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Deterministic and Random Pebbling of Graphs

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Abstract

Graph pebbling is an area of graph theory in which the properties of configurations of discrete objects, pebbles, on the vertices of a graph are considered. In this work, a summary of the field is presented, beginning in its number-theoretic origins, continuing through its inherent combinatorial nature which naturally transitions to a random graph theoretic framework, and finally ends in a brief overview of modern research in the field regarding one specific conjecture in particular.

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Chapter 1

Introduction

1.1 Pebbling - an analogy

Suppose one were to consider a network of stations, with some stations connected to others by guarded passages, and that there are a few stones distributed at some of the stations. Suppose furthermore that there is a stone merchant who is assigned the task of transporting stones in such a way that some particular station, the target station, has at least one stone. This would be a simple task, were it not for the guards in the passages, who require a toll of exactly one stone for every stone that is transported along each passage. One question that may arise is whether or not the merchant, given a particular configuration of stones in the network, can fulfill the task regardless of which target station he is assigned. Continuing this idea, one may ask whether there is a number of stones such that, regardless of how they are distributed in the network, the merchant can always fulfill his task for all target stations, and whether or not there is an efficient way for the merchant to discover what this number is. These rather simple questions turn out to be deeper than at first glance, and will lay at the center of this dissertation, once properly abstracted.

1.2 Notation

A graph G is an ordered pair of sets, $(V(G), E(G))$, where $E(G)$ consists of unordered elements of $V(G) \times V(G)$. The set $V(G)$ is usually called the *vertex set* of the graph, and $E(G)$ the *edge set* of the graph. For brevity, we will denote $|V(G)| = n$ and $|E(G)| = e$, and call the corresponding sets as the *vertices* and *edges* of the graph, respectively. We will here only consider loop-free graphs without multiple edges, i.e. graphs which do not have any vertices with edges to themselves and, given any two vertices, there is at most one edge between them. We denote the *diameter* of G, the longest of the graph distances between two vertices, by d.

A *configuration* C of pebbles on the graph is a function $C : V(G) \to \mathbb{N}$ such that $C(v)$ is the number of pebbles placed on vertex v. A *pebbling move* from a vertex u to a vertex v is a function $p : V(G)^{\mathbb{N}} \to V(G)^{\mathbb{N}}$ such that $p(C(u)) = C(u) - 2$, $p(C(v)) = C(v) + 1$ and with p being the identity function for all other vertices. We denote by $|C|$ the total numbers of pebbles on the graph, i.e. $\sum_v C(v)$. Given two configurations C and D, C is said to be D-solvable if there is a set of pebbling moves $\{p_0, p_1, ..., p_n\}$ such that $(p_0 \circ p_1 \circ ... \circ p_n)(C) = D$. In particular, given a target vertex r, we denote by t_r the smallest number such that all configurations of size t_r are D-solvable for some configuration D with a pebble on vertex r. The number $\pi(G) = \max_r t_r$ is then called the *pebbling number* of the graph G. This is well-defined, as will be shown in Section 2.1.

For reasons that will be made clear in Section 2.1, we assume all graphs to be *connected*, meaning there are no subsets S of $V(G)$ such that there are no edges between $V(G)$ and $V(G) \setminus S$. For the graph, we may then also say that a graph is *k-connected* if $n \geq k$ and if $V(G)$ does not have a subset of $k-1$ vertices which would disconnect the graph were they to be removed. A graph which is 1-connected is said to contain a *cut vertex*, which is the vertex which would disconnect the graph if removed.

A list of symbols used throughout the text follows here.

K_n	The complete graph on n vertices.
P_n	The path graph on n vertices.
C_n	The cycle graph on n vertices.
Q_n	The hypercube of dimension n with 2^n vertices.
$\mathcal{K}$	The graph sequence $(K_1, K_2, ..., K_n, ...)$.
$\mathcal{P}$	The graph sequence $(P_1, P_2, ..., P_n, ...)$.
$\mathcal{C}$	The graph sequence $(C_1, C_2, ..., C_n, ...)$.
$\kappa(G)$	The connectivity of G.
$\operatorname{diam}(G)$	The diameter of G.
$\delta(G)$	The minimum degree of G.
$\pi(G)$	The pebbling number of G.
$\gamma(G)$	The cover pebbling number of G.
$O(g)$	$\{f \mid \exists N, \exists c, \forall n > N : f(n) \leq cg(n)\}$
$\Omega(g)$	$\{f \mid \exists N, \exists c, \forall n > N : f(n) \geq cg(n)\}$
$\Theta(g)$	$O(g) \cap \Omega(g)$

1.3 History

Graph pebbling is closely connected to the study of zero-sum sequences, sequences of elements of some finite group G which sum to the identity of the group. The study of zero-sum sequences began in 1956 by Erdős [1], and has found several useful applications within number theory since. By considering the sequence of partial sums of a sequence, applying the pigeonhole principle can be used to give the following brief result:

Lemma 1. *Any sequence of $|G|$ elements of a finite group G contains a zero-sum subsequence.*

This can further be reduced to where one only requires N terms, rather than $|G|$, where N is the maximum order of an element of G. It was in this area of number thoery that the first introduction of graph pebbling was by Chung [2] in 1989, as an alternate way to show the following result, independently proven the same year by Kleitman and Lemke [3]:

Theorem 1.1 (Kleitman and Lemke)**.** *For any positive integer n, every sequence $(a_k)_{k=1}^n$ of integers contains a nonempty subsequence $(a_k)_{k\in K}$ such that*

$$\sum_{k\in K} a_k \equiv 0 (\operatorname{mod} n) \quad \text{and} \quad \sum_{k\in K} \gcd(a_k, n) \leq n.$$

Chung's argument depended on finding the pebbling number of the hypercube Q_n and on showing the existence of a connection between zero-sum sequences and graph pebbling. In fact, Kleitman and Lemke also conjectured that Theorem 1.1 could be generalized to a further constraint on the elements of the zero-sum sequence, and their conjecture was proven correct in 2008 by Elledge and Hurlbert [4] using methods from graph pebbling:

Theorem 1.2. *For every sequence* $(g_k)_{k=1}^{|G|}$ *of* $|G|$ *elements of a finite abelian group* G *there is a nonempty subsequence* $(g_k)_{k\in K}$ *such that*

$$\sum_{k\in K} g_k = 0_G \quad \text{and} \quad \sum_{k\in K} \frac{1}{|g_k|} \leq 1$$

If $G = \mathbb{Z}/n\mathbb{Z}$, then Theorem 1.2 reduces to Theorem 1.1 above, as $\gcd(g_k, n) = \frac{n}{|g_k|}$. The proof of this theorem is, briefly, carried out by making a link between the problem and the pebbling of a particular graph, and then proceeding to show the theorem for groups $G = \oplus_j \mathbb{Z}_p$ by showing a link between invariant properties of a configuration under the pebbling move and the existence of a zero-sum subsequence. This then, with some further arguments, generalizes to any finite abelian group.

Since the first publications, the field of graph pebbling has been steadily expanding, and there is now a wealth of papers both within the field itself and applications of the field in other areas. Although graph pebbling may have originated within a number-theoretical framework, it has since come to find intersection with many other areas, some of which will be presented in this work.

Chapter 2

Deterministic Pebbling

Graph pebbling is in essence a combinatorial problem as much as it is a graph theoretical one. Therefore, most arguments contain a mix of arguments from each field, and combined they form an area of graph pebbling of that will henceforth be referred to as *deterministic pebbling*; this term is used both to separate it from the various pebbling properties of random graphs (see Section 3) as much as for the fact that it accurately portrays the the often-times forcing nature of the proofs of the theorems in the area.

