Fluctuations in the CMB through inflation

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Abstract

The goal of this project is to gain a better understanding of the temperature fluctuations observed in the CMB. This goal is reached with the aid of a literature study.

The report touches upon the origin of the CMB as well as how it is measured, mentioning and explaining relevant concepts such as recombination, photon decoupling, black body radiation, angular power spectrum etc. It is also specified that the temperature of the CMB is not uniform but varies slightly. The fact that these fluctuations show inhomogeneities at the early universe is emphasized as well as the need for a satisfying theory that explains said fluctuations.

By presenting and employing inflation theory in combination with quantum mechanics, we show how such a theory can be obtained. Through extensive calculation we show how the primordial power spectrum for zero-point fluctuations during inflation is obtained and how it can be related to the time of recombination through a transfer function, thus explaining the existence of fluctuations in the CMB.

Sammanfattning

Målet med detta projekt är att få en bättre förståelse om observerade temperaturfluktuationer i CMB. Detta mål är uppnått med hjälp av en litteraturstudie.

Rapporten nämner uppkomsten av CMB samt hur den mäts, nämner och förklarar relevanta koncept som rekombination, foton frikoppling, svartkroppstrålning, vinkelkraftspektrum etc. Det är också specificerat att temperaturen av CMB inte är enhetlig utan varierar en aning. Faktumet att dessa fluktuationer visar på inhomogeniteter i det tidiga universum är betonat samt behovet av en tillfredställande teori som förklarar dessa fluktuationer.

Genom att presentera och använda inflationsteori i kombination med kvantmekanik visar vi hur en sådan teori kan erhållas. Genom omfattande beräkningar visar vi hur det ursprungliga kraftspektrumet för nollpunktsfluktuationer under inflation är erhållet och hur det kan relateras till tiden för rekombination genom en överföringsfunktion, som således förklarar existensen av fluktuationer i CMB.

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Part I
Introduction and background

1 Introduction

This project will focus on the Cosmic Microwave Background (CMB). The CMB is sometimes described as a snapshot of the universe around the time of recombination, the time when the atoms of the universe went from being ionized to neutral. It was during this time the universe became transparent through photon decoupling. The photons that decoupled during recombination are those that make up the CMB. They have travelled across the universe undisturbed for years before being detected by us on Earth.

The CMB was first detected accidentally by Penzias and Wilson in 1965. Since then it has been measured with more accuracy and precision. Some of the most successful detectors are those placed on satellites. Three satellites that have measured the CMB with increasing precision are Cobe, WMAP and Planck.

The CMB radiates as an almost perfect black body and it is therefore possible to extract its temperature. This temperature is almost equal, no matter which direction you look. There are, however, small temperature fluctuations.

The temperature fluctuations of the CMB are often presented as an angular power spectrum, as seen in figure 1. With the more precise measurements of the Planck satellite, it is possible to accurately fit a curve to the data points. The exact shape of the curve contains information about the density fluctuations of the universe around the time of recombination.

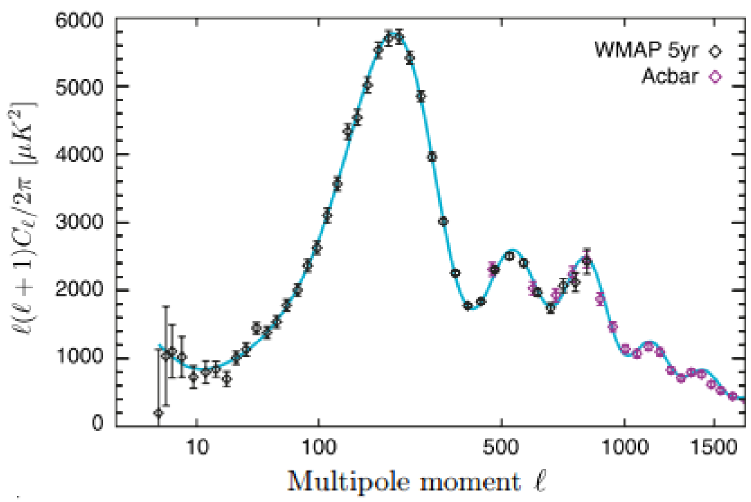

Figure 1: The angular power spectrum of the temperature fluctuations of the CMB [1].

The origin of the temperature fluctuations, and equivalently, the appearance of the angular power spectrum can be explained by inflation in combination with quantum mechanics. The inflation theory states that the universe went through a phase of rapid expansion in the very early universe, thus expanding microscopic quantum perturbations to macroscopic scales.

2 Method

This project is based on literature studies of other works on the subject with emphasis on presenting theory and facts in a less summarized way.

3 Background

A detailed review of the simplest inflation model has been given by D.Bauman in his article "TASI Lectures on Inflation" [1]. Here he shows, with extensive calculations, how this particular model would expand the energy density perturbations in the universe, which in turn explains the temperature fluctuations in the CMB.

A less in depth discussion on the subject is found in "Introduction to Cosmology" by B.Ryden [2]. The discovery and origin of the CMB as well as the motivation for the inflation theory are being discussed along with some brief but relevant calculations.

Part II
Origin and detection of the CMB

The CMB is the observable background of the universe. It is almost uniform across the sky and can be thought of as a large sphere surrounding the Earth. It is the furthest distance away from Earth observers can see and, equivalently, the furthest back in time observers can see.

This "furthest back time" is called the time of recombination. The reason why it is impossible to see beyond this background is simple, the universe was not transparent at times before recombination. The following sections will describe why the universe was opaque as well as how and why the recombination made the universe transparent and also how we detect CMB photons today.

4 Origin of the CMB

It is commonly known that the universe started in a hot, dense state that gradually cooled down due to expansion of the volume to the temperatures and densities we see today. Equivalently, the thermal energy of the universe decreased as it expanded. Around 10^5 years [1] after the creation of the universe, the universe was still very hot and dense compared to current standards, but it had cooled down to a point where the atoms of the universe went through a transition. This transition is the transition from being ionized to being neutral. This transition is called recombination.

Before recombination, the thermal energy of the universe was so high that even if a proton would combine with a free electron to become a neutral atom, a high energy photon would instantly break this bond due to the following interaction [2]

$$H + \gamma \rightarrow p + e^-, \tag{1}$$

where H stand for a neutral hydrogen atom, γ for a photon, p for a proton and e^- for an electron.

Even when not breaking neutral atoms, photons still underwent interactions with free electrons due to scattering

$$\gamma + e^- \rightarrow \gamma + e^-. \tag{2}$$

In this way the photons could not travel very far without interacting with matter and were therefore coupled to it. This is how the universe was opaque before the time of recombination.

However, since the universe most certainty is transparent today, this coupling did not last. The reason why is because of the expansion of the universe. First of all, the expansion of the universe's volume decreases its thermal energy, as mentioned. This means that radiation of the universe gradually becomes less energetic. At a certain point the photons will lose enough energy to not be able to break up neutral atoms.

But that is not the whole story as photons would keep interacting with electrons due to scattering, i.e. they were still coupled. The free electron density was, however, severely reduced when electrons could permanently combine with protons to make neutral atoms, not to mention the natural reduction in electron density from the expansion of the universe. The rate of photon scattering on electrons was therefore severely reduced. As such, the photon decoupling took place shortly after recombination.

There are concrete time steps defined for when each step actually happened [2]. The time of recombination is defined as the time where the number density of neutral and ionized atoms were the same. The photon decoupling is defined as the time when the rate for electron scattering is the same as the rate of universe expansion.

There is another time step associated with recombination and photon decoupling called the time of last scattering. This time step indicates the time when the average photon underwent its last scattering. After the last scattering, these photons travel across the universe undisturbed until they are detected by us at Earth. It is by detecting these photons that we are able to see the CMB.

5 Detection of the CMB

The CMB was first detected accidentally by Penzias and Wilson in the 1960s [2], since then it has been measured with more accuracy and precision. Penzias and Wilson were measuring signals in the microwave part of the electromagnetic spectrum when they received "noise" that turned out to be photons from the CMB[1]. After that accidental encounter more deliberate measurements were

[1] Since these photons have travelled a large distance since the time of last scattering, they are heavily redshifted.

made, and as technology became more advanced and better measurement devices became available, the accuracy of these measurements improved. Today, the most accurate measurements have been made using satellites. Three satellites that have measured the CMB with increasing precision are Cobe, WMAP and Planck [2]. A measurement taken by the Planck satellite can be seen in figure 2.

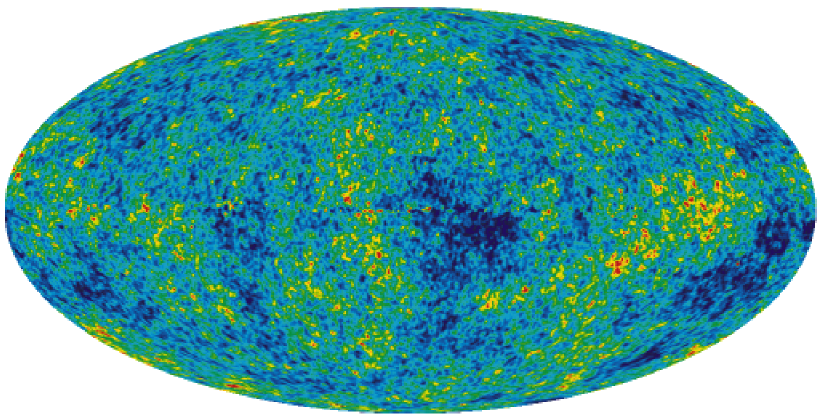

Figure 2: This figure can be seen as a photo taken by a satellite on the surrounding space but with its sensitivity set to only capture photons in the microwave part of the electromagnetic spectrum. The elliptical shape can be seen as a compression of the otherwise spherical shape of the photo. Since the CMB radiates as a black body, the photons captured have a frequency dependent on the temperature of the CMB. It is these different temperatures that the different colors represents with the blue color representing temperature slightly colder than the average and the yellow/red colors representing slightly hotter temperatures [1]

The radiation from the CMB is isotropic and very close to that of an ideal black body [2] [2]. This means that it is possible to extract the temperature of the CMB given the frequency of the CMB photons. By using measured data the average temperature of the CMB is calculated to be $\langle T \rangle = 2.7255$ K [2]. Since this is the average value it means that there are some deviations from this temperature on the CMB. These deviation, or fluctuations, are very small but the fact that they exists hints of some type of inhomogeneity in the pre-recombination universe.

The temperature fluctuations of the CMB is commonly presented in an angular power spectrum, an example of such a spectrum can be seen in figure 3. To construct a angular power spectrum it is necessary to first define a quantity that represents the fluctuations in temperature. This is done in the following

[2]In fact, the CMB is the most ideal black body that is known.

way

$$\frac{\delta T}{T}(\theta,\phi) \equiv \frac{T(\theta,\phi) - \langle T\rangle}{\langle T\rangle}, \tag{3}$$

where $T(\theta,\phi))$ is the temperature on a point of the CMB in spherical coordinates with the radius being constant[3]. The division with the average temperature ensures that this quantity is a dimensionless number.

It is possible to expand this in terms of spherical harmonics

$$\frac{\delta T}{T}(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi), \tag{4}$$

where Y_{lm} are the usual Laplace spherical harmonics and a_{lm} are constants. Since $\frac{\delta T}{T}(\theta,\phi)$ is independent of radius, the usual radial part of the spherical harmonics are implemented in to a_{lm}.

