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Tilting modules in *d*-cluster tilting subcategories

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Abstract

In this paper we look at generalised tilting modules of projective dimension at most d, introduced by Miyashita in [1]. These should satisfy an Ext-vanishing property and a generating property. The latter is usually difficult to verify. In 1989 Rickard states a conjecture saying that we can replace the generating property with the condition that the module should have the same number of non-isomorphic indecomposable direct summands as the regular module. We will confirm this conjecture by classifying all tilting modules inside a d-cluster tilting subcategory of the module category of A, where A is either in a family of acyclic Nakayama algebras or $A = KQ \otimes KQ$ where Q is the two subspace quiver.

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Contents

1 Introduction

The history of tilting theory starts with the introduction of reflection functors by Bernstein, Gelfand and Ponomarev in [2] and the interpretation of these by Auslander, Platzeck and Reiten in [3]. The reflection functors are used to compare module-categories. Brenner and Butler continued this work in [4] and introduced tilting functors and tilting modules, but with a more restrictive definition than we will use. However, the definition of tilting modules that is most commonly used today is due to Happel and Ringel in [5]. The definition looks as follows:

Definition 1.1. Let A be a K-algebra over a field K and $T_A \in$ mod-A. T is called a *tilting module* if it satisfies the following conditions:
(P) There exists an exact sequence

$$0 \to P_1 \to P_0 \to T \to 0$$

where P_0, P_1 are projective A-modules. That is, $\mathrm{pdim}(T) \leq 1$.
(E) $\mathrm{Ext}^1_A(T, T) = 0$
(G) There exists an exact sequence

$$0 \to A \to T' \to T'' \to 0$$

with $T', T'' \in \mathrm{add}(T)$. If T only satisfies (P) and (E) we call T a *partial tilting module*.

The notion of tilting modules has been generalised in different ways, for example by Rickard in [6] and Miyashita in [1]. Miyashita chooses to generalise tilting modules to higher projective dimension. These will be studied further in Section 4. Jeremy Rickard looks at what he calls tilting complexes instead of tilting modules.

Definition 1.2. Let T be an object in $K^b(\text{proj-}A)$, the homotopy category of bounded complexes of finitely generated projective modules over a K-algebra A. T is called a *tilting complex* if it satisfies the following two conditions:

1. $\mathrm{Hom}(T, T[i]) = 0$ for $i \neq 0$
2. $\mathrm{add}(T)$ generates $K^b(\text{proj-}A)$ as a triangulated category.

These tilting complexes are used to formulate a 'Morita theorem' for derived categories.

Theorem 1.3. [6, Theorem 6.4] Let A and B be two K-algebras. The following conditions are equivalent:

1. $K^-(\text{Proj-}A)$ and $K^-(\text{Proj-}B)$ are equivalent as triangulated categories
2. $D^b(\text{Mod-}A)$ and $D^b(\text{Mod-}B)$ are equivalent as triangulated categories
3. $K^b(\text{Proj-}A)$ and $K^b(\text{Proj-}B)$ are equivalent as triangulated categories
4. $K^b(\text{proj-}A)$ and $K^b(\text{proj-}B)$ are equivalent as triangulated categories
5. B is isomorphic to $\text{End}(T)$, where T is a tilting complex in $K^b(\text{proj-}A)$.

In [7], Bongartz proves the following theorem about classical tilting modules.

Theorem 1.4. [7, Section 2.1] Let A be a finite-dimensional K-algebra and let $T = T_1^{m_1} \oplus T_2^{m_2} \oplus T_3^{m_3} \oplus \cdots \oplus T_r^{m_r}$ be a module over A with $T_i \ncong T_j$ for $i \neq j$ and T_i indecomposable for all i. The following statements are equivalent:

1. T is a tilting module
2. $\text{pdim}(T) \leq 1$, $\text{Ext}_A^1(T, T) = 0$ and $r =$ number of isomorphism classes of simple A-modules.

In [6], Rickard asks if this theorem holds for tilting complexes as well: "Are the tilting complexes for a finite-dimensional algebra A precisely those objects T if $K^b(\text{proj-}A)$ such that

1. $\text{Hom}(T, T[i]) = 0$ for $i \neq 0$
2. the number of isomorphism classes of indecomposable direct summands of T is equal to the rank of $K_0(A)$?"

If T is a module we can see T as a complex

$$\cdots \to 0 \to T \to 0 \to \dots$$

That is, a complex with T in position zero and 0 in all other positions. This lets us see a generalised tilting module as a tilting complex. The opposite will work as well, if we have a tilting complex with 0 in all positions except 0, we can see this as a generalised tilting module. Thus Rickard's question can be specialised to generalised tilting modules.

In higher dimensional Auslander-Reiten theory, first introduced by Iyama in [8], one studies d-cluster tilting subcategories instead of the whole module-category. It is therefore interesting to study tilting modules contained in such subcategories of mod-A. Below we define the notion of a d-cluster tilting subcategory.

Definition 1.5. A functorially finite subcategory $\mathcal{M} \subseteq$ mod-A is called *d-cluster tilting* if the following holds.

$$\begin{aligned}\mathcal{M} &= \{X \in \text{mod-}A \mid \operatorname{Ext}^i_A(X, M) = 0 \text{ for all } M \in \mathcal{M}, 0 < i < d\}\\ &= \{X \in \text{mod-}A \mid \operatorname{Ext}^i_A(M, X) = 0 \text{ for all } M \in \mathcal{M}, 0 < i < d\}.\end{aligned}$$

This question has been studied in [9] for higher Auslander algebras of a linearly oriented quiver of type A_n. The primary goal of the article [9] is to study the relation between tilting modules over these algebras and triangulations of even-dimensional cyclic polytopes. In order to do this it becomes necessary to confirm Rickard's conjecture for these algebras. We will study this question for a different set of algebras.

2 Preliminaries

Throughout this paper we let A be a finite-dimensional algebra over an algebraically closed field K. We denote by Mod-A (A-Mod) the category of right (left) A-modules and by mod-A (A-mod) the category of finitely generated right (left) A-modules. Furthermore, Proj-A and proj-A are the subcategories of Mod-A and mod-A, respectively, containing the projective A-modules. Similarly, Inj-A and inj-A are the subcategories of Mod-A and mod-A, respectively, containing the injective A-modules. At last, let $\operatorname{add}(M) \subseteq$ mod-A denote the full subcategory of all finite direct sums of direct summands of M.

For a subcategory $\mathcal{M} \subseteq$ mod-A, closed under direct sums and summands, let $|\mathcal{M}|$ denote the number of isomorphism classes of indecomposable modules in $\mathcal{M}$ (which may be infinite). For $M \in$ mod-A set $|M| = |\operatorname{add}(M)|$ and $|A| = |A_A|$. Note that for $M \in$ mod-A, $|M|$ will be the number if non-isomorphic indecomposable direct summands of M.

2.1 Basic algebras

Now follows some useful definitions and theorems about algebras and modules in general. We follow the notation of [10].

Definition 2.1. We call an element $e \in A$ an *idempotent* if it satisfies $e^2 = e$. Two idempotents e_1, e_2 are *orthogonal* if $e_1e_2 = e_2e_1 = 0$. The idempotent e is said to be *primitive* if it cannot be written as a sum $e = e_1 + e_2$ of two non-zero orthogonal idempotents e_1, e_2.

Let $e \in A$ be an idempotent. Then eA will be a submodule of A_A, which is indecomposable if and only if e a primitive idempotent. Now let

$\{e_1, \dots, e_n\}$ be a set of primitive pair-wise orthogonal idempotents such that $1 = e_1 + \dots + e_n$. This is called a *complete set of primitive orthogonal idempotents*. It follows that $A_A = e_1A \oplus \dots \oplus e_nA$ is a decomposition of A_A of indecomposable A-modules.

Definition 2.2. Let A be a finite-dimensional algebra and $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of A. A is called *basic* if $e_iA \ncong e_jA$ for all $i \neq j$.

We denote by D the *standard duality* $D(-) = \operatorname{Hom}_K(-, K)$. This does in fact define two functors

$$\text{mod-}A \xrightarrow{D} \text{mod-}(A^{op}) \xrightarrow{D^{op}} \text{mod-}A$$

such that $D^{op} \circ D \cong 1_{\text{mod-}A}$ and $D \circ D^{op} \cong 1_{\text{mod-}(A^{op})}$. We will omit the superscript and just write D for D^{op}.

Theorem 2.3. [10, Theorem I.5.13] Let A be a finite-dimensional K-algebra. Then the following hold.

(a) A sequence $0 \to L \xrightarrow{f} M \xrightarrow{f} N \to 0$ in mod-A is exact if and only if the induced sequence $0 \to D(N) \xrightarrow{D(g)} D(M) \xrightarrow{D(f)} D(L) \to 0$ is exact in mod-A^{op}.

(b) A module I in mod-A is injective if and only if the module $D(I)$ is projective in mod-A^{op}. A module P in mod-A is projective if and only if the module $D(P)$ is injective in mod-A^{op}.

(c) A module S in mod-A is simple if and only if the module $D(S)$ is simple in mod-A^{op}.

Corollary 2.4. [10, Corollary I.5.17] Suppose that $A_A = e_1A \oplus \dots \oplus e_nA$ is a decomposition of A into indecomposable submodules.

(a) Every simple right A-module is isomorphic to one of the following modules

$$S(1) = \operatorname{top}(e_1A), \dots, S(n) = \operatorname{top}(e_nA),$$

where $\operatorname{top}(M) := M/\operatorname{rad}(M)$ and $\operatorname{rad}(M)$ denotes the intersection of all maximal submodules of M.

(b) Every indecomposable projective right A-module is isomorphic to one of the modules

$$P(1) = e_1A, \dots, P(n) = e_nA.$$

Moreover, $e_iA \cong e_jA$ if and only if $S(i) \cong S(j)$.

(c) Every indecomposable injective right A-module is isomorphic to one of the modules

$$I(1) = D(Ae_1), \dots, I(n) = D(Ae_n).$$

A consequence of this theorem is that proj-A = add(A_A) and inj-A = $D(\text{add}({}_AA)) = D(A\text{-proj})$.

As we will see in Theorem 2.8, a useful tool to visualize basic algebras are quivers. We continue to follow the notation of [10] and define a quiver in the following way.

Definition 2.5. 1. A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of points, a set Q_1 of arrows and two functions $s, t : Q_1 \to Q_0$. The function s associates to each arrow $\alpha \in Q_1$ a source $s(\alpha) \in Q_0$ and t associates a target $t(\alpha) \in Q_0$ to $\alpha \in Q_1$.

2. Let Q be a quiver. The *path algebra* KQ is the algebra with K-basis the set of all paths in Q. The product of two paths $\alpha_1\alpha_2 \dots \alpha_l$ and $\beta_1\beta_2 \dots \beta_k$ is $\alpha_1\alpha_2 \dots \alpha_l\beta_1\beta_2 \dots \beta_k$ if the target of α_l is equal to the source of β_1. Otherwise the product is zero.

Definition 2.6. Let Q be a finite quiver and R_Q the arrow ideal, that is, the KQ-ideal generated by the arrows of Q. An *admissible* ideal is a two-sided ideal $\mathcal{I}$ of KQ such that

$$R_Q^l \subseteq \mathcal{I} \subseteq R_Q^2$$

for some $l \geq 2$.

It follows directly from the definition that an admissible ideal is a two-sided ideal that does not contain any arrows of Q but contains all paths of length at least l.

We can also define a quiver from an algebra. Let A be a basic and connected finite-dimensional K-algebra and let $e_1, \dots, e_n$ be a complete set of primitive orthogonal idempotents in A. The quiver of A, denoted Q_A, is defined as follows. First, let there be a bijective correspondence between the idempotents $e_1, \dots, e_n$ and the points $1, \dots, n$ of Q_A. Then, for two points a, b of Q_A, let the arrows between a and b be in bijective correspondence with the basis of the K-vector space $e_a(\text{rad}(A)/\text{rad}(A)^2)e_b$. Here $\text{rad}(A)$ denotes the *Jacobson radical* of A, which is defined as the intersection of all maximal right ideals of A.

Theorem 2.7. [10, Proposition II.1.10] Let Q be a finite connected acyclic quiver and R_Q the arrow ideal of KQ. Then $R_Q = \text{rad}(KQ)$.

Note that in this case $(\mathrm{rad}(KQ))^l$ is the ideal generated by all paths of length l.

Theorem 2.8. [10, Theorem II.3.7] Let A be a basic finite-dimensional K-algebra. If K is algebraically closed there exists an admissible ideal $\mathcal{I}$ of KQ_A such that $A \cong KQ_A/\mathcal{I}$.

If $A = KQ/\mathcal{I}$, then an A-module M can be described by the representation (Me_a, φ_β) of Q, where for each arrow $\beta : a \to b$ the linear map $\varphi_\beta : Me_a \to Me_b$ is given by multiplication by β. Moreover, we get an easy and useful description of the projective and injective A-modules.

Theorem 2.9. [10, Lemma III.2.4] Let Q be a quiver, $\mathcal{I} \subseteq KQ$ an admissible ideal, $A = KQ/\mathcal{I}$, and $P(a) = e_aA$, where $a \in Q_0$.

(a) If $P(a) = (P(a)_b, \varphi_\beta)$, then $P(a)_b$ is the K-vector space spanned by the set of all the $\overline{w} = w + \mathcal{I}$, with w a path from a to b and, for an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : P(a)_b \to P(a)_c$ is given by the right multiplication by $\overline{\beta} = \beta + \mathcal{I}$.

(b) Let $\mathrm{rad}(P(a)) = (P'(a)_b, \varphi'_\beta)$. Then $P'(a)_b = P(a)_b$, for $b \neq a$, $P'(a)_a$ is the K-vector space spanned by the set of all $\overline{w} = w + \mathcal{I}$, with w a non-stationary path from a to a, $\varphi'_\beta = \varphi_\beta$ for any arrow β of source $b \neq a$ and $\varphi'_\alpha = \varphi_\alpha|_{P'(a)_a}$ for any arrow α of source a.

Theorem 2.10. [10, Lemma III.2.6] Let Let Q be a quiver, $\mathcal{I} \subseteq KQ$ an admissible ideal, $A = KQ/\mathcal{I}$, and $I(a) = D(Ae_a)$, where $a \in Q_0$.

(a) Given $a \in Q_0$, the simple module $S(a)$ is isomorphic to the simple socle of $I(a)$.

(b) If $I(a) = (I(a)_b, \varphi_\beta)$, then $I(a)_b$ is the dual of the K-vector space spanned by the set of all $\overline{w} = w + \mathcal{I}$, with w a path from b to a and, for an arrow $\beta : b \to c$, the K-linear map $\varphi_\beta : I(a)_b \to I(a)_c$ is given by the dual of the left multiplication by $\overline{\beta} = \beta + \mathcal{I}$.

(c) Let $I(a)/S(a) = (L_b, \phi_\beta)$. L_b is the quotient space of $I(a)_b$ spanned by the residual classes of paths from b to a of length at least one, and ϕ_β is the map induced by φ_β.

The following theorem will later be used to connect tilting A-modules and the Grothendieck group of A.

Theorem 2.11. [10, Theorem VI.3.1] Let T be an A-module and $B = \mathrm{End}(T_A)$. The functor $\mathrm{Hom}_A(T, -) : \text{mod-}A \to \text{mod-}B$ induces an equivalence of categories between $\mathrm{add}(T)$ and $\text{proj-}B$.

Corollary 2.12. Let T be a right A-module. Then $|\operatorname{End}(T_A)| = |T_A|$.

Proof. $|\operatorname{End}(T_A)| = |T_A|$ is a consequence of Theorem 2.11 since $|T_A| = |\operatorname{add}(T_A)|$ and $|\operatorname{End}(T_A)| = |\operatorname{proj-}B|$. □

We can now define the notion of a basic module. A module $T \in \text{mod-}A$ is called *basic* if $\operatorname{End}(T_A)$ is basic as an algebra. It follows that this is equivalent to $T = T_1 \oplus \cdots \oplus T_r$, with $T_i \ncong T_j$ for all $i \neq j$.

2.2 The Grothendieck group

In this section we will define the Grothendieck group of an algebra and look at some useful results. For proofs and further reading we refer to [10].

Definition 2.13. Let $\mathcal{F}$ be the free abelian group with basis the set of isomorphism classes, $\widetilde{M}$, of modules in mod-A. The elements of the form $\widetilde{M} - \widetilde{L} - \widetilde{N}$, where

$$0 \to L \to M \to N \to 0$$

is a short exact sequence, generates a subgroup $\mathcal{F}'$ of $\mathcal{F}$. We define the *Grothendieck group* of an algebra A to be the abelian group $K_0(A) = \mathcal{F}/\mathcal{F}'$. Let $[M]$ denote the equivalence class of $\widetilde{M}$ in $\mathcal{F}/\mathcal{F}'$.

