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Entanglement and the black hole information paradox

Abstract

The black hole information paradox arises when quantum mechanical effects are considered in the vicinity of the event horizon of a black hole. In this report we describe the fundamental properties of quantum mechanical systems and black holes that lead to the information paradox, with focus on quantum entanglement. While first presented in 1976, the information paradox is as of yet an unsolved problem. Two of the proposed solutions, *black hole complementarity* and *firewalls*, are discussed.

Sammanfattning

Svarta hålets informationsparadox uppkommer när man tar hänsyn till kvantmekaniska effekter i närheten av händelsehorisonten av ett svart hål. I denna rapport beskrivs de grundläggande egenskaper hos kvantmekaniska system och svarta hål som leder till informationsparadoxen, med fokus på kvantintrassling. Paradoxen, som presenterades 1976 men än idag är ett olöst problem, förklaras sedan. Två av de förslagna lösningarna till paradoxen, *svarta hål-komplementaritet* och *firewalls* diskuteras.

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1 Introduction

Physics is all about describing and understanding the world around us, usually using mathematics. In pursuit of this goal we have developed some amazing and very successful theories, such as quantum mechanics and general relativity. For a theory to be successful it needs to be not only descriptive but also predictive. Sometimes, when the predictions of one theory contradict that of another theory a so called paradox arises. The focus of this paper is on the black hole information paradox. This paradox involves phenomena from both quantum mechanics and general relativity, with the main ones being quantum entanglement and black holes.

Quantum entanglement originates from quantum mechanics, a theory a bit older than 100 years. It is safe to say that quantum mechanics has had a radical effect on physics and how we see the world. By introducing the wave-particle duality, Heisenberg's uncertainty principle and the fundamentally probabilistic nature of physics quantum mechanics has in some ways made physics more unintuitive. However, the theory is very successful since its principles and predictions have been experimentally verified many times. Entanglement is one of the strange phenomena of quantum mechanics, one which may seem both unintuitive and contradictory to standard physics.

In short, entanglement is the ability of, for example particles, to have interdependent properties. It relates to the fact that sometimes completely determining the state of a physical system of multiple components is possible while completely determining the individual states of each component is not. As the components together have to make up the entire system the individual properties of one component may depend on the same properties of the other components. The components of the system are said to be entangled if some of their properties are correlated. The strange aspect of this correlation is that it is independent of spatial separation. Two particles on opposite ends of the universe might have properties that for each particle depend on the other particle.

Black holes on the other hand, are a phenomenon predicted by general relativity. Also a theory about 100 years old, general relativity describes gravity as an effect of the geometry of spacetime. A key feature is that the presence of mass or energy curves spacetime.

A black hole is curved region of spacetime from which nothing can escape. This region contains a so called singularity where the curvature goes to infinity. This singularity is hidden behind an event horizon, which is the boundary of the black hole. Everything that falls through the event horizon ends up in the singularity. There are several peculiar aspects about black holes, one of which is the fact that they contain singularities despite the fact that infinities are something physicists usually tend to avoid.

Usually one does not use both quantum mechanics and general relativity when it comes to describing one system. If the system is small enough quantum mechanics is used as it applies to extremely small systems, at the scale of atoms or subatomic particles. On the other end of the spectrum is general relativity, which applies to large scale systems such as planets, stars and even the universe. So what happens when one does consider both general relativity and quantum mechanics at the same time? One example is the information paradox. The goal of this paper is to explain why this paradox occurs, which it does when one considers quantum mechanical effects at the event horizon of a black hole. Two of the proposed solutions to the paradox will also be discussed.

1.1 Method & layout

This project is a literature study focused on describing the information paradox and two of the proposed solutions to it. This description focuses mainly on the quantum mechanical aspect of the paradox but also considers some of the basic features of general relativity.

The source material was chosen based on how relevant and central it is to the topic. Determining which sources to use was done critically and with respect to how generally recognized the material is, as the information paradox and its possible solutions is a debated topic.

Section 2 deals with the basic formalism of quantum mechanics. The sources for this section are the two textbooks [1] and [2]. A few of the key concepts are unitary time evolution, and pure and mixed states. Sections 3 and 4 present some central concepts from quantum mechanics such as quantum entanglement and entropy.

The basics of general relativity is then covered in Section 5 with Section 6 focusing on the phenomena of black holes and Hawking radiation.

Using these concepts from both theories the information paradox is stated in Section 7. As the paradox is a problem under active research it does not yet have one solution. Two of the proposed solutions to the paradox are black hole complementarity and firewalls, both of which are reviewed and discussed in Section 8.

The final section (Section 9) gives a short review of the status of the paradox today, as well as comments on the controversy of the two proposed solutions that were discussed.

2 Fundamental concepts of quantum mechanics

Quantum mechanics is a physical theory describing systems of very small scales, such as atoms and subatomic particles. One of the main differences compared to classical theory is that in quantum theory particles are not thought of as point particles. Instead they are more like waves and described by wave functions. These wave functions embody the probabilistic nature of quantum mechanics. For example, in classical theory the position of a particle is precisely determined in terms of coordinates. In quantum theory the position is determined only in terms of probabilities of where it is most likely to be and the position wave function contains these probabilities. For more information, the books [3] and [2] cover the basics of quantum mechanics quite well.

A quantum mechanical system is said to be in a state. These states can be described by one or several wave functions, depending on what is known about the system. The wave functions are elements of a Hilbert space, which is the space quantum mechanics operates in. For a single particle, in one dimension, the Hilbert space is defined as

$$\mathcal{H} = \left\{ \psi(x) \middle| \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx < \infty \right\}.$$

with the elements of it being normalized

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1.$$

Hilbert space is a vector space that can have infinite dimensions, since the wave function describes the particle or system for every point in space. The dimension of the space depends on which variable we want to describe. Continuous variables, e.g. position, have infinite dimensional Hilbert spaces while discrete variables, e.g. spin, have finite dimensional Hilbert spaces.

The vectors in a Hilbert space are represented by ket ($|u\rangle$) vectors, which can be represented by column vectors with u_i being wave functions.

$$|u\rangle = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

The conjugate transpose of ket vectors are bra vectors ($\langle v|$)

$$|v\rangle^\dagger = \langle v|$$

which can be represented as row vectors[1]

$$\langle v| = (v_1^*, v_2^*, \ldots, v_N^*).$$

The inner product of two wave functions $\psi_1(x)$ and $\psi_2(x)$ in a Hilbert space is defined as

$$\langle \psi_1|\psi_2\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)dx.$$

A property of the inner product is that it is bilinear

$$\langle \psi_1|a\psi_2 + b\psi_3\rangle = a\,\langle \psi_1|\psi_2\rangle + b\,\langle \psi_1|\psi_3\rangle$$

$$\langle a\psi_1 + b\psi_2|\psi_3\rangle = a^*\,\langle \psi_1|\psi_3\rangle + b^*\,\langle \psi_2|\psi_3\rangle\,.$$

Elements of the Hilbert space are said to be orthogonal if their inner product is zero

$$\langle \psi_1|\psi_2\rangle = 0.$$

A set of wave functions $\{\psi_n\}$ is orthonormal if the functions are normalized and orthogonal,

$$\langle \psi_n|\psi_m\rangle = \delta_{nm}.$$

If a set of orthonormal wave functions span the entire Hilbert space it can be used as a basis. These basis sets are usually formed from eigenstates of operators.

A system can be described in terms of many different variables, with wave functions that include all or only a few of them. For example, a particle can be described in terms of its position and spin, which means the wave function $|\Psi\rangle$ is a product of the position and spin wave functions, $|\psi\rangle$ and $|S\rangle$,

$$|\Psi\rangle = |\psi\rangle \otimes |S\rangle\,.$$

The Hilbert space $\mathcal{H}$ for the particle is then a tensor product of the Hilbert spaces for its position and spin

$$\mathcal{H} = H_{position} \otimes H_{spin}.$$

[1] We get the conjugate transpose of a vector by taking the complex conjugate of each element and the transpose of the vector.

2.1 Operators

Quantum mechanics have operators that act on the wave functions in the Hilbert space. As ket and bra vectors can be represented as column and row vectors operators are often represented by matrices. This representation is useful but dependent on the coordinate basis chosen. The real physical system is independent of coordinate choice which is why other representations are possible.

An operator acting on a state ψ is

$$\hat{O}\,|\psi\rangle = \left|\hat{O}\psi\right\rangle$$

and the dual of this ket-state $\left|\hat{O}\psi\right\rangle$ is calculated by acting on the bra-state $\langle\psi|$ with the adjoint of the operator

$$\langle\psi|\,\hat{O}^\dagger = \left\langle\hat{O}\psi\right|.$$

The operators $(\hat{O})$ in quantum mechanics are linear,

$$\hat{O}(|\psi_1\rangle + |\psi_2\rangle) = \hat{O}\,|\psi_1\rangle + \hat{O}\,|\psi_2\rangle\,.$$

An operator $\hat{O}$ has the eigenvalue α_n and the corresponding eigenstate $|\psi_n\rangle$ if

$$\hat{O}\,|\psi_n\rangle = \alpha_n\,|\psi_n\rangle\,. \tag{1}$$

For the dual vector and adjoint operator the eigenvalue is the complex conjugate of α_n

$$\langle\psi_n|\,\hat{O}^\dagger = \alpha_n^*\,\langle\psi_n|\,. \tag{2}$$

An very useful type of operator is the Hermitian operator since one of the postulates of quantum mechanics states that all physical parameters (observables) have a Hermitian operator associated with them. This is related to the fact that the eigenvalues of Hermitian operators, and therefore observables, are real. The definition of an Hermitian operators is that it is equal to its adjoint[2]

$$\hat{O}^\dagger = \hat{O}$$

which is what gives $\alpha_n^* = \alpha_n$. We can see this by applying $\langle\psi_n|$ to Eq. (1) and $|\psi_n\rangle$ to Eq. (2).

$$\langle\psi_n|\,\hat{O}\,|\psi_n\rangle = \langle\psi_n|\,\alpha_n\,|\psi_n\rangle = \alpha_n\,\langle\psi_n|\psi_n\rangle$$

[2]When $\hat{O}$ is represented as a matrix this corresponds to the complex conjugate.

$$\langle\psi_n|\, \hat{O}^\dagger \,|\psi_n\rangle = \langle\alpha_n^*\psi_n|\psi_n\rangle = \alpha_n^* \langle\psi_n|\psi_n\rangle$$

For an Hermitian operator these two equations are equal

$$\alpha_n \langle\psi_n|\psi_n\rangle = \alpha_n^* \langle\psi_n|\psi_n\rangle$$

giving $\alpha_n = \alpha_n^*$.

The expectation value of an observable $\langle\hat{O}\rangle$ is the average value of the measurement of the observable. For the state $|\psi\rangle$ this is calculated

$$\langle\hat{O}\rangle = \langle\psi|\, \hat{O} \,|\psi\rangle .$$

Observables always have real expectation values since they have real eigenvalues.

Operators with discrete spectrums of eigenvalues can be used to express any wave function $|\psi\rangle$ in the Hilbert space as

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n \,|\psi_n\rangle \tag{3}$$

where $|\psi_n\rangle$ are the eigenstates of the observable. The eigenstates are orthonormal $\langle\psi_n|\psi_m\rangle = \delta_{nm}$ and c_n are the probability amplitudes such that

$$\sum_n |c_n|^2 = 1. \tag{4}$$

As previously mentioned, the eigenstates of an operator are useful when it comes to choosing a basis for the vector space. A base is a complete set of orthogonal functions, and a set of functions is complete if a linear combination of functions from the set can be used to express any other function $|\psi\rangle$ in the Hilbert space. The eigenstates of an observable fulfill these demands as they are orthonormal and span the Hilbert space. Therefore they can make up a basis for an observable.

The set $\{|n\rangle\}$ is an orthonormal basis of the (finite dimensional) Hilbert space of the discrete observable $\hat{O}$ if it fulfills the completeness relation

$$\mathbb{I} = \sum_n |n\rangle \langle n|$$

where $|n\rangle$ are normalized eigenvectors of $\hat{O}$ and $\mathbb{I}$ is the identity matrix.

2.2 Unitary time evolution

An important feature of quantum mechanics is that the time evolution operator is unitary. An operator $\hat{U}$ is unitary if

$$\hat{U}\hat{U}^{\dagger} = \mathbb{I} = \hat{U}^{\dagger}\hat{U}. \tag{5}$$

The time evolution operator $\hat{U}_t$ describes the future evolution of a quantum state according to

$$|\psi, t\rangle = \hat{U}_t |\psi, t_0\rangle .$$

We can motivate why the time evolution operator has to be unitary. Take a state $|\psi, t_0\rangle$ at the time t_0 expressed in the basis states $|a_n\rangle$ of an observable A.

$$|\psi, t_0\rangle = \sum_{n=1}^{\infty} c_n(t_0) |a_n\rangle$$

If we apply the time evolution operator we get a new state, expressed in the same basis.

$$\hat{U}_t |\psi, t_0\rangle = |\psi, t\rangle = \sum_{n=1}^{\infty} c_n(t) |a_n\rangle$$

$|c_n(t_0)|$ and $|c_n(t)|$ are not necessarily equal but since they are normalized they obey Eq. (4) or equivalently

$$\langle\psi, t_0|\psi, t_0\rangle = 1 = \langle\psi, t|\psi, t\rangle . \tag{6}$$

By expanding the right side of Eq. (6) we can see that $\hat{U}_t$ is unitary [1].

$$\langle\psi, t_0|\psi, t_0\rangle = \langle\psi, t_0| \hat{U}_t^{\dagger}\hat{U}_t |\psi, t_0\rangle$$

$$\hat{U}_t^{\dagger}\hat{U}_t = \mathbb{I}$$

The fact that the time evolution operator is unitary means that the sum of the probabilities of all possible outcomes of a measurement is one, regardless of when in time we make the measurement. This may seem trivial but it is an important part of quantum mechanics and, as we shall see later, one that relates to the information paradox.