Using equation (4) it is now possible to construct a correlation function, $C(\Theta)$

$$\begin{aligned} C(\Theta) &= \langle \frac{\delta T}{T}(\theta_1,\phi_1)\frac{\delta T}{T}(\theta_2,\phi_2)\rangle \\ &= \langle \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m=-l'}^{l'} a_{lm}a^*_{l'm'}Y_{lm}(\theta_1,\phi_1)Y^*_{l'm'}(\theta_2,\phi_2)\rangle. \end{aligned} \tag{5}$$

The $\langle ...\rangle$ means averaging over all possible ensembles. θ_1 and ϕ_1 are the coordinates for a certain point on the CMB while θ_2 and ϕ_2 are the coordinates for another point. An observer on Earth could draw a vector, $\hat{\imath}$, between him or herself to the one of the two points on the CMB. Another vector, $\hat{\jmath}$, could also be drawn to the remaining point. Between these vectors an angle would appear. This angle is defined as the following

$$\Theta = \arccos(\hat{\imath}\cdot\hat{\jmath}). \tag{6}$$

It is possible to rewrite equation (5) to obtain the following expression

$$C(\Theta) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l=m}\langle a^*_{lm}a_{lm}\rangle P_l(\cos(\Theta)), \tag{7}$$

where $\langle a^*_{lm}a_{lm}\rangle$ is a so-called power spectrum for the a_{lm}. This can be thought of as a combination of the different multipole moments of a_{lm}, i.e. the different values of l and m. Exactly what a power spectrum entails will be presented further down in the report. It is assumed that the distribution of the multipole moments are Gaussian [1], i.e.

$$\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'}, \tag{8}$$

[3] The CMB can be thought of as a spherical shell around Earth from which the CMB photons emerge. This is why it is practical to define each point on the CMB in spherical coordinates.

where

$$C_l \equiv \frac{1}{2l+1} \sum_{m=-l}^{m=l} \langle a^*_{lm} a_{lm} \rangle. \tag{9}$$

It is by assuming Gaussian distribution that the primed sums in equation (5) disappears.

P_l are the Legendre polynomials and are obtained through

$$\sum_{m=-l}^{l} Y_{lm}(\hat{i}) Y^*_{lm}(\hat{j}) = \frac{1}{4\pi} \sum_{m=-l}^{l} P_l(\hat{i} \cdot \hat{j}) = \frac{1}{4\pi} \sum_{m=-l}^{l} P_l(\cos(\Theta)). \tag{10}$$

More often than not, the correlation function is written in terms of its different multipole moments, C_l, given by equation (9). This makes it so that equation (7) becomes

$$C(\Theta) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) C_l P_l(\cos(\Theta)). \tag{11}$$

The plot of C_l versus l is what is known as the angular power spectrum, but it is more customary to use a slightly different quantity on the y-axis

$$\Delta\varpi = \frac{l(l+1)C_l}{2\pi}. \tag{12}$$

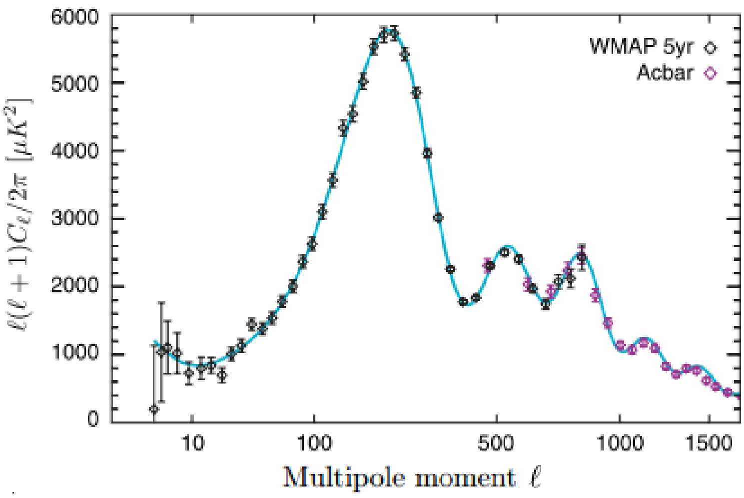

Figure 3: The angular power spectrum of the temperature fluctuations of the CMB. The dots represents actual measurements while the line represents a theoretical prediction [1].

Besides presenting the fluctuations in temperature of the CMB, the angular power spectrum contains information about the universe at times before recombination. For instance, this shows that the universe was not completely homogeneous at the time of recombination, and any theory that aims to explain the universe before the time of recombination needs to take this into consideration. Furthermore, such a theory also needs to explain why the angular power spectrum assumes its peculiar appearance.

Part III
Inflation

The theory of inflation is a theory that tries to explain the dynamics of the universe during a short time interval at a very early stage of history. Although the exact time for the start of inflation varies between different iterations of the inflation theory, the usual start occurs around 10^{-34} s [1] after the beginning of the universe. Inflation ends shortly thereafter in a process called "reheating".

In casual terms inflation is usually referred to as a time of rapid, accelerated expansion, which aptly describes what the universe is going through during this period.

The motivation for this theory are some unexplained phenomena that are present in standard Big Bang cosmology. These phenomena are called the flatness problem, the horizon problem and the monopole problem.

It also turns out that the inflation theory explains the temperature fluctuations in the CMB if quantum mechanics is taken into account. This will be shown further down in the report.

6 The horizon problem

Of the three Big Bang problems mentioned above, only the horizon problem will be touched upon in this report. This is because the solution for this problem will naturally lead to the definition for inflation. Hopefully, this will be sufficient in giving some clarification on what the inflation theory is trying to achieve as well as giving some context when the actual physics of inflation is presented.

6.1 Causal contact

Observations of the universe shows an extreme homogeneity and isotropy over large scales [1]. For example, the density and temperature of matter across the universe is very similar in every direction. This homogeneity implies that every point in the universe are in causal contact, which would explain the similarities[4]. However, this is impossible, which is why this is seen as a problem. This impossibility will be shown below.

[4]To understand why consider for example simple heat flow presented in thermodynamics. Heat always flow from a hot source to a cold sink which eventually evens the different temperatures. If all temperatures in the universe are more or less equal, this would mean that this process have occurred between every point in space [3].

Before defining causal contact in a four dimensional space-time, an expression for distance needs to be introduced

$$\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu = -\Delta t^2 + a^2(t)(\Delta x^2 + \Delta y^2 + \Delta z^2), \tag{13}$$

where Δs^2 is meant to be seen as the distance, or separation, between two events in a four-dimensional space time. The coordinates t, x, y and z are the usual space-time coordinates. $g_{\mu\nu}$ is the metric tensor which has the following appearance

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}. \tag{14}$$

The factor a on all the spatial parts is called the scale factor. It characterizes the relative size of spatial distances between different times. Mathematically it can be seen in the following way

$$D(t) = D_0 a(t), \tag{15}$$

where D_0 is a distance at present time, t_0 ($a(t_0) = 1$), and $D(t)$ is the same distance at time t.

The separation between two events can be reduced to a infinitesimal distance, which will make equation (13) change to the following

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{16}$$

This is a simplification of the so called Friedmann-Robertson-Walker (FRW) metric [1]. The simplifications consists of only considering a flat geometry of space-time as well as considering units where the speed of light, c, is set to 1.

The FRW metric can be expressed in spherical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2)). \tag{17}$$

It is not uncommon to differentiate between different separations of events depending on the sign of $\mathrm{d}s^2$ as seen in table 1 [4]. To understand the names given

$\mathrm{d}s^2 < 0$	"timelike"
$\mathrm{d}s^2 = 0$	"lightlike"
$\mathrm{d}s^2 > 0$	"spacelike"

Table 1: Different kinds of event separations.

to each separation it is useful to observe equation (16) or (17). If $\mathrm{d}s^2$ is less than zero, i.e. the case with timelike separation, then the spatial part needs to be smaller than the temporal part. This corresponds to the case where there is more time than space that separates two events. This means that it is possible for a signal, with speed less than c, to reach the second event if it starts from the first event. Conversely, the signal needs to be faster than c in the case of spacelike separation and equal to c in the case of lightlike separation. If two events in the universe are causally connected then they are timelike or lightlike separated.

Now the fact that needs to be shown is that all points in the universe can not be causally connected[5]. To do this, a conformal time is first defined

$$\begin{aligned} \tau &= \int \frac{\mathrm{d}t}{a(t)} \\ \rightarrow \mathrm{d}\tau &= \frac{\mathrm{d}t}{a(\tau)}. \end{aligned} \tag{18}$$

Using this as the time coordinate, and by assuming there is only radial propagation of signals, equation (17) can be rewritten to

$$\mathrm{d}s^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}r^2). \tag{19}$$

Assuming that the signal in question is a photon, the allowed separation between events is lightlike separated. Using the condition of lightlike separation seen in table 1, a condition on r emerges

$$\begin{aligned} \mathrm{d}s^2 &= a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}r^2) = 0 \\ \rightarrow r(\tau) &= \pm\tau + \Upsilon, \end{aligned} \tag{20}$$

where Υ is a constant. If two events in the universe is connected through a radially moving photon, then this condition must be fulfilled.

Assuming that Υ is zero, then the graph for the function $r(\tau)$ is simply two straight lines passing through origin, as seen in figure 4. From this graph is is easy to see which events are causally connected and which who are not. If one events starts at the origin, for instance, than a causally connected point would be placed within the two lines.

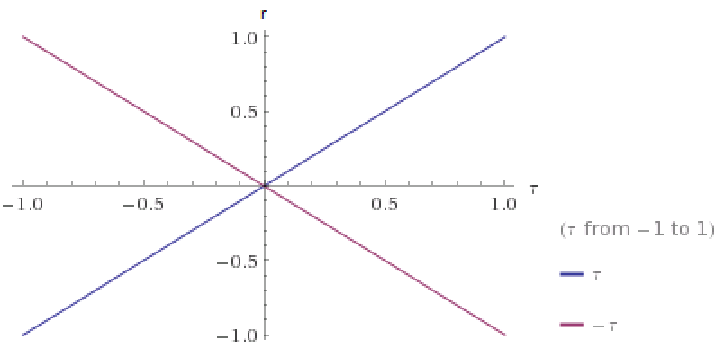

Figure 4: The plot for $r(\tau) = \pm\tau$

Now, still considering a photon signal and using conformal time, it is possible to construct a so-called "particle horizon". This horizon is the maximum comoving distance that the signal can travel given a certain time interval. By taking

[5]Assuming that we do not yet take inflation into account.

the condition in equation (20) into account, this particle horizon is defined as

$$r_{ph} \equiv \tau_f - \tau_i = \int_{t_i}^{t_f} \frac{\mathrm{d}t}{a(t)}, \tag{21}$$

where t_i is the initial time and t_f is the final time. If $t_i = 0$ and $t_f = t_0$ then the particle horizon becomes the distance that a photon could have travelled from the beginning of the universe to present time.

It is possible to use equation (21) to reformulate the horizon problem. At present time, $t = t_0$, the particle horizon is a finite distance. That means that it currently exists points in the universe that are spatially separated with a distance larger than the particle horizon. This means that these points have not been in causal contact at any time in the history of the universe[6]. Despite this fact, these points act as if they are in causal contact[7].

This confusing fact is one of the motivations for the inflation theory, as inflation explains the apparent impossible causal connection between these kinds of points.

6.2 Compatibility with standard Big Bang cosmology

It should be mentioned that the horizon problem does not falsify standard Big Bang cosmology. In fact, the horizon problem might not even be a problem at all. There is nothing in the universe that state that two point that are not in causal contact can not be very similar. The initial conditions in the creation of the universe might have been such that every point in the universe just happened to be similar.

This reasoning relies heavy on pure chance instead of a physical process or processes. The universe is a big place and a lot of points in it are not in causal contact at present time. For each pair of similar non-connected points, the more unlikely the chance that the horizon problem is a pure coincidence. Therefore, a more accepted explanation is the inflation theory [2].

6.3 The Friedmann Equation

The Friedmann equation is an expression that links the density of the universe with the scale factor. In its Newtonian form it looks like the following

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho(t)}{3}, \tag{22}$$

where $\rho(t)$ is the density of the components that make up the universe. It is time dependent since density changes as the universe expands. For convenience, this expression will further on be written in units in which $8\pi G = 1$.

The components that make up $\rho(t)$ are divided into a matter part, $\rho_m(t)$, a radiation part, $\rho_r(t)$, and a dark energy part, $\rho_v(t)$. The reason for this grouping is rather simple; the density of each component have a different time dependence. The time dependence for each component are the following

$$\rho_m(t) = \frac{\rho_0}{a(t)^3} \tag{23}$$

[6] If only standard Big Bang cosmology is considered.
[7] For example, their temperatures are very similar.

$$\rho_r(t) = \frac{\rho_0}{a(t)^4} \tag{24}$$

$$\rho_v(t) = \rho_0, \tag{25}$$

where ρ_0 is a constant initial density. As the components have different dependence on t, different eras of the universe are defined by which component that dominates the density of the universe during that time. In each respective era, the factor of ρ in Friedmann's equation is replaced by the dominant component.

