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Farey Fractions

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1 Introduction

The Farey sequence of order n is the sequence of all reduced fractions between 0 and 1 with denominator less than or equal to n, arranged in order of increasing size. The properties of this sequence have been thoroughly investigated over the years, out of intrinsic interest. The Farey sequences also play an important role in various more advanced parts of number theory.

In the present treatise we give a detailed development of the theory of Farey fractions, following the presentation in Chapter 6.1-2 of the book

MNZ = I. Niven, H. S. Zuckerman, H. L. Montgomery, "An Introduction to the Theory of Numbers", fifth edition, John Wiley & Sons, Inc., 1991,

but filling in many more details of the proofs.

Note that the definition of "Farey sequence" and "Farey fraction" which we give below is apriori different from the one given above; however in Corollary 7 we will see that the two definitions are in fact equivalent.

2 Farey Fractions and Farey Sequences

We will assume that a fraction is the quotient of two integers, where the denominator is positive (every rational number can be written in this way). A reduced fraction is a fraction where the greatest common divisor of the numerator and denominator is 1. E.g. $\frac{3.5}{4}$ is not a fraction, but $\frac{7}{8}$ is both a fraction and a reduced fraction (even though we would normally say that $\frac{3.5}{4} = \frac{7}{8}$). Also $\frac{-1}{-2}$ and $\frac{0}{-1}$ are not fractions, since their denominators are negative.

We will construct a table in the following way, where the fractions in each row of the table are in the specified order. The 1st row only contains the fractions $\frac{0}{1}$ and $\frac{1}{1}$. If the nth row has been constructed, then the $(n+1)$st row is constructed by copying the nth row and then for each pair of consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ that exist in the nth row that satisfy $b + b' \leq n+1$, the fraction $\dfrac{a+a'}{b+b'}$ is inserted in the $(n+1)$st row between $\dfrac{a}{b}$ and $\dfrac{a'}{b'}$. For example, the 2nd row is constructed by copying the 1st row and then inserting $\dfrac{0+1}{1+1} = \dfrac{1}{2}$ between $\dfrac{0}{1}$ and $\dfrac{1}{1}$, so that the 2nd row is $\dfrac{0}{1}, \dfrac{1}{2}, \dfrac{1}{1}$. When constructing the 3rd row, $\dfrac{0+1}{1+2} = \dfrac{1}{3}$ is inserted between $\dfrac{0}{1}$ and $\dfrac{1}{2}$ and $\dfrac{1+1}{2+1} = \dfrac{2}{3}$ is inserted between $\dfrac{1}{2}$ and $\dfrac{1}{1}$, so that the 3rd row becomes $\dfrac{0}{1}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{1}{1}$. When constructing the 4th row, $\dfrac{0+1}{1+3} = \dfrac{1}{4}$ is inserted between $\dfrac{0}{1}$ and $\dfrac{1}{3}$ and $\dfrac{2+1}{3+1} = \dfrac{3}{4}$ is inserted between $\dfrac{2}{3}$ and $\dfrac{1}{1}$, but $\dfrac{1+1}{3+2} = \dfrac{2}{5}$ is not inserted between $\dfrac{1}{3}$ and $\dfrac{1}{2}$ because $5 \not\leq 4$ and $\dfrac{1+2}{2+3} = \dfrac{3}{5}$ is not inserted between $\dfrac{1}{2}$ and $\dfrac{2}{3}$ because $5 \not\leq 4$. So the 4th row is $\dfrac{0}{1}, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{1}{1}$. The first six rows in the table are:

$$
\begin{array}{ccccccccccccc}
\frac{0}{1} & & & & & & & & & & & & \frac{1}{1} \\
\frac{0}{1} & & & & & & \frac{1}{2} & & & & & & \frac{1}{1} \\
\frac{0}{1} & & & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & & & \frac{1}{1} \\
\frac{0}{1} & & & \frac{1}{4} & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & \frac{3}{4} & & & \frac{1}{1} \\
\frac{0}{1} & & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & & \frac{1}{1} \\
\frac{0}{1} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \frac{1}{1}
\end{array} .
$$

Definition 1 (Farey sequence)**.** *The sequence of fractions in the nth row in the above table is called the Farey sequence of order n.*

Definition 2 (Farey fraction of order n)**.** *A Farey fraction of order n is a fraction in the Farey sequence of order n.*

Definition 3 (Farey fraction)**.** *A Farey fraction is a Farey fraction of some order, i.e. a fraction in the Farey table.*

Theorem 1 (Theorem 6.1 and Corollary 6.3 in MNZ)**.** *If* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive fractions in the* n*th row with* $\frac{a}{b}$ *to the left of* $\frac{a'}{b'}$*, then* $a'b - ab' = 1$*. The fractions in the* n*th row are also listed in order of their size (in strictly ascending order).*

Proof. Base case ($n = 1$)*:* $1 \cdot 1 - 0 \cdot 1 = 1$ and the fractions in the first row are clearly listed in order of their size (in strictly ascending order).
Induction hypothesis: Assume for some $n \in \mathbb{N}_{>0}$ that if $\frac{a}{b}$ and $\frac{a'}{b'}$ are fractions in the nth row with $\frac{a}{b}$ to the left of $\frac{a'}{b'}$, then $a'b - ab' = 1$. Also assume the fractions in the nth row are listed in order of their size (in strictly ascending order).
Induction step: We want to show that the fractions in the $(n+1)$st row are listed in order of their size (in strictly ascending order). We know that the $(n+1)$st row is constructed by copying the nth row and then for each pair of consecutive fractions in the nth row, insert 1 or 0 fractions between those fractions and we also know by the induction hypothesis that the fractions in the nth row are listed in order of their size (in strictly ascending order). So it's sufficient to show that for each pair of consecutive fractions $\frac{p}{q}, \frac{r}{s}$ in the nth row with $\frac{p}{q}$ to the left, $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$. So let $\frac{p}{q}$ and $\frac{r}{s}$ be consecutive fractions in the nth row with $\frac{p}{q}$ to the left. Then

$$\begin{aligned} \frac{p}{q} < \frac{p+r}{q+s} &\iff \frac{p(q+s)}{q(q+s)} < \frac{q(p+r)}{q(q+s)} \\ &\iff p(q+s) < q(p+r) \\ &\iff rq - ps > 0. \end{aligned}$$

But the last inequality holds because it follows from the induction hypothesis that $rq - ps = 1$. Hence $\frac{p}{q} < \frac{p+r}{q+s}$. Similarly it can be shown that

$\frac{p+r}{q+s} < \frac{r}{s}$. So the fractions in the $(n+1)$st row are listed in order of their size (in strictly ascending order) and from that it follows that no fraction can appear twice in the $(n+1)$st row.

Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be 2 consecutive fractions in the $(n+1)$st row with $\frac{a}{b}$ to the left of $\frac{a'}{b'}$. We want to show that $a'b - ab' = 1$. If they are also consecutive fractions in the nth row with $\frac{a}{b}$ to the left, then it follows from the induction hypothesis that $a'b - ab' = 1$. So assume they are not consecutive fractions in the nth row with $\frac{a}{b}$ to the left.

We want to show it can't be the case that both $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the nth row. Assume for contradiction that both $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the nth row. If they are not consecutive fractions in the nth row, then there is some fraction in the nth row somewhere between them, say $\frac{p}{q}$. But because of how rows are constructed, $\frac{p}{q}$ must be between $\frac{a}{b}$ and $\frac{a'}{b'}$ in the $(n+1)$st row, which contradicts $\frac{a}{b}$ and $\frac{a'}{b'}$ being consecutive fractions in the $(n+1)$st row. Assume instead for contradiction that $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive fractions in the nth row, but with $\frac{a'}{b'}$ to the left. Either no fraction was added between them when constructing the $(n+1)$st row, in which case $\frac{a}{b}$ and $\frac{a'}{b'}$ will be consecutive fractions in the $(n+1)$st row, but with $\frac{a'}{b'}$ to the left. This leads to a contradiction. If instead a fraction was added between $\frac{a}{b}$ and $\frac{a'}{b'}$ when constructing the $(n+1)$st row, then $\frac{a}{b}$ and $\frac{a'}{b'}$ are not even consecutive fractions in the $(n+1)$st row, which is a contradiction. So it can't be the case that both $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the nth row when they are consecutive fractions in the $(n+1)$st row with $\frac{a}{b}$ to the left and they are not consecutive fractions in the nth row with $\frac{a}{b}$ to the left.

We want to show that it can't be the case that neither of the fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the nth row. Assume or contradiction that neither of the fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the nth row. Since they exist in the $(n+1)$st row, it must be the case that they were both inserted between fractions in the nth row. But because at most 1 fraction is inserted between each pair of consecutive fractions, they must have been inserted between distinct pairs of fractions in the nth row. But then it's clear that some fraction must exist between $\frac{a}{b}$ and $\frac{a'}{b'}$ in the $(n+1)$st row, which contradicts them being consecutive fractions in the $(n+1)$st row. So it's impossible that neither of the fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the nth row when they are consecutive fractions in the $(n+1)$st row with $\frac{a}{b}$ to the left.

The remaining case is when one of the fractions exist in the nth row but the other doesn't. Assume it's the left fraction $\frac{a}{b}$ that exists in the nth row (if instead it's the right fraction that exists in the nth row, then the proof can be done analogously). The right fraction $\frac{a'}{b'}$ must have been inserted in the $(n+1)$st row between $\frac{a}{b}$ and the fraction in the nth row directly to the right of $\frac{a}{b}$, say $\frac{p}{q}$. Then the nth and $(n+1)$st row will look something like this:

Row n:	...	$\frac{a}{b}$		$\frac{p}{q}$	...
Row $n+1$:	...	$\frac{a}{b}$	$\frac{a+p}{b+q} = \frac{a'}{b'}$	$\frac{p}{q}$	...

