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Pathological functions and the Baire category theorem

Pouya Ashraf

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Handledare: Gunnar Berg
Examinator: Jörgen Östensson
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Department of Mathematics
Uppsala University

Abstract

In this document, we will present and discuss a small piece of the mathematical developments of the 19^{th} century, specifically regarding problems in analysis and the concept of functions which are everywhere continuous and at the same time nowhere differentiable. Several early examples of such functions will be presented. We will also, in relation to this, present the Baire Category Theorem, and later utilize this to show that functions possessing the aforementioned properties on a closed interval in fact constitute the majority of continuous functions on that interval.

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1 Introduction

> *Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.*
>
> - Henri Poincaré [21, p. 132]

The above quote sheds some light on the current state of affairs in mathematics in the year of 1908. These 'bizarre' functions mentioned are functions with the property of being continuous everywhere, and differentiable nowhere. In the context of this document, we will refer to these as *pathological functions.*

When first presented with the idea of derivatives, it is generally accepted rather quickly that a continuous function may lack a finite derivative at one or more points, like the function $f(x) = |x|$, which is continuous everywhere, and possesses a finite derivative everywhere, except in the single point $x = 0$, where the left- and right-hand-side derivatives don't agree.

That there exist functions which are everywhere continuous, and *nowhere* differentiable, makes much less intuitive sense, and up until the latter part of the 19^{th} century, were widely believed not to exist. When examples of these kinds of functions first began to crop up, many were initially shocked. They directly contradicted proofs and mathematical reasoning that had been accepted as correct for many years.

In this document we will discuss the following: The history of pathological functions, starting with Ampère in the early 19^{th} century. We will then present some examples of interesting pathological functions through the years, up until the beginning of the 20^{th} century (intentionally leaving out Weierstrass' function, since it is by far the most famous and commonly studied example). Finally, we will present the Baire Category Theorem and utilize it to show that the set of pathological functions not only contains an infinite number of elements, but that these functions in fact constitute the vast majority of all functions. We will also show one of the theorem's applications in functional analysis.

2 Background

2.1 Ampère, and the differentiability of continuous functions

In the mathematical community, the discussion on the existence of the derivative of an arbitrary function originates in the proposed reformation of the foundations of calculus by the 18^{th} century Italian mathematician Joseph Louis Lagrange. His proposition was to base the differential calculus on

$$f(x+i) = f(x) + pi + qi^2 + \dots, \tag{1}$$

where $i \in \mathbb{R}$ and p, q are functions of x. From (1), he deduced the form of the derivatives of f through purely formal reasoning, avoiding many difficulties that arose when dealing with infinitesimal quantities in the 18^{th} century [12, p. 43].

A corollary to (1) is the proposed property that any function (in the context of 19^{th} century mathematics) possesses a derivative in general, i.e. everywhere except the points in which the equation above does not hold.

Ampère tried to prove the corollary without the use of Lagrange's equation, which he then wanted to use to prove said equation, without Lagrange's arguments. This proof [2] paved the way for subsequent proofs, but was unfortunately faulty in its conclusion. Nevertheless, in the years following the publication of Ampère's paper up until the first example of a pathological function was presented by Karl Weierstrass in the 1870:s, the proposed property that a given function is differentiable in general was both stated and proved in, what was at the time, leading literature on analysis e.g. Lacroix's *Traité* [17, p. 241-242]:

In *Traité*, a proof of the existence of the derivative due to professor Binet at Lycée de Rennes is presented. The proof goes as follows:

Proof. Let $x' - x = h$. If we consider the interval of length h partitioned into n intervals of equal length, and if we let x' and x be two consecutive elements in the sequence

$$\left\{x + \frac{kh}{n}\right\}_{k=0}^{n} = x,\ x + \frac{h}{n},\ x + \frac{2h}{n},\ \dots\ ,\ x + \frac{nh}{n}$$

the fraction $\frac{f(x')-f(x)}{x'-x}$, for each element in the sequence attains the values

$$\frac{f(x+\frac{h}{n})-f(x)}{\frac{h}{n}}, \frac{f(x+\frac{2h}{n})-f(x+\frac{h}{n})}{\frac{h}{n}}, \ldots, \frac{f(x+\frac{nh}{n})-f(x+\frac{(n-1)h}{n})}{\frac{h}{n}} \tag{2}$$

which all have the same sign [sic], if the function is either increasing or decreasing, which must be the case if we choose $x'-x$ to be sufficiently small. Under this hypothesis, the above sequence of values is reduced to the following, if we calculate the sum of the sequence and eliminate terms of opposite sign:

$$\frac{f(x+h)-f(x)}{\frac{h}{n}} = n\frac{f(x+h)-f(x)}{h} = n\frac{f(x')-f(x)}{h}.$$

From this it follows that, no matter how large n, and consequently how small the difference $\frac{h}{n}$ between two consecutive values of x' and x is, the quantities in (2) cannot all be strictly smaller or larger than $\frac{f(x')-f(x)}{x'-x}$, since in the first case, a sum of n of these values cannot be as large as $n\frac{f(x')-f(x)}{x'-x}$, and in the second case constitute as small a sum as this quantity.

But the quantity is subject to the laws of continuity and cannot vanish without becoming arbitrarily small, or be infinite without beforehand becoming arbitrarily large. Thus all the values $\frac{f(x')-f(x)}{x'-x}$ cannot vanish or become infinite in the interval between x' and x, regardless of how small this interval is chosen. As such, the ratio has a finite and non-vanishing limit. □

The above proof relies on the statement that a function must be monotonic on an arbitrary interval h, if h is sufficiently small. This, however is not the case. The function $x^2 \sin(x^{-1})$, which is not monotonic in any neighbourhood of 0.

2.2 Mathematics in the 19^{th} century

During the very first years of the 19^{th} century, Joseph Fourier published his solution to the heat equation [10], in which he used his theory on modeling periodic functions by trigonometric series.

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\tfrac{2\pi mx}{T}) + b_m \sin(\tfrac{2\pi mx}{T})$$

This technique proved to be extremely useful, and gave rise to an entire field of study, now called Fourier Analysis, and has since been applied on countless problems in a diverse set of disciplines, including (but not limited to) electrical engineering, signal analysis, quantum mechanics, acoustics, and optics.