2.3 Complementarity

Another key feature of quantum mechanics, which marks one of its differences to classical mechanics, is complementarity. Complementarity means that operators that do not commute cannot have simultaneous eigenvalues and therefore not be in simultaneous eigenstates. Two operators $\hat{A}$ and $\hat{B}$ commute if

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0.$$

An example of operators that do not commute are the spin operators. In practise this menas that if for example the spin of a particle is measured along the z-direction of a coordinate system nothing can be said about the spin in the x- or y-direction.

This relates to the Heisenberg's uncertainty principle which puts a limit on how well these complementary properties can be known simultaneously. Position and momentum are two non-commuting operators and therefore their values for, for example a particle, cannot be known with equal precision simultaneously. This makes sense since the more accurately one measures the position of a particle the more one disturbs its momentum, and vice versa.

2.4 Pure and mixed states

Later on we will study systems consisting of two particles where we will have to determine if they are in a pure or mixed state. Here we define what pure and mixed states are.

The state of a system is called **pure** if the system can be described using only one wave function, i.e. one ket-vector $|\Psi\rangle$.

Take for example a fermion which is a particle with spin-1/2, where the spin is in units of the reduced Planck constant $\hbar$. The spin is measured along one axis, here we choose the z-axis. This particle may have either spin up, spin down or be in a superposition of the two. The basis states, which are pure, are $|\uparrow\rangle$ and $|\downarrow\rangle$ but even the superposition of the basis states

$$|\Psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$$

is a pure state. Here c_1 and c_2 are the probability amplitudes and $|c_1|^2 + |c_2|^2 = 1$.

A **mixed state** on the other hand, is a state that cannot be described by solely one wave function. To describe a mixed state a collection of wave functions is needed.

A collection of identical systems, an ensemble, can be in a pure or mixed state. If all the systems have the same quantum state then ensemble is pure and if the systems are in different quantum states then the ensemble is mixed.

A pure ensemble of particles that all have the same spin can be created by putting the particles through an apparatus that filters out particles of one spin direction. Mixed ensembles are created all the time in processes where the states of the particles are determined randomly.

An example of a mixed states is a collection of particles where 30% have spin up and 70% spin down, in the z-direction. To describe this system we need the two wave functions $|\uparrow\rangle$ and $|\downarrow\rangle$ and the aforementioned probabilities.

Mixed states are described by density matrices whose coefficients correspond to the probabilities of the system being in each pure state. Since mixed states are collections of pure states, pure states are a subset of mixed states. Therefore, the density operator can be used to identify if a state is pure or mixed.

2.4.1 Density matrix

Let us now see how to calculate a density matrix for a system. Take a collection of particles that consist of several smaller groups of particles. There are N groups of particles, each group corresponding to a different pure state and containing a fraction of the particles in the collection.

A fraction with relative population W_i is in the pure state $\left|\alpha^{(i)}\right\rangle$. The states $\left|\alpha^{(i)}\right\rangle$ are normalized but not necessarily orthogonal to each other. They are expressed in terms of the basis vectors $|n\rangle$, $n \in \{e_1, e_2, \dots\}$, of the Hilbert space,

$$\left|\alpha^{(i)}\right\rangle = \sum_n c_n^{(i)} \left|n\right\rangle . \tag{7}$$

From Eq. (7) we know the coefficients can be calculated

$$c_n^{(i)} = \left\langle n \middle| \alpha^{(i)} \right\rangle \qquad c_{n'}^{(i)*} = \left\langle \alpha^{(i)} \middle| n' \right\rangle . \tag{8}$$

The density matrix of a state will reflect the fraction of particles that are in each pure state in terms of the probabilities W_i, where W_i are real, $0 \le W_i \le 1$ and $\sum_i W_i = 1$. The definition of the density operator ρ is

$$\rho = \sum_{i=1}^{N} \left|\alpha^{(i)}\right\rangle W_i \left\langle \alpha^{(i)}\right| . \tag{9}$$

The density matrix can be expressed in several ways, it may for example be written in terms of the $\left|\alpha^{(i)}\right\rangle$ states if they span the Hilbert space, even if they are not a basis. The elements of the density matrix, in the $|n\rangle$ basis, are

$$\rho_{nn'} = \langle n| \rho \left|n'\right\rangle = \sum_{i=1}^{N} \left\langle n \middle| \alpha^{(i)} \right\rangle W_i \left\langle \alpha^{(i)} \middle| n' \right\rangle = \sum_{i=1}^{N} W_i c_{n'}^{(i)*} c_n^{(i)} \tag{10}$$

where we used Eqs. (8) and (9).

The density operator is Hermitian[3], $\rho^\dagger = \rho$, and Hermitian matrices can be diagonalized. The diagonal elements of the density matrix are

$$\rho_{nn} = \langle n| \rho |n\rangle = \sum_{i=1}^{N} W_i |c_n^{(i)}|^2. \tag{11}$$

ρ_{nn} is the probability to find one of the particles of the mixed state in the pure state $|n\rangle$,

$$0 \le \rho_{nn} \le 1. \tag{12}$$

One often deals with reduced density matrices which are density matrices containing information about one or a few parameters instead of complete wave functions. For example, a density matrix concerning only spin is a reduced density matrix.[4] Let us calculate some examples of density matrices, following [2].

[3]Which can be seen by taking the transpose of (9) and remembering that $(AB)^\dagger = B^\dagger A^\dagger$.

[4]From here on the word reduced will be dropped as all density matrices we will deal with are reduced.

Example 1

An unpolarized beam of particles is a mixture of particles with the states spin up and spin down, in the z-direction. The probability weight for each state is 0.5 since any one particle is equally likely to have spin up as spin down, $W_\uparrow = W_\downarrow = 0.5$. The density matrix for this state is

$$\begin{aligned}\rho &= \left|\uparrow\right\rangle W_\uparrow \left\langle\uparrow\right| + \left|\downarrow\right\rangle W_\downarrow \left\langle\downarrow\right| \\ &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\end{aligned}$$

in the basis $\{\left|\uparrow\right\rangle, \left|\downarrow\right\rangle\}$.

Example 2

The density matrix can also describe mixed states where the known polarizations are in different directions. Take a beam of particles that contains 30% spin-z up and 70% spin-x up particles. The spin-z density matrix is

$$\rho_z = \left|\uparrow\right\rangle \left\langle\uparrow\right| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the spin-x density matrix is

$$\begin{aligned}\rho_x = \left|S_x, +\right\rangle \left\langle S_x, +\right| &= \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)\frac{1}{\sqrt{2}}(\left\langle\uparrow\right| + \left\langle\downarrow\right|) \\ &= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.\end{aligned}$$

The density matrix for the complete system is

$$\begin{aligned}\rho &= 0.3\rho_z + 0.7\rho_x \\ &= 0.3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0.7\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0.65 & 0.35 \\ 0.35 & 0.35 \end{pmatrix}\end{aligned}$$

expressed in the basis $\{\left|\uparrow\right\rangle, \left|\downarrow\right\rangle\}$.

2.4.2 Expectation values

As we have seen, for a pure state the expectation value of an operator can be calculated using its wave function. For the pure state $\left|\alpha^{(i)}\right\rangle$ and operator A the calculation is

$$\begin{aligned}
\langle A\rangle_{\alpha^{(i)}} &= \left\langle\alpha^{(i)}\right| A \left|\alpha^{(i)}\right\rangle \\
&= \sum_{n'}\sum_{n} c_{n'}^{(i)*} c_{n}^{(i)} \left\langle n'\right| A \left|n\right\rangle \\
&= \sum_{n'}\sum_{n} \left\langle n\middle|\alpha^{(i)}\right\rangle \left\langle\alpha^{(i)}\middle|n'\right\rangle \left\langle n'\right| A \left|n\right\rangle
\end{aligned} \tag{13}$$

where Eqs. (7) and (8) were used.

Calculating $\langle A\rangle$ for a mixed state is different since the probabilities for each wave function needs to be taken into account. For a mixed state the average value of A is calculated by summing over all the possible states $\left|\alpha^{(i)}\right\rangle$.

$$\langle A\rangle = \sum_{i=1}^{N} W_i \langle A\rangle_{\alpha^{(i)}} \tag{14}$$

Using Eqs. (13) and (10) the calculation is

$$\begin{aligned}
\langle A\rangle &= \sum_{i=1}^{N}\sum_{n'}\sum_{n} W_i c_{n'}^{(i)*} c_{n}^{(i)} \left\langle n'\right| A \left|n\right\rangle \\
&= \sum_{i=1}^{N}\sum_{n'}\sum_{n} \left\langle n\middle|\alpha^{(i)}\right\rangle W_i \left\langle\alpha^{(i)}\middle|n'\right\rangle \left\langle n'\right| A \left|n\right\rangle \\
&= \sum_{n'}\sum_{n} \left\langle n\right| \rho \left|n'\right\rangle \left\langle n'\right| A \left|n\right\rangle \\
&= \sum_{n} \left\langle n\right| \rho A \left|n\right\rangle .
\end{aligned} \tag{15}$$

Eq. (15) can be written using the trace operator, Tr(). The trace Tr() of a square matrix is the sum of the diagonal elements, which also means that it is the sum of the eigenvalues λ_j of the matrix if the matrix is diagonalizable (which Hermitian matrices are),

$$\mathrm{Tr}(\rho) = \sum_{i=1}^{N} \rho_{ii} = \sum_{j} \lambda_j . \tag{16}$$

The average of A, Eq. (15), can then be written as

$$\langle A\rangle = \mathrm{Tr}(\rho A). \tag{17}$$

A property of the density matrix is that if A is the identity operator and the $\left|\alpha^{(i)}\right\rangle$ states are normalized then $\mathrm{Tr}(\rho) = 1$. This is true for all normalized density matrices, regardless if they are representing pure or mixed states.

The diagonalized density matrix for a pure state has only one non-zero element which is a 1 as a diagonal element, meaning the eigenvalues are 1 and 0 [1]. This is because if the system is in the pure (normalized) state $\left|\alpha^{(i)}\right\rangle$ then $W_i = 1$ and the density operator is

$$\rho = \left|\alpha^{(i)}\right\rangle \left\langle\alpha^{(i)}\right| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

From Eq. (18) it is clear that $\rho^2 = \rho$ for a pure state. Eq. (12) says that the diagonal elements ρ_{nn} of a density matrix are probabilities. This means that $\rho_{nn}^2 \leq \rho_{nn}$. Combining this with Eq. (16) we can conclude that

$$\mathrm{Tr}\big(\rho^2\big) \leq \mathrm{Tr}(\rho) = 1$$

which gives us a condition for when a density matrix ρ is pure or mixed. For pure states $\mathrm{Tr}(\rho^2) = 1$ and for mixed states $\mathrm{Tr}(\rho^2) < 1$ [1]. We demonstrate this and how to calculate the expectation value in the following example.

Example 3

We can calculate the spin-z expectation value of the density matrix

$$\rho = 0.3\rho_z + 0.7\rho_x = \begin{pmatrix} 0.65 & 0.35 \\ 0.35 & 0.35 \end{pmatrix}$$

for the mixed state from Example 2 by using the operator

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

σ_z is a Pauli matrix which measures the spin in the z-direction in units of $\hbar/2$.

$$\begin{aligned} \langle\sigma_z\rangle &= \mathrm{Tr}(\rho\sigma_z) \\ &= \mathrm{Tr}\left(\begin{pmatrix} 0.65 & 0.35 \\ 0.35 & 0.35 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\ &= \mathrm{Tr}\begin{pmatrix} 0.65 & -0.35 \\ 0.35 & -0.35 \end{pmatrix} \\ &= 0.65 - 0.35 = 0.3 \end{aligned}$$

Since 30% of the particles in the mixed ensemble have spin-z this expectation value is correct. To verify that this is a mixed state $\mathrm{Tr}(\rho^2)$ is calculated. It is

$$\mathrm{Tr}\left(\rho^2\right) = 0.79$$

which is less than one, as it should be for a mixed state.

3 Quantum entanglement

One very important phenomenon concerning the information paradox is quantum entanglement. Simply put entanglement is an interdependence of properties between several particles regardless of their spatial separation. Consider a system consisting of two or more particles. The particles are said to be entangled if the system can be described completely by only one wave function that cannot be separated into a wave function for each subsystem. Normally, composite quantum systems can be decomposed into subsystems. Each subsystem is then a system of its own which can be completely described by a wave function.

For example, a system of two particles can be described completely by combining two separate wave functions. If A and B are two non-entangled particles the wave function $|\Psi\rangle$ of the complete system can be written as a tensor product of the two separate pure states $|\psi_A\rangle$ and $|\psi_B\rangle$. The non-entangled state, also called a product or separable state, $|\Psi\rangle$ is

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

and the Hilbert space for the composite system is given by

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

where $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$.