By introducing an equation of state, ω, it is possible to write $\rho(t)$ in a compact way

$$\rho(t) = \frac{\rho_0}{a^{3(1+\omega)}}, \tag{26}$$

where $\omega = -1$ corresponds to dark energy domination, $\omega = 0$ corresponds to matter domination and $\omega = \frac{1}{3}$ corresponds to radiation domination.

6.4 The inflation explanation

A useful cosmological quantity is the Hubble parameter, $H(t)$

$$H(t) \equiv \frac{\dot{a}}{a}. \tag{27}$$

Using this quantity the Friedmann equation becomes

$$H^2 = \frac{\rho(t)}{3} = \frac{\rho_0}{3a^{3(1+\omega)}}, \tag{28}$$

where $H(t)$ describes the rate at which the universe expands and have the unit s^{-1}. Using this quantity, it is possible to make the following change of variables to the integral in equation (21)

$$r_{ph} = \int_{t_i}^{t_f} \frac{\mathrm{d}t}{a(t)} = \int_{a_i}^{a_f} \frac{\mathrm{d}a}{Ha^2} = \int_{a_i}^{a_f} (\frac{1}{Ha})\mathrm{d}\ln(a). \tag{29}$$

This expression is written in terms of the quantity $(\frac{1}{Ha})$, which is called the comoving Hubble radius. This quantity is a distance that represents causal contact at a certain time step. If two points are separated by a larger distance, they are not causally connected at that time step.

One can imagine the universe as being dominated by a fluid that consists of all components of the universe. If this fluid has an equation of state ω, then the comoving Hubble radius can be written in the following way using the Friedmann equation

$$\begin{aligned} H^2 &= \frac{\rho_0}{3a^{3(1+\omega)}} \\ \rightarrow H &= \sqrt{\frac{\rho_0}{3}} a^{-\frac{3}{2}(1+\omega)} \\ \rightarrow \frac{1}{Ha} &= \sqrt{\frac{3}{\rho_0}} a^{\frac{3}{2}(1+\omega)-1} = \sqrt{\frac{3}{\rho_0}} a^{\frac{1}{2}(1+3\omega)}. \end{aligned} \tag{30}$$

During standard Big Bang cosmology ω is positive, meaning that the comoving Hubble radius, and consequently the particle horizon, will increase as a increases with time.

This is basically a repetition of the horizon problem. Points separated by a spatial distance larger than the particle horizon are not in causal contact, but might be at a later time as the particle horizon grows. The fact that these points are very similar before being in causal contact is what the inflation theory is trying to explain.

The proposed solution by inflation is rather simple; the inclusion of a time interval a long time ago where the comoving Hubble radius decreased [1].

Imagine that two points in the universe a long time ago were widely separated but still in casual contact. By including the time interval of comoving Hubble radius decrease, these points would at some point be separated by a distance larger than the now smaller comoving Hubble radius and thus no longer be in causal contact. Yet the fact that they had been in causal contact would mean that their physical properties would be very similar even when no longer connected.

As this time interval passes, the usual Bing Bang cosmology proceeds and the comoving Hubble radius grows as normal. During this time of comoving Hubble radius growth, observers in the universe can observe these two points and notice that they, at the time at least, are not in causal contact but have very similar properties. This is the situation that present time observers on Earth are in.

The idea of a period of comoving Hubble radius reduction is the inflation explanation for the Horizon problem, and it comes with an interesting property of the universe. The property that certain points in the universe leaves the comoving Hubble radius at a certain time and re-enters it a later time. These two events will be discussed further down and are called horizon exit and horizon re-entry.

7 The physics of inflation

7.1 The conditions for inflation

Using the inflation explanation to the horizon problem it is possible to construct the first of three condition for inflation. This condition is that during inflation the comoving Hubble radius decreases

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{Ha}) < 0. \tag{31}$$

This condition naturally leads to the second condition, which states that the second time derivative of the scale factor, a, is positive. This would correspond to an accelerated expansion of the universe. Using equation (27) it is possible to compute the time derivative in equation (31)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{Ha}) = -(\frac{1}{Ha})^2\frac{\mathrm{d}}{\mathrm{d}t}(Ha) = -(\frac{1}{Ha})^2\frac{\mathrm{d}}{\mathrm{d}t}(\dot{a}) = -(\frac{1}{Ha})^2\ddot{a}. \tag{32}$$

Inserting the result of equation (32) into equation (31) and solving for $\ddot{a}$ gives

$$\begin{aligned} -(\frac{1}{Ha})^2\ddot{a} &< 0 \\ \rightarrow \ddot{a} &> 0. \end{aligned} \tag{33}$$

This is the second condition for inflation.

Through consulting the Friedmann equation in combination with equation (27) it is possible to extract the pressure needed to fulfil the second condition

$$\begin{aligned} H^2 &= \frac{\rho_0}{3} a^{-3(1+\omega)} \\ \rightarrow \dot{a} &= \sqrt{\frac{\rho_0}{3}} a^{-\frac{1}{2}(1+3\omega)} \\ \rightarrow \ddot{a} &= -\sqrt{\frac{\rho_0}{3}} \frac{(1+3\omega)}{2} a^{-\frac{3}{2}(1+\omega)} \dot{a} \\ \rightarrow \ddot{a} &= -\sqrt{\frac{\rho_0}{3}} \frac{(1+3\omega)}{2} a^{-\frac{1}{2}(1+3\omega)} H \\ \rightarrow \frac{\ddot{a}}{a} &= -\frac{1}{6} \rho_0 a^{-3(1+\omega)} (1+3\omega). \end{aligned} \tag{34}$$

By using equation (26) as well as defining the pressure of the fluid, p, as

$$p = \omega \rho. \tag{35}$$

Equation (34) becomes

$$\frac{\ddot{a}}{a} = -\frac{\rho}{6}(1+3\omega) = -\frac{1}{6}(\rho + 3p). \tag{36}$$

Using the condition set by equation (33), equation (36) becomes

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{1}{6}(\rho + 3p) > 0 \\ \rightarrow p &< -\frac{\rho}{3}. \end{aligned} \tag{37}$$

This is the third condition for inflation.

7.2 The dynamics of the fluid

The fluid with negative pressure required by the third condition can be modelled by a scalar field, $\phi(\vec{x}, t)$, called the Inflaton field ($\vec{x}$ is a point in space). The Lagrangian, L_ϕ, for this scalar field can be written as the following when the mass is set to one[8]

$$L_\phi = \sqrt{-g}(E_{kin}(\phi) - V(\phi)) = \sqrt{-g}(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)), \tag{38}$$

where g is the determinant of the metric tensor, defined in equation (14)

$$g = \det \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix} = -a^6, \tag{39}$$

where $E_{kin}(\phi)$ is the kinetic energy of the inflaton field, V is the potential term corresponding to self interaction of the scalar field and $g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ is

$$g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -\partial_t \phi \partial_t \phi + \partial_x \phi \partial_x \phi + \partial_y \phi \partial_y \phi + \partial_z \phi \partial_z \phi. \tag{40}$$

[8] The mass will be set to one in every following equation for simplicity sake.

The factor of $\sqrt{-g}$ in the Lagrangian is there so that the corresponding action is coordinate invariant.

Using this Lagrangian it is possible to write the equation of motion for the scalar field as well as the stress energy tensor for the fluid. The equation of motion extracted from the Lagrangian is written as

$$\frac{\partial}{\partial x^{\mu}}(\frac{\partial L_{\phi}}{\partial(\frac{\partial \phi}{\partial x^{\mu}})}) - \frac{\partial L_{\phi}}{\partial \phi} = 0, \tag{41}$$

where

$$\frac{\partial}{\partial x^{\mu}} = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}). \tag{42}$$

Expanding equation (41) gives

$$\begin{aligned} &\frac{\partial}{\partial t}(\frac{\partial L_{\phi}}{\partial(\frac{\partial \phi}{\partial t})}) + \frac{\partial}{\partial x}(\frac{\partial L_{\phi}}{\partial(\frac{\partial \phi}{\partial x})}) + \frac{\partial}{\partial y}(\frac{\partial L_{\phi}}{\partial(\frac{\partial \phi}{\partial y})}) + \frac{\partial}{\partial z}(\frac{\partial L_{\phi}}{\partial(\frac{\partial \phi}{\partial z})}) - \frac{\partial L_{\phi}}{\partial \phi} = 0 \\ \rightarrow &\frac{1}{\sqrt{-g}}(-\frac{\partial}{\partial t}\sqrt{-g}\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\sqrt{-g}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\sqrt{-g}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\sqrt{-g}\frac{\partial \phi}{\partial z}) + \frac{dV}{d\phi} = 0, \end{aligned} \tag{43}$$

which is more easily written as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\sqrt{-g}\frac{\partial \phi}{\partial x_{\mu}} + \frac{dV}{d\phi} = 0. \tag{44}$$

The stress energy tensor, on the other hand, is given by the following expression

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta L_{\phi}}{\delta g^{\mu\nu}}, \tag{45}$$

where the derivative can be written as

$$\begin{aligned} \frac{\delta L_{\phi}}{\delta g^{\mu\nu}} &= \frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}(\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - V(\phi))) \\ &= \sqrt{-g}(-\frac{1}{2}\partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{4}g_{\mu\nu}g^{\sigma\lambda}\partial_{\sigma}\phi\partial_{\lambda}\phi + \frac{1}{2}g_{\mu\nu}V(\phi)), \end{aligned} \tag{46}$$

which makes it so that equation (45) can be written as

$$T_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - g_{\mu\nu}(\frac{1}{2}g^{\sigma\lambda}\partial_{\sigma}\phi\partial_{\lambda}\phi + V(\phi)). \tag{47}$$

In a frame that is comoving with the fluid, the stress energy tensor becomes diagonalized. The time component then correspond to the energy density of the fluid while the spatial components correspond to the pressure. By further assuming that the inflaton field is homogeneous, i.e. that it only depends on time ($\phi(\vec{x}, t) = \phi(t)$), the pressure components all become the same. The then energy density becomes

$$T_{00} = \rho = \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{48}$$

The pressure becomes

$$T_{ii} = a^2 p = a^2(0 + \frac{1}{2}\dot{\phi}^2 - V(\phi)) = a^2(\frac{1}{2}\dot{\phi}^2 - V(\phi)). \tag{49}$$

As before, an equation of state, ω, is defined using ρ and p

$$\omega = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{50}$$

This shows that the pressure of fluid can be sufficiently negative to uphold the third condition if the potential, V, is much larger than the kinetic energy, $\frac{1}{2}\dot{\phi}^2$.

Using this notation in combination with equation (39) and equation (27) it is possible to rewrite the equation of motion for the scalar field, i.e equation (44). The assumption that the field is only time dependent is still in effect

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\sqrt{-g}\frac{\partial \phi}{\partial x_\mu} + \frac{dV}{d\phi} &= 0 \\
\rightarrow \frac{1}{a^3}\frac{d}{dt}(a^3\dot{\phi}) + \frac{dV}{d\phi} &= 0 \\
\rightarrow \frac{3a^2\dot{a}\dot{\phi} + a^3\ddot{\phi}}{a^3} + \frac{dV}{d\phi} &= 0 \\
\rightarrow \ddot{\phi} + 3H\phi + \frac{dV}{d\phi} &= 0.
\end{aligned} \tag{51}$$

7.3 Slow-roll condition

Using equation (48) and the Friedmann equation is is possible to get the following expression

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{\rho}{3} = \frac{1}{3}(\frac{1}{2}\dot{\phi}^2 + V(\phi)). \tag{52}$$

It is possible to define a so called slow-roll parameter, ε

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) = H^2(1 - \frac{3}{2}(\frac{p}{\rho} + 1)) = H^2(1 - \varepsilon). \tag{53}$$

The slow-roll parameter can be written in terms of the time derivative of the inflaton field using equation (48) and (49)

$$\varepsilon = \frac{3}{2}(\frac{p}{\rho} + 1) = \frac{3}{2}(\frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} + \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}) = \frac{3}{2}\frac{\dot{\phi}^2}{\rho} = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}. \tag{54}$$

The slow roll parameter contain information about the second and third inflation conditions as accelerated expansion only occurs if $\varepsilon < 1$. This fact is shown below

$$\begin{aligned}
\varepsilon &= \frac{3}{2}(\frac{p}{\rho} + 1) \\
\rightarrow \frac{2\varepsilon}{3} - 1 &= \frac{p}{\rho}.
\end{aligned} \tag{55}$$

Adding the third inflation condition given by equation (37) gives

$$\begin{aligned} \frac{2\varepsilon}{3} - 1 = \frac{p}{\rho} &< -\frac{1}{3} \\ \rightarrow \frac{2\varepsilon}{3} &< 1 - \frac{1}{3} \\ \rightarrow \varepsilon &< 1. \end{aligned} \tag{56}$$

This is one of two slow-roll condition.

