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U.U.D.M. Project Report 2017:26

Khovanov Homology of Knots

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Examensarbete i matematik, 15 hp
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Juni 2017

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Contents

Abstract

In this thesis we provide the construction of an invariant for knots called the Khovanov homology. We work out a general formula for the Jones polynomial which Khovanov homology is built upon, and provide an example of Khovanov homology of the trefoil knot. This is later used to deduce the general formula for the torus knots $T_{2,k}$.

Sammanfattning

I den här uppsatsen konstruerar vi en invariant för knutar som kallas för Khovanov homologi. Vi härleder en allmän formel för Jones polynomet som Khovanov homologin baseras på, och förse med ett exempel för Khovanov homologi på treklöver knuten. Det här exemplet är användbart då vi senare härleder en allmän förmel för torus knutarna $T_{2,k}$.

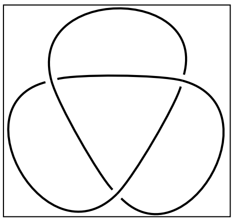
1 Introduction

A mathematical theory for knots was first developed in the year 1771 by Alexandre-Théophile Vandermonde. In his work he expressed that this theory could be useful when discussing the properties of knots related to the geometry of position. In 1984 Vaughan Jones discovered a link invariant which is now called the Jones polynomial [1], defined by a skein relation

$$\begin{aligned} \langle \times \rangle &= \langle \asymp \rangle - q \langle)(\rangle, \\ \langle \emptyset \rangle &= 1, \\ \langle \bigcirc L \rangle &= (q + q^{-1}) \langle L \rangle, \end{aligned} \tag{1}$$

this is explained in section three. Later in 1999 Mikhail Khovanov discovered a link invariant which was strictly stronger and more general than Jones polynomial and the idea was to construct a chain complex of a knot in a way that the Euler characteristic is equal to the Jones polynomial [2]. In this project we summarize his work by showing how to define it without category theory and then compute it for the general torus knot. We also explain why it is constructed the way it is by comparing it to a general formula for the unnormalized Jones polynomial.

In the second section we state the definition of knots being equal by their diagrams using the Reidemeister theorem. A diagram of a knot is a two dimensional representation of it. For example, the diagram for the trefoil knot is



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In the third section we follow the steps in [3] and it is about defining the Jones polynomial from the bracket polynomial as well as to work out a general formula for computing the Jones polynomial for an arbitrary knot. In section four we follow [4] and we define the cochain complex (which is a chain of vector spaces with a differential between every space) of a knot and also relate those to the Jones polynomial with the Euler characteristics of the complex. In the fifth section we also follow [4] and we state the main theorem and the Poincaré polynomial as well as give an example where we compute the Khovanov homology for the trefoil. In section six we study [2] and we derive a more efficient way to compute the Khovanov homology. Finally in section seven we use this method to compute the Khovanov homology for an arbitrary torus knot with coefficients in R-modules.

2 Definition of Knots

Definition 1. *Let N and M be manifolds and $g, h : N \to M$ be embeddings of N in M. A continuous map*

$$F : M \times [0,1] \to M \tag{2}$$

is called an ambient isotopy from g to h if $F(x,0) = Id$ and each map $F(x,t)$ is a homeomorphism from M to itself, and $F(x,1) \circ g = h$.

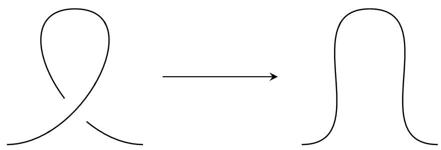
Definition 2. *A knot is an embedding of the circle (S^1) into three-dimensional Euclidean space ($\mathbb{R}^3$), or the 3-sphere (S^3). Two knots are defined to be equivalent if there is an ambient isotopy between them.*

A link is defined as being n-disjoint knots in $\mathbb{R}^3$, or S^3. In this paper we also call links for knots. The orientation of a knot is given by the standard orientation on S^1 (counter clockwise). We can think of knots being the image of the embedding with a fixed orientation. One can project a knot onto a plane $\mathbb{R}^2$ such that the projection is almost always regular, meaning that the projection is injective everywhere except for at a finite number of points where the projected curve intersects itself in a standard way. This is defined to be the diagram of a knot.

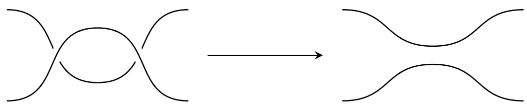
Theorem 1. *(Reidemeister's Theorem) Two links can be continuously deformed into each other if and only if any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves.*

These Reidemeister moves are defined as follows.

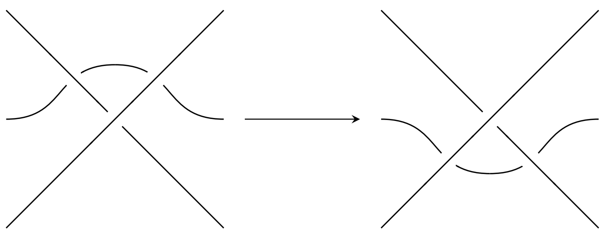
$R1$:



$R2$:



$R3$:



Note that we did not write out the orientation here because these moves are orientation preserving. We will not prove this theorem, but we will use it to define invariants for knots.

3 Jones Polynomial

In this section we define the Jones polynomial using the definition of the bracket polynomial. We also provide a general formula for computing the Jones polynomial for an arbitrarily knot. Here we follow the steps in [3].

Definition 3. *Given a knot diagram, and a crossing* $\times$ *in the knot diagram. The zero resolution and the one resolution of the crossing are when we replace the crossing* $\times$ *with* $\asymp$ *and* $\rangle\langle$ *respectively.*

Note that this is the same as the zero resolution and one resolution are given by going left and right respectively from the overcrossing down to the undercrossing. This uniquely defines the zero resolution and the one resolution without the orientation of the knot.

Definition 4. *Given a knot, then at each crossing we define the positive crossing and the negative crossing to be*

(a) Positive crossing (b) Negative crossing

We denote the number of crossings of a knot to be n*, the number of positive crossings as* n_+ *and the number of negative crossings as* n_-.

The bracket polynomial is usually defined as:

$$\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle \rangle\langle \rangle, \tag{3}$$

$$\langle \emptyset \rangle = 1, \tag{4}$$

$$\langle \bigcirc L \rangle = (-A^2 - A^{-2})\langle L \rangle. \tag{5}$$

where L is a knot, $\langle L \rangle$ is the polynomial of the knot and A a variable. The polynomial is evaluated on a diagram where the knot is kept constant outside of the crossing. To get the polynomial of the knot we have to replace each crossing with its zero and one resolutions until we arrive at disjoint unions of circles and use (5). If we replace $\langle L \rangle$ with $A^n\langle L \rangle$ and A^2 with $-q^{-1}$ we get a new polynomial with a different relation like this

$$A\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle \rangle\langle \rangle. \tag{6}$$

If we divide by A on both sides, then we can rewrite the bracket (6) in the following form:

$$\begin{aligned} \langle \times \rangle &= \langle \asymp \rangle - q\langle)(\rangle, \\ \langle \emptyset \rangle &= 1, \\ \langle \bigcirc L \rangle &= (q + q^{-1})\langle L \rangle. \end{aligned} \tag{7}$$

This defines the invariant Kauffman bracket. From this bracket we can define the unnormalized Jones polynomial of a knot to be $\widetilde{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$. To get the Jones polynomial we divide by the factor $q + q^{-1}$.

Proposition 1. *The unnormalized Jones polynomial is well-defined.*

Proof. The general formula later in this section proves that it is well-defined on a single diagam D, and we now prove that it is also invariant under the three Reidemeister moves.

$$\langle \text{ዴ} \rangle = \langle \text{ዱ} \rangle - q\langle \frown \rangle = (q + q^{-1})\langle \frown \rangle - q\langle \frown \rangle = q^{-1}\langle \frown \rangle \tag{8}$$

$$\langle \text{ዲ} \rangle = \langle \frown \rangle - q\langle \text{ዱ} \rangle = \langle \frown \rangle - q(q + q^{-1})\langle \frown \rangle = -q^2\langle \frown \rangle \tag{9}$$

Both of these cancel out by the factor $(-1)^{n_-} q^{n_+ - 2n_-}$, thus it is invariant under R1. For the R2 case we have

$$\begin{aligned} \langle \rangle\!\!<\!\!\!\!>\!\!\langle \rangle &= \langle \asymp\!\circ\,)(\rangle - q\langle)\circ(\rangle - q\langle \asymp\asymp \rangle + q^2\langle)\asymp \rangle = \\ &= \langle)(\rangle - q(q + q^{-1})\langle)(\rangle + q^2\langle)(\rangle - q\langle \asymp\asymp \rangle = -q\langle \asymp\asymp \rangle. \end{aligned} \tag{10}$$

The factor in front of $\langle \asymp\asymp \rangle$ also cancels out with $(-1)^{n_-} q^{n_+ - 2n_-}$. Left to show is that it is invariant under R3.

We have to show that the Kauffman bracket for both cases coincide. To prove this we only have to look at one of the crossings and use the fact that it is invariant under R2. The crossing we consider is the most centred one. By the definition of Kauffman bracket we consider the following two cases

These generate the same polynomial under the Kauffman bracket since we can create an isotopy between them without changing the number of crossings.

