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Bimodules over dual numbers

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Abstract

Let D be the dual numbers $D = \mathbb{C}[X]/(X^2)$. In this document the indecomposable finite deimensional D-D-bimodules are described: except for $D \otimes_{\mathbb{C}} D$, any indecomposable D-D-bimodule belongs to either one of four families of so called string bimodules, or the family of band bimodules. The main part of this document is then dedicated to calculating the multiplication table of these bimodules with respect to the tensor product over D. This can be understood as a description of the $\mathbb{Z}$-algebra $[D\text{-mod-}D]_{\oplus}$, i.e. the split Grothendieck ring. It is spanned by the set of isomorphism classes of indecomposable D-D-bimodules and has $\otimes_D$ as its multiplication. The tensor product over D is not commutative, but the band bimodules span a maximal commutative subring of the split Groethendieck ring. Finally, with left, right, and two-sided cells defined as in [4] , the cell structure is examined. The two-sided cells will turn out to be linearly ordered, whereas not all one-sided cells are related.

Acknowledgements

First and foremost I would like to thank my supervisor Volodymyr Mazorchuk for entrusting me with this interesting problem and guiding me whenever I needed. I am deeply grateful for his enthusiasm, patience and time. I also want to thank my friend Elin Persson Westin for our many rewarding discussions.

Contents

1 Introduction

When we want to describe an algebra A, a natural approach is to choose a basis $\underline{a}$ of A and give the multiplication table for the basis elements. This choice of basis is usually in some sense arbitrary. However, some algebras are equipped with a canonical basis. One main example is the group algebra KG of a group G, with basis being the group elements and multiplication as in the group. Another example is a certain $\mathbb{Z}$-algebra induced by A-mod-A - the category of finite dimensional bimodules over an algebra A. It is called the split Grothendieck ring and will be defined below. In this document we describe the basis of this algebra in the case where A is the dual numbers $D = \mathbb{C}[X]/(X^2)$, and give the multiplication table.

Definition. Let $\mathcal{C}$ be a category.

a) A *skeleton* $\underline{\mathcal{C}}$ of $\mathcal{C}$ is a full subcategory such that any object in $\mathcal{C}$ is isomorphic to some object in $\underline{\mathcal{C}}$, and any two isomorphic objects in $\underline{\mathcal{C}}$ are equal.

b) $\mathcal{C}$ is *essentially small* if it has a small skeleton $\underline{\mathcal{C}}$, i.e. if the objects of $\underline{\mathcal{C}}$ form a set, and each $\mathrm{Hom}_{\underline{\mathcal{C}}}(X, Y)$ is a set.

c) $\mathcal{C}$ is *additive* if all its Hom-sets are abelian groups, composition of morphisms is bilinear, and $\mathcal{C}$ admits all finitary biproducts.

Given a K-algebra A, the category A-mod-A of finite-dimensional A-A-bimodules is essentially small. It is also additive with biproduct direct sum. In general, given an essentially small additive category $\mathcal{C}$ with skeleton $\underline{\mathcal{C}}$, define an equivalence relation $\sim$ on $\underline{\mathcal{C}}$ by $[X] \sim [Y] + [Z]$ if $X \simeq Y \oplus Z$. Then the *split Grothendieck group* $[\mathcal{C}]_\oplus$ of $\mathcal{C}$ is defined as the free abelian group

$$[\mathcal{C}]_\oplus = \langle [X] \mid [X] \in \underline{\mathcal{C}} \rangle / \sim .$$

The split Grothendieck group $[A\text{-mod-}A]_\oplus$ has basis consisting of isomorphism classes of indecomposable A-A-bimodules. Being an abelian group, it is a $\mathbb{Z}$-module. Not only that: it is an algebra over $\mathbb{Z}$, due to the following lemma.

Lemma 1. *The multiplication $\otimes_A$ turns the abelian group $[A\text{-mod-}A]_\oplus$ into a ring.*

Proof. The regular bimodule ${}_AA_A$ serves as identity with respect to the tensor product $\otimes_A$:

$${}_AA_A \otimes_A {}_AM_A \simeq {}_AM_A \simeq {}_AM_A \otimes_A {}_AA_A$$

for any $M \in A$-mod-A. Tensor products are always associative, as well as distributive over direct sums. Since $[A\text{-mod-}A]_\oplus$ is an abelian group under direct summation, this concludes the proof. □

We call $[A\text{-mod-}A]_\oplus$ equipped with $\otimes_A$ the *split Grothendieck ring*. For the $\mathbb{Z}$-multiplication, we shall write $M^{\oplus k}$ instead of kM for $k \in \mathbb{Z}$ and $M \in [A\text{-mod-}A]_\oplus$. Let us begin by a basic example.

Example. Consider $\mathbb{C}$ as a $\mathbb{C}$-algebra. Since $\mathbb{C}$-$\mathbb{C}$-bimodules are just $\mathbb{C}$-modules, the split Grothendieck group $[\mathbb{C}\text{-mod-}\mathbb{C}]_\oplus$ is one-dimensional, spanned by ${}_\mathbb{C}\mathbb{C}_\mathbb{C}$. As a group and ring, this is isomorphic to $\mathbb{Z}$.

Already as we proceed to the category of bimodules over a two-dimensional complex algebra the split Grothendieck ring is more interesting. Consider the *dual numbers* $D := \mathbb{C}[X]/(X^2)$ and the category D-mod-D. The main goal of this document is to describe $[\mathrm{D\text{-}mod\text{-}D}]_\oplus$ in the sense that we describe its canonical basis, i.e. classify the indecomposable D-D-bimodules, and give the multiplication table with respect to $\otimes_\mathrm{D}$. We shall also investigate the cell combinatorics in the sense of [4].

2 Indecomposable D-D-bimodules

The first aim of this section is to classify the indecomposable D-D-bimodules. For D-modules, the problem is not at all difficult: 1 always acts as the identity, and X must square to 0. Since we work over $\mathbb{C}$ we can write the matrix of X in Jordan normal form, and the only Jordan blocks that square to 0 are $[0]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Consequently the only indecomposable D-modules are ${}_\mathrm{D}\mathbb{C}$ and ${}_\mathrm{D}\mathrm{D}$. For D-D-bimodules, the solution to the classification problem is more complicated. It will be presented in 2.1. Once the classification is done, we prepare to start calculating tensor products.

2.1 Classification

The category D-mod-D is equivalent to $D \otimes_\mathbb{C} D^{\mathrm{op}}$-mod. Since D is commutative, $D \simeq D^{\mathrm{op}}$, so D-mod-D and $D \otimes_\mathbb{C} D$-mod are equivalent as categories. In the following we omit the $\mathbb{C}$ and write only $D \otimes D$. Let us examine this algebra.

D has basis $\{1, X\}$, so $D \otimes D$ has basis $\{1 \otimes 1,\ 1 \otimes X,\ X \otimes 1,\ X \otimes X\}$. The radical $\mathrm{Rad}(D \otimes D)$ can be characterised as the unique nil ideal I of $D \otimes D$ such that $D \otimes D/I$ is semisimple. From this it is clear that $\mathrm{Rad}(D \otimes D) = (1 \otimes X,\ X \otimes 1,\ X \otimes X)$. The socle $\mathrm{soc}(D \otimes D)$ can in turn be characterised as the annihilator of the radical, so $\mathrm{soc}(D \otimes D) = (X \otimes X)$. Furthermore, the algebra $D \otimes D$ has no nontrivial idempotents: assume that $a(1 \otimes 1) + b(1 \otimes X) + c(X \otimes 1) + d(X \otimes X)$ is idempotent. This element has square

$$a^2(1 \otimes 1) + 2ab(1 \otimes X) + 2ac(X \otimes 1) + 2(ad + bc)(X \otimes X)$$

so the idempotency yields the system of equations

$$\begin{cases} a^2 = a \\ 2ab = b \\ 2ac = c \\ 2(ad+bc) = d \end{cases}.$$

The first equation has solutions $a = 0$ and $a = 1$. Both imply, together with the second respectively third equation, that $b = c = 0$. Then the fourth equation becomes $2ad = d$, so either value of a implies $d = 0$ as well. Hence the only idempotents are 0 and $1 \otimes 1$. Consequently, the left regular module $D \otimes D$ is the sole principal module. In particular, it is indecomposable. Obviously, the regular module is also projective. Less trivially, $D \otimes D$ is injective. To see this, we introduce the following notion.

Definition. Let A be a K-algebra. We say that A is *Frobenius*, if there is a non-degenerate bilinear form $\sigma : A \times A \to K$ such that $\sigma(ab, c) = \sigma(a, bc)$ for any $a, b, c \in A$. Such σ is called a *Frobenius form* on A.

The form $\sigma : D \times D \to \mathbb{C}$ which maps a pair $(a, b) \in D \times D$ to the coefficient of X in ab is a Frobenius form on D, so D is Frobenius. Tensor products of Frobenius algebras are again Frobenius, so $D \otimes D$ is Frobenius. Any Frobenius algebra is (left) self-injective, i.e. if A is Frobenius, then ${}_AA$ is injective. Hence we conclude that the left regular module $D \otimes D$ is injective. We refer to modules that are both projective and injective simply as *projective-injective.*
The following two lemmas, which can be found in [3, (9.2)], now yield another step towards the classification of indecomposable $D \otimes D$-modules .

Lemma 2 (Separation lemma)**.** *Let A be an algebra and N a projective-injective A-module. Then there is a nonzero ideal $I \subseteq A$ such that any A-module U decomposes into a direct sum $U \simeq U_1 \oplus U_2$, where $I \subseteq \operatorname{Ann} U_1$, and any indecomposable direct summand of U_2 is isomorphic to a direct summand of N. In particular, if N is indecomposable, then $U \simeq U_1 \oplus N^{\oplus k}$, where U_1 is a module over A/I.*

I can be set to the intersection of the annihilators of all indecomposable A-modules nonisomorphic to N. Given an indecomposable projective-injective A-module N, denote the corresponding quotient algebra A/I by $A^-(N)$.

Lemma 3. *Let N be an indecomposable projective-injective A-module, and $A \simeq U_1 \oplus U_2$, where $U_1 \simeq N^{\oplus k}$, and U_2 has no direct summand isomorphic to N. Then $\operatorname{soc} U_1 \subseteq A$ is an ideal, and $A^-(N) \simeq A/\operatorname{soc} U_1$.*

As we have seen, the principal $D \otimes D$-module is indecomposable and projective-injective. Hence, by Separation lemma any $D \otimes D$-module decomposes into a direct sum $M_1 \oplus (D \otimes D)^{\oplus k}$, where M_1 is a module over the quotient of $D \otimes D$ with some nonzero ideal. Lemma 3 then tells us that this nonzero ideal is $\mathrm{soc}(D \otimes D) = (X \otimes X)$. In conclusion, if M is an indecomposable $D \otimes D$ module, then either $M = D \otimes D$, or M is a module over $D \otimes D/(X \otimes X)$. The latter is a so called *special biserial algebra.* Its indecomposable modules have been completely classified, see [1] and [6]. They are either string modules or band modules, as specified below. Now we go back to the language of D-D-bimodules: the action of $X \otimes 1$ is the left X-action, while $1 \otimes X$ is right X-action. Both string- and band bimodules can be nicely described using diagrams.
For the string bimodules, we use the following notation. An arrow pointing from a to b means that b is the image of a under X-action; down-left means left action, down-right is right action. The absence of an arrow in a certain direction means that the corresponding X-action is 0. For example, the diagram

$$\begin{array}{ccccc} a_1 & & & & a_2 \\ & \searrow & & \swarrow & \\ & & b_1 & & \end{array}$$

stands for the bimodule with basis $\{a_1, a_2, b_1\}$ and X-actions

$$\begin{array}{ll} Xa_1 = 0, & a_1X = b_1 \\ Xa_2 = b_1, & a_2X = 0 \\ Xb_1 = 0, & b_1X = 0. \end{array}$$

We will use a for the basis elements on which there is a nontrivial X-action, and b for the others, as suggested above [1]. Then all bimodules considered have bases of the form $\{a_1, \ldots, a_n, b_1, \ldots, b_l\}$,[2] and we use n to specify the dimension, which will be uniquely determined by n within each family of bimodules. There are four families of string bimodules, indexed by $n \geq 1$:

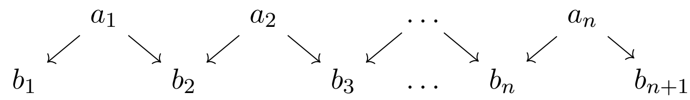
- W_n, of dimension $2n - 1$.

$$\begin{array}{ccccccccc} a_1 & & & & a_2 & \ldots & a_{n-1} & & & & a_n \\ & \searrow & & \swarrow & & \searrow \quad \swarrow & & \searrow & & \swarrow & \\ & & b_1 & & & \ldots & & & b_{n-1} & & \end{array}$$

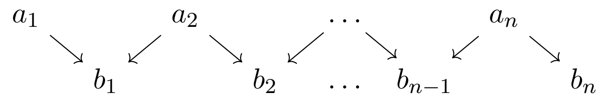
[1]The string bimodule $W_1 \simeq {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}$ has no nontrivial X-action, but we call its basis $\{a\}$. This is the only exception to this rule.

[2]When we shall calculate tensor products $U \otimes V$, we will in V use r, s instead of a, b, respectively.

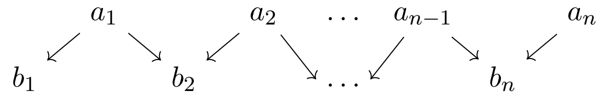
- M_n, of dimension $2n+1$.