For the second slow-roll condition, some more work needs to be put into the equations presented above. First of all, by observing the equation of motion of the inflaton field written in the form of equation (51) it is seen that it exists a term containing a second time derivative of the inflaton field. If accelerated expansion were to occur, it would not be sustained for a very long time if this term were to be large compared to the other terms. As such, the following condition needs to be uphold if accelerated expansion is to be sustained

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |\frac{dV}{d\phi}|. \tag{57}$$

This condition is met if the absolute value of the following slow-roll parameter, η, is less than one

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{58}$$

The second slow roll condition is therefore $|\eta| < 1$. Inflation ends when the slow roll conditions are violated.

To get a sense of how much the universe is expanded during inflation it could be useful to introduce the concept of e-folds. An e-fold is the time interval that it takes to increase the volume of the universe by a factor of e. The number of e-folds in a certain time interval is defined as

$$N \equiv \int_{t_i}^{t_f} H dt. \tag{59}$$

Using certain facts associated with slow-roll inflation, it is possible to rewrite this integral. First of all, as mentioned above, the first slow-roll condition is only valid as long as the pressure p is sufficiently negative. p is only sufficiently negative as long as the potential, V, is larger than the kinetic energy, term $\frac{1}{2}\dot{\phi}^2$. This means that equation (52) approximates to

$$H^2 \approx \frac{V(\phi)}{3}. \tag{60}$$

Furthermore, for the second slow-roll condition equation (57) needs to be satisfied. When this condition is satisfied the equation of motion can be approximated as

$$\begin{aligned} \ddot{\phi} + 3H\phi + \frac{dV}{d\phi} \approx 3H\phi + \frac{dV}{d\phi} &= 0 \\ \rightarrow \dot{\phi} &\approx -\frac{1}{3H}\frac{dV}{d\phi}. \end{aligned} \tag{61}$$

Using equation (54), (60) and (61) the integral in equation (59) can be rewritten in many ways

$$N \equiv \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi = \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\varepsilon}} d\phi \approx \int_{\phi_f}^{\phi_i} \frac{V}{(\frac{dV}{d\phi})} d\phi. \tag{62}$$

If index i stands for the start of inflation and index f stands for the end of inflation[9], the number of e-folds is given.

After inflation, a process called reheating commences. A brief explanation of this process is that "the inflationary energy density is converted into standard model degrees of freedom" [1].

7.4 Different models of inflation

A point that is worth brining up is that the inflationary process presented above[10] is just one of several possible models of inflation [1].

For starters, no explicit potential have been presented in the equations above, which directly affects the inflationary process. Depending on which potential a model chooses to take, the inflation will behave differently.

Furthermore, it is possible that the fluid of the universe can be modelled by more than one scalar field. If there are more than one scalar field present, the calculations above will instantly become more complex.

Part IV
Perturbations and fluctuations

By observing the CMB it is possible to see small fluctuations in temperature. The fluctuations represents inhomogeneity in the energy density of the universe at the time of recombination. To have a completely satisfactory cosmological theory concerning the early universe, this inhomogeneity needs to be explained. When considering inflation in combination with quantum fluctuations, it is possible to get one such explanation.

When inflation rapidly expands the universe to a larger scale, microscopic quantum fluctuation will also be expanded. This results in larger scale cosmological inhomogeneities.

8 Cosmological perturbations

Before going in to the quantum mechanical part of large scale perturbations, an introduction into cosmological perturbations is required.

Throughout section 7 no heed was paid to any perturbation when introducing different quantities. This makes it so that each quantity presented was not in its most general form. As such, when including perturbations, some alteration to the quantities needs to be included.

[9]This is the time when the slow-roll conditions are violated.

[10]This particular take on inflation is called single-field slow-roll inflation and is one of the simpler iterations.

Imagine an arbitrary quantity X[11]. When taking perturbations into account, this quantity gets separated into two parts, a homogeneous background part and a perturbation part. The background part is homogeneous in the sense that it only depends on time. It can be seen as the background from which the perturbation will measured. The perturbation part is generally a small term which makes it so the quantity differs from the homogeneous background. Mathematically, the resulting expression for X becomes

$$X(\vec{x}, t) = \bar{X}(t) + \delta X(\vec{x}, t), \tag{63}$$

where $\bar{X}$ is the homogeneous background of the quantity and δX is the perturbation.

To give a concrete example, the quantity X might be the inflaton field. In this case, the field would be rewritten as

$$\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t), \tag{64}$$

where $\bar{\phi}(t)$ was encountered in the previous section.

8.1 Perturbations in the FRW metric

An important quantity to consider when taking perturbations into account is the Friedmann-Robertson-Walker metric, defined in equation (16). By including perturbations, this metric can be written as

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + 2aB_i \mathrm{d}x^i \mathrm{d}t + a^2((1-2\Psi)\delta_{ij} + E_{ij})\mathrm{d}x^i \mathrm{d}x^j, \tag{65}$$

where

$$B_i = \partial_i B - S_i \tag{66}$$

$$E_{ij} = 2\partial_{ij} E + 2\partial_{(i} F_{j)} + h_{ij}, \tag{67}$$

and

$$\partial^i S_i = 0 \tag{68}$$

$$\partial^i F_i = 0 \tag{69}$$

$$h_i^i = \partial^i h_{ij} = 0, \tag{70}$$

where Φ, B_i, Ψ and E_{ij} are perturbations.

This way of writing the perturbations to the metric might seem unnecessarily complicated, but it comes with an advantage. This advantage might be more easily seen by rewriting equation (65) in matrix form

$$\mathrm{d}s^2 = \begin{bmatrix} -(1+2\Phi) & aB_1 & aB_2 & aB_3 \\ aB_1 & a^2(1-2\Psi) + E_{1,1} & a^2 E_{2,1} & a^2 E_{3,1} \\ aB_2 & a^2 E_{1,2} & a^2(1-2\Psi) + E_{2,2} & a^2 E_{3,2} \\ aB_3 & a^2 E_{1,3} & a^2 E_{2,3} & a^2(1-2\Psi) + E_{3,3} \end{bmatrix} \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{71}$$

[11]This quantity can be a scalar, vector or a tensor.

First of all, the metric is no longer diagonalized as perturbations might be added to any component of the metric, not only the diagonal. Secondly, the primary advantage to writing the perturbations Φ, B_i, Ψ and E_{ij} in this way is that they become divided into scalar parts, vector parts and a tensor parts depending on which component of the metric they are added to.

Φ only contain a scalar part since it only act on a single component of the metric. Ψ also only contain a scalar part and act on every diagonal, spatial component of the metric. It is not necessary to also associate a tensor part to it since E_{ij} also acts on the diagonal. B_i contain a vector part and a scalar part since it acts on the components that both depend on time and space. The spatial part is three dimensional, i.e. it has three components, which is why B_i will need the three dimensional vector part. E_{ij} acts on every spatial component of the metric, and as such will need a 3-tensor part, a vector part and a scalar part.

The vector parts of B_i and E_{ij} is seen in equation (66) and (67) as S_i and $2\partial_{(i}F_{j)}$ respectively. The tensor part of E_{ij} is seen as h_{ij} in equation (67). Equation (68) to (70) basically states that these parts are traceless.

The reason why it is advantageous to divide the metric into scalar, vector and tensor parts is because at linear level these parts evolve independently of each other and can thus be treated separately [1].

8.1.1 Perturbations in the FRW metric with change in coordinates

The FRW metric need to be invariant to coordinate change [4], i.e

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \tilde{g}_{\mu\nu}\mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}^\nu, \tag{72}$$

where the tilde indicate the same expression in different coordinates. This property makes it so that the scalar parts of the perturbations mentioned above changes necessarily with change in coordinates.

For example, imagine the following change in coordinates

$$\begin{aligned} t &\to t + \alpha \\ x^i &\to x^i + \delta^{ij}\partial_j\beta, \end{aligned} \tag{73}$$

where α and β are considered to be very small functions. This change in coordinates leads to the following

$$\begin{aligned} &\mathrm{d}t \to (1+\dot{\alpha})\mathrm{d}t + \partial_k\alpha \mathrm{d}x^k \\ &\mathrm{d}x^i \to \mathrm{d}x^i + \mathrm{d}(\delta^{ij}\partial_j\beta(x^k,t)) = \mathrm{d}x^i + \delta^{ij}\partial_j\dot{\beta}\mathrm{d}t + \delta^{ij}\partial_j\partial_k\beta\mathrm{d}x^k. \end{aligned} \tag{74}$$

Since a is time dependent there will be a change in the expression for a. The general expression for a differs depending on which epoch of the universe one considers. For example, it looks different in a radiation dominated universe compared to a matter dominated universe. For a inflating universe the dark energy is dominant (this will be shown below) and an expression for a can be extracted from the Friedmann equation using $\rho(t) = \rho_v(t) = \rho_0$

$$\begin{aligned} &H^2 = (\frac{\dot{a}}{a})^2 = \frac{\rho}{3} = \frac{\rho_0}{3} \\ &\to \dot{a} = \sqrt{\frac{\rho_0}{3}}a \\ &\to a = a_0 e^{\sqrt{\frac{\rho_0}{3}}t} = a_0 e^{Ht}, \end{aligned} \tag{75}$$

where a_0 is a constant. When using the different coordinates presented in equation (73) the expression becomes:

$$a = a_0 e^{H(t+\alpha)}. \tag{76}$$

Because of equation (74) and (76), $\mathrm{d}s^2$ would potentially look quite different compared to when the unchanged coordinates are used. However, since equation (72) is supposed to hold despite the changes presented in equation (74) and (76), something must compensate for these changes. This "something" is the perturbations Φ, B_i, Ψ and E_{ij}, or more specifically, their scalar parts. If these perturbations change in a certain way with the change in coordinates, then equation (72) will hold. To see what changes are required by the perturbations, it is necessary to rewrite equation (65) combined with equation (74) and (76)

$$\begin{aligned}
\mathrm{d}s^2 &= -(1+2\Phi)((1+\dot\alpha)\mathrm{d}t + \partial_k\alpha \mathrm{d}x^k)^2 \\
&+2a_0 e^{H(t+\alpha)} B_i(\mathrm{d}x^i + \delta^{ij}\partial_j\dot\beta \mathrm{d}t + \delta^{ij}\partial_j\partial_k\beta \mathrm{d}x^k)((1+\dot\alpha)\mathrm{d}t + \partial_k\alpha \mathrm{d}x^k) \\
&+(a_0 e^{H(t+\alpha)})^2((1-2\Psi)\delta_{ij} + E_{ij})(\mathrm{d}x^i + \delta^{ij}\partial_j\dot\beta \mathrm{d}t + \delta^{ij}\partial_j\partial_k\beta \mathrm{d}x^k)(\mathrm{d}x^j \\
&+\delta^{ji}\partial_i\dot\beta \mathrm{d}t + \delta^{ji}\partial_i\partial_k\beta \mathrm{d}x^k) \\
\rightarrow \mathrm{d}s^2 &= -(1+2\Phi)((1+\dot\alpha)\mathrm{d}t + \partial_k\alpha \mathrm{d}x^k)^2 \\
&+2a_0 e^{H(t+\alpha)} B_i(\mathrm{d}x^i + \partial^i\dot\beta \mathrm{d}t + \partial^i\partial_k\beta \mathrm{d}x^k)((1+\dot\alpha)\mathrm{d}t + \partial_k\alpha \mathrm{d}x^k) \\
&+(a_0 e^{H(t+\alpha)})^2((1-2\Psi)\delta_{ij} + E_{ij})(\mathrm{d}x^i + \partial^i\dot\beta \mathrm{d}t + \partial^i\partial_k\beta \mathrm{d}x^k)(\mathrm{d}x^j \\
&+\partial^j\dot\beta \mathrm{d}t + \partial^j\partial_k\beta \mathrm{d}x^k).
\end{aligned} \tag{77}$$