By using the second Reidemeister move twice we can go from the left figure to the right figure and because the Kauffman bracket is invariant under R2 we have that these generate the same polynomial. The Kauffman bracket must be invariant under R3. The Jones polynomial is then invariant under all of the Reidemeister moves and thus is an invariant for knots by the Reidemeister theorem. □

We want to find a general formula for the Jones polynomial. From the definition of Kauffman bracket we can see that all the possible ways we can resolve all the crossing corresponds to the vertices in an n-cube. Let $\alpha \in \{0,1\}^n$, then every possible resolutions of all the crossings for a knot corresponds to one of those α's. Let us explain the notation: first we fix an order of the crossings then a zero in α means the crossing at that vertex is replaced with $\asymp$ (zero resolution) and if it is a one in α it is replaced with $\succ\!\prec$ (one resolution). From this we can derive a formula for the Jones polynomials. Let us denote given an α the diagram with all the chosen resolutions is a properly embedded curve, we call this Γ_α:

$$r_\alpha = \text{number of circles in } \Gamma_\alpha, \tag{11}$$
$$k_\alpha = \text{number of ones in } \alpha. \tag{12}$$

Lemma 1. *The unnormalized polynomial of a knot L is*

$$\sum_{\alpha \in \{0,1\}^n} (-1)^{k_\alpha + n_-} q^{n_+ - 2n_- + k_\alpha} (q + q^{-1})^{r_\alpha}. \tag{13}$$

Proof. We can rewrite the formula as:

$$\sum_{\alpha \in \{0,1\}^n} (-1)^{k_\alpha + n_-} q^{n_+ - 2n_- + k_\alpha} (q+q^{-1})^{r_\alpha} = (-1)^{n_-} q^{n_+ - 2n_-} \sum_{\alpha \in \{0,1\}^n} (-1)^{k_\alpha} q^{k_\alpha} (q+q^{-1})^{r_\alpha}. \tag{14}$$

The factor in front of the sum is just the normalisation factor for the Jones polynomial. By the definition of Kauffman bracket we get the factor $(-1)^{k_\alpha} q^{k_\alpha} (q + q^{-1})^{r_\alpha}$, since k_α represents the number of one-crossings at α and the fact that r_α is the number of disjoint circles at α. □

4 Graded Vector Spaces

Here we follow [4, 5]. Let us now define the cochain complex with graded vector spaces and show that the Euler characteristic of that complex is equal to the unnormalized Jones polynomial. We also define the differentials for the complex as well as proving that it really is a complex.

Definition 5. *A graded vector space V is the direct sum $V = \bigoplus_n V_n$, where the elements in V_n are called homogeneous components. The graded dimension of a graded vector space is defined as the power series $q\dim V = \sum_{n\in\mathbb{Z}} q^n \dim V_n$.*

We will only work with graded vector spaces which has finitely many non-zero homogeneous components and all of them are finite dimensional.

Definition 6. *Let $V = \bigoplus_n V_n$ be a graded vector space. The "degree shift" operator $\cdot\{l\}$ is defined such that $V_n\{l\} = V_{n-l}$.*

Note that with the above definition we can see that $q\dim V\{l\} = q^l q\dim V$. Now we show some properties of $q\dim$. Let V and $\widetilde{V}$ be two graded vector spaces, then

$$\begin{aligned} q\dim(V\otimes\widetilde{V}) &= \sum_{n,m} q^{n+m}\dim(V_n\otimes\widetilde{V}_m) = \sum_{n,m} q^{n+m}\dim(V_n)\dim(\widetilde{V}_m) \\ &= \sum_n q^n\dim(V_n)\sum_m q^m\dim(\widetilde{V}_m) = q\dim(V)q\dim(\widetilde{V}), \qquad (15) \\ q\dim(V\oplus\widetilde{V}) &= \sum_n q^n\dim(V_n) + \sum_m q^m\dim(\widetilde{V}_m) = q\dim(V) + q\dim(\widetilde{V}). \qquad (16) \end{aligned}$$

Definition 7. *A chain complex is a sequence of abelian groups $\ldots, A_1, A_0, A_{-1}, \ldots$ with maps $d_n : A_n \to A_{n-1}$ such that for every $n \in \mathbb{Z}$, $d_n \circ d_{n+1} = 0$. It is usually written as*

$$\ldots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \ldots. \qquad (17)$$

A cochain complex is defined as a normal chain complex where the maps go in the opposite direction with respect to the grading, i.e.

$$\ldots \xrightarrow{d_{n-2}} A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} \ldots. \qquad (18)$$

Definition 8. *Let $\widetilde{C}$ be a chain complex. The "height shift" operator $\cdot[s]$ lowers the index by s. If $C = \widetilde{C}[s]$, then $C^r = \widetilde{C}^{r-s}$.*

Note that in the above definition the differentials are shifted accordingly. When defining our complex we will use an abelian group to associate with the unknot, that is a graded vector space V where $q\dim V = q + q^{-1}$ which is the Jones polynomial for the unknot. In order to achieve this we define V to have two basis elements $(1, x)$ such that $\deg 1 = 1$ and $\deg x = -1$. Note that we are working over $\mathbb{Q}$, so $V = span_{\mathbb{Q}}(1, x)$. If we have k number of disjoint unknots then the Jones polynomial becomes $(q + q^{-1})^k$. By equation (15) the graded vector space that associates with k number of disjoint unknots is $V^{\otimes^k}$.

If we are at vertex α in a cube we also need to shift the vector space to get the right polynomial, thus we get that the vector space at α's position is
$V_\alpha = V^{\otimes r_\alpha}\{n_+ - 2n_- + k_\alpha\}$, where k_α and r_α are the same as in the previous chapter. Let the i^{th} position of the complex be defined as

$$C^{i,*}(L) = \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ k_\alpha = i+n_-}} V_\alpha. \tag{19}$$

An element from $C^{i,j}(L)$ is said to have homological grading i and q-grading j. A natural way of grading the complex is letting $i = k_\alpha$ and $j = \deg v$, where $v \in V_\alpha \subset C^{i,j}(L)$, but in this case we also have some degree shifts in the definition of V_α so we will use the following gradings

$$\begin{aligned} i &= k_\alpha - n_-, \\ j &= \deg(v) + i + n_+ - n_-. \end{aligned} \tag{20}$$

Next we want to define something that relates the whole complex with the Jones polynomial.

Definition 9. *The Euler characteristic of a complex is:*

$$\sum_i (-1)^i q\dim(C^{i,*}(L)). \tag{21}$$

Theorem 2. *The Euler characteristic of the complex above is the unnormalized Jones polynomial.*

Proof. We prove this by sum manipulations using equation (15) and (16):

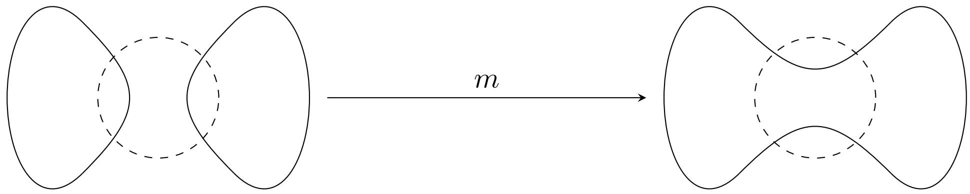
$$\begin{aligned} \sum_i (-1)^i q\dim(C^{i,*}(L)) &= \sum_i (-1)^i q\dim(\bigoplus_{\substack{\alpha\in\{0,1\}^n \\ k_\alpha = i+n_-}} V_\alpha) = \sum_i (-1)^i \sum_{\substack{\alpha\in\{0,1\}^n \\ k_\alpha = i+n_-}} q\dim V_\alpha = \\ &= \sum_{\alpha\in\{0,1\}^n} (-1)^{k_\alpha - n_-} q\dim V_\alpha = \sum_{\alpha\in\{0,1\}^n} (-1)^{k_\alpha - n_-} q^{n_+ - 2n_- + k_\alpha}(q+q^{-1})^{r_\alpha} \end{aligned} \tag{22}$$

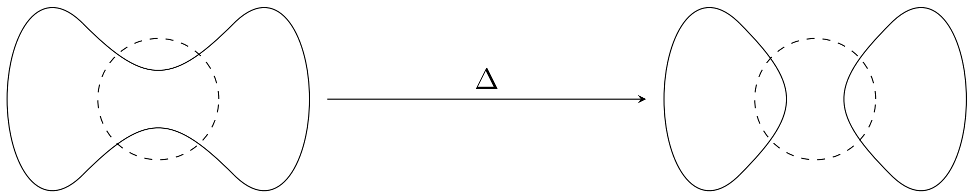
And this is exactly the unnormalized Jones polynomial. The only difference is the exponent of -1, but we see that $(-1)^{2n_+} = 1$ so we can multiply with this constant without changing the expression to get the desired result. □

Left to define for the complex is the differentials. We need two different functions, one that "collapses" two vector spaces and one that "creates" two vector spaces from one vector space. Recall that we have a tensor factor per circle in Γ_α, since the number of circles at α were r_α. We call the "collapse" function for m and the "split" function for Δ and define them to be linear functions which are identity on all tensor-factors except at the vector spaces they are acting on. There they are defined as

$$
\begin{aligned}
m\colon\ & V \otimes V \to V \\
& 1 \otimes 1 \mapsto 1 \\
& 1 \otimes x, x \otimes 1 \mapsto x \\
& x \otimes x \mapsto 0, \\
\Delta\colon\ & V \to V \otimes V \\
& 1 \mapsto 1 \otimes x + x \otimes 1 \\
& x \mapsto x \otimes x.
\end{aligned} \tag{23}
$$

We use m that flips a crossing so that two circle components become one, and we use Δ that flips a crossing so that one circle component is separated into two. This is illustrated in figures below.



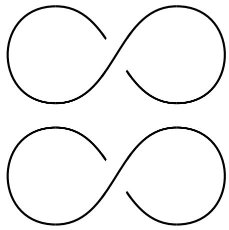


We can also think of the vector space as $V = \mathbb{Q}[1, x]/span_{\mathbb{Q}}(x^2)$. From this we get a natural algebra so that we can rewrite the definition of m as $m(a \otimes b) = ab$, which we will use later. In order to make $C^{*,*}(L)$ a complex we need to define the differential d. Recall that for every knot we have a collection of circles Γ_α at every vertex $\alpha \in \{0, 1\}^n$. We name all the edges $\xi \in \{0, 1, *\}^n$, where we only have one $*$ in each edge. For example if $\alpha_1 = 010$ and $\alpha_2 = 011$, then the edge between them is $\xi = 01*$. Think of it as an arrow where its tail is the vertex if you replace $*$ with 0, and the tip of the arrow is the vertex when we replace $*$ with 1. Now let d_ξ be the arrow on the edge ξ. By the definition of m and Δ we get that d_ξ is a linear function which is identity on all tensor-factors except at the vector spaces it is acting on. With all the d_ξ's defined we can now define the differential for the complex as

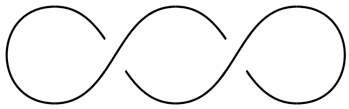
$$d^i = \sum_{|\xi|=i} (-1)^{sign(\xi)} d_\xi, \tag{24}$$

where $|\xi|$ = number of ones in ξ, and $sign(\xi)$ =number of ones before $*$. The sign is there just so the commutativity we have becomes anti-commutative, which forces the differential to make $(C^{*,*}(L), d)$ into a complex. Let us prove this.