2.1 Basic Results

First of all, it is clear that any configuration on a graph which is not connected (i.e. a disconnected graph) can be given arbitrarily many pebbles on one vertex, which would make the configuration unsolvable. Thus, we may assume that all graphs G in all sections below are connected. Now, given a graph G on n vertices, if one were to have $n-1$ pebbles, then the configuration in which one places exactly one pebble on all except for one vertex is not solvable for that vertex. Furthermore, if we denote the diameter $\mathrm{diam}(G)$ by d, then the configuration wherein one places $2^d - 1$ pebbles on a vertex w at distance d from some vertex r is not r-solvable. Thus, one immediate result is the following lower bound.

$$\pi(G) \geq \max\{n, 2^d\}$$

In order to show that the pebbling number is well-defined, it is hence only necessary to show that it has an upper bound. Consider a configuration which has 2^d pebbles on some vertex. Clearly, any vertex is solvable in that configuration, as no vertex is further than d from the given vertex, and the number of pebbles requires to move a pebble the distance of d is just 2^d. Hence, if we place $2^d - 1$ pebbles on every vertex except the target vertex, and then add one more pebble, we are guaranteed to have 2^d pebbles on some vertex, and the configuration will be solvable for all target vertices. We record this fact in the following upper bound:

$$\pi(G) \leq (2^d - 1)(n-1) + 1$$

Hence, the pebbling number always exists.

2.2 K_n, P_n and C_n

Before moving on to more general results in graph pebbling, it can be good to run through a few brief examples of pebbling numbers for specific graphs, in order to make concrete the main concepts of deterministic pebbling. The complete graph on n vertices, K_n, is a particularly simple graph to find the pebbling number for. By the two inequalities in the previous section, we have that its pebbling number must be exactly n, as $\text{diam}(K_n) = 1$. This is one of many examples of graphs for which $\pi(G) = n$; see section 2.4 for further elaboration on such graphs. To calculate the pebbling number for the path graph on n vertices, P_n, we use induction.

Theorem 2.1. $\pi(P_n) = 2^{n-1}$

Proof. Proceed by induction. For a base case, it is clear that $\pi(P_2) = 2$. Assume that $\pi(P_k) = 2^{k-1}$. We wish to show that, in this case, $\pi(P_{k+1}) = 2^k$. In order to move a pebble from one endpoint to the other, we wish to move 2^{k-1} to the neighbor of the first endpoint, from which any root can be solved by the induction hypothesis. To do this, we need 2^k pebbles on an endpoint of P_{k+1}. In fact, since moving a pebble from one endpoint to the other clearly is the hardest pebbling move to do in a path, this completes the claim by induction. □

While these two cases might seem trivial, the pebbling number for the third basic example, C_n, is a lot more difficult to show, and demonstrates the inherent combinatorial nature of graph pebbling. The proof of this claim is from [5]:

Theorem 2.2. $\pi(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$ *and* $\pi(C_{2k}) = 2^k$.

Proof. The claim is showed for odd cycles. The proof for even cycles is very similar, with further simplifications that can be made along the way.

Denote the cycle C_{2k-1} as $xa_{k-1}a_{k-2}...a_2a_1vb_1b_2...b_{k-1}yx$ and let the target vertex be v. Denote by P_A the path $va_1a_2...a_{k-2}a_{k-1}$ and by P_B the path $vb_1b_2...b_{k-2}b_{k-1}$. Let C be a configuration of pebbles. As a reminder, we denote by $C(v)$ the number of pebbles on the vertex v. By Theorem 2.1, $\pi(P_A) = \pi(P_B) = 2^{k-1}$. First, we show that the pebbling number is larger than or equal to the claim. Suppose we only have $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$ pebbles; place half on x, and the other half on y. Then, the number of pebbles that can be moved to either a_{k-1} (or b_{k-1} by swapping x and y) can be readily seen to only be at most $2^{k-1} - 1$. Hence, C would not be v-solvable, since $\pi(P_A) = 2^{k-1}$, so the pebbling number is larger than $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$.

Suppose now that we have $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$ pebbles on the graph, but that C is not v-solvable. Let j_A be the total number of pebbles on $a_1a_2...a_{k-1}$ and similarly for j_B. Then the following claim holds:

$$j_A + \left\lfloor \frac{C(x) + \left\lfloor \frac{C(y)}{2} \right\rfloor}{2} \right\rfloor \leq 2^{k-1} - 1$$

and identically for j_B, but swapping $C(x)$ by $C(y)$. This holds by a very similar reasoning as for the necessity argument above; move as many pebbles as possible from x and y to a_{k-1} (or b_{k-1}). Hence,

$$j_A + j_B + \left\lfloor \frac{C(x) + \left\lfloor \frac{C(y)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{C(y) + \left\lfloor \frac{C(x)}{2} \right\rfloor}{2} \right\rfloor \leq 2^k - 2$$

If we wish to minimize the left-hand side of this expression, since we have that $j_A + j_B + C(x) + C(y) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$, it is sufficient to assume that $j_A = j_B = 0$.

Thus, all pebbles are on x and y. Since $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$ is odd, exactly one of $C(x)$ and $C(y)$ must be even. Without loss of generality, assume that it is $C(x)$. Let us move all pebbles from x and y to a_{k-1}. If $C(y) \equiv 3 \pmod 4$, then when we are done, we would have one pebble left at x and y. However, since $C(x)$ is even, these pebbles can both be thought of as coming from y. Similarly, if we move all pebbles from x and y to b_{k-1}, then x will have no pebbles, and y will have at most one pebble, which happens if $C(x) \equiv 0 \pmod 4$; this pebble can also be thought of as coming from y. Now, the combined inequality above gives us

$$\frac{3}{4}C(x) + \frac{3}{4}C(y) - \frac{5}{4} \leq 2^k - 2$$

The additional $-\frac{5}{4}$ is subtracted due to the possible pebbles left on x and y. However,

$$\begin{aligned}\frac{3}{4}\left(C(x) + C(y)\right) &= \frac{3}{4}\left(2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1\right) \\ &\geq \frac{3}{4}\left(2\left(\frac{2^{k-1}-2}{3}\right) + 1\right) \\ &= (2^k - 1) + \frac{3}{4}\end{aligned}$$

But then

$$2^k - \frac{3}{2} = (2^k - 1) + \frac{3}{4} - \frac{5}{4} = \frac{3}{4}\left(C(x) + C(y)\right) - \frac{5}{4} \leq 2^k - 2$$

which is clearly a contradiction. Hence, C must be v-solvable, and we are done. □

2.3 Q_n

The pebbling number of the hypercube Q_n is an interesting case for several reasons including, but not limited to, being the *raison d'être* of the first paper on pebbling [2], where it was shown that $\pi(Q_n) = 2^n$ as part of the proof of Theorem 1.2. One common construction of Q_n is by setting $V(Q_n)$ to be the set of all binary n-tuples and $E(Q_n)$ to be the set of all pairs of vertices whose tuples differ in exactly one position. Another construction is to let each vertex be an n-digit binary number, and add an edge between two vertices whose numbers differ in a single digit. Each construction of the hypercube has its own benefits; in the latter of the two above constructions, for example, it is rather straightforward to prove that Q_n is Hamiltonian. The hypercube also be constructed by assigning a vertex to each subset of the set $\{1, 2, ..., n\}$, and adding an edge between two subsets which differ by precisely one element. Regardless of which construction is used, the pebbling number of the graph obviously remains the same. In the proof where the pebbling number of Q_n is found, the following lemma is needed.

Lemma 2. *Let q be the number of vertices which have an odd number of pebbles, and let p be the size of the configuration. If $p > 2^{n+1} - q$, then two pebbles can be moved to any vertex.*

The proof of this lemma is by induction, and proceeds through various inequalities to demonstrate the claim.

