Uppsala university

Institution of Physics & Astronomy

Department of Theoretical Physics

Geometric Quantization

Author:
William Hedlund

Supervisor:
Konstantina Polydorou

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Abstract

We formulate a process of quantization of classical mechanics, from a symplectic perspective. The Dirac quantization axioms are stated, and a satisfactory prequantization map is constructed using a complex line bundle. Using polarization, it is determined which prequantum states and observables can be fully quantized. The mathematical concepts of symplectic geometry, fibre bundles, and distributions are exposed to the degree to which they occur in the quantization process. Quantizations of a cotangent bundle and a sphere are described, using real and Kähler polarizations, respectively.

Sammanfattning

Vi formulerar en kvantiseringsprocess av klassisk mekanik från ett symplektisk-geometriskt perspektiv. Diracs kvantiseringsaxiom läggs fram, och en duglig förkvantisering konstrueras medelst ett komplext linjeknippe. Med polarisering avgörs vilka förkvantiserade tillstånd och mätbara storheter som kan kvantiseras till fullo. De matematiska koncepten symplektisk geometri, fiberknippen, och distributioner redogörs för till den grad till vilken de används i kvantiseringsprocessen. Kvantiseringar av ett kotangentknippe och en sfär beskrivs, med reell respektive Kähler-polarisering.

Contents

1 Introduction & Historical Background

The theory of quantum mechanics developed in the first part of the 20^{th} century in response to a series of theoretical problems, and observed phenomena which defied description by the previously established physical theories. One these was the "ultraviolet catastrophe", the prediction by classical theory that black bodies should radiate infinite amounts of energy. Another, the "photoelectric effect", that light hitting an object causes electrons to be emitted in a way that does not agree with a 'continuous' description of light. A third, the "double-slit experiment", which demonstrated that light shining through two parallell slits interacts with itself, creating an interference pattern. The last two phenomena would together demonstrate that light acts as if composed of discrete particles *and* as if being a continuous wave. The first was solved by Planck ([7]) by supposing that the possibly emitted energies came in discrete steps: *quanta* of energy. This discretisation or *quantization* is what earned the new theory its name.

Different models of quantum mechanics were developed, e.g. the matrix mechanics of Heisenberg et al. ([4]), and the wave mechanics of Schrödinger ([8]). These are shown to be equivalent in a more unified picture. What bears mentioning is that these models were not the result of any axiomatic process of quantization. It became apparent, in any case, that quantum mechanics was to be modeled on some Hilbert space - that is, a vector space with an inner product inducing a metric under which the space is complete. The possible states of the system are then the normalized vectors in the space, and the observables - the measurable quantities - represented by self-adjoint operators on the space. An exposition on these and other axioms for what mathematical objects describe quantum mechanics can be found in [10].

An early attempt at geometric quantization was set forth in 1927 by Weyl [11]. There he used the following formula, called the *Weyl transform*, to map a function $f(q,p)$ on phase space (of position and momentum coordinates) to an operator on Hilbert space:

$$\Phi_\rho[f] = \frac{1}{(2\pi)^2} \iiiint f(q,p)\rho\left(e^{i(a(Q-q)+b(P-p))}\right) dq\ dp\ da\ db.$$

Here Q and P are generators of the Heisenberg Lie algebra, with $[Q,P] = i\hbar$, and ρ is a representation of the Heisenberg group on a Hilbert space. The idea here is thus that the observables of classical mechanics - functions of phase space - are mapped to the observables of quantum mechanics - the self-adjoint operators. General axioms for what constitutes a quantization of a classical mechanical system were put forth by e.g. Dirac ([3]). In this thesis we use the three axioms stated in section 4.1.

The aim of this thesis is to develop a method of quantizing a classical mechanical system, in a way which emphasizes the geometrical aspects of the system. There are several ways to quantize a given system. Here we start off along a certain avenue - geometric quantization - which itself branches into different paths as we make certain choices. Our geometric approach puts at the fore the topological and geometrical properties of the systems we wish to quantize, on which we impose certain constraints which need to be satisfied for the system to be quantizable. For this reason we begin with section 2, where we introduce the mathematical tools which will be put into practice later on. In section 3 we demonstrate a geometrical view of classical mechanics, so that we are well aquainted with the classical model of the systems we aim to quantize.

The quantization axioms are introduced in section 4, where we demonstrate how to obtain a *prequantization map* given a system, and what conditions need to be satisfied by that system for this to be possible. We show how the question of local structure versus global structure is important, and how local notions of prequantization are patched together into a global one. What we arrive at then is not satisfactory, however. In section 5 we use *polarization* to remedy a problem with the prequantization. We describe the two main kinds of polarization, real and Kähler. In section 6 we mend one last issue with what we have done so far, and arrive at a somewhat sensible notion of quantization. At that point many things still remain to improve, and we mention some of them, but satisfy ourselves with the process developed so far. Lastly, we go through two examples in section 7, where we show how one might quantize the cotangent bundle of some manifold, and the sphere, using a real polarization in the first case and a Kähler one in the second.

2 Mathematical Preliminaries

In this first section we introduce the bulk of the mathematical machinery which will be put into action in the latter sections of this thesis. This material is arranged in three subsections, which by and large pertain to one step each of the quantization process.

- **Symplectic geometry** is the language in which we most beautifully may formulate classical mechanics, and it is from this mathematical conception of the subject we shall set out in an effort to quantize.
- **Fibre bundles** are the geometrical objects which make up the background of the quantum world; much in the same way that symplectic manifolds do for the classical world. They can be thought of as a way of "gluing" together locally defined concepts to something which makes global sense.
- **Foliations & Distributions** describe how one can disassemble a manifold into leafs - in the way one might the layers of an onion - and how a set of commuting vector fields gives rise to a set of coordinates on those onion-layers. This machinery will be used to "cut in half" the space of quantum states, once we find that we have quantized more states than we bargained for.

2.1 Symplectic Geometry

Symplectic geometry studies manifolds equipped with an antisymmetric tensor field - the symplectic form. This stands in contrast to Riemannian geometry which studies manifold with a nondegenerate *symmetric* tensor field. As opposed to Riemannian geometry which has local invariants, e.g. curvature, all symplectic manifolds of equal dimension look the same locally - this being the content of the main theorem of this section, theorem 2.1. An upshot of symplectic geometry is that the functions on a symplectic manifold - the observables of classical mechanics - are blessed with the structure of a Poisson algebra, permitting one to speak of commuting functions, and so on.

2.1.1 Definitions and constructions

The following basic constructions can be found in e.g. [2].

Definition 2.1 ([2]). *A* **symplectic vector space** *is a pair* (V, ω) *of a vector space* V *over a field* $\mathbb{F}$ *(we will have* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*) and a* **symplectic form** ω*, that is a map*

$$\omega : V \times V \to \mathbb{F},$$

which is bilinear, alternating and non-degenerate.

Proposition 2.1. *The symplectic form induces an isomorphism between* V *and its dual space* V^**, defined by* $v \mapsto \lambda_v : u \mapsto \omega(v, u)$*.*

Proof. This is an isomorphism since if $\lambda_v = 0$ then, by nondegeneracy of ω, $v = 0$. □

Proposition 2.2. *Every finite-dimensional symplectic vector space is of even dimension* $2n$ *and has a basis* $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ *such that*

$$\begin{aligned} \omega(e_i, e_j) &= 0 \\ \omega(f_i, f_j) &= 0 \\ \omega(e_i, f_j) &= \delta_{ij}. \end{aligned} \tag{2.1}$$

Proof. Unless we have the trivial case $V = 0$, picking any nonzero vector e_1, the map $\lambda_{e_1} : V \to \mathbb{F}$ is a surjective homomorphism, and thus injective on some 1-dimensional subspace L. Let f_1 be the unique vector in L with $\omega(e_1, f_1) = 1$. We can decompose the space as

$$V = \operatorname{span}\{e_1, f_1\} \oplus \ker \lambda_{e_1} \cap \ker \lambda_{f_1} = E_1 \oplus V'.$$

$\omega|_{V' \times V'}$ is symplectic: bilinearity and alternaticity are obvious on the subspace V'. Now suppose for some $X \in V'$ that $\omega(X, -)|_{V'} = 0$. Then since ω is nondegenerate we must have $\omega(X, e_1) \neq 0$ or $\omega(X, f_1) \neq 0$. But since X is in the kernels of both λ_{e_1} and λ_{f_1} this is a contradiction.

Thus $(V', \omega|_{V' \times V'})$ is symplectic. Then unless it is the zero space, we can decompose V' in the same way, and so on inductively. Since V is finite-dimensional this process ends at some point where $V' = 0$ and we have

$$V = \bigoplus_{i=1}^{n} \operatorname{span}\{e_i, f_i\},$$

and the basis satisfies (2.1) by contruction. □

As a consequence of this last expression, symplectic vector spaces have even dimension.

Definition 2.2. *The* **symplectic orthogonal** *of a subspace $U \subset V$ of a symplectic vector space is defined as*

$$U^\omega = \{v \in V \mid \omega(v, u) = 0 \ \forall u \in U\}.$$

Definition 2.3. *A subspace I of a symplectic vector space (V, ω) is* **isotropic** *if $\omega|_I = 0 \Leftrightarrow I \subset I^\omega$. It is* **Lagrangian** *if $I^\omega = I$.*

For example, any disjoint subsets $J, K \subset \{1, \dots, n\}$ give an isotropic subspace by $\operatorname{span}\{e_J, f_K\}$. If the subsets are a partition, the subspace is Lagrangian. The comments above we state as:

Proposition 2.3. *An isotropic subspace $I \subset V$ has dimension at most $\frac{1}{2}\dim V$, in which case it is Lagrangian.*

Definition 2.4. *A (linear)* **symplectic map** *is a linear map $f : (V, \omega) \to (W, \Omega)$ between symplectic spaces such that the pullback $f^*\Omega$ equals ω.*

A symplectic map must be injective since the symplectic form is nondegenerate. For $(W, \Omega) = (V, \omega)$ we then see that the symplectic maps are invertible, and form a group, since $(fg)^*\omega = g^*f^*\omega = g^*\omega = \omega$ and $id^* = id$, so that $(f^{-1})^*\omega = (ff^{-1})^*\omega = \omega$.

This group is the **Symplectic group** $Sp(V)$ or $Sp(V, \omega)$.

Definition 2.5. *A* **symplectic manifold** *is a pair (M, ω) of a smooth manifold M and a closed 2-form $\omega \in \Omega^2(M) = \Gamma(T^2M)$ such that $\omega_p = \omega|_{T_pM}$ is a symplectic form for every $p \in M$. That is, $\iota_X\omega = 0$ if and only if X is the zero vector field.*

It follows that the dimension of M is even.

Definition 2.6. *A* **symplectomorphism** *is a differmorphism $f : (M, \omega) \to (N, \Omega)$ between symplectic manifolds such that $f^*\Omega = \omega$.*

It follows that the differential $df_p : T_pM \to T_{f(p)}N$ is a linear symplectic map for any p.

The symplectomorphisms $(M, \omega) \to (M, \omega)$ form a group, the **symplectomorphism group** $\operatorname{Symp}(M, \omega)$.

Proposition 2.4. *On a symplectic manifold (M, ω), ω defines an isomorphism between each tangent and cotangent space, which we denote $\widetilde{\omega}_p : T_pM \to T_p^*M$. This gives an isomorphism between the tangent and cotangent bundles, $\widetilde{\omega} : TM \to T^*M$, taking vector fields to 1-forms.*

Proof. Let $X \in T_pM$. Define $\widetilde{\omega}_p(X) = \omega_p(X, \cdot\,)$. This map is clearly a homomorphism. Now suppose $\widetilde{\omega}_p(X) = \widetilde{\omega}_p(Y)$. Then $\omega_p(X - Y, \cdot\,) = 0$. But since ω is nondegenerate this means $X = Y$. Thus $\widetilde{\omega}_p$ is an invertible homomorphism, i.e. an isomorphism. □

Proposition 2.5. *On a 2n-dimensional symplectic manifold (M, ω), ω^n is a nondegenerate top form, i.e. a volume form.*

Proof. Peek ahead at theorem 2.1 and write $\omega = dq^i \wedge dp_i$, so that, summing over permutations of $\{1, \ldots, n\}$ (of which there are $n!$),

$$\omega^n = \sum_\sigma dq^{\sigma_1} \wedge dp_{\sigma_1} \wedge \ldots \wedge dq^{\sigma_n} \wedge dp_{\sigma_n} = n!\; dq^1 \wedge dp_1 \wedge \ldots \wedge dq^n \wedge dp_n,$$

where all terms with repeating factors have disappeared by skew-symmetry. □

As a corollary, symplectic structures exist on a manifold only if it is orientable.

Example: Orientable surfaces
Any 2-dimensional orientable manifold admits a nondegenerate top form, i.e. an area form. Since it is of top degree it is closed as well as nondegenerate, so it is symplectic.

Counterexample: No spheres
The $2n$-sphere, for $n > 1$, has De Rham cohomology $H^2_{dR}(S^{2n}, \mathbb{R}) = 0$. Thus a symplectic form would be exact: $\omega = d\alpha$. Then $\omega^n = (d\alpha)^n$ is a volume form. But $(d\alpha)^n$ is exact: $(d\alpha)^n = d(\alpha \wedge (d\alpha)^{n-1})$, so its integral over the boundaryless sphere is zero, which is impossible for a volume form. Thus no symplectic form can exist on S^{2n} for $n > 1$.

2.1.2 The cotangent bundle

A standard example of a symplectic manifold is the cotangent bundle $Q = T^*M$ of any manifold M. We construct the symplectic form as follows: (cf. [2] for a similar exposition)

We construct coordinates on Q, from a chart on a patch $U \subset M$ with coordinates $\{q^i\}$. Any point (x, ψ) of T^*U, where $\psi = p_i dq^i$ is then given the **canonical coordinates** $(q^i(x), p_i)$. There is a unique 1-form $\theta \in \Omega(Q)$, such that for any section $\lambda : M \to Q$, $\lambda^*\theta = \lambda$. To show uniqueness, suppose α and β are two such forms. Then $\lambda^*(\alpha - \beta) = 0$ for any section λ. We can, by simply picking its coordinate expression, always choose a 1-form which maps a given tangent vector to M to a tangent vector to Q. So if $\alpha(\lambda_* X) = \beta(\lambda_* X)$ for any λ and X, we must have $\alpha = \beta$. We see also that in these coordinates the form θ is given by $\theta(p_i, q^i) = p_i dq^i$. This hold for any coordinates constructed in this way from coordinates on M since θ was defined in an objective (coordinate-independent) way.
The form θ thus constructed is called the **tautological 1-form**.

Our symplectic form is then the canonical (symplectic) form $\omega = d\theta$. In coordinates, $\omega = dp_i \wedge dq^i$. In any tangent space T_xQ a symplectic basis is given by $\{\partial_{p_i}, \partial_{q^j}\}$ since

$$\sum_k dp_k \wedge dq^k(\partial_{p_i}, \partial_{q^j}) = \sum_k \delta_{ki}\delta_{kj} = \delta_{ij},$$

and the other two combinations are clearly zero. Thus ω is symplectic in each tangent space. Since the derivative commutes with pullback, for any section $\lambda \in \Omega(M)$ we have

$$\lambda^* \omega = d\lambda. \tag{2.2}$$

Proposition 2.6. *The pullback of any diffeomorphism $\psi : N \to M$ is a symplectomorphism between the canonical symplectic cotangent bundle structures.*

Proof. Define coordinates on N by $Q^i = q^i \circ \psi$ for any chart $\{q^i\}$ on M. Then $\psi^* dq^i = dQ^i$, so $\psi^*(p_i dq^i) = p_i dQ^i$, which means that in canonical coordinates $\psi^* : T^*M \to T^*N$ is given by $Q^i \circ \psi^* = q^i,\ P_i \circ \psi^* = p_i$, so $dp_i \wedge dq^i = (\psi^*)^*(dP_i \wedge dQ^i)$. □

2.1.3 Darboux's theorem

The constructions in the cotangent bundle are very natural since the tautological form at any point is just the 1-form which *is* that point. The tautological and canonical forms have simple coordinate expressions. This symplectic manifold (T^*M, ω) is also the one which describes the classical mechanics of a system with configuration space M.

As it turns out, any symplectic manifold looks locally like a cotangent space, thanks to Darboux's theorem.

Theorem 2.1 ([2])**.** *For any point p of a symplectic manifold (M, ω) there is a neighbourhood U of p on which coordinates (p_i, q^i) are defined, such that $dp_i \wedge dq^i = \omega|_U$.*

The coordinates thus defined are called **Darboux coordinates**.

Proof. To prove this we construct an open set U with coordinates (y_i, x^i) and a diffeomorphism $\phi : U \to U'$ such that $\phi^*(dy_i \wedge dx^i) = \omega$. Then the functions $q^i = x^i \circ \phi$ and $p_i = y_i \circ \phi$ give the Darboux coordinates.

Take some coordinates (y_i, x^i) and define $\omega_0 = \omega,\ \omega_1 = dy_i \wedge dx^i,\ \omega_t = t\omega_1 + (1-t)\omega_0$.
We then have $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ and $d\omega_t = 0$ since ω_1 and ω_0 are both closed.

ϕ is constructed as the time-one flow of a time-dependent vector field X_t. To this end we quote the following:

Lemma 2.1. *For any smooth vector field $V : M \times I \to TM$ for some interval I containing 0 there is some $\epsilon > 0$ and a smooth isotopy $\phi : M \times (-\epsilon, \epsilon) \to M$ such that $\phi_0 = id_M$ and $\frac{d}{dt}\phi_t(x) = V_t(\phi_t(p))$.*

This lemma of course applies to time-independent fields, where $V_t = V_s$ for all t, s.
The equation we want to solve to obtain ϕ_t is $\frac{d}{dt}(\phi_t^* \omega_t) = 0$. Since $\phi_0^* \omega_0 = \omega$ this would mean that $\phi_1^* \omega_1 = \omega$, which is the identity we wish to prove. Viewing $\phi_t^* \omega_t$ as a function of two variables at the point (t, t) we get

$$\frac{d}{dt}(\phi_t^* \omega_t) = \frac{d}{ds}\phi_s^* \omega_t \bigg|_{s=t} + \frac{d}{ds}\phi_t^* \omega_s \bigg|_{s=t}. \tag{2.3}$$

Since ϕ_t^* is a linear map, the second term on the right is $\phi_t^* \frac{d}{ds}\omega_s\big|_{s=t}$.

We have the Lie derivative $\mathcal{L} : \mathcal{X}(M) \otimes \mathcal{T}(M) \to \mathcal{T}(M)$, $\mathcal{T}(M)$ being the tensor algebra over M, which is defined as: for any vector field X, let $\phi_t : M \to M$ be the flow along X which exists for some time and let

$$\mathcal{L}_X T = \frac{d}{dt}\phi_t^* T\bigg|_{t=0}$$

for any tensor field T. Now we wish to show the following relation, for the flow ϕ_t generated by the time-dependent vector field X_t:

$$\frac{d}{dt}\phi_t^*\mu = \phi_t^*\left(\mathcal{L}_{X_t}\mu\right), \tag{2.4}$$

where now μ is any differential form. We show that $\mathcal{D}_1 = \frac{d}{dt}\phi_t^*$ and $\mathcal{D}_2 = \phi_t^*\mathcal{L}_{X_t}$ agree on all of $\Omega(M) = \bigoplus_{n=0}^{\dim M}\Omega^n(M)$, by induction on n. First note that they both commute with d, since d commutes with any pullback and any differentiation with respect to an external parameter t (since the continuous second partial derivatives commute).