Then

$$\begin{aligned} a'b - ab' = (a+p)b - a(b+q) &= pb - aq \\ &= 1. \qquad \{\text{by induction hypothesis}\} \end{aligned}$$

□

Corollary 2 (Corollary 6.2 in MNZ)**.** *Every fraction* $\frac{a}{b}$ *in the table is in reduced form, i.e.* $gcd(a,\ b) = 1$*.*

Proof. Let $n \in \mathbb{N}_{>0}$. We want to show every fraction $\frac{a}{b}$ in the nth row has $gcd(a, b) = 1$.
Let $\frac{a}{b}$ be a fraction in the nth row. Since there is more than 1 fraction in the nth row, there is a fraction, say $\frac{p}{q}$ next to $\frac{a}{b}$ in the nth row. Assume $\frac{p}{q}$ is to the right of $\frac{a}{b}$ (if it is to the left of $\frac{a}{b}$, then the proof can be done analogously). By Theorem 1, $pb-aq = 1$. Consider the diophantine equation $bx + ay = 1$. It has a solution iff $gcd(a,\ b) = 1$. But $x = p$, $y = -q$ is a solution, so $gcd(a, b) = 1$. □

Theorem 3. $\forall n \in \mathbb{N}_{>0}$*: if* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive fractions in the* n*th row, then* $b + b' \geq n + 1$*.*

Proof. For $n = 1$, $1 + 1 = 2 \geq 1 + 1$, so it's true for $n = 1$.
Induction hypothesis: Assume it's true for some $n \in \mathbb{N}_{>0}$.
Induction step: We want to show it's true for $n + 1$.
Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be 2 consecutive fractions in the $(n+1)$st row with $\frac{a}{b}$ to the left. If they are also consecutive fractions in the nth row then by the induction hypothesis, $b + b' \geq n + 1$. It can't be the case that $b + b' = n + 1$, because if that were the case then $b + b' \leq n + 1$, so in the $(n + 1)$st row a fraction should have been inserted between $\frac{a}{b}$ and $\frac{a'}{b'}$, but then $\frac{a}{b}$ and $\frac{a'}{b'}$ can't be consecutive fractions in the $(n + 1)$st row, a contradiction. So $b + b' > n + 1$, i.e. $b + b' \geq n + 2$.
Otherwise if they are not consecutive fractions in the nth row, exactly 1 of the fractions appear in the nth row like in Theorem 1. Say $\frac{a}{b}$ appears in the nth row and $\frac{p}{q}$ is the fraction in the nth row directly to the right of $\frac{a}{b}$. Then $\frac{a'}{b'}$ is the fraction "between" $\frac{a}{b}$ and $\frac{p}{q}$, so $b' = b + q$. By the induction hypothesis, $b+q \geq n+1$. And hence $b+b' = b+b+q \geq 1+b+q \geq 1+(n+1) = n+2$. □

Theorem 4. *Let $n \in \mathbb{N}_{>0}$ and let $\frac{p}{q}$ be a fraction in the nth row. Then $q \leq n$ with equality iff $\frac{p}{q}$ does not exist in any previous row.*

Proof. We will begin by showing the first part, i.e. $\forall n \in \mathbb{N}_{>0}$: if $\frac{p}{q}$ is a fraction in the nth row, then $q \leq n$.
For $n = 1$ the only fractions are $\frac{0}{1}$ and $\frac{1}{1}$, and $1 \leq 1$.
Induction hypothesis: Assume for some $n \in \mathbb{N}_{>0}$ that for every fraction $\frac{p}{q}$ in the nth row, $q \leq n$.
Induction step: Let $\frac{p}{q}$ be a fraction in the $(n+1)$st row. We want to show that $q \leq n+1$.
Case 1: $\frac{p}{q}$ exists in the nth row as well. Then by induction hypothesis, $q \leq n < n+1$.
Case 2: $\frac{p}{q}$ does not exist in the nth row. Then $\frac{p}{q}$ must lie between 2 fractions that exist in the nth row, say $\frac{a}{b}$ and $\frac{a'}{b'}$. Then $q = b + b'$ and because of the rules of how rows are constructed, $b + b' \leq n+1$, so $q \leq n+1$.

Now we want to prove the second part, i.e. $\forall n \in \mathbb{N}_{>0}$: if $\frac{p}{q}$ is a fraction in the nth row, then $q = n$ iff $\frac{p}{q}$ does not exist in any previous row.
For $n = 1$, we see that $q = 1$ and $\frac{p}{q}$ does not exist in any previous row, so the theorem holds for $n = 1$.
Say $n > 1$ and let $\frac{p}{q}$ be a fraction in the nth row. If $\frac{p}{q}$ exists in a previous row, then since rows are constructed by copying the previous row, $\frac{p}{q}$ must exist in the $(n-1)$st row. This means we are in case 1 of the proof of the previous part of this theorem, so $q < n$, i.e. $q \neq n$.
If instead $\frac{p}{q}$ doesn't exist in a previous row, then we are in case 2 of the proof of the previous part of this theorem, i.e. $\frac{p}{q}$ lies between the fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ that exist in the $(n-1)$st row. Then $q = b + b' \leq n$. But because of

Theorem 3, $b + b' \geq n$. Since $q \leq n$ and $q \geq n$ we must have that $q = n$. $\square$

Lemma 5. *Let $n \in \mathbb{N}_{>0}$ and let $x \in \mathbb{N}$ be such that $0 \leq x \leq n+1$ and $gcd(x,\ n+1) = 1$. Let $\frac{a}{b}$ be the greatest fraction in the nth row that is less than $\frac{x}{n+1}$ and let $\frac{a'}{b'}$ be the smallest fraction in the nth row that is greater than $\frac{x}{n+1}$. Then $a + a' = x$ and $b + b' = n + 1$.*

Proof. It can't be the case that $\frac{x}{n+1}$ is fraction in the nth row, since that would contradict Theorem 4 ($n + 1 \not\leq n$). So then $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive fractions in the nth row with $\frac{a}{b}$ to the left and

$$\frac{a}{b} < \frac{x}{n+1} < \frac{a'}{b'}.$$

We see that $\frac{a}{b} < \frac{x}{n+1} \iff bx - a(n+1) > 0 \iff bx - a(n+1) \geq 1$. Consider the diophantine equation

$$bu - av = k,$$

where $k := bx - a(n+1)$ and u and v are the unknowns. It has a particular solution $\langle u,\ v\rangle = \langle ka',\ kb'\rangle$ since

$$\begin{aligned} bka' - akb' &= k(a'b - ab') \\ &= k. && \{\text{by Theorem 1}\} \end{aligned}$$

We have that $gcd(a,\ b) = 1$ because of Corollary 2, so it has the general solution $\langle u,\ v\rangle = \langle ka' + ae,\ kb' + be\rangle$, $e \in \mathbb{Z}$. But we also know it has a particular solution $\langle u,\ v\rangle = \langle x,\ n+1\rangle$, so $\exists m \in \mathbb{Z} : x = ka' + am \wedge n + 1 = kb' + bm$. Let $m \in \mathbb{Z}$ be such that

$$kb' + bm = n + 1 \tag{1}$$

and

$$ka' + am = x. \tag{2}$$

We want to show that $k = m = 1$.
Since $0 \leq x \leq n+1$ and $gcd(x,\ n+1) = 1$, it must be the case that $x > 0$.
We have that

$$\begin{aligned}
\frac{x}{n+1} < \frac{a'}{b'} &\iff \frac{n+1}{x} > \frac{b'}{a'} \\
&\iff (n+1)a' - b'x > 0 \\
&\iff (n+1)a' - b'x \geq 1 \\
&\iff (kb' + bm)a' - b'(ka' + am) \geq 1 && \{\text{by (1) and (2)}\} \\
&\iff (a'b - ab')m \geq 1 \\
&\iff m \geq 1. && \big\{\text{by Theorem 1}\big\}
\end{aligned}$$

This shows that $m \geq 1$. We also know that $k \geq 1$. If we assume for contradiction that $k > 1$ or $m > 1$, then

$$\begin{aligned}
b + b' &< bm + kb' \\
&= n + 1, && \{\text{by (1)}\}
\end{aligned}$$

but that's impossible, since by Theorem 3, $b + b' \geq n + 1$. This shows that $k \leq 1$ and $m \leq 1$. Since also $m \geq 1$ and $k \geq 1$, it follows that $k = m = 1$. Equation (1) now becomes $b + b' = n + 1$ and equation (2) becomes $a + a' = x$. □

Theorem 6 (Theorem 6.5 in MNZ)**.** *If* $0 \leq x \leq y$, $gcd(x, y) = 1$, *then the fraction* $\frac{x}{y}$ *appears in the* y*th and all later rows.*

Proof. It is clear that if $\frac{x}{y}$ appears in the yth row then it also appears in all later rows, so it is enough to show that $\frac{x}{y}$ appears in the yth row.
If $y = 1$ then either $x = 0$ or $x = 1$. Both $\frac{0}{1}$ and $\frac{1}{1}$ appear in the 1st row, so the theorem holds for $y = 1$.
We want to show the theorem holds when $y \geq 2$, i.e. when $y = n + 1$ for some $n \in \mathbb{N}_{>0}$.
Let $n \in \mathbb{N}_{>0}$, $0 \leq x \leq n+1$ and $gcd(x,\ n+1) = 1$. If $x = 0$ or $x = n+1$ then $gcd(x, n+1) = n + 1 > 1$, so it must be the case that $0 < x < n + 1$.