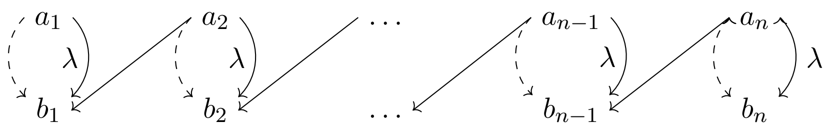
- R_n, of dimension $2n$.



- L_n, of dimension $2n$.



In addition, there is the family of band bimodules, also they indexed but $n \geq 1$ but in addition depending on an eigenvalue $\lambda \neq 0$. The band bimodule $H_n(\lambda)$ has dimension $2n$. For this diagram, a slightly different notation will be used: dashed arrows mean left action, continuous arrows right action. Then $H_n(\lambda)$ looks as follows.



Note that although the straight arrows point to the left, they indicate right action.

2.2 $\mathbb{C}$-split bimodules

Some D-D-bimodules are lucky enough to be the tensor product over $\mathbb{C}$ of a left and a right D-module. These are the $\mathbb{C}$-split D-D-bimodules, see further [5].

Definition. Let T be a D-D-bimodule. Then T is $\mathbb{C}$-*split* if $T \simeq U \otimes_{\mathbb{C}} V$, where $U \in D$-mod and $V \in$ mod-D.

Among the indecomposable D-D-bimodules, the $\mathbb{C}$-split ones are exactly $D \otimes_{\mathbb{C}} D$, $D \otimes_{\mathbb{C}} \mathbb{C} \simeq L_1$, $\mathbb{C} \otimes_{\mathbb{C}} D \simeq R_1$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \simeq {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}} \simeq W_1$. Their multiplication table with respect to $\otimes_D$ looks as follows. For readability we write $\otimes$ for $\otimes_{\mathbb{C}}$ in this table.

	$\mathbb{C}$	$D \otimes \mathbb{C}$	$\mathbb{C} \otimes D$	$D \otimes D$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \otimes D$	$\mathbb{C} \otimes D$
$D \otimes \mathbb{C}$	$D \otimes \mathbb{C}$	$D \otimes \mathbb{C}$	$D \otimes D$	$D \otimes D$
$\mathbb{C} \otimes D$	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C} \otimes D$	$(\mathbb{C} \otimes D) \oplus (\mathbb{C} \otimes D)$
$D \otimes D$	$D \otimes \mathbb{C}$	$(D \otimes \mathbb{C}) \oplus (D \otimes \mathbb{C})$	$D \otimes D$	$(D \otimes D) \oplus (D \otimes D)$

As for other tensor products involving these bimodules, for any D-D-bimodule U

$$\begin{aligned} D \otimes_{\mathbb{C}} D \otimes_{\mathrm{D}} U &\simeq D \otimes_{\mathbb{C}} U \\ D \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathrm{D}} U &\simeq D \otimes_{\mathbb{C}} U \\ \mathbb{C} \otimes_{\mathbb{C}} D \otimes_{\mathrm{D}} U &\simeq \mathbb{C} \otimes_{\mathbb{C}} U \end{aligned}$$

and similarly $U \otimes_{\mathrm{D}} D \otimes_{\mathbb{C}} D \simeq U \otimes_{\mathbb{C}} D$ etc. In the first cases, U can be viewed as a right D-module. The case $U = D \otimes_{\mathbb{C}} D$ is already done above, so assume that U is a string- or band bimodule. When left X-action is forgotten, U is a direct sum of summands isomorphic to D_D and $\mathbb{C}_{\mathrm{D}}$. (In the case $U = H_n(\lambda)$ a straightforward change of basis is needed to make this obvious.) Similarly, in the second case we can view U as a left D-module, and as such, it is a direct sum with summands ${}_{\mathrm{D}}\mathrm{D}$ and ${}_{\mathrm{D}}\mathbb{C}$. In detail, we have the following.

	Without left action	Without right action
W_n	$D_D^{\oplus n-1} \oplus \mathbb{C}_{\mathrm{D}}$	${}_{\mathrm{D}}\mathrm{D}^{\oplus n-1} \oplus {}_{\mathrm{D}}\mathbb{C}$
M_n	$D_D^{\oplus n} \oplus \mathbb{C}_{\mathrm{D}}$	${}_{\mathrm{D}}\mathrm{D}^{\oplus n} \oplus {}_{\mathrm{D}}\mathbb{C}$
R_n	$D_D^{\oplus n}$	${}_{\mathrm{D}}\mathrm{D}^{\oplus n-1} \oplus {}_{\mathrm{D}}\mathbb{C}^{\oplus 2}$
L_n	$D_D^{\oplus n-1} \oplus {\mathbb{C}_{\mathrm{D}}}^{\oplus 2}$	${}_{\mathrm{D}}\mathrm{D}^{\oplus n}$
$H_n(\lambda)$	$D_D^{\oplus n}$	${}_{\mathrm{D}}\mathrm{D}^{\oplus n}$

Now any tensor product involving a $\mathbb{C}$-split bimodule can be calculated using the above multiplication table. In particular we note that the $\mathbb{C}$-split bimodules span an ideal in $[D\text{-mod-}D]_{\oplus}$.

Remark. Already here we see that the tensor product over D is not commutative.

2.3 Dimensions of tensor products

If we choose U, V from the above bimodules and consider the tensor product $U \otimes V$, left X-action is inherited from U and right from V. If we instead think of $U \otimes V$ as a $\mathbb{C}$-vector space, we can forget these actions and think of U and V right and left D-modules, as in the previous section. To calculate dimensions we then need only the following multiplication table with respect to $\otimes_{\mathrm{D}}$.

	${}_{\mathrm{D}}\mathbb{C}$	${}_{\mathrm{D}}\mathrm{D}$
$\mathbb{C}_{\mathrm{D}}$	$\mathbb{C}$	$\mathbb{C}$
D_D	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$

.

For example

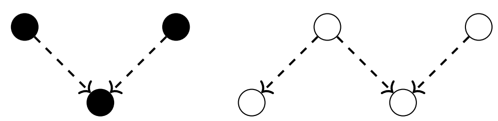
$$\begin{aligned}
&\dim(W_n \otimes L_m) = \\
&= \dim\left((D_D^{\oplus n-1} \oplus \mathbb{C}_D) \otimes {}_{\mathrm{D}}\mathrm{D}^{\oplus m} \right) \\
&= \dim\left(D_D^{\oplus n-1} \otimes {}_{\mathrm{D}}\mathrm{D}^{\oplus m} \right) + \dim\left(\mathbb{C}_{\mathrm{D}} \otimes {}_{\mathrm{D}}\mathrm{D}^{\oplus m} \right) = \\
&= m(n-1)\dim(D_D \otimes {}_{\mathrm{D}}\mathrm{D}) + m\dim(\mathbb{C}_D \otimes {}_{\mathrm{D}}\mathrm{D}) = \\
&= m(n-1) \cdot 2 + m \cdot 1 = \\
&= 2nm - m.
\end{aligned}$$

2.4 Intuition for tensor products of string bimodules

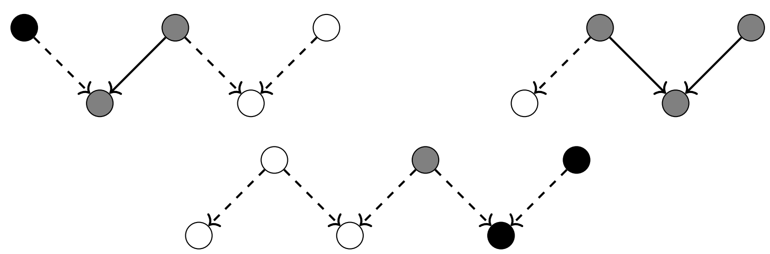
From now on, $\otimes$ means $\otimes_D$, except when we write $D \otimes D$, which as before is $D \otimes_{\mathbb{C}} D$. We shall first give the multiplication table for the string bimodules. Their tensor products are easy to describe explicitly in the the standard basis. The following will prove very useful.

Think of each string bimodule (nonisomorphic to ${}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}$) simply as the corresponding digraph, each basis element being a node and each arrow indicating X-action an edge. Then each node is either a source having outdegree 1 or 2, or a sink or having indegree 1 or 2. We always place sources above sinks. Multiplication of two graphs yields a direct sum of all graphs it is possible to obtain as follows: place the digraphs over each other, sources on sources, sinks on sinks, and take the new digraph formed by the overlapping nodes and edges.

For example, in $W_2 \otimes L_2$ we have the digraphs

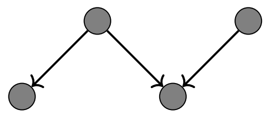


which we can place on each other in the following ways.

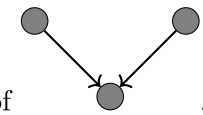


As we shall see, $W_2 \otimes_D L_2 \simeq L_1 \oplus W_2 \oplus {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}$.

The cases when this rule is entirely correct are $W \otimes W$, $L \otimes L$, $R \otimes R$, $W \otimes L$, $R \otimes W$, $R \otimes L$, $M \otimes L$, and $R \otimes M$. Deviations from it are of two types. The first type is that some extra sink appears: we expect $L_2 \otimes W_2 \simeq {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}} \oplus W_2 \oplus L_1$, but calculations show instead $L_2 \otimes W_2 \simeq {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}} \oplus L_2 \oplus L_1$. Here the middle summand is



instead of



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This extra sink can appear in $U \otimes V$ if the leftmost node of U is a sink, but that of V is a source, as in $L \otimes W$, $L \otimes R$, $M \otimes W$ and $M \otimes R$. The extra sink will also be a leftmost node: we can get L instead of W, or M instead of R. In the other direction, if the rightmost node of V is a sink, and that of U is a source, an extra (rightmost) node can appear. The possible cases are $W \otimes R$, $W \otimes M$, $L \otimes R$, $L \otimes M$.

The other type is when $D \otimes D$ appears as a summand. It can then replace an expected $M_1 \oplus {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}$, or, as in the case of $L_1 \otimes R_1$, an expected ${}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}{}^{\oplus 2}$. In terms of the graphs, this happens in $U \otimes V$ exactly when both the leftmost node of U's graph and the rightmost node of V's graph are sinks: $L \otimes R$, $L \otimes M$, $M \otimes R$, $M \otimes M$.

Remark. $W_1 \simeq {}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}$ is one-dimensional, both X-actions trivial, so its corresponding graph has only one node and no edges. To fit it into this pattern, think of this node as being a source, or upper node - recall that we denote its basis $\{a\}$. The notations W_1 and $\mathbb{C}$ will both be used in the following.

3 Multiplication table for string bimodules

In this section all formulas for tensor products between string bimodules will be explicitly presented. When we calculate them, we can explicitly find the indecomposable summands by following the intuition from the previous section. A certain symmetry saves us some - but not much - of this tedious work, as we will see in 3.12. Throughout this section, m and n are positive integers.

3.1 $W_n \otimes W_m$

$$W_n \otimes W_m \simeq \bigoplus_{k=1}^{\min(n,m)-1} W_k^{\oplus 2} \oplus W_{\min(n,m)}^{\oplus |n-m|+1}.$$

Dimension of the left-hand side:

$$2(n-1)(m-1) + n - 1 + m - 1 + 1 = 2nm - n - m + 1.$$

Dimension of the right-hand side (calculated when $n \leq m$, the case $n > m$ is symmetrical).

$$\begin{aligned} 2\sum_{i=1}^{n-1}(2i-1) + (m-n+1)(2n-1) = \\ = (n-1)(2n-2) + (m-n+1)(2n-1) = \\ = 2nm - n - m + 1. \end{aligned}$$

In the bimodule W_n we have for $i = 1, \ldots, n-1$ that $b_i = a_i X$. Therefore, in $W_n \otimes W_m$,

$$b_i \otimes s_j = a_i X \otimes s_j = a_i \otimes X s_j = a_i \otimes 0 = 0$$

for all $i = 1, \ldots, n-1$ and all $j = 1, \ldots, m-1$. Furthermore

$$b_i \otimes r_1 = a_i X \otimes r_1 = a_i \otimes X r_1 = a_i \otimes 0 = 0$$

for all i. Similarly $s_j = X r_{j+1}$ for $j = 1, \ldots m-1$, so that also $a_n \otimes s_j = 0$ for all j. No other basis elements are zero, but some of them can be identified with each other, namely

$$a_i \otimes s_j = a_i \otimes X r_{j+1} = a_i X \otimes r_{j+1} = b_i \otimes r_{j+1}$$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$. A basis for $W_n \otimes W_m$ is hence the union of the sets

$$\{a_i \otimes r_j \mid i = 1, \ldots, n,\ j = 1, \ldots, m\}$$

$$\{a_i \otimes s_j \mid i = 1, \ldots, n-1,\ j = 1, \ldots, m-1\}.$$

Since both left- and right action of X on all b_i and s_j is 0, it is clear that it is so also on $a_i \otimes s_j = b_i \otimes r_{j+1}$. It is also clear that no $a_i \otimes r_j$ is the result of X acting on any basis element, since neither a_i nor r_j is. We shall prove the isomorphism stated above by considering first all $a_i \otimes r_1$, and then all $a_1 \otimes r_j$. The left X-action is 0 on all such elements - in the latter case since it is so on a_1, in the former since

$$X a_i \otimes r_1 = b_{i-1} \otimes r_1 = a_{i-1} X \otimes r_1 = 0$$

for any $i > 1$.

We shall only consider $n \leq m$. The case $n > m$ is similar.