Then we have

$$\begin{aligned}\mathcal{D}_1(\alpha\wedge\beta) &= \frac{d}{dt}(\phi_t^*\alpha\wedge\phi_t^*\beta) = \left(\frac{d}{dt}\phi_t^*\alpha\right)\wedge\phi_t^*\beta + \phi_t^*\alpha\wedge\left(\frac{d}{dt}\phi_t^*\beta\right) = \mathcal{D}_1(\alpha)\wedge\phi_t^*\beta + \phi_t^*\alpha\wedge\mathcal{D}_1(\beta)\\ \mathcal{D}_2(\alpha\wedge\beta) &= \phi_t^*(\mathcal{L}_{X_t}(\alpha)\wedge\beta + \alpha\wedge\mathcal{L}_{X_t}(\beta)) = \mathcal{D}_2(\alpha)\wedge\phi_t^*\beta + \phi_t^*\alpha\wedge\mathcal{D}_2(\beta),\end{aligned} \tag{2.5}$$

where we used that ϕ_t^* is a homomorphism and that $\mathcal{L}_{X_t}$ is a derivation, and that the product rule of course applies for the wedge product which is distributive. Now suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ agree on all forms of degree $\leq n$, and let μ be any $(n+1)$-form expressed locally as $\mu = \mu_{J,i}dx^J\wedge dx^i$ where J is an n-multiindex. Then

$$\mathcal{D}_1(\mu) = \mathcal{D}_1(\mu_{J,i}dx^J)\wedge\phi_t^*dx^i + \phi_t^*(\mu_{J,i}dx^J)\wedge\mathcal{D}_1(dx^i) = \mathcal{D}_2(\mu)$$

by (2.5), so that they agree on $(n+1)$-forms as well.

Now we prove the base case of 0-forms, i.e. functions.

$$\begin{aligned}(\mathcal{D}_1 f)(p)|_{t=t_0} &= \frac{d}{dt}\,(\phi_t^* f)(p)|_{t=t_0} = \frac{d}{dt}\,f(\phi_t(p))|_{t=t_0} = df\left(\frac{d}{dt}\phi_t(p)\bigg|_{t=t_0}\right) =\\ &= df\,(X_{t_0}(\phi_{t_0}(p)) = (\phi_{t_0}^* df(X_{t_0}))(p) = (\mathcal{D}_2 f)(p)|_{t=t_0},\end{aligned}$$

since the Lie derivative $\mathcal{L}_X$ acts on functions by $f \mapsto df(X)$. Thus we prove (2.4).

Returning to (2.3) with this information in hand we get

$$\frac{d}{dt}(\phi_t^*\omega_t) = \phi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right).$$

If we find a vector field X_t with flow ϕ_t such that this is zero and such that ϕ_1 is defined on some neighborhood U of p, we will be done. Since the flow ϕ_t is injective by isotopy, we must have $\mathcal{L}_{X_t}\omega_t = -\frac{d}{dt}\omega_t = \omega_0 - \omega_1$.

We can now note a few things:

By Cartan's "magic" formula, $\mathcal{L}_{X_t}\omega_t = d(\iota_{X_t}\omega_t) + \iota_{X_t}d\omega_t = d(\iota_{X_t}\omega_t)$, since ω_t is closed. As $\omega_0 - \omega_1$ is closed there is a neighborhood U_1 of p where it is exact, so that $\omega_0 - \omega_1 = d\mu$. $\omega_t|_p = \omega_0|_p = \omega_1|_p$, which is nondegenerate, so ω_t is nondegenerate in some neighborhood U_2 of p (nondegeneracy is an open condition).

In $U_1 \cap U_2$ our condition on X_t reads $d(\iota_{X_t}\omega_t) = d\mu$, which is solved by $\iota_{X_t}\omega_t = \mu$, i.e. $\omega_t(X_t, -) = \mu$, and we may assume that $\mu|_p = 0$ by adding some closed form to it. By proposition 2.4 there is an isomorphism $\widetilde{\omega_t}$ between the tangent and cotangent spaces at each point, defined by $\omega_t(\widetilde{\omega_t}^{-1}(\mu), -) = \mu$, and we define $X_t = \widetilde{\omega_t}^{-1}(\mu)$. Also, since we then have $X_t|_p = 0$, $\phi_t(p) = p$ for all $t \in \mathbb{R}$, there is some neighborhood U_3 of p where ϕ_t is defined for all $t \in [0, 1]$.

Take $U = U_1 \cap U_2 \cap U_3$ and $\phi = \phi_1$. As stated, the flow is isotopic, so it is a diffeomorphism onto $\phi(U)$, which contains p. Thus we can restrict U so that the coordinates (y_i, x^i) are defined on $\phi(U)$. Set $q^i = x^i \circ \phi$, $p_i = y_i \circ \phi$. In these coordinates we have

$$dp_i \wedge dq^i = \phi^*(dy_i \wedge dx^i) = \phi^*\omega_1 = \omega, \tag{2.6}$$

so they are indeed Darboux coordinates. □

2.1.4 Lagrangian submanifolds, generating functions

Definition 2.7. *An* **Isotropic submanifold** *of a symplectic manifold (M, ω) is a submanifold $I \overset{i}{\hookrightarrow} M$ such that $i^*\omega = 0$.*

Definition 2.8 ([2])**.** *A* **Lagrangian submanifold** *of a symplectic manifold M is an isotropic submanifold of maximal dimension. By proposition 2.3, this dimension is $\frac{1}{2}\dim M$.*

For example, given any disjoint subsets $J, K \subset \{1, \ldots, n\}$, an isotropic submanifold is given by the Darboux coordinates $\{p_J, q^K\}$. If J, K is a partition, the submanifold is Lagrangian.

Proposition 2.7. *The image Λ of a 1-form $\lambda : M \to T^*M$ is a submanifold of the canonical symplectic manifold (T^*M, ω) diffeomorphic to M, and Lagrangian if and only if λ is closed. Conversely, any submanifold such that the projection π is a (local) diffeomorphism gives (locally) a 1-form which is closed if and only if the submanifold is Lagrangian.*

Proof. The projection $\pi : T^*M \to M$ restricts to a smooth map on Λ, and its smooth inverse is given by λ (considered as a map onto Λ). The inclusion of Λ factors as $i = \lambda \circ \pi$. Now let ω be the canonical symplectic form on T^*M. Then $i^*\omega = \pi^*\lambda^*\omega = \pi^* d\lambda$ by (2.2). Since π is a diffeomorphism, π^* is an isomorphism, so Λ is Lagrangian iff λ is closed.

For the converse, let U be an open set in Λ (all of Λ if possible) such that $\pi|_U$ is a diffeomorphism. Then $\lambda = \pi^{-1}$ is a 1-form on $\pi(U)$. U is the image of λ and as shown Lagrangian iff λ is closed. Of course Λ is Lagrangian iff every open subset is Lagrangian. □

Now since every closed 1-form can be written as the derivative of a function locally, any Lagrangian submanifold on which π is a local diffeomorphism can be written locally as the image of dS for some function S defined on an open subset of M. Of course any function S defines a lagrangian submanifold as the image of dS.

Definition 2.9. *A function which (locally) defines a Lagrangian submanifold is called a* **generating function** *of the submanifold.*

2.1.5 Hamiltonian vector fields, Poisson brackets

Definition 2.10 ([2])**.** *For any smooth function H on a symplectic manifold (M, ω), the* **Hamiltonian vector field** *with* **Hamiltonian function** *H is defined as the vector field X_H satisfying:*

$$\iota_{X_H}\omega = -dH. \tag{2.7}$$

In Darboux coordinates this vector field is given by

$$X_H = (D^i H)\partial_i - (\partial_i H)D^i, \tag{2.8}$$

where $\partial_i = \frac{\partial}{\partial q^i}$, $D^i = \frac{\partial}{\partial p_i}$.

Definition 2.11. *The* **Poisson bracket** $\{\cdot,\cdot\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ *is defined as*

$$\{f, g\} = \omega(X_f, X_g).$$

By construction we also have $\{f, g\} = dg(X_f) = \mathcal{L}_{X_f} g$.

We compute these things in Darboux coordinates. Write a vector field generally as $X = X^i\partial_i + x_i D^i$. In Darboux coordinates, $\omega(X, Y) = x_i Y^i - X^i y_i$, so $\iota_X\omega = x_i dq^i - X^i dp_i$. Now $df = (D^i f)dp_i + (\partial_i f)dq^i$, so

$$X_f = (D^i f)\partial_i - (\partial_i f)D^i \tag{2.9}$$

satisfies (2.7) and we get

$$\{f, g\} = (D^i f)(\partial_i g) - (\partial_i f)(D^i g). \tag{2.10}$$

And then

$$\begin{aligned}X_{\{f,g\}} &= ((D^{ij}f)(\partial_j g) + (D^j f)(D^i\partial_j g) - (D^i\partial_j f)(D^j g) - (\partial_j f)(D^{ij}g))\partial_i - \dots \\ &\dots - ((D^j\partial_i f)(\partial_j g) + (D^j f)(\partial_{ij} g) - (\partial_{ij} f)(D^j g) - (\partial_j f)(D^j\partial_i g))D^i \\ &= (X_f[D^i g] - X_g[D^i f])\partial_i + (X_f[-\partial_i g] - X_g[-\partial_i f])D^i = [X_f, X_g]\,.\end{aligned} \tag{2.11}$$

Proposition 2.8. *$(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra over $\mathbb{R}$.*

Proof. Bilinearity follows since d is linear, $\widetilde{\omega}$ is linear, and ω is bilinear.

Alternaticity follows since ω is alternating.

By (2.11),

$$\{\{f,g\},h\} = X_{\{f,g\}}h = X_f[X_g h] - X_g[X_f h] = \{f,\{g,h\}\} - \{g,\{f,h\}\}. \tag{2.12}$$

Rearranging by alternaticity we get

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0, \tag{2.13}$$

which is the Jacobi identity. Therefore this is a Lie algebra. □

From (2.11) we also see that the map $f \mapsto X_f$ is a Lie algebra homomorphism.

Every Hamiltonian function f defines a **Hamiltonian flow** $\Phi_f^t : M \to M$, the flow of the vector field X_f. Any curve $t \mapsto \Phi_f^t(x)$ is an **integral curve** of f starting at x. For any function g,

$$\frac{d}{dt}\left(g \circ \Phi_f^t\right)\Big|_{t=0} = dg(X_f) = \{f,g\}, \tag{2.14}$$

which is a very important identity in mechanics especially. In particular, two functions Poisson-commute if and only if their flows preserve each other, i.e. they are constant along each other's integral curves. A set of functions which is pairwise Poisson-commuting is said to be **in involution**.

Proposition 2.9. *The pullback of a symplectomorphism $\psi : (N,\Omega) \to (M,\omega)$ is a Lie algebra isomorphism $C^\infty(M) \to C^\infty(N)$. ψ_* takes the hamiltonian vector field of $\psi^* f$ to that of f.*

Proof. Suppose $X = \widetilde{\omega}^{-1}\alpha$, i.e. $\alpha = \omega(X,\cdot)$. Then

$$\psi^*\alpha = (\psi^*\omega)(\psi^* X,\cdot) = \Omega(\psi^* X,\cdot),$$

so

$$\psi^* \circ \widetilde{\omega}^{-1} = \widetilde{\Omega}^{-1} \circ \psi^*.$$

Now

$$\{\psi^* f, \psi^* g\} = \Omega\Big(\widetilde{\Omega}^{-1}(d\psi^* g), \Omega^{-1}(d\psi^* f)\Big) = \Omega\Big(\psi^*\widetilde{\omega}^{-1}(dg), \psi^*\widetilde{\omega}^{-1}(df)\Big) = \omega\Big(\widetilde{\omega}^{-1}(dg), \widetilde{\omega}^{-1}(df)\Big) = \{f, g\},$$

thus ψ^* is a homomorphism with inverse ψ_*.

Consequently we have the identity $(\psi_* X_{\psi^* f})[g] = dg(\psi_* X_{\psi^* f}) = (d\psi^* g)(X_{\psi^* f}) = \{\psi^* f, \psi^* g\} = \{f, g\} = X_f[g]$. ☐

As a consequence the diffeomorphism ψ takes integral curves of $\psi^* f$ to those of f, and vice versa.

Definition 2.12. *A vector field X on a symplectic manifold (M, ω) is called* **locally Hamiltonian** *if there exists a cover $\{U_i\}$ of M with functions $H_i \colon U_i \to \mathbb{R}$ such that*

$$\iota_X \omega\big|_{U_i} = -dH_i.$$

Proposition 2.10. *The Lie bracket of two locally Hamiltonian vector fields is a globally Hamiltonian vector field.*

Proof. Let X and Y be the fields, with coverings $\{U_i, f_i\}$, $\{U_i, g_i\}$ as in the above definition. Take any (not necessarily distinct) indices i, j, k and put $V = U_i \cap U_j \cap U_k$. Since $df_i = -\iota_X \omega = df_j$ on V, the local Hamiltonian functions differ only by constant functions.

What this means is that $\{f_i, g_j\} = \{f_i, g_k\}$, et.c., i.e. the Poisson brackets of any local Hamiltonian functions for X and Y agree. Patching together yields a globally defined function which is locally the Poisson bracket of the local Hamiltonian functions, and thus by (2.11) a Hamiltonian function for $[X, Y]$. ☐

Proposition 2.11. *A vector field X is locally Hamiltonian if and only if the Lie derivative $\mathcal{L}_X \omega$ is zero.*

Proof. By Cartan's magic formula, $\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega = d(\iota_X \omega)$. If X is locally Hamiltonian, $\iota_X \omega$ is locally exact, so the expression vanishes. Conversely, if $\mathcal{L}_X \omega = 0$, $\iota_X \omega$ is closed, so locally exact. Its local antiderivative is the negtive of a local Hamiltonian function. ☐

2.1.6 The Hamilton-Jacobi and WKB methods

The Hamilton-Jacobi method is a geometrical method of solving the Hamiltonian flow in a symplectic manifold M by finding a particular submanifold. In the special case of a cotangent bundle this problem can be solved analytically, by finding a function which generates the submanifold. The method rests on the following:

Lemma 2.2. *If $\Lambda \overset{i}{\hookrightarrow} M$ is a Lagrangian submanifold of a symplectic manifold lying in a level set of H, then the Hamiltonian vector field X_H is tangent to Λ at all its points.*

Proof. Since H is constant on Λ, $\omega(Y, X_H) = dH(Y) = 0$ for all Y tangent to Λ. Then we have $X_H \in (T_q\Lambda)^\omega = T_q\Lambda$ since Λ is Lagrangian. □

The consequence is that we only need solve the differential equations of the Hamiltonian flow on a manifold of half the original dimension. Now in the particular case of a cotangent bundle structure, we can try to get such a manifold from a generating function, as described in section 2.1.4.

The equation in question is the **Hamilton-Jacobi equation**:

$$H \circ i \circ dS = E. \tag{2.15}$$

Or in coordinates $\{x_i\}$ on M:

$$H\left(x_i, \frac{\partial S}{\partial x_i}\right) = E$$

Where H is the Hamiltonian function of the system, i the inclusion of the image of dS, S is some function on M, and E is some regular value of the Hamiltonian. By proposition 2.7 the image of dS is a Lagrangian submanifold of T^*M. In coordinates dS is the map $x_i \mapsto \left(x_i, \frac{\partial S}{\partial x_i}\right)$. The coordinate vector fields are

$$\frac{\partial}{\partial x_i} = \partial_i + \frac{\partial^2 S}{\partial x_i \partial x_j} D^j.$$

By (2.8) we see that

$$X_H = (D^i H \circ dS)\frac{\partial}{\partial x_i},$$

since the lemma tells us that X_H can be expressed in this basis. Thus the differential equations to solve here are

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}(dS(x)).$$

An application of the Hamilton-Jacobi equation to quantum mechanics is in the **WKB method** of solving the Schrödinger equation,

$$i\hbar\frac{\partial \psi}{\partial t} = \widehat{H}\psi,$$

for a function ψ on $\mathbb{R}^n \times \mathbb{R}$ where $\widehat{H}$ is the Hamilton operator

$$\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(x),$$

acting on and depending on the space coordinates only. With an ansatz $\psi(x,t) = \varphi(x)e^{-i\omega t}$ this is transformed into the time-independent Schrödinger equation,

$$\left(\widehat{H} - E\right)\varphi = 0, \tag{2.16}$$

where $E = \hbar\omega$ is some energy eigenvalue. Now we make the WKB ansatz $\varphi(x) = e^{iS(x)/\hbar}$, which puts (2.16) into the form

$$\left(\frac{|\nabla S(x)|^2}{2m} + (V(x) - E) - \frac{i\hbar}{2m}\Delta S(x)\right) e^{iS(x)/\hbar}.$$

But given the standard Hamiltonian on $\mathbb{R}^n$,

$$H(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}),$$

we see that if S satisfies the Hamilton-Jacobi equation our ansatz φ satisfies

$$\left(\widehat{H} - E\right)\varphi = -\varphi\frac{i\hbar}{2m}\Delta S. \tag{2.17}$$

So it is an approximate solution to the time-independent Schrödinger equation, with an error of order $O(\hbar)$. Such an S is called an **admissible phase function**: it only defines a phase since $|\varphi| = 1$. As described earlier, the geometric generalization of a solution to the Hamilton-Jacobi equation is a Lagrangian submanifold of a symplectic manifold lying in a level set of the Hamiltonian function. Since this first approximation has constant amplitude, a more educated guess might be to multiply it by some amplitude function $a\colon \mathbb{R}^n \to \mathbb{C}$ for an ansatz

$$\varphi = e^{iS/\hbar}a,$$

where S is still admissible. By (2.17) and the product rule we get

$$\begin{aligned}\left(\widehat{H} - E\right)\varphi &= - e^{iS/\hbar}a\frac{i\hbar}{2m}\Delta S - \frac{\hbar^2}{2m}e^{iS/\hbar}\Delta a - \frac{i\hbar}{m}e^{iS/\hbar}\nabla S \cdot \nabla a \\ &= - \frac{1}{2m}\left[i\hbar\left(a\Delta S + 2\nabla S \cdot \nabla a\right) + \hbar^2\Delta a\right] e^{iS/\hbar}.\end{aligned}$$

This is a solution up to second order in $\hbar$ if a satisfies the equation

$$a\Delta S + 2\nabla S \cdot \nabla a = 0. \tag{2.18}$$

This is known as the **homogeneous transport equation**. If S is an admissible phase function and a satisfies the transport equation, the function $\varphi = e^{iS/\hbar}a$ is called the **semi-classical approximation**. If we multiply (2.18) by a we can rewrite it as

$$\nabla \cdot \left(a^2 \nabla S\right) = 0,$$

Which means that the vector field $a^2\nabla S$ is divergence-free for the euclidean metric on $\mathbb{R}^n$. Now denote by $L = \text{im}(dS)$ the image of the differential of S, which is a Lagrangian submanifold (cf. 2.7) and $\pi\colon L \to \mathbb{R}^n$ the projection, which is a diffeomorphism. The Hamiltonian vector field of the standard Hamiltonian on $T^*\mathbb{R}^n$ is

$$X_H = p_i\partial_i - \partial_i V D^i.$$

But restricting to L we have $p_i = \frac{\partial S}{\partial x_i}$, and projecting to $\mathbb{R}^n$ we get

$$\pi_* X_H = \frac{\partial S}{\partial x_i}\partial_i = \nabla S. \tag{2.19}$$

The divergence of a vector field X is defined by the equation:

$$(\text{div}X)\, dV = d\left(\iota_X dV\right),$$

where dV is the volume form given by the metric. The zero-divergence condition on $a^2\nabla S$ is then equivalent to

$$d\left(\iota_{a^2\nabla S} dV\right) = 0 \Leftrightarrow d\left(\iota_{\nabla S}(a^2 dV)\right) = 0 \Leftrightarrow \mathcal{L}_{\nabla S}(a^2 dV) = 0.$$

By (2.19) this is equivalent to saying that the projection pulls the form a^2dV back to a form which the Lie derivative along the Hamiltonian flow annihilates. Conversely, any such n-form on L is pushed forward by π to some form a^2dV on $\mathbb{R}^n$, since top forms differ only by C^∞-coefficients. When this defines a function a, $a\nabla S$ is divergence-free, i.e. a satisfies (2.18). However, this may not always define a function a, since it involves taking a square root, which need not always be well-defined. To overcome this issue, we use half-densities:

Definition 2.13. *An α-**density** on a manifold M is a smooth association to each point $p \in M$ of a map $\mu_p\colon \mathcal{B}(T_pM) \to \mathbb{C}$ from the set of bases of T_pM such that $\mu_p(Ab) = |\det(A)|^\alpha \mu_p(b)$, b being a basis of T_pM and A an invertible matrix transforming that basis into another.*

α-densities are smooth sections of an α-density bundle over M; the set of all α-densities is a C^∞-module. Multiplying an α-density and a β-density pointwise gives an $\alpha\beta$-density. Pushforwards and pullbacks of densities are defined by those of the tangent bundles. Since the multiplication is just done pointwise, it commutes with pullback/pushforward, and in particular with the Lie derivative.