If A and B are entangled this is not the case. To completely describe a system of entangled particles we can still use only one wave function but it cannot be separated into a part for each particle. Trying to describe the particles separately yields an incomplete description as the properties of one particle is dependent on the properties of the other. For an entangled state $\nexists\, |\psi_A\rangle \in \mathcal{H}_A$, $\nexists\, |\psi_B\rangle \in \mathcal{H}_B$ such that $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ [4].

This is true for the generalized case as well. Take N states in N Hilbert spaces; $|\psi_n\rangle \in \mathcal{H}_n$, where $n = 1, 2, \ldots, N$. The state $|\Psi\rangle$ of the complete system is in the joint Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$. If $|\Psi\rangle$ cannot be written in the form

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots |\psi_N\rangle$$

it is an entangled state [5]. If we study only one of the particles of an entangled state we will think the particle is in a regular superposition of states. In other words, behaving exactly like a non-entangled particle. The entanglement can only be discovered if we study all the particles and notice the correlation of their properties. To give a concrete example of entanglement we will study a system of two entangled spin-$\frac{1}{2}$ particles.

3.1 Example: spin-$\frac{1}{2}$ system

Consider two fermions (spin-$\frac{1}{2}$ particles), a and b. Each particle can have spin up $|\uparrow\rangle$ or spin down $|\downarrow\rangle$ in the z-direction. This leads to four possible states for the system. The states are

$$\begin{aligned} |\psi_1\rangle &= |\uparrow\rangle|\uparrow\rangle \\ |\psi_2\rangle &= |\downarrow\rangle|\downarrow\rangle \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \end{aligned} \tag{19}$$

and

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \tag{20}$$

where the first three states each has a total spin of one (such a state is known as a triplet state) and the final state has spin zero (known as a singlet state).

If the information we have about the system is that the total spin is one and we decide to measure the spins of the two particles the possible results are

S_a	S_b
$+1$	-1
-1	$+1$
$+1$	$+1$
-1	-1

where S_i is the spin of particle $i \in a, b$. $+1$ is spin up and -1 spin down. The results are completely uncorrelated since the particles can be in any of the states in Eq. (19). The spin of a has no dependence on the spin of b and the particles are not entangled.

If instead we have a particle of spin zero that decays, though a process that conserves angular momentum and spin, into the two fermions we know that the final state is the spin-0 state. In this case the possible results when measuring the spins are

S_a	S_b
$+1$	-1
-1	$+1$

which shows a clear correlation between the spins. If a has spin up then b has spin down, and vice versa. Measuring the spin of a directly determines the spin of b and the particles are said to be entangled.

This system of two spin-$\frac{1}{2}$ particles is analogous to a system of two polarised photons. The polarisation direction of the photon would correspond to the spin of the particle.

3.2 Monogamy of entanglement

A property of entanglement is the so called monogamy of entanglement. It is the property that an entangled state cannot be shared between an arbitrary number of particles. For example, if the particles A and B are fully entangled then neither of them can be entangled to particle C. If particle A through some interaction becomes somewhat entangled to C then it will loose some of its entanglement with B [6].

3.3 EPR-paradox

The theory of quantum mechanics differ in several significant ways from classical physics. Before quantum mechanics physics was deterministic. If enough information was known about a physicals system then it would be possible to predict the future evolution of the system precisely. However, as we have seen the wave function description of a state is not deterministic but probabilistic. This means that a system is not in a predetermined state and before the system is measured one cannot claim to know its specific qualities with 100% certainty. This is a fundamental part of quantum mechanics but it was questioned by many that thought that the physical properties of a system exist regardless of the system having been measured or not. Thus, since quantum mechanics could not give this complete deterministic description something must be missing from the theory. This was what Einstein, Podolsky and Rosen proposed in their paper [7]. In this paper they presented what is known as the EPR-paradox. In the article [7] the existing physical properties of a system are called **elements of reality**, and it is argued that they exist regardless of measurements.

The EPR-paradox, explained in the form of a thought experiment [3], considers two entangled particles, that are created or interact to get their entanglement, and are then separated by an arbitrary distance in space. An example is the entangled spin-$\frac{1}{2}$ system discussed in Section 3.1. By measuring, for example the spin, of one of them we immediately know the spin of the other. The claim of EPR is that, as the separation of these particles in space is arbitrary and nothing can travel faster than light, the results of the measurements must be determined before the particles are separated. That is, if we measure one particle to have spin up then it had that spin even before the measurement was made. This also means that entanglement is nothing but a consequence of the already determined properties of the particles, no superluminal communication needed. The EPR argument is that since quantum mechanics cannot predict the spin of the particle with 100% certainty something must be missing form the theory.

A relatively straightforward way to correct the theory would be to claim the existence of so called local hidden variables. The idea here is that there could be local variables not found and described by quantum mechanics. These variables would make the theory deterministic and therefore complete. However, this has been disproven multiple times. First by Bell in 1964 [8] (the main source used here is [3]) and later by Greenberg, Horne and Zeilinger (GHZ) (for their proof the source used here is [9]).

3.4 Bell's inequality

Bell's inequality is a proof that shows that hidden variables cannot be used to "complete" quantum mechanics, or explain entanglement. As we shall see, the use of local hidden variables is not compatible with quantum mechanics as it leads to an inequality which it is easy to show does not hold. To explain/derive Bell's inequality we will consider two entangled particles; an electron and a position (both fermions), in the spin singlet state.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle).$$

These particles are traveling in opposite directions trough space, each towards a spin-detector.

The two detectors, separated in space by an arbitrary distance, each measures the spin in the direction of a unit vector $\bar{a}$ or $\bar{b}$. One will measure the spin of the electron in the direction $\bar{a}$ and the other the spin of the positron in the direction $\bar{b}$. The possible results of those measurements are

$S_{e^-}(\bar{a})$	$S_{e^+}(\bar{b})$	$S_{e^-} * S_{e^+}$
$+1$	-1	-1
-1	$+1$	-1
$+1$	$+1$	$+1$
-1	-1	$+1$

where the spins directions, $\bar{a}$ and $\bar{b}$ are not necessarily parallel or antiparallel.

We call the average product of the spins, for a set of detector directions, P such that the general formula for P is

$$P(\bar{a}, \bar{b}) = -\bar{a} \cdot \bar{b} = -ab\cos(\theta)$$

where $|\bar{a}| = a$, $|\bar{b}| = b$ and θ is the angle between the vectors $\bar{a}$ and $\bar{b}$.

Which means that when $\bar{a}$ and $\bar{b}$ are parallel, $\bar{a} = \bar{b}$,

$$P(\bar{a}, \bar{b}) = P(\bar{a}, \bar{a}) = -1$$

since the particles have opposite spin when measured in the same direction. For the same reason, when $\bar{a}$ and $\bar{b}$ are antiparallel, $\bar{a} = -\bar{b}$,

$$P(\bar{a}, \bar{b}) = P(\bar{a}, -\bar{a}) = 1.$$

One assumption we need to make is the locality assumption which states that the measurement of one particle does not depend on the orientation of the other particle's detector. Hence measuring one particle should not affect the other particle in any way.

We now introduce the hidden variable λ. λ can be one or several variables, it is not important, the only thing that is known about λ is that it is a local variable, so that the properties of system A does not depend on what is done to system B.

If we measure the spin of the electron and the positron the results are given by the functions A and B respectively. They depend on λ and the chosen directions of the detectors. The possible values of the A and B are

$$A(\bar{a}, \lambda) = \pm 1$$

$$B(\bar{b}, \lambda) = \pm 1.$$

From the functions it is clear that the square of each will always give the value 1, for example $A(\bar{a}, \lambda)^2 = 1$.

The spins for the particles are opposite one another if the detectors measure in the same direction, giving the following equation

$$A(\bar{a}, \lambda) = -B(\bar{a}, \lambda). \tag{21}$$

We can now calculate P, the average value of the product of the measurements, using the hidden variable λ. λ has a probability density $\rho(\lambda)$ such that $\rho(\lambda) \geq 0$ and $\int \rho(\lambda) d\lambda = 1$.

$$P(\bar{a}, \bar{b}) = \int \rho(\lambda) A(\bar{a}, \lambda) B(\bar{b}, \lambda) d\lambda \tag{22}$$

Using Eq. (21) we can rewrite Eq. (22) to get rid of one of the functions by replacing $B(\bar{b}, \lambda)$ with $-A(\bar{b}, \lambda)$.

$$P(\bar{a}, \bar{b}) = -\int \rho(\lambda) A(\bar{a}, \lambda) A(\bar{b}, \lambda) d\lambda \tag{23}$$

and now P only depends on $\rho(\lambda)$ and A.

We introduce a third unit vector $\bar{c}$, which has a different direction compared to $\bar{a}$ or $\bar{b}$. $P(\bar{a}, \bar{c})$ is then given by an equation of the same form as Eq. (23)

$$P(\bar{a}, \bar{c}) = -\int \rho(\lambda) A(\bar{a}, \lambda) A(\bar{c}, \lambda) d\lambda. \tag{24}$$

The following calculations lead to Bell's inequality.

$$
\begin{aligned}
P(\bar{a}, \bar{b}) - P(\bar{a}, \bar{c}) &= -\int \rho(\lambda) A(\bar{a}, \lambda) A(\bar{b}, \lambda) d\lambda + \int \rho(\lambda) A(\bar{a}, \lambda) A(\bar{c}, \lambda) d\lambda \\
&= -\int \rho(\lambda)(A(\bar{a}, \lambda) A(\bar{b}, \lambda) - A(\bar{a}, \lambda) A(\bar{c}, \lambda)) d\lambda \\
&= -\int \rho(\lambda)[1 - A(\bar{b}, \lambda) A(\bar{c}, \lambda)] A(\bar{a}, \lambda) A(\bar{b}, \lambda) d\lambda. \qquad (25)
\end{aligned}
$$

It is known that $\rho(\lambda) \geq 0$ and that

$$-1 \leq A(\bar{a}, \lambda) A(\bar{b}, \lambda) \leq 1$$

which combined lead to

$$\rho(\lambda)[1 - A(\bar{b}, \lambda) A(\bar{c}, \lambda)] \geq 0$$

which leads to

$$
\begin{aligned}
|P(\bar{a}, \bar{b}) - P(\bar{a}, \bar{c})| &\leq \int \rho(\lambda)[1 - A(\bar{b}, \lambda) A(\bar{c}, \lambda)] d\lambda \\
&= 1 + P(\bar{b}, \bar{c}).
\end{aligned}
$$

This is **Bell's inequality**

$$|P(\bar{a}, \bar{b}) - P(\bar{a}, \bar{c})| \leq 1 + P(\bar{b}, \bar{c}).$$

It may seem harmless at first but it is easy to use an example and show that it does not hold. Take for example the three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$, all in the same plane, where $\bar{a} \perp \bar{b}$ and $\bar{c}$ is at an angle of 45^o to both of them.

$$P(\bar{a}, \bar{b}) = -\bar{a} \cdot \bar{b} = 0$$

$$P(\bar{a}, \bar{c}) = -ac \cos(\pi/4) = -\frac{1}{\sqrt{2}} \approx -0.707$$

$$P(\bar{b}, \bar{c}) = -bc \cos(\pi/4) = -\frac{1}{\sqrt{2}} \approx -0.707$$

Then the inequality becomes

$$|0 - 0.707| \leq 1 - 0.707$$

$$0.707 \leq 0.293$$

which clearly does not hold.

We notice that to show that Bell's inequality does not hold we need to make measurements on several pairs of entangled particles, since P is an average value. Bell's inequality has been tested in many experiments, the results of which have shown that it does not hold.

Thus using local hidden variables to improve the theory of quantum mechanics does not work. Neither can it explain quantum entanglement. This suggests that quantum mechanics is complete as it is. The properties of a system are not determined before a measurement is made, which is what EPR argued against.

3.5 GHZ-argument

Bell's inequality is one way to disprove the EPR argument for elements of reality. The definition of an **element of reality** that EPR gives is that it is a real physical quantity of an object that can be predicted with certainty without disturbing the system in question.

Take, for example, an entangled pair of fermions. By measuring the spin, in this case in the z-direction, of one of the particles we can with certainty predict what the spin of the other particle is without disturbing it. Thus the spin-z of the second particle is an element of reality, meaning that the particle has this property regardless of measurements.

GHZ offers a new version of the EPR thought experiment, one which demolishes the notion of elements of reality and, unlike Bell's inequality, does so in a way which does not depend on probabilities.

The GHZ thought experiment involves three entangled fermions. The three particles are traveling in three different directions in the same plane. The spin state of the system (all three particles) is $|\Psi\rangle$, hence $|\Psi\rangle$ is an eigenstate of a complete set of commuting Hermitian spin operators. The spin operators in question depend on the spin operators for each particle i, where $i = 1, 2, 3$, which are defined as follows; σ_z^i is the spin of the particle along its direction of motion, σ_x^i is the spin along the vertical direction and σ_y^i the spin along the horizontal direction (orthogonal to the trajectory).