Step 1

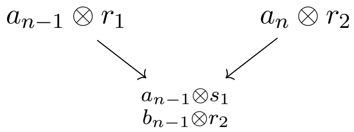
Consider $a_n \otimes r_1$. Since

$$a_n \otimes r_1 X = a_n \otimes s_1 = a_n \otimes X r_2 = a_n X \otimes r_2 = 0$$

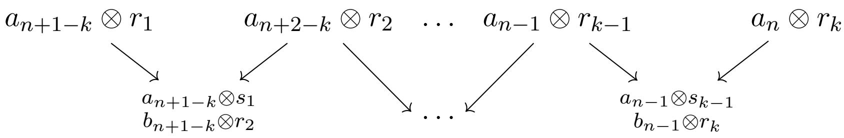
this element spans a one-dimensional bimodule isomorphic to ${}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}} \simeq W_1$. Next, look at $a_{n-1} \otimes r_1$. For the right action,

$$a_{n-1} \otimes r_1 X = a_{n-1} \otimes s_1 = b_{n-1} \otimes r_2 = X a_n \otimes r_2$$

given that $m \geq 2$. Since $a_n \otimes r_2 X = a_n \otimes s_2 = 0$ we get the following:



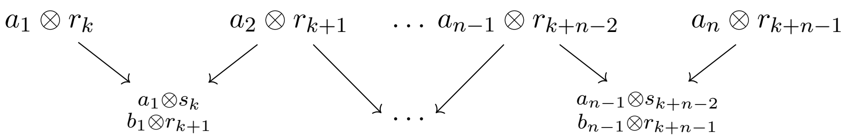
which we identify with W_2. Continuing like this we get for each $1 \leq k \leq n-1$ exactly one bimodule isomorphic to W_k, of the following form.



We note that $a_n \otimes r_k X = a_n \otimes s_k = a_n \otimes X r_{k+1} = 0$ for $k = 1, \dots, n-1$, so that there really is no more nontrivial X-action here.

Step 2

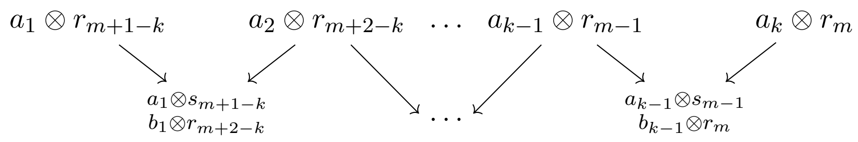
Next we reach $a_n \otimes r_1$ and get, by the same pattern as above, the bimodule W_n. In fact, this bimodule will appear $m - n + 1$ times, with $k = 1, \dots, m - n + 1$ below.



Note that for $1 \leq k \leq m - n + 1$, the index $k + n - 1$ runs from n to m, so all r_{k+n-1} exist.

Step 3

What will happen now is more or less the same thing as in the first step: each of the bimodules W_k, $1 \leq k \leq n-1$, occurs once in the following form.



Again we point out that $a_k \otimes r_m X = a_k \otimes 0 = 0$ always.

If we collect what we have now, we find two copies each of $W_1, \dots, W_{n-1}$ (from Step 1 and Step 3), and $m - n + 1$ copies of W_m (from Step 2).

Remark. In all formulas above we have that $a_i \otimes r_j$ and $a_k \otimes r_l$ occur in the same bimodule if and only if $i - j = k - l$. In Step 1, these differences range from $n-1$ to $n-m+1$. In Step 2, they range from 0 to $n-m$, and in Step 3 from $1-m$ to -1. Hence we have actually covered all possible differences in indices. It is easily seen that each nonzero $a_i \otimes s_j = b_i \otimes r_{j+1}$ has exactly one preimage in each direction, so we only need to care about $a \otimes r$-terms. This reasoning can be applied to all tensor products between string bimodules.

3.2 $M_n \otimes M_m$

$$M_n \otimes M_m \simeq \begin{cases} \mathbb{C} \oplus (D \otimes D) \oplus M_1 \oplus \bigoplus_{i=2}^{\min(n,m)-1} M_i^{\oplus 2} \oplus M_{\min(n,m)}^{\oplus |n-m|+1}, & \min(n,m) > 1 \\ \mathbb{C} \oplus (D \otimes D) \oplus M_1^{\oplus \max(n,m)-1}, & \min(n,m) = 1 \end{cases}.$$

Dimension of left-hand side:

$$2nm + n + m + 1.$$

Dimension of right-hand side in case $m \geq n > 1$:

$$\begin{aligned} &8 + 2\sum_{i=2}^{n-1}(2i+1) + (m-n+1)(2n+1) = \\ &= 8 + (n-2)(2n+4) + (m-n+1)(2n+1) = \\ &= 2nm + m + n + 1 \end{aligned}$$

and in case $m \geq n = 1$:

$$5 + 3(m-1) = 3m + 2 = 2nm + m + n + 1.$$

The cases where $m < n$ are symmetrical.

In M_n, not all b_i's have preimages in both directions, which was the case in W_n. Therefore we have fewer identifications in the tensor product. We have the relations

$$\begin{aligned} b_1 &= Xa_1 \\ b_i &= a_{i-1}X = Xa_i \text{ for } i = 2, \dots, n \\ b_{n+1} &= a_n X. \end{aligned}$$

Together with $Xb_i = 0 = b_i X$ for all i, they imply that in the tensor product, all $b_i \otimes s_j$ are 0 except $b_1 \otimes s_{m+1}$. Furthermore we can identify

$$a_i \otimes s_j = a_i \otimes Xr_j = b_{i+1} \otimes r_j$$

for $i = 1, \dots, n$ and $j = 2, \dots, m$. Now, a basis is constituted by the union of the sets

$$\begin{aligned} &\{a_i \otimes r_j \mid i = 1, \dots, n,\ j = 1, \dots, m\}, \\ &\{a_i \otimes s_j \mid i = 1, \dots, n,\ j = 1, \dots, m+1\}, \\ &\{b_1 \otimes r_j \mid j = 1, \dots, m\} \\ &\qquad \text{and } \{b_1 \otimes s_{m+1}\}. \end{aligned}$$

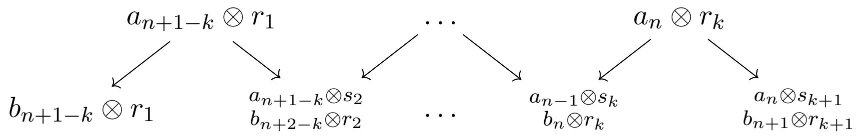
Again we do only the case $n \leq m$ in detail and divide the procedure into three steps.

Step 1

First we have the element $b_1 \otimes r_1$ with

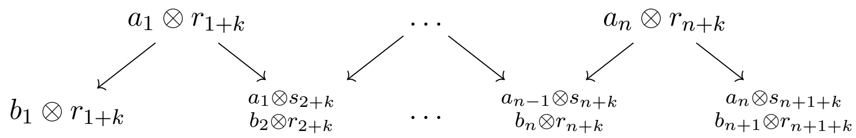
$$\begin{aligned} &Xb_1 \otimes r_1 = 0 \\ &b_1 \otimes r_1 X = b_1 \otimes s_2 = a_1 X \otimes s_2 = 0. \end{aligned}$$

We see that $b_1 \otimes r_1$ spans a bimodule isomorphic to ${}_{\mathrm{D}}\mathbb{C}_{\mathrm{D}}$. Then we have one copy each of M_k, $k = 1, \dots, n-1$, as below. Note that if $n = 1$, this part is empty.

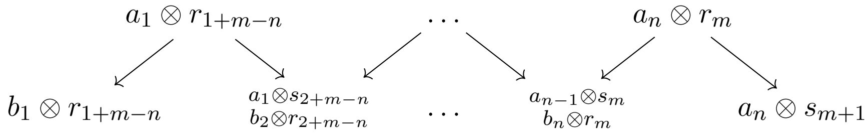


Step 2

$m - n$ copies of M_n, with $k = 0, \dots, m-n-1$ below.

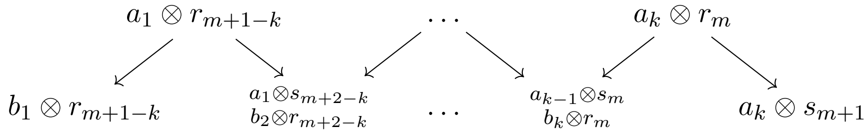


In case $n > 1$, we can let k run up to $m - n$ above, and have yet another copy of M_n as below.

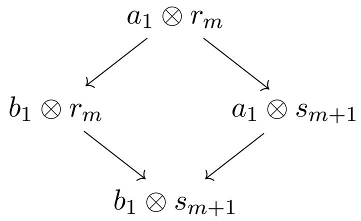


Step 3

One copy each of M_k, $k = 2, \dots, n-1$.



Lastly, we reach the element $a_1 \otimes r_m$ and get $D \otimes D$.



In total we have the following

	Step 1	Step 2	Step 3
$n > 1$	$\mathbb{C} \oplus \bigoplus_{k=1}^{n-1} M_k$	$M_n^{\oplus m-n+1}$	$\bigoplus_{k=2}^{n-1} M_k \oplus D \otimes D$
$n = 1$	$\mathbb{C}$	$M_1^{\oplus m-1}$	$D \otimes D$

3.3 $L_n \otimes L_m$

$$L_n \otimes L_m \simeq \bigoplus_{i=1}^{\min(n,m)-1} L_i^{\oplus 2} \oplus L_{\min(n,m)}^{\oplus |n-m|+1}.$$

Dimension of left-hand side: $2nm$. Dimension of right-hand side in case $n \leq m$:

$$\begin{aligned} &2\sum_{i=1}^{n-1} 2i + (m-n+1)2n = \\ &= 2n(n-1) + 2n(m-n+1) = \\ &= 2n(n-1+m-n+1) = \\ &= 2nm \end{aligned}$$

and in case $m > n$ it is by symmetry $2mn = 2nm$.
Identifications: for any $j = 1, \dots m$,

$$\begin{aligned} &b_i \otimes s_j = b_i \otimes X r_j = b_i X \otimes r_j = 0, \quad \forall\, i \\ &a_i \otimes s_j = a_i \otimes X r_j = a_i X \otimes r_j = b_{i+1} \otimes r_j, \quad i = 1, \dots n-1 \\ &a_n \otimes s_j = 0. \end{aligned}$$

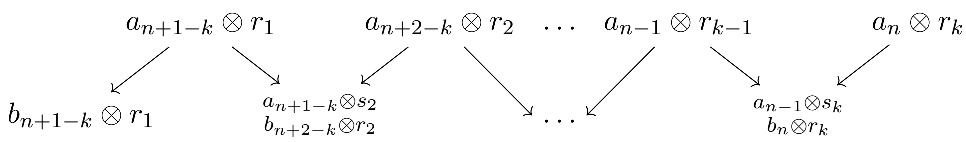
Furthermore, for $i = 1, \dots, n-1$

$$b_i \otimes r_m = a_{i+1} X \otimes r_m = a_{i+1} \otimes X r_m = 0.$$

Direct summands in case $n \leq m$:

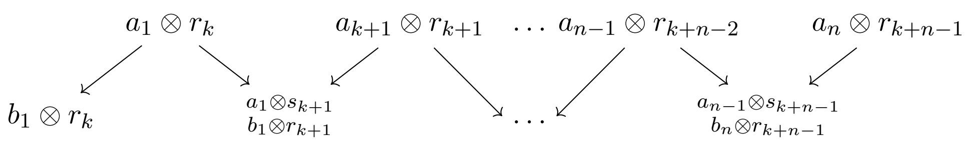
Step 1

L_k, $k = 1, \dots n-1$, of the following form.



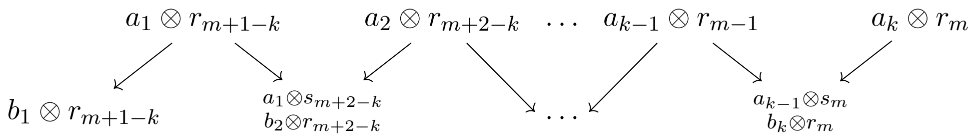
Step 2

$m - n + 1$ copies of L_n, with $k = 1, \dots, m-n+1$ below.



Step 3

Another L_k, $k = 1, \dots n-1$, now of the following form.



3.4 $R_n \otimes R_m$

$$R_n \otimes R_m \simeq \bigoplus_{i=1}^{\min(n,m)-1} R_i^{\oplus 2} \oplus R_{\min(n,m)}^{\oplus |n-m|+1}.$$

Dimension of left-hand side:

$$2n(m-1) + 2n = 2nm.$$

Dimension of right-hand side in case $n \leq m$:

$$\begin{aligned} & 2\sum_{i=1}^{n-1} 2i + (m-n+1)2n = \\ & = 2(n-1)n + (m-n+1)2n = \\ & = 2n(n-1+m-n+1) = \\ & = 2nm. \end{aligned}$$

The case $n > m$ is the same by symmetry.
Identifications: for all i, j

$$b_i \otimes s_j = 0$$
$$b_i \otimes r_1 = 0$$

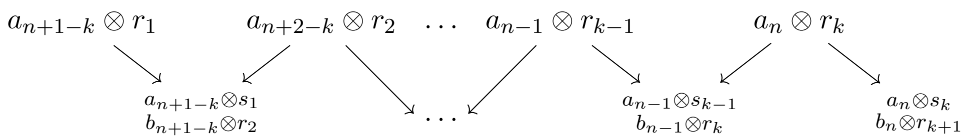
and for $j = 1, \dots, m-1$

$$a_i \otimes s_j = a_i \otimes X r_{j+1} = b_i \otimes r_{j+1}.$$

Direct summands in case $n \leq m$:

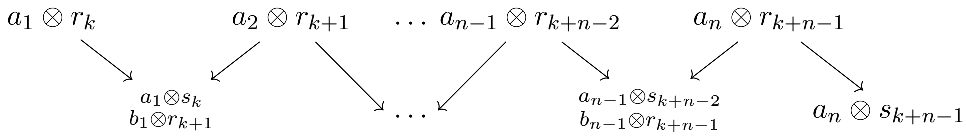
Step 1

R_k, $k = 1, \dots, n-1$.