We want to show that $\frac{x}{n+1}$ appears in the $(n+1)$st row. Let $\frac{a}{b}$ be the greatest fraction in the nth row that is less than $\frac{x}{n+1}$ and let $\frac{a'}{b'}$ be the smallest fraction in the nth row that is greater than $\frac{x}{n+1}$. By Lemma 5 we know that $a + a' = x$ and $b + b' = n + 1$. From the first few lines of the proof of Lemma 5 we know that $\frac{x}{n+1}$ does not exist in the nth row, so $\frac{a}{b}$ and $\frac{a'}{b'}$ must be consecutive fractions in the nth row. Because $\frac{a+a'}{b+b'} = \frac{x}{n+1}$ and how rows are constructed, we know that $\frac{x}{n+1}$ must be inserted in the $(n+1)$st row between $\frac{a}{b}$ and $\frac{a'}{b'}$. □

Corollary 7 (Corollary 6.6 in MNZ)**.** *The nth row consists exactly of all reduced fractions $\frac{a}{b}$ such that $0 \leq \frac{a}{b} \leq 1$ and $0 < b \leq n$. The fractions are listed in order of their size.*

Proof. Let $n \in \mathbb{N}_{>0}$ and let $\frac{a}{b}$ be a reduced fraction such that $0 \leq \frac{a}{b} \leq 1$ and $0 < b \leq n$. We want to show that $\frac{a}{b}$ appears in the nth row. Clearly $0 \leq a \leq b \leq n$. By Theorem 6 we have that $\frac{a}{b}$ appears in the bth and all later rows, so $\frac{a}{b}$ appears in the nth row. So every reduced fraction $\frac{a}{b}$ such that $0 \leq \frac{a}{b} \leq 1$ and $0 < b \leq n$ exist in the nth row, and they are listed in order of their size by Theorem 1. We want to show that the nth row doesn't contain any other fractions than these. Let $\frac{p}{q}$ be a fraction in the nth row. The fraction $\frac{p}{q}$ must be reduced by Corollary 2. By Theorem 4 we have that $q \leq n$ and $q > 0$ by the definition of fractions. So $0 < q \leq n$. Because rows are constructed by copying the previous row and then inserting fractions between the fractions and the fact that rows are ordered by size, we can never get a row with a fraction that is strictly greater than $\frac{1}{1}$ or strictly less than $\frac{0}{1}$. So $0 \leq \frac{p}{q} \leq 1$. □

Theorem 8. *If* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are Farey fractions with* $a'b - ab' = 1$ *(and thus* $\frac{a}{b} < \frac{a'}{b'}$*), then* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive Farey fractions of some order.*

Proof. Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be Farey fractions with

$$a'b - ab' = 1 \tag{3}$$

(and thus $\frac{a}{b} < \frac{a'}{b'}$). Let $\frac{c}{d}$ be a Farey fraction such that $\frac{a}{b} < \frac{c}{d} < \frac{a'}{b'}$. We want to show that $d > \max(b,\ b')$. From $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{a'}{b'}$ it follows that

$$\begin{cases} cb - da > 0 \\ da' - cb' > 0. \end{cases}$$

Let $k := cb - da$ and $l := da' - cb'$. Then k and l are positive integers and

$$\begin{cases} cb - da = k \\ da' - cb' = l. \end{cases} \tag{4}$$

Consider the diophantine equation

$$bx - ay = k. \tag{5}$$

We know that

$$\begin{aligned} b(ka') - a(kb') &= k(a'b - ab') \\ &= k. && \{\text{by (3)}\} \end{aligned}$$

Hence $\langle x,\ y\rangle = \langle ka',\ kb'\rangle$ is a particular solution to (5). Because $\gcd(a,\ b) = 1$ (this follows from Corollary 2), the general solution to (5) is $\langle x,\ y\rangle = \langle ka' + am,\ kb' + bm\rangle = k\langle a',\ b'\rangle + m\langle a,\ b\rangle$, $m \in \mathbb{Z}$.
Consider the diophantine equation

$$-b'x + a'y = l. \tag{6}$$

Since

$$\begin{aligned} -b'(al) + a'(bl) &= l(a'b - ab') \\ &= l && \{\text{by (3)}\} \end{aligned}$$

it follows that $\langle x,\ y\rangle = \langle al,\ bl\rangle$ is a particular solution to (6). We have that $\gcd(a',\ b') = 1$ because of Corollary 2, so the general solution to (6) is $\langle x,\ y\rangle = \langle al + a'n,\ bl + b'n\rangle = l\langle a,\ b\rangle + n\langle a',\ b'\rangle$, $n \in \mathbb{Z}$. We know that $\left\{\begin{pmatrix} a \\ b \end{pmatrix},\ \begin{pmatrix} a' \\ b' \end{pmatrix}\right\}$ is a basis for the vector space $\mathbb{R}^2$, since

$$\begin{aligned} \det \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} &= ab' - a'b \\ &= -(a'b - ab') \\ &= -1 \qquad\qquad \{\text{by } (3)\} \\ &\neq 0. \end{aligned}$$

But then every vector $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$ can be written in a unique way as $\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} a \\ b \end{pmatrix} + c_2 \begin{pmatrix} a' \\ b' \end{pmatrix}$, where $c_1, c_2 \in \mathbb{R}$. From (4) it follows that $\langle x,\ y\rangle = \langle c,\ d\rangle$ is a particular solution to both (5) and (6). By looking at the general solution of (5) we see that $\begin{pmatrix} c \\ d \end{pmatrix} = k \begin{pmatrix} a' \\ b' \end{pmatrix} + m \begin{pmatrix} a \\ b \end{pmatrix}$ for some $m \in \mathbb{Z}$. By looking at the general solution of (6) we see that $\begin{pmatrix} c \\ d \end{pmatrix} = l \begin{pmatrix} a \\ b \end{pmatrix} + n \begin{pmatrix} a' \\ b' \end{pmatrix}$ for some $n \in \mathbb{Z}$. By comparing these 2 ways of writing $\begin{pmatrix} c \\ d \end{pmatrix}$ and using the fact that $\left\{\begin{pmatrix} a \\ b \end{pmatrix},\ \begin{pmatrix} a' \\ b' \end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$ we conclude that $l = m$ and $k = n$. Thus $\begin{pmatrix} c \\ d \end{pmatrix} = k \begin{pmatrix} a' \\ b' \end{pmatrix} + l \begin{pmatrix} a \\ b \end{pmatrix}$. But then

$$d = kb' + lb \geq b' + b > \max(b,\ b') =: m.$$

In the above equations we used the fact that k and l are positive integers. By Corollary 7 it follows that $\frac{c}{d}$ does not exist in the mth row, since $d > m$. But $\frac{c}{d}$ was an arbitrary Farey fraction lying strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$. So no Farey fraction lying strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the mth row, but both $\frac{a}{b}$ and $\frac{a'}{b'}$ exist in the mth row. So $\frac{a}{b}$ and $\frac{a'}{b'}$ must be consecutive Farey fractions in the mth row. □

Corollary 9. *Let* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *be Farey fractions such that* $\frac{a}{b} < \frac{a'}{b'}$. *Then* $a'b - ab' = 1$ *iff* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive Farey fractions of some order.*

Proof. " $\Longrightarrow$ ": Assume $a'b - ab' = 1$. Then it follows from Theorem 8 that $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive Farey fractions of some order.
" $\Longleftarrow$ ": Assume $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive Farey fractions of some order. Then it follows from Theorem 1 that $a'b - ab' = 1$. □

Theorem 10 (Theorem 6.4 in MNZ)**.** *If* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive fractions in the nth row (with* $\frac{a}{b}$ *to the left), then among all fractions* $\frac{p}{q}$ *such that* $\frac{a}{b} < \frac{p}{q} < \frac{a'}{b'}$, $\frac{p}{q} = \frac{a+a'}{b+b'}$ *is the unique fraction with smallest denominator.*

Proof. Let $n \in \mathbb{N}_{>0}$ and let $\frac{a}{b}$ and $\frac{a'}{b'}$ be consecutive fractions in the nth row, with $\frac{a}{b}$ to the left. There are 2 possibilities.
Case 1: $b + b' \leq n + 1$. Then $b + b' \geq n + 1$ by Theorem 3, so $b + b' = n + 1$. We have that $\frac{a+a'}{b+b'}$ will be inserted in the $(n+1)$st row between $\frac{a}{b}$ and $\frac{a'}{b'}$.
Case 2: $b + b' > n + 1$. Then for all integers k such that $n \leq k < b + b'$: $\frac{a}{b}$ and $\frac{a'}{b'}$ will be consecutive fractions in the kth row and $\frac{a+a'}{b+b'}$ will be inserted between $\frac{a}{b}$ and $\frac{a'}{b'}$ in the $(b+b')$th row. This is because the kth row is constructed by copying the $(k-1)$st row and inserting fractions between the fractions. If $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive fractions in the $(k-1)$st row and $\frac{a+a'}{b+b'}$ was not inserted between them in the kth row (i.e. $b + b' \nleq k$), then $\frac{a}{b}$ and $\frac{a'}{b'}$ will be consecutive fractions in the kth row as well. If $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive fractions in the $(k-1)$st row then a fraction is only inserted between them in row k if $b + b' \leq k$.

In both cases we get that $\frac{a}{b}$, $\frac{a+a'}{b+b'}$ and $\frac{a'}{b'}$ are 3 consecutive fractions in the $(b+b')$th row and that $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive fractions in the $(b+b'-1)$st row.

