, however, the minimal geodesic manifold has many disconnected components with varying dimensions, which makes for technical difficulties. We can obviate these challenges by restricting attention to $SU(2n)$, which singles out a single connected component.

Indeed, taking even dimension guarantees that $-I \in SU(2n)$, and the Lie algebra is now the space of antihermitian, traceless $2n \times 2n$ matrices. All the above arguments go through, except that the traceless condition implies that $X \in T_I SU(2n)$ corresponds to a minimal geodesic only if its eigenvalues are $\pm i\pi$ *and* that they sum to zero, in particular $\dim \operatorname{Eigen}(i\pi) = \dim \operatorname{Eigen}(-i\pi)$. As a result, the space of minimal paths corresponds to $Gr_n(\mathbb{C}^{2n})$.

Therefore, the minimal geodesic index theorem, Theorem 3.22, allows us to calculate the lower dimensional homotopy groups of $\Omega U(n)$ from those of $Gr_n(\mathbb{C}^{2n})$, with the meaning of "lower dimensional" dictated by the index of non-minimal geodesics. Thus, the next step is to calculate a lower bound on the index of any non-minimal geodesic from I to $-I$ in $SU(2n)$. For this, we need some machinery.

Let $K_V : T_pM \to T_pM$ be the linear operator defined by $K_V(W) = R(V,W)V$, where $R(X,Y)Z := [D_X, D_Y]Z - D_{[X,Y]}Z$ and $V, W \in T_pM$. Using Theorem A.11, we have that $K_V W = \frac{1}{4}[[V,W],V]$, and K_V is adjoint with respect to the Riemannian structure, due to Theorem A.5.

Theorem 4.1 ([8], Theorem 20.5)**.** *Let $\gamma \in \Omega(G; e, q)$ a geodesic in a Lie group. The points conjugate to p along the geodesic γ occur at the values $t = \pi k/\sqrt{e_i}$, for $k > 0$ integer, e_i a positive eigenvalue of $K_{\dot\gamma(0)}$. The multiplicity of the conjugate point is the sum of the multiplicities of the eigenvalues e_i such that t is some multiple of $\pi/\sqrt{e_i}$.*

In light of the index theorem, Theorem 3.21, this gives a very explicit method of calculating the index of non-minimal geodesics. It should be noted that this theorem holds more generally for "locally symmetric" spaces, but the result as stated here is sufficient for the needs of this thesis.

Proof. Self-adjointness of the operator $K_V : T_eG \to T_eG$ implies that there is an orthonormal eigenbasis, say $X_1 \dots X_k$, $K_V X_i = e_i X_i$. Extend to vector fields along γ by parallel transport, i.e. left-translation: $X_{i,g} = L_{g*}X_i$. It can be checked that the Riemannian curvature is left-invariant,so that

$$(K_V X)_g = R(V_g, X_g)V_g = L_{g*}R(V_e, X_e)V_e\,, \tag{4.9}$$

and therefore, on the eigenvector fields:

$$(K_V X_i)_g = e_i X_{i,g}\,. \tag{4.10}$$

In particular, the fields $\{X_i\}$ form an orthonormal frame of γ, so any vector field over γ, say Y, can be expanded

$$Y_t = \sum_{i=1}^{k} \alpha_i(t) X_{i,t}\,. \tag{4.11}$$

Such a vector field is Jacobi if it satisfies $D_t^2 Y + R(V,Y)V = 0$ or, in terms of the coefficients:

$$\ddot\alpha_i + e_i\alpha_i = 0\,, \tag{4.12}$$

for all $i = 1, \dots, k$. Imposing the initial condition that the field vanishes at $t = 0$, implies the solutions are of the form:

$$\alpha_i(t) = \begin{cases} A\sin(\sqrt{e_i}t) & \text{if } e_i > 0 \\ A\sinh(\sqrt{|e_i|}t) & \text{if } e_i < 0 \\ At & \text{if } e_i = 0 \end{cases}\,. \tag{4.13}$$

Now, $\gamma(t')$ is a conjugate point to p if and only if $\alpha_i(t') = 0$, for all $i = 1, \dots, k$. Therefore, the space of Jacobi fields, vanishing at $\gamma(0)$ and $\gamma(t')$ is spanned by the vector fields $\alpha_i X_i$, where α_i as above and $\alpha_i(t') = 0$. Thus conjugate points arise from positive eigenvalues e_i, and $\alpha_i(t') = 0$ implies $t' = \frac{n\pi}{\sqrt{e_i}}$. Imposing that $t \in (0,1)$, gives the result. □

Proposition 4.2. *The index of a non-minimal geodesic $\gamma \in \Omega(SU(2n); I, -I)$ is at least $2(n+1)$.*

Proof. This is essentially an application of Theorem 4.1.

Using that $SU(2n)$ is a matrix group, the operator $K_V : T_I SU(n) \to T_I SU(n)$, can be expressed $K_X Y = \frac{1}{4}[[X,Y]X]$, (see Theorem A.11) and the eigenvalues of the right-hand side can be computed directly. It can be assumed that X is diagonal, as the adjoint action, being induced by a group action, is a Lie algebra isomorphism,

$$Ad_S[X,Y] = [Ad_S X, Ad_S Y]\,.$$

It therefore preserves eigenvalues and induces an isomorphism of the eigenbasis of the operator $[X,[X,\cdot]]$. Indeed, let $X' = Ad_S X$ be diagonal and suppose Y' is an eigenvector of $[X',\cdot]$ with eigenvalue α implying

$$\begin{aligned}
\alpha Y' &= [X',[X',Y']] \\
&= Ad_S[X,[X,Ad_{S^{-1}}Y']] \\
\implies \alpha Ad_{S^{-1}}Y' &= [X,[X,Ad_{S^{-1}}Y']]\,.
\end{aligned}$$

Thus, assume X is diagonal with entries $i\pi p_1 \geq i\pi p_2 \geq \ldots \geq i\pi p_{2n}$, and let $Y = (y_{ij}) \in T_I SU(2n)$. Then,

$$\begin{aligned}
[X,Y]_{jl} &= \sum_{k=1}^{2n} i\pi p_j \delta_{jk} y_{kl} - \sum_{k=1}^{2n} y_{jk} i\pi p_k \delta_{kl} \\
&= i\pi(p_j - p_l)y_{jl}
\end{aligned}$$

and repeating the calculation gives

$$(K_X Y)_{jl} = -\tfrac{1}{4}[X,[X,Y]] = \frac{\pi^2}{4}(p_j - p_l)^2 y_{jl} \tag{4.14}$$

The following constitutes an eigenbasis:

- For any $j < l$ the matrix E_{jl} which has a $+1$ in the jl-th entry, -1 in the lj-th entry and zeros everywhere else; E_{jl} has eigenvalue $\frac{\pi^2}{4}(p_j - p_l)^2$;
- For any $j < l$ the matrix $\tilde{E}_{jl}$ which has $+i$ in the jl-th and lj-th entry, zeros elsewhere; $\tilde{E}_{jl}$ has eigenvalue $\frac{\pi^2}{4}(p_j - p_l)^2$;
- Any diagonal matrix in $T_I SU(2n)$ is an eigenvector with eigenvalue 0.

Thus, for each pair $(j < l)$ such that $p_j \gneqq p_l$ we have a positive eigenvalue $e_{jl} = \frac{\pi^2}{4}(p_j - p_l)^2$ with multiplicity 2. It follows that for each such pair (j,l) there is a conjugate point of multiplicity 2 at the values

$$t = \frac{2}{p_j - p_l}, \frac{4}{p_j - p_l}, \ldots, \frac{2k}{p_j - p_l}\,, \tag{4.15}$$

where k is the greatest integer such that $\frac{2k}{p_j - p_l} < 1$. That is, $k = \frac{p_j - p_l}{2} - 1$ and so it follows that each $p_j \gneqq p_l$ contributes $p_j - p_l - 2$ to the index. Thus, for the geodesic $\gamma_X(t) = \exp(tX)$ we have the index $\lambda = \sum_{p_j > p_l}(p_j - p_l - 2)$.

Now assuming the geodesic associated to X is non-minimal, the p_i are not all ± 1. There are the following two cases to consider:

Case 1: There are exactly n positive p_j and n negative p_j; by the ordering convention, we have $p_j > 0$ for $1 \leq j \leq n$ and $p_j < 0$ for $n+1 \leq j \leq 2n$, and being non-minimal we have that $p_1 \geq 3$ and $p_{2n} \leq -3$. Therefore, we can estimate:

$$\begin{aligned}
\lambda = \sum_{p_j > p_l}(p_j - p_l - 2) &\geq \sum_{i=1}^{n}(p_i - (-3) - 2) + \sum_{j=n+1}^{2n-1}(3 - p_j - 2) \\
&\geq \sum_{i=1}^{n}(1 + 4 - 2) + \sum_{j=n+1}^{2n-1}(3 - (-1) - 2) \\
&\geq 4n \geq 2(n+1)\,.
\end{aligned}$$

Case 2: There are strictly greater than n of the p_j that are positive. In this case, the traceless condition ensures at least $p_{2n} \le -3$. We can make the estimate:

$$\lambda \ge \sum_{i=1}^{n+1} (p_i - (-3) - 2) \ge 2 \sum_{i=1}^{n+1} 1 = 2(n+1)\,.$$

The same conclusion, for the same reason, follows when we take strictly less than n of the p_i positive.