In a series of publications starting with the *Cours d'Analyse* in 1821 [5], Cauchy started gradually developing his theory of functions, and also applied it to a diverse array of problems; the solutions to certain differential equations, the existence theorems for implicit functions, and many others. Around the same time, another French mathematician Évariste Galois, observed that every algebraic equation had a unique group of substitutions related to it, and through this it was later proved that there is no algebraic method for solving a general polynomial of degree greater than or equal to five [18]. His work laid the foundation to what would later become the field of abstract algebra, encompassing the theory of different algebraic structures, algebraic geometry, and vector spaces, to name a few.

In what is present day Germany, Carl Friedrich Gauss (sometimes called 'The prince of mathematics') made meaningful contributions to en extensive number of fields of study including, but not limited to, analysis, algebra, celestial mechanics, differential geometry, optics, number theory and statistics [9]. Apart from making meaningful contributions of his own, Gauss was also the advisor of many students, some of whom would go on to make breakthroughs in their own respective fields of study. One of these students were Bernhard Riemann [1] who independently from Cauchy and Karl Weierstrass developed his own theory of complex functions, based on the system of partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The above examples of mathematical discoveries and advancements in the 19^{th} century, though substantial and groundbreaking in their own right, represent merely a fraction of the total amount of development that took place. During this time, most (if not all) areas of mathematics saw an unprecedented amount of development, both in breadth and complexity. Not only did areas that had been developed in the previous centuries see substantial refinement and new areas of application, but entirely new branches of mathematics were born, sometimes as a result of the discoveries of individual mathematicians.

The main subjects of this document rely heavily on these developments, specifically the rigour and precision developed within the field of analysis, and the concept of set-theoretical thinking; the concept of considering classes of objects with similar characteristics and properties, instead of individual entities.

2.3 Some Useful Definitions and Theorems

Throughout the rest of this document, there are a number of theorems and definitions used to justify the convergence and continuity of sequences of numbers and functions, as well as some topological properties that will be presented and discussed in Section 4 and onwards.

Series and Convergence

We begin by defining the notions of convergence and uniform convergence that will be used:

Definition 1 (Pointwise and Uniform Convergence)**.** Let $\{f_n\}$ be a sequence of functions $f_n : A \to \mathbb{R}$. f_n is said to converge pointwise to f on A if

$$\lim_{n\to\infty} f_n(x) = f(x).$$

f_n is said to converge *uniformly* to f on A if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon,\ \forall x \in A.$$

When the domain of the functions in the sequence is understood, we often say that f_n converges to f uniformly, instead of uniformly on A.

Uniform convergence is a particularly useful property of a sequence, since it guarantees properties of the individual elements of a sequence are also properties of the limit of said sequence. The following theorems can be of use when establishing uniform convergence of a sequence.

Definition 2 (Uniformly Cauchy Sequence)**.** A sequence $\{f_n\}$ of functions $f_n : A \to \mathbb{R}$ is uniformly Cauchy on A if $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$m, n > N \Rightarrow |f_m(x) - f_n(x)| < \varepsilon,\ \forall x \in A.$$

Theorem 1. *The sequence $\{f_n\}$ of functions $f_n : A \to \mathbb{R}$ converges uniformly on A if and only if it is a uniformly Cauchy sequence on A.*

Proof. Suppose that $\{f_n\}$ converges uniformly to f on A. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall x \in A \text{ if } n > N.$$

If $m, n > N$, it follows that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \varepsilon, \ \forall x \in A$$

and hence, $\{f_n\}$ is uniformly Cauchy on A.

Suppose, conversely that $\{f_n\}$ is uniformly Cauchy on A. Then for each $x \in A$, the sequence of real numbers $\{f_n(x)\}$ is Cauchy, so it converges by the completeness of $\mathbb{R}$. We define $f : A \to \mathbb{R}$ as

$$f(x) = \lim_{n \to \infty} f_n(x),$$

and then f_n converges to f pointwise.

To prove that f_n converges to f uniformly, let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly Cauchy, we can choose some $N \in \mathbb{N}$ such that

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2}, \ \forall x \in A \text{ if } m, n > N.$$

Let $n > N$ and $x \in A$. Then for every $m > N$ we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + |f_m(x) - f(x)|.$$

Since $f_m(x) \to f(x)$ as $m \to \infty$, we can choose $m > N$ such that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2}.$$

It follows then, that if $n > N$:

$$|f_n(x) - f(x)| < \varepsilon, \ \forall x \in A$$

□

Theorem 2 (Weierstrass M-test). *Let $\{f_n\}$ be a sequence of functions $f_k : A \to \mathbb{R}$ such that*

$$\sup_{x \in A} |f_k(x)| \leq M_k, \ \forall n \in \mathbb{N}.$$

If $\sum_{k=1}^{\infty} M_k < \infty$, then the series $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on A.

Proof. Let $m, n \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned}
\sup_{x \in A} |S_n(x) - S_m(x)| &= \sup_{x \in A} \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x) \right| \\
&= \sup_{x \in A} \left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1}^{n} \sup_{x \in A} |f_k(x)| \\
&\leq \sum_{k=m+1}^{n} M_k = \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k.
\end{aligned}$$

Since $M = \sum_{k=1}^{\infty} M_k < \infty$ we get that

$$\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k \to M - M = 0, \quad m, n \to \infty.$$

This results in $\{S_n\}$ being a uniformly Cauchy sequence on A. Utilizing Theorem 1, we obtain that the series $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on A. □

This is a rather powerful result, since it allows us to show that a series is convergent without having to explicitly state what the series converges to. Finally, the following theorem gives a connection between the notions of convergence and continuity.