If A is a finite-dimensional algebra then for every $M \in \text{mod-}A$ there exists a finite *composition series*, that is, a chain of submodules of M

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$$

such that M_{i+1}/M_i is simple for $i = 1, \ldots, m-1$. See for example [11, Theorem VIII.8.8].

Theorem 2.14 (Jordan-Hölder theorem)**.** [10, Theorem I.3.10] If A is a finite-dimensional K-algebra and

$$\begin{aligned} 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M, \\ 0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_n = M \end{aligned}$$

are two composition series of a module M in mod-A, then $m = n$, and there exists a permutation σ of $\{1, \ldots, m\}$ such that, for any $i \in \{0, 1, \ldots, m-1\}$, there is an A-isomorphism $M_{i+1}/M_i \cong N_{\sigma(i+1)}/N_{\sigma(i+1)-1}$.

Definition 2.15. Let Q be a quiver, $\mathcal{I} \subseteq KQ$ an admissible ideal, $A = KQ/\mathcal{I}$, and let $M \in \text{mod-}A$. We define the *dimension vector* of M to be

$$\underline{\dim}\, M = \begin{bmatrix} \dim_K Me_1 \\ \vdots \\ \dim_K Me_n \end{bmatrix}$$

where $e_1, \dots, e_n$ are the primitive orthogonal idempotents corresponding to the points of Q.

Theorem 2.16. [10, Theorem III.3.3] Let Q be a quiver, $\mathcal{I} \subseteq KQ$ an admissible ideal, $A = KQ/\mathcal{I}$, and $0 \to L \to M \to N \to 0$ is a short exact sequence of A-modules. Then $\underline{\dim}\, M = \underline{\dim}\, L + \underline{\dim}\, N$.

Theorem 2.17. Let A be a finite-dimensional K-algebra and $\{S_1, \dots, S_n\}$ a classification of simple A-modules, up to isomorphism. Then $\{[S_1], \dots, [S_n]\}$ constitutes a basis for the Grothendieck group $K_0(A)$.

Proof. First we show that $\{[S_1], \dots, [S_n]\}$ generates $K_0(A)$. Take any A-module M and look at its composition series $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$. Since for every $j = 1, \dots, m-1$ there is a short exact sequence

$$0 \to M_j \to M_{j+1} \to M_{j+1}/M_j \to 0,$$

we have the equality $[M_{j+1}/M_j] = [M_{j+1}] - [M_j]$ in $K_0(A)$. We thus get $[M] = \sum_{j=0}^{m-1} [M_{j+1}/M_j] = \sum_{i=1}^{n} c_i(M)[S_i]$, where $c_i(M)$ is the number of composition factors M_{j+1}/M_j that is isomorphic to S_i. It thus follows that $\{[S_1], \dots, [S_n]\}$ generates $K_0(A)$.

Now we want to show that $\{[S_1], \dots, [S_n]\}$ is linearly independent. We do this by showing that there is a group homomorphism from $K_0(A)$ to $\mathbb{Z}^n$ such that the image of $\{[S_1], \dots, [S_n]\}$ is a linearly independent set in $\mathbb{Z}^n$. Define $\varphi : \mathcal{F} \to \mathbb{Z}^n$ by $\varphi([M]) = \sum_{i=1}^{n} c_i(M)e_i$. We need to show that $\mathcal{F}' \subseteq \ker(\varphi)$ to set an induced homomorphism $\tilde{\varphi} : K_0(A) \to \mathbb{Z}^n$. But if $0 \to L \to M \to N \to 0$ is a short exact sequence then $c_i(M) = c_i(L) + c_i(N)$ by Proposition 1.1 in [12], so $\varphi([M] - [L] - [N]) = 0$. Thus $\mathcal{F}' \subseteq \ker(\varphi)$. The image of $\{[S_1], \dots, [S_n]\}$ under $\tilde{\varphi}$ is $\{e_1, \dots, e_n\}$, which proves that $\{[S_1], \dots, [S_n]\}$ is indeed a basis of $K_0(A)$.

□

Note that for a basic algebra $\tilde{\varphi} = \underline{\dim}$.

Corollary 2.18. Let A be a finite-dimensional algebra. Then $\operatorname{rank}(K_0(A)) = |A|$.

Proof. The fact that $\operatorname{rank}(K_0(A)) = |A|$ follows from Corollary 2.4 and Theorem 2.17. □

Definition 2.19. A *cochain complex* $C_\bullet$ of A-modules is a family $(C_i)_{i\in\mathbb{Z}}$ of A-modules together with a family of morphisms $(d_C^i : C^i \to C^{i+1})_{i\in\mathbb{Z}}$ such that $d_C^i \circ d_C^{i+1} = 0$ for all i. If there are no ambiguities we shall write $d_C^i = d^i$. We say that $C_\bullet$ is bounded above if for some n, $C_i = 0$ for all $i \geq n$.

Similarly, $C_\bullet$ is bounded below if there is a n such that for all $i \leq n$, $C_i = 0$. If $C_\bullet$ is both bounded above and below, we say that $C_\bullet$ is a bounded cochain complex. If $\mathrm{im}(d_C^i) = \ker(d_C^{i+1})$ for all i, then we call $C_\bullet$ and *exact sequence.*

Theorem 2.20. Let $C_\bullet$ be a bounded exact sequence in mod-A. Then $\sum_{i\in\mathbb{Z}}(-1)^i[C_i] = 0$.

Proof. Without loss of generality we assume that $C_i = 0$ for $i < 0$ and $i > n$, for some n. We will show this by induction on n. We start with the case $n = 2$. Then our bounded exact sequence is a short exact sequence and the result follows immediately from the definition of the Grothendieck group.

Now look at the case $n > 2$ and assume that $\sum_{i\geq 0}(-1)^i[C_i] = 0$ holds for all bounded exact sequences of length $< n$.

$$\begin{tikzcd}
0 \arrow[rr] & & C_0 \arrow[rr] & & C_1 \arrow[rr] & & \dots \arrow[rr] & & C_{n-2} \arrow[rr] \arrow[dr] & & C_{n-1} \arrow[rr] & & C_n \arrow[rr] & & 0 \\
& & & & & & & & & X_{n-1} \arrow[ur] \arrow[dr] & & & & & \\
& & & & & & & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

By assumption $\sum_{i=0}^{n-2}(-1)^i[C_i] + (-1)^{n-1}[X_{n-1}] = 0$, since the length is less than n. But since $[X_{n-1}] = [C_{n-1}] - [C_n]$ we get $\sum_{i=0}^{n}(-1)^i[C_i] = 0$. $\square$

Theorem 2.21. Let T be a right A-module with finite projective dimension d and $B = \mathrm{End}(T)$. Then the following are two well-defined group morphisms between $K_0(A)$ and $K_0(B)$.

$$\begin{aligned}
\mathrm{Ext} : K_0(A) \to K_0(B), &\quad [X_A] \mapsto \sum_{i\geq 0}(-1)^i[\mathrm{Ext}_A^i({}_BT, X)_A] \\
\mathrm{Tor} : K_0(B) \to K_0(A), &\quad [Y_B] \mapsto \sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^B(Y, T_A)_B].
\end{aligned}$$

Proof. To prove that the morphisms are well-defined we need to show that for every short exact sequence $0 \to M \to X \to N \to 0$ in mod-A, $\mathrm{Ext}([X]) = \mathrm{Ext}([M]) + \mathrm{Ext}([N])$ and for every short exact sequence $0 \to M' \to Y \to N' \to 0$ in mod-B, $\mathrm{Tor}([Y]) = \mathrm{Tor}([M']) + \mathrm{Tor}([N'])$.

To see this, apply $\mathrm{Hom}_A({}_BT, -)$ to the short exact sequence $0 \to M \to$

$X \to N \to 0$. We then get

$$0 \longrightarrow \mathrm{Hom}_A(T,M) \longrightarrow \mathrm{Hom}_A(T,X) \longrightarrow \mathrm{Hom}_A(T,N) \longrightarrow \dots$$

$$\dots \longrightarrow \mathrm{Ext}^1_A(T,M) \longrightarrow \mathrm{Ext}^1_A(T,X) \longrightarrow \mathrm{Ext}^1_A(T,N) \longrightarrow \dots$$

$$\dots$$

$$\dots \longrightarrow \mathrm{Ext}^d_A(T,M) \longrightarrow \mathrm{Ext}^d_A(T,X) \longrightarrow \mathrm{Ext}^d_A(T,N) \longrightarrow 0.$$

Since T has projective dimension at most d the sequence will be bounded. By Theorem 2.20 the alternating sum of the equivalence classes of the modules in the sequence is zero. We can split this sum into three terms:

$$\begin{aligned}0 = \sum_{i\geq 0}(-1)^i[C_i] &= \sum_{i=0}^{d}(-1)^i[\mathrm{Ext}^i_A(T,M)] \\ &\quad - \sum_{i=0}^{d}(-1)^i[\mathrm{Ext}^i_A(T,X)] + \sum_{i=0}^{d}(-1)^i[\mathrm{Ext}^i_A(T,N)] \\ &= \mathrm{Ext}(M) - \mathrm{Ext}(X) + \mathrm{Ext}(N).\end{aligned}$$

This shows that indeed $\mathrm{Ext}(X) = \mathrm{Ext}(M) + \mathrm{Ext}(N)$, so the morphism Ext is well-defined. By the same argument we can show that Tor is well defined. $\square$

2.3 d-representation finite algebras

This section contains a brief introduction to d-representation finite algebras.

Definition 2.22. (a) A module $M \in$ mod-A is called a *d-cluster tilting module* if

$$\begin{aligned}\mathrm{add}(M) &= \{X \in \text{mod-}A \mid \mathrm{Ext}^i_A(X,M) = 0 \quad \forall i \in \{1,\dots,d-1\}\} \\ &= \{X \in \text{mod-}A \mid \mathrm{Ext}^i_A(M,X) = 0 \quad \forall i \in \{1,\dots,d-1\}\}.\end{aligned}$$

add(M) is then called a *d-cluster tilting subcategory* of mod-A and is usually denoted by $\mathcal{M}$.

(b) An algebra A is called *d-representation finite* if it has a d-cluster tilting module, and moreover gl.dim$(A) \leq d$.

Note that if A is d-representation finite, then $(\mathrm{P})_d$ is satisfied for all A-modules.

Proposition 2.23. [13, Proposition 0.2] Let A be a basic d-representation finite algebra. Let

$$\begin{aligned} \tau_d &:= \operatorname{Tor}_d^A(DA, -) \cong D\operatorname{Ext}_A^d(-, A) : \text{mod-}A \to \text{mod-}A \\ \tau_d^{-1} &:= D\operatorname{Tor}_d^A(D-, DA) \cong \operatorname{Ext}_A^d(DA, -) : \text{mod-}A \to \text{mod-}A\,. \end{aligned}$$

(a) There exists a permutation σ of $\{1, \dots, n\}$ and positive integers $l_1, \dots, l_n$ such that $\tau_d^{l_i - 1} I(i) \cong P(\sigma(i))$ for any i.

(b) There exists a unique basic d-cluster tilting A-module M, which is given as the direct sum of the following mutually non-isomorphic indecomposable A-modules.

$$\begin{array}{llllll} I(1), & \tau_d I(1), & \tau_d^2 I(1), & \dots & \tau_d^{l_1-2} I(1), & \tau_d^{l_1-1} I(1) \cong P(\sigma(1)), \\ I(2), & \tau_d I(2), & \tau_d^2 I(2), & \dots & \tau_d^{l_2-2} I(2), & \tau_d^{l_2-1} I(2) \cong P(\sigma(2)), \\ \dots & \dots & \dots & \dots & \dots & \dots \\ I(n), & \tau_d I(n), & \tau_d^2 I(n), & \dots & \tau_d^{l_n-2} I(n), & \tau_d^{l_n-1} I(n) \cong P(\sigma(n)). \end{array}$$

(c) We have mutually quasi-inverse equivalences $\tau_d : \operatorname{add}(M/A) \xrightarrow{\sim} \operatorname{add}(M/DA)$ and $\tau_d^{-1} : \operatorname{add}(M/DA) \xrightarrow{\sim} \operatorname{add}(M/A)$.

3 Classical tilting modules

Now we introduce the concept of classical tilting modules as in [5]. We use the same definition for tilting modules as in the introduction.

Definition 3.1. Let $\mathcal{T}, \mathcal{F}$ be two full subcategories of mod-A. $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if

(a) $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$.

(b) $\operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies that $M \in \mathcal{T}$.

(c) $\operatorname{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies that $N \in \mathcal{F}$.

Equivalently, $(\mathcal{T}, \mathcal{F})$ is a torsion pair if

$$\mathcal{T} = \{M \in \text{mod-}A \mid \operatorname{Hom}_A(M, N) = 0 \text{ for all } N \in \mathcal{F}\}$$

and

$$\mathcal{F} = \{N \in \text{mod-}A \mid \operatorname{Hom}_A(M, N) = 0 \text{ for all } M \in \mathcal{T}\}.$$

Definition 3.2. Let $T_A \in \text{mod-}A$. T is called a *tilting module* if it satisfies the following conditions:
(P) There exists an exact sequence

$$0 \to P_1 \to P_0 \to T \to 0$$

where P_0, P_1 are projective A-modules. That is, $\text{pdim}(T) \leq 1$.
(E) $\text{Ext}^1_A(T,T) = 0$
(G) There exists an exact sequence

$$0 \to A \to T' \to T'' \to 0$$

with $T', T'' \in \text{add}(T)$. If T only satisfies (P) and (E) we call T a *partial tilting module.*

Let T_A be a tilting module. We can now define a torsion pair using the module T in the following way:

$$\mathcal{T}(T) := \{M_A \in \text{mod-}A \,|\, \text{Ext}^1_A(T,M) = 0\}$$
$$\mathcal{F}(T) := \{M_A \in \text{mod-}A \,|\, \text{Hom}^1_A(T,M) = 0\}$$

We call this the *torsion pair induced by* T. For the proof that this is indeed a torsion pair see [10, Theorem VI.2.5]

Now follows a number of theorems about properties of tilting modules. For the proofs we refer to [10].

Theorem 3.3. [10, Theorem VI.3.3] Let T_A be a tilting module and $B = \text{End}(T_A)$. Then ${}_BT$ is a tilting left B-module.

Theorem 3.4. [10, Theorem VI.3.6] Let A be a finite-dimensional algebra. Any tilting A-module T_A induces a torsion pair $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ in the category B-mod, where $B = \text{End}(T_A)$ and

$$\mathcal{X}(T_A) = \{{}_BX \,|\, \text{Hom}_B(X, DT) = 0\} = \{{}_BX | X \otimes_B T = 0\},$$
$$\mathcal{Y}(T_A) = \{{}_BY \,|\, \text{Ext}^1_B(Y, DT) = 0\} = \{{}_BY \,|\, \text{Tor}^B_1(Y,T) = 0\}.$$

Theorem 3.5 (Brenner-Butler tilting theorem)**.** [10, Theorem VI.3.8] Let A be an algebra, T_A be a tilting module, $B = \text{End}(T_A)$, and $(\mathcal{T}(T_A), \mathcal{F}(T_A))$, $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ be the induced torsion pairs in mod-A and mod-B, respectively. Then T has the following properties:

(a) The canonical K-algebra homomorphism $A \to \text{End}({}_BT)^{op}$ defined by $a \mapsto (t \mapsto at)$ is an isomorphism.

(b) The functors $\mathrm{Hom}_A(T,-)$ and $(-\otimes_B T)$ induce quasi-inverse equivalences between $\mathcal{T}(T_A)$ and $\mathcal{Y}(T_A)$.

(c) The functors $\mathrm{Ext}^1_A(T,-)$ and $\mathrm{Tor}^B_1(-,T)$ induce quasi-inverse equivalences between $\mathcal{F}(T_A)$ and $\mathcal{X}(T_A)$.

A generalisation of the Brenner-Butler tilting theorem will be proven in Section 4.

Theorem 3.6 (Bongartz Lemma)**.** [10, Theorem VI.2.4] Let T be a partial tilting module. Then there exists an A-module E such that $T\oplus E$ is a tilting module.

The following theorem will be proven in Section 4 for generalised tilting modules.

Theorem 3.7. [10, Theorem VI.4.3] Let A be a finite-dimensional algebra, T_A be a tilting module, and $B=\mathrm{End}(T_A)$. Then the Grothendieck groups of A and B, respectively, are isomorphic.

Theorem 3.8. [10, Theorem VI.4.4] Let A be a finite-dimensional algebra and T_A a partial tilting module. Then T_A is a tilting module if and only if $|T_A|=|A|$.

Proof. Assume first that T_A is a tilting module and let $B=\mathrm{End}(T_A)$. Theorem 3.7 implies that $\mathrm{rank}(K_0(A))=\mathrm{rank}(K_0(B))$ and by Corollary 2.12 and 2.18 we then have that $|T_A|=|B|=\mathrm{rank}(K_0(B))=\mathrm{rank}(K_0(A))=|A|$.