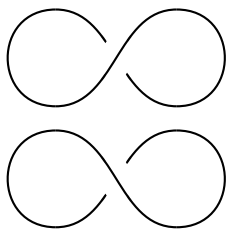
Proof. Note that we only have to check what is happening locally, we divide the cases into how many circles are changing. There is only one case that involves four circles.



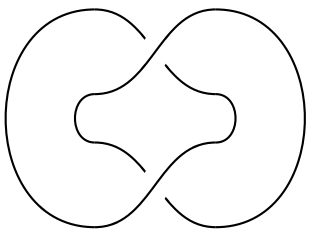
In this proof we always refer to the zero-resolution, i.e. you replace each crossing with its zero resolution. We also denote d_1 and d_2 as the functions when we change the first crossing respectively the second crossing from zero to one. In this case we see that the operations are disjoint which makes them commute naturally. Now for the cases with three circles.



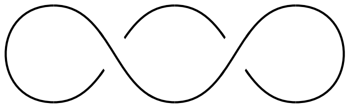
Here we can see that both of these functions are m. With the natural algebra defined on V we get that $m(a \otimes b) = ab$. Let $a, b, c \in \{1, x\}$, then $d_1 \circ d_2(a \otimes b \otimes c) = d_2(ab \otimes c) = abc = d_1(a \otimes bc) = d_2 \circ d_1(a \otimes b \otimes c)$. Hence d_1 and d_2 commutes in this case.



In this case we see that they are disjoint which makes them commute naturally. Now for the cases with two circles.



Here $d_2 \circ d_1 = \Delta \circ m = d_1 \circ d_2$ by definition of d_1 and d_2 using m and Δ. For the next case we have to do some calculations.



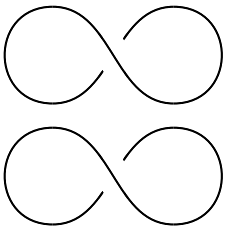
For $d_2 \circ d_1$ we have

$$\begin{aligned} d_2 \circ d_1(1 \otimes 1) &= 1 \otimes m(x \otimes 1) + x \otimes m(1 \otimes 1) = 1 \otimes x + x \otimes 1, \\ d_2 \circ d_1(1 \otimes x) &= 1 \otimes m(x \otimes x) + x \otimes m(1 \otimes x) = x \otimes x, \\ d_2 \circ d_1(x \otimes 1) &= x \otimes m(x \otimes 1) = x \otimes x, \\ d_2 \circ d_1(x \otimes x) &= x \otimes m(x \otimes x) = 0. \end{aligned} \tag{25}$$

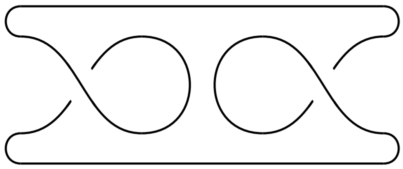
And for $d_1 \circ d_2$ we have

$$\begin{aligned} d_1 \circ d_2(1 \otimes 1) &= \Delta 1 = 1 \otimes x + x \otimes 1, \\ d_1 \circ d_2(1 \otimes x) &= \Delta(x) = x \otimes x, \\ d_1 \circ d_2(x \otimes 1) &= \Delta(x) = x \otimes x, \\ d_1 \circ d_2(x \otimes x) &= \Delta(0) = 0. \end{aligned} \tag{26}$$

Comparing these two d_1 and d_2 commutes.



Here they are also disjoint so they commute, and lastly we have one case that involves one circle



First we compute the results of $d_2 \circ d_1$

$$\begin{aligned} d_2 \circ d_1(1) &= 1 \otimes \Delta(x) + x \otimes \Delta(1) = 1 \otimes x \otimes x + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \\ d_2 \circ d_1(x) &= x \otimes \Delta(x) = x \otimes x \otimes x. \end{aligned} \tag{27}$$

And for $d_1 \circ d_2$ we have

$$\begin{aligned} d_1 \circ d_2(1) &= 1 \otimes \Delta(x) + x \otimes \Delta(1) = 1 \otimes x \otimes x + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \\ d_1 \circ d_2(x) &= x \otimes \Delta(x) = x \otimes x \otimes x. \end{aligned} \tag{28}$$

Note that these look very similar but they should not be compared as written since the second term does not live in the same vector space. We only have that the first term is the same, the second and third terms are actually switched in our identification of the V factors. But we can see that the results are the same in both cases making them commute in this case as well. The sign makes this commutativity into anti-commutativity. This completes the proof. □

5 Homology Groups

In this section we state the main theorem which tells us that the Khovanov homology is an invariant and define the Poincaré polynomial, see [4]. Before the proof of the main theorem we give an example of Khovanov homology for the trefoil knot.

Definition 10. *Let $f : X \to Y$ be a function. Then the cokernel of f is defined as $coker(f) = Y/f(X)$.*

Definition 11. *Suppose R is a commutative ring and 1_R its multiplicative identity. A R-module M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for every $r, s \in R$ and for every $x, y \in M$ we have*

$$\begin{aligned} r \cdot (x + y) &= rx + ry, \\ (r + s) \cdot x &= r \cdot x + r \cdot s, \\ (rs) \cdot x &= r \cdot (s \cdot x), \\ 1_R \cdot x &= x. \end{aligned} \tag{29}$$

The multiplication is often written as rx instead of $r \cdot x$.

Lemma 2. *(Snake lemma) Given the following commutative diagram of R-modules where each row is exact.*

$$\begin{array}{ccccccccc} & & M & \xrightarrow{f} & N & \xrightarrow{g} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & M' & \xrightarrow{f'} & N' & \xrightarrow{g'} & P' & & \end{array}$$

Then there is an exact sequence

$$\ker(\alpha) \xrightarrow{\widetilde{f}} \ker(\beta) \xrightarrow{\widetilde{g}} \ker(\gamma) \xrightarrow{\delta} coker(\alpha) \xrightarrow{\widetilde{f'}} coker(\beta) \xrightarrow{\widetilde{g'}} coker(\gamma), \tag{30}$$

where $\widetilde{f} = f|_{\ker(\alpha)}$, $\widetilde{g} = g|_{\ker(\beta)}$ and the maps $\widetilde{f'}$ and $\widetilde{g'}$ is given by $\widetilde{f'}(m + Im(\alpha)) = \widetilde{f'}(m) + Im(\beta)$ and $\widetilde{g'}(m + Im(\beta)) = \widetilde{g'}(m) + Im(\gamma)$.

This lemma shows the existence of a function δ that makes this sequence exact.

Proof. First we must show that every function is well-defined. The function $\widetilde{f}$ is well-defined because if $m \in \ker(\alpha)$ then $0 = f'(\alpha(m)) = \beta(f(m))$ by the commutativity of the diagram, thus $f(m) \in \ker(\beta)$. By the same reason if $m \in \ker(\beta)$ then $0 = g'(\beta(m)) = \gamma(g(m))$ which implies that $g(m) \in \ker(\gamma)$, hence g is well-defined. Now if $m + Im(\alpha) = m' + Im(\alpha)$ we have that $m - m' = \alpha(x)$ for some $x \in M$. Then $f'(\alpha(x)) = \beta(f(x))$ which implies that $f'(m - m') \in Im(\beta)$. Thus $\widetilde{f'}(m) + Im(\beta) = \widetilde{f'}(m') + Im(\beta)$ and this shows that $\widetilde{f'}$ is well-defined. A similar argument can be used for showing that $\widetilde{g'}$ is well-defined. Now we want to construct the function δ. If $x \in \ker(\gamma)$ then there exists an element $n \in N$ such that $g(n) = x$, this we know since we have exact rows. By the commutativity of the diagram we get $\beta(n) \in \ker(g') = Im(f')$ which implies that there exists $m \in M'$ such that $f'(m) = \beta(n)$. Now let us define δ such that

$$\delta(x) = m + Im(\alpha). \tag{31}$$

We can think of the function as $\delta(x) = f'^{-1}(\beta(g^{-1}(x))) + Im(\alpha)$. To prove that δ is well-defined we only need to check that if $x \in \ker(\gamma)$, $g(n) = g(n') = x$, $f'(m) = \beta(n)$ and $f'(m') = \beta(n')$ then $m - m' \in Im(\alpha)$. This is because the only place that is not well-defined is the lift from P to N. By the assumption on n and n' we have that $n - n' \in \ker(g) = Im(f)$. So there exists $z \in M$ such that $f(z) = n - n'$. Now we have that $\beta(n - n') = \beta(f(z)) = f'(\alpha(z)$. By injectivity of f' we have that $\alpha(z) = m - m'$, therefore $m - m' \in Im(\alpha)$. Hence δ is well-defined. What is left to show is that this sequence is exact at all places.

Exactness at $\ker(\beta)$**:** The composition $\widetilde{g} \circ \widetilde{f} = 0$ follows from the fact that $\widetilde{f}$ and $\widetilde{g}$ are restrictions of the functions f and g where $g \circ f = 0$. If $n \in \ker(\widetilde{g})$ then $g(n) = 0$ and since $\ker(g) = Im(f)$ we have that there exists $m \in M$ such that $f(m) = n$. We also need to show that $m \in \ker(\alpha)$. By commutativity of the diagram we have $f'(\alpha(m)) = \beta(f(m)) = \beta(n) = 0$ and since f' is injective $\alpha(m) = 0$ i.e $m \in \ker(\alpha)$. So $\ker(\widetilde{g}) = Im(\widetilde{f})$.