Theorem 2.3. (Chung) $\pi(Q_n) = 2^n$

Proof. Proceed by induction on n. The base case of Q_0 is immediate, so assume that we can pebble $\pi(Q_{n-1})$ with 2^{d-1} or fewer pebbles. Note that Q_n is the Cartesian product of two Q_{n-1}, which in this case means that each vertex in Q_{n-1} is attached to its copy in the other. Denote these two smaller hypercubes as G_1 and G_2. Assume there are p_1 and p_2 pebbles on G_1 and G_2, respectively. Furthermore, let q_1 and q_2 be the number of vertices in G_1 and G_2 which have been assigned an odd number of pebbles in some given configuration. Denote the target vertex by r. Since the problem is symmetric, we may locate r in G_1, which means that there will be exactly one vertex v in G_2 which is connected to it. We will now show that if we place more than 2^n pebbles on the entire graph, then it will be possible to solve any configuration. Suppose that the configuration has $p = p_1 + p_2 \geq 2^n$ pebbles. If $p_2 > 2^n - q_2$, then in G_2, by the above lemma, two pebbles can be moved to v. Hence, one pebble can be moved to r, and the configuration is solvable. Suppose thus instead that $p_2 \leq 2^n - q_2$. Then we wish to move as many pebbles from G_2 into G_1 in order to satisfy the inductive hypothesis. Every pair of pebbles on some vertex in G_2 gives rise to the possibility to move exactly one pebble onto the vertex' neighbor in G_1. This amounts to $(p_2 - q_2)/2$ pebbles that can be moved into G_1. By the assumption above, this means that this number is

$$\frac{p_2 - q_2}{2} \geq \frac{p_2 - (2^n - p_2)}{2} = p_2 - 2^{n-1}$$

Moving all these pebbles, we thus have at least

$$p_1 + (p_2 - 2^{n-1}) = (p_1 + p_2) - 2^{n-1} \geq 2^n - 2^{n-1} = 2^{n-1}$$

pebbles in G_1. Thus, r is solvable, and by induction we have that $\pi(Q_n) \geq 2^n$, which means that $\pi(Q_n) = 2^n$ by the lower bound on the pebbling number. □

The hypercube is thus, together with K_n, also a Class 0 graph, a class of graphs which is further explored and defined in Section 2.4. A more general result involving the *Cartesian product* $G \times H$ of two graphs is also noteworthy. The product is the vertex set $V(G) \times V(H)$, and two vertices (u, u') and (v, v') in this graph are adjacent if and only if $u = v$ and $(u', v') \in E(H)$ or vice versa for G. Clearly, $Q_n = P_2 \times P_2 \times ... \times P_2$, with n factors, which is usually denoted as P_2^n. The following theorem regarding the Cartesian product of paths was shown by Chung in the same paper as the above, and neatly generalizes Theorem 2.1:

Theorem 2.4. *For nonnegative integers $d_1, ..., d_m$, it holds that*

$$\pi(P_{d_1+1} \times P_{d_2+1} \times ... \times P_{d_m+1}) = 2^{d_1+d_2+...+d_m}$$

Since $\pi(P_n) = 2^{n-1}$, the product of paths is thus one of several examples of when $\pi(G \times H) \leq \pi(G) \times \pi(H)$ holds. Whether this results holds in general for the product of arbitrary graphs is an open problem known as Graham's conjecture, and some partial results on this topic are discussed in Section 4.

2.4 Class 0 Graphs

If $\pi(G) = n$, then G is said to be a Class 0 graph, or Class 0 for short. Recall the lower bound in Section 2.1, where it is shown that $\pi(G) \geq n$; hence, Class 0 graphs are the graphs with the smallest possible pebbling numbers. For completion, a graph which is not a Class 0 graph is called a Class 1 graph. Two initial examples of Class 0 graphs are K_n and Q_n, as mentioned before, but Class 0 graphs are not exceedingly rare; out of the twenty-one connected graphs on 5 vertices, ten of these are Class 0. In fact, in the probabilistic sense, almost every graph is Class 0; this is Theorem 3.2., and Section 3 in general deals with the probabilistic properties of graph pebbling. Returning to deterministic pebbling, a brief result which can be shown by elementary means is the following:

Lemma 3. *If G has a cut vertex, then it is not a Class 0 graph.*

Proof. We explicitly construct a configuration of pebbles of size n which is not solvable. Call the cut vertex x, and the two components of $G - x$ as A and B; after, pick two vertices $v \in A$ and $r \in B$, which is possible since neither A nor B are empty (as otherwise x wouldn't be a cut vertex). Now, placing 3 pebbles on v, 0 pebbles on x and r, and 1 pebble on the remaining pebbles will use exactly n pebbles. However, although it is clearly possible to reach x from v, it is not possible to reach r, as all of $B - r$ and x all just have one pebble each. Hence, the configuration is not solvable, so $\pi(G) > n$. Thus, G is not Class 0. □

To further classify Class 0 graphs, we may divide them into sets based on their diameter, and study each case. The only graph with diameter 1 is K_n, and hence all diameter 1 graphs are Class 0. Since, in the probabilistic sense, almost all graphs have diameter 2, it may seem that a reasonable starting point would be to study these, and see which of them are Class 0; to do this, we may start with the following Lemma (the proof of this, with additional details added for clarity, and the subsequent theorem are all from [5]).

Lemma 4. *Let G be a graph with $\text{diam}(G) = 2$ and let $n \geq 6$. Given any configuration of pebbles on G which has at least three vertices given two or more pebbles each, then only n pebbles are needed in total in order to make the configuration solvable.*

Proof. Assume the conditions of the lemma are fulfilled. Assume that G has a cut vertex; then, since the graph has diameter 2, the graph must consist of two complete graphs linked together by the cut vertex. Hence, if there are at least two vertices with two or more pebbles on each, we can pebble any target vertex v in the graph. Now assume G is 2-connected, and that the target vertex is v. If any of the neighbors of v contains two or more pebbles, then we are done; hence, assume each of the neighbors of v has either 0 or 1.

Now, let $x_1, x_2, ..., x_k$, with $k \geq 3$, be the vertices in the configuration which receive 2 or more pebbles. Hence, none of the x_i are neighbors of v, and since the graph has diameter 2, each must be adjacent to some neighbor of v; call this neighbor y_i. Now, if $y_i = y_j$ for some $i \neq j$, then it is easy to pebble v. Assume further that $y_i \neq y_j$ if $i \neq j$. Now, if any y_i has one pebble, then v can be pebbled (move one pebble from x_i to y_i, and then one to v). Any neighbor of some x_i which is not y_i must have 0 or 1 pebbles; if it had 1, then pebbling is easy, as this vertex must be a neighbor of v; otherwise, the graph would have diameter 3. Hence, we proceed with the case when all vertices except the x_i's have 0 pebbles. If we distribute n pebbles, then one of the $x_i's$ (call it x_1 for brevity) must have 3 pebbles, since if all of them had 2, there would only be $2k$ pebbles in total, whereas $n \geq 2k + 1$ (the $x_i's, y_i's$ and v), and if any of them had 4 pebbles, then it is trivial to solve for v, so we proceed again with the remaining case. Assuming x_1 is not adjacent to any other x_i (as otherwise we could pebble), there are a further $k - 1$ distinct vertices

$z_{12}, z_{13}, ..., z_{1k}$, where z_{1i} acts as a bridge between x_1 and x_i to ensure diameter 2, and as before, we assume none of them have any pebbles.

We now have that any x_j, with $j \geq 2$, is adjacent to at least two vertices, namely y_j and z_{1j}, and neither of these have any pebbles. Thus, we can assume that each x_i must have exactly 3 pebbles by a similar counting argument as above (assume fewer than 4, since otherwise solvable, and each x_j contributes to the total vertex count by 3), and thus neither of the x_i can be adjacent to one another. Finally, we then have that there is a vertex z_{23} distinct from all of $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k, z_{12}, z_{13}, ..., z_{1k}$ and v, which is adjacent to both x_2 and x_3. Assuming this has no pebbles (as otherwise we can easily pebble from, say, x_2 to z_{23} and then to v), we then must have another pebble to add to one of the x_i's, and thus we have a contradiction to the previous assumption that none of the pebbles have 4 pebbles. Hence all cases either result in us reaching v with a pebble or a contradiction; by *ex falso quodlibet*, the proof of the claim is thus complete. □

Now, with this Lemma, we have the following theorem; this proof is also given by the previous authors, with added details for clarity.