Step 2

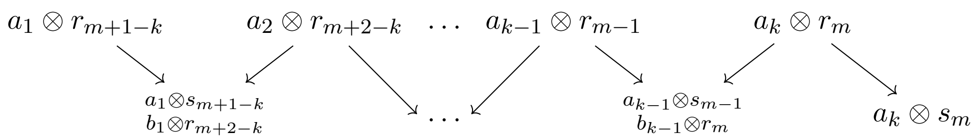
$m-n+1$ copies of R_n, with $k=1,\dots,m-n+1$



For all k except $k=m-n+1$, the identification $a_n \otimes s_{k+n-1} = b_n \otimes r_{k+n}$ is valid, but this is not important. What matters is that for none of the values of k, this element has a preimage under the left X-action (or nonzero image under either X-action).

Step 3

Another copy of R_k, $k=1,\dots,n-1$.



3.5 $W_n \otimes L_m$

$$W_n \otimes L_m \simeq \begin{cases} \bigoplus_{j=1}^{n-1} W_j \oplus W_n^{\oplus m-n+1} \oplus \bigoplus_{i=1}^{n-1} L_i, & n \le m \\ \bigoplus_{j=1}^{m} W_j \oplus L_m^{\oplus n-m} \oplus \bigoplus_{i=1}^{m-1} L_i, & n > m \end{cases}.$$

Note that if $n=m$ we have $W_n \otimes L_n \simeq \bigoplus_{j=1}^{n} W_j \oplus \bigoplus_{i=1}^{n-1} L_i$, so we could have included the equality in the case $n>m$ instead, given that we define $L_n^{\oplus 0}=0$.
Dimension of left-hand side:

$$2(n-1)m+m=2nm-m.$$

Dimension of right-hand side in case $n \le m$:

$$\begin{aligned} &\sum_{j=1}^{n-1}(2j-1)+(m-n+1)(2n-1)+\sum_{i=1}^{n-1}2i= \\ &=(n-1)(n-1)+(m-n+1)(2n-1)+n(n-1)= \\ &=(2n-1)(n-1)+(m-n+1)(2n-1)= \\ &=(2n-1)m= \\ &=2nm-m. \end{aligned}$$

and in case $n > m$:

$$\sum_{j=1}^{m}(2j-1)+(n-m)2m+\sum_{i=1}^{m-1}2i =$$
$$= m^2+2m(n-m)+m(m-1) =$$
$$= 2nm-m.$$

Now, we identify

$$b_i \otimes s_j = 0$$
$$a_i \otimes s_j = a_i \otimes Xr_j = b_i \otimes r_j$$

so a basis is

$$\{a_i \otimes r_j \mid i=1,\dots n, j=1,\dots m\} \cup \{b_i \otimes r_j \mid i=1,\dots n-1, j=1,\dots m\}.$$

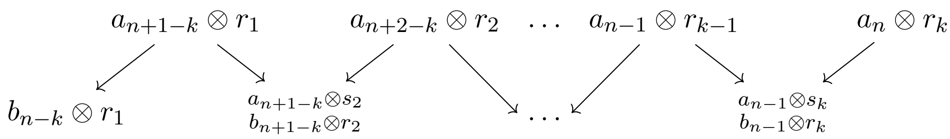
Since the result depends on the relationship between n and m, we consider two cases.

Case 1

$n \leq m$

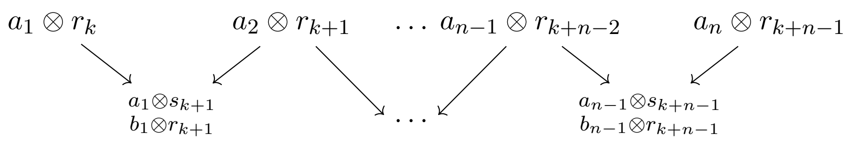
Step 1

From the right, W_n looks like L_n. Therefore we can imitate Step 1 and from $L_n \otimes L_m$ above (with a slight change of indices) and find L_k, $k=1,\dots,n-1$.



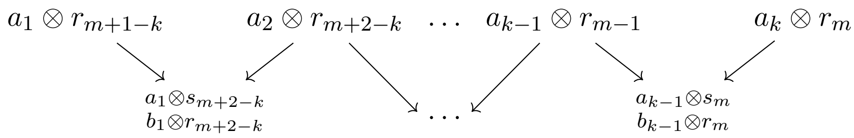
Step 2

$m-n+1$ copies of W_n, with $k=1,\dots,m-n+1$ below.



Step 3

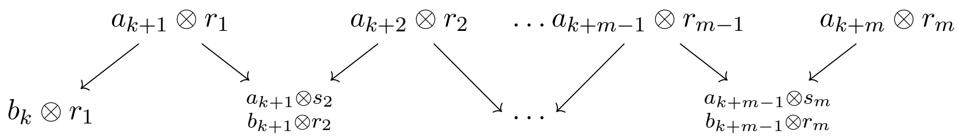
From the right, L_m looks like W_m, so we imitate Step 3 from $W_n \otimes W_m$, yet again changing the indices slightly. Then we find W_k, $k = 1, \ldots, n-1$.



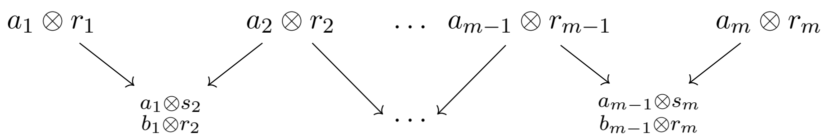
Case 2

$n > m$

Step 1 can again be imitated from $L_n \otimes L_m$, with $n > m$. Similarly, Step 3 is imitated from $W_n \otimes W_m$ with $n > m$. In Step 2, we find $n - m$ copies of L_m, with $k = 1, \ldots, n - m$ below.



Lastly, there is also one W_m.



Remark. Due to the similarity of the cases $n \leq m$ and $m > n$, we shall from now on only find the direct summands in case $n \leq m$.

3.6 $W_n \otimes R_m$

$$W_n \otimes R_m \simeq \begin{cases} \bigoplus_{i=1}^{n} R_i \oplus W_n^{\oplus m-n} \oplus \bigoplus_{j=1}^{n-1} W_j, & n \leq m \\ \bigoplus_{i=1}^{m-1} R_i \oplus R_m^{\oplus n-m+1} \oplus \bigoplus_{j=1}^{m-1} W_j, & n > m \end{cases}.$$

Dimension of left-hand side:

$$2(n-1)(m-1) + 2(n-1) + (m-1) + 2 = 2nm - m + 1.$$

Dimension of right-hand side in case $n \leq m$:

$$\sum_{i=1}^{n} 2i + (m-n)(2n-1) + \sum_{j=1}^{n-1}(2j-1) =$$
$$= n(n+1) + (m-n)(2n-1) + (n-1)(n-1) =$$
$$= 2nm - m + 1$$

and in case $n > m$:

$$\sum_{i=1}^{m-1} 2i + (n-m+1)2m + \sum_{j=1}^{m-1}(2j-1) =$$
$$= (m-1)m + 2m(n-m+1) + (m-1)(m-1) =$$
$$= 2nm - m + 1.$$

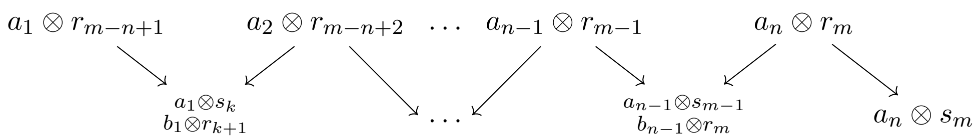
Identifications: for all i, j

$$b_i \otimes s_j = 0$$
$$b_i \otimes r_1 = 0$$

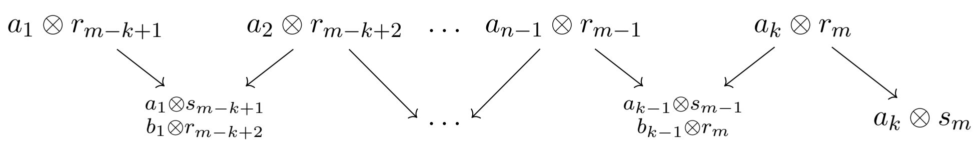
and for $j = 1, \ldots, m-1$

$$a_n \otimes s_j = 0$$
$$a_i \otimes s_j = a_i \otimes X r_{j+1} = b_i \otimes r_{j+1}$$

Direct summands in case $n \leq m$.
Step 1 and Step 2 can be copied from $W_n \otimes W_m$, with the alteration that the last summand in Step 2 is



i.e. R_n. Step 3 yields R_k, $k = 1, \ldots, n-1$ as below.



3.7 $M_n \otimes W_m$

$$M_n \otimes W_m \simeq \begin{cases} \bigoplus_{i=1}^{n} L_i \oplus M_n^{\oplus m-n} \oplus \bigoplus_{j=1}^{n-1} R_j, & n \le m \\ \bigoplus_{i=1}^{m} L_i \oplus W_m^{\oplus n-m} \oplus \bigoplus_{j=1}^{m-1} R_j, & n > m \end{cases}.$$

Dimension of left-hand side:

$$2(m-1)n + m + (n-1) + 1 = 2mn + m - n.$$

Dimension of right-hand side in case $n \le m$:

$$\begin{aligned} &\sum_{i=1}^{n} 2i + (m-n)(2n+1) + \sum_{j=1}^{n-1} 2j = \\ &= n(n+1) + (m-n)(2n+1) + n(n-1) = \\ &= 2n^2 + (m-n)(2n+1) = \\ &= 2mn + m - n \end{aligned}$$

and in case $n > m$ it is

$$\begin{aligned} &\sum_{i=1}^{m} 2i + (n-m)(2m-1) + \sum_{j=1}^{m-1} 2j = \\ &= m(m+1) + (n-m)(2m-1) + m(m-1) = \\ &= 2m^2 + (n-m)(2m-1) = \\ &= 2mn + m - n. \end{aligned}$$

Identifications: for any $i = 1, \dots, n+1$ and $j = 1, \dots, m-1$

$$b_i \otimes s_j = b_i \otimes X r_{j+1} = b_i X \otimes r_{j+1} = 0$$

and for $i = 2, \dots, n+1$

$$b_i \otimes r_1 = a_{i-1} X \otimes r_1 = a_{i-1} \otimes X r_1 = 0.$$

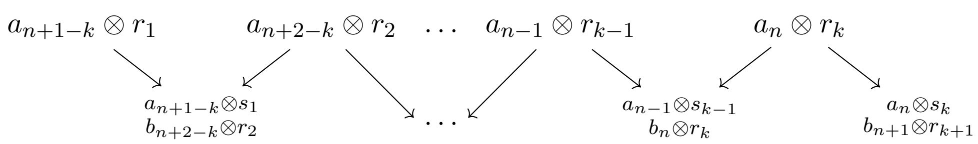
Also

$$a_i \otimes s_j = a_i \otimes X r_{j+1} = a_i X \otimes r_{j+1} = b_{i+1} \otimes r_{j+1}.$$

Direct summands in case $n \le m$.

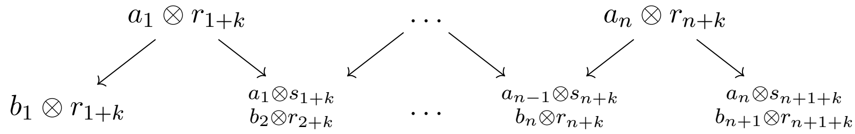
Step 1

R_k, $k = 1, \dots, n-1$.



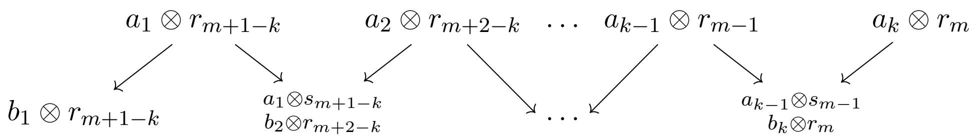
Step 2

$m-n$ copies of M_n, with $k = 0, \dots, m-n-1$ below.



Step 3

L_k, $k = 1, \dots, n$.



3.8 $L_n \otimes R_m$

$$L_n \otimes R_m \simeq \begin{cases} (D \otimes D) \oplus \bigoplus_{i=1}^{n-1} W_i \oplus L_n^{\oplus m-n} \oplus \bigoplus_{j=2}^{n} M_j, & n \leq m \\ (D \otimes D) \oplus \bigoplus_{i=1}^{m-1} W_i \oplus R_m^{\oplus n-m} \oplus \bigoplus_{j=2}^{m} M_j, & n > m \end{cases}.$$

Dimension of left-hand side:

$$2(n-1)(m-1) + 2(n-1) + 2(m-1) + 4 = 2nm + 2.$$

Dimension of right-hand side in case $n \leq m$:

$$\begin{aligned} &4 + \sum_{i=1}^{n-1}(2i-1) + (m-n)2n + \sum_{j=2}^{n}(2j+1) = \\ &= 4 + n(n-1) - (n-1) + 2n(m-n) + (n-1)(n+2) + n - 1 = \\ &= 4 + (2n+2)(n-1) + 2n(m-n) = \\ &= 2nm + 2 \end{aligned}$$

and by symmetry, the dimension in case $n > m$ is the same.
Identifications: for $i = 2, \dots, n$

$$b_i \otimes s_j = a_{i-1}X \otimes s_j = a_{i-1} \otimes Xs_j = 0$$
$$b_i \otimes r_1 = a_{i-1}X \otimes r_1 = a_{i-1} \otimes Xr_1 = 0$$

and for $j = 1, \dots, m-1$

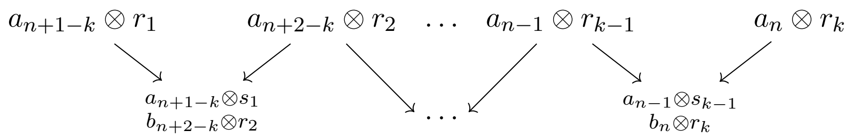
$$b_1 \otimes s_j = b_1 \otimes Xr_{j+1} = b_1X \otimes r_{j+1} = 0$$
$$a_n \otimes s_j = a_n \otimes Xr_{j+1} = a_nX \otimes r_{j+1} = 0.$$

Finally for $i = 1, \dots, n-1$ and $j = 1, \dots m-1$,

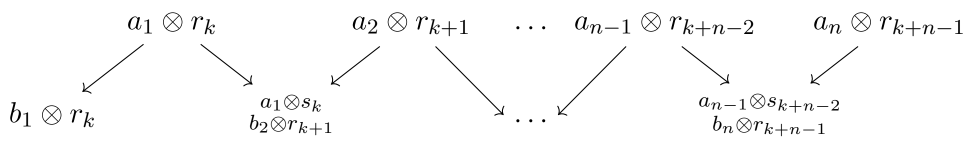
$$a_i \otimes s_j = a_i \otimes X r_{j+1} = b_{i+1} \otimes r_{j+1}.$$

Direct summands in case $n \le m$:

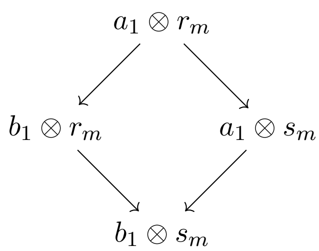
- W_k, $k = 1, \dots, n-1$.