Let $\frac{x}{y}$ be a reduced fraction such that $\frac{a}{b} < \frac{x}{y} < \frac{a'}{b'}$. Since $\frac{a}{b}$ and $\frac{a'}{b'}$ are fractions in the nth row, it follows from Corollary 7 that $0 \leq \frac{a}{b} \leq 1$ and $0 \leq \frac{a'}{b'} \leq 1$. Since $\frac{a}{b} < \frac{x}{y} < \frac{a'}{b'}$, we have that $0 \leq \frac{x}{y} \leq 1$. If we assume for contradiction that $y < b+b'$, then by Theorem 6, $\frac{x}{y}$ appears in the $(b+b'-1)$st row. But that's impossible since we know from Theorem 1 that fractions are listed in order of their size and there is no fraction strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ in the $(b+b'-1)$st row. So $y \geq b+b'$, i.e. a reduced fraction that lies strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ can't have a denominator smaller than $b+b'$. Suppose $\frac{x}{y}$ has minimal denominator, i.e. $y = b+b'$. By Theorem 6, $\frac{x}{y}$ exists in the $(b+b')$th row. But since $\frac{a}{b}$, $\frac{a+a'}{b+b'}$ and $\frac{a'}{b'}$ are 3 consecutive fractions in the $(b+b')$th row and the $(b+b')$th row is ordered by size by Theorem 1, the only fraction in the $(b+b')$th row that lies strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ is $\frac{a+a'}{b+b'}$. So $\frac{x}{y}$ must be the fraction $\frac{a+a'}{b+b'}$.

If you have a fraction $\frac{p}{q}$ that lies strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ that is not reduced, then it can't be the case that $q \leq b+b'$, since then you could simplify $\frac{p}{q}$ into a reduced fraction where the denominator is strictly less than $b+b'$, but that's impossible.

Say you have a fraction $\frac{r}{s}$ that lies strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$. Then $s \geq b+b'$, since if $\frac{r}{s}$ is not reduced, then $s > b+b'$ and if $\frac{r}{s}$ is reduced, then $s \geq b+b'$. Also since $\frac{a+a'}{b+b'}$ exists in the $(b+b')$th row it must be a reduced fraction because of Corollary 2. So the smallest denominator $\frac{r}{s}$ can have is $b+b'$, and in that case $\frac{r}{s}$ must be a reduced fraction. So if you have a fraction that lies strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ with smallest denominator, then it must be a reduced fraction with denominator $b+b'$ (since $b+b'$ is the smallest denominator). But the only reduced fraction with denominator $b+b'$ that lies strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$ is $\frac{a+a'}{b+b'}$. So among all fractions that lie strictly between $\frac{a}{b}$ and $\frac{a'}{b'}$, $\frac{a+a'}{b+b'}$ is the unique fraction with smallest denominator. □

Proposition 11 (Problem 6.1.1 in MNZ)**.** *Let n be a positive integer such that $n > 1$ and let $\frac{a}{b}$ and $\frac{a'}{b'}$ be the Farey fractions immediately to the left and the right of $\frac{1}{2}$ respectively in the Farey sequence of order n. Then $b = b' = 1 + 2 \cdot \left\lfloor \frac{n-1}{2} \right\rfloor$, i.e. b is the greatest odd integer $\leq n$. It is also true that $a + a' = b$.*

Proof. By Corollary 7 we know that $\frac{1}{2}$ exists in the Farey sequence of order n since $n \geq 2$. We will prove the Proposition by induction.
Base case ($n = 2$): Then we see from the Farey table on page 3 that $\frac{a}{b}$ is $\frac{0}{1}$ and $\frac{a'}{b'}$ is $\frac{1}{1}$. We have that $1 = 1$, 1 is the greatest odd integer ≤ 2 and $0 + 1 = 1$.
Induction hypothesis: Assume for some $n \geq 2$ that if $\frac{a}{b}$ and $\frac{a'}{b'}$ are the Farey fractions immediately to the left and the right of the fraction $\frac{1}{2}$ respectively in the Farey sequence of order n, then $b = b'$ and b is the greatest odd integer $\leq n$ and $a + a' = b$.
Induction step: Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be the Farey fractions immediately to the left and the right of the fraction $\frac{1}{2}$ respectively in the Farey sequence of order n. Let $\frac{c}{d}$ and $\frac{c'}{d'}$ be the Farey fractions immediately to the left and the right of the fraction $\frac{1}{2}$ respectively in the Farey sequence of order $n+1$. We want to

show that $d = d'$ and that d is the greatest odd integer $\leq n+1$ and $c+c' = d$.
Case 1: $b+2 = b'+2 = n+1$ ($b = b'$ by induction hypothesis). Then when constructing the $(n+1)$st row the fraction $\frac{a+1}{b+2}$ is inserted between $\frac{a}{b}$ and $\frac{1}{2}$ and also the fraction $\frac{a'+1}{b'+2}$ is inserted between $\frac{1}{2}$ and $\frac{a'}{b'}$. Then it must be the case that $\frac{c}{d}$ is the fraction $\frac{a+1}{b+2}$ and $\frac{c'}{d'}$ is the fraction $\frac{a'+1}{b'+2}$, so then $d = b+2$ and $d' = b'+2$. Also from the induction hypothesis we see that b and b' are odd, and since $b+2 = n+1$ we have that $n = b+1$, i.e. n is even. From the induction hypothesis we know that b is the greatest odd integer $\leq n$. Since $d = b+2$ we need to show that $b+2$ is the greatest odd integer $\leq n+1$, but this is obvious, since we know that n is even.
Since $d = b+2$ and $d' = b'+2$ and $b = b'$ by induction hypothesis, it follows that $d = d'$, so

$$d = d'$$

and d is the greatest odd integer $\leq n+1$. We also see that

$$\begin{aligned} c + c' &= a + 1 + a' + 1 && \{c = a+1 \text{ and } c' = a'+1\} \\ &= b + 2 && \{\text{by induction hypothesis}\} \\ &= d. \end{aligned}$$

Case 2: $b+2 = b'+2 > n+1$ ($b = b'$ by induction hypothesis). Then no fraction will be inserted between $\frac{a}{b}$ and $\frac{1}{2}$ or between $\frac{1}{2}$ and $\frac{a'}{b'}$ when constructing the $(n+1)$st row. So $\frac{c}{d}$ is the fraction $\frac{a}{b}$ and $\frac{c'}{d'}$ is the fraction $\frac{a'}{b'}$. We know that

$$\begin{aligned} n + 1 &< b + 2 \\ &\leq n + 2, && \{b \leq n \text{ by Theorem 4}\} \end{aligned}$$

so it follows that $b+2 = n+2$, i.e. $b = n$. But b is odd by the induction hypothesis, so n must be odd. We know from the induction hypothesis that b is the greatest odd integer $\leq n$ and $d = b$. We need to show that b is the greatest odd integer $\leq n+1$ But that is obvious.

Since $b = b'$ by induction hypothesis, $b = d$ and $b' = d'$ it follows that $d = d'$. So

$$d = d'$$

and d is the greatest odd integer $\leq n + 1$.
We also see that

$$\begin{aligned} c + c' &= a + a' && \{a = c \text{ and } a' = c'\} \\ &= b && \{\text{by induction hypothesis}\} \\ &= d. \end{aligned}$$

There is no case where $b + 2 = b' + 2 < n + 1$ because of Theorem 3. □

Theorem 12 (Problem 6.1.9 in MNZ)**.** *For each Farey fraction* $\frac{a}{b}$ *let* $\mathscr{C}\left(\frac{a}{b}\right)$ *denote the circle in the plane of radius* $(2b^2)^{-1}$ *and center* $\left(\frac{a}{b},\ (2b^2)^{-1}\right)$. *These circles are called the Ford circles. The interior of a Ford circle contains no point of any other Ford circle and two Ford circles* $\mathscr{C}\left(\frac{a}{b}\right)$ *and* $\mathscr{C}\left(\frac{a'}{b'}\right)$ *are tangent if and only if* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive Farey fractions of some order.*

Proof. First assume that $\frac{a}{b}$ and $\frac{a'}{b'}$ are distinct Farey fractions such that $\frac{a}{b} < \frac{a'}{b'}$. We want to show that neither of the Ford circles $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ contain a point that is an interior point of the other Ford circle. $\mathscr{C}\left(\frac{a}{b}\right)$ has radius

$$r := \frac{1}{2b^2}$$

and center

$$\left(\frac{a}{b},\ \frac{1}{2b^2}\right) = \left(\frac{a}{b},\ r\right).$$

$\mathscr{C}\left(\frac{a'}{b'}\right)$ has radius

$$s := \frac{1}{2(b')^2}$$

and center

$$\left(\frac{a'}{b'}, \frac{1}{2(b')^2}\right) = \left(\frac{a'}{b'}, s\right).$$

Let d be the distance between the center of $\mathscr{C}\left(\frac{a}{b}\right)$ and the center of $\mathscr{C}\left(\frac{a'}{b'}\right)$. Then

$$d^2 = \left(\frac{a'}{b'} - \frac{a}{b}\right)^2 + (s-r)^2 = \frac{(a'b-ab')^2}{b^2(b')^2} + s^2 + r^2 - 2rs. \tag{7}$$

In order to show that the Ford circles $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ don't contain any interior points of the other Ford circle, it is sufficient to show that $d \geq r+s$, i.e. $d^2 \geq r^2 + s^2 + 2rs$. Using (7) we see that we must show that

$$\begin{aligned} &\frac{(a'b-ab')^2}{b^2(b')^2} + s^2 + r^2 - 2rs \geq r^2 + s^2 + 2rs \\ \iff &\frac{(a'b-ab')^2}{b^2(b')^2} \geq 4rs = \frac{1}{b^2(b')^2} \\ \iff &(a'b-ab')^2 \geq 1. \end{aligned} \tag{8}$$

It can't be the case that $a'b - ab' = 0$, since then

$$\begin{aligned} &a'b - ab' = 0 \\ \iff &a'b = ab' \\ \iff &\frac{a'}{b'} = \frac{a}{b}, \qquad \{\text{divide both sides by } bb'\} \end{aligned}$$

which contradicts the Farey fractions being distinct. But then $a'b - ab'$ is a non-zero integer, hence its square must be greater than or equal to 1, i.e. (8) is true. So the Ford circles $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ don't contain any interior points of the other Ford circle and $d \geq r + s$.