Theorem 3. *If $\{S_n\}$ is a sequence of continuous functions on A and S_n converges uniformly to S on A, then S is a continuous function on A.*

Proof. Let $x_0 \in A$ be arbitrary. By assumption we have

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, \sup_{x \in A} |S_n(x) - S(x)| < \frac{\varepsilon}{3}$$

and

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow |S_n(x) - S_n(x_0)| < \frac{\varepsilon}{3}.$$

Let $\varepsilon > 0$ be given, $x \in A$, $n \in \mathbb{N}$ with $n > N$ and $|x - x_0| < \delta$. Then

$$|S(x) - S(x_0)| \leq |S(x) - S_n(x)| + |S_n(x) - S_n(x_0)| + |S_n(x_0) - S(x_0)| < 3\frac{\varepsilon}{3} = \varepsilon$$

and thus S is continuous at x_0. Since the choice of $x_0 \in A$ was arbitrary, S is continuous on A. □

Metric Spaces and Topology

Definition 3 (Metric space, metric)**.** A metric space is a pair (X, d), where X is a set and d is a metric (or distance function) on X, i.e. a function $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Definition 4 (Supremum Norm)**.** Let $C[a, b]$, $(a < b)$ be the normed (real) vector space of all continuous functions $f : [a, b] \to \mathbb{R}$. The Supremum norm is given by

$$||f|| = \sup_{x \in [a,b]} |f(x)|.$$

Definition 5 (Completeness)**.** A metric space M is said to be complete if every Cauchy sequence of points has a limit that is also in M.

Definition 6 (Dense set)**.** A subset A of a topological space T is dense if $T = \bar{A}$

The set of rational numbers $\mathbb{Q}$, and the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, are dense subsets of $\mathbb{R}$, since the irrationals are the set of limit points of the rationals (and vice versa), and the union of them form $\mathbb{R}$.

Definition 7 (Nowhere dense set)**.** A subset A of a topological space T is nowhere dense if $(\bar{A})^\circ = \emptyset$.

It is important to note that 'nowhere dense' is not the negation of 'dense'. The positive rationals, for example, is a set which is neither dense *nor* nowhere dense in the reals, $\mathbb{R}$ (they are, however somewhere dense, specifically in the positive reals).

Theorem 4 (Weierstrass Approximation Theorem)**.** *The set W of all polynomials with real coefficients is dense in the real space $C[a, b]$. Hence for every $x \in C[a, b]$ and a given $\varepsilon > 0$ there exists a polynomial p such that $|x(t) - p(t)| < \varepsilon \, \forall t \in [a, b]$.*

Proof. Every $x \in C[a, b]$ is uniformly continuous on $[a, b]$ since it is a compact interval. Hence for each $\varepsilon > 0$ there is a y whose graph is the arc of a polygon such that

$$\max_{t\in[a,b]} |x(t) - y(t)| < \frac{\varepsilon}{3}. \tag{3}$$

We first assume that $x(a) = x(b)$ and $y(a) = y(b)$. Since y is piecewise linear and continuous, its Fourier coefficients have bounds of the form $|a_0| < k$, $|a_m| < \frac{k}{m^2}$, $|b_m| < \frac{k}{m^2}$. This can be seen by applying integration by parts to the formulas for the Fourier coefficients a_m and b_m. Hence for the Fourier series of y, we have

$$\left|\sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{b-a}\right) + b_m \sin\left(\frac{2\pi mt}{b-a}\right)\right| \leq 2k\left(1 + \sum_{m=1}^{\infty}\frac{1}{m^2}\right) = 2k\left(1 + \frac{\pi^2}{6}\right).$$

This shows that the series converges uniformly on $[a, b]$. Consequently, for the n^{th} partial sum s_n with sufficiently large n,

$$\max_{t\in[a,b]} |y(t) - s_n(t)| < \frac{\varepsilon}{3}. \tag{4}$$

The Taylor series of cos and sin in s_n also converge uniformly on $[a, b]$, so that there is a polynomial p such that

$$\max_{t\in[a,b]} |y(t) - p(t)| < \frac{\varepsilon}{3}.$$

From this, (3), (4) and

$$|x(t) - p(t)| \leq |x(t) - y(t)| + |y(t) - s_n(t)| + |s_n(t) - p(t)|$$

we have

$$\max_{x\in[a,b]} |x(t) - p(t)| < \varepsilon. \tag{5}$$

This takes care of every $x \in C[a, b]$ such that $x(a) = x(b)$. If $x(a) \neq x(b)$, take $u(t) = x(t) - \gamma(t-a)$ with γ such that $u(a) = u(b)$. Then for u there is a polynomial q satisfying $|u(t) - q(t)| < \varepsilon$ on $[a, b]$. Hence $p(t) = q(t) + \gamma(t-a)$ satisfies (5) because $x - p = u - q$. Since $\varepsilon > 0$ was arbitrary, we have shown that W is dense in $C[a, b]$. $\square$

3 Early counterexamples

3.1 Bolzano Function

The Czech mathematician Bernard Bolzano is credited as being the first person to construct a nowhere differentiable function, which he did in 1830. His results were not published, however, until an entire century later in 1930.

What makes the Bolzano function interesting compared to other pathological functions is the fact that it is based on a geometrical construction, as opposed to the more common infinite series approach. The function in question is constructed in the following manner:

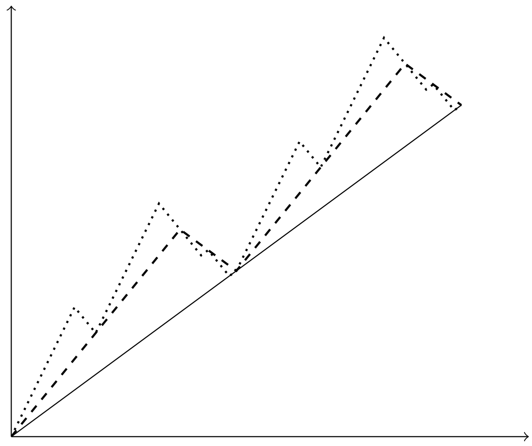

Figure 1: The first three steps in the construction of the Bolzano function

Definition 8 (the Bolzano function)**.** The Bolzano function is the limit of a sequence of piecewise continuous linear functions $B = \{B_k\}_{k=1}^{\infty}$, where each B_k i constructed using the following procedure:

1. Let the interval $[a, b]$ be the desired domain of B, and let the interval $[A, B]$ be the desired range. Let $B_1(x) = A + \frac{B-A}{b-a}(x - a)$.