Therefore, all non-minimal geodesics in $SU(2n)$, from I to $-I$ have index $\ge 2(n+1)$. □

Corollary 4.3. *There is an isomorphism of homotopy groups*

$$\pi_{i+1}(SU(2n)) \cong \pi_i(Gr_n(\mathbb{C}^{2n}))\ \forall 0 \le i \le 2n+1\,, \tag{4.16}$$

induced by the inclusion of the minimal geodesics $Gr_n(\mathbb{C}^{2n}) \hookrightarrow \Omega SU(2n)$.

Proof. First of all, recall that the pathspace $\Omega^*(M;p,q)$ is homotopic to the loop space $\Omega^*(M;p,p)$, both with compact-open topology. To see this, simply let γ any path from p to q, and $\bar{\gamma}$ denote the same path traced backwards, $\bar{\gamma}(t) = \gamma(1-t)$. Then the maps $F : \Omega^*(M;p,q) \to \Omega^*(M;p,p)$ and $G : \Omega^*(M;p,p) \to \Omega^*(M;p,q)$, given by concatenating paths: $F(\sigma) = \bar{\gamma} * \sigma$, and $G(\tau) = \gamma * \tau$ are homotopy inverses.

Using that $\pi_i(\Omega^* X) \cong \pi_{i+1}(X)$ and the fact ΩM is homotopy equivalent to $\Omega^* M$ (by Proposition 3.10), we know that $\pi_{i+1}(SU(2n))$ is isomorphic to $\pi_i\big(\Omega(SU(2n);I,-I)\big)$. Further, by Theorem 3.22, the groups $\pi_i(\Omega(SU(2n);I,-I),\Omega_{min})$ vanish for $i \le 2m+1$, so the long exact sequence for relative homotopies implies the groups $\pi_i\Omega(SU(2n);I,-I)$ are isomorphic $\pi_i\Omega_{min} \cong \pi_i Gr_n(\mathbb{C}^{2n})$. □

The next step is to make some observations about the homotopy groups of $SU(n)$ and $U(n)$. We can do this because we have the fibre bundle $SU(n) \to U(n) \to S^1$. That this is in fact a fibre bundle is a result of the following theorem and discussion.

Theorem 4.4 ([13], p.30)**.** *Suppose B a topological group, $G \subset B$, $H \subset G$ closed subgroups. If G admits of a local section, then the natural map $p : B/H \to B/G$ has a bundle structure. The fibre of the bundle is G/H.*

Note that a local section of G in B is a map $f : V \to B$, for $V \subset B/G$ a neighbourhood of the coset eG, such that $p \circ f = \mathrm{Id}$.

The proof is omitted; the idea is that a local section gives us a local trivial principal bundle structure, which we can transport over B/G via the group action. We can use this result with H trivial to obtain some well-known bundles. In fact, we will use a stronger result, which states that for any Lie group B with closed subgroup G, the hypotheses of Theorem 4.4 with H trivial hold, see [13], p. 33.

In particular:

Lemma 4.5. *The following are fibre bundles:*

$$U(n) \longrightarrow U(n+1) \longrightarrow S^{2n+1} \tag{4.17}$$
$$U(n) \longrightarrow U(2n) \longrightarrow V_n(\mathbb{C}^{2n}) \tag{4.18}$$
$$U(n) \longrightarrow V_n(\mathbb{C}^{2n}) \longrightarrow Gr_n(\mathbb{C}^{2n}) \tag{4.19}$$
$$SU(n) \longrightarrow U(n) \longrightarrow S^1\,. \tag{4.20}$$

Proof. Starting with (4.17), the above discussion implies that it is sufficient to recognise S^{2n+1} as the coset space $U(n+1)/U(n)$. This is a result of the fact that $U(n+1)$ has a transitive group action on S^{2n+1} with the stabilizer of a point, p, being the $U(n)$ corresponding to the unitary transformations of the equatorial sphere orthogonal to p. This implies the identification.

The sequence (4.18) is similar. The action of $U(2n)$ on $V_n(\mathbb{C}^{2n})$ has as its stabilizer the subgroup of $U(2n)$ which fixes an n-basis, and hence corresponds to $U(n)$, implying $V_n(\mathbb{C}^{2n}) \cong U(2n)/U(n)$.

The third case has essentially been dealt with already in Example 2.16, so only (4.20) remains. The identification of S^1 with the quotient $U(n)/SU(n)$ is a result of the fact that the determinant map is a group morphism $\det : U(n) \to S^1$ and the kernel is precisely $SU(n)$. Hence the first group isomorphism theorem immediately gives the result. □

These fibre bundles, along with the long exact sequence of homotopy groups allows us to infer the following isomorphisms, using $\pi_i(S^k) = 0$ for all $i < k$, $\pi_i(S^1) = 0$ for $i \neq 1$:

$$\pi_i U(m) \cong \pi_i U(m+1) \qquad i \leq 2m - 1 \tag{4.21}$$

$$\pi_i Gr_m(\mathbb{C}^{2m}) \cong \pi_{i-1} U(m) \qquad i \leq 2m \tag{4.22}$$

$$\pi_i SU(m) \cong \pi_i U(m) \qquad i \neq 1 \tag{4.23}$$

It follows from (4.21) that $\pi_{i-1}U(n) \cong \pi_{i-1}U(n+1) \cong \pi_{i-1}U(n+2)$ for all $i \leq 2n$ and so, for $U = \operatorname{colim} U(n)$ we have $\pi_{i-1}U \cong \pi_{i-1}U(n)$ for $i \leq 2n$; we call U the stable unitary group, and its homotopy groups are the *stable homotopy groups* of the unitary group. We similarly let $SU = \operatorname{colim} SU(n)$. As (4.22) relates the homotopy groups of the Grassmann spaces and the unitary group, we can use it in conjunction with (4.21) to observe that the homotopy groups of $\mathbf{B}\mathrm{U}$ can be calculated from those of a finite Grassmann space.

Theorem 4.6 (Bott Periodicity Theorem). *The stable unitary group is homotopy equivalent to its second loop space, $U \simeq \Omega^2 U$, and similarly, the space $\mathbf{B}\mathrm{U} \times \mathbb{Z}$ is homotopy equivalent to its second loop space, $\mathbf{B}\mathrm{U} \times \mathbb{Z} \simeq \Omega^2(\mathbf{B}\mathrm{U} \times \mathbb{Z})$.*

The symbol ΩM here means the based loop space, with compact-open topology, in contrast with the notation of Chapter 3. In light of 3.10, this should not cause any confusion.

Proof. It will be important to the proof that the direct limits U, $\mathbf{B}\mathrm{U}$, SU and $V = \operatorname{colim} V_n$ (where V_n is the universal bundle, defined in Example 2.16) each have the homotopy type of a CW-complex, which will allow us to deduce homotopy equivalence from *weak* homotopy equivalence, using Whitehead's Theorem. The demonstration of a CW-structure is similar in all four cases; for concreteness, I focus on $\mathbf{B}\mathrm{U}$. This space was defined as the colimit of the spaces $\mathbf{B}\mathrm{U}(n)$, but we could equally well define it to be the colimit of $Gr_n(\mathbb{C}^{2n})$ (more precisely the colimits are homeomorphic). It is a fact, which I will not prove here, that these manifolds have a CW-structure with the further property that the image under the inclusion $Gr_n(\mathbb{C}^{2n}) \hookrightarrow Gr_{n+1}(\mathbb{C}^{2(n+1)})$ is a subcomplex. The CW-structure of the colimits is then immediate. This statement also holds for the sequences of $U(n)$, $SU(n)$ and $V_n(\mathbb{C}^{2n})$ and hence implies the colimits each have a CW-structure.

We will also use the fact that the loop space of a CW-complex has the homotopy type of a CW-complex (see [7]), and therefore ΩU, $\Omega\,\mathbf{B}\mathrm{U}$ and ΩSU have the homotopy type of CW-complexes.

To begin with, it will be shown $\mathbf{B}\mathrm{U} \times \mathbb{Z}$ is homotopy equivalent to the loop space ΩU.

The inclusion $Gr_n(\mathbb{C}^{2n}) \hookrightarrow Gr_{n+1}(\mathbb{C}^{2n+2})$ induces a commutative square

$$\begin{array}{ccc} Gr_n(\mathbb{C}^{2n}) & \longrightarrow & \Omega SU(2n) \\ \downarrow & & \downarrow \\ Gr_{n+1}(\mathbb{C}^{2n+2}) & \longrightarrow & \Omega SU(2(n+1)) \end{array} \quad . \tag{4.24}$$

Commutativity implies there is an induced map $\mathbf{B}\mathrm{U} \to \Omega SU$ that, as a consequence of Corollary 4.3, induces isomorphisms on every homotopy group . Therefore $\mathbf{B}\mathrm{U} \simeq_W \Omega SU$, and the above remark imply this is in fact a strong homotopy equivalence.