We want to show that $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are tangent iff $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive Farey fractions of some order. If $\frac{a}{b}$ and $\frac{a'}{b'}$ are the same Farey fraction, then the Ford circles $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are the same and are therefore not tangent and $\frac{a}{b}$ and $\frac{a'}{b'}$ are not consecutive Farey fractions of some order, so the Theorem holds in that case. So assume $\frac{a}{b}$ and $\frac{a'}{b'}$ are distinct. Without loss of generality assume that $\frac{a}{b} < \frac{a'}{b'}$ like before. We know that $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are tangent if and only if $d = r + s$ or $d = |r - s|$. Since r and s are positve, it follows that $|r - s| \leq \max(r,\ s) < r + s$. But before we showed that $d \geq r + s$, so it can't be the case that $d = |r - s|$. Therefore $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are tangent if and only if $d = r + s$. But $d = r + s$ iff $d^2 = r^2 + s^2 + 2rs$. Using (7) we see that

$$
\begin{aligned}
& d^2 = r^2 + s^2 + 2rs \\
\iff & \frac{(a'b - ab')^2}{b^2(b')^2} + s^2 + r^2 - 2rs = r^2 + s^2 + 2rs \\
\iff & \frac{(a'b - ab')^2}{b^2(b')^2} = 4rs = \frac{1}{b^2(b')^2} \\
\iff & (a'b - ab')^2 = 1 \\
\iff & a'b - ab' = \pm 1.
\end{aligned}
$$

But it can easily be shown that $\frac{a}{b} < \frac{a'}{b'}$ iff $a'b - ab' > 0$, and since we assumed that $\frac{a}{b} < \frac{a'}{b'}$, it must be the case that $a'b - ab' > 0$. Therefore

$$a'b - ab' = \pm 1 \iff a'b - ab' = 1.$$

From Corollary 9 it follows that $a'b - ab' = 1 \iff \frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive

Farey fractions of some order. So $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are tangent if and only if $\frac{a}{b}$ and $\frac{a'}{b'}$ are consecutive Farey fractions of some order. $\square$

Remark 1. *We will now use an alternative method to show that if* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are consecutive Farey fractions of some order, then* $\mathscr{C}\left(\frac{a}{b}\right)$ *and* $\mathscr{C}\left(\frac{a'}{b'}\right)$ *are tangent. When using this method we will find explicitly the coordinates for the intersection of the Ford circles.*

Proof. Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be consecutive Farey fractions of order n, with $\frac{a}{b} < \frac{a'}{b'}$. We want to show that $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are tangent. The equation for $\mathscr{C}\left(\frac{a}{b}\right)$ is

$$\left(x - \frac{a}{b}\right)^2 + \left(y - \frac{1}{2b^2}\right)^2 = \frac{1}{4b^4} \tag{9}$$

and the equation for $\mathscr{C}\left(\frac{a'}{b'}\right)$ is

$$\left(x - \frac{a'}{b'}\right)^2 + \left(y - \frac{1}{2(b')^2}\right)^2 = \frac{1}{4(b')^4}. \tag{10}$$

Then the x-coordinate of center of $\mathscr{C}\left(\frac{a}{b}\right)$ is less than (i.e. to the left of) the x-coordinate of the center of $\mathscr{C}\left(\frac{a'}{b'}\right)$. Let L be the line that goes through the center of $\mathscr{C}\left(\frac{a}{b}\right)$ and the center of $\mathscr{C}\left(\frac{a'}{b'}\right)$. It's slope is

$$\begin{aligned}
&\left(\frac{1}{2(b')^2} - \frac{1}{2b^2}\right) / \left(\frac{a'}{b'} - \frac{a}{b}\right) \\
&= \left(\frac{b^2 - (b')^2}{2b^2(b')^2}\right) / \left(\frac{a'b - ab'}{bb'}\right) \\
&= \left(\frac{b^2 - (b')^2}{2b^2(b')^2}\right) / \left(\frac{1}{bb'}\right) \qquad \{\text{by Theorem 1}\} \\
&= \frac{b^2 - (b')^2}{2bb'}.
\end{aligned}$$

The equation for L is $y - \frac{1}{2(b')^2} = \frac{b^2 - (b')^2}{2bb'}\left(x - \frac{a'}{b'}\right)$. We want to see where L intersects $\mathscr{C}\left(\frac{a'}{b'}\right)$, so we plug the equation for L into (10) and get

$$\begin{aligned}
&\left(x - \frac{a'}{b'}\right)^2 + \left(\frac{b^2 - (b')^2}{2bb'}\right)^2 \left(x - \frac{a'}{b'}\right)^2 = \frac{1}{4(b')^4} \\
\iff &\left(1 + \left(\frac{b^2 - (b')^2}{2bb'}\right)^2\right) \left(x - \frac{a'}{b'}\right)^2 = \frac{1}{4(b')^4} \\
\iff &\left(\frac{4b^2(b')^2}{4b^2(b')^2} + \frac{(b^2)^2 - 2b^2(b')^2 + ((b')^2)^2}{4b^2(b')^2}\right) \left(x - \frac{a'}{b'}\right)^2 = \frac{1}{4(b')^4} \\
\iff &\left(\frac{b^2 + (b')^2}{2bb'}\right)^2 \left(x - \frac{a'}{b'}\right)^2 = \frac{1}{4(b')^4} \\
\iff &\left(x - \frac{a'}{b'}\right)^2 = \frac{b^2}{(b')^2\left(b^2 + (b')^2\right)^2} \\
\iff &x = \frac{a'}{b'} \pm \sqrt{\frac{b^2}{(b')^2(b^2 + (b')^2)^2}} \\
\iff &x = \frac{a'}{b'} \pm \frac{b}{b'(b^2 + (b')^2)}.
\end{aligned}$$

We only care about the left intersection, i.e. where $x = \frac{a'}{b'} - \frac{b}{b'(b^2 + (b')^2)}$.

If we plug this x-value into the equation for L and solve for y we get

$$\begin{aligned} y &= \frac{1}{2(b')^2} + \frac{b^2-(b')^2}{2bb'} \cdot \frac{-b}{b'(b^2+(b')^2)} \\ &= \frac{1}{2(b')^2} - \frac{b^2-(b')^2}{2(b')^2(b^2+(b')^2)} \\ &= \frac{b^2+(b')^2}{2(b')^2(b^2+(b')^2)} - \frac{b^2-(b')^2}{2(b')^2(b^2+(b')^2)} \\ &= \frac{1}{b^2+(b')^2}. \end{aligned}$$

So $x = \dfrac{a'}{b'} - \dfrac{b}{b'(b^2+(b')^2)}$ and $y = \dfrac{1}{b^2+(b')^2}$ is the left intersection of L and $\mathscr{C}\left(\dfrac{a'}{b'}\right)$. We want to show that this point also belongs to $\mathscr{C}\left(\frac{a}{b}\right)$. Plugging the x-value and y-value into the left hand side of (9) gives

$$
\begin{aligned}
&\left(\frac{a'}{b'} - \frac{b}{b'(b^2+(b')^2)} - \frac{a}{b}\right)^2 + \left(\frac{1}{b^2+(b')^2} - \frac{1}{2b^2}\right)^2 \\
&= \left(\frac{a'b - ab'}{bb'} - \frac{b}{b'(b^2+(b')^2)}\right)^2 \\
&+ \left(\frac{2b^2}{2b^2(b^2+(b')^2)} - \frac{b^2+(b')^2}{2b^2(b^2+(b')^2)}\right)^2 \\
&= \left(\frac{1}{bb'} - \frac{b}{b'(b^2+(b')^2)}\right)^2 + \left(\frac{b^2-(b')^2}{2b^2(b^2+(b')^2)}\right)^2 \qquad \{\text{by Theorem 1}\} \\
&= \left(\frac{b^2+(b')^2}{bb'(b^2+(b')^2)} - \frac{b^2}{bb'(b^2+(b')^2)}\right)^2 \\
&+ \left(\frac{b^2-(b')^2}{2b^2(b^2+(b')^2)}\right)^2 \\
&= \left(\frac{2b(b')^3}{2b^2(b')^2(b^2+(b')^2)}\right)^2 + \left(\frac{b^2(b')^2-(b')^4}{2b^2(b')^2(b^2+(b')^2)}\right)^2 \\
&= \frac{4b^2(b')^6 + b^4(b')^4 - 2b^2(b')^6 + (b')^8}{4b^4(b')^4(b^2+(b')^2)^2} \\
&= \frac{(b')^4(b^2+(b')^2)^2}{4b^4(b')^4(b^2+(b')^2)^2} \\
&= \frac{1}{4b^4},
\end{aligned}
$$

which means that $\left(\frac{a'}{b'} - \frac{b}{b'(b^2+(b')^2)}, \frac{1}{b^2+(b')^2}\right)$ belongs to $\mathscr{C}\left(\frac{a}{b}\right)$, $\mathscr{C}\left(\frac{a'}{b'}\right)$ and L, so it must be the case that $\mathscr{C}\left(\frac{a}{b}\right)$ and $\mathscr{C}\left(\frac{a'}{b'}\right)$ are tangent. □

Theorem 13 (Problem 6.1.2 in MNZ)**.** *The number of Farey fractions* $\frac{a}{b}$ *of order* n *satisfying the inequalities* $0 \le \frac{a}{b} \le 1$ *is* $1 + \sum_{j=1}^{n} \phi(j)$ *and their sum is exactly half this value.*

Proof. We want to prove the first part of the theorem. Let n be a positive integer. Every Farey fraction $\frac{a}{b}$ of order n must satisfy the inequalities

$0 \leq \frac{a}{b} \leq 1$ by Corollary 7, so the Farey fractions $\frac{a}{b}$ of order n satisfying the inequalities $0 \leq \frac{a}{b} \leq 1$ are precisely the Farey fractions of order n.