2. Define $B_2(x)$ as the piecewise linear function on the subintervals

$$I_1 = [a,\ a + \tfrac{3}{8}(b-a)] \qquad I_2 = [a + \tfrac{3}{8}(b-a), \tfrac{1}{2}(a+b)]$$
$$I_3 = [\tfrac{1}{2}(a+b), a + \tfrac{7}{8}(b-a)] \qquad I_4 = [a + \tfrac{7}{8}(b-a), b]$$

having the endpoint values

$$\begin{aligned} &B_2(a) = A \\ &B_2(a + \tfrac{3}{8}(b-a)) = A + \tfrac{5}{8}(B-A) \\ &B_2(\tfrac{1}{2}(b+a)) = A + \tfrac{1}{2}(B+A) \\ &B_2(a + \tfrac{7}{8}(b-a)) = B + \tfrac{1}{8}(B-A) \\ &B_2(b) = B \end{aligned}$$

3. $B_3(x)$ is constructed with the same procedure as in (2) applied on each of the subintervals I_i. This is repeated for $k = 4, 5, 6, \ldots$. The limit of the sequence $\{B_k\}_{k=1}^{\infty}$ as $k \to \infty$ is the Bolzano function $B(x)$.

The complete, rather lengthy proof of continuity and nowhere differentiability will be omitted here, but can be found in [15].

3.2 Cellérier Function

In 1860, the mathematician Charles Cellérier constructed a function that was unfortunately only published posthumously in 1890 [6]. By that time, Weierstrass had already published a function of his own construction with the very same properties as Cellérier's function, namely being continuous and nowhere differentiable. The function in question is given by the infinite sum

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{a^k} \sin(a^k x). \tag{6}$$

What is particularly curious about (6) is its striking similarity to Weierstrass' function, which was not published until 1875 [7]; 15 years after Cellérier constructed (6).

The conditions originally placed on a by Cellérier was that $a > 1000$, where an even integer would result in nowhere differentiability, and an odd integer would result in no intervals of growth or decay. In [11], G. H. Hardy shows that for the case of nowhere differentiability, the condition on a can be weakened to $a > 1$.

Theorem 5. *The function*

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{a^k} \sin(a^k x), \quad a > 1$$

is continuous and nowhere differentiable.

Proof. To prove the continuity of the function, we first observe that $a > 1$ gives $0 < \frac{1}{a} < 1$, which in turn implies

$$\sum_{k=0}^{\infty} \frac{1}{a^k} = \frac{1}{1-a} < \infty.$$

This, together with $\sup_{x \in \mathbb{R}} |\frac{1}{a^k} \sin(a^k x)| \leq a^k$ gives, with Theorem 2, that (6) converges uniformly, and is thus also continuous by Theorem 3.

For the proof of nowhere differentiability, we refer to [14, p. 35-36] □

3.3 Space-Filling Curves

The Peano function (sometimes called the Peano curve) is an interesting example of, not only a pathological function, but also a space filling curve. This function was originally published in 1890 in [20]. After his original publications, other mathematicians constructed new examples of space filling curves. The most notorious of those were the Hilbert curve, seen in Figure 2 (published in 1891 [13]). It is not generally the case that space filling curves are pathological. Lebesgue's space filling curve, for example, is differentiable everywhere, except for points in the Cantor set [22, p. 76-79].

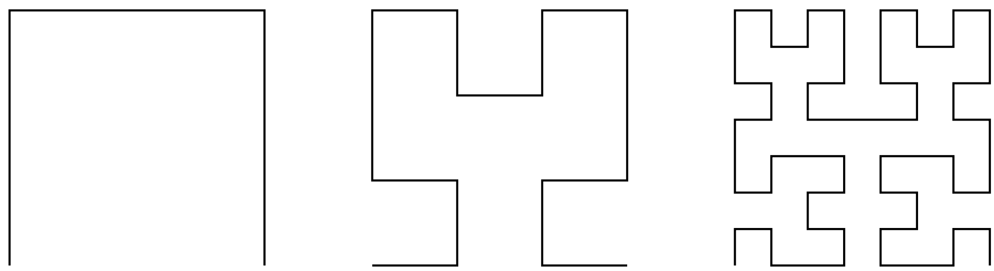

Figure 2: Three iterations of the construction of the Hilbert Curve.

It is important to note that a space filling curve is not a particular one of these curves, but the curve that is generated when taking the limit of the sequence constructing them. In other words, if we let P_n be the n^{th} curve in the sequence, it will only be 'the' Peano curve when n approaches infinity.

Remark 1. At the end of the above paragraph, we discuss the behaviour of a sequence of curves at infinity (i.e. the behaviour in the limit of the sequence), while we never *actually reach* said limit.

Definition 9 (The Peano Function)**.** Let $t = (t_1 t_2 t_3 \ldots)_3$ be a ternary representation of $t \in [0,1]$ i.e.

$$t = \sum_{k=1}^{\infty} t_k 3^{-k}, \quad t_k \in \{0,1,2\}.$$

Then Peano's function $P : [0,1] \to [0,1] \times [0,1]$ is expressed as:

$$(t_1 t_2 t_3 \ldots)_3 \to \begin{pmatrix} (t_1 (k^{t_2} t_3)(k^{t_2+t_4} t_5)(k^{t_2+t_4+t_6} t_7)\ldots)_3 \\ ((k^{t_1} t_2)(k^{t_1+t_3} t_4)(k^{t_1+t_3+t_5} t_6)\ldots)_3 \end{pmatrix} \tag{7}$$

where k is defined as

$$kt_j = 2 - t_j, \quad t_j = 0, 1, 2$$

and $k^l t_j$ is the l^{th} element of the sequence $\{kt_j, k(kt_j), k(k(kt_j)), \ldots\}$.

It can be shown [22, p. 32-33] that P is independent of ternary representation, and that P is surjective, i.e. it covers its entire range. Figure 3 illustrates the first three iterations of the construction of the Peano curve.

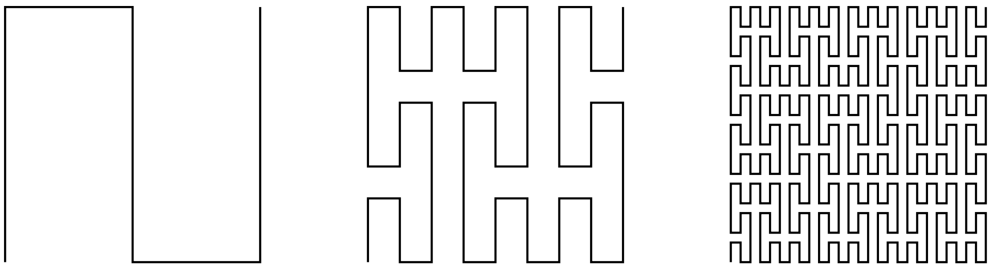

Figure 3: Three iterations of the geometrical construction of the Peano curve.