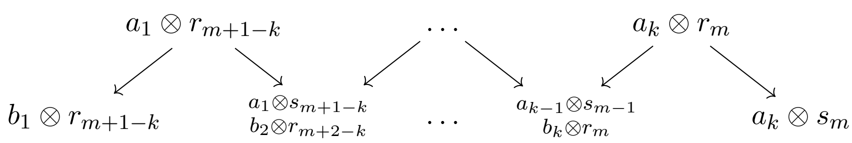
- $m-n$ copies of L_n, with $k = 2, \dots, m-n+1$ below.



- $D \otimes D$.



- M_k, $k = 2, \dots n-1$.



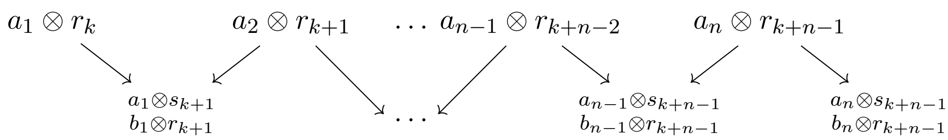
3.9 $R_n \otimes L_m$

$$R_n \otimes L_m \simeq \begin{cases} \mathbb{C}^{\oplus 2} \oplus \bigoplus_{i=1}^{n-1} M_i \oplus R_n^{\oplus m-n} \oplus \bigoplus_{j=2}^{n} W_j, & n \le m \\ \mathbb{C}^{\oplus 2} \oplus \bigoplus_{i=1}^{m-1} M_i \oplus L_m^{\oplus n-m} \oplus \bigoplus_{j=2}^{m} W_j, & n > m \end{cases}.$$

Dimension of left-hand side is $2nm$. Dimension of right-hand side in case $n \leq m$

$$\begin{aligned}
&2 + \sum_{i=1}^{n-1}(2i+1) + 2n(m-n) + \sum_{j=2}^{n}(2j-1) = \\
&= 2 + (n-1)(n+1) + 2n(m-n) + (n-1)(n+1) = \\
&= 2mn.
\end{aligned}$$

By symmetry and $\dim R_n = \dim L_n$ dimension is the same in case $n > m$. Direct summands in case $n \leq m$: imitate Step 1 from $M \otimes M$ and Step 3 from $W \otimes W$ to find $\mathbb{C} \oplus \bigoplus_{i=1}^{n-1} M_i$ resp. $\mathbb{C} \oplus \bigoplus_{j=2}^{n} W_j$. In Step 2 we find $m-n$ copies of R_n, with $k = 1, \ldots, m-n$.



3.10 $M_n \otimes L_m$

$$M_n \otimes L_m \simeq \begin{cases} \mathbb{C} \oplus \bigoplus_{i=1}^{n} L_i \oplus M_n^{\oplus m-n} \oplus \bigoplus_{j=1}^{n-1} M_j, & n \leq m \\ \mathbb{C} \oplus \bigoplus_{i=1}^{m-1} L_i \oplus L_m^{\oplus n-m+1} \oplus \bigoplus_{j=1}^{m-1} M_j, & n > m \end{cases}.$$

Dimension of left-hand side: $2nm+m$. Dimension of right-hand side in case $n \leq m$:

$$\begin{aligned}
&1 + \sum_{i=1}^{n} 2i + (m-n)(2n+1) + \sum_{j=1}^{n-1}(2j+1) \\
&= 1 + n(n+1) + (m-n)(2n+1) + (n-1)(n+1) = \\
&= 2nm + m
\end{aligned}$$

and in case $n > m$:

$$\begin{aligned}
&1 + \sum_{i=1}^{m-1} 2i + (n-m+1)2m + \sum_{j=1}^{m-1}(2j+1) = \\
&= 1 + (m-1)m + 2m(n-m+1) + (m-1)(m+1) = \\
&= 2mn + m.
\end{aligned}$$

Direct summands in case $n \leq m$: copy Step 1 and Step 2 from $M \otimes M$ directly, and Step 3 from $M \otimes W$ with shifted indices.

3.11 $M_n \otimes R_m$

$$M_n \otimes R_m \simeq \begin{cases} (D \otimes D) \oplus M_1^{\oplus m-1}, & n = 1 \\ (D \otimes D) \oplus \bigoplus_{i=2}^{n-1} M_i \oplus M_n^{\oplus m-n+1} \oplus \bigoplus_{j=1}^{n-1} R_j, & 1 < n \leq m \\ (D \otimes D) \oplus \bigoplus_{i=2}^{n} M_i \oplus R_m^{\oplus n-m} \oplus \bigoplus_{j=1}^{m-1} R_j, & n > m \end{cases}.$$

Dimension of left-hand side:

$$2n(m-1) + 2n + (m-1) + 2 = 2mn + m + 1.$$

Dimension of right-hand side in case $n = 1$:

$$4 + 3(m-1) = 3m + 1 = 2mn + m + 1$$

and in case $1 < n \leq m$:

$$\begin{aligned} &4 + \sum_{i=2}^{n-1}(2i+1) + (m-n+1)(2n+1) + \sum_{j=1}^{n-1} 2j = \\ &= 4 + (n-2)(n+2) + (m-n+1)(2n+1) + (n-1)n = \\ &= 2nm + m + 1 \end{aligned}$$

and in case $n > m$:

$$\begin{aligned} &4 + \sum_{i=2}^{m}(2i+1) + (n-m)2m + \sum_{j=1}^{m-1} 2j = \\ &= 4 + (m-1)(m+3) + 2m(n-m) + (m-1)m = \\ &= 2nm + m + 1. \end{aligned}$$

Direct summands in case $1 < n \leq m$: copy Step 1 and Step 2 from $M \otimes W$, but in Step 2, let k run up to $m - n$, yielding one extra copy of M_n. Then copy Step 3 from $M \otimes M$, with shifted indices.

3.12 Remaining cases

As already seen, tensor product of string bimodules is not commutative, but it still has certain symmetry properties. The remaining tensor products can be calculated using such, and in particular the lemma formulated below.

Definition. Let V be one of the string modules over $D \otimes D$, written as a D-D-bimodule, and let $\underline{v}$ be its basis. Then the *involution* of V is the string bimodule V' with basis $\underline{v'} = \underline{v}$ and X-actions $Xv' = vX$ and $v'X = Xv$ for any $v' \in \underline{v'}$.

In words, the involution exchanges left- and right actions.

Fix $n \geq 1$ and consider the $\mathbb{C}$-linear function $\alpha : W_n \to W_n$ defined via

$\alpha(a_i) = a_{n-i+1}$, $\alpha(b_i) = b_{n-i}$. It is clearly an isomorphism of complex vector spaces, but it also satisfies

$$\begin{aligned}\alpha(Xa_i) &= \alpha(b_i) = \\ &= b_{n-i} = \\ &= a_{n-i+1}X = \\ &= \alpha(a_i)X\end{aligned}$$

and similarly $\alpha(a_iX) = X\alpha(a_i)$. Also

$$\alpha(Xb_i) = \alpha(0) = 0 = b_{n-i}X = \alpha(b_i)X$$

and $\alpha(b_iX) = X\alpha(b_i)$. Hence, as D-D-bimodule morphism, α is an isomorphism $W_n \to W'_n$. Similarly we can exhibit, for any n, isomorphisms $M_n \simeq M'_n$ and $L_n \simeq R'_n$. It is clear that $(V')' \simeq V$ for any V, and that $(U \oplus V)' = U' \oplus V'$. In addition we have the following result.

Lemma 4. *Let U, V be string bimodules and U', V' their involutions. Then $U' \otimes V' \simeq (V \otimes U)'$.*

Proof. Let $\varphi : U' \to U$ be the $\mathbb{C}$-linear map with $\varphi(u') = u$, so that $\varphi(Xu') = \varphi(u')X$ and $\varphi(u'X) = X\varphi(u')$, and $\psi : V' \to V$ the corresponding map for V, V'. Define $\Phi : U' \otimes V' \to (V \otimes U)'$ by $\Phi(u' \otimes v') = \Big(\psi(v') \otimes \varphi(u')\Big)'$. First of all, Φ is well-defined, since

$$\begin{aligned}\Phi(u'X \otimes v') &= \Big(\psi(v') \otimes \varphi(u'X)\Big)' = \\ &= \Big(\psi(v') \otimes X\varphi(u')\Big)' = \\ &= \Big(\psi(v')X \otimes \varphi(u')\Big)' = \\ &= \Big(\psi(Xv') \otimes \varphi(u'X)\Big)' = \\ &= \Phi(u' \otimes Xv').\end{aligned}$$

Furthermore

$$\begin{aligned}\Phi(Xu' \otimes v') &= \Big(\psi(v') \otimes \varphi(Xu')\Big)' = \\ &= \Big(\psi(v') \otimes \varphi(u')X\Big)' = \\ &= X\Big(\psi(v') \otimes \varphi(u')\Big)' = \\ &= X\Phi(u' \otimes v')\end{aligned}$$

and similarly $\Phi(u' \otimes v'X) = \Phi(u' \otimes v')X$. Since φ and ψ are $\mathbb{C}$-module isomorphisms, this proves that Φ is a D-D-bimodule isomorphism. $\square$

Now the multiplication table for the string bimodules can be completed with the following results.

- $R_m \otimes W_n \simeq \begin{cases} \bigoplus_{j=1}^{n-1} W_j \oplus W_n^{\oplus m-n+1} \oplus \bigoplus_{i=1}^{n-1} R_i, & n \leq m \\ \bigoplus_{j=1}^{m} W_j \oplus R_m^{\oplus n-m} \oplus \bigoplus_{i=1}^{m-1} R_i, & n > m \end{cases}.$

- $L_m \otimes W_n \simeq \begin{cases} \bigoplus_{i=1}^{n} L_i \oplus W_n^{\oplus m-n} \oplus \bigoplus_{j=1}^{n-1} W_j, & n \leq m \\ \bigoplus_{i=1}^{m-1} L_i \oplus L_m^{\oplus n-m+1} \oplus \bigoplus_{j=1}^{m-1} W_j, & n > m \end{cases}.$

- $W_m \otimes M_n \simeq \begin{cases} \bigoplus_{i=1}^{n} R_i \oplus M_n^{\oplus m-n} \oplus \bigoplus_{j=1}^{n-1} L_j, & n \leq m \\ \bigoplus_{i=1}^{m} R_i \oplus W_m^{\oplus n-m} \oplus \bigoplus_{j=1}^{m-1} L_j, & n > m \end{cases}.$

- $R_m \otimes M_n \simeq \begin{cases} \mathbb{C} \oplus \bigoplus_{i=1}^{n} R_i \oplus M_n^{\oplus m-n} \oplus \bigoplus_{j=1}^{n-1} M_j, & n \leq m \\ \mathbb{C} \oplus \bigoplus_{i=1}^{m-1} R_i \oplus R_m^{\oplus n-m+1} \oplus \bigoplus_{j=1}^{m-1} M_j, & n > m \end{cases}.$

- $L_m \otimes M_n \simeq \begin{cases} (D \otimes D) \oplus M_1^{\oplus m-1}, & n = 1 \\ (D \otimes D) \oplus \bigoplus_{i=2}^{n-1} M_i \oplus M_n^{\oplus m-n+1} \oplus \bigoplus_{j=1}^{n-1} L_j, & 1 < n \leq m \\ (D \otimes D) \oplus \bigoplus_{i=2}^{n} M_i \oplus L_m^{\oplus n-m} \oplus \bigoplus_{j=1}^{m-1} L_j, & n > m \end{cases}.$

Remark. It can be noted that

$$R_n \otimes R_m \simeq L_n' \otimes L_m' \simeq (L_m \otimes L_n)',$$

which makes the explicit calculations for $R_n \otimes R_m$ redundant. However, they were made early on and are used for reference in the (avoidance of) later calculations, so the section for $R_n \otimes R_m$ is kept for practical reasons.

4 Tensoring by band bimodules

In this section the tensor products involving band bimodules are calculated. For tensor products between two band bimodules we give a recursive formula, see 4.7 and forth. Again m and n are positive integers.