Theorem 2.5. *If* $\text{diam}(G) = 2$*, then* $\pi(G) = n$ *or* $n + 1$.

Proof. Assume that v is the target vertex, and that we have a configuration of $n+1$ pebbles on G. As in the proof of Lemma 4, we assume that G is 2-connected and that $n \geq 6$; there are only a few simple cases to check for $n = 2, 3, 4$ and 5. Now, if v or any of its neighbors has 2 vertices, then the configuration is trivially solvable. Hence, assume that all except v have exactly one pebble each; then, since there are $n + 1$ pebbles in total, at least one neighbor of the neighbors of v must have at least two pebbles by the diameter being 2, and thus the configuration would be solvable again. Hence, assume the case when v and at least one neighbor u of v both have zero pebbles. Now, if three or more vertices in G have 2 or more pebbles, then that case is taken care of by Lemma 4, so assume the case when at most two vertices in G have 2 or more pebbles; call these vertices x_1 and x_2. The case when either has 4 pebbles is trivially solvable, so we go to the other and assume that they either have 2 or 3 pebbles each. Since we have at least two vertices (u and v) without pebbles, and since all others except x_1 and x_2 has at most one pebble each, we have that, in order to make the configuration have $n + 1$ pebbles, we must distribute at least 2+2+1 (two from u and v, two from x_1 and x_2, one from the fact that we need $n+1$ in total) pebbles on x_1 and x_2, so one of them must have exactly 3 pebbles, and the other 2 or 3 (again disregarding the case when one has at least 4); let x_1 have 3, and x_2 the other case.

Now, if x_1 and x_2 are adjacent to the same neighbor of v, then we are done, so assume they are not, and call these neighbors y_1 and y_2, and that both are free of pebbles (otherwise it is solvable). Then, to ensure diameter 2, there must be a vertex $x_{12} \neq y_1, y_2$ which is adjacent to both x_1 and x_2; if this vertex has a pebble, then x_1 can receive another pebble, and then have four, so assume x_{12} has zero pebbles. Then, by the same counting argument above, we now have a total of five extra pebbles from the $n+1$, v,y_1, y_2 and z_{12}, so we must place 7 pebbles in total on x_1 and x_2; hence, one has at least 4, and this means that all cases lead to the graph being solvable for v, since we have a contradiction to the assumption that neither x_1 nor x_2 has 4 or more vertices. □

Corollary. *If diam*$(G) = 2$ *and* G *has a cut vertex, then* G *is Class 1.*

In [6] Clarke et. al. showed the following result, further involving the connectivity $\kappa(G)$:

Theorem 2.6. *If diam$(G) = 2$ and $\kappa(G) \geq 3$, then G is Class 0.*

The proof of the theorem is rather technical, and involves a great deal of theoretical machinery, and is thus omitted from this discussion for reasons of brevity. As will be shown in Section 3, almost all graphs are Class 0; this is hinted at by the observation that graphs with few edges often have large pebbling numbers, and that graphs with many edges tend to have smaller pebbling numbers. Since random graphs often tend to have many edges, the result is hence hinted at. This topic is developed further and investigated much more rigorously in Section 3.

2.5 Diameter d Graphs

There are many noteworthy graphs which do not have diameter 2, and it is thus of interest to ask whether or not Theorem 2.6 can be generalized to all graphs of diameter $d > 2$. The following conjecture, as presented in [6] attempts to answer this question:

Conjecture 1. *For every $d \geq 1$, there exists a least positive integer $k(d)$ such that all graphs of diameter d and connectivity at least $k(d)$ are Class 0.*

As the only graphs of diameter 1 are the complete graphs, which are all Class 0 (see Section 2.4), the conjecture clearly holds for $d = 1$. For $d = 2$ we know, from Theorem 2.6, that $k(2) = 3$. In fact, the conjecture is answered affirmatively in [7], where the following theorem was shown:

Theorem 2.7. *Let d be a positive integer and let $k = 2^{2d+3}$. If G is a graph of diameter $\leq d$ and connectivity at least k, then G is Class 0.*

The proof of the theorem is both rather lengthy as well as technically involved, and is thus omitted. Similarly to how diameter 2 graphs have $\pi(G) \leq n + 1$, as was shown in Section 2.4, it is also interesting to see whether graphs with larger diameters also have similar bounds. To this effect, we have the following theorem, proven in 2006 in [8]:

Theorem 2.8. *If $d = 3$, then $\pi(G) \leq \frac{3}{2}n + O(1)$.*

The constant is described by the author as "huge"; the bound is therefore more useful in the asymptotic rather than the practical sense.

2.6 Cover Pebbling

Consider a variant of the pebbling number, described as follows: what is the minimum number of pebbles needed to place a pebble on every vertex of the graph using a sequence of pebbling moves, regardless of what the initial configuration may be? Call this number the *cover pebbling number* $\gamma(G)$ of G. As is done in [9], this has quite a natural real-world realization as, for instance, an information network, in which all parties must obtain some piece of relevant information, but information is lost along the way. This problem may be extended to include weighted vertices, in which the end configuration must have, for some weighting function w, exactly $w(v)$ pebbles on vertex v instead of simply one. Some results regarding this cover pebbling number are immediate:

Theorem 2.9. $\gamma(K_n) = 2n - 1$

Proof. Place $2n-2$ pebbles on some vertex v, then, doing as efficient pebbling moves as possible to cover the graph, we will use 2 pebbles to end up with exactly one pebble on each of the other $n-1$ vertices. However, that would leave no pebble on v. Hence, $\gamma(K_n) \geq 2n-1$. Suppose now that at least $2n-1$ pebbles are place on the vertices. Assume at least one vertex does not have a pebble on it. Call this vertex v. By the pigeonhole principle, at least one other vertex has at least two pebbles on it. Thus, we can use these to cover v. Now there are at least $2n-3$ pebbles on the remaining $n-1$ vertices; hence, by induction, with the base case of $\gamma(K_1) = 1$, we have that this smaller graph can be covered as well. Hence we have that $\gamma(K_n) \leq 2n-1$, and thus $\gamma(K_n) = 2n-1$. □

We may use a similar style of proof to show the cover pebbling for path graphs. Denote P_n as $v_1 v_2 \cdots v_n$. Then the following theorem holds.

Theorem 2.10. $\gamma(P_n) = 2^n - 1$

Proof. If we have $2^n - 2$ pebbles on v_n, then we will need 2^{n-1} pebbles to cover v_1, we will need 2^{n-2} pebbles to cover v_2, ..., and finally, we will need 2 pebbles to cover v_{n-1}. But then, summing these contributions, we will have no pebbles remaining to cover v_n. Hence, $\gamma(P_n) \geq 2^n - 1$. Now consider a configuration with $2^n - 1$ pebbles. If there are no pebbles on v_n, then, since $\pi(P_n) = 2^{n-1}$, we require at most 2^{n-1} pebbles to cover v_n. Using induction, with the immediate base case of $\gamma(P_1) = 1$, the remaining $2^{n-1} - 1$ or more pebbles may be used to cover P_{n-1}, and the result would follow. If there are pebbles on v_n, then moving as many of them as possible to v_{n-1}, we have 1 or 2 remaining on v_n. Either there are at least $2^{n-1} - 1$ pebbles moved to v_{n-1}, or at most $2^{n-1} - 2$ moves have been made at at most two pebbles would be remaining on v_n. In both of these cases, we have at least $2^{n-1} - 1$ pebbles remaining on P_{n-1}, and induction again gives that $\gamma(P_n) \leq 2^n - 1$, and the result follows. □

Consider again the weighted covering mentioned briefly above. If we have a weight function w with $w(v) > 0$ for all vertices, then we may ask what the hardest configuration to solve would be. For instance, as we saw above, both the proof of the cover pebbling numbers of K_n and P_n proceeded by considering all pebbles stacked on one vertex, and it was shown that these cases were the hardest to solve. Is this the case in general, even for arbitrary w as above? Sjöstrand [10] answered this affirmatively with the following theorem.