Composing with the inclusion $\Omega SU \hookrightarrow \Omega U$ gives a map $\jmath : \mathbf{BU} \to \Omega U$, such that the induced map on homotopy groups, $\jmath_* : \pi_i(\mathbf{BU}) \to \pi_i(\Omega U)$, is an isomorphism for all $i \geq 1$, however $\pi_0(\mathbf{BU}) = \{*\}$, while $\pi_1(U) \cong \mathbb{Z}$, using that $\pi_1(U) \cong \pi_1(S^1)$. Therefore, define $\tilde{\jmath} : \mathbf{BU} \times \mathbb{Z} \to \Omega U$ by $\tilde{\jmath}(x,r) = \jmath_r(x)$ and the curve $\jmath_r(x)$ is defined by $\jmath_r(x)(z) = D^r(z)\jmath(x)(z)$, for $z \in S^1$ and $D^r(z)$ is the diagonal matrix $\operatorname{diag}(z^r, 1, 1 \ldots)$. We can therefore write $\jmath_r(x) = D^r \cdot \jmath(x)$, where $\cdot$ indicates pointwise-multiplication of loops in ΩU.

The induced map, $\tilde{\jmath}_{*,0}$, is bijective as $\tilde{\jmath}_{*,0}(\mathbf{BU} \times \{r\}) \mapsto [z^r] \in \pi_0(\Omega U) \cong \pi_1(U) = \pi_1(S^1)$, where the last equality used stability of U, (4.21). Therefore, to conclude that $\tilde{\jmath}$ is a weak equivalence it remains to show that the induced morphisms $\tilde{\jmath}_{*,i} : \pi_i(\mathbf{BU}, (x_0, r)) \to \pi_i(\Omega U, \jmath_r(x_0))$ are isomorphisms for each $i \geq 1$ and all $r \in \mathbb{Z}$, $x_0 \in \mathbf{BU}$. In fact, since $\mathbf{BU}$ is path-connected, it is sufficient to fix any $x_0 \in \mathbf{BU}$, and vary $r \in \mathbb{Z}$.

Using that the pointwise multiplication of two paths is homotopic to the concatenation of paths, it follows that for any $[f] \in \pi_i(\mathbf{BU} \times \mathbb{Z}, (x_0, r))$

$$\tilde{\jmath}_{*,i}[f] = [D^r] \cdot \jmath_*[f], \tag{4.25}$$

where $\jmath_*$ is an isomorphism for each $i \geq 1$ and multiplication by $[D^r]$ has inverse $[D^{-r}]$. Thus, we can conclude that $\tilde{\jmath}_{*,i} : \pi_i(\mathbf{BU}, (x_0, r)) \to \pi_i(\Omega U, \jmath_r(x_0))$ is an isomorphism for all $i \geq 1$ and a bijection for $i = 0$, so Whitehead's theorem implies that $\mathbf{BU} \times \mathbb{Z} \simeq \Omega U$.

This is half of the result we are aiming at. The other half follows from a general result in homotopy theory, which states that for a fibre bundle over paracompact base space[2] $E \to X$ with E contractible, there is a weak homotopy equivalence between the fibre and the loop space of the base, ΩX (see e.g. [5], Proposition 4.66). This result is applicable because $\mathbf{BU}$ is paracompact, being a colimit of compact spaces.

In order to use this result, we need to exhibit a bundle over $\mathbf{BU}$ with fibre U and a contractible total space. The naïve approach is to take the colimit of the principal bundles V_n along with the map induced by the bundle surjections $V_n \to \mathbf{BU}(n)$. I will sketch the reason that this does indeed yield a bundle. The key fact is that we can choose trivialisations of the bundles V_n, V_{n+1} which are compatible under the inclusion $V_n \hookrightarrow V_{n+1}$, in the sense that the following diagram commutes:

$$\begin{array}{ccccc} U^n \times U(n) & \xleftarrow{\psi^n} & \pi^{-1}(U^n) & \longrightarrow & U^n \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ U^{n+1} \times U(n+1) & \xleftarrow{\psi^{n+1}} & \pi^{-1}(U^{n+1}) & \longrightarrow & U^{n+1} \end{array} \tag{4.26}$$

for trivialising neighbourhoods, $U^n \subset \mathbf{BU}(n)$.

Therefore, choosing arbitrary $[P] \in \mathbf{BU}$, we can define a neighbourhood in $\mathbf{BU}$ as the colimit of these compatible neighbourhoods of representatives, loosely speaking $N = \bigcup_{i \in I} U^i \subset \mathbf{BU}$, with the index set I defined by the requirement that there exists a representative $P \in \mathbf{BU}(i)$ for all $i \in I$ (i.e. I is the natural numbers with a some set $\{0, 1, \ldots, k\}$ removed). Then we can define $\Psi : \pi^{-1}(N) \to N \times U$, by choosing a representative in V_n and evaluating using ψ^n. Thanks to the compatibility condition, the equivalence class of the result does not depend on the representative chosen and one can check that the resulting map is indeed a trivialisation.

Therefore, $\operatorname{colim}_n V_n = V \to \mathbf{BU}$ is a fibre bundle with fibre U. It remains to show that V is contractible. In fact, contractibility of each V_n implies that $\pi_i(V) = 0$ for all i, so the fact that V is a CW-complex implies that V is contractible.

We can conclude that there is a weak homotopy equivalence $U \simeq_W \Omega\,\mathbf{BU}$, and as both of these spaces are CW-complexes, we can conclude that this is in fact a strong homotopy equivalence.

Therefore:

$$U \simeq \Omega\,\mathbf{BU} \cong \Omega(\mathbf{BU} \times \mathbb{Z}) \simeq \Omega(\Omega U) = \Omega^2 U \tag{4.27}$$

[2]The precise result concerns fibrations, examples of which are fibre bundles with a paracompact base space, see [12], Chapter 2.7, Corollary 14

and similarly:

$$\mathbf{B}\mathrm{U} \times \mathbb{Z} \simeq \Omega U \simeq \Omega^2(\mathbf{B}\mathrm{U} \times \mathbb{Z}), \tag{4.28}$$

which establishes the theorem. □

With this periodicity theorem, we now have the tools necessary to return to the study of K-theory.

Chapter 5

K-Theory as a Cohomological Theory

We begin this chapter investigating some consequences of Theorem 4.6 on the K-rings, including an explicit calculation on the spheres. Along the way, we will use some point-set topology operations to define a negatively-supported sequence of functors, generalising K. Using Bott periodicity we can extend this sequence to the positive integers, and we will show that this defines a generalised cohomology theory, based on the Eilenberg-Steenrod axioms. Finally, we can utilise the tools developed in this chapter to calculate the K-groups of some relatively simple topological spaces.

5.1 Bott Periodicity and K-Theory

Recalling the isomorphism $\widetilde{K}(X) \cong [X, \mathbb{Z} \times \mathbf{B}\mathrm{U}]_0$, the homotopy equivalences of the previous chapter have as a consequence the following:

Theorem 5.1. *There is an isomorphism* $\widetilde{K}(\Sigma^2 X) \to \widetilde{K}(X)$.

Recall that the suspension of a space SX is the quotient of $X \times I$ with $X \times \{0\}$ identified to a point and $X \times \{1\}$ identified with a second point. If (X, x_0) is a pointed space, then the reduced suspension ΣX further identifies the line segment $\{x_0\} \times I$, which has a canonical choice of basepoint compatible with the basepoint of X, namely the point corresponding to $\{x_0\} \times I$. The reduced suspension will be seen many times in this chapter, in particular because it is left adjoint to the loopspace functor in the category of pointed spaces i.e. there is a natural bijection

$$\mathrm{Hom}_0(\Sigma X, Y) \cong \mathrm{Hom}_0(X, \Omega Y)\,. \tag{5.1}$$

This bijection descends to homotopy classes and makes the above theorem a corollary of Bott periodicity. In particular

Proof of 5.1:

$$\begin{aligned}\widetilde{K}(\Sigma^2 X) &\cong [\Sigma^2 X, \mathbf{B}\mathrm{U} \times \mathbb{Z}]_0 \\ &\cong [X, \Omega^2(\mathbf{B}\mathrm{U} \times \mathbb{Z})]_0 \\ &\cong [X, \mathbf{B}\mathrm{U} \times \mathbb{Z}]_0 \cong \widetilde{K}(X)\,.\end{aligned}$$

□

Bott periodicity also allows for the immediate calculation of the group structure of $\widetilde{K}(S^n)$, although the multiplicative structure is not a consequence, and requires more work.