By Corollary 7 the Farey fractions of order n are precisely the reduced fractions $\frac{a}{b}$ such that $0 \leq \frac{a}{b} \leq 1$ and $0 < b \leq n$. For integers j such that $1 \leq j \leq n$, define

$$A(j) := \text{the number of reduced fractions } \frac{a}{j} \text{ such that } 0 \leq \frac{a}{j} \leq 1.$$

Also let

$$F(n) := \text{the number of Farey fractions of order } n.$$

Then clearly $F(n) = \sum_{j=1}^{n} A(j)$. It is clear that $A(j)$ is the number of Farey fractions of order n with denominator j. We have that $A(1) = 2$, because $\frac{0}{1}$ and $\frac{1}{1}$ are the only reduced fractions with denominator 1 that lie between 0 and 1. The definition of $A(j)$ can be rewritten by multiplying the last inequalities by j and using the definition of reduced fraction to obtain

$$A(j) = \text{the number of fractions } \frac{a}{j} \text{ such that } gcd(a,\ j) = 1 \text{ and } 0 \leq a \leq j.$$

Different values for a give different fractions $\frac{a}{j}$, so we can rewrite $A(j)$ as

$$A(j) = \text{the number of integers } a \text{ such that } gcd(a,\ j) = 1 \text{ and } 0 \leq a \leq j.$$

For $j \geq 2$ we have that $gcd(0,\ j) = j \neq 1$, so for $j \geq 2$ we can rewrite $A(j)$ as

$$\begin{aligned} A(j) &= \text{the number of integers } a \text{ such that } gcd(a,\ j) = 1 \text{ and } 1 \leq a \leq j \\ &= \phi(j). \end{aligned}$$

So

$$\begin{aligned} F(n) = \sum_{j=1}^{n} A(j) = A(1) + \sum_{j=2}^{n} A(j) &= 2 + \sum_{j=2}^{n} \phi(j) \\ &= 1 + \phi(1) + \sum_{j=2}^{n} \phi(j) \\ &= 1 + \sum_{j=1}^{n} \phi(j). \end{aligned}$$

We want to prove the second part of the theorem. Let $S(n)$ be the sum of the Farey fractions of order n. We want to show that $S(n) = \frac{1}{2}\left(1 + \sum_{j=1}^{n} \phi(j)\right)$.

$$S(n) = \sum_{\frac{a}{b} \,\middle|\, \frac{a}{b} \text{ is a Farey fraction of order } n} \frac{a}{b}.$$

If $\frac{a}{b}$ is a Farey fraction of order n, then $1 - \frac{a}{b} = \frac{b-a}{b}$ is also a Farey fraction of order n. This is because if $\frac{a}{b}$ is Farey fraction of order n then by Corollary 7 we have that $0 \leq a \leq b$, so $0 \leq b - a \leq b$ and if $gcd(a,\ b) = 1$, then $gcd(b-a,\ b) = 1$ and hence by Corollary 7, $\frac{b-a}{b}$ is also a Farey fraction of order n. If $b \geq 3$, then it can't be the case that $\frac{a}{b} = \frac{b-a}{b}$ if $\frac{a}{b}$ is a Farey fraction of order n, since that would imply $2a = b$, i.e. $a|b$, but also $a = \frac{b}{2} > 1$, so that a is a common divisor of a and b that is strictly greater than 1, which means that $gcd(a,\ b) \neq 1$. So

$$S(n) = \sum_{\frac{a}{b} \,\middle|\, \frac{a}{b} \text{ is a Farey fraction of order } n \text{ and } b \leq 2} \frac{a}{b} + \sum_{\frac{a}{b} \,\middle|\, \frac{a}{b} \text{ is a Farey fraction of order } n \text{ and } b > 2} \frac{a}{b}.$$

The terms in the right sum can be paired up as $\left(\frac{a}{b},\ \frac{b-a}{b}\right)$. The terms in each pair add up to 1. If we assume $n \geq 2$ then the only Farey fractions of order n with denominator ≤ 2 are $\frac{0}{1}$, $\frac{1}{1}$ and $\frac{1}{2}$, so there are $1 + \sum_{j=1}^{n} \phi(j) - 3$ terms being added in the right sum, so there are $\frac{1}{2}\left(\sum_{j=1}^{n} \phi(j) - 2\right)$ pairs being summed in the second sum. Thus the right sum equals $\frac{1}{2}\left(\sum_{j=1}^{n} \phi(j) - 2\right)$. The left sum equals $\frac{3}{2}$, so

$$S(n) = \frac{3}{2} + \frac{1}{2}\left(\sum_{j=1}^{n} \phi(j) - 2\right) = \frac{1}{2}\left(1 + \sum_{j=1}^{n} \phi(j)\right).$$

If $n = 1$, then $S(n) = \frac{0}{1} + \frac{1}{1} = 1 = \frac{1}{2}\left(1 + \sum_{j=1}^{1} \phi(j)\right)$, so the theorem holds for all n. $\square$

3 Rational Approximations

Theorem 14 (Theorem 6.7 in MNZ)**.** *Let* $\frac{a}{b}$ *and* $\frac{c}{d}$ *be Farey fractions of order* n *such that no other Farey fraction of order* n *lies between them. Then*

$$\left|\frac{a}{b} - \frac{a+c}{b+d}\right| = \frac{1}{b(b+d)} \leq \frac{1}{b(n+1)}$$

and

$$\left|\frac{c}{d} - \frac{a+c}{b+d}\right| = \frac{1}{d(b+d)} \leq \frac{1}{d(n+1)}.$$

Proof. For the first formula we have

$$\begin{aligned}
\left|\frac{a}{b} - \frac{a+c}{b+d}\right| &= \left|\frac{a(b+d) - b(a+c)}{b(b+d)}\right| \\
&= \frac{|ad - bc|}{b(b+d)} \\
&= \frac{1}{b(b+d)} && \{\text{by Theorem 1}\} \\
&\leq \frac{1}{b(n+1)}. && \{\text{since } b+d \geq n+1 \text{ by Theorem 3}\}
\end{aligned}$$

The second formula is obtained in a similar way. $\square$

Theorem 15 (Theorem 6.8 in MNZ)**.** *Let* $n \in \mathbb{N}_{>0}$ *and* $x \in \mathbb{R}$*. Then there is a rational number* $\frac{a}{b}$ *such that* $0 < b \leq n$ *and*

$$\left|x - \frac{a}{b}\right| \leq \frac{1}{b(n+1)}.$$

Proof. Let $n \in \mathbb{N}_{>0}$ and $x \in \mathbb{R}$. Let k be the unique integer such that $0 \leq x + k < 1$. There are 2 cases.

Case 1: $x+k$ has the same value as some Farey fraction of order n. Let $\frac{p}{q}$ be the Farey fraction of order n that $x+k$ simplifies to. Let the rational number $\frac{a}{b}$ be $\frac{p-k\cdot q}{q}$. Since $\frac{p}{q}$ is a Farey fraction of order n, $q \leq n$ by Theorem 4. So $\frac{a}{b}$ is a rational number such that $0 < b \leq n$ and

$$\begin{aligned}\left|x - \frac{a}{b}\right| = \left|x - \frac{p-k\cdot q}{q}\right| = \left|x + k - \frac{p-k\cdot q}{q} - \frac{k\cdot q}{q}\right| &= \left|\frac{p}{q} - \frac{p}{q}\right| \\ &= 0 \\ &\leq \frac{1}{b(n+1)}.\end{aligned}$$

Case 2: $x+k$ does not have the same value as some Farey fraction of order n. It can't be the case that $x+k = 0$, since $\frac{0}{1}$ is a Farey fraction of order n. So $0 < x+k < 1$. Let $\frac{p}{q}$ be the greatest Farey fraction of order n that is smaller than $x+k$ and let $\frac{r}{s}$ be the smallest Farey fraction of order n that is greater than $x+k$. There is no Farey fraction of order n that lies between $\frac{p}{q}$ and $\frac{r}{s}$. Since $\frac{p}{q}$ and $\frac{r}{s}$ are Farey fractions of order n, it follows from Theorem 4 that $q \leq n$ and $s \leq n$. There are 2 cases now.