Theorem 2.11 (The Stacking Theorem). *For every graph G and weight function w with $w(v) > 0$ for all vertices v, it holds that*

$$\gamma(G) = \max_v \sum_u 2^{\text{dist}(u,v)}$$

Where $\text{dist}(u, v)$ *denotes the graph distance between u and v. Furthermore, if $\gamma_w(G)$ denotes the cover pebbling number of G when each vertex v requires not one, but $w(v)$ pebbles, then it holds that*

$$\gamma_w(G) = \max_v \sum_u w(u) 2^{\text{dist}(u,v)}$$

This implies that all proofs regarding the cover pebbling number, both weighted and unweighted, will almost certainly be based in resolving a "stacked" configuration into a lower and an upper bound, as was done in the above proofs of K_n and P_n.

Chapter 3

Random Pebbling

Random pebbling is a sub-area of the topic of random graphs, an area which concerns the asymptotic frequency of properties of graphs and how conditions change as $n \to \infty$. One of the more common models for a random graph is the graph $G_{n,p}$, where $|V(G_{n,p})| = n$ and each of the $\binom{n}{2}$ possible edges is added to the graph with probability $p = p(n)$. Generally, analysis is done on the function $p(n)$ in order to deduce for which such functions certain properties of the graph would appear with probability 1 as $n \to \infty$. To do this, we require a way to order functions according to their growth rate. For any two functions $f = f(n)$ and $g = g(n)$ we write $f \ll g$ if, as $n \to \infty$, $f/g \to 0$. If a property P appears in a graph with probability 1 if $p(n) \gg \tau(n)$, and with probability 0 if $p(n) \ll \tau(n)$, then we call $\tau(n)$ a *threshold function* for the property P. We may, for simplicity, that all function values considered here are integers; were they not, they could simply be rounded off to the nearest integer without changing any of the vital arguments. With the basic definitions of random graphs settled, it is possible to begin deriving some elementary results about random graphs and how their pebbling numbers behave; this is the area known as random pebbling.

3.1 Basic Results

An important theorem from random pebbling is that regarding the asymptotic density of Class 0 graphs, due to their prevalence in deterministic pebbling theory. In order to attack this problem, it is necessary to first consider results from random graph theory. As mentioned earlier, a disconnected graph cannot be Class 0, so with the following Lemma from the seminal paper [11] by Erdős and Rényi, we immediately have a result regarding Class 0 graphs:

Lemma 5. *The threshold function for connectivity is* $\Theta(\log n/n)$.

Corollary. *The threshold function for being* Class 0 *is* $\Omega(\log n/n)$.

In order to show further results, it is necessary to restate two more important results regarding the asymptotic properties of both the connectivity and diameter of a random graph; these Lemmas are proven in [12] and [13], respectively:

Lemma 6. *The threshold function for* $\kappa(G) \geq k$ *is* $\Omega((\log n + k \log\log n)/n)$.

Lemma 7. *The threshold function for diam*$(G) \leq d$ *is* $\Omega((n \log n)^{1/d}/n)$.

With these two lemmas, it is possible to prove the following theorem, as is done in [7], which settles the aforementioned problem:

Theorem 3.1. *For all $d > 0$, the threshold for being* Class 0 *is $O((n \log n)^{1/d}/n)$.*

Proof. Since $\Omega((\log n + k \log\log n)/n) \subseteq \Omega((n \log n)^{1/d}/n)$ for all k, we also have, by Lemma 6 and 7 above, that $\kappa(G) \geq k$ for all k whenever $p \gg (n \log n)^{1/d}/n)$. Thus, for any fixed d, and for some $k \geq 2^{2d+3}$, then if $p \gg (n \log n)^{1/d}/n)$ we have that the probability that $G_{n,p}$ has diameter at most d and connectivity at least k must tend to 1. Thus, by Theorem 2.7, the probability that $G_{n,p}$ is Class 0 tends to 1. □

In fact, further characterization of how frequent Class 0 graphs are is possible; it is well-known that almost all graphs are Hamiltonian, from which it follows that almost all graphs are 3-connected. It is also an elementary result in random graph theory that almost all graphs have diameter 2. Hence we have the following remarkable result:

Theorem 3.2. *Almost all graphs are* Class 0*.*

Proof. Immediate from Theorem 2.5 and the probabilistic facts above. □

3.2 Pebbling Thresholds

The previous section considered the asymptotic pebbling properties of a random graph; however, it is also of interest to consider, say, the probability that a randomly selected configuration of some size t on a fixed graph G is solvable. To formalize this, the following construction from [14] is useful. Consider the probability space $\mathcal{D}(G_n, t(n))$, for a sequence of graphs $\mathcal{G} = (G_n)$ where $|V(G_n)| = n$, of all configurations of size $t(n)$ on the graph G_n, where each configuration has the same probability $1/\binom{n+t(n)-1}{t(n)}$ of being chosen. Then, letting $P(G_n, t(n))$ denote the probability that a configuration C on G_n chosen uniformly at random from the above space is solvable. Then we denote a *threshold* for $\mathcal{G}$ for some $\alpha \in (0, 1)$ to be

$$\tau_\alpha(n) = \tau_\alpha(G_n) = \min\{t(n) : P(G_n, t(n)) \geq \alpha\}$$

It is clear that this function always exists, as there always is the identity function satisfying the condition that the probability be over α - indeed, we have $P(G_n, n) = 1$ for all graph sequences. Consider the following set of functions for some sequence of functions $\omega(n)$ tending to infinity:

$$th(\mathcal{G}) := \{t = t(n) : P(G_n, t\omega) \to 1 \text{ and } P(G_n, t/\omega) \to 0 \text{ as } n \to \infty\}$$

It is not entirely clear a priori that this set is non-empty; however, this is the case, as is shown by the following theorem from [15] together with the above reasoning:

Theorem 3.3. $\tau_{1/2}(n) \in th(\mathcal{G})$

This means that we can consider the set $th(\mathcal{G})$ for some graph sequence, and determining which functions belong to this set will give us information regarding asymptotic qualities of the graph sequence. For this reason, we will call the set the *threshold functions* for $\mathcal{G}$; for brevity, the full set will often be referred to as the *pebbling threshold* for $\mathcal{G}$. Consider two graphs H and G such that $E(H) \subseteq E(G)$. Naturally, we have that the probability that a randomly chosen configuration on G is solvable is at least as large as the probability that the same configuration is solvable on H; after all, adding more edges only adds more possibilities, and removes none. Hence, we have the following lemma, which will prove useful:

Lemma 8. *If $E(\mathcal{H}) \subseteq E(\mathcal{G})$ for all n and two graph sequences $\mathcal{H}$ and $\mathcal{G}$, then $th(\mathcal{G}) \subseteq O\left(th(\mathcal{H})\right)$.*

Before finding specific examples of thresholds, the following theorem is useful, as it provides a bound on where it is possible to find the threshold functions:

Theorem 3.4. *For any $\epsilon > 0$ and any graph sequence $\mathcal{G}$ as previously considered, $th(\mathcal{G}) \subseteq o(n^{1+\epsilon})$.*

Proof. This proof is the one given in [16]. Let $\epsilon > 0$. We wish to show that $t(n) \in \Omega(n^{1+\epsilon})$ implies that $P(G_n, t(n)) \to 1$ as $n \to \infty$. First of all, note that for any graph H_l on l vertices, we have $\pi(H_l) < 2^l$; this is because the pebbling number of a graph is smaller than or equal to the pebbling number of a spanning tree of the graph, which is smaller than the pebbling number of a path on l vertices, which is strictly smaller than 2^l. Now, let $\delta > 0, t = cn^{1+\epsilon}$ for some $c > 0, l = (1+\delta)/\epsilon$, and $k = 2^l$. Fix n, and consider $G = G_n$. For all vertices of G, choose $G(v)$ to be a connected subgraph of G on l vertices, including v. We call $G(v)$ an *l-neighborhood* of G. Denoting by $|D_{G_v}|$ the number of pebbles on vertices of $G(v)$, we say that $G(v)$ is *k-bounded* if $|D_{G_v}| < k$.