These operators are directly related to the Pauli matrices, the operators of spin in the x, y and z directions, and since we calculate spin in units of $\frac{\hbar}{2}$ they are identical to the Pauli matrices. For σ_z the eigenstates are spin up and spin down

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{26}$$

and the Pauli matrices are

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{27}$$

What is known, and that a few simple calculations will show, is that the spin operators do not commute with one another, for example $[\sigma_x, \sigma_y] = i\sigma_z$, which means that the spin of a particle cannot be known for several directions simultaneously. If, for example, the spin-z for a particle is measured then the spin-x or spin-y of the particle is ± 1 with equal probability. The eigenvalues of the operators are $\lambda = \pm 1$.

We can also note that $\sigma_x\sigma_y = -\sigma_y\sigma_x$ as this will be useful later.

$$\begin{aligned} \sigma_x\sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \sigma_y\sigma_x &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \end{aligned}$$

$$\Rightarrow \sigma_x\sigma_y = -\sigma_y\sigma_x \tag{28}$$

In this thought experiment the spin-x and spin-y of the particles will be measured. The set of spin operators for the system is

$$\sigma_x^1\sigma_y^2\sigma_y^3 \qquad \sigma_y^1\sigma_x^2\sigma_y^3 \qquad \sigma_y^1\sigma_y^2\sigma_x^3 \tag{29}$$

and they all commute. For example, using Eq. (28) we can see that the two first operators in (29) commute.

$$\begin{aligned} [\sigma_x^1\sigma_y^2\sigma_y^3, \sigma_y^1\sigma_x^2\sigma_y^3] &= \sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3 - \sigma_y^1\sigma_x^2\sigma_y^3\sigma_x^1\sigma_y^2\sigma_y^3 \\ &= \sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3 - (-\sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^1\sigma_x^2\sigma_y^3) \\ &= \sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3 - (\sigma_x^1\sigma_y^2\sigma_y^3\sigma_y^1\sigma_x^2\sigma_y^3) \\ &= 0 \end{aligned}$$

The square of each of the operators (29) is unity since $(\sigma_x^i)^2 = (\sigma_y^i)^2 = \mathbb{I}$

$$(\sigma_x^1\sigma_y^2\sigma_y^3)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}. \tag{30}$$

The fact that the three operators in (29) commute means that we can apply them to a state and know which eigenstate the state is in for each of the operators simultaneously. The state will be in three eigenstates, each for one of the operators.

The entangled state of the particles we will focus on is the GHZ state.

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) \tag{31}$$

The eigenvalue for the GHZ state is 1 for each of the three operators.

$$\begin{aligned}
\sigma_x^1\sigma_y^2\sigma_y^3|GHZ\rangle &= \sigma_x^1\sigma_y^2\sigma_y^3\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) && (32)\\
&= \frac{1}{\sqrt{2}}((1*i*i)|\uparrow\uparrow\uparrow\rangle - (1*-i*-i)|\downarrow\downarrow\downarrow\rangle) && (33)\\
&= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) && (34)\\
&= 1|GHZ\rangle && (35)
\end{aligned}$$

If the particles are in the GHZ state and we choose to measure the spin-x of one particle and the spin-y of the other two then the product of those measurements always has to be 1, since the particles are in an eigenstate of (29) with the eigenvalues 1.

S_x^1	S_y^2	S_y^3	$S_x^1S_y^2S_y^3$
$+1$	-1	-1	$+1$
-1	-1	$+1$	$+1$
-1	$+1$	-1	$+1$
$+1$	$+1$	$+1$	$+1$

S_α^i is the result of the measurement of the spin in the direction α ($\alpha = x, y$), it is ± 1. We also denote this with m_α^i, which we say is the corresponding element of reality.

We make the same locality assumption as for Bell's inequality; measuring one of the particles does not disturb the other. Thus by measuring the spin for two of the particles one will know the spin of the third. According to the EPR criterion for an element of reality this means that the spin of the third particle is real, but since this can be applied to all of the particles by changing which ones we choose to measure all the six elements of reality m_x^1, m_x^2, m_x^3, m_y^1, m_y^2 and m_y^3 must exist.

This is clearly a violation of quantum mechanics since m_x^i and m_y^i cannot be known simultaneously, due to their operators not commuting.

For now we ignore this and assume the elements of reality do exist. From Eq. (35) we know

$$\begin{aligned}
\sigma_x^1\sigma_y^2\sigma_y^3|GHZ\rangle &= m_x^1m_y^2m_y^3|GHZ\rangle = 1|GHZ\rangle\\
\sigma_y^1\sigma_x^2\sigma_y^3|GHZ\rangle &= m_y^1m_x^2m_y^3|GHZ\rangle = 1|GHZ\rangle\\
\sigma_y^1\sigma_y^2\sigma_x^3|GHZ\rangle &= m_y^1m_y^2m_x^3|GHZ\rangle = 1|GHZ\rangle
\end{aligned}$$

that is, the product of the three m^i_α is always one. The product of the three different combinations of measurements will also be 1.

$$1^3 = m^1_x m^2_y m^3_y \cdot m^1_y m^2_x m^3_y \cdot m^1_y m^2_y m^3_x = m^1_x m^2_x m^3_x \tag{36}$$

As we can see, since $(m^i_y)^2 = 1$, this product is equal to an operator measuring the spin-x for each particle, which then would have the eigenvalue 1 for the GHZ state. This means that the operator $\sigma^1_x\sigma^2_x\sigma^3_x$ should commute with the operators (29).

It is easy to check if it commutes, which it does.

$$[\sigma^1_x\sigma^2_x\sigma^3_x, \sigma^1_x\sigma^2_y\sigma^3_y] = \sigma^1_x\sigma^2_x\sigma^3_x\sigma^1_x\sigma^2_y\sigma^3_y - \sigma^1_x\sigma^2_y\sigma^3_y\sigma^1_x\sigma^2_x\sigma^3_x = 0$$

Next we check to see if it has the eigenvalue 1,

$$\sigma^1_x\sigma^2_x\sigma^3_x|GHZ\rangle = \sigma^1_x\sigma^2_x\sigma^3_x\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) \tag{37}$$

$$= \frac{1}{\sqrt{2}}(|\downarrow\downarrow\downarrow\rangle - |\uparrow\uparrow\uparrow\rangle) = -|GHZ\rangle \tag{38}$$

which it does not. It has the eigenvalue -1. By assuming the reality criterion is correct we have found a contradiction. The GHZ thought experiment has thus shown that the elements of reality do not exist [9].

3.6 No-cloning theorem

The no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown state. This theorem follows from the linearity of quantum mechanics and shows that using entanglement to perform superluminal communication, for example in the EPR thought experiment, is impossible.

The mathematical statement of the the theorem is as follows: given two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ in $\mathcal{H}$, there exists no unitary transformation $\hat{U}$, defined as $\hat{U} : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$, such that

$$\hat{U}(|\psi_i\rangle|0\rangle) = |\psi_i\rangle|\psi_i\rangle$$

where $|\psi_i\rangle \in \{|\psi_1\rangle, |\psi_2\rangle\}$ but is unknown. $|0\rangle$ is an unprepared state which is determined when the cloning operator acts on it, i.e. the cloning process is conducted.

The proof of this theorem is fairly simple. Suppose the operator $\hat{U}$ is linear and can clone known the known states $|\psi_1\rangle$ and $|\psi_2\rangle$.

$$\hat{U}(|\psi_1\rangle|0\rangle) = |\psi_1\rangle|\psi_1\rangle$$

$$\hat{U}(|\psi_2\rangle |0\rangle) = |\psi_2\rangle |\psi_2\rangle$$

What would happen if $\hat{U}$ acted on an unknown state? For example a superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$.

$$|\psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$$

$$\begin{aligned}\hat{U}((a|\psi_1\rangle + b|\psi_2\rangle)|0\rangle) &= \hat{U}(a|\psi_1\rangle |0\rangle) + \hat{U}(b|\psi_2\rangle |0\rangle)\\ &= a|\psi_1\rangle |\psi_1\rangle + b|\psi_2\rangle |\psi_2\rangle\end{aligned}$$

The resulting state is not a clone of the initial state. Had the cloning operator succeeded the final state would have been

$$\begin{aligned}\hat{U}((a|\psi_1\rangle + b|\psi_2\rangle)|0\rangle) &= (a|\psi_1\rangle + b|\psi_2\rangle)(a|\psi_1\rangle + b|\psi_2\rangle)\\ &= a^2|\psi_1\rangle |\psi_1\rangle + b^2|\psi_2\rangle |\psi_2\rangle + ab(|\psi_1\rangle |\psi_2\rangle + |\psi_2\rangle |\psi_1\rangle).\end{aligned}$$

If cloning were possible superluminal communication would have been possible in the EPR thought experiment. For if one of the particles, say the electron, was measured the person measuring the other particle, the positron, would be able to know if the electron was or was not measured. The person measuring the positron could simply clone it and measure the spin of each of the clones. If all the clones had the same spin it is very likely that the electron was measured. If half of the cloned positrons had one spin and the other half the opposite spin the person would know that the electron had not been measured. Thus by simply choosing to measure or not measure the electron one could communicate information to the person measuring the positron, and vice versa [10].

4 Entropy of quantum systems

Classically entropy is a measurement of the number of microstates that would constitute the same macrostate. The thermodynamical definition of entropy is

$$S = k_B \ln \Omega$$

where every microstate is equally likely to occur, k_B is the Boltzmann constant and Ω the number of microstates a specific macrostate could have. Increasing the number of microstates leads to an increase in entropy of the system.

For example, a box containing a gas has an entropy. Increasing the volume of the box, without changing the other parameters, gives a higher entropy since the gas particles now have more possible positions to occupy and therefore more available microstates.

Entropy is also used to quantify information. Calculating the entropy for a system depends on if the system is classical or quantum mechanical. Two types of entropy in information theory are Shannon entropy and von Neumann entropy, each applicable to one type of system.

The main source for Sections 4.1 and 4.2 is [5], any other sources are cited in the text.

4.1 Shannon entropy

Shannon entropy is used to describe classical systems and is formally equal to the thermodynamical definition of entropy. It quantifies the amount of information, in terms of bits, of a random variable X, where X belongs to a classical probability distribution. The definition of Shannon entropy is

$$H(X) \equiv -\sum_i p(x_i) \log_b p(x_i)$$

where b is the basis used for the measurement and $p(x_i)$ is the probability of the outcome x_i, such that

$$\sum_i p(x_i) = 1.$$

In other words, the Shannon entropy is the average information gained from a measurement, or equivalently the average uncertainty of the result before the measurement. The usual basis is $b = 2$ for which the Shannon entropy is measured in bits. Take a collection of possible outcomes of a measurement where one outcome is more probable than others. By observing the more likely outcome we gain less information than if we were to observe an outcome with very low probability.

If we have a system with only one possible outcome then the Shannon entropy is zero. This can be thought of as that the information we gain by measuring this outcome is zero, or that the uncertainty of the result is zero.

Example 4

Let X be a variable with two possible outcomes. For a system with two possible outcomes the maximum Shannon entropy is one and occurs when the two outcomes are equally probable.

$$H(X) = -2(0.5 \log_2 0.5) = 1 \text{ bit}.$$

The base used here is $b = 2$. When the two outcomes are equally likely predicting the outcome of the system is as difficult as it gets. The uncertainty of the measurement, or information gained from the result, of this system is one.

If the two outcomes have different probabilities the Shannon entropy is less than one. Because now measuring one of the outcomes yields more information. Or equivalently, there is less uncertainty in the result since one of the two outcomes is more likely than the other.

For example, for the probabilities $p(x_1) = 0.1$ and $p(x_2) = 0.9$ the Shannon entropy is

$$H(X) = -(0.1 \log_2 0.1 + 0.9 \log_2 0.9) \approx 0.469 \text{ bits.}$$

Example 4 shows that the Shannon entropy is larger if the probabilities are equal than if they are not. This is true in the general case as well. If there are N possible outcomes that all are equally likely then $p_i = \frac{1}{N}$ and

$$H(X) = -\sum_{i=1}^{N} \frac{1}{N} \log_b \frac{1}{N} = \log_b N.$$

With only one possible outcome, $p_i = 1$, the Shannon entropy has its minimum value

$$H(X) = -\log 1 = 0.$$

The Shannon entropy can be used to describe classical systems but not quantum systems. The Shannon entropy of two variables, X and Y, is always larger than the entropy for one of them. The joint probability of two independent random variables is the product of the individual probabilities, $p(x, y) = p(x)p(y)$, and the joint entropy $H(X, Y)$ is simply the sum of the entropies, $H(X, Y) = H(X) + H(Y)$, which leads to $H(X, Y) \geq H(X)$ and $H(X, Y) \geq H(Y)$.

For entangled systems, described by the von Neumann entropy, this is not the case [11].

4.2 Von Neumann entropy

The von Neumann entropy is used to describe the entropy of quantum systems. It is defined as

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho)$$

where $S(\rho)$ is the entropy or information content in a quantum random variable. ρ is the density operator, it was defined in Section 2.4.1 but can also be written as a sum of the possible states ρ_{xi}.

$$\rho = \sum_i p(x_i)\rho_{xi}$$

were $p(x_i)$ is the probability that the ensemble is in the state given by ρ_{xi}.

The unit for the von Neumann entropy is the qubit. A qubit is $|\psi\rangle = c_1|0\rangle + c_2|1\rangle$ where $|0\rangle$ and $|1\rangle$ are basis states. One bit can have one of two values; 1 or 0. A qubit can be in the state $|0\rangle$, $|1\rangle$ or a superposition of both. Therefore a qubit can contain more information than a bit.