Corollary 5.2 (The K-groups of spheres). *We have the following group isomorphisms, for all $n \in \mathbb{N}$:*

$$\widetilde{K}(S^{2n}) \cong \mathbb{Z}, \tag{5.2}$$
$$\widetilde{K}(S^{2n+1}) \cong 0; \tag{5.3}$$
$$K(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}, \tag{5.4}$$
$$K(S^{2n+1}) \cong \mathbb{Z}. \tag{5.5}$$

Proof. Recall that a homotopy group, $\pi_i(X)$, for $i \geq 0$ denotes pointed homotopy classes of maps $S^i \to X$. Therefore, $\widetilde{K}(S^k) \cong \pi_k(\mathbf{B}\mathrm{U}) \cong \pi_{k+1}(U)$ so we need to compute the two distinct homotopy groups of the stable unitary group, U.

$$\pi_1(U) = \pi_1(U(1)) = \pi_1(S^1) \cong \mathbb{Z}$$
$$\pi_2(U) = \pi_2(U(2)) \cong \pi_2(SU(2)) \cong \pi_2(S^3) = 0\,.$$

By Bott periodicity, we have $\pi_{2n}(U) \cong 0$ and $\pi_{2n+1}(U) \cong \mathbb{Z}$, for arbitrary natural numbers, n. In particular, we have $\widetilde{K}(S^{2n}) \cong \pi_{2n+1}(U) \cong \mathbb{Z}$, whereas $\widetilde{K}(S^{2n+1}) = 0$.

The unreduced K groups follow immediately, using the splitting of the short exact sequence (2.17). □

The generator of $\widetilde{K}(S^2)$ can be found using the isomorphism $\widetilde{K}(S^2) \cong [S^2, \mathbf{B}\mathrm{U} \times \mathbb{Z}]_0 = [S^2, \mathbb{C}\mathrm{P}^1]_0$. The latter equality follows from (4.22). As $\mathbb{C}\mathrm{P}^1 \cong S^2$ we know that the generator of the group is the identity $[\mathrm{Id}] \in [S^2, S^2]_0$. Therefore, the generator of $\widetilde{K}(S^2)$ corresponds to the equivalence class of vector bundles corresponding to the pullback of the inclusion morphism $\mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}\mathrm{P}^\infty$, which is simply the tautological line bundle $H = \{(l, v) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2 : v \in l\}$. That is, $\widetilde{K}(S^2)$ is the subgroup generated by $[H] - 1$ in $K(S^2)$.[1] Now that we have an explicit representative as a vector bundle, we can compute the tensor product of this bundle and learn about the product in the rings $\widetilde{K}(S^2)$ and $K(S^2)$.

In particular, we will show that there is an isomorphism of vector bundles $(H \otimes H) \oplus \Theta^1 \cong H \oplus H$. We can prove this by recalling that $\mathbb{C}\mathrm{P}^1$ can be identified as a quotient S^3/S^1, where we think of $S^3 \subset \mathbb{C}^2$ and quotient out the S^1-action $\xi \cdot (z_1, z_2) := (\xi z_1, \xi z_2)$. The quotient map $h : S^3 \to \mathbb{C}\mathrm{P}^1$ is the famous Hopf fibration. Using this observation, we can work over S^3, where the bundles are trivial. By constructing an explicit, S^1-equivariant isomorphism we will be able to conclude that there is a well-defined isomorphism induced on the quotient bundles.

As a first step, we can identify $h^*H = \{(\vec{z}, v) \in S^3 \times \mathbb{C}^2 : v \in \mathrm{span}_{\mathbb{C}}(\vec{z})\}$, which has a non-vanishing section, $\vec{z} \mapsto (\vec{z}, \vec{z})$ and is therefore trivial. We can use this trivialisation to construct a morphism $\Psi : h^*H \oplus h^*H \to (h^*H \otimes h^*H) \oplus \Theta^1$ by:

$$\Psi(\vec{z}, v, w) := (\vec{z}, v \otimes \vec{z}, w/\vec{z})\,,$$

where $w/\vec{z}$ is the unique complex number, κ, such that $w = \kappa\vec{z}$. Observe that Ψ is a lift of the identity and a fibrewise linear isomorphism, hence a vector bundle isomorphism; it only remains to check S^1-equivariance. The S^1-action induced on the respective bundles by the action on S^3 is as follows:

$$h^*H \oplus h^*H : \qquad \xi \cdot (\vec{z}, v, w) = (\xi\vec{z}, \xi v, \xi w)$$
$$(h^*H \otimes h^*H) \oplus \Theta^1 : \quad \xi \cdot (\vec{z}, v \otimes w, \alpha) = (\xi\vec{z}, \xi^2(v \otimes w), \alpha)$$

so we can check:

$$\Psi(\xi \cdot (\vec{z}, v, w)) = (\xi\vec{z}, \xi v \otimes \xi\vec{z}, \xi\xi^{-1} w/\vec{z}) = \xi \cdot (\vec{z}, v \otimes \vec{z}, w/\vec{z}) = \xi \cdot \Psi(\vec{z}, v, w)\,. \tag{5.6}$$

[1] In this chapter, we will often write n for Θ^n.

Therefore, Ψ is an S^1-equivariant isomorphism and there is an induced isomorphism on S^1-equivalence classes. It can be seen that $(h^*H \oplus h^*H)/S^1 \cong H \oplus H$, and similarly for $(h^*H \otimes h^*H) \oplus \Theta^1$, so there is an induced isomorphism $H \oplus H \cong (H \otimes H) \oplus \Theta^1$. It is actually the case that all vector bundles over S^3 are trivial. We can regard this calculation as demonstrating the general fact that the S^1-action on the bundle determines the bundle on the quotient $\mathbb{C}\mathrm{P}^1$.

In terms of $K(S^2)$ this isomorphism translates to $[H]^2 + 1 = 2[H]$, implying that $([H]-1)^2 = 0$. We can therefore conclude that $K(S^2) \cong \mathbb{Z}[H]/(H-1)^2$ as a ring, and $\widetilde{K}(S^2) = (H-1)K(S^2)$. In particular, the multiplication in $\widetilde{K}(S^2)$ is trivial.

One is able to use these generators to express the generators of higher spheres, $K(S^{2n})$. A key ingredient in that calculation will be a product defined between K-rings of different spaces.

Definition 5.3 (External Product). *For arbitrary compact, Hausdorff X_1, X_2, there is a natural map $\mu : K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$, induced by the projection maps, $p_i : X_1 \times X_2 \to X_i$:*

$$\mu(x \otimes y) = p_1^* x p_2^* y \,; \tag{5.7}$$

here, juxtaposition indicates multiplication in $K(X_1 \times X_2)$, x, y are virtual bundles in $K(X_1), K(X_2)$ respectively and the tensor product between K-rings is the tensor product as $\mathbb{Z}$-modules. This has the ring structure $(x \otimes y)(x' \otimes y') := xx' \otimes yy'$. It is straightforward to check that this map is a well-defined ring morphism.

We sometimes write $\mu(x, y) = x \star y$.

There is a similar product on reduced groups. To see how it works we need to review some topological constructions and will introduce some exact sequences of K-groups. With these tools, we will be able to decompose the external product into trivial parts and the sought after reduced external product.

For now, we restrict attention to the reduced K-rings. Let $A \subset X$ a closed subspace and consider $\widetilde{K}(X/A)$. Observe that as A is closed, the quotient is again compact and Hausdorff, so $\widetilde{K}(X/A)$ is indeed defined.

Proposition 5.4 ([4], Proposition 2.9). *The inclusion and quotient maps $A \to X \to X/A$ induce an exact sequence of rings:*

$$\widetilde{K}(X/A) \overset{q^*}{\to} \widetilde{K}(X) \overset{\iota^*}{\to} \widetilde{K}(A) \,. \tag{5.8}$$

Proof. This proof largely follows [4]. The inclusion $\operatorname{Im} q^* \subset \ker \iota^*$ can be seen by observing $\iota^* q^* = (q\iota)^*$ and $q\iota$ is the composition $A \to A/A \to X/A$, where $\widetilde{K}(A/A) = 0$.

Thus, we have to show that $\ker \iota^* \subset \operatorname{Im} q^*$. Suppose, then, that $[E] \in \widetilde{K}(X)$ is such that $\iota^*[E] = 0$, i.e. $[E|_A] = 0$, implying that $E|_A \oplus \Theta^n(A) \cong \Theta^{n+m}(A)$. Therefore, in $\widetilde{K}$ we may as well assume that $E|_A$ is trivial, with a trivialisation, say $\psi : E|_A \to A \times \mathbb{C}^n$. Let E/ψ denote the quotient space $E/\sim$, where $\psi^{-1}(x, v) \sim \psi^{-1}(y, v)$ for all $x, y \in A$. There are two things to ascertain, firstly that E/ψ is indeed a bundle, and secondly that $q^*(E/\psi) = E$.