Taking (pointwise) the square root of the absolute value of the volume form of $\mathbb{R}^n$ gives us a half-density: $|dV|^{1/2}$. We rephrase the problem above as finding a half-density on L, annihilated by the Lie derivative along the Hamiltonian flow. This is then pushed forward to $a|dV|^{1/2}$, and we have

$$\mathcal{L}_{\nabla S}\left(a|dV|^{1/2}\right) = 0 \Leftrightarrow \mathcal{L}_{\nabla S}\left(a^2|dV|\right) = 0 \Leftrightarrow \mathcal{L}_{\nabla S}\left(a^2 dV\right) = 0,$$

since the Lie derivatie commutes with density multiplication, and $|dV|$ differs from dV locally by a sign. Finding such a half-density on L then gives an amplitude satisfying the transport equation (2.18), given a phase function S.

The geometric generalization of the semi-classical approximation is then first finding a Lagrangian submanifold of the symplectic space lying in a level set of the Hamiltonian, and second finding a half-density on that submanifold which is annihilated by the Lie derivative along the Hamiltonian flow.

2.2 Fibre Bundles

A fibre bundle is, roughly, a space which looks locally like the product of some *base space* with a certain *fibre space*. Part of the data of a fibre bundle is a *structure group* in which certain *transition functions* take their values. An example would be the famous Möbius band, which looks locally like the product of (some subset of) the circle with an interval. However, it can not be described as a product globally; there is some sort of twisting and *gluing together* going on in its construction. The transition functions describe the way in which this gluing occurs and, as we shall see, determine the bundle exactly. The upshot of fibre bundles is this relation of the global and local perspectives: we shall use them to turn a local notion of quantization into one which makes sense globally.

2.2.1 Definitions

Definition 2.14 ([9]). *A **fibre bundle** is a topological space E, called the **bundle space** or the **total space**, along with additional data:*

- *A map $p\colon E \to X$ called the **projection** onto the **base space** X.*
- *A space F called the **fibre** of the bundle.*
- *A topological group G, called the **structure group**, of homeomorphisms of F.*
- *An open cover $\{U_i\}$ of X, such that to each U_i is associated a homeomorphism*

$$\varphi_i\colon\ p^{-1}(U_i) \to U_i \times F,$$

  *Called a **local trivialization**, which satisfies:*

  - $p \circ \varphi_i = \mathrm{Id}_X$.
  - *If $U_i \cap U_j \neq \emptyset$, the map $\varphi_i \circ \varphi_j^{-1}$ is a homeomorphism of $U_i \cap U_j \times F$, the restriction of which to $\{x\} \times F$ is an element of G. This defines a map $g_{ij}\colon U_i \cap U_j \to G$ called the **transition function** which is required to be continuous.*

*We say that the bundle E is an F**-fibred** bundle **over** X.*

As we shall see later on, the transition functions of the bundle determine its structure entirely, given base and fibre spaces. They describe how the different fibres of the bundle are glued together. The most basic example of a fibre bundle is the **trivial bundle** $E = X \times F$. This bundle admits a global trivialization, which is just the identity, and so has just a single transition function, whose value at every point is the identity homeomorphism. By definition, a bundle admitting a global trivialization is equivalent to it being homeomorphic to the trivial bundle.

Now we want to define a notion of well-behaved maps between bundles. All bundles are topological spaces, so there is of course a notion of coninuity. However, bundles carry extra structure, and we would like to look at maps which respect this structure.

Definition 2.15 ([9]). *Given fibre bundles E and P over spaces X and Y with the same fibre F and structure group and G, and trivializing covers $\{U_i, \phi_i\}$ and $\{V_i, \varphi_i\}$, a* **bundle map** *or* ***bundle homomorphism*** *is a continuous map $\Psi\colon\ E \to P$ such that:*

- *There is a continuous map ψ making the following diagram commutative:*

$$\begin{array}{ccc} E & \xrightarrow{\Psi} & P \\ {\scriptstyle\pi_E}\downarrow & & \downarrow{\scriptstyle\pi_P} \\ X & \xrightarrow{\psi} & Y \end{array}$$

- *The composition $\varphi_j \circ \Psi \circ \phi_i^{-1}$, where defined, takes the form $(x, f) \mapsto (\psi(x), \eta_{ij}(x)f)$, where η_{ij} is a continuous G-valued function on $U_i \cap \psi^{-1}(V_j)$. That is, the action of Ψ on a fibre of the bundle is that of an element of G, continuously chosen, acting on F.*

Note that if ψ is a homeomorphism, the second criterion shows that Ψ is one too, and Ψ^{-1} is a bundle map as well.

2.2.2 Equivalence, Reconstruction

We establish a notion of equivalence between fibre bundles, and show some ways in which it can be determined that two bundles are equivalent. We prove a basic classification theorem, in an effort to forshadow the major result of this section.

Definition 2.16. *Two bundles E and P over the same base space X are called* **equivalent** *or* **isomorphic** *if there is a bundle map $\Psi\colon\ E \to P$ such that the induced map $\psi\colon\ X \to X$ is the identity.*

A space B homeomorphic to a fibre bundle by $\Psi\colon\ B \to E$ is given a fibre bundle structure in a natural way, such that the homeomorphism is a bundle isomorphism. Let the projection be $\pi_P = \pi_E \circ \Psi$, with trivializations $\phi_i = \varphi_i \circ \Psi$. Under this fibre bundle structure, Ψ is an isomorphism. Thus topological equivalence gives bundle equivalence, up to a choice of the additional data.

Proposition 2.12. *Two bundles E and P over X are equivalent if and only if their transition functions $\{g_{ij}\}$ and $\{h_{ij}\}$, for the same cover $\{U_i\}$ of X, are related by*

$$h_{ji} = \lambda_j g_{ji} \lambda_i^{-1} \tag{2.20}$$

for some functions $\lambda_i \colon U_i \to G$.

Note that, given covers $\{U_i\}$ and $\{V_i\}$, we can reduce to the case of a single cover by taking $\{W_{ij} = U_i \cap V_j\}$, defining new trivialization by restriction of the old.

Proof. For the "only if" part, let Ψ be a bundle isomorphism, we then have that

$$\varphi_i \circ \Psi \circ \phi_i^{-1} \colon (x, f) \mapsto (x, \eta_{ii}(x) f)$$
$$\Updownarrow$$
$$\phi_i \circ \Psi^{-1} \circ \varphi_i^{-1} \colon (x, f) \mapsto (x, \eta_{ii}(x)^{-1} f),$$

and we simply put $\lambda_i = \eta_{ii}$ to get

$$\left(\varphi_j \circ \Psi \circ \phi_j^{-1}\right) \circ \left(\phi_j \circ \phi_i^{-1}\right) \circ \left(\phi_i \circ \Psi^{-1} \circ \varphi_i^{-1}\right) : (x, f) \mapsto \left(x, \lambda_j(x) g_{ji}(x) \lambda_i(x)^{-1} f\right).$$

But this map is the same as

$$\varphi_j \circ \varphi_i^{-1} \colon (x, f) \mapsto (x, h_{ji}(x) f),$$

which implies (2.20).

For the "if" part, we construct a bundle isomorphism:

Define Ψ_k on $\pi_E^{-1}(U_k)$ as the map $\varphi_k^{-1} \circ \Lambda_k \circ \phi_k$. Where $\Lambda_k \colon (x, f) \mapsto (x, \lambda_k(x) f)$. Then, where it is defined,

$$\begin{aligned} \varphi_j \circ \Psi_k \circ \phi_i^{-1} \colon (x, f) &\mapsto (x, h_{jk}(x) \lambda_k(x) g_{ki}(x) f) \\ &= (x, \lambda_j(x) g_{jk}(x) \lambda_k(x)^{-1} \lambda_k(x) g_{ki}(x) f) \\ &= (x, \lambda_j(x) g_{ji}(x) f), \end{aligned}$$

where we used (2.20) and the cyclic property of the transition functions. Notably, this map does not depend on the chosen k. Since the trivializations are homeomorphisms on the intersection of their domains, we conclude that the definitions of Ψ_k for different k agree on their common domain.

This also shows that the second condition to being a bundle map is satisfied, because λ_k is a G-valued function. Since the induced map ψ is clearly the identity, Ψ is a bundle isomorphism. □

As a corollary, any bundles with the same fibre and structure group over the same space are equivalent if their transition functions are the same, a fact which is of course very intuitive.

Reconstruction:
Any fibre bundle E with base X, fibre F and structure group G is defined by its transition functions $\{g_{ij}\}$, for a trivializing cover $\{U_i, \phi_i\}$ of X. That is: from the data of the covering $\{U_i\}$ and the functions $\{g_{ij}\}$ we can construct a fibre bundle over X as follows:

Start with the total space

$$\widetilde{E} = \bigsqcup_i U_i \times F,$$

and impose an equivalence relation in $\widetilde{E}$:

$$(x, f) \sim (x', f') \Leftrightarrow U_i \ni x = x' \in U_j \wedge g_{ij}(x)f = f'.$$

Then define the fibre bundle

$$E = \widetilde{E}/\sim$$

with projection:

$$\pi\colon\ [(x, f)] \mapsto x.$$

The trivializing functions are defined by their inverses,

$$\begin{aligned} \phi_i^{-1}\colon\ U_i \times F &\to \pi^{-1}(U_i) \\ (x, f) &\mapsto [(x, f)]. \end{aligned}$$

This is surjective since $\pi([(x, f)]) \in U_i$ means $x \in U_i$ means $(x, f) \in U_i \times F$. It is injective since $g_{ii} = \mathrm{Id}_F$ and thus every element of $U_i \times F$ has its own equivalence class in E. Furthermore this is a right inverse of π. We give E the quotient topology (of the natural product topology of $\widetilde{E}$), so that the ϕ_i are homeomorphisms. This completes the construction of the bundle E from the given data.

Now throughout this construction we could equally well insert some other fibre space F' of which G is a homeomorphism group. This would produce a bundle with different fibre but the same base and transition functions. Since the transition functions determine the bundle up to isomorphism, this exchange of fibre gives a bijection between equivalence classes of F-fibred bundles and F'-fibred bundles.

This is the construction of **associated bundles**, where one associates one bundle to another by exchanging the fibre. Typically, one associates a vector bundle to a principal G-bundle by exchanging the group fibre for a vector space V such that $G < \mathrm{End}(V)$. Conversely, one can associate a principal $\mathrm{End}(V)$-fibred bundle to a V-fibred bundle.

Definition 2.17. *Given a map $f\colon\ X \to Y$ and a F-fibred G-bundle E over Y, the* **pullback bundle** *f^*E is defined as follows: take a covering $\{V_i\}$ of Y, with corresponding G-valued functions $\{g_{ij}\}$. Covering X with $\{f^{-1}(V_i)\}$ with corresponding transition functions $\{g_{ij} \circ f\}$, we reconstruct a new bundle over X, which we call f^*E.*

As a set, we have

$$f^*E = \{(x,e) \in X \times E \mid \pi_E(e) = f(x)\}.$$

We now establish another way of finding equvalence between bundles, quoting first the following theorem from [9]:

Theorem 2.2. Covering Homotopy Theorem *Let E and P be fibre bundles over X and Y, respectively, and $\widetilde{f}\colon E \to P$ a bundle map inducing $f\colon X \to Y$. Let also $H\colon X \times I \to Y$ be a homotopy of f. Then there is a covering homotopy of H by a bundle map $\widetilde{H}$, so that the following diagram commutes:*

$$\begin{array}{ccc} E \times I & \xrightarrow{\widetilde{H}} & P \\ \pi_E \downarrow & & \downarrow \pi_P \\ X \times I & \xrightarrow{H} & Y \end{array}$$

From this theorem we proceed to prove another:

Theorem 2.3. *Let E be a fibre bundle over Y, and $f_0, f_1\colon X \to Y$ homotopic maps.*
Then the pullback bundles f_0^*E and f_1^*E are isomorphic.

Proof. $f_0^*E \times I$ is a bundle over $X \times I$, with projection $\pi_{f_0^*E} \times \mathrm{Id}$. By the covering homotopy theorem there is a map $\widetilde{H}\colon f_0^*E \times I \to E$ such that $\pi_E \circ \widetilde{H} = H \circ \pi_{f_0^*E \times I}$. $\widetilde{H}$ induces a bundle map $f_0^*E \times I \to H^*E$ by the formula $(p,t) \mapsto (\pi_{f_0^*E \times I}(p), \widetilde{H}(p,t))$, which we see is a bundle map since $\widetilde{H}$ is one.

The induced map on $X \times I$ is the identity, meaning that the bundles are isomorphic. Restricting this isomorphism to $f_0^*E \times \{1\}$, we get an isomorphism onto the pullback of E along $H|_{X \times \{1\}} = f_1$, which is f_1^*E. □

An application of this is the following on trivializing neighborhoods:

Lemma 2.3. *The restriction of a bundle E with fibre F to the bundle $E|_U$ over a contractible neighborhood U is trivial.*

Proof. Let $P = E|_U$. Since U is contractible, let r_t be a homotopy from the identity of U to a constant map. Then by theorem 2.3, the bundles r_0^*P and r_1^*P are equvalent. But r_1^*P is trivial since r_1 is constant. Thus there exists a bundle isomorphism $P \to U \times F$, which furnishes a trivialization of E over U. □

This lemma also follows from theorem 2.3:

Lemma 2.4. *If two bundles with the same fibre and structure group over the same space have homotopic transition maps, they are equivalent.*

Proof. Call the base space X and the bundles E_0 and E_1, with transition functions g_{ij}^0 and g_{ij}^1, respectively. Let the homotopy of transition functions be g_{ij}^t. These functions define the transition functions of a bundle P on $X \times I$, $g_{ji}(x,t) = g_{ij}^t(x)$. Let ι_0 and ι_1 be the inclusions of X at $X \times \{0\}$ and $X \times \{1\}$, respectively. These maps are homotopic, so $\iota_0^* P$ is isomorphic to $\iota_1^* P$. But $g_{ij} \circ \iota_t = g_{ij}^t$, so by the definition of the pullback bundle, $E_0 = \iota_0^* P$ is isomorphic to $E_1 = \iota_1^* P$. □

Which we apply to prove:

Theorem 2.4. *If the group G is homotopy equivalent to a subgroup H, any fibre bundle E with structure group G is homeomorphic to a fibre bundle with structure group H.*

Proof. Let $r_t \colon G \to G$ be a homotopy from the identity to a map into H, and g_{ij} the transition functions of E. Then $r_t \circ g_{ij}$ is a homotopy of the transition functions to transition functions taking values in H. Now, reconstructing a bundle isomorphic to E from the transition functions $r_1 \circ g_{ij}$, it does not enter into anything whether the group is G or H, since the values of g_{ij} lie in both. The construction must yield the same space in either case, which is then isomorphic to E. □

An application of this is in the case of vector bundles: the most naive choice of a group to act on the vector space fibre V would be $\mathrm{End}(V)$, which is usually $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$. However, one can demonstrate that they have the homotopy type of compact subgroups: $O(n)$ and $U(n)$, respectively.

We can use this to prove a simple classification theorem which is not really relevant, but which shows the idea of the classification theorem we will be aiming for regarding prequantizable line bundles.

Theorem 2.5. *The real line bundles over S^1 are partitioned into two equivalence classes: that of the cylinder, and that of the Möbius strip.*

Proof. The structure group of a real line bundle is $GL_1(\mathbb{R})$ which is homeomorphic to $\mathbb{R} \setminus \{0\}$, and contractible to $O(1) = \{1, -1\}$. We cover S^1 with two arcs A and B, which intersect in two disjoint arcs a and b. A and B are contractible, so we can trivialize over them.

Let g_a and g_b be the transition functions, going from the trivialization over A to that over B, on a and b respectively. Let g'_a and g'_b be the corresponding functions for another line bundle trivialized in the same way. Let λ_A and λ_B be any continuous $O(1)$-valued functions on the arcs A and B respectively. Note that the domains of all these functions are path-connected, while the points of $O(1)$ are not, so they must all be constant. Since $O(1)$ is Abelian we have

$$\lambda_A g_a \lambda_B^{-1} = g_a \quad \text{and} \quad \lambda_A g_b \lambda_B^{-1} = g_b$$
$$\text{OR}$$
$$\lambda_A g_a \lambda_B^{-1} = -g_a \quad \text{and} \quad \lambda_A g_b \lambda_B^{-1} = -g_b.$$

By proposition 2.12 the bundles are equivalent if and only if the g':s both agree with or both differ from the g:s. Every bundle is thus equivalent to a bundle with transition functions $g_a = 1,\ g_b = 1$, or one with $g_a = 1,\ g_b = -1$.

Since the cylinder and the Möbius strip are both line bundles over S^1 which are not homeomorphic - the cylinder being orientable while the Möbius strip is not - they fall into different equivalence classes of line bundles and thus characterize those classes. □

Another corollary is that any nontrivial fibre bundle must have transition functions taking values in a noncontractible group: since otherwise we can contract all transition functions to the identity, so that all trivializations of the resulting bundle agree, and we can patch them together to a global trivialization.

2.2.3 Principal Bundles

Principal bundles are the bundles in which the fibre is the same space as the structure group, of which an example would be the *frame bundle* of a manifold, the set of all frames for all of its tangent spaces. We introduce sections, connection, and gauge potentials.

Definition 2.18. *A* **principal G-bundle** *is a smooth fibre bundle P with structure group G and a smooth right action $\sigma\colon P \times G \to P;\ (p, g) \mapsto \sigma_g(p) = p \cdot g$ which preserves the fibres, and acts freely and transitively on them.*

As a consequence of the definition, the fibres of P are diffeomorphic to G, but lack any group structure, since there is no canonical choice of identity. A principal G-bundle P over a space M is often denoted as $G \hookrightarrow P \to M$. A relevant example of a principal bundle is the set of all pointwise bases for a vector bundle. The automorphism group of the vector space fibre acts freely and transitively on the set of bases for the fibre over a certain point. A section of such a principal bundle is the same as a local frame of sections of the vector bundle.

Given $p \in P$, the action $\sigma_p\colon G \to P;\ g \mapsto p \cdot g$ is a diffeomorphism of G onto the fibre in which p sits. The pushforward σ_{p*} embeds the Lie algebra of G into the tangent space T_pP. This subspace is called the **vertical subspace** at p: $V_pP = \sigma_{p*}\mathfrak{g} \subset T_pP$. The pushforward of $A \in \mathfrak{g}$ by this action is denoted $A^{\#}(p) = \sigma_{p*}A$. Given $A \in \mathfrak{g}$ these vectors form a smooth tangent vector field: including P in $P \times G$ at $P \times \{e\}$ and mapping via the smooth tangent section $(0, A)$ over $P \times \{e\}$ and then pushing it forward by σ_* gives the section which is $A^{\#}$.

Definition 2.19. *A* **connection** *on a principal bundle $G \hookrightarrow P \to M$ is a $\mathfrak{g}$-valued 1-form ω on P satisfying:*

- $\sigma_g^*\omega = \mathrm{ad}_{g^{-1}} \circ \omega$
- $\omega(A^{\#}) = A$

The second condition means that all connections coincide on the vertical subspaces, and that they are in fact the inverse of σ_{p*} on V_pP. Thus the rank of ω_p is the dimension of $\mathfrak{g}$, which is the dimension of V_pP, so that ω_p is completely determined if one specifies what its kernel is. This kernel is called the **horizontal subspace** at p: $H_pP = \ker \omega_p \subset T_pP$.