To avoid this huge expression, it would be beneficial to identify terms that can be approximated to zero. For starters, α, β and all the perturbations are very small. This means that any multiplication between these quantities are even a smaller number, so small that it can be approximated to zero. By removing all terms containing a multiplication of two of these quantities, the following expression will remain

$$\begin{aligned}
\mathrm{d}s^2 &= -(1+2\Phi)((1+2\dot\alpha)\mathrm{d}t^2 + 2\partial_k\alpha \mathrm{d}x^k \mathrm{d}t) + 2a_0 e^{H(t+\alpha)} B_i \mathrm{d}x^i \mathrm{d}t \\
&+(a_0 e^{H(t+\alpha)})^2((1-2\Psi)\delta_{ij} + E_{ij})(\mathrm{d}x^i \mathrm{d}x^j + \partial^j\dot\beta \mathrm{d}t \mathrm{d}x^i + \partial^j\partial_k\beta \mathrm{d}x^k \mathrm{d}x^i \\
&+\partial^i\dot\beta \mathrm{d}t \mathrm{d}x^j + \partial^i\partial_k\beta \mathrm{d}x^k \mathrm{d}x^j) \\
\rightarrow \mathrm{d}s^2 &= -((1+2\dot\alpha)\mathrm{d}t^2 + 2\partial_i\alpha \mathrm{d}x^i \mathrm{d}t + 2\Phi \mathrm{d}t^2) + 2a_0 e^{H(t+\alpha)} B_i \mathrm{d}x^i \mathrm{d}t \\
&+(a_0 e^{H(t+\alpha)})^2((1-2\Psi)\delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \partial_i\dot\beta \mathrm{d}t \mathrm{d}x^i + \partial_i\partial_j\beta \mathrm{d}x^j \mathrm{d}x^i + \partial_j\dot\beta \mathrm{d}t \mathrm{d}x^j \\
&+\partial_j\partial_i\beta \mathrm{d}x^i \mathrm{d}x^j + E_{ij}\mathrm{d}x^i \mathrm{d}x^j) \\
\rightarrow \mathrm{d}s^2 &= -((1+2\dot\alpha)\mathrm{d}t^2 + 2\partial_i\alpha \mathrm{d}x^i \mathrm{d}t + 2\Phi \mathrm{d}t^2) + 2a_0 e^{H(t+\alpha)} B_i \mathrm{d}x^i \mathrm{d}t \\
&+(a_0 e^{H(t+\alpha)})^2((1-2\Psi)\delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + 2\partial_i\dot\beta \mathrm{d}t \mathrm{d}x^i \\
&+2\partial_j\partial_i\beta \mathrm{d}x^i \mathrm{d}x^j + E_{ij}\mathrm{d}x^i \mathrm{d}x^j) \\
\rightarrow \mathrm{d}s^2 &= \mathrm{d}t^2(-(1+2\Phi) - 2\dot\alpha) \\
&+\mathrm{d}t \mathrm{d}x^i(2a_0 e^{H(t+\alpha)} B_i + 2\partial_i((a_0 e^{H(t+\alpha)})^2\dot\beta - \alpha) \\
&+\mathrm{d}x^i \mathrm{d}x^j (a_0 e^{H(t+\alpha)})^2((1-2\Psi)\delta_{ij} + 2\partial_i\partial_j\beta + E_{ij}).
\end{aligned} \tag{78}$$

Furthermore, it is beneficial to further narrow down the expression by making the following approximations due to the smallness of α

$$a_0 e^{H(t+\alpha)} \approx a_0 e^{Ht}, \tag{79}$$

and

$$(a_0 e^{H(t+\alpha)})^2 \approx (a_0 e^{Ht})^2(1+2H\alpha). \tag{80}$$

Putting in these approximations changes equation (78) to

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}t^2(-(1+2\Phi) - 2\dot{\alpha}) \\ &+\mathrm{d}t\mathrm{d}x^i(2a_0e^{Ht}B_i + 2\partial_i((a_0e^{Ht})^2\dot{\beta} - \alpha) \\ &+\mathrm{d}x^i\mathrm{d}x^j(a_0e^{Ht})^2((1-2\Psi)\delta_{ij} + 2\partial_i\partial_j\beta + E_{ij} + 2H\alpha\delta_{ij}). \end{aligned} \tag{81}$$

The last step is to see how the perturbations Φ, B_i, Ψ and E_{ij} must change in order for equation (72) to hold. Through equation (65) it is known how the expression for $\mathrm{d}s^2$ looks without the change in coordinates. For the metric to remain invariant under coordinate change, the factors in front of $\mathrm{d}t^2$, $\mathrm{d}x^i\mathrm{d}t^i$ and $\mathrm{d}x^i\mathrm{d}x^j$ must be equal under the change in coordinates. Thus, it is known what condition the change in perturbation must uphold. Mathematically it can be written as

$$-(1+2\Phi) = (-(1+2(\Phi+X_1)) - 2\dot{\alpha}) \tag{82}$$

$$2a_0e^{Ht}B_i = 2a_0e^{Ht}(B_i + X_{2,i}) + 2\partial_i((a_0e^{Ht})^2\dot{\beta} - \alpha \tag{83}$$

$$\begin{aligned} (a_0e^{Ht})^2((1-2\Psi)\delta_{ij} + E_{ij}) = (a_0e^{Ht})^2(&(1-2(\Psi+X_3))\delta_{ij} + (E_{ij} + X_{4,ij}) \\ &+ 2\partial_i\partial_j\beta + 2H\alpha\delta_{ij}), \end{aligned} \tag{84}$$

where X_1, $X_{2,i}$, X_3 and $X_{4,ij}$ are the change in perturbations due to change in coordinates.

The change in B_i and E_{ij} differs a bit from the other perturbations since they contain other parts other than scalar parts. However, because of the derivatives that act on the scalar parts in B_i and E_{ij}, it will be seen that only the scalar parts needs to change.

X_1 can be found by solving for it in equation (82) and it is

$$X_1 = -\dot{\alpha}, \tag{85}$$

which makes it so that Φ changes in the following way due to change in coordinates

$$\Phi \to \Phi + X_1 = \Phi - \dot{\alpha}. \tag{86}$$

$X_{2,i}$ can be found in a similar way

$$X_{2,i} = \partial_i(-a_0e^{Ht}\dot{\beta} + \frac{\alpha}{a_0e^{Ht}}). \tag{87}$$

Because of the derivative, ∂_i, it is seen that that the perturbation only changes in the scalar part, B. The change in B_i due to coordinate change becomes

$$B_i = \partial_iB - S_i \to \partial_iB - S_i + X_{2,i} = \partial_i(B - a_0e^{Ht}\dot{\beta} + \frac{\alpha}{a_0e^{Ht}}) - S_i. \tag{88}$$

X_3 and $X_{4,ij}$ are a bit trickier to solve for since they are both part of the same expression. However, since E_{ij} contains derivatives while Ψ does not, it is possible to separate equation (84) into two expressions, one with derivatives and one without

$$(1-2\Psi)\delta_{ij} = (1-2(\Psi + X_3))\delta_{ij} + +2H\alpha\delta_{ij} \tag{89}$$

$$E_{ij} = (E_{ij} + X_{4,ij}) + 2\partial_i\partial_j\beta. \tag{90}$$

Now it is easier to solve for X_3 and $X_{4,ij}$

$$X_3 = H\alpha \tag{91}$$

$$X_{4,ij} = -2\partial_i\partial_j\beta, \tag{92}$$

which in turn makes it so that the perturbations Ψ and E_{ij} change as the following due to change in coordinates

$$\Psi \to \Psi + X_3 = \Psi + H\alpha \tag{93}$$

$$\begin{aligned} E_{ij} &= 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij} \\ \to 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij} + X_{4,ij} &= 2\partial_{ij}(E-\beta) + 2\partial_{(i}F_{j)} + h_{ij}. \end{aligned} \tag{94}$$

Again it is seen that only the scalar part of E_{ij} change.

8.2 Perturbation in the stress energy tensor

Just as the FRW metric can be rewritten to incorporate perturbation, the same goes for the stress energy tensor [1]

$$T^0_0 = -(\bar{\rho} + \delta\rho) \tag{95}$$

$$T^0_i = (\bar{\rho} + \bar{p})av_i \tag{96}$$

$$T^i_0 = -(\bar{\rho} + \bar{p})\frac{(v^i - B^i)}{a} \tag{97}$$

$$T^i_j = \delta^i_j(\bar{p} + \delta p) + \Sigma^i_j. \tag{98}$$

Where Σ^i_j corresponds to anisotropic stress. As in the case of the FRW metric, the stress energy tensor is no longer diagonalized.

If the same change in coordinates presented in equation (73) would be imposed on the stress energy tensor, its components would change. More specifically, the perturbations $\delta\rho$ and δp would change. The perturbations changes as follows

$$\delta\rho \to \delta\rho - \dot{\bar{\rho}}\alpha \tag{99}$$

$$\delta p \to \delta p - \dot{\bar{p}}\alpha \tag{100}$$

$$\delta q \to \delta q - (\bar{\rho} + \bar{p})\alpha, \tag{101}$$

where δq is the scalar part of the 3-momentum density $T^0_i = \partial_i\delta q$ and

$$\partial_i\delta q \equiv (\bar{\rho} + \bar{p})av_i. \tag{102}$$

8.3 Variables invariant to coordinate change

As the previous subsections showed, different choices of coordinates have a big impact on certain quantities. This being the case, it is useful to introduce variables that are invariant to change in coordinates. As before, the focus will be on perturbations.

These coordinate invariant perturbation variables are constructed by combining perturbations from the FRW metric and the stress energy tensor. The first of these variables is the "curvature perturbation on uniform-density hypersurfaces" and is constructed by combining Ψ with $\delta\rho$

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho. \tag{103}$$

It is possible to check if this expression is coordinate invariant by using the the coordinates from equation (73) and the corresponding expressions for Ψ and $\delta\rho$

$$-\tilde{\zeta} = (\Psi + H\alpha) + \frac{H}{\dot{\bar{\rho}}}(\delta\rho - \dot{\bar{\rho}}\alpha) = \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho = -\zeta. \tag{104}$$

The tilde represent ζ in different coordinates.

The purpose of ζ is to measure the spatial curvature of constant-density hypersurfaces [1]. A crucial property of ζ is that it remains constant outside the comoving Hubble radius. The ζ are frozen outside the Hubble scale because of causality. More precisely, they are frozen out if we assume the fluctuations are adiabatic. This means that once this perturbation leaves the radius during inflation, it does not change until it gets reabsorbed into the comving Hubble radius during standard Big Bang cosmology.

During slow-roll inflation ζ can be rewritten in terms of ϕ. This is seen by first computing the time derivative of $\bar{\rho}$ using equation (48)

$$\dot{\bar{\rho}} = \frac{d}{dt}(\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}) = \dot{\bar{\phi}}\ddot{\bar{\phi}} + \frac{dV}{d\bar{\phi}}\dot{\bar{\phi}}. \tag{105}$$

Using equation (51), the expression becomes

$$\dot{\bar{\rho}} = \dot{\bar{\phi}}\ddot{\bar{\phi}} + \frac{dV}{d\bar{\phi}}\dot{\bar{\phi}} = -3H\dot{\bar{\phi}}^2. \tag{106}$$

Furthermore, the perturbation $\delta\rho$ can be written as

$$\delta\rho = \delta(\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}) = \dot{\bar{\phi}}\delta\dot{\phi} + \frac{dV}{d\bar{\phi}}\delta\phi, \tag{107}$$

which again can be rewritten using equation (51)

$$\delta\rho = \dot{\bar{\phi}}\delta\dot{\phi} + (\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}})\delta\phi \approx -3H\dot{\bar{\phi}}\delta\phi. \tag{108}$$

The first two terms disappears in the approximation due to the conditions set by inflation, in (57), as well as the fact that the perturbation in ϕ is assumed to be adiabatic. Using equation (108) and (108) ζ becomes

$$-\zeta \approx \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi. \tag{109}$$

Another coordinate invariant variable to consider is the so-called "comoving curvature perturbation"

$$\mathcal{R} \equiv \Psi + \frac{H}{\bar{\rho}+\bar{p}}\delta q. \tag{110}$$

It is easy to check that this variable is coordinate invariant in the same way as in equation (104)

$$\tilde{\mathcal{R}} = \Psi + H\alpha + \frac{H}{\bar{\rho}+\bar{p}}(\delta q - (\bar{\rho}+\bar{p})\alpha) = \mathcal{R}. \tag{111}$$

The purpose of $\mathcal{R}$ is to measure the spatial curvature of comoving hypersurfaces [1].