The first point amounts to showing that there is an open neighbourhood of A, $A \subset U \subset X$ such that $E|_U$ is trivial. Let $\sigma_i : A \to E$, $i = 1 \ldots, n$ a basis of sections over A. Using that E is a bundle and X compact, we can choose a finite open cover $\{U_\alpha\}$ such that $E|_{U_\alpha}$ is trivial for each α. The trivialisation induces $\tilde{\sigma}_{i\alpha} : A \cap U_\alpha \to \mathbb{C}^n$, and we can extend this to a map $\tilde{\sigma}_{i\alpha} : U_\alpha \to \mathbb{C}^n$, by Tietze's theorem, in turn inducing a section $\sigma_{i\alpha} : U_\alpha \to E$. Let ξ_α a partition of unity subordinate to U_α, and define $\sigma_i = \sum_\alpha \xi_\alpha \sigma_{i\alpha}$. These are n sections over $\bigcup_\alpha U_\alpha$, which are linearly independent over $A \subset \bigcup_\alpha U_\alpha$. These sections must therefore be linearly independent in a neighbourhood of A; we can see this by observing that the determinant is continuous, so $\det(\sigma_1 | \ldots | \sigma_n) \neq 0$ at some point[2] implies nonzero determinant in a neighbourhood of that point. In particular, E/ψ is a vector bundle over X/A.

[2]It is sufficient to calculate the determinant in a local trivialisation, so this notation means the determinant of the matrix with columns being the coordinate-vectors of the sections.

Now, to deduce that $q^*(E/\psi) \cong E$, we use that the natural map $E \to E/\psi$ is a lift of the quotient map, q, and also an isomorphism on the fibres. This is sufficient to conclude that E is the pullback. □

It is worth remarking on the interpretation of the inclusion $\ker \iota^* \subset \operatorname{Im} q^*$ in terms of the classifying space picture. The inclusion says that if f is a classifying map of some bundle over X, such that $f|_A$ is nullhomotopic, then there is an open neighbourhood $U \supset A$ such that $f|_U$ is nullhomotopic. Note that all homotopies are basepoint preserving.

Corollary 5.5. *If A is contractible, then $q^* : \widetilde{K}(X/A) \to \widetilde{K}(X)$ is an isomorphism.*

Proof. This can be shown by constructing an inverse.

Indeed, if A is contractible, then for any map $f : X \to \mathbf{B}\mathrm{U}$, the restriction $f|_A$ is nullhomotopic. By the above, there is an open neighbourhood U such that $f|_U$ is also nullhomotopic with, say, $F : U \times I \to \mathbf{B}\mathrm{U}$, $F_0 = f|_U$ and $F_1 = b_0$, for b_0 the basepoint in $\mathbf{B}\mathrm{U}$. Let $\chi : X \to [0,1]$ a continuous function such that $\chi|_A = 1$ and $\operatorname{supp}(\chi) \subset U$, which exists because X is normal, so Urysohn's lemma holds. We can note that the fact that all maps and homotopies are basepoint preserving is important for uniqueness

Define $Q(f) : X/A \to \mathbf{B}\mathrm{U}$ by

$$Q(f) = \begin{cases} F_{\chi(x)}(x) & x \in U \\ f(x) & x \notin U \end{cases}$$

which is continuous by the choice of χ and F. As $Q(f)$ is constant on A, we can think of it as defining a map $X/A \to \mathbf{B}\mathrm{U}$. It needs to be shown that the homotopy type of Q depends only on the homotopy type of f. So, suppose that $g_t : X \to \mathbf{B}\mathrm{U}$ is some homotopy with $g_0 = f$ and $g_1 = g$, for some g. By contractibility there is a neighbourhood of $A \times I \subset X \times I$, say $V \times I$ such that $g|_{V \times I}$ is nullhomotopic. Letting G any such homotopy, we can now essentially repeat the construction of Q to yield a homotopy between $Q(f)$ and $Q(g)$. Namely, let $\eta : X \to [0,1]$ such that $\eta|_A = 1$ and $\operatorname{supp}(\eta) \subset V$, then

$$Q(g_t) = \begin{cases} G_{\eta(x)}(x,t) & x \in V \\ f(x) & g_t(x) \notin V \end{cases}$$

Therefore, $Q : [X, \mathbf{B}\mathrm{U}]_0 \to [X/A, \mathbf{B}\mathrm{U}]_0$ is well-defined and, by construction, is an inverse to q^*, implying q^* is a bijective homomorphism, and hence an isomorphism. □

It is in fact true that $\mathrm{Vect}^n(X/A) \cong \mathrm{Vect}^n(X)$, for contractible A, which can be useful, though more than needed here, c.f. [4], Lemma 2.10.

This corollary allows one to naturally extend the short exact sequence (5.8) into a long exact sequence. The starting point is to observe that the quotient X/A can be understood as the composition $X \hookrightarrow X \cup CA \to X/A$, where CA denotes the cone, which is contractible, and the second arrow quotients out the cone. In particular, the quotient is the composition of an inclusion, and the quotient of a contractible space. At the level of $\widetilde{K}$-rings only the inclusion is meaningful. This motivates the long sequence:

$$\begin{array}{ccccccccc} A & \hookrightarrow & X & \hookrightarrow & X \cup CA & \hookrightarrow & (X \cup CA) \cup CX & \hookrightarrow & ((X \cup CA) \cup CX) \cup C(X \cup CA) \dots \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & X/A & & SA & & SX \end{array} \tag{5.9}$$

where the spaces in the first row are obtained by adding the cone of the space two steps back in the sequence, and the vertical maps contract out the recently added cone; the S here denotes the suspension of the space. This construction is known as the cofibration sequence, or Puppe sequence

([5], p. 398) and induces a long exact sequence on the $\widetilde{K}$-rings, by repeated iteration of the short exact sequence. For our purposes, it is more clearly seen by iterating vertically:

$$\begin{array}{ccccccccccc}
A_0 & \hookrightarrow & A_1 & \hookrightarrow & A_1 \cup CA_0 & \hookrightarrow & A_2 \cup CA_1 & \hookrightarrow & \dots & & \\
 & & & & \downarrow & \searrow & \downarrow & \searrow & & & \\
 & & & & A_1/A_0 & & SA_0 & \hookrightarrow & SA_1 & \hookrightarrow & SA_1 \cup C(A_1/A_0) \hookrightarrow \dots \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & SA_1/SA_0 \longrightarrow \dots
\end{array} \tag{5.10}$$

where the dashed arrows mean they are simply the composition of the known horizontal and vertical maps. Then, applying the $\widetilde{K}$ functor:

$$\widetilde{K}(A_0) \longleftarrow \widetilde{K}(A_1) \longleftarrow \widetilde{K}(A_1/A_0) \dashleftarrow \widetilde{K}(SA_0) \longleftarrow \widetilde{K}(SA_1) \longleftarrow \dots . \tag{5.11}$$

By being careful with the above diagram, one can check that the non-dashed arrows are simply the relevant inclusion or quotient map, while the dashed arrow is the composition of the map induced by $A_1 \cup CA_0 \to SA_0$ and the isomorphism $\widetilde{K}(A_1/A_0) \cong \widetilde{K}(A_1 \cup CA_0)$, and similarly in higher places. This implies that any sequence of three groups in (5.11) is exact and so the entire sequence is exact.

More over, this construction is functorial: any map between pairs $f : (X, A) \to (Y, B)$ induces a map of complexes on the long exact sequence. This is straightforward to see, though messy to explicate, so I omit doing so.

Recalling that the reduced suspension is obtained from the suspension by simply contracting an interval, we can replace the suspension with the reduced suspension and maintain exactness. It is convenient to do so, because the reduced suspension has several beneficial properties, for instance there is a homeomorphism $\Sigma(X/A) \cong \Sigma X/\Sigma A$ and the functor Σ is adjoint to the loop space functor, as has already been seen. Another property of the reduced suspension is the existence of the homeomorphism $\Sigma^n X \cong S^n \wedge X$, which will be useful. Here $\wedge$ denotes the smash product of pointed spaces, which is the quotient of the product space $S^n \times X$ by the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$. This latter subspace is denoted $X \vee Y$, the wedge sum.