Now assume that T_A is a partial tilting module such that $|T|=|A|$. By Theorem 3.6 there exists a module E such that $T\oplus E$ is tilting. By what we just proved $|T\oplus E|=|A|$. But from our assumption we get that $|T|=|A|=|T\oplus E|$. Thus $E\in\mathrm{add}(T)$ and T must then be a tilting module. $\square$

4 Generalised tilting modules

In this paper we will study the generalisation of tilting modules first defined by Miyashita in [1]. Our aim of this section is to prove that the result that $K_0(A)\cong K_0(B)$, where $B=\mathrm{End}(T_A)$, holds for generalised tilting modules. From this result it follows that the number of indecomposable direct summands of a generalised tilting module T must equal the rank of the Grothendieck group. Some result will only be stated and we refer to [1] for the proofs. Throughout the section, let A be a finite-dimensional algebra over an algebraically closed field K.

Definition 4.1. Let T_A be a right A-module.

- T_A satisfies $(\mathrm{P})_d$ if there is a projective resolution

$$0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to T \to 0$$

  where each $(P_i)_A$ is finitely generated.

- T_A satisfies $(\mathrm{E})_d$ if

$$\mathrm{Ext}^i_A(T,T) = 0 \quad \text{for all } i = 1 \dots d.$$

- T_A satisfies $(\mathrm{G})_d$ if there is an exact sequence

$$0 \to A_A \to T_0 \to T_1 \to \cdots \to T_d \to 0$$

  where each $T_i \in \mathrm{add}(T)$.

We call T_A a generalised tilting module of projective dimension at most d if T_A satisfies $(\mathrm{P})_d, (\mathrm{G})_d$ and $(\mathrm{E})_d$. Note that for $d = 1$, T is exactly a classical tilting module defined in Section 3.

Since A is finite-dimensional, A_A is the direct sum of all projective A-modules. Thus confirming $(\mathrm{G})_d$ is equivalent to finding an $\mathrm{add}(T)$-resolution of each projective A-module.

Now we define two subcategories of Mod-A and A-Mod that are generalisations of the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$. Let T_A be an A-module and e a fixed non-negative integer. In almost all cases d will be the projective dimension of the module T_A.

Definition 4.2. Let e be a non-negative integer. We define the subcategories $\mathrm{KE}_e(T_A)$ and $\mathrm{KT}_e(T_A)$ of Mod-A in the following way.

$$\mathrm{KE}_e(T_A) := \{M_A \in \text{Mod-}A \,|\, \mathrm{Ext}^i_A(T,M) = 0 \text{ if } 0 \leq i \leq d+e \text{ and } i \neq e\}$$

$$\mathrm{KT}_e(T_A) := \{{}_AM \in A\text{-Mod} \,|\, \mathrm{Tor}^A_i(T,M) = 0 \text{ if } 0 \leq i \leq d+e \text{ and } i \neq e\}$$

We can similarly define $\mathrm{KE}_e({}_AT)$ and $\mathrm{KT}_e({}_AT)$ for a left A-module ${}_AT$.

Theorem 4.3. [1, Theorem 1.1] Let T be a right A-module, T' a left A-module, let $e, d \geq 1$ be integers and take an exact sequence in Mod-A:

$$\begin{tikzcd}
0 \arrow[r] & Y_e \arrow[r] & X_{e-1} \arrow[rr] \arrow[dr] & & \dots \arrow[rr] \arrow[dr] & & X_0 \arrow[r] & Y_0 \arrow[r] & 0 \\
& & & Y_{e-1} \arrow[ur] & & Y_1 \arrow[ur] & & &
\end{tikzcd}$$

(a) If $\mathrm{Ext}^i_A(T, X_j) = 0$ for all $i = 1, \ldots, d+e$ and for all X_j, then

$$\mathrm{Ext}^i_A(T, Y_0) \xrightarrow{\sim} \mathrm{Ext}^{i+1}_A(T, Y_1) \xrightarrow{\sim} \ldots \xrightarrow{\sim} \mathrm{Ext}^{i+e}_A(T, Y_e)$$

for all $i = 1, \ldots, d$, by connecting homomorphisms.

(b) If $\mathrm{Ext}^i_A(X_j, T) = 0$ for all $i = 1, \ldots, d+e$ and for all X_j, then

$$\mathrm{Ext}^i_A(Y_e, T) \xrightarrow{\sim} \mathrm{Ext}^{i+1}_A(Y_{e-1}, T) \xrightarrow{\sim} \ldots \xrightarrow{\sim} \mathrm{Ext}^{i+e}_A(Y_0, T)$$

for all $i = 1, \ldots, d$, by connecting homomorphisms.

(c) If $\mathrm{Tor}_i^A(X_j, T') = 0$ for all $i = 1, \ldots, d+e$ and for all X_j, then

$$\mathrm{Tor}^A_{i+e}(Y_0, T') \xrightarrow{\sim} \mathrm{Tor}^A_{i+e-1}(Y_1, T') \xrightarrow{\sim} \ldots \xrightarrow{\sim} \mathrm{Tor}^A_i(Y_e, T')$$

for all $i = 1, \ldots, d$, by connecting homomorphisms.

Theorem 4.4. 1. Let T_A be a right tilting module of projective dimension at most d. Then $\mathrm{KE}_0(T_A)$ generates $K_0(A)$.

2. Let ${}_BT$ be a left tilting B-module of projective dimension at most d. Then $\mathrm{KT}_0({}_BT)$ generates $K_0(B)$.

Proof. 1. Let X be a module in Mod-A and take an injective resolution of X:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & \ldots \arrow[r] & I_{d-1} \arrow[r] \arrow[dr] & & I_d \arrow[r] & \ldots \\
& & & & & & \Omega^{-d}X \arrow[ur] \arrow[dr] & & \\
& & & & & & & 0 &
\end{tikzcd}$$

Since all injective modules lies in $\mathrm{KE}_0(T_A)$, Theorem 4.3 implies that $\mathrm{Ext}^i_A(T, \Omega^{-d}X) \cong \mathrm{Ext}^{i+d}_A(T, X) = 0$ and thus $\Omega^{-d}X \in \mathrm{KE}_0(T_A)$. By Lemma 2.20 we have $[X] = -\sum_{i=0}^{d-1}(-1)^i[I_i] + (-1)^d[\Omega^{-d}X]$. All terms in the sum in the right-hand side lies in $\mathrm{KE}_0(T_A)$, which means that $\mathrm{KE}_0(T_A)$ generates $K_0(A)$.

2. We can dualize the proof of (a) using a projective resolution instead. □

The following theorem shows that we can recover the algebra A from the endomorphism algebra of ${}_BT$, where $B = \mathrm{End}(T_A)$.

Theorem 4.5. [1, Theorem 1.5] Let T_A be a tilting module of projective dimension at most d and let $B := \mathrm{End}(T_A)$. Then ${}_BT$ is a tilting module of projective dimension at most d and

$$A \xrightarrow{\sim} \mathrm{End}({}_BT)^{op}, a \mapsto (t \mapsto ta)$$

is an isomorphism.

Proof. We start by checking that ${}_BT$ satisfies $(\mathrm{G})_d$. Since T_A is tilting there exists a $(\mathrm{P})_d$-sequence of T_A:

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & P_d \arrow[rr] \arrow[dr] & & P_{d-1} \arrow[rr] \arrow[dr] & & \dots \arrow[rr] \arrow[dr] & & P_0 \arrow[rr] \arrow[dr] & & T \arrow[rr] & & 0 \\
& & & Y_d \arrow[ur] & & Y_{d-1} \arrow[ur] & & Y_1 \arrow[ur] & & Y_0 \arrow[ur] & & &
\end{tikzcd}$$

If we apply $\mathrm{Hom}_A(-,T)$ to the sequence we get

$$0 \to \mathrm{Hom}_A(T,T) \to \mathrm{Hom}_A(P_0,T) \to \cdots \to \mathrm{Hom}_A(P_r,T) \to 0.$$

We claim that this is a $(\mathrm{G})_d$-sequence. This sequence is exact if

$$0 \to \mathrm{Hom}_A(Y_{i+1},T) \to \mathrm{Hom}_A(P_i,T) \to \mathrm{Hom}_A(Y_i,T) \to 0$$

is exact for every $i = 1, \dots, d-1$. We know that it is left exact since $\mathrm{Hom}_A(-,T)$ is left exact. The sequence is right exact if $\mathrm{Ext}^1_A(Y_{i+1},T) = 0$. This fact follows from Theorem 4.3:

$$\mathrm{Ext}^1_A(Y_i,T) \xrightarrow{\sim} \mathrm{Ext}^{i+1}_A(T,T) = 0 \text{ for } i = 1, \dots, d-1.$$

Now we need to show that $\mathrm{Hom}_A(P_i,T) \in \mathrm{add}({}_BT)$ for every $i = 1, \dots, d$. Since P_i is projective there exists $Q_i \in \text{Mod-}A$ such that $P_i \oplus Q_i \cong A^n$. Thus

$$\mathrm{Hom}_A(P_i,T) \oplus \mathrm{Hom}_A(Q_i,T) \cong \mathrm{Hom}_A(P_i \oplus Q_i,T) \cong \mathrm{Hom}_A(A^n,T) \cong T^n.$$

So $\mathrm{Hom}_A(P_i,T)$ does indeed lie in $\mathrm{add}({}_BT)$. ${}_BT$ does therefore satisfy $(\mathrm{G})_d$.

To prove that ${}_BT$ satisfies $(\mathrm{P})_d$ we take a $(\mathrm{G})_d$ sequence of T_A:

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & A \arrow[rr] & & T_0 \arrow[rr] \arrow[dr] & & \dots \arrow[rr] \arrow[dr] & & T_{d-1} \arrow[rr] & & T_d \arrow[rr] & & 0 \\
& & & & & X_1 \arrow[ur] & & X_{d-1} \arrow[ur] & & & & &
\end{tikzcd}$$

Since $T_i \in \mathrm{add}(T_A)$ and T_A satisfies $(\mathrm{E})_d$ we get that $\mathrm{Ext}^1_A(X_i, T) = 0$ for every $i = 1, \ldots, d-1$, by Theorem 4.3. This implies that the sequence

$$0 \to \mathrm{Hom}_A(T_d, T) \to \cdots \to \mathrm{Hom}_A(T_0, T) \to \mathrm{Hom}_A(A, T) \to 0 \tag{4.1}$$

that we get from applying $\mathrm{Hom}_A(-, T)$ is exact. Note that $\mathrm{Hom}_A(A, T)$ and T are isomorphic as B-modules. By Theorem 2.11 the B-modules $\mathrm{Hom}_A(T_i, T)$ are projective.

First we show that there are canonical isomorphisms

$$T_i \xrightarrow{\sim} \mathrm{Hom}_B(\mathrm{Hom}_A(T_i, T), T)$$

for all $T_i \in \mathrm{add}(T)$. This follows from the isomorphism $T \cong \mathrm{Hom}_B(B, T) = \mathrm{Hom}_B(\mathrm{Hom}_A(T, T), T)$ and the additivity of the functors.

Now look at the following diagram, obtained by applying $\mathrm{Hom}_B(-, T)$ to the sequence 4.1.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{End}({}_BT) \arrow[r] & \mathrm{Hom}_B(\mathrm{Hom}_A(T_0, T), T) \arrow[r] & \ldots \arrow[r] & \mathrm{Hom}_B(\mathrm{Hom}_A(T_r, T), T) \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\varphi"] & T_0 \arrow[r] \arrow[u, "\sim"] & \ldots \arrow[r] & T_d \arrow[r] \arrow[u, "\sim"] & 0
\end{tikzcd}$$

Here φ is defined by $a \mapsto (t \mapsto ta)$. Since the squares commute we get that A and $\mathrm{End}({}_BT)$ are isomorphic as A-modules. What remains to prove in order for φ to be an algebra isomorphism is that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. This follows as $\varphi(ab)$ sends t to tab and $\varphi(a) \circ \varphi(b)$ sends t to $(tb)a = t(ba)$. We therefore see that we need to use the opposite product. Thus,

$$\varphi : A \to \mathrm{End}({}_BT)^{op}, \quad t \mapsto (t \mapsto ta)$$

is indeed an algebra isomorphism. $\square$

Next we have two theorems that will be used to prove that the subcategories $\mathrm{KE}_e(T_A)$ and $\mathrm{KT}_e({}_BT)$ are equivalent. We will only state the theorems and refer to [1] for the proofs and further reading.

Theorem 4.6. [1, Theorem 1.7] Let ${}_BT_A$ be a tilting module of projective dimension at most d, where $B = \mathrm{End}(T_A)$.

(a) Let $I \in \text{Mod-}A$ be injective, then there is a canonical isomorphism

$$\begin{aligned} \mathrm{Hom}_A({}_BT, I) \otimes_B T &\xrightarrow{\sim} I \\ f \otimes t &\mapsto f(t) \end{aligned}$$

(b) For any integer $i \geq 1$ we have $\mathrm{Tor}_i^B(\mathrm{Hom}_A(T, I), T) = 0$ for all injective modules I_A.

Theorem 4.7. [1, Theorem 1.13] Assume that T_A be a right A-module that satisfies $(\mathrm{P})_d, (\mathrm{E})_d$, and let $e \geq 1$ be an integer. Let X be a left A-module, and assume that there exists an exact sequence

$$0 \to V_e \to \cdots \to V_0 \to X \to 0$$

with $V_i \in \mathrm{KT}_0(A)$. Then the following are equivalent.

(i) ${}_AX \in \mathrm{KT}_e(T_A)$

(ii) The induced sequence

$$T \otimes_A V_e \to T \otimes_A V_{e-1} \to \cdots \to T \otimes_A V_0$$

is exact.

In this case, $\mathrm{Tor}_e^A(T, X) \xrightarrow{\sim} \ker(T \otimes_A V_e \to T \otimes_A V_{e-1})$ holds.

Theorem 4.8. [1, Theorem 1.16] Let ${}_BT_A$ be a tilting module of projective dimension at most d and let $0 \leq e \leq d$ be an integer. Then the functors $\mathrm{Ext}_A^e(T, -)$ and $\mathrm{Tor}_e^B(-, T)$ induce quasi-inverse equivalences between $\mathrm{KE}_e(T_A)$ and $\mathrm{KT}_e({}_BT)$.

Proof. We will show that if $Y_A \in \mathrm{KE}_e(T_A)$ then $\mathrm{Ext}_A^e({}_BT, Y) \in \mathrm{KT}_e({}_BT)$ and that there is a natural isomorphism

$$\mathrm{Tor}_e^B(\mathrm{Ext}_A^e({}_BT, Y), {}_BT) \xrightarrow{\sim} Y_A.$$

The other part of the proof is dual.

We start with the case $e = 0$. Take an injective resolution of Y_A:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & I_0 \arrow[rr] \arrow[dr] & & I_1 \arrow[r] & \ldots \arrow[r] & I_{d-1} \arrow[rr] \arrow[dr] & & I_d \arrow[r] & \ldots \\
& & & \Omega^{-1}Y \arrow[ur] & & & & \Omega^d Y \arrow[ur] & &
\end{tikzcd}$$

and apply $\mathrm{Hom}_A({}_BT, -)$. We then have an exact sequence in Mod-A:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(T, Y) \arrow[r] & \mathrm{Hom}_A(T, I_0) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_A(T, I_1) \arrow[r] & \ldots \\
& & & \mathrm{Hom}_A(T, \Omega^{-1}Y) \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{Hom}_A(T, I_{e-1}) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_A(T, I_e) \arrow[r] & \ldots \\
& & \mathrm{Hom}_A(T, \Omega^{-e}Y) \arrow[ur] & &
\end{tikzcd}$$

Exactness follows from Theorem 4.3 since $Y, I_0, \ldots, I_d$ lies in $\mathrm{KE}_0(T)$: $\mathrm{Ext}^1_A(T, \Omega^{-i}Y) = \mathrm{Ext}^{1+i}_A(T, Y) = 0$, so $0 \to \mathrm{Hom}_A(T, \Omega^{-i}Y) \to \mathrm{Hom}_A(T, I_i) \to \mathrm{Hom}_A(T, \Omega^{-(i+1)}Y) \to 0$ is exact. Now we apply $- \otimes_B T$ to the sequence:

$$\begin{tikzcd}
 & & \mathrm{Hom}_A(T, I_0) \otimes T \arrow[rr] \arrow[dr] & & \mathrm{Hom}_A(T, I_1) \otimes T \\
 & \mathrm{Hom}_A(T, Y) \otimes T \arrow[ur] & & \mathrm{Hom}_A(T, \Omega^{-1}Y) \otimes T \arrow[ur] \arrow[dr] & \\
\mathrm{Tor}^B_1(\mathrm{Hom}_A(T, \Omega^{-1}Y), T) \arrow[ur] & & \mathrm{Tor}^B_1(\mathrm{Hom}_A(T, \Omega^{-2}Y), T) \arrow[ur] & & 0
\end{tikzcd}$$

To show that

$$0 \to \mathrm{Hom}_A(T, Y) \otimes T \to \mathrm{Hom}_A(T, I_0) \otimes T \to \mathrm{Hom}_A(T, I_1) \otimes T$$

is exact we need to prove that

$$\mathrm{Tor}^B_1(\mathrm{Hom}_A(T, \Omega^{-1}Y), T) = 0 = \mathrm{Tor}^B_1(\mathrm{Hom}_A(T, \Omega^{-2}Y), T).$$

By Theorem 4.6 we have that $\mathrm{Hom}_A(T, I_i)$ lies in $\mathrm{KT}_0(T)$ for all $i \geq 0$, and therefore, by the corollary of Lemma 1.3 in [1], we have that

$$\mathrm{Hom}_A(T, Y), \mathrm{Hom}_A(T, \Omega^{-1}Y), \mathrm{Hom}_A(T, \Omega^{-2}Y) \in \mathrm{KT}_0({}_BT).$$

In particular, $\mathrm{Tor}^B_1(\mathrm{Hom}_A(T, \Omega^{-1}Y), T) = 0 = \mathrm{Tor}^B_1(\mathrm{Hom}_A(T, \Omega^{-2}Y), T)$. Furthermore, from Theorem 4.6 we have that $\mathrm{Hom}_A(T, I_i) \otimes_B T \xrightarrow{\sim} I_i$.