Case 2.1: $\frac{p+r}{q+s} \leq x+k$. Let the rational number $\frac{a}{b}$ be $\frac{r-k\cdot s}{s}$. We have that $\frac{a}{b}$ is a rational number such that $0 < b \leq n$ and

$$\begin{aligned}
\left|x - \frac{a}{b}\right| = \left|x - \frac{r - k \cdot s}{s}\right| &= \left|x + k - \frac{r}{s}\right| \\
&= \left|\frac{r}{s} - (x + k)\right| \\
&\leq \left|\frac{r}{s} - \frac{p + r}{q + s}\right| \qquad \left\{\text{since } \frac{p + r}{q + s} \leq x + k < \frac{r}{s}\right\} \\
&\leq \frac{1}{s(n + 1)} \qquad \{\text{by Theorem 14}\} \\
&= \frac{1}{b(n + 1)}.
\end{aligned}$$

Case 2.2: $\frac{p + r}{q + s} > x + k$. Let the rational number $\frac{a}{b}$ be $\frac{p - k \cdot q}{q}$. We have that $\frac{a}{b}$ is a rational number such that $0 < b \leq n$ and

$$\begin{aligned}
\left|x - \frac{a}{b}\right| = \left|x - \frac{p - k \cdot q}{q}\right| &= \left|x + k - \frac{p}{q}\right| \\
&\leq \left|\frac{p + r}{q + s} - \frac{p}{q}\right| \qquad \left\{\text{since } \frac{p}{q} < x + k < \frac{p + r}{q + s}\right\} \\
&= \left|\frac{p}{q} - \frac{p + r}{q + s}\right| \\
&\leq \frac{1}{q(n + 1)} \qquad \{\text{by Theorem 14}\} \\
&= \frac{1}{b(n + 1)}.
\end{aligned}$$

□

Theorem 16 (Theorem 6.9 in MNZ)**.** *If $\xi \in \mathbb{R} \setminus \mathbb{Q}$, then there are infinitely many distinct rational numbers $\frac{a}{b}$ such that*

$$\left|\xi - \frac{a}{b}\right| < \frac{1}{b^2}.$$

Proof. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$. For each $n \in \mathbb{N}_{>0}$, let

$$A_n := \left\{ \frac{a}{b} \in \mathbb{Q} \middle| \ 0 < b \leq n \ \wedge \ \left| \xi - \frac{a}{b} \right| \leq \frac{1}{b(n+1)} \right\}.$$

Let $X := \{A_n | \ n \in \mathbb{N}_{>0}\}$ By Theorem 15, each A_n is non-empty, so that X is a set of non-empty sets. By the axiom of choice, there exists a choice function $f : X \to \cup X$ such that $\forall A \in X : f(A) \in A$. So let $f : X \to \cup X$ be a function such that $\forall A \in X : f(A) \in A$. For each $n \in \mathbb{N}_{>0}$, let $\frac{a_n}{b_n}$ be the rational number $f(A_n)$. Then

$$\begin{aligned} \left| \xi - \frac{a_n}{b_n} \right| &\leq \frac{1}{b_n(n+1)} \\ &< \frac{1}{b_n^2} && \{\text{since } n+1 > b_n\} \end{aligned}$$

for all $n \in \mathbb{N}_{>0}$.

We want to show there are infinitely many distinct $\frac{a_n}{b_n}$. Assume for contradiction there are only finitely many distinct $\frac{a_n}{b_n}$. Then there are only finitely many distinct values for $\left| \xi - \frac{a_n}{b_n} \right|$. Let $d := \min_{n \in \mathbb{N}_{>0}} \left| \xi - \frac{a_n}{b_n} \right|$. Clearly $d \geq 0$ and $d \neq 0$, since $d = 0$ would imply that $\xi = \frac{a_k}{b_k}$ for some $k \in \mathbb{N}_{>0}$, but that contradicts ξ being irrational. So $d > 0$. Then for all $n \in \mathbb{N}_{>0}$ we have that

$$\begin{aligned} d \leq \left| \xi - \frac{a_n}{b_n} \right| &\leq \frac{1}{b_n(n+1)} \\ &\leq \frac{1}{n+1}. && \{\text{since } b_n \geq 1\} \end{aligned}$$

But if we let n be sufficiently large ($n \geq \frac{1}{d}$ will work), then $\frac{1}{n+1} < \frac{1}{n} \leq d$. But then $d < d$, a contradiction. □

Lemma 17 (Lemma 6.10 in MNZ)**.** *If x and y are positive integers then not both of the inequalities* $\frac{1}{xy} \geq \frac{1}{\sqrt{5}} \left(\frac{1}{x^2} + \frac{1}{y^2} \right)$ *and* $\frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}} \left(\frac{1}{x^2} + \frac{1}{(x+y)^2} \right)$ *can hold.*

Proof. We will rewrite the inequalities. We see that

$$\begin{aligned}
&\frac{1}{xy} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{y^2}\right) \\
&\iff \frac{1}{xy} \geq \frac{1}{\sqrt{5}} \cdot \frac{y^2+x^2}{(xy)^2} \\
&\iff \sqrt{5}xy \geq y^2+x^2 \qquad \left\{\text{multiply both sides by } \sqrt{5}(xy)^2\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{(x+y)^2}\right) \\
&\iff \frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}} \cdot \frac{(x+y)^2+x^2}{(x(x+y))^2} \\
&\iff \sqrt{5}x(x+y) \geq (x+y)^2+x^2. \qquad \left\{\text{multiply both sides by } \sqrt{5}(x(x+y))^2\right\}
\end{aligned}$$

Assume for contradiction that both

$$\sqrt{5}xy \geq y^2+x^2 \tag{11}$$

and

$$\sqrt{5}x(x+y) \geq (x+y)^2+x^2 \tag{12}$$

are true. By adding the inequalities (11) and (12) we get

$$\begin{aligned}
&\sqrt{5}(x^2+2xy) \geq 3x^2+2xy+2y^2 \\
&\iff 3x^2+2xy+2y^2 \leq \sqrt{5}(x^2+2xy) \\
&\iff (3-\sqrt{5})x^2 - 2(\sqrt{5}-1)xy + 2y^2 \leq 0 \\
&\iff (5-2\sqrt{5}+1)x^2 - 4(\sqrt{5}-1)xy + 4y^2 \leq 0 \quad \{\text{multiply by } 2\} \\
&\iff (2y-(\sqrt{5}-1)x)^2 \leq 0 \\
&\iff 2y-(\sqrt{5}-1)x = 0 \qquad \{\text{since squares are non-negative}\} \\
&\iff \sqrt{5} = \frac{2y+x}{x}, \qquad \left\{\text{solve for } \sqrt{5}\right\}
\end{aligned}$$

but that contradicts $\sqrt{5}$ being irrational. □

Lemma 18. *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions of order n with $\frac{a}{b}$ to the left, then $\frac{c}{d} - \frac{a}{b} \leq \frac{1}{n}$.*

Proof. The fractions

$$\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n} \tag{13}$$

are Farey fractions of order n if they are simplified by Corollary 7 (the simplified fractions lie between 0 and 1 and the denominator of each simplified fraction lies between 1 and n). It can't be the case that $\frac{a}{b} < \frac{j}{n}$ and $\frac{c}{d} > \frac{j}{n}$ for some $\frac{j}{n}$ in the sequence (13), since then $\frac{j}{n}$ would be a Farey fraction of order n (if simplified) that lies strictly between $\frac{a}{b}$ and $\frac{c}{d}$, which would contradict $\frac{a}{b}$ and $\frac{c}{d}$ being consecutive Farey fractions of order n. So $\frac{a}{b}$ and $\frac{c}{d}$ must both lie between 2 consecutive elements in (13), which means $\left|\frac{a}{b} - \frac{c}{d}\right| \leq \frac{1}{n}$, since the distance between consecutive elements of (13) is always $\frac{1}{n}$. □

Theorem 19 (Theorem 6.11 in MNZ). *Given any $\xi \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many different rational numbers $\frac{h}{k}$ such that*

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}.$$

Proof. Let μ be the unique integer such that $0 \leq \xi + \mu < 1$. It can't be the case that $\xi + \mu = 0$, since that would imply that ξ is rational, which is not the case. So $0 < \xi + \mu < 1$. Let $\lambda := \xi + \mu$. Then $0 < \lambda < 1$. Let n be a positive integer. Let $\frac{a_n}{b_n}$ be the greatest Farey fraction of order n that is less than λ and let $\frac{c_n}{d_n}$ be the smallest Farey fraction of order n that is greater than λ. Then $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ are consecutive fractions in the nth row. We want to show that at least one of the numbers $\frac{a_n}{b_n}$, $\frac{c_n}{d_n}$ and $\frac{a_n+c_n}{b_n+d_n}$ will work as $\frac{h}{k}$ in the theorem (when $\xi = \lambda$). Assume for contradiction this is not the case. Then

$$\lambda - \frac{a_n}{b_n} \geq \frac{1}{\sqrt{5}b_n^2}, \tag{14}$$

$$\frac{c_n}{d_n} - \lambda \geq \frac{1}{\sqrt{5}d_n^2} \tag{15}$$

and

$$\left|\lambda - \frac{a_n+c_n}{b_n+d_n}\right| \geq \frac{1}{\sqrt{5}(b_n+d_n)^2}. \tag{16}$$

Case 1: $\frac{a_n+c_n}{b_n+d_n} < \lambda$. Then (16) becomes

$$\lambda - \frac{a_n+c_n}{b_n+d_n} \geq \frac{1}{\sqrt{5}(b_n+d_n)^2}.$$

By adding the inequalities (14) and (15) we get

$$\begin{aligned}
&\frac{c_n}{d_n} - \frac{a_n}{b_n} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{d_n^2}\right) \\
&\iff \frac{b_nc_n - a_nd_n}{b_nd_n} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{d_n^2}\right) \\
&\iff \frac{1}{b_nd_n} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{d_n^2}\right) \qquad \{\text{by Theorem 1}\}
\end{aligned} \tag{17}$$

and by adding the inequalities (15) and (16) we get

$$\begin{aligned}
&\frac{c_n}{d_n} - \frac{a_n+c_n}{b_n+d_n} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{d_n^2} + \frac{1}{(b_n+d_n)^2}\right) \\
&\iff \frac{c_n(b_n+d_n) - d_n(a_n+c_n)}{d_n(b_n+d_n)}) \geq \frac{1}{\sqrt{5}}\left(\frac{1}{d_n^2} + \frac{1}{(b_n+d_n)^2}\right) \\
&\iff \frac{b_nc_n - a_nd_n}{d_n(b_n+d_n)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{d_n^2} + \frac{1}{(b_n+d_n)^2}\right) \\
&\iff \frac{1}{d_n(b_n+d_n)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{d_n^2} + \frac{1}{(b_n+d_n)^2}\right). \qquad \{\text{by Theorem 1}\}
\end{aligned} \tag{18}$$