Observe that the space obtained by quotienting one of the factors in the wedge sum is canonically homeomorphic to the other space, i.e. $(X \vee Y)/X \cong Y$, and $(X \vee Y)/Y \cong X$. Therefore, the pair $(X \vee Y, Y)$ yields a short exact sequence:

$$\widetilde{K}(X) \to \widetilde{K}(X \vee Y) \to \widetilde{K}(Y) \tag{5.12}$$

in which the first map is a composition of the isomorphism $\widetilde{K}(X) \cong \widetilde{K}((X \vee Y)/Y)$ and the map induced by the quotient. This sequence in fact splits, as the sequence obtained from the pair $(X \vee Y, X)$ yields a sequence of the same rings, with arrows in the other direction. We can observe that the maps are inverse to each other. Therefore, $\widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y)$. This implies $\widetilde{K}(\Sigma(X \vee Y)) \cong \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y)$, because there is a homeomorphism $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

Now we consider the compact pair $(X \times Y, X \vee Y)$; by definition $(X \times Y)/(X \vee Y) = X \wedge Y$, and so we have an induced long exact sequence:

$$\cdots \widetilde{K}(\Sigma(X \times Y)) \longrightarrow \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \longrightarrow \widetilde{K}(X) \oplus \widetilde{K}(Y) . \tag{5.13}$$

Let $p_1 : X \times Y \to X$ the projection, similarly for p_2, and consider the morphism $\widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \times Y)$ given by $(x, y) \mapsto p_1^* x + p_2^* y$. Observe that for arbitrary points $x_0 \in X$, $y_0 \in Y$ we have a natural isomorphism $p_1^* x|_{\{x_0\} \times Y} \cong x|_{\{x_0\}} \times Y$ and similarly for $p_2^* y|_{X \times \{y_0\}}$, implying that the morphism $\widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \times Y)$ splits in the sequence above. In particular $\widetilde{K}(X \times Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y)$ is surjective. The same argument shows that $\widetilde{K}(\Sigma(X \times Y)) \to \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y)$ is surjective. Therefore, the long exact sequence yields a split short exact sequence

$$0 \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \longrightarrow \widetilde{K}(X) \oplus \widetilde{K}(Y) \longrightarrow 0 , \tag{5.14}$$

implying the isomorphism:

$$\widetilde{K}(X \times Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \widetilde{K}(X \wedge Y) . \tag{5.15}$$

We can also observe that the following sequence is exact

$$0 \longrightarrow \widetilde{K}(\Sigma(X \wedge Y)) \longrightarrow \widetilde{K}(\Sigma(X \times Y)) \longrightarrow \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \longrightarrow 0 , \tag{5.16}$$

where Bott periodicity is used to show injectivity of the homomorphism $\widetilde{K}(\Sigma(X \wedge Y)) \to \widetilde{K}(\Sigma(X \times Y))$. The above comments show that this sequence also splits.

With this decomposition in hand we can finally define a reduced external product. To do so, use the splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$ and recall that $K(x_0)$ is a ring isomorphic to the integers, therefore:

$$\begin{aligned} K(X) \otimes K(Y) &\cong (\widetilde{K}(X) \oplus K(x_0)) \otimes (\widetilde{K}(Y) \oplus K(y_0)) \\ &\cong \big(\widetilde{K}(X) \otimes \widetilde{K}(Y)\big) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} , \end{aligned}$$

having used the general fact that $M \otimes_R R \cong R$, for any ring R, and R-module M.

Combining the external product and these decompositions, we have

$$\begin{array}{ccc} K(X) \otimes K(Y) & \xrightarrow{\sim} & \big(\widetilde{K}(X) \otimes \widetilde{K}(Y)\big) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \\ \downarrow & & \downarrow \\ K(X \times Y) & \xrightarrow{\sim} & \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} . \end{array} \tag{5.17}$$

The key observation here is that summand $\big(\widetilde{K}(X) \otimes \widetilde{K}(Y)\big)$ maps to $\widetilde{K}(X \wedge Y)$. For, recalling the definition of the reduced groups as kernels, $x \in \widetilde{K}(X)$, $y \in \widetilde{K}(Y)$, then $p_1^* x p_2^* y$ is in the kernel of the restriction $K(X \times Y) \to K\big((\{x_0\} \times Y) \cup (X \times \{y_0\})\big)$, so that $\mu(x \otimes y) \in \widetilde{K}(X \wedge Y)$. In particular, the righthand arrow can be decomposed as $(\mu, \mathrm{Id}_{\widetilde{K}(X)}, \mathrm{Id}_{\widetilde{K}(Y)}, \mathrm{Id}_{\mathbb{Z}})$.

This product, as well as the unreduced version, can be used to explicitly compute the Bott isomorphism as demonstrated by the following two results. Unfortunately, I have no proof of this fact from the perspective of Bott periodicity developed here, although such a proof should exist and I aim to find it in the future.

Lemma 5.6. *For any compact X, the composition $\widetilde{K}(X) \to \widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(S^2 \wedge X)$, $x \mapsto \mu(x, H-1)$ is the isomorphism of Theorem 5.1.*

Corollary 5.7. *The external product $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism.*

Proof. The diagram 5.17 and the discussion following it, along with Lemma 5.6 gives the result immediately. ☐

Assuming Lemma 5.6, we can explicitly state the generators of the K-rings of the higher spheres. Indeed, the generator of $\widetilde{K}(S^{2n})$ is simply the n-th power external product of the generator $\widetilde{K}(S^2)$, i.e. $\widetilde{K}(S^{2n}) \cong ([H]^{\star n} - 1)$, and has trivial multiplication. Then, using that $K(S^{2n}) \cong \widetilde{K}(S^{2n}) \oplus \mathbb{Z}$, we can conclude that $K(S^{2n}) \cong \mathbb{Z}[H^{\star n}]/(1 - H^{\star n})^2$.

5.2 Eilenberg-Steenrod Axioms of a Reduced Cohomology Theory

In this section we use the long exact sequence (5.20) and Bott periodicity to develop a sequence of functors describing a cohomology theory.

We begin by reviewing the axiomatic characterisation of cohomology theories, due to Eilenberg and Steenrod, [14]. Let $\mathcal{C}$ some category of pairs of topological spaces $A \subset X$. A cohomology theory is the data of a sequence of functors into the category of Abelian groups

$$F^k : \mathcal{C} \to \mathbf{Ab}, \qquad k \in \mathbb{Z}$$

along with natural transformations

$$\delta : F^k(A) \to F^{k+1}(X, A),$$

(where $F^k(A) := F^k(A, \varnothing)$) satisfying the following axioms:

Axiom 1: If $f \simeq g : (X, A) \to (Y, B)$, then $f^* = g^* : F^k(Y, B) \to F^k(X, A)$ for all $k \in \mathbb{Z}$, where $f^* := F^k(f)$;

Axiom 2: For a pair (X, A), the above maps induce a long exact sequence

$$\dots \longrightarrow F^k(X, A) \longrightarrow F^k(X) \longrightarrow F^k(A) \longrightarrow F^{k+1}(X, A) \longrightarrow \dots; \tag{5.18}$$

Axiom 3 (Excision): If $U \subset X$ is an open subset such that the closure $\bar{U}$ is in the interior of A, $\bar{U} \subset A^o$, then $F^k(X \smallsetminus U, A \smallsetminus U) \cong F^k(X, A)$;

Axiom 4 $F^k(*) = 0$, for all $k \neq 0$, where $*$ is the one-point space.

Naturality of δ, along with Axiom 2, implies that the maps $\{F^k(f)\}$ form a map of chain complexes between the long exact sequences induced by any two pairs (X, A), (Y, B).

It is known that these four axioms uniquely specify a cohomology theory, however there are many inequivalent theories satisfying the first three axioms alone. If we need to distinguish, a cohomology theory without the dimension axiom is *extraordinary*, or *generalised*.

We will use the long exact sequences defined above to show that the groups $K(X)$ give rise to a generalised cohomology theory. The long exact sequence (5.11) suggests the definition $\widetilde{K}^{-n}(X) := \widetilde{K}(\Sigma^n X)$, for all $n \geq 0$. Observe that the Bott periodicity isomorphisms developed in the previous section imply $\widetilde{K}^{-2n}(X) \cong \widetilde{K}^0(X)$, and $\widetilde{K}^{-2n-1}(X) \cong \widetilde{K}^{-1}(X)$. Therefore, we can reasonably extend the sequence to positive integers by setting $\widetilde{K}^{2n}(X) \cong \widetilde{K}(X)$, $\widetilde{K}^{2n+1}(X) \cong \widetilde{K}(\Sigma X)$.

This periodicity can be expressed more concretely by wrapping the long exact sequence in on itself:

$$\begin{gathered}\widetilde{K}^{-2}(A) \to \widetilde{K}^{-1}(X/A) \to \widetilde{K}^{-1}(X) \longrightarrow \widetilde{K}^{-1}(A) \longrightarrow \widetilde{K}^0(X/A) \to \widetilde{K}^0(X) \to \widetilde{K}^0(A) \xrightarrow{\cong} \\ \widetilde{K}^{-2}(A) \to \widetilde{K}^{-1}(X/A) \dots\end{gathered} \tag{5.19}$$

We can lift these results on the reduced rings to statements about the full K-rings by observing that $K(X)$ can be identified with the reduced group $\widetilde{K}(X_+) = \ker(+ \to X_+) \cong K(X)$, where $X_+ := X \coprod \{+\}$. For any closed $A \subset X$, we have an induced $A_+ = A \coprod \{+\}$, and $X/A \cong X_+/A_+$. It can also be shown that $\widetilde{K}(\Sigma X_+) \cong \widetilde{K}(\Sigma X)$ using the six-term exact sequence obtained from the wrapped long exact sequence (5.19).