Exactness at $\ker(\gamma)$**:** Let $n \in \ker(\beta)$. Then let $\delta(g(n)) = m$. By the definition of the delta function we get that $f'(m) = \beta(n) = 0$ by assumption, and since f' is injective implies that $m = 0$. Hence $\delta \circ \widetilde{g} = 0$. Conversely, let $p \in \ker(\delta)$ (also $p \in \ker(\gamma)$). Now by surjectivity of g there exists $n \in N$ such that $g(n) = p$. Then $\beta(n) = f'(m)$ and $m \in Im(\alpha)$ since $p \in \ker(\delta)$, so there exists $x \in M$ such that $\alpha(x) = m$. Now $f'(\alpha(x)) = \beta(f(x)) = \beta(n)$ thus $n - f(x) \in \ker(\beta)$ and $\widetilde{g}(n - f(x)) = \widetilde{g}(n) - \widetilde{g}(f(x)) = \widetilde{g}(n) = p$. This follows from the fact that $g \circ f = 0$, so $\ker(\delta) = Im(\widetilde{g})$.

Exactness at $\boldsymbol{coker}(\alpha)$**:** Let $p \in \ker(\gamma)$. Then there exists $n \in N$ such that $g(n) = p$ and $\delta(p) = m$. Now $f'(m) = \beta(n)$ which implies that $f'(m) \in Im(\beta)$ so $\widetilde{f'} \circ \delta = 0$. Conversely, let $m \in \ker(f')$. Then there exists $n \in N$ such that $\beta(n) = f'(m)$. Now since $g' \circ f' = 0$ we get that $0 = g'(f'(m)) = g'(\beta(n)) = \gamma(g(n)) = 0$ which says that $g(n) \in \ker(\gamma)$. So $\delta(g(n)) = m$ and $\ker(\widetilde{f'}) = Im(\delta)$.

Exactness at $\boldsymbol{coker}(\beta)$**:** Let $m + Im(\alpha) \in coker(\alpha)$. $\widetilde{g'}(\widetilde{f'}(m + Im(\alpha)) = g'(f'(m)) + Im(\gamma)$ and since the second row is exact $g'(f'(m)) = 0$, thus $\widetilde{g'} \circ \widetilde{f'}$. Conversely, Let $q + Im(\beta) \in \ker(\widetilde{g'})$. We have to show that there exists $m \in coker(\alpha)$ such that $\widetilde{f'} = q$. Now $g'(q) \in Im(\gamma)$ which implies there exists $p \overset{\gamma}{\mapsto} g'(q)$ and $n \in N$ such that $g(n) = p$. Note that if we replace q with $q - \beta(n)$ then it would not change the coset. So $g'(q - \beta(n)) = g'(q) - \gamma(g(n)) = 0$. We did not change the coset by subtracting $\beta(n)$, so we can assume that $g'(q) = 0$. Now $q \in \ker(g') = Im(f')$ so there exists $m \in M'$

such that $n + Im(\beta) = \widetilde{f}'(m + Im(\alpha))$. This shows exactness at $coker(\beta)$.

Left to show is that $\widetilde{f}$ is injective and that $\widetilde{g}'$ is surjective. $\widetilde{f}$ is injective since it is a restriction of an injective function. If g' is surjective and $p + Im(\gamma) \in coker(\gamma)$ with $p = g'(n)$ for some $n \in N$, then $p + Im(\gamma) = \widetilde{g}'(n + Im(\beta))$ which implies that $\widetilde{g}'$ is surjective. □

Lemma 3. *Let C be a chain complex and let $C' \subset C$ be a subchain complex.*

- *If C' is acyclic (has no homology), then it can be cancelled in that case the homology $H(C)$ of C is equal to the homology $H(C/C')$ of C/C'.*
- *Likewise, if C/C' is acyclic then $H(C) = H(C')$.*

Proof. We have the long exact sequence of homology groups

$$\cdots \to H^r(C') \to H^r(C) \to H^r(C/C') \to H^{r+1}(C') \to \ldots, \tag{32}$$

associated with the short exact sequence $0 \to C' \to C \to C/C' \to 0$. This is a short exact sequence of complexes which are modules. Consider the following diagram.

$$\begin{array}{ccccccc}
 & & C'_n/\, Im\,\alpha_{n-1} & \longrightarrow & C_n/\, Im\,\beta_{n-1} & \longrightarrow & (C_n/C'_n)/\, Im\,\gamma_{n-1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha_n} & & \Big\downarrow{\scriptstyle \beta_n} & & \Big\downarrow{\scriptstyle \gamma_n} & & \\
0 & \longrightarrow & \ker\alpha_{n+1} & \longrightarrow & \ker\beta_{n+1} & \longrightarrow & \ker\gamma_{n+1} & &
\end{array}$$

Here α_n, β_n and γ_n are differentials in the complexes. These rows are exact and thus we can apply the Snake lemma to get the following short exact sequence

$$H_n(C') \to H_n(C) \to H_n(C/C') \xrightarrow{\delta} H_{n+1}(C') \to H_{n+1}(C) \to H_{n+1}(C/C'). \tag{33}$$

We apply the Snake lemma for all $n \in \mathbb{Z}$ to transform the short exact sequence $0 \to C' \to C \to C/C' \to 0$ into a long exact sequence of homology groups. If C' is acyclic then $0 \to H(C) \xrightarrow{f} H(C/C') \to 0$ for every $r \in \mathbb{Z}$. Since this is an exact sequence f becomes bijective, hence $H(C) = H(C/C')$. If C/C' is acyclic, then we get the sequence $0 \to H(C') \xrightarrow{f} H(C) \to 0$ and by the same argument we have that f is also bijective here, thus $H(C) = H(C')$. □

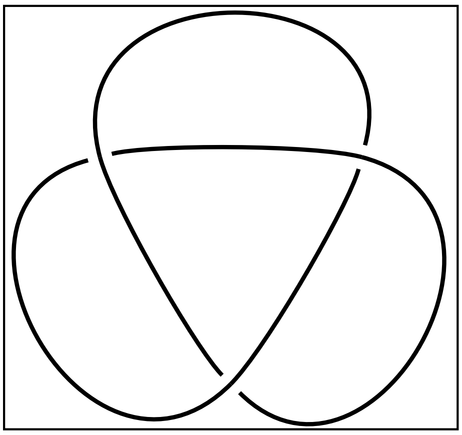
Let $H^r(L)$ be the rth cohomology of the complex $C(L)$. Now we can define $Kh(L)$ as

$$Kh(L) := \sum_r t^r q \dim(H^r(L)). \tag{34}$$

This is called the Poincaré polynomial.

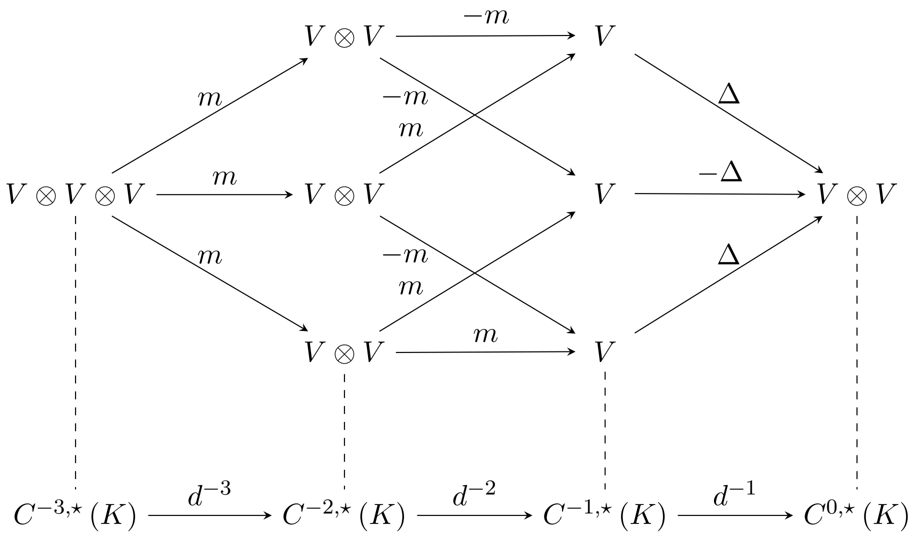
Theorem 3. *(Khovanov [Kh1]) The graded dimensions of the homology groups $H^r(L)$ are link invariants, and hence $Kh(L)$, a polynomial in the variables t and q, is a link invariant that specializes to the unnormalized Jones polynomial at $t = -1$.*

Before we prove this theorem we compute the Khovanov homology for the trefoil.



.

First we create a diagram of the complex. Recall that we can think of all the different cases for the crossings as a n-dimensional cube, where n is the number of crossings. Here we have a picture representing that case on each of the vertices. If we do this we get the following figure.