Theorem 6. *The components ϕ_p and ψ_p of the Peano function P are pathological on the interval* $[0, 1]$.

Peano left the proofs of his statements about the continuity and nowhere differentiability of the functions out of his original paper. The proof presented below is due to [22, p. 33-34]

Proof. We begin by proving that ϕ_p is continuous in two steps; the first of which is showing that it is continuous from the right.

For $t_0 \in [0, 1)$, let $t_0 = (t_1 t_2 \ldots t_{2n} t_{2n+1} \ldots)_3$ be the ternary representation of t_0 that does not end in infinitely many 2's. Choose

$$\delta = 3^{-2n} - (00 \ldots 2_{2n+1} t_{2n+2} \ldots)_3$$

which converges to zero as n tends to infinity. Using this definition of δ gives

$$\begin{aligned} t_0 + \delta &= (t_1 t_2 \ldots t_{2n} t_{2n+1} \ldots)_3 + 3^{-2n} - (00 \ldots 2_{2n+1} t_{2n+2} \ldots)_3 \\ &= (t_1 t_2 \ldots t_{2n} 00 \ldots)_3 + 3^{-2n} = (t_1 t_2 \ldots t_{2n} 22 \ldots)_3. \end{aligned}$$

Thus we see that for any $t \in [t_0, t_0 + \delta)$, the first $2n$ digits in the ternary expansion are the same, that is $t = (t_1 t_2 \ldots t_{2n} \tau_{2n+1} \tau_{2n+2} \ldots)_3$. Now, let

$$\varepsilon_n = \sum_{i=1}^{n} t_{2i}.$$

We have

$$
\begin{aligned}
|\phi(t) - \phi_p(t_0)| &= |(t_1(k^{t_2}t_3)\dots(k^{\varepsilon_n}\tau_{2n+1})\dots)_3 - (t_1(k^{t_2}t_3)\dots(k^{\varepsilon_n}t_{2n+1})\dots)_3| \\
&\leq \sum_{i=n}^{\infty} \frac{1}{3^{i+1}} |k^{\varepsilon_i}\tau_{2i+1} - k^{\varepsilon_i}t_{2i+1}| \leq \sum_{i=n}^{\infty} \frac{2}{3^{i+1}} \\
&= \frac{2}{3^{n+1}} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{1}{3^n} \to 0 \text{ as } n \to \infty
\end{aligned}
$$

Hence ϕ_p is continuous from the right. To show that ϕ_p is continuous from the left follows a similar procedure.

To prove that ϕ_p is nowhere differentiable on $[0,1]$, let $t = (t_1t_2\dots t_{2n}t_{2n+1}\dots)_3$ be a ternary representation of an arbitrarily chosen $t \in [0,1]$. Define the sequence $\{t_n\}$ by $t_n = (t_1t_2\dots t_{2n}\tau_{2n+1}\tau_{2n+2}\dots)_3$ where $\tau_{2n+1} \equiv t_{2n+1} + 1 \mod 2$. This implies

$$
|t - t_n| = \frac{1}{3^{2n+1}}.
$$

From (7) and the definition of t_n, $\phi_p(t)$ and $\phi_p(t_n)$ only differ at the $(n+1)^{\text{th}}$ ternary place, and we have

$$
|\phi(t) - \phi_p(t_n)| = \frac{1}{3^{n+1}} |k^{\varepsilon_n}t_{2n+1} - k^{\varepsilon_n}\tau_{2n+1}| = \frac{1}{3^{n+1}}
$$

and hence

$$
\frac{|\phi_p(t) - \phi_p(t_n)|}{|t - t_n|} = 3^n \to \infty.
$$

The continuity and nowhere differentiability of $\psi_p(t)$ follows from the fact that $\psi_p(t) = 3\phi_p(t/3)$. $\square$

3.4 van der Waerden Function

Van der Waerden was a Dutch mathematician who contributed to many different branches of mathematics, e.g. number theory, abstract algebra, quantum mechanics, and more [23, p. 309-320].

The particular function we are interested in is defined as follows: A rational number that can be written in decimal notation with n decimals, i.e. a number on the form of $m/10^n$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$ is called a n-digit decimal fraction. We now define the function $f_n(x)$ as the distance between x and the closest n-digit decimal number. In other words, we have:

$$f_n(x) = \inf_{m \in \mathbb{Z}} |x - m \cdot 10^{-n}|.$$

The individual $f_n(x)$:s are rather easy to visualize. The graphs for $f_1(x)$ and $f_2(x)$ can be seen in Figure 4. Further, define the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \inf_{m \in \mathbb{Z}} |x - m \cdot 10^{-n}|$$

as the van der Waerden function.

We will present a proof, due to [24], that $f(x)$ is everywhere continuous and nowhere differentiable.

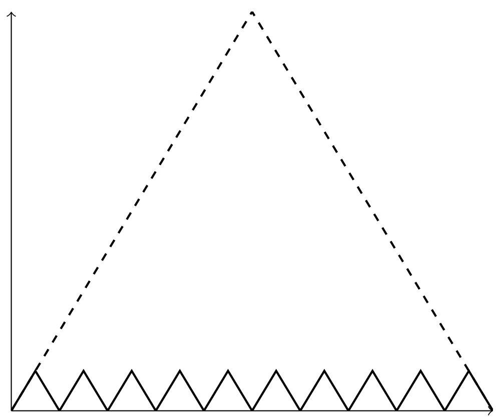

Figure 4: The two components $f_1(x)$ (dashed) and $f_2(x)$ (black) of the van der Waerden function

Theorem 7. *Let $f_n(x)$ be the magnitude of the difference between $x \in \mathbb{R}$ and the nearest n-digit decimal number:*

$$f_n(x) = \inf_{m \in \mathbb{Z}} |x - m \cdot 10^{-n}|.$$

The function

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \tag{8}$$

is everywhere continuous and possesses no finite derivative.