We wish to show that the probability that there is a k-bounded l-neighborhood tends to zero. If this is shown, then with probability tending to 1 we have that every l-neighbourhood of G contains at least $k = 2^l$ pebbles, which means that every subgraph of G is solvable with probability 1, i.e. $P(G_n, t(n)) \to 1$, which is the upper bound to be shown. To show it, we use a series of combinatorial identities.

$$
\begin{aligned}
P\Big[\exists\, k\text{-bounded } l\text{-neighborhood}\Big] &\le nP\Big[G(v) \text{ is } k\text{-bounded}\Big] \\
&= n\sum_{i=0}^{k-1} P\Big[|D_{G(v)} = i|\Big] \\
&= n\sum_{i=0}^{k-1} \frac{\binom{l+i-1}{i}\binom{n-l+t-i-1}{t-i}}{\binom{n+t-1}{t}} \\
&= \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\binom{n-l+t-1}{t}\prod_{j=0}^{i-1}\left(\frac{t-j}{n-l+t-j-1}\right) \\
&\le \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\binom{n-l+t-1}{t}\left(\frac{t}{n-l+t-1}\right)^i \\
&= \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\left(\frac{t}{n-l+t-1}\right)^i \binom{n+t-1}{t}\prod_{j=1}^{l}\left(\frac{n-j}{n+t-j}\right) \\
&\le n\sum_{i=0}^{k-1} \binom{l+i-1}{i}\left(\frac{t}{n-l+t-1}\right)^i \left(\frac{n}{t}\right)^l \\
&= n\left(\frac{n}{t}\right)^l \sum_{i=0}^{k-1} \binom{l+i-1}{i}\left(\frac{t}{n-l+t-1}\right)^i \\
&\le n(n^{-\epsilon l}) \sum_{i=0}^{k-1} \binom{l+i-1}{i} \\
&\le n^{1-\epsilon l} k \binom{l+i-1}{i} \\
&= Cn^{-\delta} \to 0.
\end{aligned}
$$

Thus, by the argument above, we have the bound. □

Finally, one may ask how dense pebbling thresholds are in the spectrum of functions. The following conjecture was posed in [16]:

Conjecture 2. *For every $\Omega(n^{1/2}) \ni t_1 \ll t_2 \in O(n)$ there exists a graph sequence $\mathcal{G} = (G_1, ..., G_n, ...)$ such that $th(\mathcal{G}) \subset \Omega(t_1) \cap O(t_2)$.*

This conjecture, in clearer terms, asks whether for any subsection of the given interval of functions there always is a pebbling threshold dominating the lower bound of the section as well as being dominated by the upper bound of the section. An affirmative answer to the conjecture was proven in [17] wherein the authors show that the thresholds for sequences of *brooms* cover the entire spectrum. A broom $B_{m,n}$ is a star graph S_{n-m+1} combined with a path on m vertices, as the name suggests. Defining the sequence $\mathcal{B}_m = (B_{m,1}, ..., B_{m,n}, ...)$ with $m = m(n)$, we have the following theorem:

Theorem 3.5. *Let $\epsilon = \epsilon(n) > 1/2$ be any function satisfying $n^\epsilon \ll n$. Then for $m = (2\epsilon - 1)\log_2 n$ we have $th(\mathcal{B}_m) = \Theta(n^\epsilon)$.*

This shows that the conjecture is true, as for any $t \in \Omega(t_1) \cap O(t_2)$, then if $m = \log_2 \frac{t^2}{n}$, we have that

$$\begin{aligned} \log_2 \frac{t^2}{n} &= (2\epsilon - 1)\log_2 n \\ \frac{t^2}{n} &= n^{2\epsilon-1} \\ t &= n^{\epsilon} \end{aligned}$$

and so, by Theorem 3.5 above,

$$th(\mathcal{B}_m) = \Theta(n^{\epsilon}) = \Theta(t)$$

Hence, we have a sequence with threshold t.

3.3 $\mathcal{K}, \mathcal{P}, \mathcal{C}$

In order to use Lemma 8 to its fullest extent, it would be of use to calculate the threshold function for the sequence of complete graphs, $\mathcal{K}$, as this would provide a lower bound for the threshold functions of all graph sequences. In fact, this problem is not very different from the famous Birthday Problem, in which is demanded the smallest number of people such that the probability that any two of them were to share a birthday is at least 1/2; these problems are very similar, as any configuration on a graph in $\mathcal{K}$ is solvable if it has two pebbles on any vertex due to the diameter being 1.

Theorem 3.6. *$th(\mathcal{K}) = \Theta(\sqrt{n})$.*

Proof. The size of the probability space of configurations of size t is, as mentioned earlier, $\binom{n+t-1}{t}$. We wish to calculate $\tau(n)$ such that for any $t(n) \gg \tau(n)$, the probability that any configuration with $t(n)$ vertices is solvable tends to 1, and to 0 if the domination is reversed. The probability in question is of course simply $1 - P(A)$, where A is the set of unsolvable configurations on K_n. Clearly, $|A|$ is just the number of combinations of t elements onto n nodes; hence, $|A| = \binom{n}{t}$. We thus have

$$P(A) = \frac{\binom{n}{t}}{\binom{n+t-1}{t}} = \frac{\frac{n!}{(n-1)!\,t!}}{\frac{(n+t-1)!}{(n-1)!\,t!}} = \frac{n(n-1)\cdots(n-t+1)}{(n+t-1)(n+t-2)\cdots n}$$

Hence, we have that

$$\left(\frac{n-t+1}{n}\right)^t < P(A) < \left(\frac{n}{n+t-1}\right)^t$$

As the terms are written in descending order of size. Consider $\tau = \tau(n) = c\sqrt{n}$. Then

$$\begin{aligned} \left(\frac{n-c\sqrt{n}+1}{n}\right)^{c\sqrt{n}} &< P(A) < \left(\frac{n}{n+c\sqrt{n}-1}\right)^{c\sqrt{n}} \\ \left(\frac{x^2-cx+1}{x^2}\right)^{cx} &< P(A) < \left(\frac{x^2}{x^2+cx-1}\right)^{cx} \\ \left(\left(1-\frac{c}{x}+\frac{1}{x^2}\right)^x\right)^c &< P(A) < \left(\left(1-\frac{c}{x}+O\left(\frac{1}{x^2}\right)\right)^x\right)^c \end{aligned}$$

Now, as $n \to \infty$, we have $x \to \infty$, and using L'Hôpital's rule and finding the limits we find that both bounds become e^{-c^2}. Thus, the probability that a configuration with $\tau(n)$ pebbles is solvable tends to $1 - e^{-c^2}$. This tends to 1 if $c \to \infty$, and 0 if $c \to 0$. But

$$c = \frac{\tau(n)}{\sqrt{n}}$$

So we have that $\tau(n)$ is a threshold function for $\mathcal{K}$ if and only if $\tau(n) \in \Theta(\sqrt{n})$. Hence, $th(\mathcal{K}) = \Theta(\sqrt{n})$. □

From this theorem together with Theorem 3.4 and Lemma 8 we have the following bounds on where threshold functions can be found.

Theorem 3.7. *For any $\epsilon > 0$ and any graph sequence $\mathcal{G}$ as previously considered, $th(\mathcal{G}) \subseteq \Omega(\sqrt{n}) \cap o(n^{1+\epsilon})$.*

We may also extract some information about the threshold for the sequence of paths, $\mathcal{P}$. The following result is from [16]:

Theorem 3.8. *For every $\epsilon > 0$, we have that $th(\mathcal{P}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$.*

Proof. The upper bound is from Theorem 3.7. We wish to show that if $t(n) \ll n$, then $P(P_n, t(n)) \to 0$ as $n \to \infty$.