But not both the inequalities (17) and (18) can be true by Lemma 17 (if we let $x = d_n$ and $y = b_n$ in the lemma), so we get a contradiction.
Case 2: $\dfrac{a_n + c_n}{b_n + d_n} > \lambda$. Then (16) becomes

$$\frac{a_n + c_n}{b_n + d_n} - \lambda \geq \frac{1}{\sqrt{5}(b_n + d_n)^2}.$$

Just like in case 1 we can add the inequalities (14) and (15) to get (17). By adding the inequalities (14) and (16) we get

$$\begin{aligned}
&\frac{a_n + c_n}{b_n + d_n} - \frac{a_n}{b_n} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{(b_n + d_n)^2}\right)\\
&\iff \frac{b_n(a_n + c_n) - a_n(b_n + d_n)}{b_n(b_n + d_n)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{(b_n + d_n)^2}\right)\\
&\iff \frac{b_n c_n - a_n d_n}{b_n(b_n + d_n)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{(b_n + d_n)^2}\right)\\
&\iff \frac{1}{b_n(b_n + d_n)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_n^2} + \frac{1}{(b_n + d_n)^2}\right). \qquad \{\text{by Theorem 1}\}
\end{aligned} \tag{19}$$

But not both inequalities (17) and (19) can be true by Lemma 17 (if we let $x = b_n$ and $y = d_n$ in the lemma), so we get a contradiction.

Thus one of the numbers $\dfrac{a_n}{b_n}$, $\dfrac{c_n}{d_n}$ and $\dfrac{a_n + c_n}{b_n + d_n}$ will work as $\dfrac{h}{k}$ in the theorem (when $\xi = \lambda$). So for each $n \in \mathbb{N}_{>0}$, let $\dfrac{h_n}{k_n}$ be one of the fractions $\dfrac{a_n}{b_n}$, $\dfrac{c_n}{d_n}$ and $\dfrac{a_n + c_n}{b_n + d_n}$ such that

$$\left|\lambda - \frac{h_n}{k_n}\right| < \frac{1}{\sqrt{5}k_n^2},$$

where $\dfrac{a_n}{b_n}$ is the greatest Farey fraction of order n that is less than λ and $\dfrac{c_n}{d_n}$ is the smallest Farey fraction of order n that is greater than λ.

We want to show that for every $\epsilon > 0$ there exists a Farey fraction $\dfrac{h_n}{k_n}$ (as defined above) such that $\left|\lambda - \frac{h_n}{k_n}\right| < \epsilon$. Let $\epsilon > 0$ and let n be a positive integer such that $\frac{1}{n} < \epsilon$ (i.e. $n > \frac{1}{\epsilon}$). We know that $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ are consecutive fractions of order n, so we know by Lemma 18 that $\frac{c_n}{d_n} - \frac{a_n}{b_n} \leq \frac{1}{n} < \epsilon$. Clearly $\left|\lambda - \frac{a_n}{b_n}\right| < \epsilon$ and $\left|\lambda - \frac{c_n}{d_n}\right| < \epsilon$. Also we know that $\frac{a_n}{b_n} < \frac{a_n + c_n}{b_n + d_n} < \frac{c_n}{d_n}$, so $\left|\lambda - \frac{h_n}{k_n}\right| < \epsilon$, since $\frac{h_n}{k_n}$ is one of the fractions $\frac{a_n}{b_n}$, $\frac{c_n}{d_n}$ and $\frac{a_n + c_n}{b_n + d_n}$.

We want to show that there are infinitely many $\frac{h_n}{k_n}$ with distinct values. Assume for contradiction that there are only finitely many $\frac{h_n}{k_n}$. Then let $A := \left\{ \frac{h_n}{k_n} \middle| n \in \mathbb{N}_{>0} \right\}$ and $\epsilon := \min_{\frac{h_n}{k_n} \in A} \left|\lambda - \frac{h_n}{k_n}\right|$. But as we've shown there is a Farey fraction $\frac{h_n}{k_n}$ such that $\left|\lambda - \frac{h_n}{k_n}\right| < \epsilon$, which contradicts ϵ being minimal. So there are infinitely many $\frac{h_n}{k_n}$ with distinct values.

For every $n \in \mathbb{N}_{>0}$:

$$\begin{aligned}
&\left|\lambda - \frac{h_n}{k_n}\right| < \frac{1}{\sqrt{5}k_n^2} \\
\iff &\left|\lambda - \mu - \frac{h_n - \mu \cdot k_n}{k_n}\right| < \frac{1}{\sqrt{5}k_n^2} \\
\iff &\left|\xi - \frac{h_n - \mu \cdot k_n}{k_n}\right| < \frac{1}{\sqrt{5}k_n^2}
\end{aligned}$$

and there are infinitely many $\frac{h_n - \mu \cdot k_n}{k_n}$ with distinct values since there are infinitely many $\frac{h_n}{k_n}$ with distinct values. This is because $\frac{h_n - \mu \cdot k_n}{k_n} = \frac{h_n}{k_n} - \mu$ and the function $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto x - \mu$ is bijective.

□

Theorem 20 (Theorem 6.12 in MNZ)**.** *The constant* $\sqrt{5}$ *in Theorem 19 is the best possible, i.e. Theorem 19 does not hold if* $\sqrt{5}$ *is replaced by any larger value.*

Proof. It's enough to find one ξ such that $\sqrt{5}$ cannot be replaced by any larger value in Theorem 19. Let $\xi := \frac{1+\sqrt{5}}{2}$. We see that

$$(x-\xi)\left(x-\frac{1-\sqrt{5}}{2}\right)=\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right)=x^2-x-1.$$

So if we let h and k be integers with $k>0$, then

$$\begin{aligned}&\left|\frac{h}{k}-\xi\right|\left|\frac{h}{k}-\xi+\sqrt{5}\right| &(20)\\ &=\left|\left(\frac{h}{k}-\xi\right)\left(\frac{h}{k}-\frac{1-\sqrt{5}}{2}\right)\right|\\ &=\left|\frac{h^2}{k^2}-\frac{h}{k}-1\right|\\ &=\frac{1}{k^2}\left|h^2-hk-k^2\right|. &(21)\end{aligned}$$

We know that (20) can't be zero since that would imply $\xi=\frac{h}{k}$ or $\frac{h}{k}=\xi-\sqrt{5}$, but neither ξ nor $\xi-\sqrt{5}$ are rational. We know that $\left|h^2-hk-k^2\right|$ is a non-negative integer. It can't be the case that $\left|h^2-hk-k^2\right|=0$, since then equation (21) is zero and hence equation (20) is zero. Since $\left|h^2-hk-k^2\right|\geq 1$ it must be the case that $\frac{1}{k^2}\left|h^2-hk-k^2\right|\geq\frac{1}{k^2}$ and because equation (21) is equal to equation (20) it follows that

$$\left|\frac{h}{k}-\xi\right|\left|\frac{h}{k}-\xi+\sqrt{5}\right|\geq\frac{1}{k^2}. \tag{22}$$

Let m be a positive real number and assume we have an infinite sequence of rational numbers $\frac{h_j}{k_j}$, $k_j>0$ such that

$$\left|\frac{h_j}{k_j}-\xi\right|<\frac{1}{mk_j^2}. \tag{23}$$

If we multiply by k_j on both sides of (23) we get

$$|h_j - k_j\xi| < \frac{1}{mk_j}.$$

From this we can see that

$$k_j\xi - \frac{1}{mk_j} < h_j < k_j\xi + \frac{1}{mk_j},$$

so that for each rational number $\frac{h_j}{k_j}$ in the sequence there can only be finitely many other fractions in the sequence with the same denominator, but different numerator. It follows that $\lim_{j\to\infty} k_j = \infty$.

We see that

$$\begin{aligned}
&\frac{1}{k_j^2} \\
&\leq \left|\frac{h_j}{k_j} - \xi\right| \left|\frac{h_j}{k_j} - \xi + \sqrt{5}\right| && \{\text{by (22)}\} \\
&< \frac{1}{mk_j^2}\left|\frac{h_j}{k_j} - \xi + \sqrt{5}\right| && \{\text{by (23)}\} \\
&\leq \frac{1}{mk_j^2}\left(\left|\frac{h_j}{k_j} - \xi\right| + \sqrt{5}\right) && \{\text{by } \Delta\text{-inequality}\} \\
&< \frac{1}{mk_j^2}\left(\frac{1}{mk_j^2} + \sqrt{5}\right). && \{\text{by (23)}\}
\end{aligned}$$

Multiplying by mk_j^2 we get

$$m < \frac{1}{mk_j^2} + \sqrt{5}.$$

Therefore

$$m \leq \lim_{j\to\infty}\left(\frac{1}{mk_j^2} + \sqrt{5}\right) = \sqrt{5}. \qquad \left\{\lim_{j\to\infty} k_j = \infty\right\}$$

So if $m > \sqrt{5}$ then there is no infinite sequence of rational numbers $\frac{h_j}{k_j}$, $k_j > 0$ such that

$$\left|\frac{h_j}{k_j} - \xi\right| < \frac{1}{mk_j^2}$$

for every $\dfrac{h_j}{k_j}$ in the sequence. $\square$