Proof (due to Dr. A. Heyting). $f_n(x)$ is a continuous function with respect to x (since $|f_n(y) - f_n(x)| \leq |y - x|$). Because $f_n(x) < 10^{-n}$, the series converges uniformly due to Theorem 2, and this makes it a continuous function of x. Now, let x be an arbitrary real decimal fraction. If the q^{th} decimal of x is either 4 or 9, we assign $y = x - 10^{-q}$, and otherwise $y = x + 10^{-q}$. Then for $n < q$, the nearest n-digit decimal fraction to x and y, and x and y lie on the same side of the fraction (this is found by breaking off the decimal expansion at the n^{th} decimal place, and adding a unit to the n^{th} place, if the $(n+1)^{\text{th}}$ decimal ≥ 5). Therefore, for $n < q$ we have:

$$f_n(y) - f_n(x) = \pm(y - x).$$

For $n \geq q$ however, we have:

$$f_n(y) - f_n(x) = 0,$$

therefore

$$f(y) - f(x) = s(y - x),$$

where s is an odd number for even q:s, and an even number for odd q:s. The difference quotient s thus takes both even and odd integer values in every neighbourhood of x. It therefore follows that a finite derivative cannot exist in any point along the function curve. □

3.5 von Koch's Snowflake

The von Koch snowflake is yet another example of a curve that is everywhere continuous and nowhere differentiable. This curve was first presented in an article by the Swedish mathematician Helge von Koch [25], and its name arises from the fact that the geometrical visualization resembles that of a snowflake. Von Koch's motivation for constructing this curve was to find a pathological function with a more easily understandable geometrical interpretation, since he felt Weierstrass' example had been somewhat lacking in this regard.

The snowflake can be constructed by the following procedure:

1. Begin with an equilateral triangle.
2. Split each line in three equally long parts.
3. Replace the middle segment of each side with a new equilateral triangle with its base removed.
4. Continue from step (2) for each straight line.

Similar to the space filling curves from Section 3.3, it is the limit of this procedure that generates the snowflake that is continuous everywhere and differentiable nowhere (or has a tangent nowhere, as von Koch put it). In Figure 5 below, we see the first iterations of its construction.

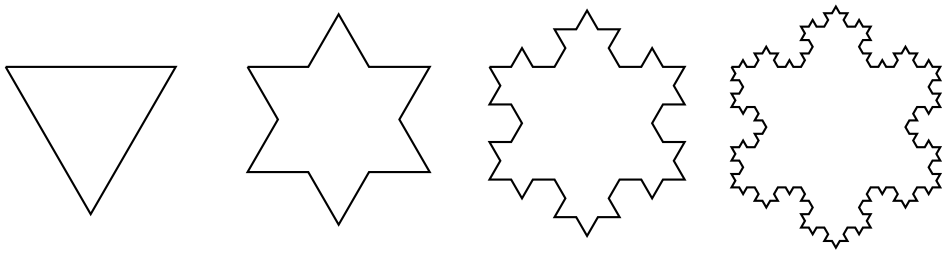

Figure 5: The first iterations of Koch's snowflake.

4 René Louis Baire

Much of the information presented here on the life of Baire is due to the biography on him in the MacTutor archives [19] and the chapter dedicated to him in *The Calculus Gallery* [8, p. 183-199].

Baire was born in Paris in the year of 1874 to a family in difficult financial circumstances. At the age of twelve, Baire won a scholarship which gave him the opportunity to, despite the poor financial state of his family, receive a good education. He enrolled in the Lycée Lakanal where he performed admirably and went on to receive honorable mentions in the Concours Général. By 1890 he had completed the advanced classes at Lakanal, and entered the special mathematics section of Lycée Henri IV. After a year of preparations at the school, he managed to pass the entrance examinations at the École Polytechnique and the École Normale Supéruire, both very well known and prestigious French universities.

It was also at this point in hos life that his delicate health started to manifest itself more clearly. He enrolled at the École Normale Supéruire in 1891, and received his licentiate degree after three years of study. He passed his *agrégation* on the second attempt, having failed the oral examination on the first, and was subsequently employed as a professor at the Lycée Bar-le-Duc, which for once gave him some financial security. At this Lycée, Baire worked on the theory of functions and limits, and it was during this time that he discovered the conditions under which a function is a limit of a sequence of continuous functions.

Shortly thereafter, he defined his classification of functions as follows:

- Functions of Class 0 are the continuous functions
- Functions of Class 1 are functions which are the (pointwise) limits of a

sequence of functions of class 0

- Generally, functions of Class n are functions which are the limit of a sequence of functions of Class $n-1$.

This procedure is continued by transfinite induction, to all ordinal numbers less than the first uncountable; ω_1.

Example 1 (A function of Class 1)**.** Any discontinuous function which may be represented as a Fourier series if of class 1, since each individual element of the sequence is a continuous function, and thus of class 0.

Example 2 (A function of Class 2)**.** The Dirichlet function $D(x)$ is of class 2, since

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad = \quad \lim_{n\to\infty} \lim_{m\to\infty} (\cos(n!\pi x))^{2m}.$$

During his time at Lycée Bar-le-Duc, Baire wrote his doctoral thesis [3] on discontinuous functions, for which he was awarded a doctorate in May, 1899. In 1901 he was appointed a position at the University of Montpellier, but was forced to leave shortly thereafter, around 1904 when his delicate health again started to prevent him from regular day-to-day activities. When the worst of his illness had passed, he returned to work, this time at the Faculty of Science of Dijon, where he started in 1905 and was promoted to Professor of Analysis in 1907. The spells of illness returned however, and in periods made him unable to work which eventually led to his leaving, in an attempt to recover.

He spent his remaining lifetime fighting with physical and mental illness, while at times dealing with quite severe poverty, not unlike what he experienced during his younger years while growing up in Paris.

4.1 The Baire Category Theorem

> *One can even say, in a general manner, that [...] any problem relative to the theory of functions leads to certain questions relative to the theory of sets, and insofar as these latter questions are or can be addressed, it is possible to resolve, more or less completely, the given problem.*
>
> \- René-Louis Baire [3, p. 121]

The usefulness and importance of Baire's theorem should not be understated in any way. His insights and intuition regarding set theory and its application to analysis paved way for, and lays at the core of, many important results in functional analysis, topology (as we shall see in Section 4.2), and many other fields of mathematics. What makes Baire's theorem particularly interesting from a set theoretic point of view is that it distinguishes different sets not based on their cardinality or denseness within an interval, but instead a combination of the two in some sense.

Next, we shall introduce the notions of category in the sense of Baire, and lastly the Baire Category Theorem, as formulated and proved in [16, p. 247-248].