This is in fact an equivalent definition of a connection: a smooth distribution of tangent subspaces H_pP such that $V_pP \oplus H_pP = T_pP$ and $\sigma_{g*}H_pP = H_{p\cdot g}P$. Given a connection 1-form, the first condition guarantees that its kernel distribution is G-invariant, and given a horizontal distribution, the 1-form defined by letting the horizontal space be its kernel satisfies the first condition if the distribution is G-invariant.

Definition 2.20. *A* **section** *(or cross-section) of a principal bundle $G \hookrightarrow P \to M$ is a map $s\colon U \subset M \to P$ such that $\pi \circ s = \mathrm{Id}_U$. If $U = M$ the section is called* **global**, *otherwise it is called* **local**.

If the bundle admits a global section s, the map $M \times G \to P$; $(x, g) \mapsto s(x) \cdot g$ from the trivial bundle is an isomorphism of bundles, so P is trivial if and only if it admits a global section. This is analogous, and sometimes equivalent, to a vector bundle being trivial if it admits a global frame. The first condition of a connection 1-form means that it is determined on each fibre by its restriction to one of its points, since the G-action is transitive, so it is determined on all of $\pi^{-1}(U)$ if we find its values on every $T_{s(x)}P$ for $x \in U$, given a section $s\colon U \to P$. The map $(x, g) \mapsto s(x) \cdot g$ is a diffeomorphism onto $\pi^{-1}(U)$, and since the pushforward of the G-action at the identity embeds the Lie algebra onto the vertical subspace, we must have $T_{s(x)}P = V_{s(x)}P \oplus s_*(T_xM)$. Since the second condition of a connection 1-form means that they are all the same on the vertical subspace, this means that it is completely determined on $T_{s(x)}P$, and then on all of the fibre over x, by its value on $s_*(T_xM)$. The final point is then that any connection ω is determined on $\pi^{-1}(U)$ by the pullback $s^*\omega$ on U.

Definition 2.21. *The pullback $s^*\omega$ of a connection ω on a principal bundle through a section $s\colon U \to P$ is called the* **gauge potential** *in* **gauge** *s and is denoted $\mathcal{A} = s^*\omega$.*

As we showed, a connection is determined completely if we cover M by neighborhoods U_i with associated gauge potentials $\mathcal{A}_i$ coming from local sections s_i. If we have section s_i with transition functions $g_{ij}\colon U_i \cap U_j \to G$ defined by $s_j = s_i \cdot g_{ij}$ we can calculate how the gauge potentials in the different gauges relate. To this end we calculate the pushforward of s_j at $x \in M$. Let γ be a curve with $\gamma(0) = x$, $\gamma'(0) = X$. To find $s_{j*}X$ we differentiate:

$$\frac{d}{dt} s_i(\gamma(t)) \cdot g_{ij}(\gamma(t)) \bigg|_{t=0} = \frac{d}{dt} s_i(\gamma(t)) \cdot g_{ij}(x) + \frac{d}{dt} s_i(x) \cdot g_{ij}(\gamma(t)).$$

Where we could use a "product" rule since the G-action is a map from the product manifold $P \times G$. The first term becomes $\sigma_{g_{ij}(x)*} s_{i*}X$, and for the second we rewrite it by inserting the identity in a slightly clever way:

$$g_{ij}(\gamma(t)) = g_{ij}(x) \left(L_{g_{ij}(x)^{-1}} \circ g_{ij}(\gamma(t)) \right).$$

The factor within the parentheses is a curve in G passing through the identity at $t = 0$ with tangent vector $L^*_{g_{ij}(x)} g_{ij*} X$. By definition of the fundamental vector field we then get that

$$\frac{d}{dt} s_i(x) \cdot g_{ij}(\gamma(t)) = \left(L^*_{g_{ij}(x)} g_{ij*} X \right)^{\#} (s_i(x) \cdot g_{ij}(x)).$$

Inserting what we calculated for $s_{j*}X$ into ω we get

$$\begin{aligned} \omega(s_{j*}X) &= \omega \left(\sigma_{g_{ij}(x)*} s_{i*} X + \left(L^*_{g_{ij}(x)} g_{ij*} X \right)^{\#} (s_i(x) \cdot g_{ij}(x)) \right) \\ &= \left(\sigma^*_{g_{ij}(x)} \omega \right) (s_{i*} X) + L^*_{g_{ij}(x)} g_{ij*} X \\ &= \mathrm{ad}_{g_{ij}(x)^{-1}} \circ (s_i^* \omega)(X) + L^*_{g_{ij}(x)} g_{ij*} X, \end{aligned}$$

which gives us the relation

$$\mathcal{A}_j = \mathrm{ad}_{g_{ij}^{-1}} \circ \mathcal{A}_i + L^*_{g_{ij}} g_{ij*} \tag{2.21}$$

between the gauge potentials in different gauges. Conversely:

Proposition 2.13. *If we cover M with neighborhoods U_i on which sections s_i are defined, and associate to each U_i a Lie algebra-valued 1-form $\mathcal{A}_i$ such that* (2.21) *is satisfied for all i, j, this defines a connection 1-form on the bundle, by imposing the two conditions of a connection form on it.*

In the case where G is a matrix Lie group, (2.21) can be written in the slightly simpler form:

$$\mathcal{A}_j = g_{ij}^{-1} \mathcal{A}_i g_{ij} + g_{ij}^{-1} dg_{ij} \tag{2.22}$$

This then shows that any connection on a principal bundle $G \hookrightarrow P \to M$ can be defined by choosing its gauge potential under one section per connected component of M, and letting it be defined on the rest of each component by (2.21), and lifting to all of P by the two conditions for connection forms. As we have shown in theorem 2.4, if $H < G$ is a homotopy equivalent subgroup, we can construct an isomorphic principal G-bundle with H-valued transition functions h_{ij}. The Lie algebra $\mathfrak{h}$ of H is of course invariant under the adjoint action of H, so if choose one $\mathfrak{h}$-valued gauge potential in each component of M, we get an $\mathfrak{h}$-valued connection 1-form on P. To summarize:

Theorem 2.6. *A connection 1-form on a principal bundle $G \hookrightarrow P \to M$ can be defined by defining its gauge potential in one gauge per connected component of M, and extending using* (2.21) *and the defining conditions of a connection. If $H < G$ is a homotopy equivalent Lie subgroup with Lie algebra $\mathfrak{h}$, P is isomorphic to a principal G-bundle admitting an $\mathfrak{h}$-valued connection.*

2.2.4 Holonomy

Holonomy measures how the bundle twists and turns when going along horizontal curves in it. We develop a formula to express the holonomy of curves in abelian principal bundles.

A connection ω in a principal bundle $G \hookrightarrow P \to X$ defines a notion of parallell transport along curves: for any curve $\alpha\colon I \to X$ with $x_0 = \alpha(0)$ and point $p_0 \in \pi_P^{-1}(x_0)$, there is a unique lift $\widetilde{\alpha}\colon I \to P$ such that $\widetilde{\alpha}(0) = p_0$, $\pi_P(\widetilde{\alpha}(t)) = \alpha(t)$, and the tangent vector to $\widetilde{\alpha}$ lies in the horizontal tangent space at each point. This amounts to an ordinary differential equation in the coordinates of P, so there is a unique solution given the initial condition p_0.

Given each pair of points x_0, x_1 and curve α between them, the parallell transport of points defines a map $\tau_\alpha\colon \pi_P^{-1}(x_0) \to \pi_P^{-1}(x_1)$. If $x_1 = x_0$ this is a map of the fibre over x_0 to itself. It is a bijection since the lift is unique given an initial point, and we can traverse the curve in either direction, i.e. an inverse is given by $\tau_{\bar{\alpha}}$, where $\bar{\alpha}$ is α traversed in the opposite direction. If we fix $p_0 \in \pi_P^{-1}(x_0)$, this map corresponds to an action of G, because G acts transitively on each fibre.

Further, we can show that τ_α commutes with the action of G. Let $\widetilde{\alpha}$ be the lift with initial point p_0. We claim that $\widetilde{\alpha} \cdot g$ is the lift with initial point $p_0 \cdot g$. Certainly this new curve projects to α. The other condition is equivalent to saying that the curve pulls the connection back to the zero form on I. Now this curve is $\sigma_g \circ \widetilde{\alpha}$, and since the connection satisfies $\sigma_g^* \omega = ad_{g^{-1}} \circ \omega$, we get

$$(\widetilde{\alpha} \cdot g)^* \omega = \widetilde{\alpha}^* \sigma_g^* \omega = \widetilde{\alpha}^*(ad_{g^{-1}} \circ \omega) = ad_{g^{-1}} \circ (\widetilde{\alpha}^* \omega) = 0.$$

Since $\widetilde{\alpha}$ satisfied this condition, so we have $\tau_\alpha \circ \sigma_g = \sigma_g \circ \tau_\alpha$.

Now we do some calculations to develop a formula for the holonomy of a given curve. First suppose that a curve α (not necessarily closed) lies in a trivializing coordinate patch U. Taking a section s over U we can write

$$\widetilde{\alpha}(t) = s(\alpha(t)) \cdot g(t).$$

To differentiate this we first do it for the action $\sigma\colon P \times G \to P$. Any tangent vector $X \in T_{(x,g)}(P \times G)$ can be decomposed as the sum of $X_P \in T_p P$ and $X_G \in T_g G$, tangent to curves β and γ, respectively, at $t = 0$. Then we get (suppressing indexes of some tangent maps throughout as the points of evaluation seem clear)

$$d\sigma_{(p,g)}(X_P) = \frac{d}{dt} \beta(t) \cdot g \bigg|_{t=0} = d\sigma_g(X_P),$$

and

$$d\sigma_{(p,g)}(X_G) = \frac{d}{dt} p \cdot \gamma(t) \bigg|_{t=0} = \frac{d}{dt} (p \cdot g) \cdot (L_{g^{-1}}(\gamma(t)) \bigg|_{t=0},$$

where we just inserted an identity factor slightly cleverly. The curve $(L_{g^{-1}}(\gamma(t))$ passes through the identity at $t = 0$ with tangent vector $dL_{g^{-1}}\left(\frac{d}{dt}\gamma(t)\big|_{t=0}\right) = L_g^* X_G$. So by the definition of the fundamental vector field as obtained from the tangent map at the identity of $\sigma_p\colon\ g \mapsto p \cdot g$ we have

$$d\sigma_{(p,g)}(X_G) = \left(L_g^* X_G\right)^{\#}.$$

This vector being meant as the value of the fundamental vector field at $p \cdot g$. The tangent map of σ is of course linear so in total we have

$$d\sigma_{(p,g)}(X_P + X_G) = d\sigma_g(X_P) + \left(L_g^* X_G\right)^{\#}.$$

With this we can proceed to differentiate $\widetilde{\alpha}$:

$$\frac{d}{dt}\widetilde{\alpha}(t) = d\sigma(ds(\alpha'(t)) + g'(t)) = d\sigma_{g(t)}(ds(\alpha'(t)) + \left(L_{g(t)}^* g'(t)\right)^{\#}.$$

The horizontal condition on $\widetilde{\alpha}$ amounts precisely to saying that $\omega\left(\widetilde{\alpha}'(t)\right) = 0$. Inserting the above into ω this condition reads

$$ad_{g^{-1}} \circ \omega(ds(\alpha'(t))) + L_{g(t)}^* g'(t) = 0,$$

where we used both defining properties of the connection, one for each term. Now let us suppose that G is a matrix Lie group (which is certainly more generality than we shall need) to write the above in a handier way:

$$g(t)^{-1}\omega(ds(\alpha'(t)))g(t) + g(t)^{-1}g'(t) = 0.$$

Multiply this from the left by $g(t)$ and note that $\mathcal{A} = s^*\omega$ is the local potential in gauge s:

$$g'(t) = -\mathcal{A}(\alpha'(t))g(t). \tag{2.23}$$

This is, finally, a quite useful expression for calculating something. One might at first suppose this would have a solution like

$$g(t) = \exp\left(-\int_0^t \mathcal{A}(\alpha'(t))\right), \tag{2.24}$$

but this does not work in general, as we may not have $\frac{d}{dt}\exp(A(t)) = A'(t)\exp(A(t))$, as one might expect, due to noncommutativity in the Lie group. This can be solved by the technique of time-ordered exponentials, but we shall skip this, as we shall in fact be interested only in the holonomy of bundles with Abelian structure groups here, so that we do have $\frac{d}{dt}\exp(A(t)) = A'(t)\exp(A(t))$. In this case, (2.24) is the solution.

Then, for a closed curve α lying in a trivializing neighborhood, the holonomy is given by

$$g_\alpha = \exp\left(-\int_\alpha \mathcal{A}\right),$$

and the form of this expression, which makes no mention of a starting point for the loop, shows that the holonomy depends only on the loop and not any chosen point on it. Now suppose we have a loop $\alpha * \beta$ made up of the concatenation of two other loops, taking the notation to mean that α is traversed first. The holonomy is then given by $g_{\alpha*\beta} = g_\alpha g_\beta$.

Suppose that a curve α is the boundary of a triangulable 2-chain in the base X. A triangulation can be refined so that each triangle lies within some trivializing neighborhood. Also, we can smooth out the internal edges of the triangulation homotopically so that all curves along the edges are smooth.

We orient the curves α_i around the smoothed triangles compatibly, i.e. so that when two edges intersect they have opposite orientation. Each internal edge can be removed by taking the concatenation $\alpha_i * \alpha_j$ of the curves intersecting there. So on, we can remove all internal edges and get

$$\alpha = \prod_i \alpha_i,$$

which implies that

$$g_\alpha = \prod_i g_{\alpha_i} = \prod_i \exp\left(-\int_{\alpha_i} \mathcal{A}_i\right). \tag{2.25}$$

This could perhaps be arrived at more simply via appeals to the commutativity of the structure- and thus also holonomy group, which we postulated for the case of interest here. However, we feel that this topological argument lends some credence and intuition to the result.

2.2.5 Abelian Curvature & Projection

We introduce the curvature of a connection and develop expressions for it and its local field strength. We show that the field strength is globally defined in Abelian bundles, and relate it to the notion of bundle equivalence. We prove theorem 2.8, giving a necessary topological condition for equivalence of Abelian bundles.

Definition 2.22. *In a principal bundle $G \hookrightarrow P \xrightarrow{\pi_P} X$ with connection ω, the* **curvature** Ω *of the connection is the derivative of ω, restricted to the horizontal subspace at each point:*

$$\Omega_p(X, Y) = d\omega_p(X^H, Y^H). \tag{2.26}$$

Any vector X^V in the vertical subspace at p can be obtained as tangent to a curve $p \cdot g(t)$, giving the form $X^V = g'(0)^{\#}$. We characterize the curvature in the following way:

Theorem 2.7.

$$\Omega_p(X,Y) = d\omega_p(X,Y) + [\omega_p(X), \omega_p(Y)]. \tag{2.27}$$

Proof. We prove this for three cases, the sum of which the above can always be decomposed into by bilinearity:

- X and Y are horizontal vectors. In this case $\omega_p(X) = \omega_p(Y) = 0$ and (2.27) reduces to (2.26) immediately.

- X and Y are vertical vectors. In this case the curvature should of course be zero. They extend to fundamental vector fields $\mathcal{X}^{\#}$ and $\mathcal{Y}^{\#}$. We have the expression for the derivative:

  $$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]),$$

  where the vectors are assumed to be extended to a neighborhood of the point of evaluation, an extension for which we use $\mathcal{X}$ and $\mathcal{Y}$. We have $\omega(X) = \mathcal{X}$ and $\omega(Y) = \mathcal{Y}$, both constant, and $[X,Y] = [\mathcal{X}^{\#}, \mathcal{Y}^{\#}] = [\mathcal{X}, \mathcal{Y}]^{\#}$, by the properties of the connection and fundamental field. We get

  $$d\omega(X,Y) = -\omega([\mathcal{X},\mathcal{Y}]^{\#}) = -[\mathcal{X},\mathcal{Y}],$$

  so that (2.27) does indeed return zero.

- X is vertical and Y is horizontal. This curvature should also be zero. Again extend X to a field $\mathcal{X}^{\#}$, and Y to a horizontal field Y^H. Since $\omega(\mathcal{X}^{\#})$ is constant and $\omega(Y^H) = 0$, we again get

  $$d\omega(X,Y) = -\omega([\mathcal{X}^{\#}, Y^H]),$$

  and (2.27) is satisfied if this is zero. To this end we show that the pullback of the flow of $\mathcal{X}^{\#}$ preserves horizontality. The flow of $\mathcal{X}^{\#}$ is by definition the map

  $$\phi_t \colon\ p \mapsto p \cdot \exp(t\mathcal{X}^{\#}),$$

  which for any fixed t is just a group action $\sigma_g \colon P \to P$. We know that the connection satisfies $\sigma_g^* \omega = ad_{g^{-1}} \circ \omega$, so if Y^H is horizontal,

  $$\omega(\phi_t^* Y^H) = (\phi_t^{-1*}\omega)(Y^H) = ad_{\exp(t\mathcal{X}^{\#})}(\omega(Y^H)) = 0,$$

  i.e. $\phi_t^* Y^H$ is horizontal, for every t. Differentiating to get the Lie derivative we thus obtain a horizontal vector field.

□

Definition 2.23. *For a section s, the* ***local field strength*** *in gauge s is:*

$$\mathcal{F} = s^*\Omega = s^*(d\omega + [\omega,\omega]) = d(s^*\omega) + [s^*\omega, s^*\omega] = d\mathcal{A} + [\mathcal{A},\mathcal{A}]$$

Lemma 2.5. *The field strengths under sections s_i and s_j are related by:*

$$\mathcal{F}_j = ad_{g_{ij}} \circ \mathcal{F}_i \tag{2.28}$$

Proof. Let s_i and s_j be any two sections with corresponding field strengths $\mathcal{F}_i$ and $\mathcal{F}_j$. Since by definition we have $s_j = \sigma_{g_{ij}} \circ s_i$, we get

$$\begin{aligned}\mathcal{F}_j =& d(s_i^*\sigma_{g_{ij}}^*\omega) + [s_i^*\sigma_{g_{ij}}^*\omega, s_i^*\sigma_{g_{ij}}^*\omega] = d(s_i^*(ad_{g_{ij}} \circ \omega)) + [s_i^*(ad_{g_{ij}} \circ \omega), s_i^*(ad_{g_{ij}} \circ \omega)]\\ =& ad_{g_{ij}} \circ d(s_i^*\omega) + [ad_{g_{ij}} \circ s_i^*\omega, ad_{g_{ij}} \circ s_i^*\omega] = ad_{g_{ij}} \circ d\mathcal{A}_i + ad_{g_{ij}} \circ [\mathcal{A}_i, \mathcal{A}_i]\\ =& ad_{g_{ij}} \circ (d\mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_i]) = ad_{g_{ij}} \circ \mathcal{F}_i.\end{aligned}$$

□

The remarkable thing here is that (2.28) implies that the local field strength does not depend on the chosen section if the group is Abelian. In this case it defines a global Lie algebra-valued 2-form on the base space!