Since the following diagram commutes and the morphisms $\mathrm{Hom}_A(T, I_i) \otimes_B T \to I_i$ are isomorphisms it follows that $\mathrm{Hom}_A(T, Y) \otimes_B T \cong Y$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(T, Y) \otimes_B T \arrow[r] \arrow[d] & \mathrm{Hom}_A(T, I_0) \otimes_B T \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_A(T, I_1) \otimes_B T \arrow[d, "\sim"] \\
0 \arrow[r] & Y \arrow[r] & I_0 \arrow[r] & I_1
\end{tikzcd}$$

The case $e = 0$ is thus proven.

Now assume that $e \geq 1$ and consider again an injective resolution of Y_A:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & Y \arrow[r] & I_0 \arrow[rr] \arrow[dr] & & I_1 \arrow[r] & \ldots \arrow[r] & I_{r-1} \arrow[rr] \arrow[dr] & & I_r \arrow[r] & \ldots \\
 & & & \Omega^{-1}Y \arrow[ur] & & & & \Omega^{-r}Y \arrow[ur] & &
\end{tikzcd}$$

Since $\mathrm{Ext}^i_A(T, \Omega^{-e}Y) \cong \mathrm{Ext}^{i+e}_A(T, Y)$ and $\mathrm{Ext}^{i+e}_A(T, Y) = 0$ for $1 \leq i \leq r$, $\Omega^{-e}Y \in \mathrm{KE}_0(T_A)$. By the first part $\mathrm{Hom}_A(T, \Omega^{-e}Y)$ then lies in $\mathrm{KT}_0({}_BT)$ and $\mathrm{Hom}_A(T, \Omega^{-e}Y) \otimes T \cong \Omega^{-e}Y$. Therefore, when applying $\mathrm{Hom}_A(T, -)$ to the injective resolution we get:

$$0 \longrightarrow \mathrm{Hom}_A(T, I_0) \longrightarrow \mathrm{Hom}_A(T, I_1) \longrightarrow \dots$$

$$\dots \longrightarrow \mathrm{Hom}_A(T, I_{e-1}) \longrightarrow \mathrm{Hom}_A(T, \Omega^{-e}Y) \longrightarrow \mathrm{Ext}^e_A(T, Y) \longrightarrow 0$$

since $\mathrm{Ext}^1_A(T, \Omega^{-(e-1)}Y) \cong \mathrm{Ext}^e_A(T, Y)$ by Theorem 4.3. Now apply $- \otimes_B T$:

$$\begin{aligned} &\mathrm{Hom}_A(T, I_0) \otimes T \to \mathrm{Hom}_A(T, I_1) \otimes T \to \dots \\ \dots \to &\mathrm{Hom}_A(T, I_{e-1}) \otimes T \to \mathrm{Hom}_A(T, \Omega^{-e}Y) \otimes T \end{aligned} \tag{4.2}$$

Recall that $\mathrm{Hom}_A(T, I) \otimes T \cong I$ for all injective A-modules and that $\mathrm{Hom}_A(T, J^eY) \otimes T \cong \Omega^{-e}Y$ since $\Omega^{-e}Y \in \mathrm{KE}_0(T_A)$. Thus the sequence above is isomorphic to

$$I_0 \to I_1 \to \dots \to I_{e-1} \to \Omega^{-e}Y \to 0$$

which we know is exact. Since the sequence 4.2 is exact Theorem 4.7 implies that $\mathrm{Ext}^e_A(T, Y) \in \mathrm{KT}_e({}_BT)$. From the same theorem we also get the following isomorphism

$$\begin{aligned} \mathrm{Tor}^B_e(\mathrm{Ext}^e_A(T, Y), T) &\cong \ker(\mathrm{Hom}_A(T, I_0) \otimes_B T \to \mathrm{Hom}_A(T, I_1) \otimes_B T) \\ &\cong \ker(I_0 \to I_1) \cong Y. \end{aligned}$$

The second isomorphism comes from Theorem 4.6 and $\ker(I_0 \to I_1)$ is isomorphic to Y since the injective resolution is exact. □

Theorem 4.9. [1, Theorem 1.19] Assume that ${}_BT_A$ is a tilting module of projective dimension at most d, A is finite-dimensional and $B := \mathrm{End}(T_A)$. Then $K_0(A) \cong K_0(B)$, where the isomorphisms between the two groups are given by

$$\mathrm{Ext} : K_0(A) \to K_0(B), \quad [X_A] \mapsto \sum_{i \geq 0} (-1)^i [\mathrm{Ext}^i_A({}_BT, X)_A]$$
$$\mathrm{Tor} : K_0(B) \to K_0(A), \quad [Y_B] \mapsto \sum_{i \geq 0} (-1)^i [\mathrm{Tor}^B_i(Y, T_A)].$$

Proof. By Theorem 2.21 the morphisms Ext and Tor are well-defined. What remains to show is that they are the inverse of each other.

By Theorem 4.8 we see that $\mathrm{Tor} \circ \mathrm{Ext}([X_A]) = [X_A]$ for all $X \in \mathrm{KE}_0(T_A)$ and $\mathrm{Ext} \circ \mathrm{Tor}([X'_A]) = [X'_A]$ for all $X' \in \mathrm{KT}_0({}_BT)$. Since, by Theorem 4.4, $\mathrm{KE}_0(T_A)$ and $\mathrm{KT}_0({}_BT)$ generates $K_0(A)$ and $K_0(B)$, respectively, this proves that the functions $\mathrm{Ext} : K_0(A) \to K_0(B)$ and $\mathrm{Tor} : K_0(B) \to K_0(A)$ are in fact isomorphisms that are inverse to each other. □

Corollary 4.10. Let T_A be a generalised tilting module. Then $|T_A| = |A|$.

Proof. Theorem 4.9 implies that $\mathrm{rank}(K_0(A)) = \mathrm{rank}(K_0(B))$, where $B = \mathrm{End}(T_A)$. By Corollary 2.12 and 2.18 we then have that $|T_A| = |B| = \mathrm{rank}(K_0(B)) = \mathrm{rank}(K_0(A)) = |A|$. □

We would like to show that an A-module T that satisfies $(\mathrm{P})_d$ and $(\mathrm{E})_d$ is a generalised tilting module if and only if $|T_A| = |A|$, to get a generalisation of Theorem 3.7. What we are missing is a generalisation of Bongartz lemma, Theorem 3.6. If A is a hereditary algebra then all tilting modules are classical tilting modules and Theorem 3.8 gives a complete answer in this case. However, if the global dimension is greater than one this is an open problem. In higher dimensional Auslander-Reiten theory, hereditary algebras and their module categories are replaced by algebras of global dimension d and d-cluster tilting subcategories of their module categories. It is then natural to try to generalise Theorem 3.8 to the case when T is a generalised tilting module contained in a d-cluster tilting subcategory of mod-A with A an algebra of global dimension d.

Since a tilting module of projective dimension at most 1 is exactly a classical tilting module defined in Section 3 some theorems from that section now follows from what we have just proven. For example is Theorem 3.5 a special case of Theorem 4.8 and Theorem 3.7 is a special case of Theorem 4.9.

5 Tilting modules over acyclic Nakayama algebras

In [14] it is proven that a certain acyclic Nakayama algebra $A_{n,l}$, see Definition 5.1, is d-representation finite for some specific values of d. In this section we will confirm Rickard's conjecture for all modules in the corresponding d-cluster tilting subcategory.

Definition 5.1. Let Q_n be the Dynkin quiver

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n-1 \leftarrow n.$$

Recall from Theorem 2.7 that $\operatorname{rad}(KQ_n)^l$ is the ideal generated by all paths of length l. We define the algebra $A_{n,l}$ to be the bounded quiver algebra $A_{n,l} = KQ_n/\operatorname{rad}(KQ_n)^l$.

Let $M(i,j)$ be the indecomposable module whose dimension vector has value 1 in position i to j, and zeros in all other positions. Note for $M(i,j)$ to be a $A_{n,l}$-module we must have $j - i \leq l - 1$. By [14] all indecomposable $A_{n,l}$ are of this form. In general it is not always true that we can recover an indecomposable module from its dimension vector, but in this case it is possible.

Therefore, by Theorem 2.9, the projective modules are

$$P(i) = \begin{cases} M(1,i), & 1 \leq i \leq l \\ M(i-l+1,i), & l \leq i \leq n \end{cases}$$

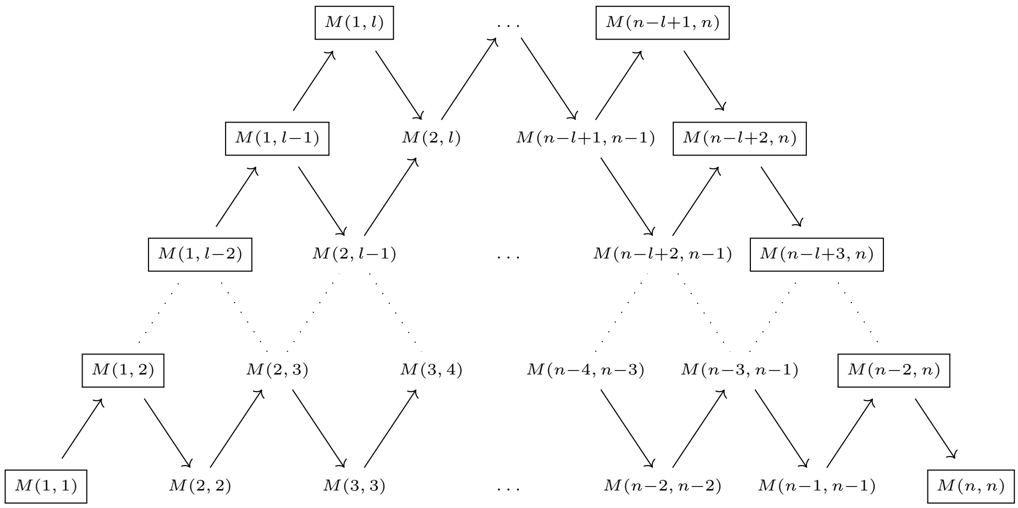
By Theorem 2.10 the injective modules are

$$I(i) = \begin{cases} M(i,i+l-1), & 1 \leq i \leq n-l+1 \\ M(i,n), & n-l+1 \leq i \leq n \end{cases}$$

Theorem 5.2. [14, Theorem 3] $A_{n,l}$ admits a d-cluster tilting subcategory $\mathcal{M}$ if and only if $l | n-1$ or $l = 2$. Moreover, in that case, $\mathcal{M} = \operatorname{add}(A_{n,l} \oplus DA_{n,l})$ and $d = 2\frac{n-1}{l}$.

From now on we assume that the conditions in Theorem 5.2 hold. To ease notation we will denote $A_{n,l}$ by A throughout this section. Note that the indecomposable modules in $\mathcal{M} = \operatorname{add}(A \oplus DA)$ are exactly the indecomposable projective and injective modules.

Below we have the Auslander-Reiten quiver of A where the indecomposable modules in the d-cluster tilting subcategory $\mathcal{M}$ are marked. This result follows from the computations in Section 4.2 in [14].



5.1 Projective resolutions

Since A is d-representation finite we know that the projective dimension of all modules is at most d, but we still want to find the projective resolutions to later calculate the Ext-groups. The only non-projective modules in $\mathcal{M}$ are $I(i)$ where $i \geq n-l+2$. Let $r := \frac{d}{2} = \frac{n-1}{l}$.

Let $i \leq j$ be integers and define $m(i,j)$ to be a module-morphism with bijections in position i to j and the zero-map everywhere else. We use this notation for all morphisms of this form, regardless of which modules it is defined between. Using this definition, the image of the module-morphism $m(i,j)$ is exactly the module $M(i,j)$. We often label morphisms by the dimension vector of their image. For example, if $n=6$, we have

$$m(2,4) : 011100.$$

Let n, l and r be defined as above and let $n-l+2 \leq j \leq n$ be a fixed integer and let $k = 1, \dots, r$. We define a collection of morphisms in the following way.

$p_0 : P(n) \to I(j), p_0 = m(j,n)$
$p_1 : P(j-1) \to P(n), p_1 = m(n-l+1, j-1)$
$p_{2k} : P((r-k)l+1) \to P(j-(k-1)l-1), p_{2k} = m(j-kl, (r-k)l+1)$
$p_{2k+1} : P(j-kl-1) \to P((r-k)l+1), p_{2k+1} = m((r-k-1)l+2, j-kl-1)$

Since

$$\mathrm{im}(p_{2k+1}) = M((r-k-1)l+2, j-kl-1) = \ker(p_{2k})$$

and

$$\operatorname{im}(p_{2k}) = M(j-kl,(r-k)l+1) = \ker(p_{2k-1})$$

for all $k = 1,\dots,r$, the following sequence is exact and is thus a projective resolution of $I(j)$.

$$0 \longrightarrow P(1) \xrightarrow{p_d} P(j-n+l) \xrightarrow{p_{d-1}} P(l+1) \xrightarrow{p_{d-2}} \dots$$

$$\dots \xrightarrow{p_{2k}} P(j-kl-1) \xrightarrow{p_{2k-1}} P((r-k)l+1) \longrightarrow \dots$$

$$\dots \xrightarrow{p_2} P(j-1) \xrightarrow{p_1} P(n) \xrightarrow{p_0} I(j) \longrightarrow 0$$

5.2 Ext-groups

Now we want to look at for which $X, Y \in \mathcal{M}$ we have $\operatorname{Ext}^d_A(X,Y) = 0$. Since $\operatorname{Ext}^d_A(X,Y) = 0$ holds if X is projective or Y injective we look at $\operatorname{Ext}^d_A(I(i),P(j))$ for $n-l+2 \le i \le n$ and $1 \le j \le l-1$. By the definition of $\operatorname{Ext}^d_A(I(i),P(j))$ we have:

$$\begin{aligned}\operatorname{Ext}^d_A(I(i),P(j)) =& H^d(P_\bullet, P(j)) \\ =& H^d(\operatorname{Hom}_A(P(n),P(j)) \to \dots \\ & \quad \dots \to \operatorname{Hom}_A(P(i-n+l),P(j)) \xrightarrow{-\circ p_d} \operatorname{Hom}_A(P(1),P(j)) \to 0) \\ =& \operatorname{Hom}_A(P(1),P(j))/\operatorname{im}(\_\circ p_d).\end{aligned}$$

This shows that $\operatorname{Ext}^d_A(I(i),P(j)) = 0$ if and only if all morphisms from $P(1)$ to $P(j)$ factors through p_d. By looking at the Auslander-Reiten quiver of A we see that this happens if and only if $i-n+l \le j \le n$. To ease notation, we define $M(i)$ in the following way:

$$M(i) = \begin{cases} P(i), & 1 \le i \le n \\ I(i-l+1), & n < i \le n+l-1) \end{cases}$$

We can then formulate the following result.

Lemma 5.3. Let $M(i)$ be defined as above, then $\operatorname{Ext}^m_A(M(i),M(j)) = 0$ for all $m > 0$ if and only if $1 \le i \le n+j-1$.

Now assume we want to find a module $T \in \mathcal{M}$ satisfying $(\mathrm{P})_d$, $(\mathrm{E})_d$ and $|T| = |A| = n$ such that $M(i) \in \operatorname{add}(T)$ but $M(j) \notin \operatorname{add}(T)$ for $j < i$. By Lemma 5.3 we can only choose $M(k)$ to lie in $\operatorname{add}(T)$ if $i \le k \le n+i-1$, but this is precisely n choices for k. This leads to the following result.