Here $C^{i,\star}(K)$ is the direct sum of all the vector spaces above it in the i^{th} place in the complex. The differentials for the complex are defined as

$$\begin{aligned} d^{-3}(v_1 \otimes v_2 \otimes v_3) &= (m(v_1 \otimes v_2) \otimes v_3, v_1 \otimes m(v_2 \otimes v_3), m(v_1 \otimes v_3) \otimes v_2), \\ d^{-2}((v_1 \otimes v_2, v_3 \otimes v_4, v_5 \otimes v_6)) &= (m(v_3 \otimes v_4) - m(v_1 \otimes v_2) \\ &\quad m(v_5 \otimes v_6) - m(v_1 \otimes v_2), m(v_5 \otimes v_6) - m(v_3 \otimes v_4)), \\ d^{-1}((v_1, v_2, v_3)) &= \Delta(v_1) - \Delta(v_2) + \Delta(v_3). \end{aligned} \tag{35}$$

We want to rewrite our vector spaces as

$$\begin{aligned} V \otimes V \otimes V =& \mathbb{Q}1 \otimes 1 \otimes 1 \oplus \mathbb{Q}x \otimes 1 \otimes 1 \oplus \mathbb{Q}1 \otimes x \otimes 1 \oplus \\ & \oplus \mathbb{Q}1 \otimes 1 \otimes x \oplus \mathbb{Q}x \otimes x \otimes 1 \oplus \mathbb{Q}x \otimes 1 \otimes x \oplus \\ & \oplus \mathbb{Q}1 \otimes x \otimes x \mathbb{Q}x \otimes x \otimes x, \\ (V \otimes V) \oplus (V \otimes V) \oplus (V \otimes V) =& \mathbb{Q}^3 1 \otimes 1 \oplus \mathbb{Q}^3 x \otimes 1 \oplus \mathbb{Q}^3 1 \otimes x \oplus \mathbb{Q}^3 x \otimes x, \\ V \oplus V \oplus V =& \mathbb{Q}^3 1 \oplus \mathbb{Q}^3 x, \\ V \otimes V =& \mathbb{Q}1 \otimes 1 \oplus \mathbb{Q}x \otimes 1 \oplus \mathbb{Q}1 \otimes x \oplus \mathbb{Q}x \otimes x. \end{aligned} \tag{36}$$

To make a matrix representation of d^{-3} one have to compute where every basis vector are mapped to.

$$\begin{aligned} d^{-3}:\ 1 \otimes 1 \otimes 1 &\mapsto (1 \otimes 1, 1 \otimes 1, 1 \otimes 1), \\ x \otimes 1 \otimes 1 &\mapsto (x \otimes 1, x \otimes 1, x \otimes 1), \\ 1 \otimes x \otimes 1 &\mapsto (x \otimes 1, 1 \otimes x, 1 \otimes x), \\ 1 \otimes 1 \otimes x &\mapsto (1 \otimes x, 1 \otimes x, x \otimes 1), \\ x \otimes x \otimes 1 &\mapsto (0, x \otimes x, x \otimes x), \\ x \otimes 1 \otimes x &\mapsto (x \otimes x, x \otimes x, 0), \\ 1 \otimes x \otimes x &\mapsto (x \otimes x, 0, x \otimes x), \\ x \otimes x \otimes x &\mapsto (0, 0, 0). \end{aligned} \tag{37}$$

If we take an ordered basis for all of the vector spaces then the matrix representation for d^{-3} becomes

$$d^{-3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{38}$$

Similarly for d^{-2} and d^{-1} we have

$$\begin{aligned} d^{-2}:\ &(1\otimes 1,0,0)\mapsto(-1,-1,0),\\ &(0,1\otimes 1,0)\mapsto(1,0,-1),\\ &(0,0,1\otimes 1)\mapsto(0,1,1),\\ &(x\otimes 1,0,0)\mapsto(-x,-x,0),\\ &(0,x\otimes 1,0)\mapsto(x,0,-x),\\ &(0,0,x\otimes 1)\mapsto(0,x,x),\\ &(1\otimes x,0,0)\mapsto(-x,-x,0),\\ &(0,1\otimes x,0)\mapsto(x,0,-x),\\ &(0,0,1\otimes x)\mapsto(0,x,x),\\ &(x\otimes x,0,0)\mapsto(0,0,0),\\ &(0,x\otimes x,0)\mapsto(0,0,0),\\ &(0,0,x\otimes x)\mapsto(0,0,0), \end{aligned} \tag{39}$$

$$\begin{aligned} d^{-1}:\ &(1,0,0)\mapsto 1\otimes x+x\otimes 1,\\ &(0,1,0)\mapsto -1\otimes x-x\otimes 1,\\ &(0,0,1)\mapsto 1\otimes x+x\otimes 1,\\ &(x,0,0)\mapsto x\otimes x,\\ &(0,x,0)\mapsto -x\otimes x,\\ &(0,0,x)\mapsto x\otimes x. \end{aligned} \tag{40}$$

The matrix representations then becomes

$$d^{-2}=\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

$$d^{-1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 1 & 0 & 0 & 0\\ 1 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}. \tag{42}$$

Before we can compute the homology we have to work out the image and the kernel for each matrix. The image is the space spanned by the column vectors and the kernel is the nullspace. With some linear algebra we work out the image and kernel of d^{-3} to be

$$Im\, d^{-3} = span_{\mathbb{Q}} \left(\begin{bmatrix} 1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\1\\0\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1\\-1\\0\\-1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix} \right), \ker d^{-3} = span_{\mathbb{Q}} \left(\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix} \right). \tag{43}$$

Next we compute the image of d^{-2}. It is seen that the first, second, fourth and fifth column vectors are linearly independent and the rank of the matrix is four, so the image is

$$Im\, d^{-2} = span_{\mathbb{Q}} \left(\begin{bmatrix} 1\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\-1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\-1 \end{bmatrix} \right). \tag{44}$$

To calculate the kernel we need to first reduce the matrix to its reduced row echelon form. The reduced row echelon form of d^{-2} is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{45}$$

Which corresponds to the system

$$\left\{ \begin{array}{ccccccccccc} x_1 & & -x_3 & & & & & & & = & 0 \\ & x_2 & -x_3 & & & & & & & = & 0 \\ & & & x_4 & & -x_6 & x_7 & & -x_9 & = & 0 \\ & & & & x_5 & -x_6 & & x_8 & -x_9 & = & 0 \end{array} \right. . \tag{46}$$

Thus the basis for the null space is

$$\ker d^{-2} = span_{\mathbb{Q}} \left(\begin{bmatrix}1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\\0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\-1\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} \right). \tag{47}$$

Lastly the kernel and image for d^{-1}. The image is spanned by first and fourth column vector

$$Im\, d^{-1} = span_{\mathbb{Q}} \left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right). \tag{48}$$

For the kernel we rewrite it to its reduced row echelon form.

$$\begin{bmatrix}1 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{49}$$

Which corresponds to the system

$$\begin{cases} x_1 \quad -x_2 \quad x_3 & = 0 \\ \qquad\qquad x_4 \quad -x_5 \quad x_6 & = 0 \end{cases}. \tag{50}$$

Thus the null space is

$$\ker d^{-1} = span_{\mathbb{Q}} \left(\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\-1\end{bmatrix} \right). \tag{51}$$

Left to compute is the cohomology groups. We begin by computing $H^{-3}(K)$. The image of d^{-4} has dimension zero since it is the zero function, thus $H^{-3}(K) = \mathbb{Q}x \otimes x \otimes x$. For the second cohomology group we have to do some work. Note that the first and the last three vectors are the same for $\ker d^{-2}$ and $Im\, d^{-3}$. We only need to work out the quotient

$$\frac{span_{\mathbb{Q}}\left(\begin{bmatrix}0\\0\\0\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\-1\\0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\-1\\0\\0\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\1\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}\right)}{span_{\mathbb{Q}}\left(\begin{bmatrix}0\\0\\0\\1\\0\\1\\0\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\-1\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\1\\1\\0\\0\\0\end{bmatrix}\right)}. \tag{52}$$

We want to rewrite it by changing to a basis formed by the numerator. Let us denote the vector in the numerator as v_1, v_2, v_3 and v_4 (in order). The bottom vectors can now be express as $v_1 - v_3$, $v_4 - v_1 - v_3$ and $v_4 - v_2 - v_3$ (in order). After this change of basis the quotient becomes

$$\frac{span_{\mathbb{Q}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)}{span_{\mathbb{Q}}\left(\begin{bmatrix}1\\0\\-1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix},\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\right)}. \tag{53}$$

Now we see that the relation induced on the generators in the numerator by the vectors in the denominator are as follows $v_1 = -v_2 = v_3 = \frac{1}{2}v_4$, thus we only need to look at v_1 in the quotient. So $H^{-2}(K) = \mathbb{Q}(x \otimes 1, x \otimes 1, x \otimes 1)$. For the next homology group we have $H^{-1}(K) = 0$, since $Im\, d^{-2}$ and $\ker d^{-1}$ are spanned by the same vectors. Lastly we have quotient

$$\frac{span_{\mathbb{Q}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)}{span_{\mathbb{Q}}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\right)} \cong span_{\mathbb{Q}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right). \tag{54}$$

Hence $H^0(K) = \mathbb{Q}1 \otimes 1 \oplus \mathbb{Q}1 \otimes x$. To create a table for the trefoil we need to compute the quantum gradings. Recall that $i = k_\alpha - n_-$ and $j = \deg(v) + i + n_+ - n_-$. In this case we have that $n_- = 3$ and $n_+ = 0$.

j \ i	-3	-2	-1	0
-1				$\mathbb{Q}$
-2				
-3				$\mathbb{Q}$
-4				
-5		$\mathbb{Q}$		
-6				
-7				
-8				
-9	$\mathbb{Q}$			

We also want to compute the unnormalized Jones polynomial using (13) and then compute it with (34) and compare them. Plugging everything into (13) gives us

$$\sum_{\alpha\in\{0,1\}^n} (-1)^{k_\alpha+n_-} q^{n_+-2n_-+k_\alpha}(q+q^{-1})^{r_\alpha} = -q^{-9} + q^{-5} + q^{-3} + q^{-1}. \tag{55}$$

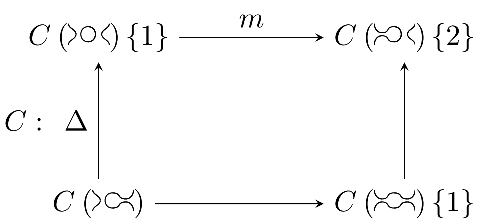
The Poincaré polynomial for the trefoil is

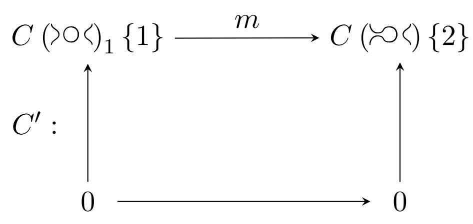
$$Kh(K) = t^{-3}q^{-9} + t^{-2}q^{-5} + q^{-3} + q^{-1}. \tag{56}$$

Note that if we set $t = -1$ then we get the unnormalized Jones polynomial, thus we are relatively sure that we have not made any mistake.