During inflation the scalar part of the 3-momentum density is equal to the following

$$T_i^0 = -\dot{\bar{\phi}}\partial_i\delta\phi, \tag{112}$$

which makes it so that equation (110) becomes

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi. \tag{113}$$

Notice that equation (109) and (113) are equal, meaning that $-\zeta$ and $\mathcal{R}$ are equal during inflation.

8.4 Power spectra

As will be seen, the power spectrum of $\mathcal{R}$ (or equivalently, ζ) is important in relating the observed temperature fluctuations in the CMB to inflation. Before calculating the power spectrum for $\mathcal{R}$, however, it would be beneficial to calculate the power spectrum for a more simple quantity.

Imagine that two non-interacting quantities exists, x and y (these are not the Cartesian coordinates that we have used so far). These quantities represent the positions of two particles. Exactly where these particles are in space is not determined, but is rather decided by probability functions

$$P(x) = \frac{1}{\Delta x\sqrt{\pi}}\exp(\frac{-x^2}{2(\Delta x)^2}) \tag{114}$$

$$P(y) = \frac{1}{\Delta y\sqrt{\pi}}\exp(\frac{-y^2}{2(\Delta y)^2}). \tag{115}$$

So the probability that the first particle occupies position x is given by the first expression and the probability that the second particle occupies position y is given by the second expression. The factors outside the exponents are normalization factors.

Since the particles have to occupy some position in space, the following is true

$$\int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} P(y)dy = 1. \tag{116}$$

The expectation value of x and y, that is the average position is given by

$$\langle x\rangle = \int_{-\infty}^{\infty} xP(x)dx \tag{117}$$

$$\langle y\rangle = \int_{-\infty}^{\infty} yP(y)dy. \tag{118}$$

However, since the integrands consist of an even and an odd function, they both becomes zero

$$\langle x\rangle = \langle y\rangle = 0. \tag{119}$$

Now, using both x and y it is possible to construct a complex quantity as well as its complex conjugate

$$z = x + iy \tag{120}$$

$$z^* = x - iy. \tag{121}$$

The probability function for this quantity is still dependent on x and y however

$$P(z) = P(x,y) = P(x)P(y) = \frac{1}{C}\exp(\frac{-(x^2+y^2)}{2(\Delta x)^2}) = \frac{1}{C}\exp(\frac{-zz^*}{2(\Delta x)^2}), \tag{122}$$

where we have assumed that $\Delta x = \Delta y$ and C is the normalization factor.

Now, for reasons that will be clear in a moment, the average of z^2 as well as zz^* will be calculated

$$\langle z^2\rangle = \langle (x+iy)(x-iy)\rangle = \langle x^2 - y^2\rangle 2i\langle x\rangle\langle y\rangle = 0 \tag{123}$$

$$\langle zz^*\rangle = \langle x^2 + y^2\rangle \neq 0. \tag{124}$$

So when it comes to complex quantities, the average value is zero when taking the square of it. However, one does get a contribution from the average of the complex quantity with its complex conjugate. This is very useful information when it comes to computing the power spectrum for $\mathcal{R}$.

As with any smooth function, $\mathcal{R}$ can be Fourier transformed

$$\hat{\mathcal{R}}(t,\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} R(t,\bar{x}) \equiv R_{\vec{k}}, \tag{125}$$

where $\vec{k}$ is the wave number. This is a complex quantity, and as such it obeys the same mathematics as z when it comes to averages. The power spectrum for $\mathcal{R}_{\vec{k}}$ is found by computing the expectation of $\mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}$ where $\vec{k}'$ is different wave number compared to $\vec{k}$. As equation (123) and (124) show, however, one only gets a non-zero contribution if $\vec{k}' = -\vec{k}$. The power spectrum for $\mathcal{R}_{\vec{k}}$ becomes

$$\langle \mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}\rangle = \int D\mathcal{R}\frac{\mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}}{C}\exp(-\int\frac{d^3k''}{(2\pi)^3}(\frac{1}{2}\mathcal{R}_{\vec{k}''}\mathcal{R}_{-\vec{k}''}f(k))). \tag{126}$$

This expression is analogous for that of zz^*, even though it might not look like it. The integral over $D\mathcal{R}$ means a separate integral over $\mathcal{R}_{\vec{k}}$ for each possible value of $\vec{k}$. The factor $\frac{1}{C}$ is the normalization factor. The integral in the exponent ensures that the exponent is evaluated for every value of $\vec{k}''$. The factor of $\frac{1}{(2\pi)^3}$ is there as this integral is evaluated in a three dimensional k-space and $f(k)$ is for the moment an unknown function.

Imagine that $J_{\vec{k}''}\mathcal{R}_{\vec{k}''}$ is added to the integral in the exponent where $J_{\vec{k}''}$ is some function. If this new function is set to zero, the integral is unchanged

$$\langle \mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}\rangle = \int D\mathcal{R}\frac{\mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}}{C}\exp(-\int\frac{d^3k''}{(2\pi)^3}(\frac{1}{2}\mathcal{R}_{\vec{k}''}\mathcal{R}_{-\vec{k}''}f(k)+J_{\vec{k}''}\mathcal{R}_{\vec{k}''}))|_{J_{\vec{k}}=0}. \tag{127}$$

This expression can be rewritten in terms of derivatives of J

$$\langle \mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}\rangle = \frac{\partial}{\partial J_{\vec{k}}}\frac{\partial}{\partial J_{\vec{k}'}}\int D\mathcal{R}\frac{1}{C}\exp(-\int\frac{d^3k''}{(2\pi)^3}(\frac{1}{2}\mathcal{R}_{\vec{k}''}\mathcal{R}_{-\vec{k}''}f(k)+J_{\vec{k}''}\mathcal{R}_{\vec{k}''}))|_{J_{\vec{k}}=0}. \tag{128}$$

It is possible to further rewrite the integral in the exponent by completing a square[12]

$$\langle \mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}\rangle = \\ \frac{\partial}{\partial J_{\vec{k}}}\frac{\partial}{\partial J_{\vec{k}'}}\int D\mathcal{R}\frac{1}{C}\exp(-\int\frac{d^3k''}{(2\pi)^3}(\frac{1}{2}(\mathcal{R}_{\vec{k}''}+\frac{J_{-\vec{k}''}}{f(k)})(\mathcal{R}_{-\vec{k}''}+\frac{J_{\vec{k}''}}{f(k)})f(k)-\frac{J_{\vec{k}''}J_{-\vec{k}''}}{2f(k)})|_{J_{\vec{k}}=0}. \tag{129}$$

From this position it is convenient to change the integral slightly by defining new integration variables

$$\mathcal{R}'_{\vec{k}''} = \mathcal{R}_{\vec{k}''}+\frac{J_{-\vec{k}''}}{f(k)} \tag{130}$$

$$\mathcal{R}'_{-\vec{k}''} = \mathcal{R}_{-\vec{k}''}+\frac{J_{\vec{k}''}}{f(k)}. \tag{131}$$

This does not change the integrals since they are evaluated over all $\mathcal{R}'_{\vec{k}''}$.

Equation (129) becomes

$$\langle \mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}\rangle = \\ \frac{\partial}{\partial J_{\vec{k}}}\frac{\partial}{\partial J_{\vec{k}'}}\int D\mathcal{R}\frac{1}{C}\exp(-\int\frac{d^3k''}{(2\pi)^3}(\frac{1}{2}\mathcal{R}'_{\vec{k}''}\mathcal{R}'_{-\vec{k}''}f(k)-\frac{J_{\vec{k}''}J_{-\vec{k}''}}{2f(k)})|_{J_{\vec{k}}=0}. \tag{132}$$

When applying the derivatives in this expression only the second term in the exponent will be affected, everything else can be considered to be constant factor. Furthermore, since everything is treated in k-space, the derivatives have the following property

$$\frac{\partial J'_{\vec{k}}}{\partial J_{\vec{k}}} = \delta^3(\vec{k}-\vec{k}'')(2\pi)^3. \tag{133}$$

[12]Notice that it is possible to change $-\vec{k}''$ to $\vec{k}''$ and vice versa since the integral is evaluated over all $\vec{k}''$.

With all this in mind, what needs to be calculated is the following

$$\begin{aligned}
&\frac{\partial}{\partial J_{\vec{k}}}\frac{\partial}{\partial J_{\vec{k}'}}\exp(\frac{1}{2}\int\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{J_{\vec{k}''}J_{-\vec{k}''}}{f(k)})\\
&\rightarrow\frac{\partial}{\partial J_{\vec{k}'}}\exp(\frac{1}{2}\int\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{J_{\vec{k}''}J_{-\vec{k}''}}{f(k)})\int\mathrm{d}^3k''\frac{J_{-\vec{k}''}}{2f(k)}\delta^3(\vec{k}''-\vec{k})\\
&\rightarrow\exp(\frac{1}{2}\int\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{J_{\vec{k}''}J_{-\vec{k}''}}{f(k)})\int\mathrm{d}^3k''\frac{J_{-\vec{k}''}}{2f(k)}\delta^3(\vec{k}''-\vec{k})\int\mathrm{d}^3k''\frac{J_{\vec{k}''}}{2f(k)}\delta^3(-\vec{k}''-\vec{k}')\\
&+\exp(\frac{1}{2}\int\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{J_{\vec{k}''}J_{-\vec{k}''}}{f(k)})\int\mathrm{d}^3k''\frac{1}{2f(k)}\delta^3(\vec{k}''-\vec{k})\delta^3(-\vec{k}''-\vec{k}')(2\pi^3).
\end{aligned}\tag{134}$$

If $J_{\vec{k}}$ is set to zero then equation (134) becomes

$$\int\mathrm{d}^3k''\frac{1}{2f(k)}\delta^3(\vec{k}''-\vec{k})\delta^3(-\vec{k}''-\vec{k}')(2\pi^3)=\frac{1}{2f(k)}\delta^3(-\vec{k}-\vec{k}')(2\pi)^3.\tag{135}$$

So the power spectrum of $\mathcal{R}_{\vec{k}}$ is

$$\langle\mathcal{R}_{\vec{k}}\mathcal{R}_{\vec{k}'}\rangle=\delta^3(\vec{k}+\vec{k}')(2\pi)^3P_{\mathcal{R}}(k),\tag{136}$$

where

$$P_{\mathcal{R}}(k)=\frac{1}{2f(k)}.\tag{137}$$

Using equation (137) it is possible to define another quantity

$$\Delta_s^2\equiv\frac{k^3}{2\pi^2}P_{\mathcal{R}}(k).\tag{138}$$

In the same way, the power spectrum for the two polarization modes of the tensor perturbation h_{ij} can be calculated

$$\langle h_{\vec{k}}h_{\vec{k}'}\rangle=\delta^3(\vec{k}+\vec{k}')(2\pi)^3P_h(k),\tag{139}$$

where a similar quantity to that of equation (138) can be defined

$$\Delta_t^2\equiv 2\frac{k^3}{2\pi^2}P_h(k),\tag{140}$$

where the factor of 2 is for the two polarizations.

Notice that the only thing that differentiate equation (136) and (139) is their respective factor of $P(k)$. As such, these are the most interesting things to compute.

9 Adding quantum mechanics

It turns out that quantum fluctuations are the source of the power spectrum of $\mathcal{R}$ and h_{ij}. Quantum mechanics adds variance for fluctuations in all the light fields presented above, such as the inflaton field or the metric perturbations.

This variance behaves similarly to the variance in amplitude of a harmonic oscillator from zero-point fluctuations around a ground state [1]. Since quantum harmonic oscillators are known systems, the idea is to promote perturbations, such as $\mathcal{R}$, to harmonic oscillators.