Corollary 5.4. Let $T \in \mathcal{M}$ satisfy $(\mathrm{P})_d$, $(\mathrm{E})_d$ and $|T| = n$. Then $\mathrm{add}(T) = \mathrm{add}(T(i))$ where $T(i) = \bigoplus_{k=i}^{n+i-1} M(k)$ for some $1 \leq i \leq l$.

Example 5.5. Let $n = 4$ and $l = 3$, then $A = A_{4,3}$ is 2-representation finite. The modules with 4 non-isomorphic indecomposable direct summands that satisfies $(\mathrm{P})_2$ and $(\mathrm{E})_2$ are A_A, DA_A and the module marked in the Auslander-Reiten quiver below:

$$
\begin{tikzcd}
 & & \mathbf{1110} \arrow[dr] & & \mathbf{0111} \arrow[dr] & & \\
 & \mathbf{1100} \arrow[ur] \arrow[dr] & & 0110 \arrow[ur] \arrow[dr] & & \mathbf{0011} \arrow[dr] & \\
1000 \arrow[ur] & & 0100 \arrow[ur] & & 0010 \arrow[ur] & & 0001
\end{tikzcd}
$$

5.3 $\mathrm{add}(T)$-resolutions of projective modules

Now that we have found all modules $T(i)$, $1 \leq i \leq l$, with n non-isomorphic direct summands that satisfies $(\mathrm{P})_d$ and $(\mathrm{E})_d$ we need to confirm that these also satisfies $(\mathrm{G})_d$. Since $A_A = \bigoplus_{i=1}^{n} P(i)$ it suffices to find a resolution of $P(j)$, $1 \leq j \leq n$, in $\mathrm{add}(T(i))$. We only need to look at $P(j)$ for $j < i$, since all other projective modules lies in $\mathrm{add}(T(i))$. Fix $1 \leq i \leq l$ and $1 \leq j < i$. Then look at the following morphisms.

$$
\begin{aligned}
&t_0 : P(j) \to P(i), t_0 = m(1, j) \\
&t_1 : P(i) \to P(j + l), t_1 = m(j + 1, i) \\
&t_{2k} : P(j + kl) \to P(i + kl), t_{2k} = m(i + (k - 1)l + 1, j + kl) \\
&t_{2k+1} : P(i + kl) \to P(j + (k + 1)l), t_{2k+1} = m(j + kl + 1, i + kl) \\
&t_d : I(n + j - l) \to I(n + i - l), t_d : m(n + i - l, n).
\end{aligned}
$$

It is easy to check that $\mathrm{im}(t_{2k}) = M(i + (k - 1)l + 1, j + kl) = \ker(t_{2k+1})$ and $\mathrm{im}(t_{2k-1}) = M(j+(k-1)l+1, i+(k-1)l) = \ker(t_{2k})$. Thus the following is a resolution of $P(j)$ in $\mathrm{add}(T(i))$ for $j < i \leq l$.

$$
\begin{aligned}
0 \longrightarrow P(j) \xrightarrow{t_0} P(i) \xrightarrow{t_1} P(j + l) \xrightarrow{t_2} \dots \\
\dots \xrightarrow{t_{2k}} P(i + kl) \xrightarrow{t_{2k+1}} P(j + (k + 1)l) \longrightarrow \dots \\
\dots \xrightarrow{t_{d-1}} I(j - l + n) \xrightarrow{t_d} I(n + i - l) \longrightarrow 0
\end{aligned}
$$

This shows that $T(i) = \bigoplus_{k=i}^{n+i-1} M(k)$ satisfies $(\mathrm{G})_d$ for all $i \leq l$. We can thus state the following theorem which confirms Rickard's conjecture for $T \in \mathcal{M}$.

Theorem 5.6. Let $\mathcal{M}$ be the d-cluster tilting subcategory of mod-$A_{n,l}$ and $T \in \mathcal{M}$. Then the following is equivalent.

1. T is a tilting module of projective dimension at most d.
2. T satisfies $(\mathrm{P})_d$, $(\mathrm{E})_d$ and $|T| = n$.
3. $\mathrm{add}(T) = \mathrm{add}(T(i))$ where $T(i) = \bigoplus_{k=i}^{n+i-1} M(k)$ for some $1 \leq i \leq l$.

6 Tilting modules over the tensor product of two 2-subspace algebras

The next algebra we want to look at is $KQ \otimes KQ$ where Q is the quiver

$$1 \to 2 \leftarrow 3.$$

For the algebra KQ, called the 2-subspace algebra, we have the following dimension vectors of projective and injective modules:

$$P(1) : 110, P(2) : 010, P(3) : 011$$
$$I(1) : 100, I(2) : 111, I(3) : 001$$

To understand the algebra $KQ \otimes KQ$ better we make use of the following lemma.

Lemma 6.1. [15, Lemma 1.3] For quivers Q, Q' there is a K-algebra isomorphism

$$KQ \otimes_K KQ' \cong K(Q \otimes Q')/\mathcal{I}$$

where $(Q \otimes Q')_0 = Q_0 \times Q'_0$, $(Q \otimes Q')_1 = (Q_0 \times Q'_1) \cup (Q_1 \times Q'_0)$ and $\mathcal{I}$ is the $K(Q \otimes Q')$-ideal generated by

$$\{(\alpha, t) \circ (p, \beta) - (r, \beta) \circ (\alpha, s) | \alpha : p \to r \in Q_1, \beta : s \to t \in Q'_1\}.$$

By Lemma 6.1 the following quiver with relations describes $KQ \otimes KQ$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, dash, dashed] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dl, dash, dashed] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u] \arrow[ur, dash, dashed] & \bullet \arrow[u] & \bullet \arrow[l] \arrow[u] \arrow[ul, dash, dashed]
\end{tikzcd}$$

where the dashed lines indicate that the squares commute. This gives us the following dimension vectors of projective and injective modules.

$$
\begin{array}{lll}
P_{11}: \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0, \\ 0 & 0 & 0 \end{matrix} & P_{12}: \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0, \\ 0 & 0 & 0 \end{matrix} & P_{13}: \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1, \\ 0 & 0 & 0 \end{matrix} \\
P_{21}: \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0, \\ 0 & 0 & 0 \end{matrix} & P_{22}: \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0, \\ 0 & 0 & 0 \end{matrix} & P_{23}: \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1, \\ 0 & 0 & 0 \end{matrix} \\
P_{31}: \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0, \\ 1 & 1 & 0 \end{matrix} & P_{32}: \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0, \\ 0 & 1 & 0 \end{matrix} & P_{33}: \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \\
I_{11}: \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0, \\ 0 & 0 & 0 \end{matrix} & I_{12}: \begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 0, \\ 0 & 0 & 0 \end{matrix} & I_{13}: \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0, \\ 0 & 0 & 0 \end{matrix} \\
I_{21}: \begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 0, \\ 1 & 0 & 0 \end{matrix} & I_{22}: \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1, \\ 1 & 1 & 1 \end{matrix} & I_{23}: \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1, \\ 0 & 0 & 1 \end{matrix} \\
I_{31}: \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0, \\ 1 & 0 & 0 \end{matrix} & I_{32}: \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0, \\ 1 & 1 & 1 \end{matrix} & I_{33}: \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}
\end{array}
$$

Proposition 6.2. [13, Section 3.1] The algebra $KQ \otimes KQ$, where Q is the quiver

$$\bullet \to \bullet \leftarrow \bullet$$

is 2-representation finite with 2-cluster tilting module $(KQ \otimes KQ) \oplus (\tau^{-1}KQ \otimes \tau^{-1}KQ)$.

Using our notation, the 2-cluster tilting subcategory corresponding to $KQ \otimes KQ$ is

$$\mathcal{M} = \mathrm{add}(\{I_{ij} | i, j = 1, 2, 3\} \cup \{P_{kl} | k, l = 1, 2, 3\}).$$

We can visualize the indecomposable modules in $\mathcal{M}$ and the morphisms between them by the following quiver.

$$
\begin{tikzcd}
 & P_{23} \arrow[r] \arrow[dr, dashed] & P_{13} \arrow[ddr] & & I_{21} \arrow[r] \arrow[dr, dashed] & I_{31} \\
 & P_{12} \arrow[ur] \arrow[ddr] & P_{33} \arrow[dr] & & I_{32} \arrow[ur] \arrow[ddr] & I_{11} \\
P_{22} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & & I_{22} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \\
 & P_{21} \arrow[r] \arrow[dr] & P_{11} \arrow[ur] & & I_{23} \arrow[r] \arrow[dr] & I_{33} \\
 & P_{32} \arrow[r] \arrow[uuur, dashed] & P_{31} \arrow[uur] & & I_{12} \arrow[r] \arrow[uuur, dashed] & I_{13}
\end{tikzcd} \tag{6.1}
$$

By Theorem 3.2 in [13] we have $\tau^{-1}P(i) = I(\sigma(i))$, where $\sigma(i) = 4 - i$. Since $P_{kl} \cong P(k) \otimes P(l)$ and $I_{ij} \cong I(i) \times I(j)$ it follows that $\tau_2^{-1} P_{kl} = I_{\sigma(k)\sigma(l)}$. Let $P(\mathcal{M})$ be the subquiver consisting of the projective modules and the arrows between them, and let $I(\mathcal{M})$ be the subquiver consisting of the injective modules and the arrows between them. Then the underlying graphs of $P(\mathcal{M})$ and $I(\mathcal{M})$ are the same. We can then see, by verifying this for all cases, that P_{kl} and $\tau_2^{-1} P_{kl}$ corresponds to the same node in the underlying graphs.

6.1 Projective resolutions

In this section we find the projective resolution of each injective module I_{ij}. Since the global dimension of $KQ \otimes KQ$ is 2, the existence of projective resolutions of length at most 2 is guaranteed. We need however the projective resolutions to calculate the Ext-groups.

We will now present the projective resolutions using dimension vectors of the modules and the dimension vector of the image of each module-morphism. For clarity we also describe the projective resolutions using direct sums of the modules P_{kl}.

Projective resolution of I_{11}:

$$
0 \longrightarrow \begin{matrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{smallmatrix}} \begin{matrix} 0&1&0 \\ 1&2&0 \\ 0&0&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&1&0 \\ 1&1&0 \\ 0&0&0 \end{smallmatrix}} \begin{matrix} 1&1&0 \\ 1&1&0 \\ 0&0&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}} \begin{matrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{matrix} \longrightarrow 0
$$

Or, equivalently:

$$
0 \longrightarrow P_{22} \xrightarrow{d_{11}} P_{12} \oplus P_{21} \longrightarrow P_{11} \longrightarrow I_{11} \longrightarrow 0
$$

Projective resolution of I_{12}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&1&0\\1&3&1\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&1&0\\1&2&1\\0&0&0\end{smallmatrix}} \begin{matrix}1&2&1\\1&2&1\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}1&1&1\\0&0&0\\0&0&0\end{smallmatrix}} \begin{matrix}1&1&1\\0&0&0\\0&0&0\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{12}} P_{12} \oplus P_{21} \oplus P_{23} \longrightarrow P_{11} \oplus P_{13} \longrightarrow I_{12} \longrightarrow 0$$

Projective resolution of I_{13}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&1&0\\0&2&1\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&1&0\\0&1&1\\0&0&0\end{smallmatrix}} \begin{matrix}0&1&1\\0&1&1\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&1\\0&0&0\\0&0&0\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{13}} P_{12} \oplus P_{23} \longrightarrow P_{13} \longrightarrow I_{13} \longrightarrow 0$$

Projective resolution of I_{21}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&1&0\\1&3&0\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&1&0\\1&2&0\\0&1&0\end{smallmatrix}} \begin{matrix}1&1&0\\2&2&0\\1&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}1&0&0\\1&0&0\\1&0&0\end{smallmatrix}} \begin{matrix}1&0&0\\1&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{21}} P_{12} \oplus P_{21} \oplus P_{32} \longrightarrow P_{11} \oplus P_{31} \longrightarrow I_{21} \longrightarrow 0$$

Projective resolution of I_{22}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&1&0\\1&4&1\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&1&0\\1&3&1\\0&1&0\end{smallmatrix}} \begin{matrix}1&2&1\\2&4&2\\1&2&1\end{matrix} \xrightarrow{\begin{smallmatrix}1&1&1\\1&1&1\\1&1&1\end{smallmatrix}} \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{22}} \begin{matrix}P_{12} \oplus P_{21} \quad \oplus\\ P_{23} \oplus P_{32}\end{matrix} \longrightarrow \begin{matrix}P_{11} \oplus P_{13} \quad \oplus\\ P_{31} \oplus P_{33}\end{matrix} \longrightarrow I_{22} \longrightarrow 0$$

Projective resolution of I_{23}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&1&0\\0&3&1\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&1&0\\0&2&1\\0&1&0\end{smallmatrix}} \begin{matrix}0&1&1\\0&2&2\\0&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&1\\0&0&1\\0&0&1\end{smallmatrix}} \begin{matrix}0&0&1\\0&0&1\\0&0&1\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{23}} P_{12} \oplus P_{23} \oplus P_{32} \longrightarrow P_{13} \oplus P_{33} \longrightarrow I_{23} \longrightarrow 0$$

Projective resolution of I_{31}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\1&2&0\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\1&1&0\\0&1&0\end{smallmatrix}} \begin{matrix}0&0&0\\1&1&0\\1&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&0&0\\1&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{31}} P_{21} \oplus P_{32} \longrightarrow P_{31} \longrightarrow I_{31} \longrightarrow 0$$

Projective resolution of I_{32}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\1&3&1\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\1&2&1\\0&1&0\end{smallmatrix}} \begin{matrix}0&0&0\\1&2&1\\1&2&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&0&0\\1&1&1\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\1&1&1\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{32}} P_{21} \oplus P_{23} \oplus P_{32} \longrightarrow P_{31} \oplus P_{33} \longrightarrow I_{32} \longrightarrow 0$$

Projective resolution of I_{33}:

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&2&1\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&1\\0&1&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&1&1\\0&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\7&0&0\\0&0&1\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\0&0&1\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \longrightarrow P_{22} \xrightarrow{d_{33}} P_{23} \oplus P_{32} \longrightarrow P_{33} \longrightarrow I_{33} \longrightarrow 0$$

6.2 Ext-groups

Now that we have found all projective resolutions we can calculate the Ext-group between each pair of indecomposable modules in $\mathcal{M}$. To ease notation we set $A = KQ \otimes KQ$ throughout this section. Since $\mathrm{Ext}^2_A(X, Y) = 0$ if X is projective or Y is injective, we only need to calculate $\mathrm{Ext}^2_A(I_{ij}, P_{kl})$. From our previous calculations we get that $\mathrm{Ext}^2_A(I_{ij}, P_{kl}) = 0$ if and only if $\_ \circ d_{ij}$ is surjective, that is, all morphisms in $\mathrm{Hom}_A(P_{22}, P_{kl})$ factors through d_{ij}.

We can visualize morphisms between the projective $KQ \otimes KQ$-modules using the following diagram.

$$\begin{tikzcd}
P_{11} & P_{12} \arrow[l] \arrow[r] & P_{13} \\
P_{21} \arrow[u] \arrow[d] & P_{22} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & P_{23} \arrow[u] \arrow[d] \\
P_{31} & P_{32} \arrow[l] \arrow[r] & P_{33}
\end{tikzcd}$$

Using this diagram it is now easy to check whether or not $\mathrm{Ext}^2_A(I_{ij}, P_{kl})$ vanishes. We present the result in the following table where "$-$" indicates a non-vanishing Ext-group.

	P_{11}	P_{12}	P_{13}	P_{21}	P_{22}	P_{23}	P_{31}	P_{32}	P_{33}
I_{11}	0	0	0	0	$-$	$-$	0	$-$	$-$
I_{12}	0	0	0	0	$-$	0	0	$-$	0
I_{13}	0	0	0	$-$	$-$	0	$-$	$-$	0
I_{21}	0	0	0	0	$-$	$-$	0	0	0
I_{22}	0	0	0	0	$-$	0	0	0	0
I_{23}	0	0	0	$-$	$-$	0	0	0	0
I_{31}	0	$-$	$-$	0	$-$	$-$	0	0	0
I_{32}	0	$-$	0	0	$-$	0	0	0	0
I_{33}	$-$	$-$	0	$-$	$-$	0	0	0	0

Next we classify all 48 modules T in $\mathcal{M}$ that satisfies $(\mathrm{P})_2$, $(\mathrm{E})_2$ and $|T| = 9 = |A|$. These are the modules containing 9 non-isomorphic indecomposable direct summands such that all pairs of direct summands have vanishing Ext-groups.