Specifically, the sequence of spaces $X \hookrightarrow X_+ \to S^0$ induces the exact rectangle:

$$\begin{array}{ccccc}\widetilde{K}(S^0) & \xrightarrow{q^*} & \widetilde{K}(X_+) & \xrightarrow{\iota^*} & \widetilde{K}(X) \\ \uparrow & & & & \downarrow \\ \widetilde{K}(\Sigma X) & \xleftarrow{(\Sigma\iota)^*} & \widetilde{K}(\Sigma X_+) & \xleftarrow{(\Sigma q)^*} & \widetilde{K}(S^1)\end{array}$$

where $\widetilde{K}(S^1) = 0$ immediately implies $(\Sigma\iota)^*$ is injective. It remains to show surjectivity of this map, which is equivalent to injectivity of q^* or, to put it another way, the existence of a left-inverse of q^*. Such a map is guaranteed by the splitting of the diagram (2.17).

Therefore, the long exact sequence on reduced groups, induced by $A_+ \hookrightarrow X_+ \to X/A$ induces a long exact sequence:

$$\ldots \to \widetilde{K}(\Sigma X) \to \widetilde{K}(\Sigma A) \to \widetilde{K}(X/A) \to K(X) \to K(A)\,. \tag{5.20}$$

Motivated by this sequence, we define the functors $K^n(X,A) := \widetilde{K}^n(X/A)$, whenever $A \neq \varnothing$, $K^{2n+1}(X) := \widetilde{K}^{2n+1}(X)$ and $K^{2n}(X) := K(X)$, for all $n \in \mathbb{Z}$.[3] This implies that we have an induced exact sequence

$$\begin{array}{ccccc} K^0(X,A) & \longrightarrow & K^0(X) & \longrightarrow & K^0(A) \\ \uparrow & & & & \downarrow \\ K^1(A) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,A) \end{array}\,. \tag{5.21}$$

Theorem 5.8. *The sequence of functors $K^n(X)$ defines a generalised cohomology theory on the category of compact pairs.*

Proof. The first axiom, homotopy invariance, follows from the interpretation of K as a space of homotopy classes of maps.

The second axiom is satisfied, due to the exact sequence (5.21), and by construction, the maps induced by any $f : (X,A) \to (Y,B)$ form a morphism of chain complexes.

It remains to check excision. Recalling $K^{-n}(X,A) := \widetilde{K}^{-n}(X/A)$, and the fact that $\bar{U} \subset A^o$ implies $(X \smallsetminus U)/(A \smallsetminus U) = X/A$, it follows that the morphism induced by the inclusion $(X \smallsetminus U, A \smallsetminus U) \to (X,A)$ is indeed an isomorphism. □

An important component of K-theory is that $K^*(X) := \bigoplus_{n\in\mathbb{Z}} K^n(X)$ is in fact a ring with product induced by the external product $\widetilde{K}(\Sigma^i X) \otimes \widetilde{K}(\Sigma^j X) \to \widetilde{K}(\Sigma^{i+j} X)$, which is related to the reduced product defined above. Unfortunately, there is no space here to discuss this aspect in further detail. Instead, the remainder of the thesis will be spent calculating examples using the machinery developed.

5.3 Examples

We will calculate the reduced groups of several simple topological spaces; the full groups are then obtained by adding a copy of the integers onto the even groups.

Example 5.9. *The first example we consider is the n-torus, $\mathbf{T}^n = (S^1)^{\times n}$*

The claim is that :

$$\widetilde{K}^i(T^n) \cong \begin{cases} \mathbb{Z}^{\oplus(2^{n-1}-1)} & i \text{ even} \\ \mathbb{Z}^{\oplus 2^{n-1}} & i \text{ odd} \end{cases}, \tag{5.22}$$

which is proved by induction.

For $n = 1$, it immediately follows from the identification $\mathbf{T}^1 = S^1$.

Assume that the claim holds for $\mathbf{T}^{n-1}$ and consider $\mathbf{T}^n = \mathbf{T}^{n-1} \times S^1$. Comparing with (5.15)*, the sequence $S^1 \vee \mathbf{T}^{n-1} \to \mathbf{T}^{n-1} \times S^1 \to \mathbf{T}^{n-1} \wedge S^1$, induces the isomorphism*

$$\widetilde{K}^i(\mathbf{T}^n) \cong \widetilde{K}^i(S^1) \oplus \widetilde{K}^i(\mathbf{T}^{n-1}) \oplus \widetilde{K}^{i+1}(\mathbf{T}^{n-1})\,. \tag{5.23}$$

By the inductive hypothesis we therefore have $\widetilde{K}^0(\mathbf{T}^n) \cong \mathbb{Z}^{\oplus(2^{n-2}+2^{n-2}-1)}$ and $\widetilde{K}^1(\mathbf{T}^n) \cong \mathbb{Z}^{\oplus(2^{n-2}+2^{n-2}-1+1)}$, so the result holds by the fact that $2^{n-2} + 2^{n-2} = 2^{n-1}$.

[3] It is reasonably natural to define $X/\varnothing := X_+$, in which case all these functors can be summarised by the definition of $K^n(X,A)$.

Example 5.10. *We next consider the 3-dimensional lens space. This is slightly more complicated, and will be achieved by applying a CW-structure to the space; this structure is taken from [5], p. 145, applied to the case of dimension 3.*

A lens space $L(k,l)$, *for* $k,l \in \mathbb{N}$ *relatively prime, is obtained from* $S^3 \subset \mathbb{C}^2$ *by quotienting with a* $\mathbb{Z}_l$ *group action* $(z_1, z_2) \sim (e^{2\pi i/l} z_1, e^{2\pi i k/l} z_2)$. *Set* $C \subset S^3$ *the circle defined by* $z_1 = 0$, *and mark the* l*-th roots of unity* $1, \lambda = e^{2\pi i/l}, \ldots, \lambda^{l-1}$. *Let* $A \subset S^3$ *the circle defined by* $z_2 = 0$ *and consider the discs* D_j^2 *formed from the great circles between the marked point,* $\lambda^j \in C$, *and the circle* A. *Similarly, define a three-disc* D_j^3 *as the points on the great circles starting at any point in the segment between* λ^j *and* λ^{j+1}. *Observe that the discs* D_j^3 *cover* S^3, *and that* D_j^3 *is bounded by* D_j^2 *and* D_{j+1}^2. *Furthermore,* $D_j^2 \mapsto D_{j+k}^2$ *(more precisely,* $(j+k) \mod l$*) under the group action and* $D_j^3 \mapsto D_{j+k}^3$. *In particular, by choosing* α *such that* $\alpha k \equiv 1 \mod l$, *we can iterate the action* α *times to obtain a map* $D_j \mapsto D_{j+1}$. *Clearly,* λ^α *is a generator of the group* $\mathbb{Z}_l$, *because* $\gcd(\alpha, l) = 1$. *Therefore, we can obtain* $L(k,l)$ *by identifying the bounding discs of* D^3 *under the action* λ^α. *This is the key in constructing the CW-complex.*

Start the construction with a 0-skeleton, $\{x_0\}$, *and attach a 1-cell such that* X_1 *is a circle; the one-skeleton corresponds to* A *in the above. The attaching map for a 2-cell,* $S^1 \to S^1$, *is given by the quotient map* $z \sim \lambda z$. *Finally, attach a 3-cell, where the attaching map identifies its bottom hemisphere with the top hemisphere after a twist of* λ^α, *which are then identified with* X_2.

We now want to calculate $\widetilde{K}^*(L)$ *using this CW-structure. The key to this calculation is the observation that* $X_3/X_2 \cong S^3$, $X_2/X_1 \cong S^2$. *In fact, in a general CW-complex,* X_n/X_{n-1} *is homeomorphic to a wedge sum of* n*-spheres, for* $n \geq 1$. *We are able to capitalise on this because the CW-structure of the lens space is quite simple, having only one cell in each level.*

Indeed, the six-term exact sequence corresponding to the pair (L, X_2) *will allow the computation of* $\widetilde{K}^*(L)$ *in terms of* $\widetilde{K}^*(X_2)$, *and the sequence corresponding to* (X_2, S^1) *will give* $\widetilde{K}^*(X_2)$. *Starting with* X_2, *we have the exact sequence:*

$$\begin{array}{ccccc} \widetilde{K}^0(S^2) & \longrightarrow & \widetilde{K}^0(X_2) & \longrightarrow & \widetilde{K}^0(S^1) \\ \uparrow{\scriptstyle \delta} & & & & \downarrow \\ \widetilde{K}^1(S^1) & \longleftarrow & \widetilde{K}^1(X_2) & \longleftarrow & \widetilde{K}^1(S^2). \end{array} \tag{5.24}$$

It follows that $\widetilde{K}^1(X_2) \cong \ker \delta$ *and* $\widetilde{K}^0(X_2) \cong \operatorname{coker} \delta$, *so that only the coboundary map* δ *needs to be identified. To do so, recall that the generator of* $\widetilde{K}^0(S^2)$ *can be identified with the generator of* $\pi_2(S^2)$, *which is the suspension of the identity map* $S^1 \to S^1$, *cf. the proof of Corollary 5.2. When we quotient* X_2/X_1 *we have the 2-disk* D_2 *with its boundary first identified with* X_1, *and then shrunk to a point. In our case, the attaching map wraps the boundary of the disk* l*-times, and thus, the identity gets mapped to the suspension of* z^l, *implying that* δ *is multiplication by* l. *It follows immediately that* $\widetilde{K}^1(X_2) = 0$ *and* $\widetilde{K}^0(X_2) \cong \mathbb{Z}_l$.