Lemma 2.6. *The global field strength of a connection ω with curvature Ω in a principal bundle $G \hookrightarrow P \to B$ is characterized by the fact that*

$$\pi_P^*\mathcal{F} = \Omega. \tag{2.29}$$

Proof. Pick a section s and look at a point $x \in B$. The tangent space at $s(x)$ can be decomposed into the pushforward $s_*(T_xB)$ and the vertical space as

$$T_{s(x)}P = s_*(T_xB) \oplus V_{s(x)}P.$$

The vertical space is the kernel of the projection π_P. Also, by definition the curvature is zero if one of its arguments is vertical. We can then decompose any vector $X \in T_{s(x)}P$ as $X = s_*X^B + X^V$. Doing this we get

$$\begin{aligned}(\pi_P^*\mathcal{F})(X,Y) &= (\pi_P^*\mathcal{F})(s_*X^B + X^V, s_*Y^B + Y^V) = \mathcal{F}(\pi_{P*}(s_*X^B), \pi_{P*}(s_*Y^B))\\ &= \mathcal{F}(X^B, Y^B) = (s^*\Omega)(X^B, Y^B) = \Omega(s_*X^B, s_*Y^B) = \Omega(X,Y),\end{aligned}$$

since, again, Ω is zero on the vertical parts. Since the pushforward π_{P*} is surjective, two forms satisfying (2.29) must be the same, so that condition determines the global field strength uniquely. □

Lemma 2.7. *The global field strength of an Abelian principal bundle is closed. Furthermore, given two connections on the bundle, the difference of their field strengths is exact.*

Proof. Closedness is immediate since the field strength is the pullback of an exact form. Let the two connections be ω_0 and ω_1. Put $\alpha = \omega_1 - \omega_0$. Then the difference of the curvatures is

$$\Omega_1 - \Omega_0 = d\omega_1 - d\omega_0 = d\alpha.$$

Since this is exact, the pullback $\mathcal{F}_1 - \mathcal{F}_0 = s^* (\Omega_1 - \Omega_0)$ is as well. □

Lemma 2.8. *Equivalent Abelian principal bundles with curvature have cohomologous global field strengths.*

Proof. Let $\Psi\colon P' \to P$ be a principal bundle diffeomorphism. Let the bundles have connections ω', ω and curvatures Ω', Ω. Let $\mathcal{F}$ be the field strength of Ω. We then have $\pi_{P'} = \pi_P \circ \Psi$. The pullback $\Psi^*\omega$ is a connection on P' with curvature $\Psi^*\Omega$. Its field strength is also $\mathcal{F}$, since $\pi_{P'}^* \mathcal{F} = \Psi^*(\pi_P^* \mathcal{F}) = \Psi^*\Omega$.

By lemma 2.7 $\mathcal{F}$ is then cohomologous with the field strength of Ω'. □

With all the above work done, we finally arrive at the main classification theorem which has been our goal throughout. It will be used to classify principal $U(1)$-bundles and complex line bundles, determining whether they are prequantizable.

Theorem 2.8. *All isomorphic Abelian principal bundles $G \hookrightarrow P \xrightarrow{\pi_P} B$ determine the same element $c_1(P) \in H^2(B, \mathbb{R}) \otimes \mathfrak{g}$ as the cohomology class of the global field strength. Furthermore, this element in fact determines an element of $H^2(B, \mathbb{Z}) \otimes \Sigma_{\mathfrak{g}}^e$, where $\Sigma_{\mathfrak{g}}^e$ is the preimage of the identity under the exponential map.*

Proof. The first statement is just lemmas 2.7 & 2.8. For the next part, recall the holonomy formula (2.25):

$$g_\gamma = \prod_i \exp\left(-\int_{\gamma_i} \mathcal{A}_i\right) = \exp\left(-\sum_i \int_{\gamma_i} \mathcal{A}_i\right).$$

Pick a 2-cycle $\mathcal{S} \in H_2(B, \mathbb{Z})$ in B, i.e. a closed oriented 2-submanifold, and let the curve γ divide it in two parts. Let $\mathcal{S}_i$ be the part of $\mathcal{S}$ enclosed by γ_i, $\mathcal{S}^+$ the part enclosed by γ, and $\mathcal{S}^-$ the part outside γ, with the induced orientation (which, since the γ's were coherently oriented, determines an orientation for $\mathcal{S}$). By Stokes' theorem the holonomy formula becomes

$$g_\gamma = \exp\left(-\sum_i \int_{\mathcal{S}_i} \mathcal{F}\right) = \exp\left(-\int_{\mathcal{S}^+} \mathcal{F}\right),$$

while by the same argument,

$$g_{\gamma^{-1}} = \exp\left(-\int_{\mathcal{S}^-} \mathcal{F}\right).$$

Since the holonomy of the backwards curve is the inverse of the holonomy, we must have

$$e = g_\gamma g_{\gamma^{-1}} = \exp\left(-\int_{\mathcal{S}^+}\mathcal{F} - \int_{\mathcal{S}^-}\mathcal{F}\right) = \exp\left(-\int_{\mathcal{S}}\mathcal{F}\right),$$

which implies that the integral of $\mathcal{F}$ over $\mathcal{S}$ is an integral multiple of something which exponentiates to the identity. Since $\mathcal{S}$ was an arbitrary element of $H_2(B,\mathbb{Z})$ this implies our result. □

In particular, for $G = U(1)$, the integral of $\mathcal{F}$ over any 2-cycle is an integer multiple of $2i\pi$.

2.2.6 Vector Bundles & Connection

Vector bundles are bundles with vector spaces as fibres, the sections of which are vector fields. We introduce the concept of connection in a vector bundle and show that it is equivalent to that of connection in a certain principal bundle.

Definition 2.24. *A* **vector bundle** *is a fibre bundle whose fibre is a vector space V and whose structure group is a subgroup of* $\mathrm{Aut}(V)$*.*

Definition 2.25. *Given a vector bundle $E \xrightarrow{p} M$, a* **connection** *in E is a $\mathbb{C}$-bilinear map*

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E),$$

or equivalently

$$\nabla : \Gamma(E) \to \Gamma(E) \times \Gamma(T^*M),$$

such that

$$\nabla_{fX} s = f\nabla_X s \quad \text{and} \quad \nabla_X(fs) = X(f)s + f\nabla_X s.$$

We usually abbreviate ∇_{∂_i} to ∇_i. The section $\nabla_X s$ is called the **covariant derivative** of s along X.

Definition 2.26. *Given local frames $\{e_i\}$ and $\{\partial_i\}$ for E and TM, the* Christoffel symbols *of the connection in these frames are the functions Γ^k_{ij} defined by*

$$\nabla_i e_j = \Gamma^k_{ij} e_k \quad \textit{or} \quad \nabla e_j = \Gamma^k_{ij} e_k dx^i.$$

If $\{f_i\}$ is another local frame, with Christoffel symbols $\widetilde{\Gamma}$, related to $\{e_i\}$ by $f_j = a^i_j e_i$, a being an $\mathrm{Aut}(V)$-valued function with inverse b, we can calculate a relation between the symbols:

$$\begin{aligned}
\widetilde{\Gamma}^\ell_{ij} f_\ell = \nabla_i f_j &= \left(\partial_i a^k_j\right) e_k + a^k_j \nabla_i e_k = \left(\partial_i a^k_j\right) b^\ell_k f_\ell + a^k_j \Gamma^s_{ik} e_s \\
&= \left(\partial_i a^k_j\right) b^\ell_k f_\ell + a^k_j \Gamma^s_{ik} b^\ell_s f_\ell \\
&\Updownarrow \\
\widetilde{\Gamma}^\ell_{ij} &= b^\ell_k \left(\partial_i a^k_j\right) + b^\ell_s \Gamma^s_{ik} a^k_j.
\end{aligned}$$

We can write this with just the matrices as

$$\widetilde{\Gamma}_i = a^{-1}\left(\partial_i a\right) + a^{-1}\Gamma_i a.$$

Meaning that the $\mathrm{End}(V)$-valued 1-forms $\widetilde{\Gamma} = \widetilde{\Gamma}_i dx^i$ and $\Gamma = \Gamma_i dx^i$ are related by

$$\widetilde{\Gamma} = a^{-1}da + a^{-1}\Gamma a.$$

We call the 1-forms $\widetilde{\Gamma}$ and Γ so defined the **Christoffel 1-forms** of the frames $\{f_i\}$ and $\{e_i\}$, respectively.

Conversely, suppose we cover M by neighborhoods $\{U_i\}$ such that we can define a local frame $E^i = \{e^i_j\}$ on each U_i, which are related by transition functions a_{ij} such that $E^j = a_{ij}E^i$. If we associate an $\mathrm{End}(V)$-valued 1-form Γ^i to each U_i, such that

$$\Gamma^j = a_{ij}^{-1}da_{ij} + a_{ij}^{-1}\Gamma^i a_{ij} \quad \forall i, j. \tag{2.30}$$

This defines a connection in E by the formula $\nabla_X e^i_j = \Gamma^i(X)^k_j e^i_k$, since any section can be written locally in some frame E^i. By the reasoning we just did above, the satisfaction of (2.30) is equivalent to the connection being globally well-defined, since the local frame expressions agree on their overlapping domains. The Christoffel symbols of the connection in this frame are defined by $\Gamma^i\left(\partial_j\right)^\ell_k e^i_\ell = \nabla_j e^i_k = \Gamma^{(i)\ell}_{jk} e^i_\ell$, so that in fact $\Gamma^i = \Gamma^{(i)\ell}_{jk} dx^j$.

Comparing what we just stated to proposition 2.13, we have the following equivalence:

Theorem 2.9. *A connection in a vector bundle E defines a connection in the frame bundle FE, and conversely. In a local section/local frame, the gauge potential/Christoffel 1-form is the same* $\mathrm{End}(V)$*-valued 1-form in either case.*

By the construction described in the theorem, the correspondance between vector and principal bundle connections follows since (2.30) is the same equation as (2.22).

Picking a local frame $\{e_i\}$ of E, we can write any section as $s = f^i e_i$, and the covariant derivative along the vector field $X^i\partial_i$ is

$$\nabla_X s = (Xf^i)e_i + f^i\nabla_X e_i = X^j\left(\partial_j f^i\right)e_i + f^i X^j \Gamma^k_{ji} e_k.$$

This expression goes to show that the covariant derivative depends, at a point, only on the value of the vector field X at that point, and the derivatives of the coefficients of s at that point. This shows that the covariant derivative $\nabla_X s$ is *local* in the argument s, and *pointwise* in X.

Definition 2.27. *A connection on a vector bundle E is* **Hermitian** *with respect to a Hermitian fiber metric $\langle\cdot,\cdot\rangle$ on E if for any X,*

$$\nabla_X\langle s, \sigma\rangle = \langle\nabla_X s, \sigma\rangle + \langle s, \nabla_X \sigma\rangle.$$

For instance, given a Riemannian metric on the tangent bundle, the unique connection which, in addition to being torsion-free, is Hermitian, is the familiar Levi-Civita connection. Using theorems 2.9 and 2.6 we derive the following existance theorem:

Theorem 2.10. *Any vector bundle with a fibre metric admits at each point a local orthonormal frame, as well as a global Hermitian connection.*

Proof. This follows since the groups $U(n)$ and $O(n)$ are homotopy equivalent to $GL(n,\mathbb{C})$ and $GL(n,\mathbb{R})$, respectively, so we can choose frames with transition functions in those groups, and a connection with its values in the space of anti-Hermitian/anti-symmetric matrices, the Lie algebra of $U(n)/O(n)$. □

Definition 2.28. *Given a connection ∇ on a vector bundle E, the* **curvature** *of the connection is a map $\Gamma(TM)\otimes\Gamma(TM)\otimes\Gamma(E)\to\Gamma(E)$ defined by*

$$R^{\nabla}(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s.$$

Proposition 2.14. *R^{∇} is skew-symmetric in X,Y and C^{∞}-linear in s, i.e. an* $\mathrm{End}(V)$*-valued 2-form. In a frame $\{e_k\}$, R^{∇} is the local field strength of the connection form in that frame.*

Proof. Skew-symmetry is clear from the definition, and that of the bracket. Now,

$$\begin{aligned}
R^{\nabla}(X,Y)(fs) &= \nabla_X(Y(f)s + f\nabla_Y s) - \nabla_Y(X(f)s + f\nabla_X s) - [X,Y](f)s - f\nabla_{[X,Y]}s \\
&= (XY(f)s + Y(f)\nabla_X s + X(f)\nabla_Y s + f\nabla_X\nabla_Y s) - \dots \\
&\dots - (YX(f)s + X(f)\nabla_Y s + Y(f)\nabla_X s + f\nabla_Y s\nabla_X s) - \dots \\
&\dots - XY(f)s + YX(f)s - f\nabla_{[X,Y]}s \\
&= f\left(\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s\right) = fR^{\nabla}(X,Y)s.
\end{aligned}$$

Thus it suffices to define the curvature on the frame $\{e_k\}$, for X and Y coordinate vector fields. Let Γ be the connection form in the frame $\{e_k\}$:

$$\begin{aligned}
R^{\nabla}(\partial_i,\partial_j)e_k &= \nabla_i\left(\Gamma(\partial_j)^{\ell}_k e_\ell\right) - \nabla_j\left(\Gamma(\partial_i)^{\ell}_k e_\ell\right) \\
&= \Gamma(\partial_i)^s_\ell\Gamma(\partial_j)^\ell_k e_s - \Gamma(\partial_j)^s_\ell\Gamma(\partial_i)^\ell_k e_s + \left(\partial_i\Gamma(\partial_j)^\ell_k\right)e_\ell - \left(\partial_j\Gamma(\partial_i)^\ell_k\right)e_\ell
\end{aligned}$$

The first two terms is $[\Gamma(\partial_i),\Gamma(\partial_j)]^s_k e_s$, and the last two is $(d\Gamma(\partial_i,\partial_j))^\ell_k\, e_\ell$. Since Γ is the gauge potential of the connection in this frame, the curvature $R^{\nabla} = d\Gamma + [\Gamma,\Gamma]$ is by definition the local field strength of the connection in this frame.

□

Proposition 2.15. *If the connection ∇ is Hermitian, then the curvature is also Hermitian. That is: for any X, Y, s, σ,*

$$\langle R^{\nabla}(X,Y)s,\sigma\rangle + \langle s, R^{\nabla}(X,Y)\sigma\rangle = 0$$

Proof. This follows from the local formula $R^\nabla = d\Gamma + [\Gamma, \Gamma]$, where Γ takes values of anti-hermitian matrices, which are closed under the Lie bracket. $\square$

Definition 2.29. *A (real or complex)* **line bundle** *is a vector bundle where each fibre is isomorphic to the base field.*

$\mathbb{C}$ is its own set of endomorphisms. The skew-Hermitian ones satisfy $\langle az_1, z_2\rangle + \langle z_1, az_2\rangle = (a+\bar{a})\langle z_1, z_2\rangle = 0$, so the skew-Hermitian maps of $\mathbb{C}$ are $i\mathbb{R}$. Then if ∇ is a Hermitian connection in a line bundle $E \to M$, $-iR^\nabla$ is a real 2-form on M.

2.3 Foliation & Distribution

Certain manifolds can be decomposed into the union of a set of layers or *leaves*, much like the peeling of an onion. Formally, one writes the manifold as the union of a set of maximal disjoint connected embedded submanifolds: this is called a *foliation*. A foliation can be obtained by declaring at each point which tangent vectors lie along the leaf of the foliation which goes through that point. This is a fact by virtue of the *Frobenius theorem*, 2.11. The upshot of this is that one can *polarize* the sections of a vector bundle by reducing to the space of sections which are covariantly constant along the vectors of a distribution, or equivalently, covariantly constant on the leaves of a foliation.

2.3.1 Foliations

Definition 2.30. *A* **p-dimensional foliation** *of an n-dimensional manifold M is a covering $\{U_i\}$ of M, with maps $\phi_i\colon U_i \to \mathbb{R}^n$ which are homeomorphisms onto their images, giving rise to transition functions $\varphi_{ij} = \phi_j \circ \phi_i^{-1}$ satisfying*

$$\varphi_{ij}(x, y) = \left(\varphi^1_{ij}(x), \varphi^2_{ij}(x, y)\right),$$

x being the first $n-p$ coordinates, and y the last p.

For instance, any coordinate atlas is both a 0-foliation and an n-foliation, similarly, any smooth foliation is also a smooth atlas. We write the maps of the foliation as $\phi_i = (\phi^1_i, \phi^2_i)$, being the first $n-p$ and last p coordinates, respectively. The set $\Lambda_i(x) = (\phi^1_i)^{-1}(x)$ is called a **plaque** of the foliation. The definition above entail that the first $n-p$ coordinates in *any* chart are constant on the plaques.

The plaques in different charts can be patched together to form a submanifold of M. To do this construction, assume all U_i are connected, pick a chart $\phi_0\colon U_0 \to V_0 \subset \mathbb{R}^n$ and a point $x_0 \in V_0$. For any i, j indexing the charts of the foliation intersecting U_0, put $x_j = \varphi^1_{ij}(x_i)$, whenever this function is defined and x_i is in its domain. To see that this formula well-defines x_j, note that $\varphi^1_{kj} \circ \varphi^1_{ik}$ is the first part of $\varphi_{kj} \circ \varphi_{ik} = \varphi_{ij}$, which imples that $\varphi^1_{kj} \circ \varphi^1_{ik} = \varphi^1_{ij}$, so

$$\varphi^1_{kj}(x_k) = \varphi^1_{kj}(\varphi^1_{ik}(x_i)) = \varphi^1_{ij}(x_i), \tag{2.31}$$

and taking $i = 0$ we see that the definition is not just entirely circular. Next, repeat this definition for any charts intersecting this first family of charts. On any chart U_i intersecting a U_j on which x_j is previously defined so that x_i can be defined, (2.31), shows that it is well-defined. Proceed with these steps, defining the x's for charts intersecting the ones on which they are defined, until it has been done for every chart in the connected component of U_0. Then define

$$\Lambda = \bigcup_i \Lambda_i(x_i) = \bigcup_i (\phi_i^1)^{-1}(x_i).$$

An atlas for Λ is given by $\{U_i \cap \Lambda, \phi_i^2|_\Lambda\}$. The restriction of a homeomorphism to a subspace is of course a homeomorphism onto its image, and since ϕ_i^1 is constant on Λ, $\phi_i^2|_\Lambda$ must be a homeomorphism. Thus Λ is a manifold. The inclusion in M is by definition of subspace topology a homeomorphism onto its image. In local coordinates ϕ_i on M and ϕ_i^2 on Λ the inclusion is just the inclusion into the hyperplane $\{x_i\} \times \mathbb{R}^p \subset \mathbb{R}^n$, which is smooth with injective differential.

The connected components of such a Λ are called the **leaves** of the foliation. They are injectively immersed connected submanifolds of M. Also they are pairwise disjoint, since if they intersect in U_i, ϕ_i^1 maps them to the same x_i, so they coincide in U_i. Then in any chart intersecting U_i and either leaf, they must intersect, and by the same agrument coincide there as well. Since each leaf is connected, this argument can be repeated for all charts connected to U_i, so the leaves coincide entirely. An equivalent definition of a foliation can be made:

Definition 2.31. *A p-dimensional foliation of a manifold M is a collection $\{\Lambda_\alpha\}$ of pairwise disjoint, connected, immersed submanifolds of M whose union is M, such that there exists an atlas of local charts $\{U_i, \phi_i\}$ such that $\phi_i(U_i \cap \Lambda_\alpha)$ is either empty or lies in a countable collection of p-dimensional hyperplanes of $\mathbb{R}^n$ whose first $n - p$ coordinates are constant.*

2.3.2 Distributions, Frobenius' Theorem

Definition 2.32. *A* **smooth distribution** *D over a manifold M is a smooth subbundle of its tangent bundle. The distribution is called* **involutive** *if for any sections X, Y of D, $[X, Y]$ is also a section of D (Lie bracket taken within TM).*

Theorem 2.11. *Every p-dimensional foliation of a n-dimensional manifold M induces an involutive distribution over it, and every involutive p-dimensional distribution over a manifold is induced by a foliation of it.*

Proof. The first part is easy:

Pick $x \in M$, and foliating coordinates $\phi_i \colon U_i \to \mathbb{R}^n$ at x. The distribution D has as fibre over each point the tangent space to the leaf at that point. The first p coordinates are constant on the leaves of the foliation, so the corresponding coordinate vector fields lie tangent to the leaves, and so in the distribution D. These smooth fields at any point show that D is a smooth distribution.

Since the Lie bracket of vector fields works by differentiating the components of the fields, the bracket of two sections of D, which are expressed in the first p coordinate vector fields must also be a combination of those coordinate fields; therefore, they are tangent to the leaves, which shows that D is involutive.