The connection between Shannon entropy and von Neumann entropy can be seen by considering a density operator where all the ρ_{xi} states are pure and orthogonal, just as they are in the classical case. From Section 2.4.2 we remember that pure states only have one non-zero element and it is a one on the diagonal. A density matrix ρ consisting of these pure and orthogonal ρ_{xi} will therefore only have the diagonal elements $p(x_i)$ as non-zero elements. For such a density matrix the von Neumann entropy becomes the Shannon entropy

$$S(\rho) = -\operatorname{Tr}(\rho \log(\rho)) = -\sum_i p(x_i)\log_b p(x_i).$$

For example, if ρ consists of a single pure state, i.e. all the particles in the ensemble are in the same pure state, the von Neumann entropy is

$$S(\rho) = -1\log(1) = 0$$

which is also the minimum value of the entropy. Just as in the Shannon entropy case, there is no uncertainty in the outcome of a measurement of this system, since only one outcome is possible.

As previously mentioned (see Section 2.4.1) ρ is Hermitian and therefore diagonalizable. The entropy is an observable quantity and therefore independent of the coordinate basis used to calculate it. Diagonalizing the density matrix will not affect the value of the entropy. The elements of the diagonalized ρ are the eigenvalues λ_j of ρ. Hence, the von Neumann entropy can be calculated using only the eigenvalues of ρ

$$S(\rho) = -\operatorname{Tr}(\rho \log(\rho)) = -\sum_j \lambda_j \log(\lambda_j). \tag{39}$$

For the Shannon entropy the entropy of a system is always larger than the entropy of its subsystems. This is also true for the von Neumann entropy as long as the system is separable, i.e. not entangled. A system described by the density matrix ρ_{AB} has the components A and B. If the system is separable

$$S(\rho_{AB}) > S(\rho_A)$$

and

$$S(\rho_{AB}) > S(\rho_B).$$

If A and B are entangled then ρ_{AB} is a pure state with von Neumann entropy zero. Therefore it is possible for entangled states to have subsystems for which the von Neumann entropies are larger than that of the total system [11]

$$S(\rho_A) > S(\rho_{AB})$$

$$S(\rho_A) > S(\rho_{AB}).$$

5 Fundamental concepts of general relativity

Einsteins theory of general relativity (GR) describes gravitation as an effect of the geometry of spacetime. This theory is successful, with ample observational support, but also complicated. In this section a few important features and tools of general relativity that we need to describe black holes in Section 6 are presented. For more details on GR, we refer the reader to the books by Carroll [12] and Schutz [13], which are the main sources for these sections (5 and 6).

5.1 Spacetime

An important aspect of general relativity is that it operates in four-dimensional spacetime, which consists of one time dimension and three spatial dimensions. Space can be though of as a three-dimensional set of points $\{x, y, z\}$. Spacetime is then a four-dimensional set of points $\{t, x, y, z\}$. The difference between the spatial dimensions and the time dimension is that one can only move in one direction in the time dimension, i.e. forwards. In the spatial dimensions one can move backwards and forwards.

But what is the difference compared to Newtonian physics? After all, calculations in Newtonian physics utilizes time and space as well. The difference is that Newtonian physics treats space and time separately while GR treats them as one.

For example, calculating the path between two points in space using classical physics does not require any consideration of the time dimension. For two points in a coordinate system the distance between them is simply the difference in their position. In GR there is no observer-independent distinction between space and time, so the length of a path must be calculated using all four dimensions and not just the three spatial ones.

A mathematical tool to describe spacetime is the manifold. Manifolds are topological spaces that locally have similar properties to flat (n-dimensional) Euclidean space. Globally, manifolds can be curved and are therefore used to describe spacetime in GR. An example of a manifold is the two-sphere, i.e. the two-dimensional surface of a three-dimensional sphere.

5.2 The metric

The metric tensor is a useful tool in describing the geometry of spacetime. A metric of spacetime contains the information to describe the curvature of the spacetime manifold.

In general a metric tensor is a function $\mathbf{g}(\ ,\ )$ that takes two vectors and turns them into a scalar. It defines an inner product

$$\bar{A} \cdot \bar{B} = A^{\mu} B^{\nu} g_{\mu\nu}$$

where $\bar{A}$ and $\bar{B}$ are vectors in the basis $\{\bar{e}_{\mu}\}$ and $g_{\mu\nu}$ are the components of the metric tensor. The definition of the metric tensor $\mathbf{g}$ is

$$\mathbf{g}(\bar{A}, \bar{B}) := \bar{A} \cdot \bar{B}$$

and it is symmetric.

Another notation for the metric is the line element ds^2

$$g_{\mu\nu} V^{\mu} W^{\nu} = g(V, W) = ds^2(V, W).$$

In the curved spacetime of general relativity the metric tensor is denoted $g_{\mu\nu}$. It is non-degenerate, $\det(g_{\mu\nu}) = |g_{\mu\nu}| \neq 0$ and has a symmetric inverse $g^{\mu\nu}$.[5] In GR a few of the properties and uses of a metric tensor are that it determines the shortest distance between two points and allows the computation of path length and proper time. It also replaces the three-dimensional scalar product with the four-dimensional inner product discussed above.

Some examples of metrics that may be familiar are the ones used in classical physics and special relativity. In classical physics, which usually operates in flat three-dimensional space, the metric is called the Euclidean metric and is given by $g_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{40}$$

or equivalently

$$ds^2 = dx^2 + dy^2 + dz^2$$

in Cartesian coordinates. In spherical coordinates (r, θ, ϕ) the Euclidean line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

From this we can see that the components of the metric are coordinate-dependent. The metric can be expressed in any coordinates, as long as they span the entire space in question, and they are used according to which describe a situation better.

[5] All the metrics in GR will be non-degenerate, and continuous (i.e. have the same number of positive and negative eigenvalues at all points).

The metric tensor in special relativity is called the Minkowski metric and is denoted $\eta_{\mu\nu}$. It describes four-dimensional flat spacetime and is defined as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

or

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dx^2 + dy^2 + dz^2 \end{aligned} \tag{42}$$

in Cartesian coordinates, where c is the speed of light in vacuum and set to 1.

In spherical coordinates $\{t, r, \theta, \phi\}$ it is

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{43}$$

An important thing to note about the Minkowski metric (41) is the sign of the components. The way the time dimension is distinguished from the spatial dimensions is that the time component has the opposite sign from the space components. Thus, the four-dimensional spacetime metrics used in special and general relativity always have one component with the opposite sign of the other components, while the components of the Euclidean metric, which only concerns the spatial dimensions, all have the same sign.

5.3 Spacetime diagram

A useful tool in special and general relativity is the spacetime diagram. In a spacetime diagram the time-dimension t (or ct, but we set $c = 1$) is plotted against a spatial dimension, for example x or r. Figure (1) shows an example of a spacetime diagram.

The grey cone in Figure (1) is a light cone. A light cone consists of all the points connected to one event by straight lines with velocity c. In the coordinates t and x for the Minkowski metric they are straight lines at 45°. The light cone has one future part and one past part, in Figure (1) the future part is for $t > 0$ and the past for $t < 0$. The events inside the light cone of an event p (in Figure (1) p is at origo) can reach p by a so called timelike path. Events outside the cone cannot reach p and are spacelike separated from it while events on the cone are lightlike separated from p.

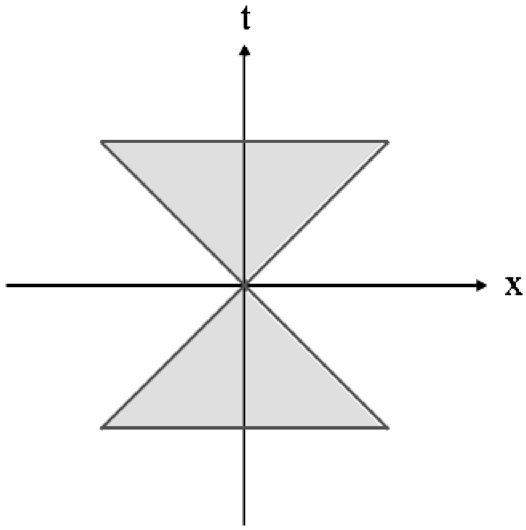

Figure 1: Spacetime diagram, plotting the time t against the distance x.

Light cones are useful and important to study. They show all the possible events that one event p can be connected to. For an event p all the possible events it can reach are within its future light cone. All the events that may have caused p are in its past light cone.

5.4 Einsteins equations

Einsteins equations of general relativity are fairly complicated. Here the equations will be stated and each parameter explained. The short explanation is that Einsteins equations are a set of non-linear second order differential equations, the solution to which is a metric tensor $g_{\mu\nu}$.

The equations can be written in the following two forms

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{44}$$

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{45}$$

where the energy-momentum tensor $T_{\mu\nu}$ is the tensor generalization of mass density. It describes the distribution of mass/energy in the chosen region of spacetime. $g_{\mu\nu}$ is the metric tensor (which replaces the gravitational potential in Newtons equation). The Ricci tensor $R_{\mu\nu}$ describes the curvature of spacetime. $R_{\mu\nu}$ involves the first and second derivatives of the metric.[6] Even the energy-momentum tensor involves the metric, which is why the equations are complicated second order differential equations.

In vacuum, for example in space outside a sun or planet, the equation is seemingly much simpler since it is

$$R_{\mu\nu} = 0 \tag{46}$$

as the energy-momentum tensor is zero in vacuum.

5.4.1 The Schwarzschild solution

A spherically symmetric solution to the vacuum Eq. (46) is the Schwarzschild metric, which is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{47}$$

in spherical coordinates, where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and G is Newton's gravitational constant. The Schwarzschild metric describes the curvature of spacetime in the vacuum outside of a spherically symmetric object of mass M. One thing to note about this solution is that when $M \to 0$ it becomes the Minkowski metric. The same occurs when $r \to \infty$, meaning that the further away from the object one is the flatter spacetime is.

Another thing to note is that this solution seems to have two singularities, one where $r = 0$ and another one where $r = 2GM$ ($r_S = 2GM$ is the Schwarzschild radius). At these points the metric components approach infinity. However, as previously mentioned the metric components are coordinate-dependent. Points that are singular in one coordinate system may not be singular in another one. For the Schwarzschild metric the points at $r = 2GM$ are not a real singularities but the one at $r = 0$ is[7].

The Schwarzschild solution is applicable in the vacuum outside of a spherically symmetric object of mass M, such as stars or planets. In this case outside means at a radius larger than $2GM$, which ignores the problem of the singularity at $r = 0$. There are, however, objects that are described by the Schwarzschild solution for all values of r, even for $r < 2GM$. These objects are called black holes.

[6] $R_{\mu\nu}$ and R (Ricci tensor and scalar) are related to the Riemann tensor $R^{\rho}_{\sigma\mu\nu}$ which is computed from the Christoffel symbols and their first derivatives. The Christoffel symbols are, in turn, constructed from the metric and its first derivatives.

[7] The way this is checked is by looking at the behaviour of scalars, which unlike components of a metric are coordinate-independent. If for example the Ricci scalar R approaches infinity at one point then that point is singular.

6 Black holes

The colloquial definition of a black hole describes it as an infinitely dense regions of space from which not even light can escape. While this is true, black holes are more complicated than that. A more precise description is to say that a black hole is a collection of events an outside observer can never see, where outside again refers to $r > 2GM$. This implies that they are not simply very dense regions of space but rather regions of spacetime.

Black holes are not just predicted by the theory of general relativity. Astrophysical observations have shown that black holes do exist in nature. For example, every major (spiral) galaxy seems to have a supermassive black hole in its center, including our own galaxy the Milky Way [14], and smaller black holes are formed in the collapse of massive stars.

6.1 Event horizon

The region in spacetime where $r = 2GM$ is called the event horizon of the black hole. The definition of an event horizon is that it is a surface where anything that has fallen within it can never escape.

To see why this is the case for black holes we can observe what happens with Eq. (47) when $r < 2GM$. For $r > 2GM$ the first term in Eq. (47) has a negative sign but for $r < 2GM$ the sign changes and

$$-\left(1 - \frac{2GM}{r}\right)dt^2 > 0.$$

The second term in the Eq. also changes sign for $r < 2GM$ and becomes negative

$$\left(1 - \frac{2GM}{r}\right)^{-1} dr^2 < 0.$$

The final two terms experience no change of sign,

$$r^2 d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) > 0.$$

Recalling from Section 5.2 that what distinguishes the time dimension from the spatial dimensions is the signs of their components in the metric, we notice that something odd happens for $r < 2GM$. In this region the radial component is the component with the negative sign and thus corresponds to the time dimension. In a way the radial direction and the time direction change places. This menas that within the event horizon moving forwards in time is the same as moving radially inwards towards the singularity at $r = 0$.

Although, in this region it is more accurate to consider the singularity a time rather than a position in space [15].

Therefore, the reason nothing can escape the event horizon is not simply due to an immense pull of gravity but rather a result of the curvature of spacetime inside the black hole. Spacetime inside the event horizon is distorted enough for there to simply be no outwards direction when moving forwards in time.

As we noted in Section 5.4.1, there is no singularity at the event horizon. But is there anything special about spacetime there at all? Some use the equivalence principle to argue that there is not. In the following section we explain why, when crossing the event horizon of a sufficiently large black hole, an observer will detect nothing out of the ordinary at the event horizon.