First of all, quantum fluctuations are created over all lengths scales inside the so-called Hubble radius [1]. The comoving Hubble radius was introduced earlier in equation (30) as $(aH)^{-1}$. If all these lengths scales are incorporated into a spectrum with wave number k, it is possible to mathematically write an expression that represents that these length scales are inside the Hubble radius

$$k \gg aH. \tag{141}$$

When talking about physical quantities inside the radius they are referred to as being subhorizon. Conversely, this also means that it is possible to refer to quantities outside the Hubble radius, called superhorizon

$$k < aH. \tag{142}$$

The fact that quantities can exist inside or outside the horizon is paramount to understanding the fluctuations in the CMB.

Recall section 6.4 where the comoving Hubble radius was introduced. It was defined as the maximum distance in which points in the universe could be causally connected. During inflation the universe is dark energy dominated, meaning that $\omega = -1$ in equation (30)[13]. After inflation the universe is radiation dominated, meaning $\omega = \frac{1}{3}$, so the comoving Hubble radius increases with time. This means that quantities, such as quantum fluctuations, that initially exists inside the horizon at some point exit the horizon during inflation[14]. Once outside the horizon, the fluctuations freeze, i.e. they remain constant. This freeze out remains until a later time when inflation has stopped and the Hubble radius have increased to a point when the fluctuations can re-enter the horizon. This horizon exit and re-entry can be visualised graphically, as seen in figure 5.

The perturbation $\mathcal{R}$ and ζ leaves the horizon at some point during inflation. However, as was seen above, these quantities are equal during inflation, meaning that at horizon exit they were equal. As they are freezed out at superhorizon scales they remain equal until horizon re-entry. Why this point is important to emphasize is because after horizon re-entry, $\mathcal{R}$ and ζ determine the perturbations of the cosmic fluid which is observable in the CMB.

9.1 Equation of motion of the Mukhanov variable

Using the Lagrangian in equation (38), we can find the corresponding action

$$S_\phi = \int L\mathrm{d}^4x = \int \frac{\sqrt{-g}}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi))\mathrm{d}^4x. \tag{143}$$

Adding the so-called gravitation Einstein-Hilbert action, $S_{EH} = \int \frac{\sqrt{-g}}{2}R\mathrm{d}^4x$ to this expression gives

$$S = \int L\mathrm{d}^4x = \int \frac{\sqrt{-g}}{2}(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi))\mathrm{d}^4x, \tag{144}$$

[13]Since the first inflation condition must hold for inflation the exponent in equation (30) must be such that a time derivative make the quantity negative. Between the three different ω presented in section (6.3), only $\omega = -1$ satisfies this condition.

[14]This was the solution to the horizon problem.

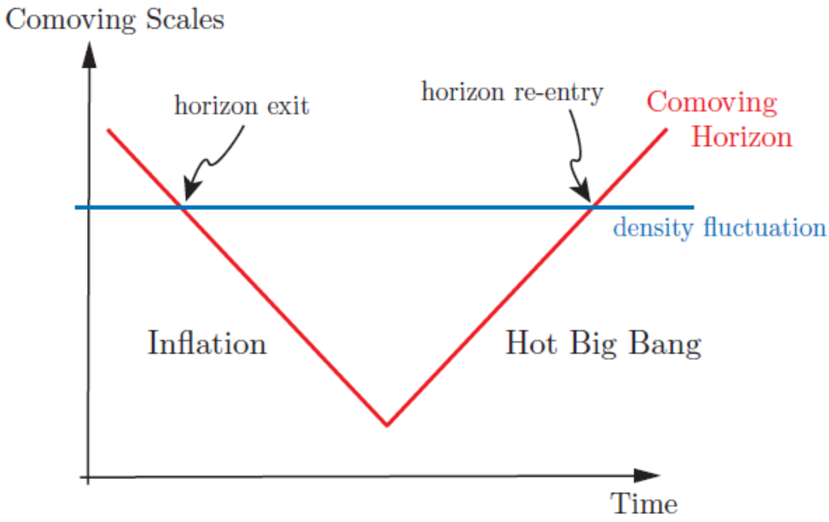

Figure 5: A graphical representation of the horizon exit and re-entry for density fluctuations [1].

where R is the Ricci scalar.

The intention here is to rewrite the new action in terms of the perturbation $\mathcal{R}$. A choice of coordinates is first picked so that the following is true:

$$\delta\phi = 0 \tag{145}$$

$$g_{ij} = a^2((1-2\mathcal{R})\delta_{ij} + h_{ij}) \tag{146}$$

$$\partial_i h_{ij} = h^i_i = 0. \tag{147}$$

The first equation states that there is no perturbation in the inflaton field. The second equation is the change in g_{ij} coming from the comoving curvature perturbation in (110). and the third equation just makes sure that h_{ij} is traceless.

With this choice of coordinates it is possible to expand equation (144) to the second order in $\mathcal{R}$

$$S_{(2)} = \int \frac{\dot{\phi}^2 a^3}{2H^2}(\dot{\mathcal{R}}^2 - \frac{(\partial_j \mathcal{R})^2}{a^2}). \tag{148}$$

Equation (148) can be rewritten further by using conformal time in addition to introducing the so-called Mukhanov variable

$$v \equiv z\mathcal{R}, \tag{149}$$

where

$$z^2 \equiv \frac{a^2\dot{\phi}^2}{H^2} = 2a^2\varepsilon. \tag{150}$$

Equation (148) then becomes

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}x^3((\partial_\tau v)^2 - (\partial_i v)^2 + \frac{\partial_\tau \partial_\tau z}{z}v^2), \tag{151}$$

where confromal time τ, defined in equation (18), is introduced.

The Mukhanov variable can be written in terms of a Fourier expansion

$$v(\tau, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} v_{\vec{k}}(\tau) e^{i\vec{k}\cdot\vec{x}}, \tag{152}$$

where $\vec{k}$ is the wave number. The equation of motion for $v_{\vec{k}}$ can be extracted from the action by varying the action with respect to $v_{-\vec{k}}$ and setting it to zero

$$\begin{aligned}
\frac{\delta S_{(2)}}{\delta v_{-\vec{k}}} &= \frac{1}{\delta v_{-\vec{k}}}(\frac{1}{2}\int \mathrm{d}\tau \frac{\mathrm{d}^3 k}{(2\pi)^3}(\partial\tau v_{\vec{k}}\partial\tau v_{-\vec{k}} - \vec{k}^2 v_{\vec{k}} v_{-\vec{k}} + \frac{\partial_\tau\partial_\tau z}{z} v_{\vec{k}} v_{-\vec{k}})) \\
&= \frac{1}{\delta v_{-\vec{k}}}(\frac{1}{2}\int \mathrm{d}\tau \frac{\mathrm{d}^3 k}{(2\pi)^3} v_{-\vec{k}}(-\partial\tau\partial\tau v_{\vec{k}} - \vec{k}^2 v_{\vec{k}} + \frac{\partial_\tau\partial_\tau z}{z} v_{\vec{k}})) \\
&= \partial_\tau\partial_\tau v_k + (k^2 - \frac{\partial_\tau\partial_\tau z}{z})v_k = 0,
\end{aligned} \tag{153}$$

where the subscript $\vec{k}$ has been dropped and k is the magnitude of $\vec{k}$. To completely solve this equation, some boundary conditions needs to be established. This will be done below.

Equation (153) is written in the form of a harmonic oscillator, i.e.

$$\ddot{x} + \omega^2 x = 0, \tag{154}$$

where x is the displacement from an equilibrium position and ω is the frequency of oscillations.

To promote $v_{\vec{k}}$ to a quantum harmonic oscillator, the variable is written in terms of annihilation and creation operators

$$\hat{v}_{\vec{k}} = v_k(\tau)\hat{a}_{\vec{k}} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\vec{k}}. \tag{155}$$

The annihilation and creation operators are defined as

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{156}$$

$$\hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle. \tag{157}$$

v_k is normalized in the following way

$$\langle v_k, v_k\rangle \equiv \frac{i}{\hbar}(v_k^*(\partial_\tau v_k) - (\partial_\tau v_k^*)v_k) = 1, \tag{158}$$

and the annihilation and creation operators follow the following relation

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k} - \vec{k}'). \tag{159}$$

To completely describe the variance in fluctuation for v_k as a quantum harmonic oscillator that oscillates around a ground state, it is crucial to define said ground state. Since the annihilation and creations operators have been introduced, it is possible to introduce a condition for this state

$$\hat{a}_{\bar{k}} |0\rangle = 0, \tag{160}$$

i.e. the ground state $|0\rangle$ should be completely annihilated by the annihilation operator.

However, just because a condition has been introduced does not mean that the ground state is chosen. The usual choice of ground state is the "Minkowski vacuum of a comoving observer in the far past" [1]. By far past it is meant at a time before inflation, i.e. a time where the comoving Hubble radius has not been reduced and all quantities exist on a subhorizon level, $k \gg aH$. Using this limit, equation (153) becomes

$$\partial_\tau \partial_\tau v_k + k^2 v_k = 0, \tag{161}$$

which has exactly the same form as equation (154). The one dimensional quantum harmonic oscillator presented in equation (154) can be solved exactly. We can write the Hamiltonian as

$$\hat{H} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2}((\partial_\tau \hat{v}_k)^2 + k^2 \hat{v}_k^2). \tag{162}$$

Treating this analogous to the one dimensional quantum harmonic oscillator case, the first term of the Hamiltonian corresponds to the momentum squared of the oscillator and the second term corresponds to the displacement factor multiplied by the frequency, i.e.

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2). \tag{163}$$

Using the expression for $\hat{v}_k$ presented in equation (155), the Hamiltonian can be rewritten as

$$\begin{aligned}
\hat{H} &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2}((\partial_\tau (v_k(\tau)\hat{a}_k + v^*_{-k}(\tau)\hat{a}^\dagger_{-k}))^2 + k^2(v_k(\tau)\hat{a}_k + v^*_{-k}(\tau)\hat{a}^\dagger_{-k})^2) \\
\hat{H} &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2}(((\partial_\tau v_k)^2 + k^2 v_k^2)\hat{a}_k\hat{a}_k + ((\partial_\tau v_{-k})^2 + k^2 v_{-k}^2)^* \hat{a}^\dagger_{-k}\hat{a}^\dagger_{-k} \\
&\quad + (|(\partial_\tau v_k)|^2 + k^2 |v_k|^2)(\hat{a}_k \hat{a}^\dagger_{-k} + \hat{a}^\dagger_{-k}\hat{a}_k)).
\end{aligned} \tag{164}$$

Let the Hamiltonian act on the ground state

$$\hat{H}|0\rangle = (\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2}(((\partial_\tau v_{-k})^2 + k^2 v_{-k}^2)^* \hat{a}^\dagger_{-k}\hat{a}^\dagger_{-k} + (|(\partial_\tau v_k)|^2 + k^2|v_k|^2)))|0\rangle. \tag{165}$$

The ground state is an eigenstate to the Hamiltonian, meaning that only the ground state can remain on the RHS. This fact gives the following condition

$$\begin{aligned}
\partial_\tau v_{-k}^2 + k^2 v_{-k}^2 &= 0 \\
\rightarrow \partial_\tau v_k &= \pm i k v_k,
\end{aligned} \tag{166}$$

which also means that equation (158) becomes

$$\langle v_k, v_k\rangle = \frac{i}{\hbar}(v_k^*(\pm ikv_k) - (\pm ikv_k)^* v_k) = \mp\frac{2k}{\hbar}|v_k|^2 = 1. \tag{167}$$

The last equality only holds if the LHS uses the plus sign, meaning that equation (166) needs to use the minus sign. Using the minus sign in equation (166) reduces it to a simple differential equation that is solvable

$$v_k(\tau) = \sqrt{\frac{\hbar}{2k}}e^{-ik\tau}, \tag{168}$$

where the factor $\sqrt{\frac{\hbar}{2k}}$ comes from the condition set by equation (167).

So assuming the normalization given by equation (167) as well as the ground state to be the Minkowski vacuum, a solution to equation (161) exists.