Now for the second part:
Let D be the involutive distribution. We proceed in steps. First:

• Any involutive distribution D has, in a neighborhood of any point $x \in M$, a local frame $Y_1, \dots Y_p$ such that $[Y_i, Y_j] = 0$.

To show this, let $X_1, \dots X_p$ be a local frame for D, expressed in a local coordinate basis as $X_i = a_i^v \partial_v$. Throughout, indices i, j, k run over the first p coordinates, while u runs over the last $n-p$ and v over all n.

Since it maps the coordinate basis to a p-frame, the matrix $[a_i^v]$ has rank p, and so, after a relabeling, the matrix $[a_i^j]$ is invertible. Call its inverse $[b_j^i]$. Define

$$Y_i = b_i^j X_j = \partial_i + g_i^u \partial_u.$$

We have, since D is involutive,

$$[Y_i, Y_j] = c_{ij}^k Y_k = c_{ij}^k \left(\partial_k + g_k^u \partial_u\right), \tag{2.32}$$

but also

$$[Y_i, Y_j] = [\partial_i + g_i^u \partial_u, \partial_j + g_j^u \partial_u] = h_{ij}^u \partial_u, \tag{2.33}$$

where there are only terms in the last $n-p$ coordinate fields, since the other fields in the bracket have constant coefficients. But for (2.32) and (2.33) to agree, all c_{ij}^k must be zero, so $[Y_i, Y_j] = 0$. Second:

• Lie-commuting vector fields have commuting flows.

If g is any diffeomorphism of M, and Φ_X^t the flow of the vector field X at time t, we see that $t \mapsto g \circ \Phi_X^t(g^{-1}(x))$ is a curve through x whose tangent vectors lie in the field $g_* X$. This entails that $g \circ \Phi_X^t \circ g^{-1} = \Phi_{g_* X}^t \Leftrightarrow g \circ \Phi_X^t = \Phi_{g_* X}^t \circ g$. Consequently, g commutes with the flow of X iff it pushes X forward to itself.

Now let X and Y be Lie-commuting vector fields, and calculate the pushforward $\Phi_{X*}^t Y$. First, $\Phi_{X*}^0 Y = \mathrm{Id}_* Y = Y$. Second,

$$\frac{d}{dt} \Phi_{X*}^t Y = \frac{d}{ds} \Phi_{X*}^t \Phi_{X*}^s Y \bigg|_{s=0} = \Phi_{X*}^t \frac{d}{ds} \Phi_{X*}^s Y \bigg|_{s=0} = \Phi_{X*}^t 0 = 0,$$

since the Lie derivative of Y along X is zero. This means that $\Phi_{X*}^t Y$ is constant in t, so always equal to Y. Then we have shown that $\Phi_X^t \circ \Phi_Y^s = \Phi_Y^s \circ \Phi_X^t$.

For any $x \in M$, define the leaf Λ_x through x to be the set of all points which can be reached by a curve whose tangent field lies in the distribution D. This definition is independent of which "basepoint" $x \in \Lambda_x$ one chooses. The leaf is by definition path-connected, and maximal in the sense that it equals any set containing it and satisfying the criteria. With the above two steps together, we construct a chart for this leaf, at any point x in it. Take a Lie-commutative frame $Y_1, \dots Y_p$ and can construct a map

$$\begin{aligned} \Psi \colon V &\to M \\ (t_1, \dots, t_p) &\mapsto \Phi^{t_1}_{Y_1} \circ \dots \circ \Phi^{t_p}_{Y_p}(x), \end{aligned}$$

V being some neighborhood of zero such that this composition of flows is defined. By the commutativity of the flows this is well-defined. Its differential maps $\frac{\partial}{\partial t_i}\Big|_{(t_1,\dots,t_p)}$ to

$$\frac{d}{dt}\Phi^{t_i+t}_{Y_i} \circ \Phi^{t_1}_{Y_1} \circ \dots \circ \widehat{\Phi^{t_i}_{Y_i}} \circ \dots \circ \Phi^{t_p}_{Y_p}\Big|_{t=0} = Y_i|_{\Psi(t_1,\dots,t_p)} . \tag{2.34}$$

Since the Y_i's are linearly independent, Ψ has an everywhere injective differential and - at least up to shrinking of V - embeds V as a submanifold of M. By construction, the tangent spaces of $\Psi(V)$ lie in the distribution D. To show that this is indeed a local chart for the leaf we need to show that $\Psi(V)$ coincides with Λ_x locally at x.

For a sufficiently small neighborhood U of x, any point $y \in U \cap \Lambda_x$ can be connected to x by a smooth curve $\gamma \colon I \to U$, with its tangent field written in terms of the Lie-commutative frame as $\gamma'(t) = f^i(t) Y_i(\gamma(t))$. Let F^i be antiderivatives of f^i with $F^i(0) = 0$. They define a map $F \colon I \to \mathbb{R}^p$, and the composition $\Psi \circ F$ is a curve starting at x with tangent vector (cf. (2.34))

$$\Psi_* F_* \partial_t = \Psi_* f^i(t) \partial_{t_i} = f^i(t) \, Y_i|_{\Psi(F(t))} ,$$

so this curve is the same as γ. Thus all of $U \cap \Lambda_x$ is in $\Psi(V) \cap U$, and the converse is certainly true since we can take the domain of Ψ to be path-connected. This establishes a local chart for the leaf through x, making it a connected embedded smooth submanifold. □

3 Classical Mechanics

The first step towards a proper quantization is of course to know what it is that we want to quantize. Here we give a brief exposition on the principles of classical mechanics, from a variational formulation through Lagrangian mechanics to Hamiltonian mechanics in the language of symplectic geometry, for a treatment of which we refer to the section on mathematical preliminaries. We work out a few example systems in detail. For a treatment of mechanics through variational principles, Lagrangian mechanics, Legendre transform, and Hamiltonian mechanics, see [1].

3.1 Lagrangian mechanics, Legendre transform

We can model a mechanical system as the tangent bundle TM of the system's configuration space M, each point corresponding to a position and a velocity. The principle of extremal action states that the trajectories of the system are extremals of the **action functional**:

$$I[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt.$$

where $L : TM \to \mathbb{R}$ is the **Lagrangian** defining the dynamics of the system. This corresponds to the **Euler-Lagrange equations**:

$$\frac{d}{dt}\frac{\partial L}{\partial v^i}(q(t)) - \frac{\partial L}{\partial q^i}(q(t)) = 0.$$

Now define the 1-form θ_L by letting $\theta_{L(p,v)}(u) = \frac{d}{dt}L(v + tu)$ (the fibre derivative of L). Then $\theta_{L(p,v)}(u) = \frac{\partial L}{\partial v^i}u^i$, so $\theta_L = \frac{\partial L}{\partial v^i}dq^i$.

Define the function $H = v^i \frac{\partial L}{\partial v^i} - L$, and $\omega_L = d\theta_L = \frac{\partial^2 L}{\partial q^i \partial v^j}dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j}dv^i \wedge dq^j$.

Proposition 3.1. *The Euler-Lagrange equations are equivalent to*

$$\iota_X \omega_L + dH = 0, \tag{3.1}$$

where X is the tangent vector of the solution curve $q(t)$.

Proof.

$$\iota_X \omega_L = \dot{q}^i \left(\frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial q^j \partial v^i} \right) dq^j - \dot{q}^i \frac{\partial^2 L}{\partial v^i \partial v^j} dv^j + \ddot{q}^i \frac{\partial^2 L}{\partial v^i \partial v^j} dq^j$$

$$dH(q, \dot{q}) = \frac{\partial L}{\partial v^i} dv^i + \dot{q}^i \frac{\partial^2 L}{\partial q^j \partial v^i} dq^j + \dot{q}^i \frac{\partial^2 L}{\partial v^j \partial v^i} dv^j - \frac{\partial L}{\partial v^i} dv^i - \frac{\partial L}{\partial q^i} dq^i$$

$$\iota_X \omega_L + dH = \left(\dot{q}^i \frac{\partial^2 L}{\partial q^i \partial v^j} + \ddot{q}^i \frac{\partial^2 L}{\partial v^i \partial v^j} - \frac{\partial L}{\partial q^j} \right) dq^j = \left(\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i} \right) dq^i.$$

□

By looking at the matrix form of ω_L we see that it is nondegenerate whenever

$$\det \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \right) \neq 0. \tag{3.2}$$

In that case the Lagrangian is called **regular**. For example, often the Lagrangian is constructed from a potential energy dependent only on which base point one is above, and a kinetic energy proportional to some fibre metric. In such a case the Lagrangian is regular whenever the metric is nondegenerate.

For a regular Lagrangian, ω_L then provides an isomorphism between the tangent and cotangent spaces of each point, meaning that we can solve (3.1) uniquely for X. Then the solution curve $q(t)$ exists for some time from each point, as the flow of X.

Definition 3.1. *The* **Legendre tranform** *given by the Lagrangian* $L : TM \to \mathbb{R}$ *is the function* $\tau_L : TM \to T^*M$ *defined by* $\tau_L(v)(u) = d(L|_{T_qM})_v(u)$*, identifying* $T_vT_qM \cong T_qM$*.*

Proposition 3.2. *This map is smooth and fiber-preserving, and a local diffeomorphism iff the Lagrangian is regular. In canonical coordinates it is given by*

$$q^i(x, v) = x_i,\ p_i(x, v) = \frac{\partial L}{\partial v^i}.$$

Also, τ_L *pulls the canonical 1-form* θ *back to* θ_L*.*

Proof. That it preserves the fibres is immediate from the definition. In coordinates (q^i, v^i) on TM we have $\tau_L(q, v)(u) = \frac{\partial L}{\partial v^i}(q, v)u^i$, so $\tau_L(q, v) = \frac{\partial L}{\partial v^i}(q, v)dx^i$, so in canonical coordinates (q^i, p_i), τ_L is given by $q^i = x_i,\ p_i = \frac{\partial L}{\partial v^i}$ which is smooth since L is. The matrix of $d\tau_L$ is

$$\big[d\tau_L\big](x, v) = \begin{pmatrix} id & 0 \\ \frac{\partial^2 L}{\partial x_i \partial v^j} & \frac{\partial^2 L}{\partial v^i \partial v^j} \end{pmatrix}.$$

Thus by the inverse function theorem, τ_L is a local diffeomorphism iff $\det\big(\frac{\partial^2 L}{\partial v^i \partial v^j}\big) \neq 0$. For the coordinate vector fields we have $d\tau_L(\partial_{q^i}) = \partial_{q^i} + \frac{\partial^2 L}{\partial v^j \partial q^i}\partial_{p_j},\ d\tau_L(\partial_{v^i}) = \frac{\partial^2 L}{\partial v^j \partial v^i}\partial_{p_j}$, so $\theta(d\tau_L(\partial_{p_i})) = 0,\ \theta(d\tau_L(\partial_{q^i})) = \frac{\partial L}{\partial v^i} = \theta_L(\partial_{q^i})$. □

Then for a regular Lagrangian, we can move the problem (3.1) to the symplectic cotangent space where we retrieve (2.7). Noting that the Hamiltonian on T^*M can be found by moving the above H by a Legendre transform in the case of a regular Lagrangian, we study the mechanics in a symplectic setting.

3.2 Hamiltonian mechanics

Hamiltonian mechanics models a mechanical system by a symplectic manifold (M, ω) wherein every point is a possible state of the system, positing that the physical observables of the system are the algebra $\mathcal{F}(M)$ of smooth functions on M, and that there is a particular function H, called the **Hamiltonian** (possibly time-dependent), whose Hamiltonian flow defines the time-evolution of the system.

Above we derived the Hamiltonian on the cotangent bundle and reasoned why it should govern the system dynamics, but these principles are suited to any symplectic manifold and Hamiltonian function. In particular, by (2.14), the evolution of an observable g is given by

$$\frac{dg}{dt} = \{H, g\}. \tag{3.3}$$

The time-evolution equations of the Darboux coordinates, given by (3.3), are called Hamilton's equations:

$$\begin{aligned} \frac{dq^i}{dt} &= \{H, q^i\} = X_H q^i \\ \frac{dp_i}{dt} &= \{H, p_i\} = X_H p_i. \end{aligned} \tag{3.4}$$

3.3 Example systems

Harmonic Oscillator

A (one-dimensional) harmonic oscillator is a system with one spatial coordinate, where the potential energy is proportional to the square of the spatial displacement of the system. This could be e.g. a weight attached to a spring, moving frictionlessly on a plane. We take the configuration manifold to be $\mathbb{R}$, and the velocity phase space $T\mathbb{R} \cong \mathbb{R}^2$. On this we have the kinetic energy $T(x,v) = \frac{m}{2}v^2$ and the potential energy $V(x,v) = \frac{1}{2}\omega x^2$, so we have the Lagrangian

$$L(x,v) = \frac{m}{2}v^2 - \frac{1}{2}\omega x^2.$$

To solve this using Hamiltonian formalism we calculate the Legendre transform τ_L. Since $\frac{\partial L}{\partial v} = mv$ we have, by proposition 3.2

$$\tau_L(x,v) = (x, mv).$$

Now we want to find a function H on $T^*\mathbb{R}$ such that $H \circ \tau_L(x,v) = v\frac{\partial L}{\partial v} - L(x,v)$. In this case we can invert τ_L, the inverse coordinate tranformation is $(q,p) \mapsto (q, p/m)$. Then the appropriate H is the function

$$H = \frac{p^2}{m} - L\left(q, \frac{p}{m}\right) = \frac{p^2}{2m} + \frac{\omega q^2}{2},$$

with Hamiltonian vector field (cf. (2.8))

$$X_H = \frac{p}{m}\partial_q - \omega q \partial_p.$$

Now solving Hamilton's equations (3.4) we get

$$\begin{aligned} \dot{q}(t) &= \frac{p(t)}{m} \\ \dot{p}(t) &= -\omega q(t), \end{aligned}$$

which yields a second-order equation for q:

$$\ddot{q}(t) = -\frac{\omega}{m}q(t),$$

with (real) solutions

$$\begin{aligned} q(t) &= q_0 \sin\left(\sqrt{\frac{\omega}{m}}t + \delta\right) \\ p(t) &= q_0\sqrt{m\omega}\cos\left(\sqrt{\frac{\omega}{m}}t + \delta\right). \end{aligned}$$

Spherical Pendulum

A spherical pendulum is a system of a point moving on a sphere subject to gravitational potential. Our configuration manifold is S^2_R with spherical coordinates (θ, ϕ) such that the embedding in $\mathbb{R}^3$ is given by $(\theta, \phi) \mapsto R(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. The induced metric g is in these coordinates $g_{\theta\theta} = R^2\sin^2\phi,\ g_{\phi\phi} = R^2,\ g_{\theta\phi} = 0$, so the kinetic energy is

$$T = \frac{mR^2}{2}\left(\sin^2(\phi)\dot{\theta}^2 + \dot{\phi}^2\right),$$

and the gravitational potential is

$$V = mgR\cos\phi,$$

so the Lagrangian is

$$L = T - V = mR\left(\frac{R}{2}\left(\sin^2(\phi)\dot{\theta}^2 + \dot{\phi}^2\right) - g\cos\phi\right),$$

with derivatives

$$\frac{\partial L}{\partial\dot{\theta}} = mR^2\sin^2(\phi)\dot{\theta}, \quad \frac{\partial L}{\partial\dot{\phi}} = mR^2\dot{\phi},$$

which gives the Legendre transform $\tau_L : TS^2_R \to T^*S^2_R$:

$$\begin{array}{l} p_\theta = mR^2\sin^2(\phi)\dot{\theta} \\ p_\phi = mR^2\dot{\phi} \end{array} \quad \Leftrightarrow \quad \begin{array}{l} \dot{\theta} = \frac{p_\theta}{mR^2\sin^2\phi} \\ \dot{\phi} = \frac{p_\phi}{mR^2} \end{array},$$

and we finally get the Hamiltonian

$$H = \dot{x}_i\frac{\partial L}{\partial x_i} - L = \frac{p_\theta^2}{2mR^2\sin^2\phi} + \frac{p_\phi^2}{2mR^2} + mgR\cos\phi,$$

with Hamiltonian vector field

$$X_H = \frac{p_\theta}{mR^2\sin^2\phi}\partial_\theta + \frac{p_\phi}{mR^2}\partial_\phi + \left(\frac{p_\theta^3}{mR^2\sin^2\phi} + mgR\sin\phi\right)\partial_{p_\theta}.$$

The equations of motion are

$$\left\{\begin{array}{l} \dot{\theta} = \frac{p_\theta}{mR^2\sin^2\phi} \\ \dot{p_\theta} = 0 \end{array}\right. \quad \left\{\begin{array}{l} \dot{\phi} = \frac{p_\phi}{mR^2} \\ \dot{p_\phi} = \frac{p_\theta^2}{mR^2\sin^3\phi} + mgR\sin\phi \end{array}\right. .$$

From the first set of equations we see that the momentum about the z-axis is constant, which corrseponds to a $SO(2)$-symmetry of the system: the rotational speed increases closer to the poles of the sphere. To find the highest points of the trajectories of the system, we set $\dot{\phi} = 0$, and $H = E$ (constant). Then $p_\phi = 0$ and we solve the equation

$$E = \frac{p_\theta^2}{2mR^2 \sin^2 \phi} + mgR \cos \phi$$

for ϕ. In the simple case of $p_\theta = 0$ we get

$$\phi = \cos^{-1}\left(\frac{E}{mgR}\right),$$

so the greater the energy of the system, the higher it swings, reasonably.

We can attempt to solve the Hamilton-Jacobi equation for this system. We make an ansatz of a generating function S depending only on ϕ. Then the HJE reads

$$\frac{\left(\frac{\partial S}{\partial \phi}\right)^2}{2mR^2} + mgR\cos\phi = E \Leftrightarrow \frac{\partial S}{\partial \phi} = R\sqrt{2m}\sqrt{E - mgR\cos\phi},$$

which is fact has a solution

$$S(\phi) = m\sqrt{8gR^3\left(\frac{E}{mgR} - 1\right)} el\left(\frac{\phi}{2}\middle| - \frac{2}{\frac{E}{mgR} - 1}\right),$$

where el is the elliptic integral of the second kind with parameter $-\frac{2}{\frac{E}{mgR}-1} = k^2$.