6.2 The equivalence principle

Classically, the equivalence principle states that the effects of gravity and acceleration on a body are indistinguishable. That is, there is no way to tell if you are accelerating or in a gravitational field. The same principle in general relativity concurs with this but adds a little exception, which is that the only effects an observer in free fall[8] can feel are those from variations in the gravitational field, so called tidal forces.

The scale of the tidal forces are determined by the Ricci tensor (curvature tensor) components. At the event horizon of a black hole the order of the components (called R) are

$$R \sim \frac{1}{(2MG)^2}$$

in the reference frame of an observer in free fall. This means that for a relatively small body in a strong gravitational field these tidal forces will be undetectable. An observer traveling past the event horizon of a large black hole[9] will not notice anything special about spacetime there, according to Susskind [15] amongst others.

6.3 Diagram representations of black holes

Black holes can be represented in spacetime diagrams in many different ways by using different sets of coordinates. For example, by studying the spacetime

[8] In general relativity an body in free fall has no force acting on it, since gravity is not a force but the curvature of spacetime. In free fall a body moves along a geodesic (the equivalent of a straight line in curved space).

[9] In this case a large black hole means a black hole with a radius much larger than even the length of the solar system [15].

diagram for black holes using the coordinates of Eq. (47) we can find out what an outside observer would see when looking at a black hole. This is done by observing how a ray of light (or anything else really) sent towards it would appear to the observer. For lightlike trajectories, also called null geodesics, θ and ϕ are constant, the line element ds^2 is zero and the slopes of the trajectories can be calculated as follows,

$$\begin{aligned} ds^2 = 0 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 \\ \Rightarrow \left(1-\frac{2GM}{r}\right)dt^2 = \left(1-\frac{2GM}{r}\right)^{-1}dr^2 \\ \Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}. \end{aligned} \tag{48}$$

From Eq. (48) we see that as $r \to 2GM$, i.e. the ray of light approaches the event horizon, the slopes approach infinity

$$\frac{dt}{dr} = \pm\infty.$$

In Figure (2) this is visualized. Far away from the black hole spacetime is the same as for Minkowski space and the light cones are at an angle of 45°. As the light approaches the event horizon the light cones become more narrow, making it seem as if the light never reaches the horizon. The same goes for anything else that might be sent into the black hole since the timelike path for any object is within the forward direction of the light cone. This is the perspective of an outside observer. An observer traveling with the infalling object would experience the situation differently, as the speed of the object falling into the black hole would increase and approach the speed of light as it nears the event horizon. However, the outside observer would see the object slow down and, due to the effects of gravitational time dilation[10], observe time to move slower for the object.

The light from the object approaching the event horizon would also become more and more redshifted. At the event horizon the light would be so redshifted as to never reach an outside observer, which is one reason why black holes are black.

The perspective of an outside observer will differ greatly from that of an observer crossing the event horizon. To show that light and other matter or

[10]Gravitational time dilation means that the weaker the gravitational potential the faster time passes. That is, the closer one is to a gravitational mass the slower time will pass.

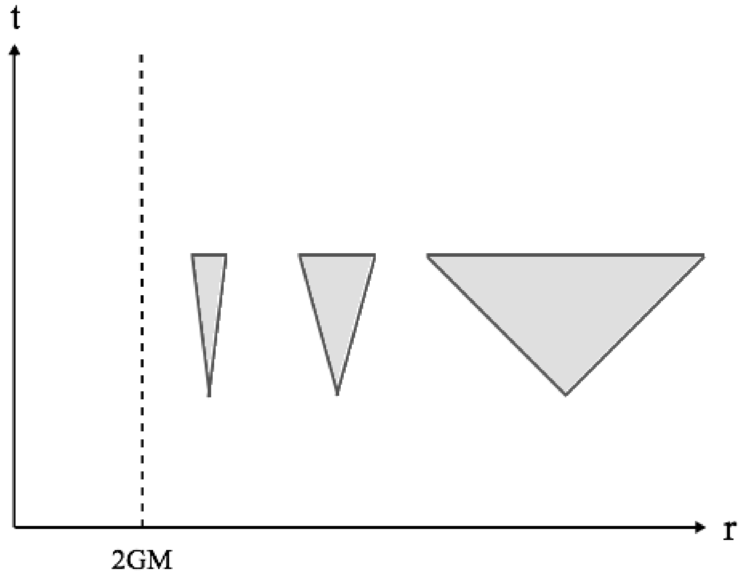

Figure 2: Spacetime diagram of a black hole, plotting the time t against the radial distance from the singularity r. $r = 2GM$ is the distance to the event horizon of the black hole.

energy actually passes the event horizon we only need to change the reference frame to one with coordinates that do not approach infinity as $r \to 2GM$. The coordinates we will use here are the Eddington-Finkelstein coordinates v and r, where v is the time coordinate and r is the standard radial coordinate. v is defined as

$$v = t + r^* \tag{49}$$

in terms of the tortoise coordinate r^*

$$r^* = r + 2GM \ln\Big(\frac{r}{2GM} - 1\Big). \tag{50}$$

The choice of these coordinates will be motivated in several steps, starting with the coordinate r^*. The benefit of using r^* as the radial coordinate is that it removes the problematic singularity at $r = 2GM$ from the metric. The metric in terms of t and r^* is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)dr^{*2} + r^2 d\Omega^2 \tag{51}$$

where $r = r(r^*)$. The negative aspect of using r^* is that at $r = 2GM$ $r^* \to -\infty$. If instead we use r^* in the time coordinate v and the radial coordinate r the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + (dvdr + drdv) + r^2 d\Omega^2. \tag{52}$$

For this metric there is no singularity at $r = 2GM$ and since we are using r again $r = 2GM$ is not infinitely far away. By showing that no coordinate approaches infinity at $r = 2GM$ we have proven that it is not a real singularity but only a coordinate singularity.

The light cones in these coordinates do close up but not at $r = 2GM$. This can be seen from calculating the slopes of the null geodesics.

$$\frac{dv}{dr} = \frac{d}{dr}(t + r^*) = \pm\left(1 - \frac{2GM}{r}\right)^{-1} + \left(1 - \frac{2GM}{r}\right)^{-1} \tag{53}$$

gives two values:

$$\frac{dv}{dr} = 0 \tag{54}$$

for infalling light rays and

$$\frac{dv}{dr} = 2\left(1 - \frac{2GM}{r}\right)^{-1} \tag{55}$$

for outgoing light rays.

Figure 3 shows the behaviour of the light cones in the coordinates (v, r). As we can see, the forward light cone at $r = 2GM$ is tilted towards $r = 0$ with no part of it being to the right of the event horizon. This shows that indeed, anything that has passed the event horizon cannot escape the black hole and will eventually reach the singularity. We also note that the light cones do not close up at the event horizon and therefore matter and energy are able to pass it.

6.4 No-hair theorem

One would expect black holes to be complicated objects to describe as they are created from huge amounts of matter and energy. Completely describing a black hole should include having to describe everything that fell into it to create it. However, black holes are in fact fairly easy to describe according to the no-hair theorem.

The no-hair theorem states that (stationary, asymptotically flat) black holes can be completely characterized by only a few parameters, which are mass, angular momentum, electric and magnetic charge [12]. This is a strange

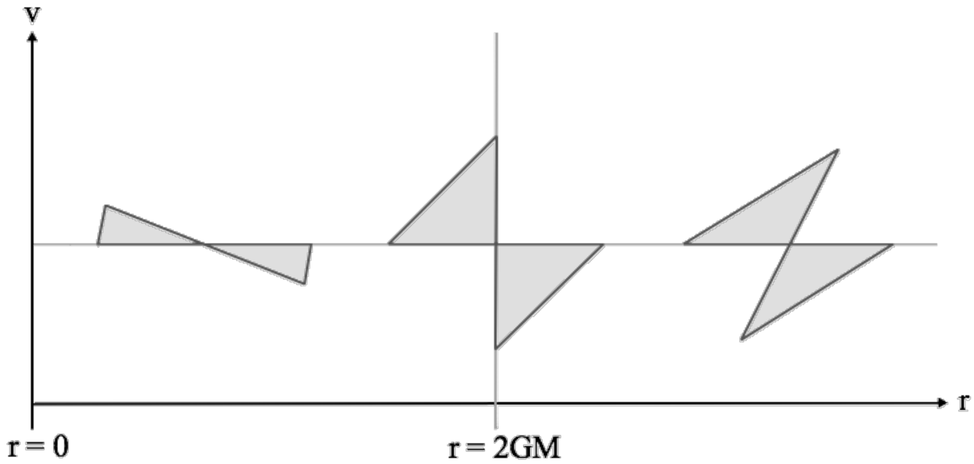

Figure 3: Light cones in a spacetime diagram of a black hole with the coordinates (v, r). $r = 2GM$ is the distance to the event horizon of the black hole.

feature since even most microscopic systems can be characterized by more features than that. Not to mention that macroscopic systems can be described using an extremely large amount of parameters. Take a planet for example, to completely describe the gravitational field outside it the entire structure and distribution of its mass would need to be known. Every mountain and trench would need to be measured and taken into account.

Black holes are created by massive stars collapsing and grow over time if they absorb matter and energy. The infalling matter and energy need many parameters to be described but when inside the black hole these parameters have no role in describing the black hole metric. Only the added mass, angular momentum and charge are relevant. The entirety of an uncharged black hole can be characterized solely by its mass and angular momentum, both of which can be measured from far away.

This means that from the perspective of classical general relativity a black hole has no internal microscopic structure. However, we shall see that when quantum mechanics becomes involved this leads to a few problems. When considering quantum effects in the vicinity of a black hole we find that a phenomenon known as Hawking radiation takes place.

6.5 Hawking radiation

Hawking radiation occurs when so called virtual particles are created in the close vicinity of the event horizon of a black hole. [12] and [13] both have good descriptions of this phenomenon.

Classically, vacuum is thought of as completely empty. In quantum field theory this is not the case. Instead there are so called vacuum fluctuations due to the fields that permeate space. These fluctuations are the rapid creation and annihilation of particles in pairs of one particle and one antiparticle. These particles are called virtual due to their very short existence as they recombine and annihilate almost immediately after they are formed. For example, fluctuations in the electromagnetic field produce pairs of virtual photons, as photons are their own antiparticles.

Particles spontaneously appearing and disappearing might seem to violate the conservation of energy but as long as they exist for a short enough time there is no problem. One may consider them in terms of Heisenbergs uncertainty principle which (in terms of energy and time) is

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

where ΔE is the energy uncertainty and Δt the time uncertainty. For these virtual particles ΔE is the energy violation (or total energy of the virtual particles) and Δt the time they can exist.

Virtual particles may form near the event horizon of a black hole. In order for them to become real particles, and not annihilate immediately, they can use/borrow energy from the black hole. Another way to formulate it is to say that one of the two particles created has a negative energy. If the particle pair is created just outside the event horizon, one of them (the one with the negative energy) may enter the black hole while the other one escapes to infinity. This radiation of particles away from the black hole is called Hawking radiation. Through Hawking radiation the black hole looses energy since half of the energy used to created the particles escapes with one of the particles. Or, if one prefers the negative-energy formulation, the black hole looses energy since it gains a particle with negative energy.[11]

Via this process the black hole looses energy over time unless matter or energy is absorbed by it. Thus, over time the black hole evaporates and

[11] Hawking [16], and others, emphasizes that this mechanism of particle creation is a simplification of the process responsible for the thermal emission of black holes. However, this explanation highlights the quantum mechanical effects of the Hawking radiation as well as is useful in explaining of the information paradox. That is why this explanation works well for the purposes of this report.

disappears. The energy required to create the particles is very small and for a black hole to completely evaporate may take an enormous amount of time. The lifetime τ of the black hole is

$$\tau \sim M^3$$

where M is the mass of the black hole. The lifetime of a black hole with the mass of the sun ($M \approx 10^{30}$ kg) is then of the order

$$\tau \approx 10^{71} \text{ s}$$

which is 10^{53} times longer than the age of the universe [12].

An important property of black holes is the entropy. The entropy, known as Bekenstein or Bekenstein-Hawking entropy, of a black hole is

$$S = \frac{A}{4G} \tag{56}$$

in natural units [12], where A is the surface area of the black hole [17] [16].

The fact that black holes have entropy is rather important since if they did not it would lead to a violation of the second law of thermodynamics. The second law of thermodynamics states that the total entropy of an isolated system must increase over time. If black holes did not have entropy then sending matter into them would reduce the over all entropy since all the entropy of the infalling matter would disappear.

The existence of black hole entropy means that black holes also have temperatures. A body with a temperature emits thermal radiation and according to Hawking the thermal radiation from a black hole is the Hawking radiation. Black holes act as black bodies in thermal equilibrium and for an observer far away from a Schwarzschild black hole the temperature T is in natural units

$$T = \frac{1}{8\pi GM}. \tag{57}$$

From Eq. (57) we see that as the mass of the black hole decreases the temperature increases. At low temperatures mostly massless particles, such as photons and gluons, are radiated away but when the temperature increases massive particles such as electrons or muons are also created and emitted. When a black hole becomes small enough, with a mass of about 10^{14} g, its temperature is so high that it can radiate away all kinds of massive particles and does so in a very short time. The black hole then disappears by creating an explosion of particles [16].