Let the vertices of P_n be labeled such that v_1 has an edge to v_2, v_2 has an edge to v_3, and so on. Choose a configuration $C = C_n$ be chosen from the standard probability space. For each v_i, let $X_i = C(v_i)$, the number of pebbles on v_i; also, let $X = \sum X_i$ and $Y = \sum X_i/2^{i-1}$. Note that the configuration is v_1-solvable if and only if $Y \geq 1$. This is since any pebble placed on v_{i-1} can be considered as two pebbles on v_i, and vice versa; hence, since 2^{n-1} pebbles are needed for the configuration to be v_1-solvable if they were all to be placed on v_n, we can weight the values of each pebble accordingly, as is done in the definition of Y. We wish to show that the probability that $Y \geq 1$ goes to 0 when $t \ll n$.

Let $t = n/\omega$ for any sequence $\omega \to \infty$. Then we have that $\mathbb{E}(X_i) = t/n = 1/\omega \to 0$. As a result, $\mathbb{E}(Y) = \sum \mathbb{E}(X_i)/2^{i-1} = \left(\sum 1/2^{i-1}\right)/\omega \leq 2/\omega \to 0$. Hence, by Markov's inequality, we have that

$$P(Y \geq 1) \leq \mathbb{E}(Y) \to 0$$

. □

The same method can be applied to the sequence of cycles $\mathcal{C}$, with the only difference more or less being a very slight modification in the weighting in Y; the modification is slight enough for the proof to be omitted.

Theorem 3.9. *For every $\epsilon > 0$, we have that $th(\mathcal{C}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$.*

In fact, better upper bounds for both cycles and paths were found in [18], and the threshold for both $\mathcal{C}$ and $\mathcal{P}$ were found, for any constant $c > 1$, to be $O\left(n2^{c\sqrt{\log_2 n}}\right)$. A much better lower bound for the sequence of paths was found in [17]), which almost matches the upper bound given, giving us the following theorem:

Theorem 3.10. *Let $\omega(n) := n2^{\sqrt{\log_2 n}}$, $\epsilon > 0$ and $c > 1$. Then*

$$th(\mathcal{P}) = \Omega(\omega^{1-\epsilon}) \cap O(\omega^c)$$

Despite these very narrow bounds, the exact pebbling threshold for the sequence of paths is not known, and remains an open problem.

3.4 $\pi(\mathcal{G})$ and $th(\mathcal{G})$

There are some further theoretical results that can be arrived at from basic principles, especially those relating the pebbling thresholds to the pebbling number.

Lemma 9. *For any sequence $\mathcal{G}$ let $\pi_n(n) = \pi(G_n)$. Then $th(\mathcal{G}) \subseteq O(\pi_n)$.*

Proof. Immediate; the probability that a randomly chosen configuration with $\pi(G_n)$ pebbles is solvable on G_n is exactly 1. □

From this Lemma, quite a few interesting facts spring out.

Corollary. *Assume $diam(G_n) = 2$ for all n. Then $th(\mathcal{G}) \subseteq O(n)$.*

Proof. By Theorem 2.5 $\pi(G_n) \leq n + 1$ for all n, so by Lemma 9 the result follows immediately. □

In fact, this corollary can be extended even further.

Corollary. *Let $d(n) := \mathrm{diam}(G_n)$. Then $th(\mathcal{G}) \subseteq O(2^{d(n)}n)$. If $d(n) \leq d$, then $th(\mathcal{G}) \subseteq O(n)$.*

Proof. By the upper bound in Section 2.1, we immediately get the result. □

Exploiting Theorem 2.7, we have a final easy corollary:

Corollary. *With $d(n)$ as above, and $\kappa(n)$ as the connectivity of G_n, if $\kappa(n) \geq 2^{2d(n)+3}$ for all n, then $th(\mathcal{G}) \subseteq O(n)$.*

One question that might appear is what the relation between the pebbling numbers and thresholds of a graph sequence might be. Hurlbert et al. posed the following very natural conjecture in, among other places, [16]:

Conjecture 3 (Hurlbert et al.)**.** *If $\mathcal{G}$ and $\mathcal{H}$ are graph sequences satisfying $\pi(G_n) \leq \pi(H_n)$ for all n, then $th(\mathcal{G}) \subseteq th(\mathcal{H})$.*

The conjecture was, however, shown to be false, as counterexamples were found in [14] by Björklund and Holmgren.

3.5 Different Models

The model of selecting a distribution of size t at random is only one of many possible. For example, one may consider a model in which each pebble is distinguishable from one another, and in which the pebbles are placed one at a time on some randomly chosen vertex. In [17] this model is called the *binomial model*, and there the binomial pebbling threshold $th_{\mathcal{B}}(\mathcal{G})$ is defined in the same way as the standard pebbling threshold. One immediate result regarding pebbling thresholds in this model is the following:

Theorem 3.11. $th_{\mathcal{B}}(\mathcal{G}) \in O(n \log n)$

Proof. An upper bound for the pebbling threshold is simply the solution to the coupon collector's problem; given an urn with n different coupons from which coupons are drawn with equal probability and replacement, what is the expected number of draws needed to obtain one of each coupon? It is immediate that this is equivalent to finding the expected number of pebbles needed to place one pebble on each vertex, which would mean that the configuration were solvable. Now, the probability of picking a coupon p_i that had not been chosen before is $p_i = \frac{n-i+1}{n}$, where $i-1$ is the number of coupons collected before. This means that the number of draws needed to collect the ith coupon is geometrically distributed, and its first moment is thus $1/p_i$. Thus, the total expected number of draws required to receive one of each coupon is

$$\begin{aligned}\mathbb{E} &= \frac{1}{p_1} + \cdots + \frac{1}{p_n} \\ &= n\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= nH_n\end{aligned}$$

Since H_n grows as $\log n$, we thus have that the pebbling threshold has to be $O(n \log n)$, as any function in $\Theta(n \log n)$ will have the probability of having one pebble on each vertex tending to 1. □

Naturally, this upper bound is rather weak, but it can be improved for the case of the sequence of paths, as is done in [17]:

Theorem 3.12. *The threshold for $\mathcal{P}$ is*

$$th_{\mathcal{B}}(\mathcal{P}) = \left(\frac{1}{2} + o(1)\right) n \frac{\log n}{\log \log n}$$

Several other models can be considered, and a few of them can be useful in showing results regarding the standard model. In [17] another model for distributions on paths, in which each vertex is given a certain number of pebbles according to a geometric distribution, is presented and used to prove several results about the pebbling thresholds for sequences of paths in the standard model. Another model, specifically for the pebbling number, rather than for how distributions are chosen, was discussed in Section 2.6.

Chapter 4

Graham's Conjecture

The Cartesian product of two graphs was mentioned briefly in Section 2.3, but is repeated here for clarity; the *Cartesian product* $G \times H$ of two graphs G and H is defined as the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if and only if

$$u = v \wedge (u', v') \in E(H) \qquad or \qquad u' = v' \wedge (u, v) \in E(G)$$

This product is of great interest, and has been studied in its own right. It was shown in 1985 by Feigenbaum et al. [19] that recognizing a Cartesian product graph can be done in polynomial time; Imrich and Peterin reduced this to linear time in 2007 [20]. All graphs are also decomposable into products of 'prime' graphs (graphs which are not reducible into products of smaller graphs) [21] although not always uniquely [22]. One of the more interesting and well-studied open problems in graph pebbling concerns this product, and was originally stated by Ronald Graham:

Conjecture 4 (Graham)**.** *For all graphs G and H, $\pi(G \times H) \leq \pi(G)\pi(H)$*

This conjecture, which lies at the center of much study, is still unproven, yet some partial results have been made. As was mentioned in Section 2.3., the conjecture holds true for arbitrarily large products of all paths, with Q_n being a particular case when all products are P_2. It has also been shown, among several other cases, to hold for the product of a tree by a tree [23], a cycle by a cycle [24], a complete bipartite graph by another [25]. When the minimum degree of G is high, then Theorem 2.7 was used by [26] to derive the following result:

Theorem 4.1. *If G_1 and G_2 are connected graphs on n vertices that satisfy $\delta(G_i) \geq k$ and $k \geq 2^{12n/k+15}$, then $\pi(G_1 \times G_2) \leq \pi(G_1) \times \pi(G_2)$.*

Graham's conjecture can also be extended to products of graph sequences, and be formulated in a probabilistic sense.