Kepler Problem

The Kepler problem is the problem of two bodies in space subject to a central (e.g. gravitational) force, proportional to the inverse square of the distance between them. We can reduce this to the problem of one body in a central potential as follows: If r_1 and r_2 are the positions of the two bodies in some coordinate system, their differential equations are

$$\ddot{r}_1 = \frac{-rk}{m_1|r|^3}, \quad \ddot{r}_2 = \frac{rk}{m_2|r|^3},$$

where k is the force's proportionality constant, m_1, m_2 the respective masses, and $r = r_1 - r_2$ the vector between the bodies. Subtracting in the above and letting $\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)$ we get

$$\ddot{r} = \frac{-rk}{\mu|r|^3},$$

which is the equation for an object of mass μ moving in a central potential $V(r) = \frac{k}{|r|}$: solving this problem gives us the relative motion of the two objects. To do this we note first that r will always lie in one plane: the one spanned by the initial position and velocity vectors, since the only force acts inwards. Therefore we adopt polar coordinates (r, θ) in this plane. In these coordinates the kinetic and potential energies are

$$T = \frac{\mu}{2}\left(r^2\dot{\theta}^2 + \dot{r}^2\right)$$
$$V = \frac{k}{r},$$

and we have the Lagrangian

$$L = T - V = \frac{\mu}{2}\left(r^2\dot{\theta}^2 + \dot{r}^2\right) - \frac{k}{r},$$

which gives the Legendre transform from its derivatives:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}, \quad p_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$
$$\Updownarrow$$
$$\dot{\theta} = \frac{p_\theta}{\mu r^2}, \quad \dot{r} = \frac{p_r}{\mu},$$

and we get the Hamiltonian

$$H = \dot{q}^i p_i - L = \frac{1}{2\mu}\left(\frac{p_\theta^2}{r^2} + p_r^2\right) + \frac{k}{r},$$

with Hamiltonian vector field

$$X_H = \frac{p_\theta}{\mu r^2}\partial_\theta + \frac{p_r}{\mu}\partial_r + \left(\frac{p_\theta^2}{\mu r^3} + \frac{k}{r^2}\right)\partial_{p_r}.$$

The equations of motion are

$$\begin{cases} \dot{\theta} = \frac{p_\theta}{\mu r^2} \\ \dot{p_\theta} = 0 \end{cases} \begin{cases} \dot{r} = \frac{p_r}{\mu} \\ \dot{p_r} = \frac{p_\theta^2}{\mu r^3} + \frac{k}{r^2} \end{cases}.$$

The first set shows that the angular momentum $l = p_\theta$ is conserved. Combining the two in the second set we get

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} + \frac{k}{r^2}.$$

Now we shall try to find r as a function of θ instead of time. To this end, note that $\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{l}{\mu r^2}\frac{d}{d\theta}$, so the above equation is

$$\mu \frac{l}{\mu r^2}\frac{d}{d\theta}\left(\frac{l}{\mu r^2}\frac{dr}{d\theta}\right) = \frac{l^2}{\mu r^3} + \frac{k}{r^2} \Leftrightarrow \frac{l}{r^2}\left(\frac{l}{\mu r^2}\frac{d^2 r}{d\theta^2} - \frac{2l}{\mu r^3}\left(\frac{dr}{d\theta}\right)^2\right) = \frac{l^2}{\mu r^3} + \frac{k}{r^2},$$

where we make the substitution $z = 1/r$, giving

$$\frac{dz}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$
$$\frac{d^2z}{d\theta^2} = \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^2}\frac{d^2r}{d\theta^2},$$

which transforms the equation to

$$z^2 l\left(-\frac{l}{\mu}\frac{d^2z}{d\theta^2}\right) = \frac{z^3l^2}{\mu} + z^2k.$$

Since $1/r \neq 0$ we can divide out $-\frac{z^2l^2}{\mu}$ to get

$$\frac{d^2z}{d\theta^2} + z = -\frac{\mu k}{l^2}.$$

The homogeneous solution is sinusoidal, and the particular solution is simply the constant in the RHS, so we have

$$z(\theta) = -\frac{\mu k}{l^2}\left(1 + e\cos(\theta - \delta)\right),$$

where the undetermined constants δ and e are a phase shift and the **eccentricity** of the orbit. For e.g. $e = 0$ the orbit is simply circular. Given such a solution we can express the energy (Hamiltonian) in θ only. Note that

$$p_r = \mu\dot{r} = \mu\frac{l}{\mu r^2}\frac{dr}{d\theta} = \mu\frac{l}{\mu r^2}\left(-r^2\frac{dz}{d\theta}\right) = -l\frac{dz}{d\theta}.$$

With this the energy is

$$\begin{aligned}
E &= \frac{1}{2\mu}\left(z^2l^2 + l^2\left(\frac{dz}{d\theta}\right)^2\right) + kz \\
&= \frac{1}{2\mu}\left(\left(\frac{\mu k}{l}\right)^2(1 + e\cos(\theta-\delta))^2 + \left(\frac{\mu k}{l}\right)^2 e^2\sin^2(\theta-\delta)\right) - \frac{\mu k^2}{l^2}(1 + e\cos(\theta-\delta)) \\
&= \frac{\mu k^2}{2l^2}\left(1 + 2e\cos(\theta-\delta) + e^2\right) - \frac{\mu k^2}{l^2}(1 + e\cos(\theta-\delta)) = \frac{\mu k^2}{2l^2}\left(e^2 - 1\right).
\end{aligned}$$

The eccentricity of the orbit can then be expressed in terms of the physical quantities:

$$e = \sqrt{\frac{2l^2E}{\mu k^2} + 1},$$

characterizing all the orbits by their energy and angular momentum.

4 Geometric Prequantization

Quantization is a mathematical way of turning a classical mechanical system into a quantum one. The concepts which one wants to carry over are for instance states of the systems, the observables of the system, and the time evolution of the states and observables. In this section we use geometrical methods to *prequantize* a mechanical system. This means that we end up with some sort of system which looks roughly like what we are after, but with some issues which we aim to fix in later seections.

4.1 The Dirac Quantization axioms

To quantize a classical Hamiltonian system (M, ω) we want to construct a map Q from the classical observables $C^\infty(M)$ to a set of Hermitian operators on a Hilbert space $\mathcal{H}$. We want such a quantization map $Q\colon\ f \mapsto Q_f$ to satisfy the **Dirac axioms**:

- $Q_{\{f,g\}} = \frac{i}{\hbar}[Q_f, Q_g]$
- $Q_{\alpha f+g} = \alpha Q_f + Q_g$
- $Q_1 = \mathrm{Id}_{\mathcal{H}}$

Axioms 1 and 2 mean that the quantization is like a Lie algebra homomorphism, with a factor $\frac{i}{\hbar}$, making the result Hermitian and inversely $\hbar$-dependent to the first order. Axiom 3 is a naturality condition: we want constant functions to quantize to multiplication operators: this will mean that for a constant c, any state vector is an eigenvector to Q_c with eigenvalue equal to the constant c: clearly the measured value of a constant observable should be that constant value, for any state. Had we not had this last condition, a satisfactory quantization would have been to let $\mathcal{H}$ be (the Cauchy-completion of) the space of smooth functions on M, and let Q_f be the Hamiltonian vector field $-i\hbar X_f$, which map is a Lie algebra homomorphism. A new guess with this in mind would be to put

$$Q_f = -i\hbar X_f + f,$$

which satisfies axioms 2 and 3. However, looking at the commutator

$$\begin{aligned}
[Q_f, Q_g]\,\psi =&(-i\hbar X_f + f)(-i\hbar X_g\psi + g\psi) - (-i\hbar X_g + g)(-i\hbar X_f\psi + f\psi) \\
=& -\hbar^2 X_f X_g\psi - i\hbar f X_g\psi - i\hbar g X_f\psi - i\hbar\psi X_f g + fg\psi - \ldots \\
&\ldots - \left(-\hbar^2 X_g X_f\psi - i\hbar g X_f\psi - i\hbar f X_g\psi - i\hbar\psi X_g f + gf\psi\right) \\
=& -\hbar^2 [X_f, X_g]\psi - 2i\hbar\{f,g\}\psi = -\hbar^2 X_{\{f,g\}}\psi - 2i\hbar\{f,g\}\psi \\
=& \left(-i\hbar Q_{\{f,g\}} - i\hbar\{f,g\}\right)\psi,
\end{aligned}$$

we see that this does not satisfy axiom 1. We need to add something which cancels the extra Poisson bracket in the commutator above. Let us try for such a term which is simply a function Φ_f, depending in some way on f. All "multiplication-by-function" operators commute, so by the bilinearity of the Lie bracket, the only new terms in the commutator will be

$$[\Phi_f, -i\hbar X_g] \quad \text{and} \quad [-i\hbar X_f, \Phi_g].$$

We have

$$\begin{aligned}[\Phi_f, -i\hbar X_g]\psi &= -i\hbar\Phi_f X_g\psi + i\hbar X_g\left(\Phi_f\psi\right) = -i\hbar\Phi_f X_g\psi + i\hbar\Phi_f X_g\psi + i\hbar\left(X_g\Phi_f\right)\psi \\ &= i\hbar\left(X_g\Phi_f\right)\psi,\end{aligned}$$

and antisymmetrically,

$$[-i\hbar X_f, \Phi_g] = -i\hbar\left(X_f\Phi_g\right).$$

What we need is then

$$\begin{gathered} i\hbar\left(X_g\Phi_f - X_f\Phi_g\right) = i\hbar\{f,g\} - i\hbar\Phi_{\{f,g\}} \\ \Updownarrow \\ X_g\Phi_f - X_f\Phi_f + \Phi_{\{f,g\}} = \{f,g\}. \end{gathered}$$

Now noting that $\{f,g\} = \omega(X_f, X_g)$, we can do the following: suppose that the symplectic form is exact, $\omega = d\theta$, so that the right hand side is $d\theta(X_f, X_g)$. Recall the formula for the exterior derivative:

$$d\theta(X_f, X_g) = X_f\theta(X_g) - X_g\theta(X_f) - \theta\left([X_f, X_g]\right).$$

Now remembering that $[X_f, X_g] = X_{\{f,g\}}$, we see that a working choice is

$$\Phi_f = -\theta(X_f).$$

Thus a quantization satisfying all axioms is given by

$$f \mapsto Q_f = -i\hbar X_f - \theta(X_f) + f. \tag{4.1}$$

However, this works only when the symplectic form is exact, and the mapping is certainly not unique: for one thing, adding any closed form to θ gives another antiderivative of ω. Such a form is again locally the derivative of some function u, so that the operator transforms to

$$Q_f' = Q_f - du(X_f) = Q_f - X_f u.$$

If we also transform the function operated upon as

$$\psi' = e^{iu/\hbar}\psi,$$

we get

$$\begin{aligned}
Q_f'\psi' &= Q_f\left(e^{iu/\hbar}\psi\right) - (X_f u)\, e^{iu/\hbar}\psi = e^{iu/\hbar}Q_f\psi - i\hbar(X_f e^{iu/\hbar})\psi - e^{iu/\hbar}(X_f u)\psi \\
&= e^{iu/\hbar}\left(Q_f\psi - i\hbar\left(X_f\frac{iu}{\hbar}\right)\psi - (X_f u)\psi\right) = e^{iu/\hbar}\left(Q_f\psi + (X_f u)\,\psi - (X_f u)\,\psi\right) \\
&= e^{iu/\hbar}Q_f\psi = (Q_f\psi)'.
\end{aligned}$$

For real u, this transformation leaves the usual L^2 inner product invariant.

To overcome the problem of the nonexactness of ω, we will patch together some operator from local data using the language of principal- and vector bundles.

4.2 Prequantizable Fibre Bundles

We can always cover M by neighborhoods on all of which ω is exact. Let $\{U_i, \alpha_i\}$ be such a covering, with $d\alpha_i = \omega$. Our goal is now to realize these local 1-forms as the local representations of a connection, either in a principal $U(1)$-bundle or, equivalently, in a complex line bundle. To start, let E be a complex line bundle over M with a Hermitian metric. By theorem 2.10, E admits an orthonormal frame, i.e. a section of length one, locally, as well as a Hermitian connection. Using a local orthonormal section s over $U \subset M$, we can locally identify $C^\infty(M,\mathbb{C})$ with $\Gamma(E)$ by $f \mapsto fs$. The connection acts on a section of E by

$$\nabla_X fs = (Xf)\,s + f\nabla_X s = (Xf)s + f\Gamma(X)s,$$

where Γ is a $\mathfrak{u}(1)$-valued 1-form representing the connection in the frame s, i.e. the gauge potential of the connection on the frame bundle in the gauge given by the section s. Using the local identification $f \mapsto fs$, we can consider the connection to operate on $C^\infty(M,\mathbb{C})$, its action is then

$$\nabla_X\colon\ f \mapsto Xf + \Gamma(X)f.$$

Comparing this to (4.1), we see that we are on the right track. Multiplying by $-i\hbar$ we get

$$-i\hbar\nabla_X\colon\ f \mapsto -i\hbar Xf - i\hbar\Gamma(X)f.$$

If $i\hbar\Gamma$ were an antiderivative of ω, we could then quantize over U by the map

$$f \mapsto Q_f = -i\hbar\nabla_{X_f} + f.$$

For this to be done globally, the connection gauge potentials must be antiderivatives of $-\frac{i}{\hbar}\omega$. Since the structure group of the line bundle is $U(1)$, which is Abelian, the derivatives of the gauge potentials coincide on their common domains, according to equation (2.28). Also, since $U(1)$ is abelian, the derivatives of the gauge potentials are simply the field strength $\mathcal{F}$, which is globally defined. We then have

$$R^{\nabla} = \mathcal{F} = -\frac{i}{\hbar}\omega,$$

which, according to theorem 2.8 means that the symplectic form ω defines an $2\pi\hbar$-integral cohomology class: $\omega \in H^2(M, 2\pi\hbar\mathbb{Z})$. This is a constraint on which manifolds can be prequantized, as the symplectic form must satisfy this condition.

4.3 The Prequantization Map & Prequantum Hilbert Space

Definition 4.1. *Given a line bundle E over M with Hermitian connection ∇, with curvature $R^{\nabla} = -\frac{i}{\hbar}\omega$, the* **prequantization map** *is the map*

$$\begin{aligned} Q : C^{\infty}(M) &\to \mathrm{End}(E) \\ f &\mapsto -i\hbar\nabla_{X_f} + f, \end{aligned}$$

where X_f is the Hamiltonian vector field of f.

Locally this map looks like (4.1), while being globally well-defined, so that it defines a quantization satisfying the Dirac axioms 4.1.

Definition 4.2. *A* **Hilbert space** *is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ of a complex vector space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ such that $\mathcal{H}$ is Cauchy-complete under the metric topology of the norm given by $||v|| = \sqrt{\langle v, v\rangle}$.*

Any complex inner product space can be completed to a Hilbert space much in the same way as one completes the rational numbers into the reals: constructing the space on the set of equivalence classes of Cauchy sequences in the starting space.

We denote by $\Gamma_c(E)$ the space of compactly supported sections of $E \to M$. When M has a symplectic form ω, and E a Hermitian fibrewise inner product we can use proposition 2.5 to define a Hermitian inner product on $\Gamma_c(E)$ by

$$\langle\langle s, \sigma\rangle\rangle = \int_M \langle s, \sigma\rangle \omega^n.$$

Definition 4.3. *Given a hermitian line bundle E over a symplectic manifold (M, ω), the* **prequantum Hilbert space** *$\mathcal{H}$ is defined as the completion of $\Gamma_c(E)$ with respect to the above inner product.*

Proposition 4.1. *The operators obtained by the prequantization map are Hermitian:*

$$\langle\langle Q_f s, \sigma\rangle\rangle = \langle\langle s, Q_f \sigma\rangle\rangle$$

Proof. First,

$$\langle fs, \sigma\rangle = f\langle s, \sigma\rangle = \langle s, f\sigma\rangle.$$

Also,

$$\langle -i\hbar\nabla_{X_f}s, \sigma\rangle - \langle s, -i\hbar\nabla_{X_f}\sigma\rangle = i\hbar\left(\langle\nabla_{X_f}s, \sigma\rangle + \langle s, \nabla_{X_f}\sigma\rangle\right) = i\hbar\mathcal{L}_{X_f}\langle s, \sigma\rangle.$$

Since X_f is a Hamiltonian vector field, $\mathcal{L}_{X_f}\omega = 0$, so we get

$$(i\hbar\mathcal{L}_{X_f}\langle s, \sigma\rangle)\omega^n = i\hbar\mathcal{L}_{X_f}(\langle s, \sigma\rangle\omega^n) = i\hbar d(\iota_{X_f}\langle s, \sigma\rangle\omega^n)$$

Where we used Cartan's formula again on a top form. In total we get:

$$\langle\langle Q_f s, \sigma\rangle\rangle - \langle\langle s, Q_f\sigma\rangle\rangle = i\hbar\int_M d(\iota_{X_f}\langle s, \sigma\rangle\omega^n) = 0.$$

□

5 Polarization

The prequantum Hilbert space constructed above is "too large", in a sense: the states depend on all the coordinates of the symplectic space of the classical system, while we know empirically, for instance, that the state of a particle in n-dimensional space is determined by a wave function depending on the n spatial coordinates of $\mathbb{R}^n$, not all $2n$ coordinates of the whole space $T^*\mathbb{R}^n$ of the classical system. In fact, the uncertainty principle does not permit simultaneous measurement of the position and momentum coordinates. What we want is therefore to cut the prequantum space "in half", and to this end we shall use the mathematical tool of polarizations.

Definition 5.1. *The complexification of the tangent bundle of a manifold M is denoted $T_{\mathbb{C}}M = TM \oplus iTM$. The fibre over each point is the complexification of that tangent space, $T_pM \oplus iT_pM \cong T_pM \otimes \mathbb{C}$.*

The module of sections of $T_{\mathbb{C}}M$ is still locally spanned by coordinate vector fields, with $C^\infty(M, \mathbb{C})$-coefficients. Differential forms and other tensor fields extend naturally to $T_{\mathbb{C}}M$, by complex-linearity - in particular the symplectic form of a symplectic manifold and a connection in a vector bundle do. There is a natural conjugation map in the complexified tangent bundle given by $v \otimes z \mapsto v \otimes \bar{z}$ or $v + iu \mapsto v - iu$. We usually write simply zv for $v \otimes z$.

Definition 5.2. *A* **polarization** *of a symplectic manifold (M, ω) is an involutive Lagrangian subbundle $\mathcal{P}$ of $T_{\mathbb{C}}M$.*

That $\mathcal{P}$ is Lagrangian means that the fibre over any point is a Lagrangian subspace of that complexified tangent space - with respect to the $\mathbb{C}$-linear extension of ω. There are two particular kinds of polarizations which are of interest:

Definition 5.3. *A* **real polarization** *is a polarization $\mathcal{P}$ satisfying $\mathcal{P} = \overline{\mathcal{P}}$.*

Proposition 5.1. *A real polarization of M induces, and is induced by, a Lagrangian foliation of M.*

Proof. Let $w_1, \dots, w_n$ be a basis for $\mathcal{P}_p$, the fibre of $\mathcal{P}$ over p, and write $w_j = u_j + iv_j$, with $u_j, v_j \in T_pM$. Since $\mathcal{P}$ is invariant under complex conjugation, $u_j - iv_j \in \mathcal{P}_p$, which means that $u_j \in \mathcal{P}_p$. Doing the same for iw_j shows also that $v_j \in \mathcal{P}_p$. Thus the vectors $u_1, \dots, u_n, v_1 \dots, v_n$ span $\mathcal{P}_p$, and so there is a linearly independent set of n of them. The real span of those n vectors is a subspace of T_pM of half dimension, on which the symplectic form is zero since the space lies in $\mathcal{P}_p$. Thus it is a Lagrangian subspace of T_pM: the fibre over p of a Lagrangian distribution in TM, the complexification of which is the polarization. This distribution is involutive since it is a subbundle of $\mathcal{P}$.

By theorem 2.11 this corresponds (in a bijective fashion) to a foliation of M, and since the distribution is Lagrangian, the leaves of the foliation are Lagrangian submanifolds. Thus one can equivalently think of a real polarization as a foliation of M by Lagrangian submanifolds. □

One way to find a real polarization of a $2n$-dimensional symplecic manifold (M, ω) is to find a set of n linearly independent functions $f_1, \dots, f_n$ in involution. Linear independence means that the differentials $df_1, \dots, df_n$ are linearly independent (in particular nonzero), and involutions means $\{f_i, f_j\} = 0$. Then the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ give a global frame (since $df \mapsto X_f$ is an isomorphism) for a distribution on M. Since $\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0$, the distribution is Lagrangian, and since $[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}} = X_0 = 0$, it is involutive. The chosen functions are constant on the leaves, and their values parametrize the leaves locally.

Definition 5.4. *A* **complex polarization** *is a polarization $\mathcal{P}$ satisfying $\mathcal{P} \cap \overline{\mathcal{P}} = \{0\}$.*

Proposition 5.2. *A complex polarization induces an almost complex structure J on TM by saying that Jv is the unique vector u such that $(v, iu) \in \mathcal{P}$. This structure is in fact integrable, making M into a complex manifold, and defines a Kähler metric g by $g(u, v) = \omega(u, Jv)$, making M Kähler. Furthermore, the polarization is given as the complex span of the complexified tangent vector fields $\frac{\partial}{\partial \bar{z}_j}$, where $\{z_j\}$ is* ***any*** *choice of holomorphic coordinates on M.*

Proof. Let $w_1 = u_1 + iv_1, \dots, w_n$ be a basis for $\mathcal{P}_p$. Suppose that all the vectors u_i, v_i are not real-linearly independent, and without loss of generality that $u_1 = a^j u_j + b^j v_j + cv_1$, $2 \leq j \leq n$. Then $w_1 - a^j w_j + ib^j w_j + icw_1$, which is nonzero by linear independence of the w_j, has only imaginary part, i.e. takes the form iV for $V \in T_pM$. But this is a contradiction since we would then have $\overline{iV} = -iV \in \mathcal{P}_p$, so the vectors u_j, v_j are linearly independent.