[12] In natural units $c = \hbar = k_B = 1$, where k_B is the Boltzmann constant and c the speed of light in vacuum.

A solar mass black hole mainly emits massless particles and has a temperature of about

$$T \approx 10^{-7}\ \mathrm{K}$$

which is very low. That the Hawking radiation is thermal means that the radiation is in a mixed state, as thermal radiation cannot carry all the information that went into the black hole.

One consequence of Hawking radiation is that the inside of the black hole becomes entangled with the outside (the Hawking radiation). This occurs since the particle pairs that are created near the event horizon are entangled. They are entangled because they are created from vacuum where all quantum numbers are zero. Thus the combined quantum numbers of the particles must be zero [18] [13].

7 The information paradox

The phenomenon of Hawking radiation is what gives rise to the black hole information paradox. The information paradox can be stated in a numbers of ways, each highlighting a contradiction between quantum mechanics and general relativity. When Hawking first presented the idea of Hawking radiation [16] and later claimed that this lead to information being lost in the black hole [20] he formulated the information paradox. The paradox really revolves around the question "What happens to information that enters a black hole?". Does the information disappear, is it sent out, or does something else occur? As we shall see, both the disappearance of the information and it being sent out in the form of radiation seem to lead to violations of either quantum mechanics or general relativity.

One way of stating the paradox is to focus on the time evolution of an entangled pair of particles. An entangled pair of particles are created just outside the event horizon of a black hole, see Figure 4. The particles are created from vacuum and their entangled state is completely known, i.e. the pair constitutes a pure state. One of the particles (A) enters the black hole while the other (B) escapes and travels away from it, as Hawking radiation. We now make the assumption that information is lost when it enters the black hole. When A enters the black hole the information contained in it is destroyed. As A and B were entangled we no longer have a pure state when only B is left. The state of B alone is mixed as the following calculations will show.

[13] This is the case for a static black hole but if the black hole has angular momentum the particles can be created with some angular momentum which is then carried away with them [19], the particles are still entangled however.

We take the entangled particles to be in the singlet state[14]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)$$

which is a pure state. The system of the entangled particles can be described by the reduced density matrix ρ_{AB}

$$\rho_{AB} = |\psi\rangle \langle\psi| . \tag{58}$$

ρ_{AB} is then calculated.

$$\begin{aligned}\rho_{AB} &= \frac{1}{2}(|\uparrow\rangle_A |\downarrow\rangle_B \langle\uparrow|_A \langle\downarrow|_B - |\uparrow\rangle_A |\downarrow\rangle_B \langle\downarrow|_A \langle\uparrow|_B \\ &- |\downarrow\rangle_A |\uparrow\rangle_B \langle\uparrow|_A \langle\downarrow|_B + |\downarrow\rangle_A |\uparrow\rangle_B \langle\downarrow|_A \langle\uparrow|_B) \\ &= \frac{1}{2}(|\uparrow\rangle_A \langle\uparrow|_A |\downarrow\rangle_B \langle\downarrow|_B - |\uparrow\rangle_A \langle\downarrow|_A |\downarrow\rangle_B \langle\uparrow|_B \\ &- |\downarrow\rangle_A \langle\uparrow|_A |\uparrow\rangle_B \langle\downarrow|_B + |\downarrow\rangle_A \langle\downarrow|_A |\uparrow\rangle_B \langle\uparrow|_B) \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

By diagonalizing ρ_{AB} we can verify that it is a pure state. ρ_{AB} is diagonalized by calculating its eigenvalues λ_i via $\det(\rho_{AB} - \lambda\mathbb{I})$. The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$ giving the matrix ρ_{AB}^{diag}.

$$\rho_{AB}^{diag} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From ρ_{AB}^{diag} we confirm that the state is pure since the only non-zero element is a one on the diagonal. This also means that the von Neumann entropy is zero, which it should be for a pure state. Eq. (39) gives

$$S(\rho_{AB}^{diag}) = -(1\ln(1)) = 0.$$

Individually A and B should be in mixed states. We can verify this using $S(\rho_A), S(\rho_B) > 0$, or by diagonalizing the matrices ρ_A and ρ_B. To

[14] Here we consider entangled spin-$\frac{1}{2}$ particles but Hawking radiation is mostly photons. As previously mentioned (in Section 3.1), studying a system of two entangled spin-$\frac{1}{2}$ particles is no different compared to studying a system of two entangled polarized photons.

find ρ_A we take the partial trace over B in the equation for ρ_{AB} (using $\mathrm{Tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$).

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(\rho_{AB}) \\ &= \frac{1}{2}(|\uparrow\rangle_A \langle\uparrow|_A \langle\downarrow|\downarrow\rangle_B - |\uparrow\rangle_A \langle\downarrow|_A \langle\uparrow|\downarrow\rangle_B \\ &- |\downarrow\rangle_A \langle\uparrow|_A \langle\downarrow|\uparrow\rangle_B + |\downarrow\rangle_A \langle\downarrow|_A \langle\uparrow|\uparrow\rangle_B) \\ &= \frac{1}{2}(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A) \\ &= \frac{1}{2}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\end{aligned}$$

ρ_A is diagonal. If we calculate ρ_B we find that $\rho_A = \rho_B$. We now calculate $S(\rho_A) = S(\rho_B)$.

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \ln(\rho_A)) = -\left(\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{1}{2}\right)\right) = \ln(2) \approx 0.693 > 0$$

Since $S(\rho_A) = S(\rho_B) > 0$ both A and B are in mixed states. Therefore B, if measured alone, is indeed in a mixed state.

Over time the black hole evaporates completely and all that is left is the Hawking radiation, which is an ensemble of particles in the mixed state B. The pure state of A and B has evolved into a mixed state. This contradicts the unitary time evolution of quantum mechanics (see Section 2.2). According to unitary evolution a pure state will not evolve into a mixed state. Once a pure state is known the time evolution operator will be able to describe its future evolution.

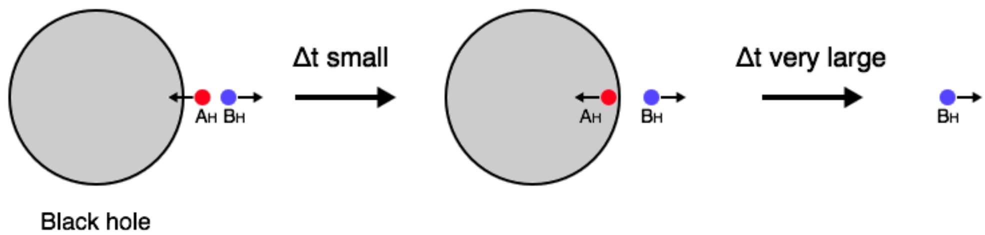

Figure 4: The Hawking particles A_H and B_H are entangled. A_H enters a black hole, where the information is assumed to be lost. The black hole evaporates. The pure state of A_H and B_H is turned to the mixed state of B_H alone, violating unitary time evolution.

This way of stating the information paradox is based on the assumption that information is forever lost if it enters the black hole. Thus a pure state is turned into a mixed state when the black hole disappears. The conclusion is that this violates the unitary time evolution of quantum mechanics.

However, there are other formulations of the paradox that focus on the violation of other aspects of quantum mechanics or general relativity.

We change the assumption that information is lost inside a black hole and instead assume that infalling information is re-emitted in the Hawking radiation. A consequence of this is a violation of general relativity. The central concept here is the equivalence principle, explained in Section 6.2. A consequence of the equivalence principle is that an observer in free fall crossing the event horizon, of a sufficiently large black hole, does not experience anything strange. There is no detectable surface there that might function as a medium to absorbs and re-emit the infalling information [15]. Information cannot escape the black hole once it has passed the event horizon, per the definition of event horizon, but as there is no surface at the horizon neither should anything happen to whatever passes it. Therefore, if the information crossing the event horizon is re-emitted, presumably by some sort of surface at the horizon, the equivalence principle is violated.

8 Proposed solutions

The previous section showed that there are several ways of stating the information paradox. However, finding a solution to it has proven difficult. Over the years many solutions have been proposed and the paradox is still an active field of research to this day.

It is important to remember that despite there being strong evidence for the existence of black holes Hawking radiation from them has yet to be observed. As previously mentioned, observations of black holes have been made. Hawking radiation has a very low intensity, which is why it is difficult to detect it from far away. However, studying black holes up close is also very difficult, for obvious reasons. The black holes we know of are far away and even if there was one right around the corner, conducting experiments just outside the event horizon of it would be incredibly difficult. Perhaps Hawking radiation could be observed from a safe distance but as of yet the concept remains theoretical.

Another important thing to remember is that black holes themselves are strange and cannot be described completely by general relativity. The existence of a singularity at the center of a black hole is difficult to deal with and indicates that the GR description is incomplete. In dealing both with black holes and potential solutions to the information paradox one may find oneself working with energies at the Planck scale or even higher. At these energies the known laws of physics seem to break down. That is one reason why solving the information paradox is an active and somewhat speculative field of research. Describing physics at Planck scale energies will probably require a new physical theory and even if solving the information paradox does not rely on these high energies it is likely that the solution will require an entirely new theory as well. This theory would be a theory of quantum gravity, which would agree with both quantum mechanics and general relativity. One thing physicists seem to agree upon when it comes to this is that, as of yet, a fully working theory of quantum gravity has not been found. Although, since Hawking first postulated Hawking radiation in 1974 [16] there has been plenty of work done on the subject.

We will now explore some of that work in the form of a few of the proposed solutions to the information paradox. There are three main alternatives as to what happens to matter thrown into a black hole.

1. The information is lost (destroyed at the singularity).
2. The information escapes with the Hawking radiation.
3. The information is stored in remnants (i.e. the black hole does not evaporate completely).

Alternative 1 means that there are some who do not even consider the information paradox a paradox and simply accept that information seems to be lost inside a black hole [21]. However, most of the solutions try to avoid information loss and do not consider the storing of information in remnants as a good way of doing so. We therefore focus on option two in the following sections. The first solution we will discuss is called **black hole complementarity** which was proposed by Susskind, Thorlacius and Uglum [22] in 1993. After that **firewalls** will be discussed, as the idea of a firewall came from critique against black hole complementarity [23].

8.1 Black hole complementarity

Black hole complementarity (BHC) solves the information paradox by suggesting that black holes have a surface membrane, a so called stretched horizon [22], that absorbs and re-emits the infalling information. This membrane only exists from the perspective of an outside observer far away from the black hole. BHC uses a complementary approach to explaining the contradictory experiences of different observers. To explain the viewpoint of BHC we will start by exactly stating the postulates used in [22].

- **Postulate 1:** The process of formation and evaporation of a black hole, as viewed by a distant observer, can be described entirely within the context of standard quantum theory. In particular, there exists a unitary S-matrix which describes the evolution from infalling matter to outgoing Hawking-like radiation.
- **Postulate 2:** Outside the stretched horizon of a massive black hole, physics can be described to good approximation by a set of semi-classical field equations.
- **Postulate 3:** To a distant observer, a black hole appears to be a quantum system with discrete energy levels. The dimension of the subspace of states describing a black hole of mass M is the exponential of the Bekenstein entropy $S(M)$.

Postulate 1 implies that information is not lost in the black hole but instead emitted in the form of Hawking radiation. Postulate 2, also known as effective field theory, states that there is nothing unexpected about the physics outside the stretched horizon of the black hole. The outside can be described by semiclassical gravity. The third postulate emphasizes that to a distant observer a black hole appears to be a normal quantum mechanical system with a finite number of states. Apart from these postulates BHC includes a crucial assumption which is that an observer in free fall crossing the event horizon does not experience anything strange, according to the equivalence principle stated in Section 6.2. But how can these postulates be unified in a solution of the information paradox when they are the very same postulates that we in Section 7 saw cause the paradox in various ways? It has to do with the perspectives of different observers and the properties of the stretched horizon of a black hole.

The key point of BHC is that it claims that even though the perspectives of several observers may differ, when looking at one perspective at a time no physical laws are violated. The concept is usually explained by studying the perspectives of two observers. Bob, one of the observers, is far away from the black hole and he is observing Alice, who is in free fall and about to enter the black hole.

Consider the perspective of Alice when she is crossing the event horizon. The black hole considered in this case is large enough so that when Alice has crossed the event horizon she is still a very long time away from the singularity. Alice is in free fall as she approaches the event horizon, i.e. no other "forces" than the gravity from the black hole is affecting her. According to the equivalence principle Alice will not experience anything special when she crosses the horizon, in fact she will not notice it at all. If the black hole is very large, which we assumed, any tidal forces she could experience are entirely negligible. Alice cannot see any radiation, with the information she contained when crossing encoded in it, go out either because if it did it would violate causality.

On the other hand, the perspective of Bob is completely different. As explained in Section 6.3 an external observer will not see anything entering the black hole. The closer an object gets to the event horizon the slower it will appear to move to the observer, as per Eq. (48). The light from the object will also become more and more redshifted as the object approaches the event horizon. Thus, Bob will see Alice approach the horizon and as she does Bob will see her move slower and slower. Eventually she will appear to be completely still and hover above the horizon. Due to the effects of special relativity, i.e. length contraction, Alice will also appear to be flattened in the direction of motion, which is towards the black hole.