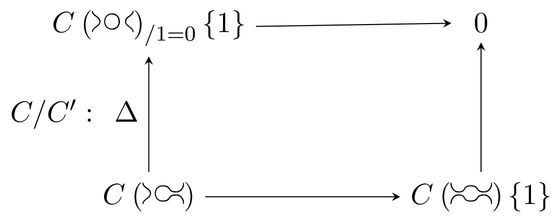
Proof. (Theorem 3)(Invariance under R1, right twist) In the case of the right twist we have the following complex $C = (C\left(\overset{\circ}{\frown}\right) \xrightarrow{m} C\left(\frown\right)\{1\})$. Let us denote with $C\left(\overset{\circ}{\frown}\right)_a$ the space where the special cycle (the circle disjoint from the rest of the knot seen in the figure) is always marked with a. We can see that $\tilde{C} = (C\left(\overset{\circ}{\frown}\right)_1 \xrightarrow{m} C\left(\frown\right)\{1\})$ is a subcomplex. The cohomology of this subcomplex is zero since $\ker m = \{0\}$ and because of the definition of the function m we get that it is the identity. In this case it is also onto, so the subcomplex is acyclic. By applying Lemma (3), the $H(C)$ is equal to $H(C/\tilde{C})$. The complex $C/\tilde{C} = (C(\cdots)_{/1=0} \xrightarrow{m} 0\{1\})$, where the subscript $C(\cdots)_{/1=0} = C(\cdots)/C(\cdots)_1$. The degree shift of the two differentials are cancelled out by $[-n_-]\{n_+ - 2n_-\}$. Note that it is one dimensional making $C(\cdots)_{/1=0}$ isomorphic to $C(\cdots)$. The proof for the left twist follows from the right twist and $R2$.

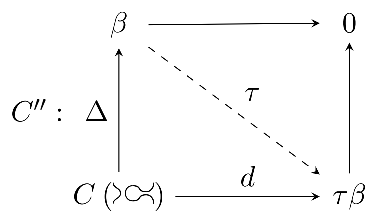
(Invariance under R2) For R2 we have the complex





We do the same trick here as we did for the R1 case. We set the special cycle to 1 in the upper left complex and keep the upper right as it is. The bottom part be set to zero. This makes the function m to be equal the identity. This subcomplex is indeed acyclic since $\ker m = \{0\}$ and its image is $C(\cdots)$. By lemma (3) we have that $H(C) = H(C/C')$. The quotient is equal to

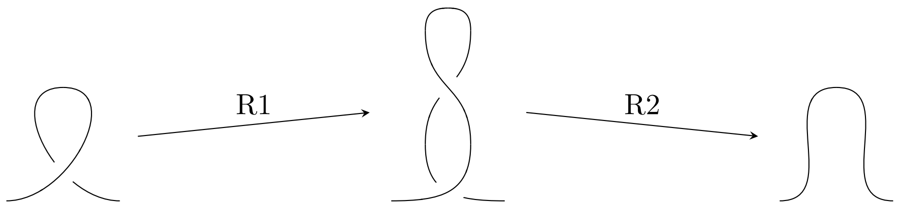




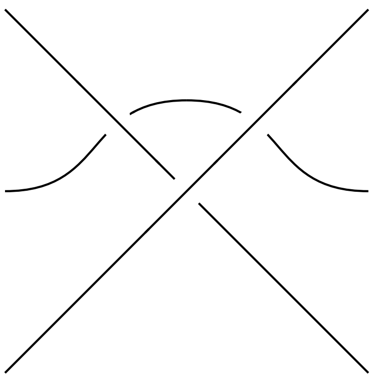
In this complex we see that $a \otimes \Delta(1) = a \otimes 1 \otimes x + a \otimes x \otimes 1 = a \otimes x \otimes 1$ and $a \otimes \Delta(x) = a \otimes x \otimes x$, where a contains all the tensor factors that remains constant. Under the quotient C/C' the function Δ just adds an x to the special cycle making it bijective. Now we want to define a function $\tau = d \circ \Delta^{-1}$ and construct the subcomplex with all pairs $(\beta, \tau\beta) \in C(\cdots)\{1\} \oplus C(\cdots)\{1\}$ as well as the whole $C(\cdots)$. Note that this function is not a differential for the complex. The function Δ has a trivial kernel since it is bijective. Thus $\ker(\Delta + d) = \{0\}$. The image is all the pairs $(\beta, \tau\beta)$ by definition of τ. Hence the subcomplex is acyclic. By lemma (3) it suffices to consider $(C/C')/C''$. When we quotient out all the pairs $(\beta, \tau\beta)$, it makes every element $(\beta, 0) = (0, -\tau\beta)$. So every $\beta \in C(\cdots)_{/1=0}$ is identified with some element in $C(\cdots)$, and we have no restriction for the elements in $C(\cdots)$. This makes the two following complexes isomorphic

$$
(C/C')/C'' : \quad
\begin{array}{ccc}
\beta & \longrightarrow & 0 \\
\uparrow & \overset{\tau}{\searrow} & \uparrow \\
0 & \xrightarrow{d_{\star 0}} & \gamma
\end{array}
\qquad\qquad
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & C\,(\asymp)
\end{array}
$$

We see that we have some shift between the above complex and the complex $C\,(\asymp)$, but this shift gets taken out with the shift $[-n_-]\{n_+ - 2n_-\}$. Thus $H(C\,(\asymp)) = H(C\,(\asymp))$.
(Invariance under R1, left twist) This follows from the right twist and R2, as shown in the figure below.



(Invariance under R3) First we write the complexes for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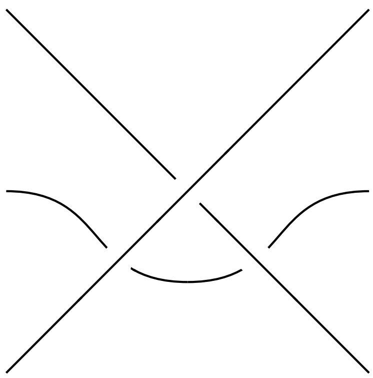
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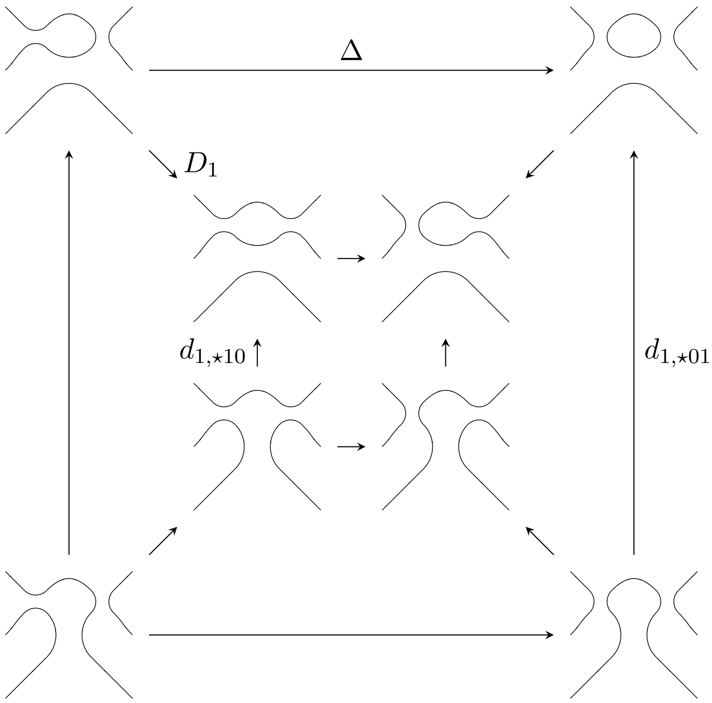