Plugging this result into the exact sequence corresponding to (L, X_2) *gives:*

$$\begin{array}{ccccc} 0 & \longrightarrow & \widetilde{K}^0(L) & \longrightarrow & \mathbb{Z}_l \\ \uparrow & & & & \downarrow{\scriptstyle \partial} \\ 0 & \longleftarrow & \widetilde{K}^1(L) & \longleftarrow & \mathbb{Z} \end{array} . \tag{5.25}$$

Observe that $\partial \equiv 0$, *because there is no other group morphism from* $\mathbb{Z}_l \to \mathbb{Z}$, *given that the latter has no zero divisors. Therefore,* $\widetilde{K}^0(L) \cong \widetilde{K}^0(X_2) \cong \mathbb{Z}_l$ *and* $\widetilde{K}^1(L) \cong \mathbb{Z}$.

We have defined K-theory using vector bundles and their formal inverses. With this geometric interpretation we were able to represent the functors as homotopy classes of maps into some explicit models of classifying space. By focusing on this interpretation we were able to use differential topology to uncover Bott periodicity, enabling us to encounter the essential features of K-theory and calculate some examples. This barely scratches the surface of topological K-theory, which has a number of generalisations and connects with many different parts of mathematics. Interesting examples include the Atiyah-Singer index theorem, which associates K-theoretic data to the index of elliptic operators and thus relates topologic and analytic information, as well as applications in physics, including string theory and condensed matter physics.

Appendix A

Some Riemannian Geometry

This is a compilation of various results in Riemannian geometry that are used in the main text. The main reference is [6].

A.1 Basic Definitions

Definition A.1. *A Riemannian manifold, (M, g) is a smooth manifold M together with a bundle metric on the tangent bundle.*

Just as complex vector bundles can always be endowed with Hermitian bundle metrics, a real vector bundle can always be endowed with a symmetric bundle metric - the proof is essentially the same. Thus, a Riemannian structure exists on any given smooth manifold.

A.2 Covariant Derivatives and Geodesics

Let $E \to M$ a smooth vector bundle, $\Gamma(E)$ the space of sections and $\Omega(TM) := \Gamma(T^*M)$, where TM denotes the tangent bundle, and T^*M its dual.

Definition A.2 (Covariant Derivative)**.** *A covariant derivative is a map $D : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$, written $D(V \otimes \sigma) = D_V \sigma$ and satisfying:*

- *D is $C^\infty(M)$ linear in $\Gamma(TM)$;*
- *D is $\mathbb{R}$ linear in $\Gamma(E)$, and satisfies a Leibnitz rule:*

$$D_V(f\sigma) = f D_V \sigma + V(f)\sigma\,. \tag{A.1}$$

Thus, a covariant derivative is essentially a means of differentiating sections. We can also use them to transport a vector over a curve, viz.

Definition A.3 (Parallel Transport)**.** *Let $V \in E_p$, and γ any smooth curve starting at p; then we can define a vector field over γ, such that $D_{\dot{\gamma}(t)} V_{\gamma(t)} = 0$, $\forall t$.*

In the text, we will often use the notation $V_t := V_{\gamma(t)}$, and $D_t := D_{\dot{\gamma}(t)}$.

A geodesic is a curve γ satisfying $D_t \dot{\gamma(t)} = 0$; i.e. the tangent vector is parallel transported along the curve.

Locally, the geodesics are length-minimizing paths.

Definition A.4. *The Levi-Civita connection on a Riemannian manifold is a connection on the tangent bundle, satisfying:*

Torsion-Free $D_X Y - D_Y X = [X, Y]$, *for all smooth vector fields X, Y;*

Metric-Preserving $X(g(Y,Z)) = g(D_X Y, Z) + g(Y, D_X Z)$ *for all smooth vector fields* X, Y, Z.

A Levi-Civita connection exists and is unique, [6], Theorem 4.3.1.

In this thesis, the connection is always assumed to be the Levi-Civita connection.

A.3 Curvature and Jacobi Fields

Theorem A.5 ([8], Lemma 9.3)**.** *For* (M,g) *a Riemannian manifold, and* X,Y,Z,W *any vector fields, the curvature*

$$R(X,Y)Z := [D_X, D_Y]Z - D_{[X,Y]}Z$$

satisfies the following identities:

$$R(X,Y)Z + R(Y,X)Z = 0 \tag{A.2}$$
$$R(X,Y)Z + R(Y,Z)X + R(Z,Y)X = 0 \tag{A.3}$$
$$g(R(X,Y)Z,W) = -g(R(X,Y)W,Z) \tag{A.4}$$
$$g(R(X,Y)Z,W) = g(X,R(Z,W)Y)\,. \tag{A.5}$$

Let γ a geodesic, V its velocity vector field and X a smooth vector field over γ. We say X is a Jacobi field over γ if it satisfies the identity:

$$D_t^2 X - R(V,X)V = 0 \tag{A.6}$$

Proposition A.6 ([6], Lemma 5.2.3)**.** *For any given* $X, Y \in T_{\gamma(0)}M$ *there exists a unique Jacobi field over* γ, W, *such that* $W(0) = X$ *and* $D_t W(0) = Y$

Proposition A.7 ([6], Theorem 5.2.1)**.** *Let* γ *a geodesic, and* α *a variation of* $\alpha : I \times (-\epsilon, \epsilon) \to M$ *through geodesics. Then,* $\partial_s \alpha(t,0)$ *is a Jacobi vector field. Conversely, any Jacobi field over* γ *gives rise to a variation through geodesics.*

Definition A.8. *Let* $\gamma : I \to M$ *a geodesic. Two points,* $t_0 \neq t_1$ *are called* conjugate *if there exists a non-zero Jacobi field,* J, *over* γ *such that* $J(t_0) = 0 = J(t_1)$.

Proposition A.9. *If* $\gamma(0), \gamma(1)$ *are not conjugate along* γ, *then for any* $X_0 \in T_{\gamma(0)}M$, $X_1 \in T_{\gamma(1)}M$ *there is a unique Jacobi field* X *such that* $X(0) = X_0$ *and* $X(1) = X_1$.

A.4 Lie Groups

Theorem A.10. *The geodesics through the identity of a Lie group equipped with a left- and right-invariant metric are precisely the one-parameter subgroups.*

As a result, the covariant derivative $D_X X$ of any left invariant vector field vanishes. This is key in the following:

Theorem A.11 ([8], Theorem 21.3)**.** *For* G *a Lie group equipped with left- right-invariant Riemannian metric and* X,Y,Z,W *left-invariant vector fields, the following hold:*

$$D_X Y = -D_Y X \tag{A.7}$$
$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle \tag{A.8}$$
$$R(X,Y)Z = \tfrac{1}{4}[[X,Y],Z] \tag{A.9}$$
$$\langle R(X,Y)Z, W\rangle = \tfrac{1}{4}\langle [X,Y],[Z,W]\rangle\,. \tag{A.10}$$

Bibliography

[1] M. F. Atiyah and I. M. Singer. The index of elliptic operators: I. *Annals of Mathematics*, 87(3):484–530, 1968.

[2] Daniel Dugger. A geometric introduction to K-theory. `http://math.uoregon.edu/~ddugger/kgeom.pdf`. Date accessed 02–03–2017.

[3] Pierre Antoine Grillet. *Abstract Algebra, 2nd Edition.* Springer, 2007.

[4] Allen Hatcher. *Vector Bundles and K-Theory.* unpublished.

[5] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.

[6] Jürgen Jost. *Riemannian Geometry and Geometric Analysis.* Springer, 2011.

[7] J. Milnor. On spaces having the homotopy type of CW-complexes. *Trans. Amer. Math. Soc*, 90(2), 1959.

[8] J. Milnor. *Morse Theory.* Princeton University Press, 1973.

[9] Stephen A. Mitchell. Notes on principal bundles and classifying spaces. `http://www.math.washington.edu/~mitchell/Quillen/prin.pdf`. Date accessed 17 – 11 – 2016.

[10] Efton Park. *Complex Topological K-Theory.* Cambridge University Press, 2008.

[11] Victor Guillemin, Alan Pollack. *Differential Topology.* AMS Chelsea Publishing, 1974.

[12] Edwin H. Spanier. *Algebraic Topology.* McGraw-Hill Book Company, 1966.

[13] Norman Steenrod. *Topology of Fibre Bundles.* Princeton University Press, 1951.

[14] Samual Eilenberg, Norman Steenrod. Axiomatic approach to homology theory. *Proc. Natl. Acad. Sci*, 31:117–120, 1945.

[15] Ravi Vakil. The rising sea: Fundamentals of algebraic geometry. `http://math.stanford.edu/~vakil/216blog/index.html`. Date accessed 29–01–2017.