0 projective modules:

$$\bigoplus_{ij} I_{ij}$$

1 projective module:

$$\left(\bigoplus_{ij\neq 33} I_{ij}\right)\oplus P_{11},\quad \left(\bigoplus_{ij\neq 31} I_{ij}\right)\oplus P_{13},\quad \left(\bigoplus_{ij\neq 13} I_{ij}\right)\oplus P_{31},\quad \left(\bigoplus_{ij\neq 11} I_{ij}\right)\oplus P_{33}.$$

2 projective modules:

$$\begin{aligned}
&\left(\bigoplus_{ij\neq 33,31} I_{ij}\right)\oplus P_{11}\oplus P_{13}, &&\left(\bigoplus_{ij\neq 33,13} I_{ij}\right)\oplus P_{11}\oplus P_{31},\\
&\left(\bigoplus_{ij\neq 33,11} I_{ij}\right)\oplus P_{11}\oplus P_{33}, &&\left(\bigoplus_{ij\neq 31,13} I_{ij}\right)\oplus P_{13}\oplus P_{31},\\
&\left(\bigoplus_{ij\neq 31,11} I_{ij}\right)\oplus P_{13}\oplus P_{33}, &&\left(\bigoplus_{ij\neq 13,11} I_{ij}\right)\oplus P_{31}\oplus P_{33}.
\end{aligned}$$

3 projective modules:

$$\begin{aligned}
&\left(\bigoplus_{ij\neq 33,31,13} I_{ij}\right)\oplus P_{11}\oplus P_{13}\oplus P_{31}, &&\left(\bigoplus_{ij\neq 33,31,11} I_{ij}\right)\oplus P_{11}\oplus P_{13}\oplus P_{33},\\
&\left(\bigoplus_{ij\neq 33,13,11} I_{ij}\right)\oplus P_{11}\oplus P_{31}\oplus P_{33}, &&\left(\bigoplus_{ij\neq 31,13,11} I_{ij}\right)\oplus P_{13}\oplus P_{31}\oplus P_{33}\\
&\left(\bigoplus_{ij\neq 33,32,31} I_{ij}\right)\oplus P_{11}\oplus P_{12}\oplus P_{13}, &&\left(\bigoplus_{ij\neq 33,23,13} I_{ij}\right)\oplus P_{11}\oplus P_{21}\oplus P_{31},\\
&\left(\bigoplus_{ij\neq 21,13,11} I_{ij}\right)\oplus P_{13}\oplus P_{23}\oplus P_{33}, &&\left(\bigoplus_{ij\neq 13,12,11} I_{ij}\right)\oplus P_{31}\oplus P_{32}\oplus P_{33}.
\end{aligned}$$

4 projective modules:

$$\left(\bigoplus_{ij\neq 11,13,31,33} I_{ij}\right) \oplus P_{11} \oplus P_{13} \oplus P_{31} \oplus P_{33},$$

$$\left(\bigoplus_{ij\neq 33,32,31,13} I_{ij}\right) \oplus P_{11} \oplus P_{12} \oplus P_{13} \oplus P_{31}, \quad \left(\bigoplus_{ij\neq 33,32,31,11} I_{ij}\right) \oplus P_{11} \oplus P_{12} \oplus P_{13} \oplus P_{33},$$

$$\left(\bigoplus_{ij\neq 33,31,23,13} I_{ij}\right) \oplus P_{11} \oplus P_{13} \oplus P_{21} \oplus P_{31}, \quad \left(\bigoplus_{ij\neq 33,23,13,11} I_{ij}\right) \oplus P_{11} \oplus P_{21} \oplus P_{31} \oplus P_{33},$$

$$\left(\bigoplus_{ij\neq 33,31,21,11} I_{ij}\right) \oplus P_{11} \oplus P_{13} \oplus P_{23} \oplus P_{33}, \quad \left(\bigoplus_{ij\neq 31,21,13,11} I_{ij}\right) \oplus P_{13} \oplus P_{23} \oplus P_{31} \oplus P_{33},$$

$$\left(\bigoplus_{ij\neq 33,13,12,11} I_{ij}\right) \oplus P_{11} \oplus P_{31} \oplus P_{32} \oplus P_{33}, \quad \left(\bigoplus_{ij\neq 31,13,12,11} I_{ij}\right) \oplus P_{13} \oplus P_{31} \oplus P_{32} \oplus P_{33}.$$

4 injective modules:

$$\left(\bigoplus_{kl\neq 32,23,22,21} P_{kl}\right) \oplus I_{12} \oplus I_{21} \oplus I_{22} \oplus I_{23}, \quad \left(\bigoplus_{kl\neq 32,23,22,12} P_{kl}\right) \oplus I_{12} \oplus I_{21} \oplus I_{22} \oplus I_{32},$$

$$\left(\bigoplus_{kl\neq 32,22,21,12} P_{kl}\right) \oplus I_{12} \oplus I_{21} \oplus I_{22} \oplus I_{32}, \quad \left(\bigoplus_{kl\neq 23,22,21,12} P_{kl}\right) \oplus I_{21} \oplus I_{22} \oplus I_{23} \oplus I_{32},$$

$$\left(\bigoplus_{kl\neq 33,32,23,22} P_{kl}\right) \oplus I_{11} \oplus I_{12} \oplus I_{21} \oplus I_{22}, \quad \left(\bigoplus_{kl\neq 32,31,22,21} P_{kl}\right) \oplus I_{12} \oplus I_{13} \oplus I_{22} \oplus I_{23},$$

$$\left(\bigoplus_{kl\neq 23,22,13,12} P_{kl}\right) \oplus I_{21} \oplus I_{22} \oplus I_{31} \oplus I_{32}, \quad \left(\bigoplus_{kl\neq 22,21,11,12} P_{kl}\right) \oplus I_{22} \oplus I_{23} \oplus I_{32} \oplus I_{33}.$$

3 injective modules:

$$\left(\bigoplus_{kl\neq 32,23,22} P_{kl}\right) \oplus I_{12} \oplus I_{21} \oplus I_{22}, \quad \left(\bigoplus_{kl\neq 31,22,21} P_{kl}\right) \oplus I_{12} \oplus I_{22} \oplus I_{23},$$

$$\left(\bigoplus_{kl\neq 31,22,12} P_{kl}\right) \oplus I_{12} \oplus I_{22} \oplus I_{32}, \quad \left(\bigoplus_{kl\neq 23,22,21} P_{kl}\right) \oplus I_{21} \oplus I_{22} \oplus I_{23},$$

$$\left(\bigoplus_{kl\neq 23,22,12} P_{kl}\right) \oplus I_{21} \oplus I_{22} \oplus I_{32}, \quad \left(\bigoplus_{kl\neq 22,21,12} P_{kl}\right) \oplus I_{22} \oplus I_{23} \oplus I_{32}.$$

2 injective modules:

$$\left(\bigoplus_{kl\neq 32,22} P_{kl}\right) \oplus I_{12} \oplus I_{22}, \quad \left(\bigoplus_{kl\neq 23,22} P_{kl}\right) \oplus I_{21} \oplus I_{22},$$
$$\left(\bigoplus_{kl\neq 22,21} P_{kl}\right) \oplus I_{22} \oplus I_{23}, \quad \left(\bigoplus_{kl\neq 22,12} P_{kl}\right) \oplus I_{22} \oplus I_{32}.$$

1 injective module:

$$\left(\bigoplus_{kl\neq 22} P_{kl}\right) \oplus I_{22}$$

0 injective modules:

$$\bigoplus_{kl} P_{kl}$$

After classifying all modules T in $\mathcal{M}$ satisfying $(\mathrm{P})_2$, $(\mathrm{E})_2$ and $|T| = 9$ we observe that if $P_{kl} \in \mathrm{add}(T)$, then all successors of P_{kl} in the quiver 6.1 lie in $\mathrm{add}(T)$. Similarly, if $I_{ij} \in \mathrm{add}(T)$ then all predecessors of I_{ij} lies in $\mathrm{add}(T)$. Given a subset P of $\{P_{kl} | k, l = 1, 2, 3\}$ that is closed under successors in the quiver 6.1, there is a unique subset $I(P)$ of $\{I_{ij} | i, j = 1, 2, 3\}$ such that the direct sum of the modules in these sets form a module in the list above. Calculations show that $I(P) = \{\tau_2^{-1} P_{kl} | P_{kl} \notin P\}$. So $P_{kl} \in \mathrm{add}(T)$ if and only if $\tau_2^{-1} P_{kl} \notin \mathrm{add}(T)$. The claim is proven by verifying this for all modules in the list above.

Example 6.3. Look at the module

$$T = \left(\bigoplus_{kl\neq 22,13,23,12} P_{kl}\right) \oplus I_{22} \oplus I_{31} \oplus I_{21} \oplus I_{32}.$$

$$\begin{tikzcd}
 & P_{23} \arrow[r] \arrow[dr, dashed] & P_{13} \arrow[ddr] & & \boxed{I_{21}} \arrow[r] \arrow[dr, dashed] & \boxed{I_{31}} \\
 & P_{12} \arrow[ur] \arrow[ddr] & \boxed{P_{33}} \arrow[dr] & & \boxed{I_{32}} \arrow[ur] \arrow[ddr] & I_{11} \\
P_{22} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & & \boxed{I_{22}} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \\
 & \boxed{P_{21}} \arrow[r] \arrow[dr] & \boxed{P_{11}} \arrow[ur] & & I_{23} \arrow[r] \arrow[dr] & I_{33} \\
 & \boxed{P_{32}} \arrow[r] \arrow[uuur, dashed] & \boxed{P_{31}} \arrow[uur] & & I_{12} \arrow[r] \arrow[uuur, dashed] & I_{13}
\end{tikzcd}$$

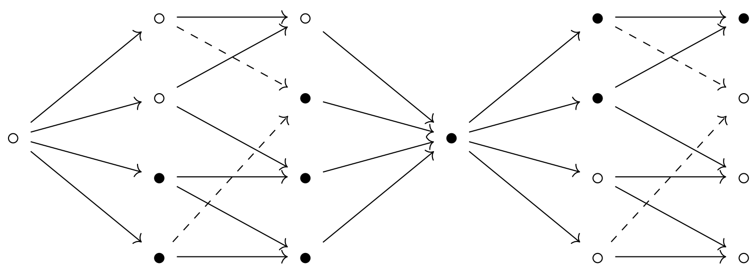
We note that $\tau_2^{-1} P_{22} = I_{22}, \tau_2^{-1} P_{23} = I_{21}, \tau_2^{-1} P_{12} = I_{32}, \tau_2^{-1} P_{13} = I_{31}$ lies in $\mathrm{add}(T)$, while $\tau_2^{-1} P_{21} = I_{23}, \tau_2^{-1} P_{32} = I_{12}, \tau_2^{-1} P_{33} = I_{11}, \tau_2^{-1} P_{11} = I_{33}, \tau_2^{-1} P_{31} = I_{13}$ does not lie in $\mathrm{add}(T)$.

6.3 add(T)-resolutions of projective modules

To prove that the modules satisfying $(\mathrm{E})_2$, $(\mathrm{P})_2$ and $|T| = 9$ are tilting modules of projective dimension at most 2 we need to find add(T)-resolutions of each projective module. Luckily enough there are a lot of symmetries between these modules. In the diagram above we have four cycles: $(P_{23}, P_{12}, P_{21}, P_{32})$, $(P_{13}, P_{11}, P_{31}, P_{33})$, $(I_{21}, I_{32}, I_{23}, I_{12})$ and $(I_{31}, I_{33}, I_{13}, I_{11})$. Moving one step to the right in these cycles corresponds to rotating the dimension vector 90 degrees counter-clockwise. Since the quiver $Q \otimes Q$ is invariant under rotations as well as horizontal and vertical reflection we get the following result.

If we have an exact sequence with modules in $\mathcal{M}$ and rotate all dimension vectors 90 degrees, or horizontally or vertically reflect all dimension vectors, we get a new sequence that is still exact.

Assume that we start with an exact sequence using, for example, the modules marked with circles in the diagram below. If we rotate the whole diagram, that is, rotating each cycle simultaneously, a new set of modules will lie in the marked positions. This yields a new exact sequence. Thus, all cases can be expressed by using the notion of "neighbours", "successors" and "predecessors". By "neighbours" we mean two modules that are adjacent in the cycles above.



Since each module in the list above corresponds to a specific subset P of $\{P_{kl} | k, l = 1, 2, 3\}$ that is closed under successors, we will sometimes refer to a module by the corresponding subset $P(T)$.

6.3.1 Resolutions of $P_{11}, P_{13}, P_{31}, P_{33}$

First we look at the modules P_{11}, P_{13}, P_{31} and P_{33}. These are special, because if P_{kl}, where $k, l \in \{1, 3\}$, does not lie in add(T), neither does P_{2l}, P_{k2} and P_{22}. By the observation above $\tau_2 P_{kl}, \tau_2 P_{2l}, \tau_2 P_{k2}$ and $\tau_2 P_{22} = I_{22}$ must then lie in add(T). Using these modules we find the add(T)-resolution of T_{kl}, where $k, l \in \{1, 3\}$:

$$0 \to P_{kl} \to I_{22} \to \tau_2 P_{2l} \oplus \tau_2 P_{k2} \to \tau_2 P_{kl} \to 0$$

Explicitly we get:

$$\begin{aligned}
0 \to P_{11} \to I_{22} \to I_{23} \oplus I_{32} \to I_{33} \to 0 \\
0 \to P_{13} \to I_{22} \to I_{21} \oplus I_{32} \to I_{31} \to 0 \\
0 \to P_{31} \to I_{22} \to I_{23} \oplus I_{12} \to I_{13} \to 0 \\
0 \to P_{33} \to I_{22} \to I_{21} \oplus I_{12} \to I_{11} \to 0
\end{aligned}$$

6.3.2 Resolution of $P_{12}, P_{21}, P_{23}, P_{32}$

Next we want to find add(T)-resolutions of P_{12}, P_{21}, P_{23} and P_{32}. Note that we must have $P_{22} \notin \text{add}(T)$, thus $I_{22} \in \text{add}(T)$. Here we will get several cases depending on which indecomposable modules that belongs to add(T).

1. no successors of P_{kl} lies in add(T),
2. one successor of P_{kl} lies in add(T),
3. both successors of P_{kl} lies in add(T).

First we look at the case when no successors of P_{kl} lies in add(T). Let $kl = 23$. The modules that we know lie in add(T) are marked with a square in the quiver below. Recall that the right-hand side is closed under predecessors.

$$\begin{tikzcd}
 & P_{23} \arrow[r] \arrow[dr, dashed] & P_{13} \arrow[ddr] & & \boxed{I_{21}} \arrow[r] \arrow[dr, dashed] & \boxed{I_{31}} \\
 & P_{12} \arrow[ur] \arrow[ddr] & P_{33} \arrow[dr] & & \boxed{I_{32}} \arrow[ur] \arrow[ddr] & \boxed{I_{11}} \\
P_{22} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & & \boxed{I_{22}} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \\
 & P_{21} \arrow[r] \arrow[dr] & P_{11} \arrow[ur] & & I_{23} \arrow[r] \arrow[dr] & I_{33} \\
 & P_{32} \arrow[r] \arrow[uuur, dashed] & P_{31} \arrow[uur] & & \boxed{I_{12}} \arrow[r] \arrow[uuur, dashed] & I_{13}
\end{tikzcd}$$

For P_{23} we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \xrightarrow{\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}} \begin{matrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{matrix} \xrightarrow{\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}} \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \to P_{23} \to I_{22} \to I_{12} \oplus I_{21} \oplus I_{32} \to I_{11} \oplus I_{31} \to 0$$

The other sequences are found using rotations of the dimension vectors.

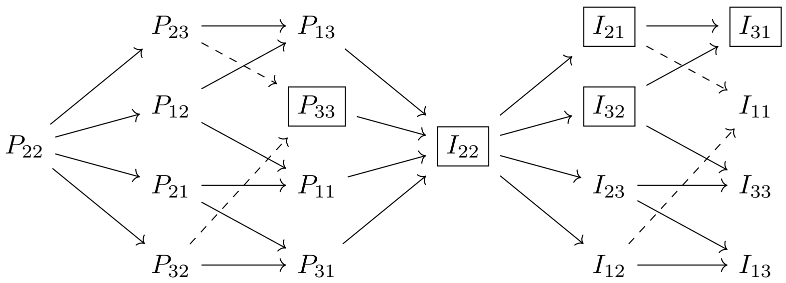
If $kl = 23$ the modules T belonging to this case are:
$P(T) = \emptyset, \{P_{11}\}, \{P_{31}\}, \{P_{11}, P_{31}\}, \{P_{11}, P_{31}, P_{21}\}$.

If $kl = 12$ the modules T belonging to this case are:
$P(T) = \emptyset, \{P_{31}\}, \{P_{33}\}, \{P_{31}, P_{33}\}, \{P_{31}, P_{32}, P_{33}\}$.

If $kl = 21$ the modules T belonging to this case are:
$P(T) = \emptyset, \{P_{13}\}, \{P_{33}\}, \{P_{13}, P_{33}\}, \{P_{13}, P_{23}, P_{33}\}$.