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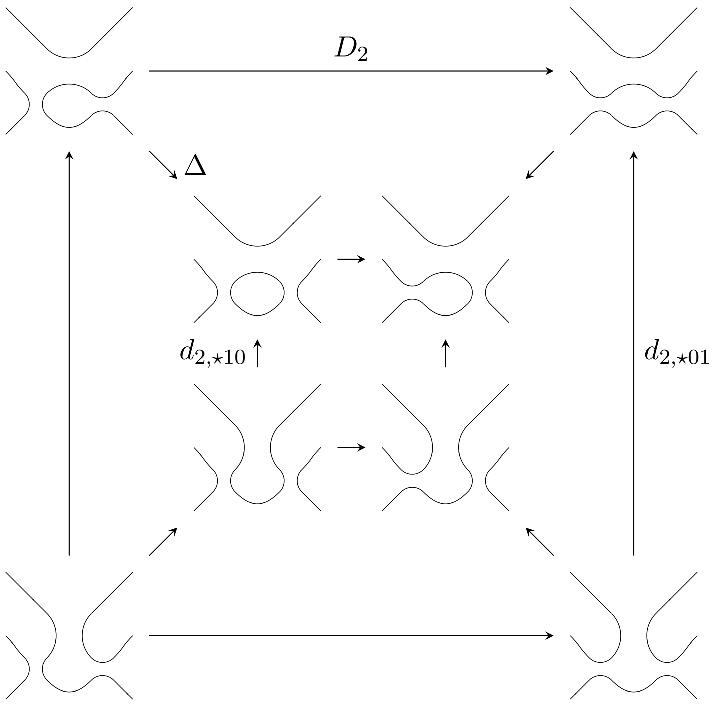
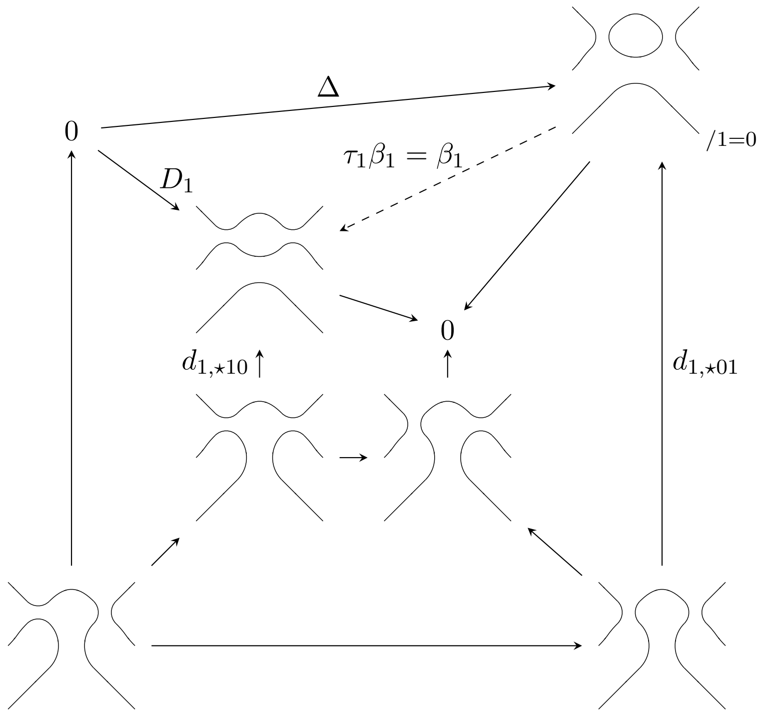
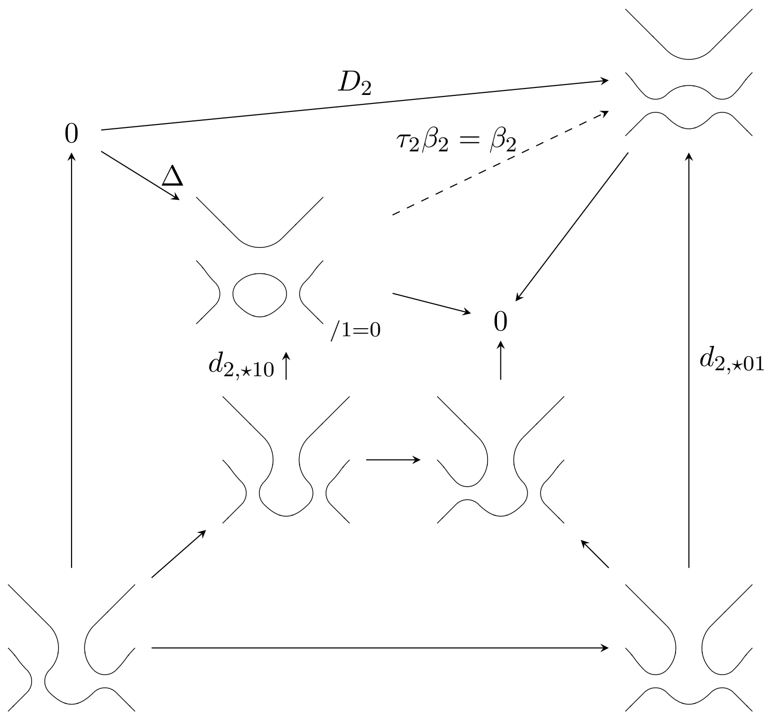


.

Note that the top sides of both of these complexes is exactly like the case of R2 as we saw earlier. If we apply R2 here then we get the following



,



.

Note that the bottom sides are isomorphic and that the top sides are mirrored. Thus we consider the isomorphism ϕ that transposes the top layer while keeping the bottom layer fixed, i. e. the top right vertex switches place with the bottom left vertex in the top layer and keeping the rest fixed. To show that this is an isomorphism between complexes we need to know that the diagrams commute with ϕ, i.e. we need to show that $\tau_1 \circ d_{1,\star 01} = d_{2,\star 01}$ and $\tau_2 \circ d_{2,\star 10} = d_{1,\star 10}$. Note that $d_{1,\star 01} = \Delta$ and $d_{2,\star 10} = \Delta$. Now since the special cycle has been set to x these functions become bijective, by the same reason as in R2 case. We first prove $\tau_1 \circ d_{1,\star 01} = d_{2,\star 01}$. By definition we have $\tau_1 \circ d_{1,\star 01} = D_2 \circ \Delta^{-1} \circ d_{1,\star 01}$, but the bottom right corner is isomorphic to the top left corner. This makes $\Delta^{-1} \circ d_{1,\star 01}$ equal to the identity. Since the bottom half of the two complexes are isomorphic, it follows that $D_1 = d_{2,\star 01}$. For the second case we use the same reasoning to say that $\Delta^{-1} \circ d_{2,\star 10}$ is the identity and here we also use the fact that the bottom sides are isomorphic to each other making $D_2 = d_{1,\star 10}$. This shows that the functions commutes with ϕ so it is an isomorphism between the complexes. The last thing we have to show is that the Poincaré polynomial specializes to the unnormalized Jones polynomial at $t = 1$. By theorem (2) we need to show the following equality

$$\sum_i (-1)^i q\dim(H^i(L)) = \sum_i (-1)^i q\dim(C^{i,\star}(L)). \tag{57}$$

We prove this by induction with base case $0 \to C^0 \to 0$. This is true for this case since the cohomology group at zero is C^0. Now assume it works for $n-1$. For the complex of length n we have

$$0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0. \tag{58}$$

Note that we can split C^n to $Im\, d^{n-1} \oplus H^n$, where H^n is the complement of $Im\, d^{n-1}$ in C^n. Now consider the following subcomplex

$$0 \to C^1 \to \cdots \to \ker d^{n-1} \to H^n \to 0. \tag{59}$$

These two have the same cohomology groups by definition. If we use the assumption here we only have to look at the end of the complex. From linear algebra we have $q\dim C^{n-1} = q\dim\ker d^{n-1} + q\dim Im\, d^{n-1}$ and $q\dim C^n = q\dim(Im\, d^{n-1})^c + q\dim H^n$. This is the dimension theorem but for the graded dimension. It holds since our functions have degree -1 so it follows when we also consider degree shifts. Plugging everything into the right side of the equation (57) we get

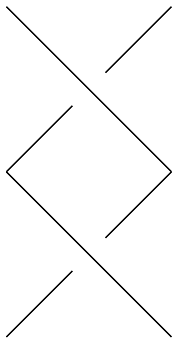
$$\begin{aligned} &\sum_i^{n-2} (-1)^i q\dim(C^{i,\star}(L)) + (-1)^{n-1}(q\dim\ker d^{n-1} + q\dim Im\, d^{n-1}) + \\ &\qquad\qquad + (-1)^n (q\dim(Im\, d^{n-1})^c + q\dim H^n) = \\ &= \sum_i^{n-2} (-1)^i q\dim(C^{i,\star}(L)) + (-1)^{n-1} q\dim\ker d^{n-1} + (-1)^n (q\dim H^n) = \\ &= \sum_i^{n} (-1)^i q\dim(H^i(L)). \end{aligned} \tag{60}$$

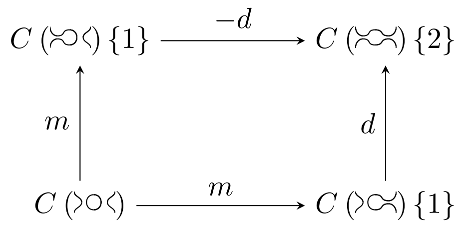
This completes the proof. □

6 Computations for $(T_{2,k})$ Torus Links

It takes time to work out the Khovanov homology of a knot, hence we want to derive a more efficient method to compute the Khovanov homology for some torus knots $(T_{2,k})$. Here we study [2].

First let us study the following case

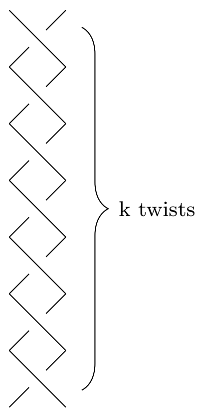




The left figure shows the knot diagram and on the right we see its complex. If we fix the special cycle in the lower left corner to 1 then, by the same argument as earlier, each m becomes the identity. Thus we have a complex that is $C(\succ\circ\prec)_1 \xrightarrow{m+m} (\beta,\beta) \to 0$, where (β,β) means $\{(\beta,\beta):\ (\beta,\beta)\in C(\succ\circ\asymp)\oplus C(\asymp\circ\prec)\}$. This is acyclic. By lemma (3) it suffices to consider the quotient. What is this quotient? First note that if we have a linear function $f: X\to X\times X$ defined by $f(x)=(f_1(x),f_2(x))$, and if we quote out the diagonal Δ in the image then $f: X\to X\times X/\Delta$ where $f(x)=(f_1(x),f_2(x))=(f_1(x)-f_2(x),f_2(x)-f_2(x))=(f_1(x)-f_2(x),0)$. This is equivalent to $f': X\to X$ where $f'(x)=f_1(x)-f_2(x)$. Also note that $C(\succ\circ\asymp)\cong C(\asymp\circ\prec)$. Applying this trick to our case yields

$$0 \longrightarrow C(\succ\circ\prec)_{/1=0} \xrightarrow{m_l-m_r} C(\asymp\circ\prec)\{1\} \xrightarrow{d} C(\asymp\circ\asymp)\{2\} \longrightarrow 0. \tag{61}$$

Here m_l and m_r collapses the special cycle with the left circle and the right circle respectively. Now we want to construct a similar complex for when we have k twists.



Definition 12. *A quasi-isomorphism is a map $A \to B$ of chain complexes (respectively, cochain complexes) such that the induced maps*

$$H_n(A_\star) \to H_n(B_\star) \; (\textit{respectively } H^n(A^\star) \to H^n(B^\star), \tag{62}$$

of homology groups (respectively, of cohomology groups) are isomorphisms for all n.

Let us denote the k twist diagram as D.