Definition 10 (A set of first category)**.** A subset A of a topological space T is said to be *meager*, or of *first category* in T if it is the union of countably many nowhere dense sets, that is

$$A = \bigcup_{k=1}^{\infty} A_k$$

where A_k is nowhere dense in T.

Definition 11 (A set of second category)**.** A subset A of a topological space T is said to be *nonmeager*, or of *second category* in T if it is not meager in T.

Theorem 8 (The Baire Category Theorem)**.** *If a metric space $X \neq \emptyset$ is complete, it is nonmeager in itself.*

Proof. Suppose the complete metric space $X \neq \emptyset$ were meager in itself. Then

$$X = \bigcup_{k=1}^{\infty} M_k$$

with each M_k is nowhere dense in X. We shall construct a Cauchy sequence $\{p_k\}$ whose limit p (which exists by completeness) is in no M_k, thereby leading to a contradiction.

By assumption, M_1 is nowhere dense in X, so that, by definition, $\bar{M}_1$ does not contain a nonempty open set. But X does (for instance X itself). This implies $\bar{M}_1 \neq X$. Hence the complement $\bar{M}_1^c = X \setminus \bar{M}_1$ of $\bar{M}_1$ is not empty and open. We may thus choose a point p_1 in $\bar{M}_1^c$ and an open ball about it, say,

$$B_1 = B(p_1; \varepsilon_1) \subset \bar{M}_1^c \qquad \varepsilon_1 < \tfrac{1}{2}.$$

By assumption, M_2 is nowhere dense in X, so that $\bar{M}_2$ does not contain a nonempty open set. Hence it does not contain the open ball $B(p_1; \frac{1}{2}\varepsilon_1)$. This implies that $\bar{M}_2^c \cap B(p_1; \frac{1}{2}\varepsilon_1)$ is not empty and open, so that we may choose an open ball in this set, say,

$$B_2 = B(p_2; \varepsilon_2) \subset \bar{M}_2^c \cap B(p_1; \frac{1}{2}\varepsilon_1) \qquad \varepsilon_2 < \tfrac{1}{2}\varepsilon_1.$$

By induction we thus obtain a sequence of balls

$$B_k = B(p_k; \varepsilon_k) \qquad \varepsilon_k < 2^{-k}$$

such that $B_k \cap M_k = \emptyset$ and

$$B_{k+1} \subset B(p_k; \frac{1}{2}\varepsilon_k) \subset B_k \qquad k = 1, 2, \ldots.$$

Since $\varepsilon_k < 2^{-k}$, the sequence $\{p_k\}$ of the centers is Cauchy and converges, say, $p_k \to p \in X$ because X is complete by assumption. Also, for every m and $n > m$ we have $B_n \subset B(p_m; \frac{1}{2}\varepsilon_m)$, so that

$$\begin{aligned} d(p_m, p) &\le d(p_m, p_n) + d(p_n, p) \\ &\le \frac{1}{2}\varepsilon_m + d(p_n, p) \to \frac{1}{2}\varepsilon_m \end{aligned}$$

as $n \to \infty$. Hence $p \in B_m$ for every m. Since $B_m \subset \bar{M}_m^c$, we now see that $p \notin M_m$ for every m, so that $p \notin \bigcup M_m = X$. This contradicts $p \in X$, and thus Baire's theorem is proved. □

Remark 2. Note that in choosing the p_k for the centers of the balls B_k we invoke the axiom of countable choice, a weaker version of the axiom of choice.

4.2 Examples and applications of Baire's theorem

Our first example of applying Baire's theorem will be to motivate a statement made at the very beginning of this document, and at the same time provide a connection between the two main subjects of the title, namely that the pathological functions constitute the vast majority of all functions (formally speaking, the pathological property is a *generic* property, meaning that an arbitrarily chosen function is more likely to be pathological than it is to possess 'nice' qualities like being differentiable in one or more points), when considering real-valued functions on a normed vector space together with the supremum norm. Formally, this is known as the *Banach-Mazurkiewicz theorem*.

The general idea for proving the statement in the preceding paragraph is to consider not a single pathological function, but instead the class of all such functions, which may very well be empty. We then give a set theoretic description of the complement of this class of functions (that is, all functions which possess a finite derivative at one or more points), and then utilize Baire's theorem to show that this set is in fact a meager subset of the set of continuous functions. The theorem and proof presented here are due to [4, p. 174-175].

Lemma 1. *The sets E_n defined as the set of functions $f \in C[0,1]$ each of which having at least one value of $x \in [0, 1 - \frac{1}{n}]$ satisfying the inequality*

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq n \qquad h \in (0, \tfrac{1}{n})$$

are closed.

Proof. Let $f \in \bar{E}_n$, and let $\{f_k\}$ be a sequence in E_n that converges uniformly to f. Since we are in a metric space, there is a corresponding sequence of numbers x_k such that, for each k,

$$0 \leq x_k \leq 1 - \tfrac{1}{n}$$

and

$$\left| \frac{f_k(x_k+h) - f_k(x_k)}{h} \right| \leq n \qquad \forall h \in (0, \tfrac{1}{n}).$$

We may also assume that

$$x_k \to x, \text{ for some } x \in [0, 1 - \tfrac{1}{n}],$$

since this condition will be satisfied if we replace $\{f_k\}$ by a suitable subsequence. If $h \in (0, \frac{1}{n})$, then the following holds for sufficiently large k

$$\begin{aligned} |f(x+h) - f(x)| &\leq |f(x+h) - f(x_k+h)| + |f(x_k+h) - f_k(x_k+h)| \\ &\quad + |f_k(x_k+h) - f_k(x_k)| + |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| \\ &\leq |f(x+h) - f(x_k+h)| + d(f, f_k) \\ &\quad + nh + d(f_k, f) + |f(x_k) - f(x)|. \end{aligned}$$

Now, letting $k \to \infty$ and using the fact that f is continuous at x and $x+h$, it follows that

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leq n \qquad \forall h \in (0, \tfrac{1}{n}).$$

Therefore $f \in E_n$, which is consequently a closed set. □

Theorem 9 (Banach-Mazurkiewicz theorem)**.** *The set $A \subset C[0,1]$ of continuous functions of x, for which in no point $x \neq 1$ the two right-hand derivatives are finite, is of second category, and its complement, A^c is of first category.*

Proof. Let the sets E_n be defined as in Lemma 1, i.e. E_n are sets of functions possessing a finite right-hand derivative in one or more points on the interval $[1, 1-\frac{1}{n}]$

$$E_n = \left\{ f \in C[0,1] \;\middle|\; \exists x \in [0, 1-\tfrac{1}{n}] : \left|\frac{f(x+h)-f(x)}{h}\right| \leq n,\ h \in (0, 1-x] \right\}.$$

The sets E_n are closed due to Lemma 1, and we have

$$A^c = \bigcup_{n=2}^{\infty} E_n.$$

The set A^c is thus an F_σ (meaning it can be expressed as the countable union of closed sets), and A must therefore be a G_δ (meaning it can be expressed as the countable intersection of open sets). It suffices to show that every E_n is nowhere dense.