4.1 $H_2(1) \otimes M_n$

$$H_2(1) \otimes M_n \simeq M_n^{\oplus 2}.$$

Dimension of left-hand side is $2 \cdot n \cdot 2 + 2 = 4n + 2$. Dimension of right-hand side is $2(2n+1) = 4n + 2$.
Identifications: first of all, for all j,

$$b_1 \otimes s_j = a_1 X \otimes s_j = a_1 \otimes X b_j = 0$$
$$b_1 \otimes r_j = a_1 X \otimes r_j = a_1 \otimes X r_j = a_1 \otimes s_j.$$

Furthermore,

$$b_1 \otimes s_j + b_2 \otimes s_j = a_2 X \otimes s_j = 0$$

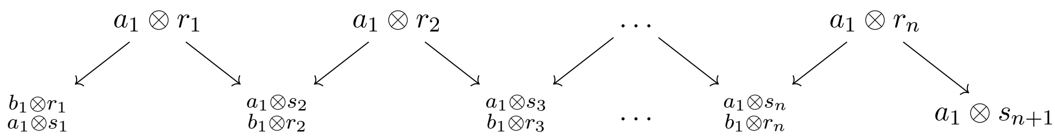
which implies that also all $b_2 \otimes s_j = 0$. Finally

$$a_2 \otimes s_j = a_2 \otimes X r_j = a_2 X \otimes r_j = (b_1 + b_2) \otimes r_j$$

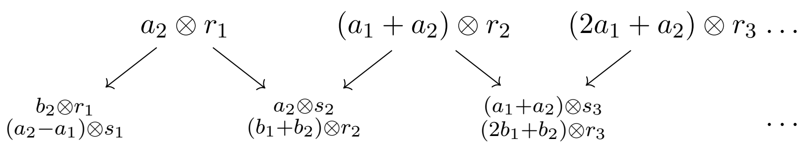
Hence, one choice of basis is the following:

$$\{a_i \otimes r_j,\, b_i \otimes r_j,\, a_i \otimes s_{n+1} \mid i = 1, 2,\, j = 1, \ldots, n\}.$$

This seems at first like a nice basis, since we immediately see one copy of M_n.



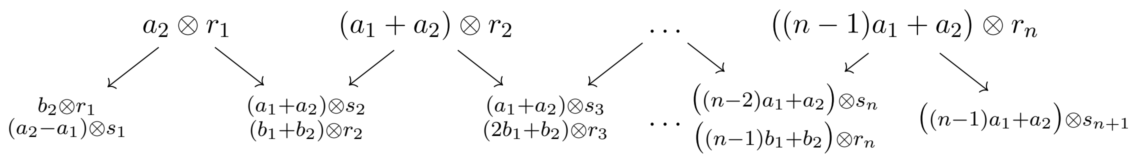
However, as we try to repeat this pattern with a_2 instead of a_1, we see the following:



Therefore we change basis to the union of the sets

$$\{a_1 \otimes r_j,\, b_1 \otimes r_j,\, a_1 \otimes s_{n+1}\}$$
$$\{\big(a_2 + (j-1)a_1\big) \otimes r_j,\, \big(b_2 + (j-1)b_1\big) \otimes r_j,\, (a_2 + n a_1) \otimes s_{n+1}\}.$$

The first set is exactly the basis of the first M_n above, which is left unchanged. Then a second copy appears as follows.



4.2 $H_2(1) \otimes W_n$

$$H_2(1) \otimes W_n \simeq W_n^{\oplus 2}.$$

Dimension of left hand side: $2(n-1)\cdot 2+2 = 4n-2$. Dimension of right-hand side: $2(2n-1) = 4n-2$. Identifications: for $i = 1, 2$ and $j = 1, \ldots, n-1$

$$\begin{aligned} b_i \otimes s_j &= b_i \otimes X r_{j+1} = 0 \\ a_1 \otimes s_j &= a_1 \otimes X r_{j+1} = b_1 \otimes r_{j+1} \\ a_2 \otimes s_j &= a_2 \otimes X r_{j+1} = (b_1 + b_2) \otimes r_{j+1}. \end{aligned}$$

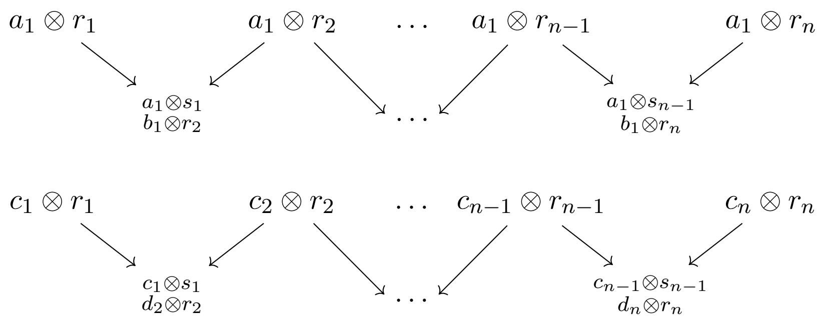
Set $c_j := (j-1)a_1 + a_2$ and $d_j := (j-1)b_1 + b_2$. Then $Xc_j = d_j$ and

$$\begin{aligned} c_j \otimes s_j &= c_j \otimes X r_{j+1} = \\ &= c_j X \otimes r_{j+1} = \\ &= \big((j-1)a_1 X + a_2 X\big) \otimes r_{j+1} = \\ &= \big((j-1)b_1 + b_1 + b_2\big) \otimes r_{j+1} = \\ &= (j b_1 + b_2) \otimes r_{j+1} = \\ &= d_{j+1} \otimes r_{j+1}. \end{aligned}$$

Basis:

$$\{a_1 \otimes r_j,\, a_1 \otimes s_j,\, c_j \otimes r_j,\, c_j \otimes s_j\}.$$

Direct summands: two copies of W_n as follows.



4.3 $H_2(1) \otimes R_n$

$$H_2(1) \otimes R_n \simeq R_n^{\oplus 2}.$$

Dimension of left-hand side: $2(n-1)\cdot 2+2\cdot 2 = 4n$. Dimension of right-hand side: $2 \cdot 2n = 4n$.

Identifications: firstly, for all j,

$$\begin{aligned} b_1 \otimes s_j &= a_1 X \otimes s_j = a_1 \otimes X s_j = 0 \\ (b_1 + b_2) \otimes s_j &= a_2 X \otimes s_j = a_2 \otimes X s_j = 0 \\ b_1 \otimes r_1 &= a_1 X \otimes r_1 = a_1 \otimes X r_1 = 0 \\ (b_1 + b_2) \otimes r_1 &= a_2 X \otimes r_1 = a_2 \otimes X r_1 = 0 \end{aligned}$$

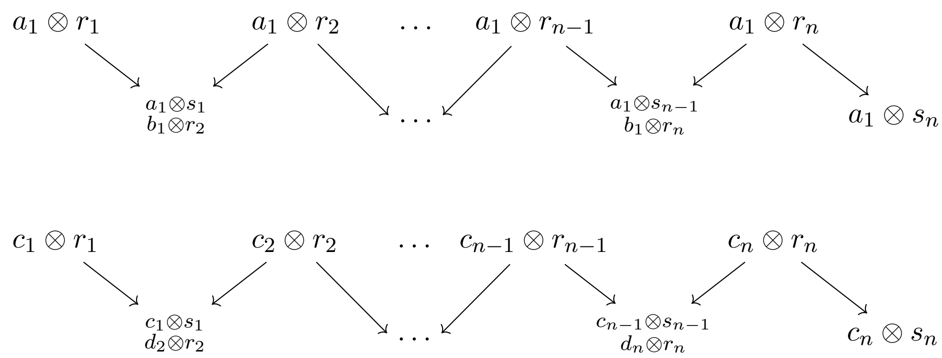
which implies that all $b_i \otimes s_j = 0$, $b_i \otimes r_1 = 0$. Also, for $j = 1, \ldots, n-1$,

$$a_1 \otimes s_j = a_1 \otimes X r_{j+1} = a_1 X \otimes r_{j+1} = b_1 \otimes r_{j+1}$$
$$a_2 \otimes s_j = a_2 \otimes X r_{j+1} = a_2 X \otimes r_{j+1} = (b_1 + b_2) \otimes r_{j+1}.$$

Set again $c_j := (j-1)a_1 + a_2$ and $d_j := (j-1)b_1 + b_2$ and choose as basis

$$\{a_1 \otimes r_j,\, a_1 \otimes s_j,\, c_j \otimes r_j,\, c_j \otimes s_j\}.$$

Direct summands: two copies of R_n as follows.



4.4 $H_2(1) \otimes L_n$

$$H_2(1) \otimes L_n \simeq L_n^{\oplus 2}.$$

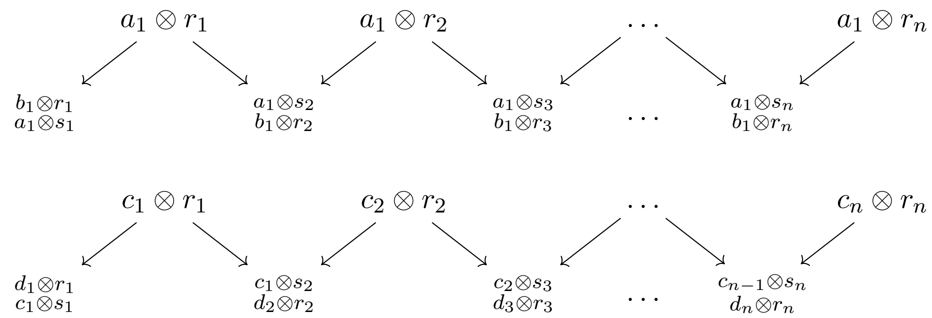
Both sides have dimension $2n \cdot 2 = 4n$.
Identifications:

$$b_i \otimes s_j = b_i \otimes X r_j = b_i X \otimes r_j = 0$$
$$a_1 \otimes s_j = a_1 \otimes X r_j = a_1 X \otimes r_j = b_1 \otimes r_j$$
$$a_2 \otimes s_j = a_2 \otimes X r_j = a_2 X \otimes r_j = (b_1 + b_2) \otimes r_j.$$

Set again $c_j := (j-1)a_1 + a_2$ and $d_j := (j-1)b_1 + b_2$ and choose as basis

$$\{a_1 \otimes r_j,\, a_1 \otimes s_j,\, c_j \otimes r_j,\, c_j \otimes s_j\}.$$

Direct summands: two copies of L_n as follows.



4.5 $- \otimes H_2(1)$

$$M_n \otimes H_2(1) \simeq M_n^{\oplus 2}.$$

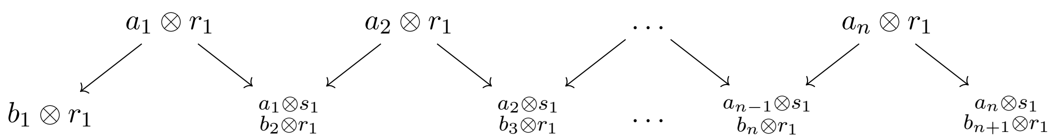
Dimension of left-hand side is $n \cdot 2 \cdot 2 + 2 = 4n + 2$. Dimension of right-hand side is $2(2n+1) = 4n + 2$.
Identifications:

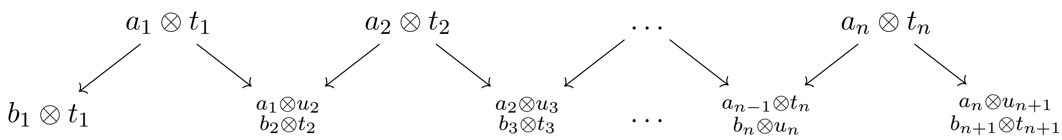
$$b_i \otimes s_j = b_i \otimes X r_j = b_i X \otimes r_j = 0$$
$$a_i \otimes s_j = a_i \otimes X r_j = a_i X \otimes r_j = b_{i+1} \otimes r_j.$$

Now set $t_i := (i-1)r_1 + r_2$ and $u_i := (i-1)s_1 + s_2$. Then $t_i X = u_{i+1}$ and $a_i \otimes u_i = b_{i+1} \otimes t_i$. Choose as basis

$$\{a_i \otimes r_1,\ b_i \otimes r_1,\ a_i \otimes t_i,\ b_i \otimes t_i\}.$$

Now we see two copies of M_n as follows.





We see that this is almost identical to $H_2 \otimes M_n$. Therefore the following results do not come as a surprise.

- $W_n \otimes H_2(1) \simeq W_n^{\oplus 2}$.
- $R_n \otimes H_2(1) \simeq R_n^{\oplus 2}$.
- $L_n \otimes H_2(1) \simeq L_n^{\oplus 2}$.

4.6 $H_2(1) \otimes H_2(1)$

$$H_2(1) \otimes H_2(1) \simeq H_1(1) \oplus H_3(1).$$

Dimension of left-hand side is $2 \cdot 2 \cdot 2 = 8$. Dimension of right-hand side is $2 + 6 = 8$.
Identifications: for $j = 1, 2$

$$b_1 \otimes s_j = a_1 X \otimes s_j = a_1 \otimes X s_j = 0$$
$$b_2 \otimes s_1 = b_1 \otimes X r_1 = b_1 X \otimes r_1 = 0$$
$$(b_1 + b_2) \otimes s_2 = a_2 X \otimes s_2 = a_2 \otimes X s_2 = 0.$$

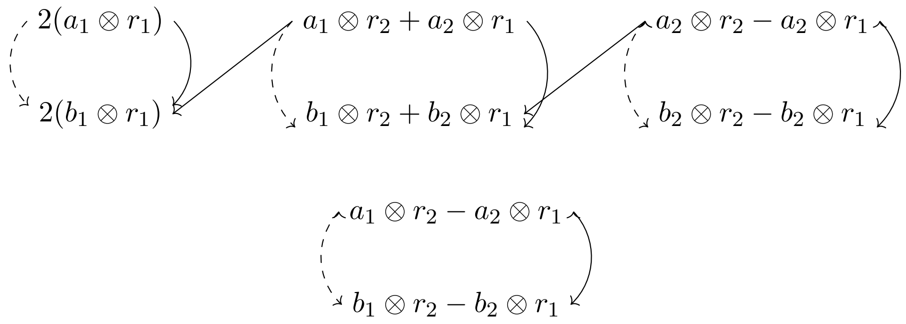
The last two lines above together imply that also $b_2 \otimes s_2 = 0$, so that all $b_i \otimes s_j = 0$. Furthermore

$$a_1 \otimes s_j = a_1 \otimes Xr_j = b_1 \otimes r_j$$
$$a_2 \otimes s_j = a_2 \otimes Xr_j = (b_1 + b_2) \otimes r_j.$$

Hence one possible choice of basis is

$$\{a_i \otimes r_j,\, b_i \otimes r_j \mid i, j = 1, 2\}.$$

Again we need to change basis. Unlike the previous cases, there is no obvious nice basis in which the decomposition is directly visible. One alternative, which is not too bad, is given below. It is easily verified that the indicated set really is a basis.