We have then proved that the vectors $u_1, \dots, u_n, v_1, \dots, v_n$ span T_pM. Taking an arbitrary $V = a^j u_j + b^j v_j$ we have $a^j w_j - ib^j w_j = V + iU$ for $U \in T_pM$. This U is nonzero, since otherwise we get $\mathcal{P}_p \ni iV = -\overline{iV} \in \overline{\mathcal{P}_p}$, and unique, since if also $V + iU' \in \mathcal{P}_p$ we have that the difference satisfies $\mathcal{P}_p \ni i\,(U - U') = -\overline{i(U - U')} \in \overline{\mathcal{P}_p}$. We define the complex structure here by setting $JV = U$. On the set u_j, v_j, J is defined by $Ju_j = v_j$, $Jv_j = -u_j$. Assuming the w's. u's and v's to be given by local frames, this shows J to be smooth, as well as $J^2 = -\mathrm{Id}$, so J is indeed an almost complex structure.

Now we paraphrase the following theorem from the section on complex geometry in [6]:

Theorem 5.1. *If a manifold M admits an almost complex structure J such that the Nijenhuis tensor*

$$N_J(X,Y) = [X,Y] + J\left([JX,Y] + [X,JY]\right) - [JX,JY]$$

is identically zero, then M admits the structure of a complex manifold, which structure induces the same almost complex structure J.

Any two sections of $\mathcal{P}$ take the form $X + iJX$ and $Y + iJY$, for tangent vector fields X and Y. By involution of $\mathcal{P}$ we have

$$0 = [X, +iJX, Y + iJY] = [X,Y] - [JX,JY] + i\left([X,JY] + [JX,Y]\right),$$

and this decomposition into a real and imaginary part, each of which must be zero, shows that N_J vanishes. Thus M has a complex structure, i.e. holomorphic coordinate functions $z_j = x_j + iy_j$, where the structure J is defined on the coordinate fields by $J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$ and $J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$. Differentiation along the coordinates z_j and $\bar{z}_j$ corresponds to the vector fields $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$ (cf. [6]). This last expression shows that the polarization is in fact given by the complexified tangent vector fields which differentiate with respect to the holomorphic coordinates $\bar{z}_j$, given *any* choice of holomorphic coordinates on M.

Next, define a tensor g on M by $g(X,Y) = \omega(X,JY)$. Since $\mathcal{P}$ is Lagrangian we have the following:

$$0 = \omega(X + iJX, Y + iJY) = \omega(X,Y) - \omega(JX,JY) + i\left(\omega(X,JY) + \omega(JX,Y)\right), \qquad (5.1)$$

which we use to show a few properties of the tensor g:

- **Symmetry** follows from the imaginary part of (5.1), and the antisymmetry of ω.
- **Nondegeneracy** follows since $0 \neq \omega(X, X+iJX) = \omega(X,X) + i\omega(X,JX) = i\omega(X,JX)$ if $X \neq 0$.
- **Hermeticity**, i.e. $g(X,Y) = g(JX,JY)$ follows from the real part of (5.1).

So we see that the g so defined is a Hermitian Riemannian metric on the complex manifold M. The Kähler form of the metric is

$$\Omega(X,Y) = g(JX,Y) = \omega(JX,JY) = \omega(X,Y),$$

wo $\Omega = \omega$, hence the Kähler form of g is closed. By definition, M is then a Kähler manifold with Kähler metric g. □

To restate the above two propositions briefly: a real polarization amounts to a Lagrangian foliation, and a complex polarization amounts to a Kähler structure.

Definition 5.5. *Given a polarization $\mathcal{P}$ of a symplectic manifold (M, ω), over which we have a complex line bundle $E \to M$ with a Hermitian metric and connection ∇ whose curvature satisfies $R^\nabla = -\frac{i}{\hbar}\omega$, the* **polarized Hilbert space** *of the quantum system is the Cauchy-completion of the space of all sections $s\colon M \to E$ satisfying $\nabla s|_\mathcal{P} = 0$, or equivalently $\nabla_X s = 0 \ \ \forall X \in \mathcal{P}$, the sections which are covariantly constant along the polarization.*

This is a vector space since sums and multiples of covariantly constant sections are covariantly constant as well, and the zero section is included. In the case of a real polarization, this means that we look only at the sections which are constant along the leaves of the polarization. Locally, a real polarization is always given by a set of Poisson-commuting functions $f_1, \ldots, f_n$. On the polarized space the corresponding operators Q_{f_i} then reduce to multiplication operators: $Q_{f_i}\colon\ s \mapsto f_i s$. Typically, this could be the coordinate functions on $\mathbb{R}^n$, giving the vertical polarization of $T^*\mathbb{R}^n$: it is a basic fact of quantum mechanics that these position operators work by multiplication on the position-space wave function.

In the case of a complex polarization, we get holomorphic coordinates z_j on M, and the polarized Hilbert space is the completion of the space of all sections satisfying

$$\nabla_{\frac{\partial}{\partial \bar{z}_j}} s = 0$$

6 Quantization

We are almost done with our quantization prodcedure. What we still need to do is to make sure that our operators are defined on the space of states we choose for the system. Since the polarized space is a subspace $\mathcal{H}_\mathcal{P} \subset \mathcal{H}$, the operators Q_f restrict to linear maps on $\mathcal{H}_\mathcal{P}$, but their ranges can in general still be all of $\mathcal{H}$. To fix this we need to make the requirement on the function f to be quantized that

$$Q_f \mathcal{H}_\mathcal{P} \subset \mathcal{H}_\mathcal{P}.$$

What this means is that if a section s is covariantly constant along $\mathcal{P}$ we must have, for $Y \in \Gamma(\mathcal{P})$,

$$0 = \nabla_Y (Q_f s) = \nabla_Y \left(-i\hbar \nabla_{X_f} s + fs\right) = -i\hbar \nabla_Y \nabla_{X_f} s + (Yf)s + f\nabla_Y s.$$

Because $\nabla_Y s = 0$, and since the connection has curvature $R^\nabla = -\frac{i}{\hbar}\omega$, this becomes

$$0 = -\omega(Y, X_f)s - i\hbar \nabla_{[Y,X_f]} s + (Yf)s = -i\hbar \nabla_{[Y,X_f]} s.$$

This is to hold for *any* polarized section s, which means that a necessary and sufficient condition on f is that

$$\mathcal{L}_{X_f} \mathcal{P} \subset \mathcal{P}.$$

This is a restriction on which observables can be quantized in a way that makes sense in our context. It depends on the choice of polarization, but not any any choice regarding the line bundle and connection.

Proposition 6.1. *The set $\mathcal{Q}$ of all quantizable observables is a Lie subalgebra of* $(C^\infty(M), \{\cdot,\cdot\})$.

Proof. Let P be any section of $\mathcal{P}$. For $f, g \in \mathcal{Q}$ we have that

$$X_{af+g} = aX_f + X_g,$$

so that the commutator is

$$[X_{af+g}, P] = a[X_f, P] + [X_g, P],$$

which is a section of $\mathcal{P}$ if f and g are quantizable. Thus $\mathcal{Q}$ is closed under addition and $\mathbb{R}$-multiplication. The Lie bracket satisfies the Jacobi identity:

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0,$$

so we can write

$$[X_{\{f,g\}}, P] = [[X_f, X_g], P] = -[[P, X_f], X_g] - [[X_g, P], X_f],$$

which is a section of $\mathcal{P}$ since $f, g \in \mathcal{Q}$. Thus $\mathcal{Q}$ is closed under the Poisson bracket. □

Note that $\mathcal{Q}$ is not a Poisson algebra, since products of quantizable observables need not be quantizable.

7 Examples

In this section we shall go through the quantization procedure described in the thesis with a few example spaces.

7.1 The Cotangent bundle

Refer to section 2.1.2 for the notation and symplectic structure of the cotangent bundle $Q = T^*M$ of some n-dimensional manifold M. We showed there that there exists a unique 1-form θ such that $\lambda^*\theta = \theta$ for every section $\lambda\colon M \to Q$, and the canonical symplectic form on Q is $\omega = d\theta$.

Prequantization

Since the symplectic form is exact we can determine immediately that it is $2\pi\hbar$-integral: for any 2-cycle $\mathcal{S}$ we have, by Stokes' theorem,

$$\int_S \omega = \int_S d\theta = \int_{\partial S} \theta = 0.$$

We can make a prequantum line bundle simply from the trivial bundle

$$E = Q \times \mathbb{C},$$

the sections of which are just $C^\infty(Q, \mathbb{C})$. Define a hermitian metric on the fibres $E_p = \{p\} \times \mathbb{C}$ by $\langle z, w \rangle = \bar{z}w$. Define a connection ∇ in E by

$$\nabla_X \psi = X\psi - \frac{i}{\hbar}\theta(X)\psi.$$

The prequantization map will in this case be $f \mapsto Q_f = -i\hbar\nabla_{X_f} + f$.

In some local base-space coordinates q^i, and fibre coordinates p_i, the tautological form θ takes the form $p_i dq^i$, implying that $\omega = dp_i \wedge dq^i$, so that those coordinates are indeed darboux coordinates. Now in the notation of (2.8), the Hamiltonian vector field of f is:

$$X_f = \left(D^i f\right)\partial_i - \left(\partial_i f\right) D^i$$

Thus the covariant derivative along a Hamiltonian vector field is:

$$\nabla_{X_f}\psi = \left(D^i f\right)\left(\partial_i \psi\right) - \left(\partial_i f\right)\left(D^i \psi\right) - \frac{i}{\hbar}p_i\left(D^i f\right)\psi = \{f, \psi\} - \frac{i}{\hbar}p_i\left(D^i f\right)\psi$$

The quantization operator then works on the sections by:

$$Q_f\psi = -i\hbar\{f, \psi\} + \left[f - p_i\left(D^i f\right)\right]\psi$$

Polarization

A real polarization is given simply by foliating Q by the fibres Q_p over $p \in M$, this polarization is spanned by the vector fields $D^i = \frac{\partial}{\partial p_i}$. The coordinates p_i are coordinates on the leaves of the foliation while the coordinates q^i parametrize the leaves locally. As we noted in the section on polarization, the polarized Hilbert space will be the completion of the sections of E - that is functions on Q - which are covariantly constant on the fibres Q_p. For our connection this is the equation:

$$0 = \nabla_{D^i}\psi = D^i\psi - \frac{i}{\hbar}\theta(D^i)\psi = D^i\psi \quad \forall i$$

That is, the polarized space consists of all functions which do not depend on the momentum coordinates.

Quantization

Now for a function f to be admissible for quantization, we have the condition

$$\mathcal{L}_{X_f}\mathcal{P} \subset \mathcal{P}.$$

What this means, for our particular polarization, is that all the ∂_i-components of X_f must be annihilated by any D^j. This is the equation

$$D^i D^j f = 0 \quad \forall\, i, j.$$

This means that all the quantizable observables have only first-order dependance on the momentum coordinates. Thus they can all be written on the form $F = f + p_j g^j$, for functions f and g of the position coordinates. For the calculation of Q_F, only the ∂_i-components of X_F matter, and that part of it is $g^j\partial_j$. What we get is that

$$\begin{aligned} Q_F\psi &= -i\hbar g^j \nabla_{\partial_j}\psi + F\psi = -i\hbar g^j(\partial_j\psi) - p_j g^j\psi + f\psi + p_j g^j\psi \\ &= \left(f - i\hbar g^j\partial_j\right)\psi. \end{aligned}$$

You could say that the quantization of F is just a multiplication by that function, though with each p_j swapped out for a $-i\hbar\partial_j$, which is a familiar rule from basic quantum mechanics. Note that this does not allow the usual Hamiltonian on the cotangent bundle to quantized, since it depends on the squares of the momentum coordinates. However, when a function can be written as a polynomial in a set of Poisson-commuting quantizable functions, we can define its quantization in a sensible way by simply letting it be the same polynomial in the quantized operators correseponding to the functions. So for $H = \sum_j p_j^2 + V(q)$, we would have

$$Q_H = \sum_j Q_{p_j}^2 + Q_V = -\hbar^2 \sum_j \partial_j + V.$$

7.2 The Kähler Sphere

Here we put a Kähler structure on the 2-sphere, and use it to illustrate a Kähler polarization. We parametrize the sphere by stereographic projection from the north and south poles, N and S, onto the complex plane. This gives complex coordinate functions z and w on $U_S = S^2 \setminus S$ and $U_N = S^2 \setminus N$, which are related by $zw = 1$. In these coordinates the volume form on S^2 given by the induced metric from $\mathbb{R}^3$ takes the form

$$dV = \frac{2i}{\left(1+|z|^2\right)^2} dz \wedge d\bar{z} = \frac{2i}{\left(1+|w|^2\right)^2} dw \wedge d\bar{w}.$$

We know that the area of the sphere is 4π, so to get a $2\pi\hbar$-integral form we can put

$$\omega_n = \frac{ni\hbar}{\left(1+|z|^2\right)^2} dz \wedge d\bar{z} = \frac{ni\hbar}{\left(1+|w|^2\right)^2} dw \wedge d\bar{w}.$$

Prequantization

To construct a complex line bundle over S^2 we need to define one $U(1)$-valued transition function g_{SN}, going from the trivialization over U_N to that over U_S. The possible line bundles over S^2 are parametrized by the homotopy classes of maps $U_S \cap U_N \to U(1) \cong S^1$. Since $U_S \cap U_N$ contracts to the equator circle, and $\pi_1(S^1) \cong \mathbb{Z}$, the line bundles are parametrized by integers. The k:th bundle is given by the transition function

$$g_{SN} = \frac{z^k}{|z|^k} = \frac{w^k}{|w|^k},$$

which winds k times about $U(1)$. This has differential

$$dg_{SN} = \frac{k}{2}\left(\frac{z^{k-1}}{|z|^k}dz - \frac{kz^{k+1}}{2|z|^{k+2}}d\bar{z}\right)$$

So that we have

$$\frac{dg_{SN}}{g_{SN}} = \frac{k}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right), \tag{7.1}$$

and identical expressions in w. Antiderivatives of $-\frac{i}{\hbar}\omega_n$ are given by

$$\Gamma_z = \frac{n}{2}\,\frac{zd\bar{z} - \bar{z}dz}{1+|z|^2}, \quad \Gamma_w = \frac{2}{n}\,\frac{wd\bar{w} - \bar{w}dw}{1+|w|^2},$$

where we get

$$\Gamma_w - \Gamma_z = \frac{n}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right). \tag{7.2}$$

Now comparing equations (7.1) and (7.2) with the condition (2.22), we see that we must have $n = k$ for the one forms Γ_z and Γ_w to define a connection in the bundle. That is, the integer coeffient classifying the cohomology class of the symplectic form must be the same as the integer defining the homotopy class of the transition function of the prequantum bundle E.

Polarization

We choose the holomorphic Kähler polarization: this puts the condition

$$\nabla_{\frac{\partial}{\partial z}} s = 0$$

on our sections. If we choose the constant function 1_z on the z-coordinate patch as a basis for the sections, we can write any section as $f \cdot 1_z$, so the polarization condition becomes a condition on f:

$$\frac{\partial}{\partial z}f + \Gamma\left(\frac{\partial}{\partial z}\right)f = \frac{\partial}{\partial z}f - \frac{n\bar{z}f}{2\left(1+|z|^2\right)} = 0 \Leftrightarrow \frac{\frac{\partial}{\partial z}f}{f} = \frac{n\bar{z}}{2\left(1+|z|^2\right)^2}.$$

This is a differential equation having solutions

$$f = Ae^{H(z,\bar{z})}$$

for H such that

$$\frac{\partial H}{\partial z} = \frac{n\bar{z}}{2\left(1+|z|^2\right)}.$$

This has solutions

$$H = \frac{n}{2}\log\left(1+|z|^2\right) + B(\bar{z}),$$

which gives solutions for f:

$$f = C(\bar{z})\left(1+|z|^2\right)^{n/2}.$$

Which solutions give the polarized sections of E.

Quantization

Finally we put the condition on the observable f that

$$\left[X_f, \frac{\partial}{\partial z}\right] \propto \frac{\partial}{\partial z}.$$

If we write f as a function of two variables $f(z, \bar{z})$, the Hamiltonian field of f is

$$X_f = \frac{\left(1+|z|^2\right)^2}{in\hbar}\left(\frac{\partial f}{\partial z}\frac{\partial}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}}\frac{\partial}{\partial z}\right)$$

and the condition becomes

$$\frac{\partial}{\partial z}\left(\frac{\left(1+|z|^2\right)^2}{in\hbar}\frac{\partial f}{\partial z}\right) = 0 \Leftrightarrow \frac{\partial f}{\partial z} = \frac{in\hbar D(\bar{z})}{\left(1+|z|^2\right)^2}.$$

This has solutions

$$f(z,\bar{z}) = \frac{zD(\bar{z})}{1+|z|^2} + E(\bar{z})$$

for any functions D and E. However, we want the observable f to be real, so that the operator Q_f is Hermitian. For this reason, we must have that $D(\bar{z})$ is a real constant times $\bar{z}$, and $E(\bar{z})$ a real constant. The quantizable observables are thus, for $a, b \in \mathbb{R}$,

$$\frac{a|z|^2}{1+|z|^2} + b.$$

8 Discussion & Recommendations

We have developed a method of quantizing a classical system, which is what we set out to do. The result is summarized as follows: given a symplectic manifold (M, ω) and a compatible choice of a complex line bundle E with connection ∇, and of a polarization $\mathcal{P} \subset T_{\mathbb{C}}M$, the quantum (polarized) Hilbert space is the completion of the space of all compactly supported sections of E which are covariantly constant along $\mathcal{P}$, with an inner product given by the integral over M of the fibrewise inner product of the sections. The quantum observables corresponding to classical observables are the self-adjoint operators $Q_f = -i\hbar\nabla_{X_f} + f$, and the observables which can be quantized in this way are the ones which satisfy $\mathcal{L}_{X_f}\mathcal{P} \subset \mathcal{P}$.

This quantization is well-defined and satisfies the axioms stated in section 4.1, as well as the uncertainty principle, thanks to the polarization. However, there are further issues which could be remedied. For instance, if we choose a real polarization by foliating M with noncompact leaves, none of the polarized sections will be square-integrable, so our Hilbert space will in fact be empty. A way to remedy this would be to construct the leaf space $M/\mathcal{P}$, and an induced bundle $E/\mathcal{P} \to M/\mathcal{P}$. And define an inner product between sections of that bundle. Another point of concern is that it is unclear how one would quantize a given *state* of the system, since we have focused on the observables of the system in this thesis, aside from the discussion of semiclassical states and the WKB approximation in section 2.1.6.

Recommended areas of further study would thus be how to construct a Hilbert space with a more sensible inner product, and how to quantize the states of the system, as these are, for the purposes of this thesis, open questions at this point. On a higher conceptual level, it bears examining how one would carry on with this sort of process to derive quantum field theory, which one might consider the "next step" from quantum mechanics.

9 Conclusions

First deriving a symplectic-geometrical formulation of classical mechanics from variational principles, we described how to quantize a classical system. We presented the Dirac quantization axioms, and showed a local formula for a prequantization. We described how to use a complex line bundle with connection to develop a global prequantization, and constructed the prequantization map from the space of classical observables to that of quantum observables. Using theory of fibre bundles developed in the thesis, we proved a necessary integral condition for the symplectic system to be prequantized.

Noting that the prequantized system violates the uncertainty principle, we used a polarization to reduce the amount of states which can be quantized. We described the two main types of polarization: real and Kähler. We showed what condition to put on the observables for them to be quantizable. At that point we obtained a sensible quantization of a classical system, but mentioned further issues with the method which should be fixed. We showed in practice how to apply the developed method to quantize two familiar spaces: the cotangent bundle of a manifold, and the sphere with a Kähler structure.

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