If $kl = 32$ the modules T belonging to this case are:
$P(T) = \emptyset, \{P_{11}\}, \{P_{13}\}, \{P_{11}, P_{13}\}, \{P_{11}, P_{12}, P_{13}\}$.

Secondly, we have the case when exactly one successor of P_{kl} lies in $\mathrm{add}(T)$. Let $kl = 23$ and we choose the case when $P_{33} \in \mathrm{add}(T)$. Again, we mark the modules we know lie in $\mathrm{add}(T)$ with a square in the quiver below.



For P_{23} we get the following $\mathrm{add}(T)$-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&1\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&1\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&1&1\\0&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&1\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\1&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&0&0\\1&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \to P_{23} \to P_{33} \to I_{32} \to I_{31} \to 0$$

The other sequences are found using rotations and reflections of the dimension vectors.

If $kl = 23$ the modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{13}\}, \{P_{33}\}, \{P_{11}, P_{13}\}, \{P_{11}, P_{33}\}, \{P_{13}, P_{31}\}, \{P_{31}, P_{33}\}, \\ &\{P_{11}, P_{13}, P_{31}\}, \{P_{11}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{12}\}, \{P_{31}, P_{32}, P_{33}\}, \\ &\{P_{11}, P_{13}, P_{31}, P_{12}\}, \{P_{11}, P_{13}, P_{31}, P_{21}\}, \{P_{11}, P_{31}, P_{33}, P_{21}\}, \\ &\{P_{11}, P_{31}, P_{33}, P_{32}\}, \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}\}, \{P_{11}, P_{21}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

If $kl = 12$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{11}\}, \{P_{13}\}, \{P_{11}, P_{31}\}, \{P_{11}, P_{33}\}, \{P_{13}, P_{31}\}, \{P_{13}, P_{33}\}, \\ & \{P_{11}, P_{31}, P_{33}\}, \{P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{21}, P_{31}\}, \{P_{13}, P_{23}, P_{33}\}, \\ & \{P_{11}, P_{21}, P_{31}, P_{33}\}, \{P_{13}, P_{23}, P_{31}, P_{33}\}, \{P_{11}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{13}, P_{31}, P_{32}, P_{33}\}, \{P_{11}, P_{21}, P_{31}, P_{32}, P_{33}\}, \{P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

If $kl = 21$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{11}\}, \{P_{31}\}, \{P_{11}, P_{13}\}, \{P_{11}, P_{33}\}, \{P_{13}, P_{31}\}, \{P_{31}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{33}\}, \{P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{12}\}, \{P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{33}\}, \{P_{11}, P_{13}, P_{23}, P_{33}\}, \{P_{13}, P_{23}, P_{31}, P_{33}\}, \\ & \{P_{13}, P_{31}, P_{32}, P_{33}\}, \{P_{11}, P_{12}, P_{13}, P_{23}, P_{33}\}, \{P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

If $kl = 32$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{31}\}, \{P_{33}\}, \{P_{11}, P_{31}\}, \{P_{11}, P_{33}\}, \{P_{13}, P_{31}\}, \{P_{13}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{31}\}, \{P_{11}, P_{13}, P_{33}\}, \{P_{11}, P_{31}, P_{21}\}, \{P_{13}, P_{23}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{31}, P_{12}\}, \{P_{11}, P_{13}, P_{31}, P_{12}\}, \{P_{11}, P_{13}, P_{33}, P_{21}\}, \\ & \{P_{11}, P_{13}, P_{31}, P_{23}\}, \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}\}, \{P_{11}, P_{12}, P_{13}, P_{23}, P_{33}\}. \end{aligned}$$

The last case we look at is when the two successors of P_{kl} lies in $\mathrm{add}(T)$. We mark the modules we know lie in $\mathrm{add}(T)$ with a square in the quiver below. Let $kl = 23$.

$$\begin{tikzcd}
 & P_{23} \arrow[r] \arrow[dr, dashed] & \boxed{P_{13}} \arrow[ddr] & & \boxed{I_{21}} \arrow[r] \arrow[dr, dashed] & I_{31} \\
 & P_{12} \arrow[ur] \arrow[ddr] & \boxed{P_{33}} \arrow[dr] & & \boxed{I_{32}} \arrow[ur] \arrow[ddr] & I_{11} \\
P_{22} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & & \boxed{I_{22}} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \\
 & P_{21} \arrow[r] \arrow[dr] & P_{11} \arrow[ur] & & I_{23} \arrow[r] \arrow[dr] & I_{33} \\
 & P_{32} \arrow[r] \arrow[uuur, dashed] & P_{31} \arrow[uur] & & I_{12} \arrow[r] \arrow[uuur, dashed] & I_{13}
\end{tikzcd}$$

For P_{23} we get the following $\mathrm{add}(T)$-resolution.

$$0 \longrightarrow \begin{matrix} 0&0&0 \\ 0&1&1 \\ 0&0&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0 \\ 0&1&1 \\ 0&0&0 \end{smallmatrix}} \begin{matrix} 0&1&1 \\ 0&2&2 \\ 0&1&1 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&1&1 \\ 0&1&1 \\ 0&1&1 \end{smallmatrix}} \begin{matrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{matrix} \xrightarrow{\begin{smallmatrix} 1&1&0 \\ 1&1&0 \\ 1&1&0 \end{smallmatrix}} \begin{matrix} 1&1&0 \\ 1&1&0 \\ 1&1&0 \end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \to P_{23} \to P_{13} \oplus P_{33} \to I_{22} \to I_{21} \to 0$$

The other sequences are found using rotations of the dimension vectors.

If $kl = 23$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{13}, P_{33}\}, \{P_{11}, P_{13}, P_{33}\}, \{P_{13}, P_{31}, P_{33}\}, \{P_{13}, P_{23}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{33}, P_{12}\}, \{P_{13}, P_{31}, P_{33}, P_{32}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{33}\}, \{P_{11}, P_{12}, P_{13}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{21}, P_{31}, P_{32}, P_{33}\}, \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

If $kl = 12$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{11}, P_{13}\}, \{P_{11}, P_{13}, P_{31}\}, \{P_{11}, P_{13}, P_{33}\}, \{P_{11}, P_{12}, P_{13}\}, \\ & \{P_{11}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{31}\}, \{P_{11}, P_{13}, P_{23}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{21}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{23}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{21}, P_{23}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{23}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

If $kl = 21$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{11}, P_{31}\}, \{P_{11}, P_{13}, P_{31}\}, \{P_{11}, P_{31}, P_{33}\}, \{P_{11}, P_{21}, P_{31}\}, \\ & \{P_{11}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{31}, P_{12}\}, \{P_{11}, P_{31}, P_{33}, P_{32}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{23}, P_{31}, P_{33}\}, \{P_{11}, P_{12}, P_{13}, P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\}, \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

If $kl = 32$ the modules T belonging to this case are:

$$\begin{aligned} P(T) =& \{P_{31}, P_{33}\}, \{P_{11}, P_{31}, P_{33}\}, \{P_{13}, P_{31}, P_{33}\}, \{P_{31}, P_{32}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{31}, P_{33}, P_{21}\}, \{P_{13}, P_{31}, P_{33}, P_{23}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{23}, P_{31}, P_{33}\}, \\ & \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{33}\}, \{P_{11}, P_{12}, P_{13}, P_{23}, P_{31}, P_{33}\}, \\ & \{P_{11}, P_{13}, P_{21}, P_{23}, P_{31}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{23}, P_{31}, P_{32}, P_{33}\}. \end{aligned}$$

6.3.3 Resolution of P_{22}

Now we want to find add(T)-resolutions for P_{22}. For this module we have 14 different cases.

1. All projective modules except P_{22} lies in add(T), then we have the add(T)-resolution

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\substack{0\ 0\ 0\\0\ 1\ 0\\0\ 0\ 0}} \begin{matrix}0&1&0\\1&4&1\\0&1&0\end{matrix} \xrightarrow{\substack{0\ 1\ 0\\1\ 3\ 1\\0\ 1\ 0}} \begin{matrix}1&2&1\\2&4&2\\1&2&1\end{matrix} \xrightarrow{\substack{1\ 1\ 1\\1\ 1\ 1\\1\ 1\ 1}} \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} \longrightarrow 0$$

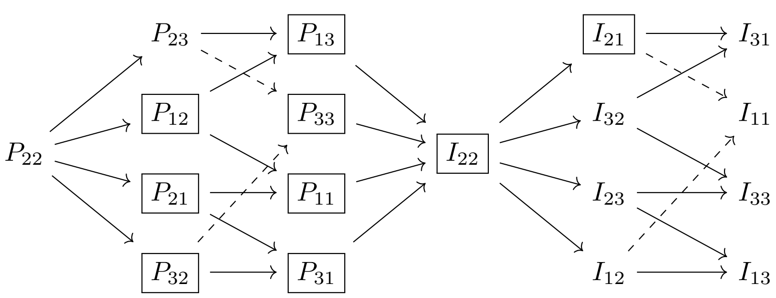
Or, equivalently:

$$0 \to P_{22} \to P_{12} \oplus P_{21} \oplus P_{23} \oplus P_{32} \to P_{11} \oplus P_{13} \oplus P_{31} \oplus P_{33} \to I_{22} \to 0$$

The module T belonging to this case is:

$$P(T) = \{P_{11}, P_{12}, P_{13}, P_{21}, P_{23}, P_{31}, P_{32}, P_{33}\}.$$

2. All projective modules except P_{22} and one immediate successor lies in add(T), we show this for the case when $P_{23} \notin$ add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\substack{0\ 0\ 0\\0\ 1\ 0\\0\ 0\ 0}} \begin{matrix}0&1&0\\1&3&0\\0&1&0\end{matrix} \xrightarrow{\substack{0\ 1\ 0\\1\ 2\ 0\\0\ 1\ 0}} \begin{matrix}1&1&0\\2&2&0\\1&1&0\end{matrix} \xrightarrow{\substack{1\ 0\ 0\\1\ 0\ 0\\1\ 0\ 0}} \begin{matrix}1&0&0\\1&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

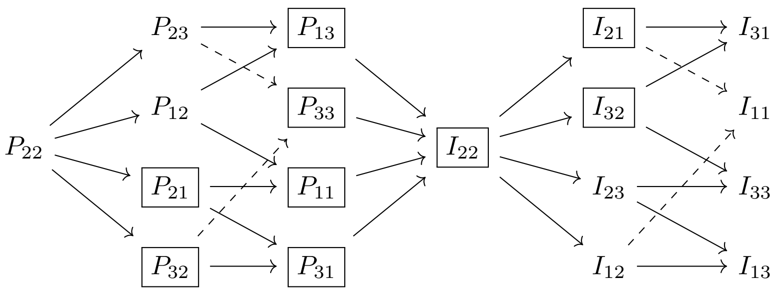
Or, equivalently:

$$0 \to P_{22} \to P_{12} \oplus P_{21} \oplus P_{32} \to P_{11} \oplus P_{31} \to I_{21} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{11}, P_{13}, P_{21}, P_{23}, P_{31}, P_{32}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{32}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{21}, P_{23}, P_{31}, P_{33}\}. \end{aligned}$$

3. All projective modules except P_{22} and two immediate successors, with a common successors, lies in add(T), we show this for $P_{23}, P_{12} \notin$ add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{smallmatrix} 0&0&0\\0&1&0\\0&0&0 \end{smallmatrix} \xrightarrow{\begin{smallmatrix} 0&0&0\\0&1&0\\0&0&0 \end{smallmatrix}} \begin{smallmatrix} 0&0&0\\1&2&0\\0&1&0 \end{smallmatrix} \xrightarrow{\begin{smallmatrix} 0&0&0\\1&1&0\\0&1&0 \end{smallmatrix}} \begin{smallmatrix} 1&1&1\\2&2&1\\2&2&1 \end{smallmatrix} \xrightarrow{\begin{smallmatrix} 1&1&1\\1&1&1\\2&1&1 \end{smallmatrix}} \begin{smallmatrix} 1&1&1\\1&1&1\\2&1&1 \end{smallmatrix} \longrightarrow 0$$

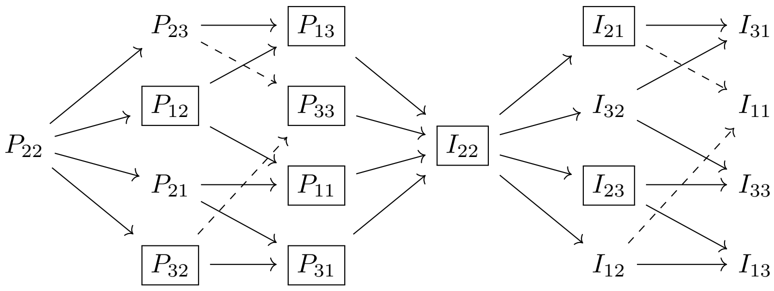
Or, equivalently:

$$0 \to P_{22} \to P_{21} \oplus P_{32} \to P_{31} \oplus I_{22} \to P_{13} \oplus I_{21} \oplus I_{32} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{11}, P_{13}, P_{21}, P_{31}, P_{32}, P_{33}\},\\ &\{P_{11}, P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{23}, P_{31}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{33}\}. \end{aligned}$$

4. All projective modules except P_{22} and two immediate successors, without a common successors, lies in add(T). We show this for $P_{23}, P_{21} \notin$ add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\1&2&0\\0&1&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\1&1&0\\0&1&0\end{smallmatrix}} \begin{matrix}1&1&1\\2&2&1\\2&2&1\end{matrix} \xrightarrow{\begin{smallmatrix}1&1&1\\1&1&1\\2&1&1\end{smallmatrix}} \begin{matrix}1&1&1\\1&1&1\\2&1&1\end{matrix} \longrightarrow 0$$

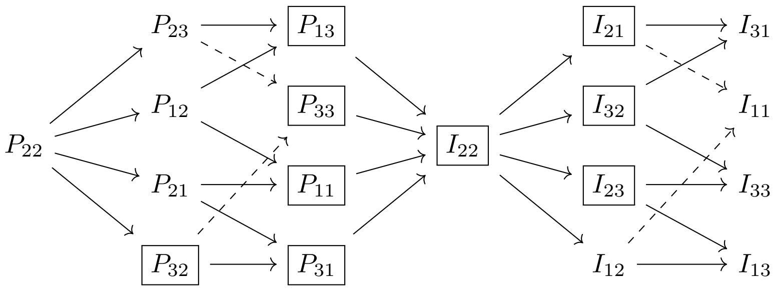
Or, equivalently:

$$0 \to P_{22} \to P_{21} \oplus P_{32} \to P_{31} \oplus I_{22} \to P_{13} \oplus I_{21} \oplus I_{32} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{11}, P_{12}, P_{13}, P_{31}, P_{32}, P_{33}\}, \\ &\{P_{11}, P_{13}, P_{21}, P_{23}, P_{31}, P_{33}\}. \end{aligned}$$

5. All projective modules except P_{22} and three immediate successors lies in add(T). We show this for $P_{23}, P_{12}, P_{21} \notin$ add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&0&0\\0&1&0\end{smallmatrix}} \begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix} \xrightarrow{\begin{smallmatrix}2&2&2\\2&2&2\\2&1&2\end{smallmatrix}} \begin{matrix}2&2&2\\2&2&2\\2&1&2\end{matrix} \longrightarrow 0$$

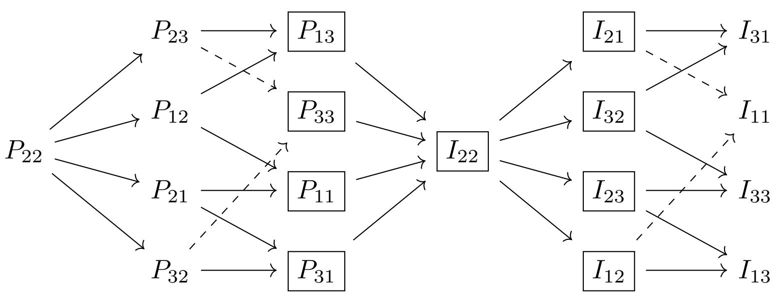
Or, equivalently:

$$0 \to P_{22} \to P_{32} \to I_{22} \oplus I_{22} \to P_{11} \oplus P_{13} \oplus I_{21} \oplus I_{23} \oplus I_{32} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{11}, P_{13}, P_{31}, P_{32}, P_{33}\},\\ &\{P_{11}, P_{13}, P_{23}, P_{31}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{31}, P_{33}\},\\ &\{P_{11}, P_{13}, P_{21}, P_{31}, P_{33}\}. \end{aligned}$$

6. All projective modules except P_{22} and all four immediate successors lies in add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}1&2&1\\2&4&2\\1&2&1\end{matrix} \xrightarrow{\begin{smallmatrix}1&2&1\\2&3&2\\1&2&1\end{smallmatrix}} \begin{matrix}3&3&3\\3&3&3\\3&3&3\end{matrix} \xrightarrow{\begin{smallmatrix}2&1&2\\1&0&1\\2&1&2\end{smallmatrix}} \begin{matrix}2&1&2\\1&0&1\\2&1&2\end{matrix} \longrightarrow 0$$

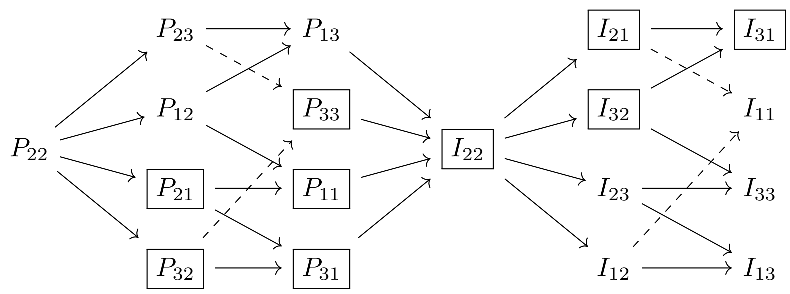
Or, equivalently:

$$0 \to P_{22} \to P_{13} \oplus P_{11} \oplus P_{31} \oplus P_{33} \to I_{22} \oplus I_{22} \oplus I_{22} \to I_{21} \oplus I_{32} \oplus I_{23} \oplus I_{12} \to 0$$

The module T belonging to this case is:

$$P(T) = \{P_{11}, P_{13}, P_{31}, P_{33}\}.$$

7. All projective modules except P_{22}, two immediate successor and one secondary successor lies in $\operatorname{add}(T)$. For this to be true we need the secondary successor to be the common successor of the other two successors. We show this for $P_{23}, P_{12}, P_{13} \notin \operatorname{add}(T)$. We mark the modules in $\operatorname{add}(T)$ with a square in the quiver below.