If E_N is not a nowhere dense set, then it must as a closed set, contain a ball K (a nowhere dense set has an empty interior, thus a set that is *not* nowhere dense must have at least one interior point). According to Theorem 4, there exists a polynomial w and a number $\varepsilon > 0$ such that every $f \in E$ for which $||f - w|| < \varepsilon$, is contained in K, and consequently contained in E_N since $K \subset E_N$.

Now let $g(x)$ denote any function in E, which for $x \in [0,1)$ has a right-hand derivative $g'_+(x)$ and satisfies the conditions

$$||g|| < \varepsilon, \quad |g'_+(x)| > \left|\left|\frac{dw}{dx}\right|\right| + N. \qquad t \in [0,1)$$

(This is a piecewise linear functions with sufficiently steep slopes for our purposes). The function $z = w + g$ is contained in E, and for $t \in [0,1)$ we have

$$|z'_+(x)| \geq g'_+(x) - \left|\frac{dw}{dx}\right| > \left|\left|\frac{dw}{dx}\right|\right| + N - \left|\left|\frac{dw}{dx}\right|\right| = N.$$

Therefore $z \notin E_N$. On the other hand, $||w - z|| = ||g|| < \varepsilon$, i.e. $z \in K \subset E_N$, which contradicts our statement about E_N not being a nowhere dense set, and we are done. □

Remark 3. In the formulation of the above theorem, we mention "the *two* right-hand derivatives". What we mean by this is the upper and lower Dini derivatives (lim sup and lim inf respectively). We will not go into more depth regarding these concepts, but judge it as necessary to clarify, as to avoid confusion.

Thus we have proven that the set of functions possessing a finite right-hand-side derivative in one or more points in a given interval is of first category. Similarly, the same is true for functions possessing a finite left-hand-side derivative in one or more points in a given interval. We now use Lemma 2 and our statement in the first paragraph of Section 4.2 follows.

Lemma 2. *The union of two sets of first category is of first category.*

Proof. Let P and R be subsets of first category, i.e.

$$P = \bigcup_{n=1}^{\infty} P_k, \qquad R = \bigcup_{n=1}^{\infty} R_k$$

where each P_k, R_k is nowhere dense. We now write

$$P \cup R = \bigcup_{n=1}^{\infty} (P_k \cup R_k).$$

To show that each $P_k \cup R_k$ is nowhere dense, we take an open set Q_1. Since P_k is nowhere dense, there is an open subset $Q_2 \subseteq Q_1$ with $Q_2 \cap P_k = \emptyset$. But Q_2 is open and R_k is nowhere dense, so there is an open subset $Q_3 \subseteq Q_2 \subseteq Q_1$ with $Q_3 \cap R_k = \emptyset$. Clearly, Q_3 is an open subset of Q_1 containing no points of P_k or R_k. Thus $Q_3 \cap (P_k \cup R_k) = \emptyset$, $P_k \cup R_k$ is nowhere dense, and the result follows. □

Another application is seen in the Banach-Steinhaus theorem (also called the uniform boundedness principle), which together with the Hahn-Banach theorem, the open mapping theorem and the closed graph theorem are regarded as the corner stones of functional analysis. The proofs of all of these rely on Baire's theorem in one way or another. The theorem and proof presented here are due to [16, p. 249-250]

Theorem 10 (The Uniform Boundedness Principle)**.** *Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y such that*

$$||T_n(x)|| \leq c_x \quad n \in \mathbb{N}_{>0},\ x \in \mathbb{X},\ c \in \mathbb{R}. \tag{9}$$

Then the sequence of the norms $||T_n||$ is bounded, that is, $\exists c$ such that

$$||T_n|| \leq c \quad n \in \mathbb{N}_{>0}. \tag{10}$$

Proof. For every $k \in \mathbb{N}$, let $A_k \subset X$ be the set of all x such that

$$||T_n(x)|| \leq k \quad \forall n.$$

The set A_k is closed. Indeed for any $x \in \bar{A}_k$ there is a sequence $\{x_j\}$ in A_k converging to x. This means that for every fixed n we have $||T_n(x_j)|| \leq k$ and obtain $||T_n(x)|| \leq k$ because T_n is continuous and so is the norm. Hence $x \in A_k$, and A_k is closed.

By (9), each $x \in X$ belongs to some A_k. Hence

$$X = \bigcup_{k=1}^{\infty} A_k.$$

Since X is complete, Baire's theorem implies that some A_k contains an open ball, say

$$B_0 = B(x_0; r) \subset A_{k_0}. \tag{11}$$

Let $x \neq 0 \in X$. We set

$$z = x_0 + \gamma x \qquad \gamma = \frac{r}{2||x||}. \tag{12}$$

Then $||z - x_0|| < r$, so that $z \in B_0$. By (11) and from the definition of A_{k_0} we thus have $||T_n(z)|| \leq k_0$ for all n. Also $||T_n(x_0)|| \leq k_0$ since $x_0 \in B_0$. From (12) we obtain

$$x = \frac{1}{\gamma}(z - x_0).$$

This yields for all n

$$||T_n(x)|| = \frac{1}{\gamma}||T_n(z - x_0)|| \leq \frac{1}{\gamma}(||T_n(z)|| + ||T_n(x_0)||) \leq \frac{4}{r}||x||k_0.$$

Hence, for all n,

$$||T_n(x)|| = \sup_{||x||=1} ||T_n(x)|| \leq \frac{4k_0}{r},$$

which is on the form of (10) with $c = 4k_0/r$. □

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