9.1.1 Solution in de Sitter space

A similar solution to equation (161) exists in the limit, $\varepsilon \to 0$. In this limit $\frac{\partial_\tau\partial_\tau z}{z}$ changes as follows

$$\frac{\partial_\tau\partial_\tau z}{z} = \frac{\partial_\tau\partial_\tau a}{a} = \frac{2}{\tau^2}, \tag{169}$$

where the identity $a = \frac{1}{H\tau}$ is used. This changes equation (161) to

$$\partial_\tau\partial_\tau v_k + (k^2 - \frac{2}{\tau^2})v_k = 0. \tag{170}$$

This has the solution

$$v_k(\tau) = \sqrt{\frac{\hbar}{2k}}e^{-ik\tau}(1 - \frac{i}{k\tau}). \tag{171}$$

9.2 Power spectrum for $\mathcal{R}$

Before computing the power spectrum for $\mathcal{R}$ it is necessary to compute the power spectrum for $\hat{\psi}_{\vec{k}} \equiv \frac{\vec{v}_{\vec{k}}}{a}$

$$\begin{aligned}
\langle 0|\,\hat{\psi}_{\vec{k}}\hat{\psi}_{\vec{k}'}\,|0\rangle &= \langle 0|\,\frac{1}{a^2}(v_k(\tau)\hat{a}_{\vec{k}} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\vec{k}})(v_k(\tau)\hat{a}_{\vec{k}'} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\vec{k}'})\,|0\rangle \\
&= \langle 0|\,\frac{1}{a^2}(v_k(\tau)\hat{a}_{\vec{k}})(v^*_{-k}(\tau)\hat{a}^\dagger_{-\vec{k}'})\,|0\rangle \\
&= \langle 0|\,\frac{1}{a^2}|v_k|^2\hat{a}_{\vec{k}}\hat{a}^\dagger_{-\vec{k}'}\,|0\rangle \\
&= (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{|v_k|^2}{a^2} = (2\pi)^3\delta^3(\vec{k}+\vec{k}')(|v_k|H\tau)^2 \\
&= (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{H^2}{2k^3}(1+k^2\tau^2) = \langle\hat{\psi}_{\vec{k}}\hat{\psi}_{\vec{k}'}\rangle.
\end{aligned} \tag{172}$$

When considering superhorizon scales, $k \ll 1$, this becomes

$$\langle\hat{\psi}_{\vec{k}}\hat{\psi}_{\vec{k}'}\rangle = (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{H^2}{2k^3}, \tag{173}$$

which is a constant.

Equation (173) allows the computation of the power spectrum for $\mathcal{R} = \frac{v}{z} = \frac{H}{\dot{\phi}}\psi$ at horizon crossing, i.e. $a(t_*)H(t_*) = k$, where t_* corresponds to the time of horizon crossing

$$\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{k}'} \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') \frac{H_*^4}{2k^3 \dot{\phi}_*^2}. \tag{174}$$

This means that $P_{\mathcal{R}}(k)$ have been found at horizon crossing. The fact that this function is only found at t_* is not a limitation because of the constant behavior of $\mathcal{R}$ at superhorizon scales. This quantity will only change after horizon re-entry. Therefore $P_{\mathcal{R}}(k)$ is

$$P_{\mathcal{R}}(k) = \frac{H_*^4}{2k^3 \dot{\phi}_*^2}. \tag{175}$$

9.3 Tensor perturbations

In a similar way that equation (148) was expanded to second order in $\mathcal{R}$, it is possible to expand equation (144) to second order in h_{ij}, giving

$$S_{(2)} = \frac{M_{pl}^2}{8} \int \mathrm{d}\tau \mathrm{d}x^3 a^2 ((\partial_\tau h_{ij})^2 - (\partial_l h_{ij})^2). \tag{176}$$

Here the Planck mass, M_{pl}, has been introduced to make h_{ij} manifestly dimensionless.

Just as with $\mathcal{R}$ (or v to be more precise), h_{ij} will be replaced by its Fourier transform

$$h_{ij} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{s=+,\times} \varepsilon_{ij}^s(k) h_{\vec{k}}^s(\tau) e^{i\vec{k}\cdot\vec{x}}, \tag{177}$$

where ε_{ij}^s are the physical polarizations with $\varepsilon_{ii} = k^i \varepsilon_{ij} = 0$ and $\varepsilon_{ij}^s(k)\varepsilon_{ij}^{s'}(k) = 2\delta_{ss'}$. The $+$ and $\times$ symbols represents the two polarizations. Using the expression for h_{ij} given by equation (177), equation (176) becomes

$$S_{(2)} = \sum_s \int \mathrm{d}\tau \mathrm{d}^3 k \frac{a^2}{4} M_{pl}^2 (\partial_\tau h_{\vec{k}}^s \partial_\tau h_{\vec{k}}^s - k^2 h_{\vec{k}}^s h_{\vec{k}}^s). \tag{178}$$

By introducing the variable

$$w_{\vec{k}}^s \equiv \frac{a}{2} M_{pl} h_{\vec{k}}^s, \tag{179}$$

equation (178) turns into

$$S_{(2)} = \sum_s \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3 k ((\partial_\tau w_{\vec{k}}^s)^2 - (k^2 - \frac{\partial_\tau a \partial_\tau a}{a})(w_{\vec{k}}^s)^2. \tag{180}$$

Each polarization of the tensor perturbation has the form

$$h_{\vec{k}}^s = \frac{2}{M_{pl}} \hat{\psi}_{\vec{k}}^s. \tag{181}$$

The power spectrum for $\hat{\psi}^s_{\vec{k}}$ was calculated in the previous section. Therefore the power spectrum for h_{ij} becomes

$$P_h(k) = \frac{2H_*^2}{k^3 M_{pl}^2}. \tag{182}$$

9.4 Primordial spectra

The expressions presented in equation (138) and (140) can now be rewritten as

$$\Delta_s^2 = \frac{k^3}{2\pi^2}\frac{H_*^4}{2k^3\dot{\phi}_*^2} = \frac{1}{8\pi^2}\frac{H_*^2}{M_{pl}^2\varepsilon_*} \tag{183}$$

$$\Delta_t^2 = \frac{k^3}{\pi^2}\frac{2H_*^2}{k^3 M_{pl}^2}. \tag{184}$$

In the last expression of equation (183) the units were changed so that $M_{pl} \neq 1$ and equation (54) was used to rewrite it in terms of ε. The ratio between these quantities is

$$r = \frac{\Delta_t^2}{\Delta_s^2} = 16\varepsilon_*. \tag{185}$$

10 Fluctuations in the CMB

Schematically, the relation between $\mathcal{R}$ and the observed temperature fluctuations can be written as

$$Q_{\vec{k}} = T_Q(k, \tau, \tau_*)\mathcal{R}_{\vec{k}}(\tau_*), \tag{186}$$

where $Q_{\vec{k}}$ represents the temperature fluctuations in the CMB and $T_Q(k, \tau, \tau_*)$ a transfer function to relate it to $\mathcal{R}$ at time τ_*.

10.1 Treatment of CMB observations

The treatment of CMB observations was touched upon in section 5. It was shown that the correlation function, C, could be written in terms of its different multipole moments, C_l. As a reminder, the expression for the multipole moments is

$$C_l = \frac{1}{2l+1}\sum_{m=-l}^{m=l}\langle a_{lm}^* a_{lm}\rangle. \tag{187}$$

If the ratio presented in equation (185) is very small ($r < 3$), then the quantity Δ_s^2 dominates. Since this quantity is related to the power spectrum for $\mathcal{R}$, it means that during horizon crossing the scalar modes, $\mathcal{R}$, are dominant compared to h_{ij}. Since the domination holds true until horizon re-entry, the scalar perturbations will be dominant during the time of recombination, which occurs sometime after re-entry. Consequently, this means that the CMB fluctuations are dominated by $\mathcal{R}$. Figure 6 illustrates this statement and shows a time lapse over important events after horizon re-entry.

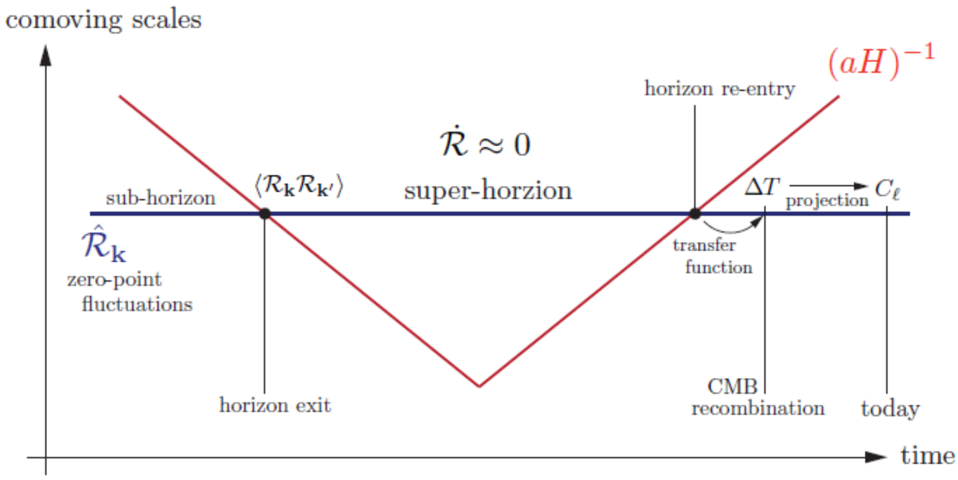

Figure 6: The history of the perturbation $\mathcal{R}$ from the sub-horizon era to recombination. The figure also shows the importance of the transfer function as it is this function alone that relates $\mathcal{R}$ to the observed inhomogeneities in the CMB observations [1].

To relate $\mathcal{R}$ and $\frac{\delta T}{T}$ to each other, a radiation[15] transfer function, $\Delta_{Tl}(k)$, is applied through the k-space integral, giving

$$a_{lm} = 4\pi(-i)^l \int \frac{\mathrm{d}^3k}{(2\pi)^3} \Delta_{Tl}(k) \mathcal{R}_{\vec{k}} Y_{lm}(\hat{k}). \tag{188}$$

Using equation (188) and the following identity

$$\sum_{m=-l}^{l} Y_{lm}(\hat{k}) Y_{lm}(\hat{k}') = \frac{2l+1}{4\pi} P_l(\hat{k} \cdot \hat{k}'), \tag{189}$$

equation (187) becomes

$$C_l = \frac{2}{\pi} \int k^2 \mathrm{d}k P_{\mathcal{R}}(k) \Delta_{Tl}(k) \Delta_{Tl}(k). \tag{190}$$

Here the identifying function for the power spectrum for $\mathcal{R}$, i.e. $P_{\mathcal{R}}(k)$ have been related to the CMB observations through the multipole moments C_l, meaning that the purpose for this report have been fulfilled. The transfer functions, Δ_{Tl}, generally have to be computed numerically.

[15]The curvature fluctuations lead to local fluctuations in the temperature through a phenomenon called the Sacks-Wolfe effect. This is implemented in (188) through the transfer function. Further explanations of this effect are beyond the scope of this report.

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Part V
Summary and conclusion

By observing CMB photons it is seen that the temperature of the CMB is not uniform but there exists fluctuations. By defining these fluctuations mathematically, it is then possible to use these to create a correlation function. This correlation function can be written in terms of its multipole moments, C_l, which can be plotted against l to give the angular power spectrum of the temperature fluctuations.

The existence of these fluctuations are explained using inflation and quantum mechanics. By defining inflation as the reduction of the comoving Hubble radius, perturbations will leave the radius during inflation which will make them freeze out. By introducing the Friedmann-Robertson-Walker metric and the Lagrangian for the inflaton field ϕ with perturbations it is possible to create a perturbation $\mathcal{R}$. By introducing quantum mechanics and calculating the power spectrum for $\mathcal{R}$ at horizon crossing, the behaviour of this perturbation is known until horizon re-entry, at which point it can be related to C_l through a transfer function, thus giving a physical explanation for the origins of C_l.

Temperature fluctuations in the CMB corresponds to inhomogeneity in energy density of the universe. This inhomogeneity is the initial seed for the large scale structure of the universe today. By understanding and explaining this inhomogeneity, a deeper insight to cosmology is obtained.

For future projects, the nature and properties of the transfer function could be examined.

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