For this case we get the following $\operatorname{add}(T)$-resolution.

$$0 \longrightarrow \begin{matrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{smallmatrix}} \begin{matrix} 0&0&0 \\ 1&2&0 \\ 0&1&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0 \\ 1&1&0 \\ 0&1&0 \end{smallmatrix}} \begin{matrix} 0&0&0 \\ 1&1&0 \\ 1&1&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{smallmatrix}} \begin{matrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{matrix} \longrightarrow 0$$

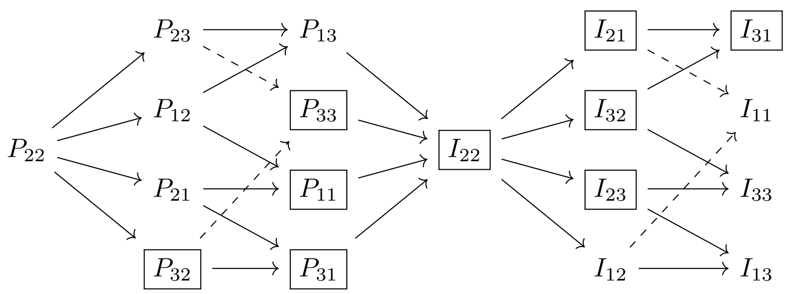
Or, equivalently:

$$0 \to P_{22} \to P_{21} \oplus P_{32} \to P_{31} \to I_{31} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{11}, P_{21}, P_{31}, P_{32}, P_{33}\}, \\ &\{P_{13}, P_{23}, P_{31}, P_{32}, P_{33}\}, \\ &\{P_{11}, P_{12}, P_{13}, P_{23}, P_{33}\}, \\ &\{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}\}. \end{aligned}$$

8. All projective modules except P_{22}, three immediate successor and one secondary successor lies in $\mathrm{add}(T)$. Note that the secondary successor must be a common successor of two of the other successors. We show this for $P_{23}, P_{12}, P_{21}, P_{13} \notin \mathrm{add}(T)$. We mark the modules in $\mathrm{add}(T)$ with a square in the quiver below.



For this case we have the following $\mathrm{add}(T)$-resolution.

$$0 \longrightarrow \begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix} \xrightarrow{\substack{0\,0\,0\\0\,1\,0\\0\,0\,0}} \begin{smallmatrix}1&1&1\\1&2&1\\0&1&1\end{smallmatrix} \xrightarrow{\substack{1\,1\,1\\1\,1\,1\\0\,1\,1}} \begin{smallmatrix}1&1&1\\1&1&1\\1&1&1\end{smallmatrix} \xrightarrow{\substack{0\,0\,0\\0\,0\,0\\1\,0\,0}} \begin{smallmatrix}0&0&0\\0&0&0\\1&0&0\end{smallmatrix} \longrightarrow 0$$

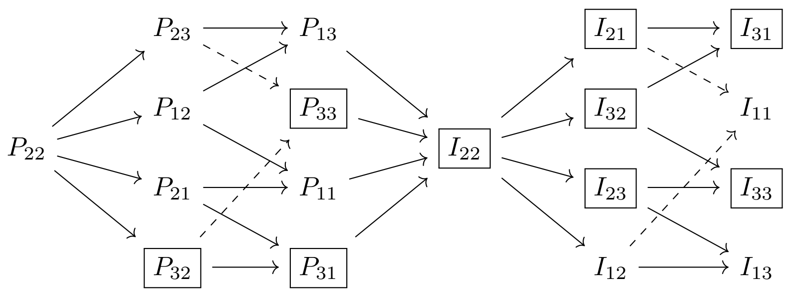
Or, equivalently:

$$0 \to P_{22} \to P_{32} \oplus P_{11} \oplus I_{23} \to I_{22} \to I_{31} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) =&\{P_{11}, P_{31}, P_{32}, P_{33}\}, \{P_{13}, P_{23}, P_{31}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{33}\}, \{P_{11}, P_{13}, P_{21}, P_{31}\},\\ &\{P_{13}, P_{31}, P_{32}, P_{33}\}, \{P_{11}, P_{13}, P_{23}, P_{33}\},\\ &\{P_{11}, P_{12}, P_{13}, P_{31}\}, \{P_{11}, P_{21}, P_{31}, P_{33}\}.\end{aligned}$$

9. All projective modules except P_{22}, three immediate successor and two secondary successor lies in $\mathrm{add}(T)$. Note that the secondary successors must be the common successors of the three other successors. We show this for $P_{23}, P_{12}, P_{21}, P_{13}, P_{11} \notin \mathrm{add}(T)$. We mark the modules in $\mathrm{add}(T)$ with a square in the quiver below.



For this case we have the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{smallmatrix}} \begin{matrix} 0&0&0\\ 0&1&0\\ 0&1&0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0\\ 0&0&0\\ 0&1&0 \end{smallmatrix}} \begin{matrix} 0&0&0\\ 0&0&0\\ 1&1&1 \end{matrix} \xrightarrow{\begin{smallmatrix} 0&0&0\\ 0&0&0\\ 1&0&1 \end{smallmatrix}} \begin{matrix} 0&0&0\\ 0&0&0\\ 1&0&1 \end{matrix} \longrightarrow 0$$

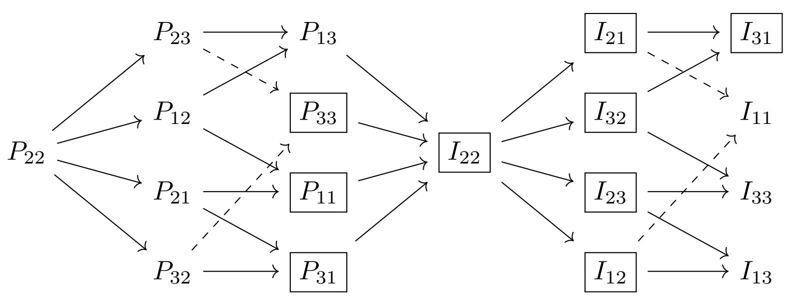
Or, equivalently:

$$0 \to P_{22} \to P_{32} \to I_{32} \to I_{31} \oplus I_{33} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{31}, P_{32}, P_{33}\}, \{P_{13}, P_{23}, P_{33}\}, \\ &\{P_{11}, P_{12}, P_{13}\}, \{P_{11}, P_{21}, P_{31}\}. \end{aligned}$$

10. All projective modules except P_{22}, four immediate successor and one secondary successor lies in add(T). We show this for P_{23}, P_{12}, P_{21}, P_{32}, $P_{13} \notin$ add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we have the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}2&2&2\\2&3&2\\1&2&2\end{matrix} \xrightarrow{\begin{smallmatrix}2&2&2\\2&2&2\\1&2&2\end{smallmatrix}} \begin{matrix}2&2&2\\2&3&2\\1&2&2\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&0&0\\1&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

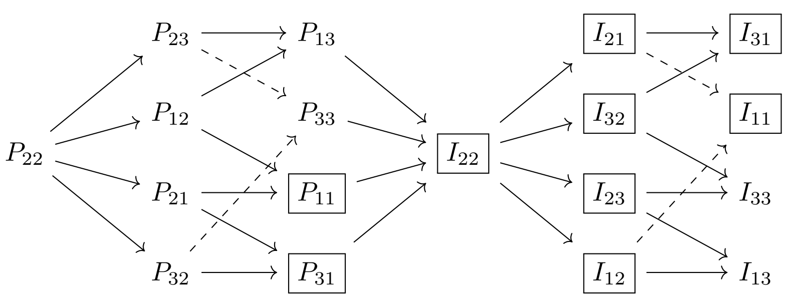
Or, equivalently:

$$0 \to P_{22} \to P_{11} \oplus P_{31} \oplus P_{33} \oplus I_{12} \oplus I_{23} \to I_{22} \oplus I_{22} \to I_{31} \to 0$$

The modules T belonging to this case are:

$$\begin{aligned} P(T) = &\{P_{11}, P_{31}, P_{33}\}, \{P_{13}, P_{31}, P_{33}\}, \\ &\{P_{11}, P_{13}, P_{33}\}, \{P_{11}, P_{13}, P_{31}\}. \end{aligned}$$

11. All projective modules except P_{22}, four immediate successors and two adjacent secondary successor lies in add(T). We show this for P_{23}, P_{12}, P_{21}, P_{32}, P_{13}, $P_{33} \notin$ add(T). We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}1&1&1\\2&2&1\\1&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}1&1&1\\2&1&1\\1&1&1\end{smallmatrix}} \begin{matrix}2&1&1\\2&1&1\\2&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}1&0&0\\0&0&0\\1&0&0\end{smallmatrix}} \begin{matrix}1&0&0\\0&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

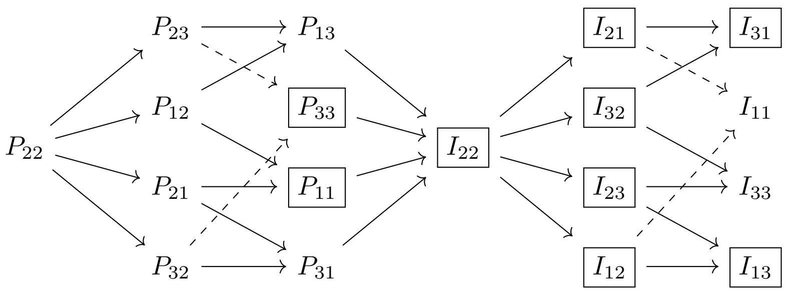
Or, equivalently:

$$0 \to P_{22} \to P_{11} \oplus P_{31} \oplus I_{23} \to I_{22} \oplus I_{21} \to I_{31} \oplus I_{11} \to 0$$

The modules T belonging to this case are:

$$P(T) = \{P_{11}, P_{31}\}, \{P_{31}, P_{33}\}, \{P_{13}, P_{33}\}, \{P_{11}, P_{13}\}.$$

12. All projective modules except P_{22}, four immediate successors and two non-adjacent secondary successors. We show this for P_{23}, P_{12}, P_{21}, P_{32}, P_{13}, $P_{31} \notin \mathrm{add}(T)$. We mark the modules in $\mathrm{add}(T)$ with a square in the quiver below.



For this case we get the following $\mathrm{add}(T)$-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}1&1&0\\1&2&1\\0&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}1&1&0\\1&1&1\\0&1&1\end{smallmatrix}} \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&1\\0&0&0\\1&0&0\end{smallmatrix}} \begin{matrix}0&0&1\\0&0&0\\1&0&0\end{matrix} \longrightarrow 0$$

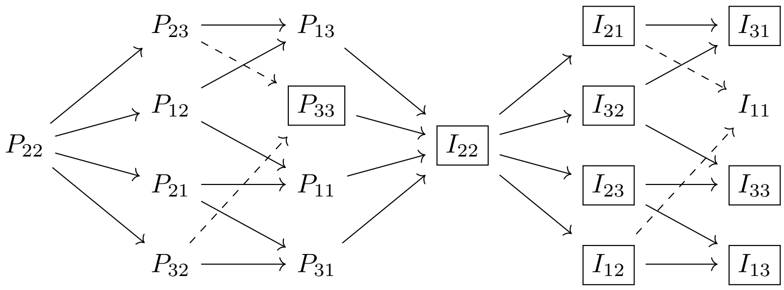
Or, equivalently:

$$0 \to P_{22} \to P_{11} \oplus P_{31} \to I_{22} \to I_{31} \oplus I_{13} \to 0$$

The modules T belonging to this case are:

$$P(T) = \{P_{11}, P_{33}\}, \{P_{13}, P_{31}\}.$$

13. All projective modules except P_{22}, four immediate successor and three secondary successor lies in $\mathrm{add}(T)$. We show this for P_{23}, P_{12}, P_{21}, P_{32}, P_{13}, P_{11}, $P_{31} \notin \mathrm{add}(T)$. We mark the modules in $\mathrm{add}(T)$ with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}} \begin{matrix}0&0&0\\0&1&1\\0&1&1\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&0\\0&0&1\\0&1&1\end{smallmatrix}} \begin{matrix}0&0&1\\0&0&1\\1&1&2\end{matrix} \xrightarrow{\begin{smallmatrix}0&0&1\\0&0&0\\1&0&1\end{smallmatrix}} \begin{matrix}0&0&1\\0&0&0\\1&0&1\end{matrix} \longrightarrow 0$$

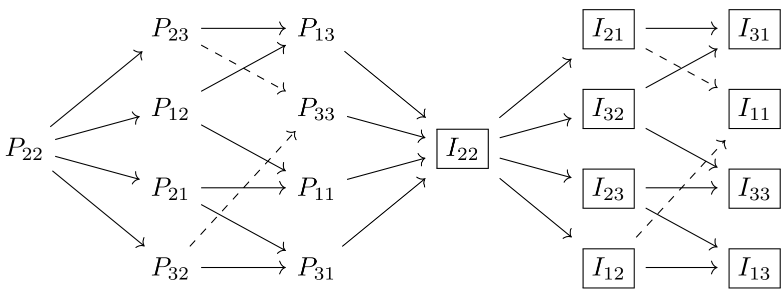
Or, equivalently:

$$0 \to P_{22} \to P_{33} \to I_{32} \oplus I_{23} \to I_{31} \oplus I_{33} \oplus I_{13} \to 0$$

The modules T belonging to this case are:

$$P(T) = \{P_{11}\}, \{P_{13}\}, \{P_{31}\}, \{P_{33}\}.$$

14. No projective modules lies in add(T). Here we get the injective resolution of P_{22}. We mark the modules in add(T) with a square in the quiver below.



For this case we get the following add(T)-resolution.

$$0 \longrightarrow \begin{matrix} 0\,0\,0 \\ 0\,1\,0 \\ 0\,0\,0 \end{matrix} \xrightarrow{\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}} \begin{matrix} 1\,1\,1 \\ 1\,1\,1 \\ 1\,1\,1 \end{matrix} \xrightarrow{\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}} \begin{matrix} 2\,1\,2 \\ 1\,0\,1 \\ 2\,1\,2 \end{matrix} \xrightarrow{\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}} \begin{matrix} 1\,0\,1 \\ 0\,0\,0 \\ 1\,0\,1 \end{matrix} \longrightarrow 0$$

Or, equivalently:

$$0 \to P_{22} \to I_{22} \to I_{21} \oplus I_{32} \oplus I_{23} \oplus I_{12} \to I_{31} \oplus I_{33} \oplus I_{13} \oplus I_{11} \to 0$$

We have now found the required add(T)-resolutions and we can thus state the following theorem.

Theorem 6.4. Let $\mathcal{M}$ be the d-cluster tilting subcategory of $KQ \otimes KQ$, where Q is the quiver $1 \to 2 \leftarrow 3$, and $T \in \mathcal{M}$. Then the following is equivalent.

1. T is a tilting module of projective dimension at most 2.
2. T satisfies $(\mathrm{P})_2$, $(\mathrm{E})_2$ and $|T| = 9$.
3. $KQ \otimes KQ = P \oplus P'$ with add(P) closed under successors in the quiver 6.1 and add$(T) = $ add$(P \oplus \tau_2^{-1} P')$.

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