Calculations for $H_3(1)$:
The actions on $2(a_1 \otimes r_1)$ are

$$X2(a_1 \otimes r_1) = 2(Xa_1 \otimes r_1) = 2(b_1 \otimes r_1),$$

$$\begin{aligned} 2(a_1 \otimes r_1)X &= 2(a_1 \otimes r_1 X) = \\ &= 2(a_1 \otimes s_1) = \\ &= 2(b_1 \otimes r_1). \end{aligned}$$

The actions on $a_1 \otimes r_2 + a_2 \otimes r_1$ are

$$X(a_1 \otimes r_2 + a_2 \otimes r_1) = b_1 \otimes r_2 + b_2 \otimes r_1$$

and

$$\begin{aligned} (a_1 \otimes r_2 + a_2 \otimes r_1)X &= a_1 \otimes (s_1 + s_2) + a_2 \otimes s_1 = \\ &= b_1 \otimes (r_1 + r_2) + (b_1 + b_2) \otimes r_1 = \\ &= 2(b_1 \otimes r_1) + b_1 \otimes r_2 + b_2 \otimes r_1. \end{aligned}$$

The actions on $a_2 \otimes r_2 - a_2 \otimes r_1$ are

$$X(a_2 \otimes r_2 - a_2 \otimes r_1) = b_2 \otimes r_2 - b_2 \otimes r_1$$

and

$$\begin{aligned}(a_2 \otimes r_2 - a_2 \otimes r_1)X &= a_2 \otimes (s_1 + s_2) - a_2 \otimes s_1 = \\ &= a_2 \otimes s_2 = \\ &= (b_1 + b_2) \otimes r_2 = \\ &= b_1 \otimes r_2 + b_2 \otimes r_1 + b_2 \otimes r_2 - b_2 \otimes r_1.\end{aligned}$$

Calculations for $H_1(1)$:

$$X(a_1 \otimes r_2 - a_2 \otimes r_1) = b_1 \otimes r_2 - b_2 \otimes r_1$$

and

$$\begin{aligned}(a_1 \otimes r_2 - a_2 \otimes r_1)X &= a_1 \otimes (s_1 + s_2) - a_2 \otimes s_1 = \\ &= b_1 \otimes (r_1 + r_2) - (b_1 + b_2) \otimes r_1 = \\ &= b_1 \otimes r_2 - b_2 \otimes r_1.\end{aligned}$$

4.7 $H_2(1) \otimes H_n(1)$

For integers $n > 1$

$$H_2(1) \otimes H_n(1) \simeq H_{n-1}(1) \oplus H_{n+1}(1).$$

Since $H_1(1) \simeq {}_{\mathrm{D}}\mathrm{D}_{\mathrm{D}}$ is the identity with respect to $\otimes_D$, we need not consider the case $n = 1$ further. The awkward change of basis in the case of $n = 2$ above inspires us to use a less explicit method in the general case. As above, the following identifications hold:

$$\begin{aligned}b_i \otimes s_j &= 0 \\ a_1 \otimes s_j &= b_1 \otimes r_j \\ a_2 \otimes s_j &= (b_1 + b_2) \otimes r_j.\end{aligned}$$

We can choose as basis

$$\{a_i \otimes r_j,\, b_i \otimes r_j \mid i = 1, 2,\, j = 1, \dots, n\}.$$

In this basis, left X-action is easily described: $Xa_i \otimes r_j = b_i \otimes r_j$ and $Xb_i \otimes r_j = 0$. From the identifications it is also clear that $b_i \otimes r_j X = 0$, and finally that

$$a_i \otimes r_j X = \sum_{i,j} \lambda_{ij}(b_i \otimes r_j)$$

where all $\lambda_{ij} \in \{0,1\}$. In words, both actions are 0 on any $b \otimes r$, and both actions on any $a \otimes r$ result in sums with summands only from the set $\{b_i \otimes r_j\}$. Let us therefore identify the elements related by left X-action, i.e. set $\overline{a_i \otimes r_j} = \overline{b_i \otimes r_j}$. In this way, we don't lose any information about actions, as long as we keep the above discussion in mind. Now the left X-action is the identity. As for the right X-action,

$$\begin{aligned} a_i \otimes r_1 X &= a_i \otimes s_1 = \\ &= a_i \otimes X r_1 = \\ &= a_i X \otimes r_1. \end{aligned}$$

Hence $\overline{a_1 \otimes r_1}X = \overline{b_1 \otimes r_1} = \overline{a_1 \otimes r_1}$, and in the same way $\overline{a_2 \otimes r_1}X = \overline{(a_1 + a_2) \otimes r_1}$. For $j \geq 2$,

$$\begin{aligned} a_i \otimes r_j X &= a_i \otimes (s_{j-1} + s_j) = \\ &= a_i \otimes X(r_{j-1} + r_j) = \\ &= a_i X \otimes (r_{j-1} + r_j). \end{aligned}$$

Thus, if we sort the basis vectors in the order

$$a_1 \otimes r_1,\ a_1 \otimes r_2, \ldots, a_1 \otimes r_n,\ a_2 \otimes r_1, \ldots, a_2 \otimes r_n$$

the matrix of the right X-action is a $2n \times 2n$-block matrix $X_R = \begin{bmatrix} J_n(1) & J_n(1) \\ 0 & J_n(1) \end{bmatrix}$ where

$$J_n(1) = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

i.e. $J_n(1)$ is a Jordan block of size n with eigenvalue 1. Now we recognise X_R as the matrix tensor product (or Kroenecker product) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \otimes J_n(1) = J_2(1) \otimes J_n(1)$. The formula for tensor products of Jordan blocks is known, see for example [2]. This yields that X_R can be decomposed to $J_{n-1}(1) \oplus J_{n+1}(1)$ in some basis of the form

$$\left\{ \sum_{i,j} \lambda_{ij}^k (\overline{a_i \otimes r_j}) \mid k = 1, \ldots, 2n,\ \lambda_{ij}^k \in \mathbb{C} \right\}.$$

Now we have the right X-action

$$\sum_{i,j} \lambda_{ij}^1 (\overline{a_i \otimes r_j}) X = \sum_{i,j} \lambda_{ij}^1 (\overline{a_i \otimes r_j}) \tag{1}$$

$$\sum_{i,j} \lambda_{ij}^n (\overline{a_i \otimes r_j}) X = \sum_{i,j} \lambda_{ij}^n (\overline{a_i \otimes r_j}) \tag{2}$$

and for $k \neq 1, n$

$$\sum_{i,j} \lambda_{ij}^k (\overline{a_i \otimes r_j}) X = \sum_{i,j} \lambda_{ij}^{k-1} (\overline{a_i \otimes r_j}) + \sum_{i,j} \lambda_{ij}^k (\overline{a_i \otimes r_j}). \tag{3}$$

Now go back to $H_2 \otimes H_n$ and choose there corresponding basis, i.e.

$$\left\{ \sum_{i,j} \lambda_{ij}^k (a_i \otimes r_j),\ \sum_{i,j} \lambda_{ij}^k (b_i \otimes r_j) \mid k = 1, \ldots, 2n,\ \lambda_{ij}^k \in \mathbb{C} \right\}.$$

Again, left action is just

$$X \sum_{i,j} \lambda_{ij}^k (a_i \otimes r_j) = \sum_{i,j} \lambda_{ij}^k (b_i \otimes r_j)$$
$$X \sum_{i,j} \lambda_{ij}^k (b_i \otimes r_j) = 0.$$

As for the right X-action,

$$\sum_{i,j} \lambda_{ij}^k (b_i \otimes r_j) X = 0.$$

Furthermore, combining (1) -(3) with the fact that $a_i \otimes r_j X$ is always a sum of $b_i \otimes r_j$-terms, we conclude that

$$\sum_{i,j} \lambda_{ij}^1 (a_i \otimes r_j) X = \sum_{i,j} \lambda_{ij}^1 (b_i \otimes r_j)$$
$$\sum_{i,j} \lambda_{ij}^n (a_i \otimes r_j) X = \sum_{i,j} \lambda_{ij}^n (b_i \otimes r_j)$$

and for $k \neq 1, n$

$$\sum_{i,j} \lambda_{ij}^k (a_i \otimes r_j) X = \sum_{i,j} \lambda_{ij}^{k-1} (b_i \otimes r_j) + \sum_{i,j} \lambda_{ij}^k (b_i \otimes r_j).$$

Hence the formula $H_2(1) \otimes H_n(1) \simeq H_{n-1}(1) \oplus H_{n+1}(1)$ is proven.

4.8 $H_n(1) \otimes H_2(1)$

For any $n \geq 1$,

$$H_n(1) \otimes H_2(1) \simeq H_2(1) \otimes H_n(1).$$

The cases $n = 1, 2$ are clear. Now fix any $n \geq 3$ and assume that the formula holds for $n-1, n-2$. For readability, we omit the 1 which specifies the eigenvalue. We shall use the recursive formula $H_2 \otimes H_n \simeq H_{n-1} \oplus H_{n+1}$, which for $n \geq 3$ is equivalent to

$$H_n \simeq H_2 \otimes H_{n-1} \oplus H_{n-2}^{\oplus -1}. \tag{4}$$

Then

$$\begin{aligned}
H_n \otimes H_2 &\overset{(4)}{\simeq} (H_2 \otimes H_{n-1} \oplus H_{n-2}^{\oplus -1}) \otimes H_2 \simeq \\
&\simeq H_2 \otimes H_{n-1} \otimes H_2 \oplus H_{n-2}^{\oplus -1} \otimes H_2 \simeq \\
&\simeq H_2 \otimes H_2 \otimes H_{n-1} \oplus H_2 \otimes H_{n-2}^{\oplus -1} \simeq \\
&\simeq H_2 \otimes (H_2 \otimes H_{n-1} \oplus H_{n-2}^{\oplus -1}) \overset{(4)}{\simeq} \\
&\simeq H_2 \otimes H_n
\end{aligned}$$

so by induction,

$$H_n \otimes H_2 \simeq H_2 \otimes H_n \simeq H_{n-1} \oplus H_{n+1}$$

for any $n \geq 1$.

4.9 Arbitrary eigenvalues

In the following, let λ and μ be any nonzero complex numbers.

4.9.1 $H_1(\lambda) \otimes H_n(\mu)$

For any $n \geq 1$ and any $\lambda, \mu \neq 0$,

$$H_1(\lambda) \otimes H_n(\mu) \simeq H_n(\lambda\mu).$$

Identifications:

$$\begin{aligned}
&a \otimes s_i = a \otimes Xr_i = aX \otimes r_i = \lambda b \otimes r_i \\
&b \otimes s_i = 0.
\end{aligned}$$

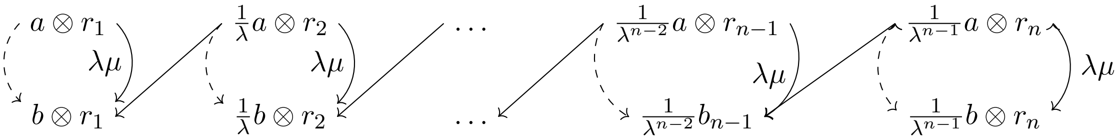
Choose as basis $\{\frac{1}{\lambda^{i-1}} a \otimes r_i,\ \frac{1}{\lambda^{i-1}} b \otimes r_i \mid i = 1, \ldots, n\}$. Then the nontrivial X-actions are

$$X \frac{1}{\lambda^{i-1}} a \otimes r_i = \frac{1}{\lambda^{i-1}} b \otimes r_i,$$
$$a \otimes r_1 X = \mu a \otimes s_1 = \lambda\mu b \otimes r_1,$$

and for $i = 2, \ldots, n$

$$\begin{aligned}
\frac{1}{\lambda^{i-1}} a \otimes r_i X &= \frac{1}{\lambda^{i-1}} a \otimes (s_{i-1} + \mu s_i) = \\
&= \frac{\lambda}{\lambda^{i-1}} b \otimes (r_{i-1} + \mu r_i) = \\
&= \frac{b \otimes r_{i-1}}{\lambda^{i-2}} + \lambda\mu \frac{b \otimes r_i}{\lambda^{i-1}}.
\end{aligned}$$

Hence the diagram is



i.e. $H_n(\lambda\mu)$.
Similarly, $H_n(\mu) \otimes H_1(\lambda) \simeq H_n(\lambda\mu)$.