Conjecture 5 (Threshold Graham)**.** *Let $\mathcal{H} = \mathcal{F} \times \mathcal{G}$ be defined as $H_{n^2} = F_n \times G_n$. Let $f(n) \in th(\mathcal{F}), g(n) \in th(\mathcal{G})$ and $h(n) \in th(\mathcal{H})$. Then $h(n) \in O(f(n)g(n))$.*

The above formulation of the conjecture assumes that F_n and G_n each have n vertices for clarity, but it is straightforward to generalize it to skip this assumption. One partial, affirmative, result is found in [26]:

Theorem 4.2. *Let n, m be positive integers. Then Graham's Threshold conjecture holds for $\mathcal{F} = \mathcal{P}^n$ and $\mathcal{G} = \mathcal{P}^m$.*

The same authors do, however, also note that they believe that counterexamples to the conjecture exist. In spite of this, the conjecture has also been verified in the case of $\mathcal{K}^2$ in [27]:

Theorem 4.3. *For $\mathcal{K}^2 = (K_1^2, K_2^2, ...)$ we have $th(\mathcal{K}^2) = \Theta(\sqrt{N})$, where $N = n^2$ is the number of vertices of K_n^2.*

Note that this result is rather extraordinary; in spite of the fact that squaring a graph makes the graph quite sparse in terms of edges, the same pebbling threshold is maintained. There number of partial results for Graham's Conjecture are quite many, but it nonetheless remains an area of graph pebbling that remains highly active. A case in point is the following result, proven in late April 2017 in [28]:

Theorem 4.4. *Let $M(G)$ denote the graph obtained by inserting a new vertex into each edge of G and joining these new vertices by an edge if and only if the two edges into which they were inserted shared a vertex in G. Let $m, n \geq 5$ and $|n - m| \geq 2$. Then*

$$\pi(M(C_{2n}) \times M(C_{2m})) \leq \pi(M(C_{2n}))\pi(M(C_{2m}))$$

There are, as mentioned earlier, different models for graph pebbling that can be chosen, which all give varying results. One such model is changing the standard pebbling number $\pi(G)$ for a target-selectable pebbling number, $\rho(G)$. This is defined as the smallest number of pebbles needed such that some chosen distribution D is reachable from every distribution with $\rho(G)$ pebbles. The difference between the standard pebbling number is clearly in that, despite the fact that the starting distribution of the pebbles is the same, after those pebbles have been placed, we are allowed to choose which distribution should be reached. This model is noted by [29] to originally have been motivated by an attempt to solve a version of Graham's conjecture. In this paper, however, they show that this pebbling number does not satisfy the analog of Graham's conjecture by constructing several counterexamples. Whether or not further progress can be made on the original conjecture using different models or not is not clear, but Graham's conjecture is certainly an open problem which continues to generate a great deal of research and interest within graph pebbling.

Bibliography

[1] P. Erdos, *On pseudoprimes and Carmichael numbers*, Publ. Math. Debrecen 4 (1956), 201–206.

[2] F. R. K. Chung. *Pebbling in hypercubes*, SIAM J. Discrete Math. 2 (4) 467-472 (1989)

[3] D. Kleitman and P. Lemke, *An addition theorem on the integers modulo n*, J. Number Theory 31 (1989), 335–345.

[4] S. Elledge, G.H. Hurlbert, *An Application of Graph Pebbling to Zero-Sum Sequences in Abelian Groups*, arXiv:math/0409588 (2008)

[5] Lior Pachter, Hunter Snevily, Bill Voxman, *On pebbling graphs*, Congr. Numer. 107 (1995), 65–8

[6] T. Clarke, R. Hochberg, G. Hurlbert, *Pebbling in Diameter Two Graphs and Products of Paths*, J. Graph Th. 25 (1997), no.2. 119-128

[7] A. Czygrinow, G. Hurlbert, H. A. Kierstad, W. Trotter, *A note on graph pebbling*, Graphs and Comb. 18 (2002), 219–225

[8] B. Bukh, *Maximum pebbling number of graphs of diameter three*, J. Graph Theory 52 (2006), 353–357

[9] B. Crull, T. Cundiff, P. Feltman, G. Hurlbert, L. Pudwell, Z. Szansiszlo, Z. Tuza, *The cover pebbling number of graphs*, Discrete Math. 296 (2005), 15-23

[10] J. Sjöstrand, *The cover pebbling theorem*, Electron. J. Combin. 12 (2005), Note 22, 5 pp.

[11] P. Erdös, A. Rényi, *On random graphs I*, Publ. Math. Debrecen 6 (1959), 290-297

[12] P. Erdös, A. Rényi, *On the strength of connectedness of a random graph*, Acta Math. Sci. Hung. 12 (1961), 261-267

[13] Y. D. Burtin, *On extreme metric characteristics of a random graph II: Limit distributions*, Theory Probab. Appl. 20, 83-101

[14] J. Björklund, C. Holmgren, *Counterexamples to a monotonicity conjecture for the threshold pebbling number*, Discrete Math. 312 (2012), 2401-2405

[15] A. Bekmetjev, G. Brightwell, A. Czygrinow, G. Hurlbert, *Thresholds for families of multisets, with an application to graph pebbling*, Discrete Math. 269 (2003) 21-34

[16] A. Czyrinow, N. Eaton, G. Hurlbert, M. Kayll, *On pebbling threshold functions for graph sequences*, Discrete Math. 247 (2002), 93-105

[17] A. Czygrinow, G. Hurlbert, *On the pebbling threshold of paths and the pebbling threshold spectrum*, Discrete Math., (to appear)

[18] A. Godbole, M. Jablonski, J. Salzman, A. Wierman, *An upper bound for the pebbling threshold of the n-path*, Discrete Math. 275 (2004), 367–373

[19] J. Feigenbaum, J. Hershberger, A. Schäffer, *A polynomial time algorithm for finding the prime factors of Cartesian-product graphs*, Disc. App. Math., 12 (2): 123–138 (1985)

[20] W. Imrich, I. Peterin, *Recognizing Cartesian products in linear time*, Disc. Math., 307 (3-5): 472–483 (2007)

[21] G. Sabidussi, *Graph multiplication*, Mathematische Zeitschrift, 72: 446–457 (1960)

[22] W. Imrich, S. Klavzar,*Product Graphs: Structure and Recognition*, Wiley (2000)

[23] D. Moews, *Pebbling graphs*, J. Combin. Theory (Ser. B) 55 (1992), 244–252

[24] D. Herscovici, *Graham's pebbling conjecture on products of cycles*, J. Graph Theory 42 (2003), 141–154

[25] R. Feng, J. Y. Kim, *Graham's pebbling conjecture on product of complete bipartite graphs*, Sci. China (Ser. A) 44 (2001), 817-822

[26] A. Czygrinow, G. Hurlbert, *Girth, pebbling, and grid thresholds*, SIAM J. Discrete Math. 20 (2006), 1–10

[27] A. Bekmetjev, G. Hurlbert, *The pebbling threshold of the square of cliques*, Discrete Math., (to appear)

[28] Z.-J. Xia, Y.-L. Pan, J.-M. Xu, X.-M. Cheng, *Graham's pebbling conjecture on Cartesian product of the middle graphs of even cycles*, arXiv:1705.00191 (2017)

[29] D. Herscovici, B. Hester, G. Hurlbert, *Generalizations of Graham's Pebbling Conjecture*, Discrete Math. 312 (2012), 2286-2293