But as Bob cannot see anything fall into the black hole should he not see everything that has "never quite" fallen into it? All the material that has ever fallen into the black hole should be visible to an outside observer. If the material is there so should its properties be, for example electric and magnetic fields. Thus, to Bob there seems to be a sort of membrane, consisting of all the infalling matter, around the black hole. This membrane has certain properties which include electric conductivity, viscosity, temperature and entropy. The Hawking temperature discussed in Section 6.5 is only the temperature measured by an observer far away from the black hole. At extremely close distances to the event horizon, of the size of an atomic nucleus, the temperature is extremely high. So Bob, when he is far away will measure the very low Hawking temperature but if he were to somehow manage to have a thermometer just outside the black hole he would measure an extremely high temperature[15]. When Bob sees Alice slow down outside the horizon he looses track of her since he observes an entire membrane with very high temperature. Bob therefore concludes that Alice was vaporized and absorbed by the membrane [24].

This decidedly gives Alice and Bob two very different experiences of the event horizon. According to Alice, she passes the event horizon without noticing anything special while to Bob she never passes it but rather becomes part of a physical membrane surrounding the black hole.

In accordance with postulate 1 proponents of BHC believe that information falling into the black hole is later emitted in the form of thermal

[15]Why the thermometer and Alice do not measure the same temperature is due to their reference frames. Alice is in free fall but the thermometer, which must remain outside the horizon, is accelerated by forces keeping it from crossing the horizon.

radiation. The information Alice contained is therefore emitted by the black hole a time after she was absorbed by the horizon. By assuming that infalling information is sent out no violation of unitarity occurs, which resolves one of the main problems stated in Section 7. But simply making this assumption does not explain the problems it leads to. One of these problems is that a violation of the no-cloning theorem seems to occur, since Alice and her Hawking radiation copy both exist. This is where the complementarity comes in.

When discussing the problem of cloning the entire system of the black hole and the space outside it is considered. It is in this system that the problems of the paradox arises. By studying the perspectives of Alice and Bob we have seen that they disagree on whether information enters a black hole or is absorbed and remitted. BHC suggests that both of these perspectives are correct and even complementary. From the perspective of each individual observer both the unitarity time evolution and the no-cloning theorem hold.

Bob, being outside the black hole, would not observe quantum cloning because he can only see the infalling matter and outgoing radiation. There is no experiment Bob can conduct which would show that Alice survived crossing the event horizon since measuring anything past the event horizon is per definition impossible for him. Even if Alice decided to help by continuously sending data as she approaches and subsequently crosses the horizon it would be of no use. The data Alice sends would become more and more redshifted as she approaches the horizon. Any data that she could sent from just outside the horizon would have to have Planck-scale energies to be useful to Bob [25].

Alice will cross the horizon with all the information she contains and believe that all the information is conserved. If the information is copied and emitted from the membrane (a membrane she cannot see) she has no way of accessing the copy and will therefore see no quantum cloning.

We can modify the experiment a bit by having Alice and Bob observe a pair of entangled particles, one of which enters the black hole and one which escapes, whilst both being outside the black hole, see Figure 5. Alice then follows the particle which enters the black hole and measures it once inside the event horizon. Bob stays outside and measures the other particle and the Hawking radiation emitted due to Alice's particle crossing the horizon. This radiation is a copy of the information Alice's particle contains. After he has measured the Hawking radiation relating to Alice's particle (which may take some time) Bob enters the black hole where Alice sends him the data she measured before he hits the singularity. Thus Bob is able to detect two copies of the same data (Alice's particle and the Hawking radiation) and observe a violation of the no-cloning theorem. Bob should also find a violation of the monogamy of entanglement since both Alice's particle and the Hawking radiation should be entangled to Bob's particle outside the black hole. However, as we will explain below, according to [25] this experiment would only work if, again, energies of the Planck-scale or beyond were used to transmit the data.

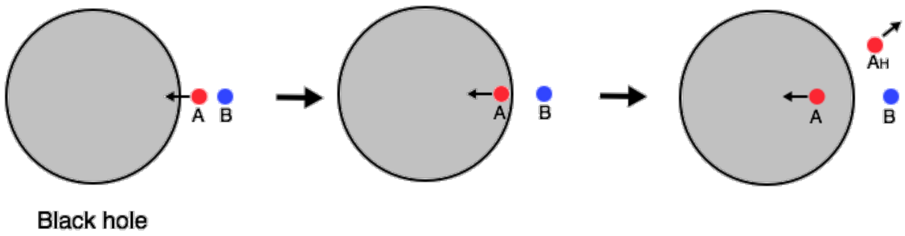

Figure 5: Particles A and B are entangled. A enters the black hole and B stays outside. Hawking radiation A_H is after some time sent out due to A entering the black hole. B is entangled to both A and A_H. Alice inside the black hole measures A and Bob outside the black hole measures both B and A_H. Bob then jumps in to get the data from Alice, thus collecting data on A, B and A_H which leads to him discovering a violation of quantum mechanics.

This is because the time Bob will have to wait outside the horizon to collect the data from the infalling particle is very long. Postulate 1 says that the entire life of a black hole can be described using the unitary time evolution operator. This means that if we form a black hole that is in a pure state and it evaporates over time the entire system, consisting of the black hole and the Hawking radiation it emits, will always be in a pure state. This does not contradict the notion that individual emitted particles of Hawking radiations are in mixed states, which they are due to their entanglement to their partner particles in the black hole. For the radiation this means that radiation emitted early in a black hole's lifetime will be entangled to radiation emitted later.

More precisely, when a black hole has emitted radiation corresponding to half its initial entropy, which happens at the so called Page time, it will start emitting radiation that is entangled to the early radiation. Therefore for an observer to be able to get information thrown into a black hole the observer has to detect both the early radiation and some of the late radiation as well, i.e. the complete system of the two entangled particles[16]. As the lifetime

[16] [26] and [27] discuss this early/late radiation entanglement.

of a black hole is very long collecting the data will take some time, at least if the entangled particles are thrown into the black hole before its Page time. Therefore Bob does not have enough time to collect both the data from outside and inside the black hole. He could only receive the data from Alice in time if she were to send it using super-Planckian frequencies. Alice's data has a limited lifetime since both she and her data are approaching the singularity where they will be destroyed.

The point is that no observer can describe both what happens inside and outside the event horizon without using Planck-scale physics, something we know next to nothing about. It is simply physically impossible much in the same way as it is impossible to simultaneously know the position and momentum of a particle. The data from measurements inside and outside the black hole can be seen as complementary: knowing one with certainty means knowing nothing about the other. This does not meant that there are two copies of the same information but instead that different observers will find the information at different events in spacetime.

In sacrificing the notion that when and where something occurs is independent of observers BHC introduces more uncertainty to physics than what already exists, due to for example the wave-particle duality of matter and the Heisenberg uncertainty principle. When and where Alice meets her doom, whether it be when she encounters the membrane or the singularity, is up to the observer.

8.2 Firewalls

A relatively recent contribution to the discussion of the information paradox is the concept of firewalls[17]. In an article published in 2012, Almheiri, Marolf, Polchinski and Sully (AMPS) [23] present the firewall as a potential solution to the information paradox as they claim to have found inconsistencies in BHC. A similar idea was proposed by Mathur [28] before AMPS proposed their firewall, but in this report we focus on the AMPS solution.

Briefly stated, the firewall solution suggests that around a black hole there is a surface of very energetic quanta which will destroy anything that tries to get past the event horizon. AMPS argue for the existence of this firewall mainly based on their claim that BHC is incomplete, which they show via a thought experiment. Let us examine the argument in [23].

[17]The reader will note that the sources for this section are not all peer-reviewed, for example some are blogs (written by physicists). Due to firewalls being a topic currently under much discussion (and somewhat unclear and difficult to understand) the sources were chosen based on which best explained the concept and controversy of firewalls.

If we assume that postulate 2 from Section 8.1 and the equivalence principle still are valid, we find a contradiction that BHC cannot explain. Consider a black hole past its Page time and a pair of entangled Hawking particles A and B. As per usual, A enters the black hole while B escapes to infinity. As B is part of the late radiation it is entangled to a particle C from the early radiation, see Figure 6. However, B cannot be entangled to both A and C according to the monogamy of entanglement.

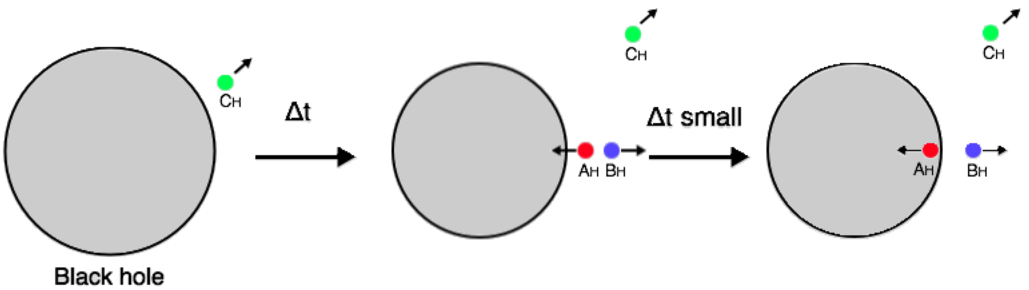

Figure 6: Particle C_H is part of the early radiation emitted from a black hole. Later, after the Page time, the entangled particles A_H and B_H are created. A_H enters the black hole while B_H escapes. As B_H is part of the late radiation it is entangled to C_H. A contradiction arises as according to the monogamy of entanglement B_H cannot be entangled to both A_H and C_H.

AMPS argue that it is possible for a single observer to measure all three particles and therefore find a violation of the monogamy of entanglement. An observer that has measured the early radiation C can enter the black hole and on the way pass the late radiation B. Once inside the event horizon the observer then measures A which is entangled to B. The observer thus sees both C and A be entangled to B [29].

To avoid this problem AMPS suggests that the infalling observer, who has found C and B to be entangled cannot also find B to be entangled to A. If A and B are not entangled the state the observer experiences at the horizon is not the vacuum state. Instead, the observer finds a firewall that consists of high energy particles and destroys everything that enters the black hole.

Not a lot is known about the details of this firewall but the following is mentioned in [30]. The exact form of this high energy barrier is not determined, the firewall could consists of high-energy photons but other processes are also possible. Exactly how the firewall is created is unclear, although a feature of it is that it has enough energy to break the entanglement between the Hawking-particles A and B, which is a huge amount of energy. As to where the firewall is located, right behind the event horizon is the current proposed location which means that an outside observer will not be able to see it, thus not changing the external properties of a black hole. Even when the firewall should form is under debate, with one of the suggestions being that it forms at the Page-time.

The firewall and the stretched horizon of BHC might seem similar but there is a difference. The stretched horizon does appear to have a very high temperature for the right observer, i.e. an observer that is just outside the horizon but in an accelerated reference frame which keeps the observer from crossing the horizon. The firewall is a surface that is behind the horizon for all observers, even though only observers that cross the horizon are able to see it. Therefore the existence of the firewall should be independent of the observer, unlike the stretched horizon whose existence relies on the observer. The main difference is that an observer may pass through the stretched horizon but not the firewall [30].

The details of this argument are subtle and difficult to interpret. In general there is much that is unclear about the firewall proposal since the authors of the original article do not propose an exact process for the creation of the firewall or many other details about its existence.

However, the main controversy of the firewall argument is that it is suggests a not at all subtle violation of the equivalence principle. To some physicists the idea that a physical barrier should appear around a black hole is too strange to accept. The authors of [23] are willing to accept a violation of the equivalence principle as they find that discarding one of the other postulates would lead to even stranger consequences. They are, like many others, reluctant to dismissing unitarity (postulate 1) since it is one of the fundamental properties of quantum mechanics. Due to this, the alternative they pose to having a firewall is the existence of new dynamics stretching out from the horizon. These new dynamics would reach macroscopic distances from the horizon and violate effective field theory (postulate 2). Based on this AMPS conclude that a firewall may be the least radical solution, violating only the equivalence principle.

9 Conclusion

As we have seen the information paradox involves several phenomena and theorems from both quantum mechanics and general relativity. Without going into the details of quantum field theory and general relativity, the key concepts used are entanglement, unitarity, black holes, the equivalence principle and Hawking radiation.

The information paradox, although stated over 40 years ago is very much still an unsolved problem. The study of the paradox has led to many different hypotheses and ideas as to what happens in and near a black hole. Hawking's initial stance, made in 1976, was that information must be lost in the process of a black hole evaporating but in 2004 his opinion changed to favour unitarity instead. Others have also changed their opinions, such as Susskind, one of the proponents of BHC who believes that wormholes may hold the solution [31].

The two solutions mentioned here have both been criticized and are in some ways controversial. BHC claims to preserve unitarity via a property of the stretched horizon to emit the radiation after a time but has not proposed a specific mechanism for this process. BHC also introduces a brand new uncertainty into physics as it gives up the notion that when and where in spacetime something occurs is independent of the observer. Firewalls on the other hand, being a very new hypotesis is in a way more controversial as it proposes a violation of the equivalence principle. The firewall argument is also lacking a precise process by which this firewall is formed and maintained, and as we mentioned even when it forms is under debate.

Beside BHC and firewalls there are many more proposed solutions to the information paradox. Describing these additional solutions are beyond the scope of this thesis. Still, there is no clear consensus as to how the information paradox will be solved exactly, it may be from one new solution or even a combination of several already proposed ideas. What physicists do seems to agree upon is that whatever the solution is it will probably involve new insights into quantum gravity and perhaps even novel physics.

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