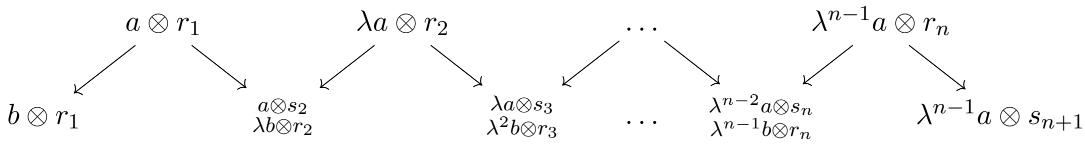
4.9.2 $H_1(\lambda) \otimes -$

For any $n \geq 1$ and $\lambda \neq 0$,

$$H_1(\lambda) \otimes M_n \simeq M_n.$$

Identifications: for $i = 1, \dots, n-1$

$$a \otimes s_i = a \otimes X r_i = aX \otimes r_i = \lambda b \otimes r_i$$
$$b \otimes s_i = b \otimes X r_i = 0.$$



In the same way, tensoring any string bimodule U with $H_1(\lambda)$, from the left or from the right, results in U again.

4.9.3 $H_n(\lambda) \otimes -$

For any $n, m \geq 1$ and $\lambda \neq 0$,

$$H_n(\lambda) \otimes M_m \simeq M_m^{\oplus n}.$$

By the previous result, it is enough to show the formula for $\lambda = 1$. For readability the 1 indicating eigenvalue is omitted. The proof is done by induction. As in 4.8 we use (4), i.e. $H_n \simeq H_2 \otimes H_{n-1} \oplus H_{n-2}^{\oplus -1}$ for $n \geq 3$. The basis of induction is $n = 1$ and $n = 2$. $H_1(1) \simeq {}_{\mathrm{D}}\mathrm{D}_{\mathrm{D}}$ is the identity, so for $n = 1$ the formula trivially holds. The case $n = 2$ is done above. Now fix any $n \geq 3$ and assume that the formula holds for $n-1$ and $n-2$. Then

$$\begin{aligned} H_n \otimes M_m &\overset{(4)}{\simeq} \left(H_2 \otimes H_{n-1} \oplus H_{n-2}^{\oplus -1}\right) \otimes M_m \simeq \\ &\simeq H_2 \otimes H_{n-1} \otimes M_m \oplus \left(H_{n-2} \otimes M_m\right)^{\oplus -1} \simeq \\ &\simeq H_2 \otimes M_m^{\oplus n-1} \oplus \left(M_m^{\oplus n-2}\right)^{\oplus -1} \simeq \\ &\simeq \left(M_m^{\oplus n-1}\right)^{\oplus 2} \oplus \left(M_m^{\oplus n-2}\right)^{\oplus -1} \simeq \\ &\simeq M_m^{\oplus 2n-2} \oplus M_m^{\oplus 2-n} \simeq \\ &\simeq M_m^{\oplus n} \end{aligned}$$

so the result follows.

Remark. We see now that the band bimodules commute with everything: for any D-D-bimodule U, any $n \geq 1$ and any $\lambda \neq 0$, $H_n(\lambda) \otimes U \simeq U \otimes H_n(\lambda)$.

5 Cells

Let $\mathcal{B}$ be the set of isomorphism classes of indecomposable D-D-bimodules. In the previous sections the multiplication table for the elements of $\mathcal{B}$ is given. This enables us to consider the cell structure of D-D-bimodules in the sense of [4].

Definition. Define the *left preorder* $\leq_{\mathrm{L}}$ on $\mathcal{B}$ by the following: $U \leq_{\mathrm{L}} V$ if V is isomorphic to a direct summand of $T \otimes U$ for some D-D-bimodule T. If $U \leq_{\mathrm{L}} V$ and $V \leq_{\mathrm{L}} U$ we say that U and V are *left equivalent* and write $U \sim_{\mathrm{L}} V$. The equivalence classes with respect to $\sim_{\mathrm{L}}$ are called *left cells*.
Similarly, define the *right preorder* $\leq_{\mathrm{R}}$, the equivalence relation $\sim_{\mathrm{R}}$, and the equivalence classes *right cells*, by tensoring from the right.
Finally, define the *two-sided preorder* $\leq_{\mathrm{J}}$ by $U \leq_{\mathrm{J}} V$ if V is isomorphic to a direct summand of $S \otimes U \otimes T$ for some $S, T \in D$-mod-D. Denote the corresponding equivalence relation by $\sim_{\mathrm{J}}$ and the equivalence classes by *two-sided cells*.

Proposition 5. *All band bimodules belong to the same minimal left-, right-, and two sided cell.*

Proof. The band bimodules span a commutative subring of the Grothendieck ring $[D\text{-mod-}D]_{\oplus}$. We have already noted that they commute with everything. Recall now that the identity ${}_{\mathrm{D}}\mathrm{D}_{\mathrm{D}} \simeq H_1(1)$ is a band bimodule, and note that the span of the band bimodules is closed under $\otimes_D$. Restricted to this subring, the notions of left, right, and two sided cells (preorder, equivalence) coincide. Therefore it suffices to show that the band bimodules belong to the same minimal left cell. The identity bimodule ${}_{\mathrm{D}}\mathrm{D}_{\mathrm{D}} \simeq H_1(1)$ is clearly minimal with respect to the left order. Now we show that for any $n \geq 1$ and

any $\lambda \neq 0$, $H_n(\lambda)$ is in the same left cell as ${}_{\mathrm{D}}\mathrm{D}_{\mathrm{D}}$.
First of all, since

$$H_n(1) \simeq H_1(\lambda^{-1}) \otimes H_n(\lambda)$$

and

$$H_n(\lambda) \simeq H_1(\lambda) \otimes H_n(1),$$

it is clear that $H_n(\lambda) \sim_{\mathrm{L}} H_n(1)$. Hence we need only consider $\lambda = 1$. Since for any $n \geq 2$ it holds that

$$H_2(1) \otimes H_n(1) \simeq H_{n-1}(1) \oplus H_{n+1}(1)$$

we always have $H_n(1) \leq_{\mathrm{L}} H_{n-1}(1)$, so by induction $H_n(1) \sim_{\mathrm{L}} {}_{\mathrm{D}}\mathrm{D}_{\mathrm{D}}$ for any n. □

Remark. The commutative subring spanned by the band bimodules is maximal in $[D\text{-mod-}D]_\oplus$, and hence constitutes the center $Z([D\text{-mod-}D]_\oplus)$.

We denote the minimal two-sided cell containing the regular bimodule D by $\mathcal{J}_D$. Now, from the previous sections it is clear that D is not left, right or two-sided equivalent to any string bimodule or $D \otimes D$. Denote by $\mathcal{S}$ the set of string bimodules together with $D \otimes D$. Since $\mathcal{S}$ spans a two-sided ideal in $[D\text{-mod-}D]_\oplus$, we can forget band bimodules for now. From 2.2 we see that the $\mathbb{C}$-split bimodules too span an ideal in $[D\text{-mod-}D]_\oplus$. As for the string bimodules, recall the intuition in 2.4 about the graphs associated to string bimodules. It tells us that if we start with a graph and tensor it by something we can never obtain a bigger graph, in the sense that the number of sources, or upper nodes, can't increase. Let us formalise this observation.

Definition. For a string bimodule $U \neq W_1$, let $U_0 = \{u \in U \mid Xu = uX = 0\}$ be the submodule on which both X-actions are always trivial. Define the *width* of U to be $\mathrm{w}(U) = \dim(U) - \dim(U_0)$. Define furthermore $\mathrm{w}(W_1) := 1$ and $\mathrm{w}(D \otimes D) := 1$.

Defined as above, the width of a string bimodule is exactly the index we use to specify it, or the number of sources in the corresponding digraph.

Lemma 6. *Let $U, V \in \mathcal{S}$ and assume that T is an indecomposable direct summand of $U \otimes V$. Then $T \in \mathcal{S}$ and $\mathrm{w}(T) \leq \min\big(\mathrm{w}(U), \mathrm{w}(V)\big)$.*

As a direct consequence fo the above lemma, if $\mathrm{w}(U) < \mathrm{w}(V)$, then $U \leq_{\mathrm{J}} V$ etc. is impossible. Hence we draw the following conclusion.

Proposition 7. *If $U, V \in \mathcal{S}$ and $\mathrm{w}(U) \neq \mathrm{w}(V)$, then U and V don't belong to the same left, right, or two-sided cell.*

For bimodules of width 1, there are the $\mathbb{C}$-split bimodules and M_1. From the discussion in 2.2 we immediately see the following.

Proposition 8. *The $\mathbb{C}$-split bimodules form a maximal two-sided cell $\mathcal{J}_{\mathbb{C}}$. M_1 does not belong to this maximal cell.*

Proposition 9. *Within $\mathcal{J}_{\mathbb{C}}$,*

(i) left cells are given by $\mathbb{C} \sim_{\mathrm{L}} D \otimes \mathbb{C}$ and $\mathbb{C} \otimes D \sim_{\mathrm{L}} D \otimes D$, and these cells are not related.

(ii) right cells are given by $\mathbb{C} \sim_{\mathrm{R}} \mathbb{C} \otimes D$ and $D \otimes \mathbb{C} \sim_{\mathrm{R}} D \otimes D$, and these cells are not related.

Since $M_1 \notin \mathcal{J}_{\mathbb{C}}$, we find that M_1 forms its own two-sided cell. Call it $\mathcal{J}_{M_1}$. Now use $M_1 \otimes M_1 \simeq \mathbb{C} \oplus (D \otimes D)$ to conclude that $\mathcal{J}_{M_1} \leq_{\mathrm{J}} \mathcal{J}_{\mathbb{C}}$. Finally, we consider each fixed $n \geq 2$ and the four string bimodules of width n.

Proposition 10. *For every $n \geq 2$, right cells are given by $W_n \sim_{\mathrm{R}} R_n$ and $M_n \sim_{\mathrm{R}} L_n$. These two cells are not related.*

Proof. $W_n \sim_{\mathrm{R}} R_n$: R_n appears as a direct summand in $W_n \otimes R_n$, so $W_n \leq_{\mathrm{R}} R_n$. Also, W_n is a direct summand of $R_n \otimes W_n$, so $R_n \leq_{\mathrm{R}} W_n$. Thus $W_n \sim_{\mathrm{R}} R_n$.
$M_n \sim_{\mathrm{R}} L_n$: look at $M_n \otimes L_n$ and $L_n \otimes M_n$ to see that $M_n \leq_{\mathrm{R}} L_n$ and $L_n \leq_{\mathrm{R}} M_n$, respectively.
After examining all formulas in Section 3, we can conclude that these two right cells are not related. □

Proposition 11. *For every $n \geq 2$, left cells are given by $W_n \sim_{\mathrm{L}} L_n$ and $M_n \sim_{\mathrm{L}} R_n$, and these two left cells are not related.*

Proof. $W_n \sim_{\mathrm{L}} L_n$: From $W_n \otimes L_n$ we see that $L_n \leq_{\mathrm{L}} W_n$. In the other direction, look at $L_m \otimes W_n$ for $m \geq n$ to see that $W_n \leq_{\mathrm{L}} L_n$.
$M_n \sim_{\mathrm{L}} R_n$: From $R_n \otimes M_n$ we see $M_n \leq_{\mathrm{L}} R_n$, and from $M_n \otimes R_n$ that $R_n \leq_{\mathrm{L}} M_n$.
Again, we examine all formulas in Section 3 and find that these left cells are not related. □

Proposition 12. *For each $n \geq 2$, all four string bimodules W_n, L_n, R_n and M_n of width n belong to the same two-sided cell $\mathcal{J}_n$.*

Proof. Since the one-sided equivalences both imply two-sided equivalence, we can combine $W_n \leq_{\mathrm{R}} R_n$, $R_n \leq_{\mathrm{L}} M_n$, $M_n \leq_{\mathrm{R}} L_n$, and $L_n \leq_{\mathrm{L}} W_n$ to obtain

$$W_n \leq_{\mathrm{J}} R_n \leq_{\mathrm{J}} M_n \leq_{\mathrm{J}} L_n \leq_{\mathrm{J}} W_n.$$

□

The cell $\mathcal{J}_n$ has the structure

M_n	R_n
L_n	W_n

where rows indicate left cells and columns right cells. In the cell $\mathcal{J}_{\mathbb{C}}$, the structure instead is as follows.

$D \otimes D$	$\mathbb{C} \otimes D$
$D \otimes \mathbb{C}$	$\mathbb{C}$

Proposition 13. *The two-sided cells are linearly ordered as follows.*

$$\mathcal{J}_{\mathbb{C}} \geq_J \mathcal{J}_{M_1} \geq_J \mathcal{J}_2 \geq_J \mathcal{J}_3 \geq_J \ldots \geq_J \mathcal{J}_D$$

Proof. We have already established that $\mathcal{J}_{\mathbb{C}}$ is maximal and $\mathcal{J}_D$ is minimal. Therefore is suffices to look at $M_n \otimes M_m$ and conclude that $M_1 \geq_J M_2 \geq_J M_3 \geq_J \ldots$ □

References

[1] Butler M.C.R, Ringel C.M. Auslander-Reiten sequences with few middle terms and applications to string algebras. Comm. Algebra 15 (1987), no. 1-2, 145–179.

[2] Darpö E, Herschend M. On the representation ring of the polynomial algebra over a perfect field. Math. Z. (2010) 265:601-615.

[3] Drozd Yu.A., Kirichenko V.V. Finite Dimensional Algebras. (Dlab V, Trans.) 1st ed. Springer, 1994.

[4] Mazorchuk V, Miemietz V. Additive versus abelian 2-representations of fiat 2-categories. Moscow Math. J. 14 (2014), No.3, 595-615.

[5] Mazorchuk V, Miemietz V, Zhang X. Characterisation and applications of $\Bbbk$-split bimodules. Preprint arXiv:1701.03025.

[6] Wald B, Waschbusch J. Tame biserial algebras, J. Algebra 95 (1985), no. 2, 480–500.