Proposition 2. *The complex $C\,(D)$ is quasi-isomorphic to the following chain complex*

$$\begin{aligned} &0 \longrightarrow C\,(\succ\prec)\,\{1-k\} \xrightarrow{\partial^{-k}} C\,(\succ\prec)\,\{3-k\} \xrightarrow{\partial^{1-k}} \dots \\ &\dots \xrightarrow{\partial^{-3}} C\,(\succ\prec)\,\{k-3\} \xrightarrow{\partial^{-2}} C\,(\succ\prec)\,\{k-1\} \xrightarrow{\partial^{-1}} C\,(\asymp)\,\{k\} \longrightarrow 0. \end{aligned} \tag{63}$$

Where all the differentials are

$$\begin{aligned} \partial^{-1} &= d, \\ \partial^{-2} &= m_{lx} - m_{rx}, \\ \partial^{-3} &= m_{lx} + m_{rx}, \\ &\dots \\ \partial^{-k} &= m_{lx} - (-1)^k m_{rx}. \end{aligned} \tag{64}$$

Cohomology groups $H(D)$ are quasi-isomorphic to the above chain complex.

First we need to explain our notation. Let $X \ : C\,(\succ\prec) \to C\,(\succ\circ\prec)$ which adds a cycle marked with x. With this we can define $m_{lx} = m_l \circ X$ and $m_{rx} = m_r \circ X$.

Proof. We prove this by induction with the base case $k = 2$, which we proved in the beginning of this section. Assume that it works for $k = n$. For the induction step we have to use this assumption to work out the case for $k = n+1$. If we apply the assumption for the last n crossings we get the complex

$$\begin{array}{ccccccc} C\,(\succ\circ\prec) & \xrightarrow{m_{lx} - (-1)^k m_{rx}} & C\,(\succ\circ\prec) & \xrightarrow{m_{lx} - (-1)^{k-1} m_{rx}} & C\,(\succ\circ\prec) & \xrightarrow{m_{lx} - (-1)^{k-2} m_{rx}} & \cdots \\ \downarrow m & & \downarrow m & & \downarrow m & & \\ C\,(\succ\prec) & \xrightarrow{-(m_{lx} - (-1)^k m_{rx})} & C\,(\succ\prec) & \xrightarrow{-(m_{lx} - (-1)^{k-1} m_{rx})} & C\,(\succ\prec) & \xrightarrow{-(m_{lx} - (-1)^{k-2} m_{rx})} & \cdots \end{array}$$

$$\begin{array}{ccccccc} \cdots & \xrightarrow{m_{lx} + m_{rx}} & C\,(\succ\circ\prec) & \xrightarrow{m_{lx} - m_{rx}} & C\,(\succ\circ\prec) & \xrightarrow{m} & C\,(\succ\prec) \\ & & \downarrow m & & \downarrow m & & \downarrow d \\ \cdots & \xrightarrow{-(m_{lx} + m_{rx})} & C\,(\succ\prec) & \xrightarrow{-(m_{lx} - m_{rx})} & C\,(\succ\prec) & \xrightarrow{-d} & C\,(\asymp) \end{array}$$

Note that we do not write what the degree shifts are. We deal with that at the end of the proof. Consider the following subcomplex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & C(\rangle\circ\langle)_1 & \xrightarrow{m} & \beta \\
\downarrow & & & & \downarrow & & \downarrow m & & \downarrow \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \beta & \longrightarrow & 0
\end{array}$$

where these $\beta \in C(\rangle\langle)$ means the diagonal, i.e $0 \to C(\rangle\circ\langle)_1 \xrightarrow{m+m} (\beta, \beta) \to 0$. When we write β we mean the diagonal. This is the same case where $k = 2$, thus this is acyclic and a subcomplex. By lemma (3) is suffices to consider that quotient. What we are left with is

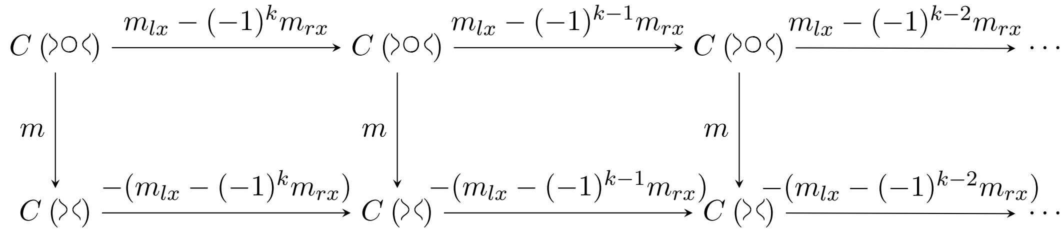
$$\begin{array}{ccccccc}
C(\rangle\circ\langle) & \xrightarrow{m_{lx} - (-1)^k m_{rx}} & C(\rangle\circ\langle) & \xrightarrow{m_{lx} - (-1)^{k-1} m_{rx}} & C(\rangle\circ\langle) & \xrightarrow{m_{lx} - (-1)^{k-2} m_{rx}} & \cdots \\
\downarrow m & & \downarrow m & & \downarrow m & & \\
C(\rangle\langle) & \xrightarrow{-(m_{lx} - (-1)^k m_{rx})} & C(\rangle\langle) & \xrightarrow{-(m_{lx} - (-1)^{k-1} m_{rx})} & C(\rangle\langle) & \xrightarrow{-(m_{lx} - (-1)^{k-2} m_{rx})} & \cdots
\end{array}$$

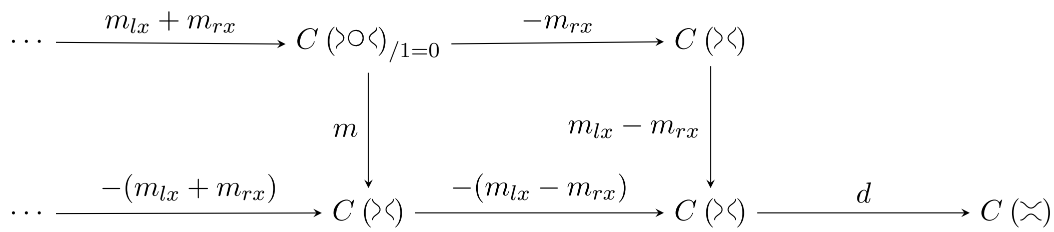
$$\begin{array}{ccccccc}
\cdots & \xrightarrow{m_{lx} + m_{rx}} & C(\rangle\circ\langle) & \xrightarrow{m_{lx} - m_{rx}} & C(\rangle\circ\langle)_{/1=0} & & \\
 & & \downarrow m & & \downarrow m_l - m_r & & \\
\cdots & \xrightarrow{-(m_{lx} + m_{rx})} & C(\rangle\langle) & \xrightarrow{-(m_{lx} - m_{rx})} & C(\rangle\langle) & \xrightarrow{d} & C(\asymp)
\end{array}$$

Where m_l and m_r collapses the special cycle with the left respectively right vector space. Here we can consider the following subcomplex

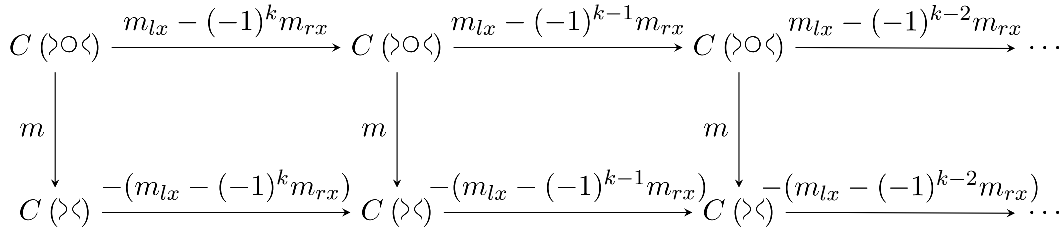
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & C(\rangle\circ\langle)_1 & \xrightarrow{m_{lx} - m_{rx}} & \beta \otimes x & & \\
\downarrow & & & & \downarrow m & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \beta & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

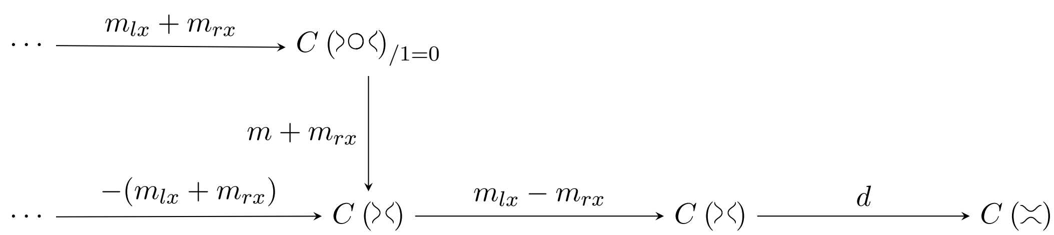
Note that here m is the identity and $m_{rx} = 0$ since the special cycle is be marked with 1, it becomes the zero function under the quotient. Also note that $m_{lx}(a \otimes 1) \mapsto a \otimes x$, so this subcomplex must be acyclic. After the quotient we can rewrite the complex as





Note that we also have the quotient with the diagonal at the end so by the same reason as before this complex becomes





We do this for every rectangle in the complex to arrive at the following complex

$$
\begin{array}{l}
C\left(\rangle\circ\langle\right) \\
\quad\Big\downarrow\, m_{lx} - (-1)^{k+1} m_{rx} \\
C\left(\rangle\langle\right) \xrightarrow{m_{lx} - (-1)^{k} m_{rx}} C\left(\rangle\langle\right) \xrightarrow{m_{lx} - (-1)^{k-1} m_{rx}} C\left(\rangle\langle\right) \xrightarrow{m_{lx} - (-1)^{k-2} m_{rx}} \cdots
\end{array}
$$

$$
\cdots \xrightarrow{m_{lx} + m_{rx}} C\left(\rangle\langle\right) \xrightarrow{m_{lx} - m_{rx}} C\left(\rangle\langle\right) \xrightarrow{d} C\left(\asymp\right)
$$

Each position in the complex has shifted its degree with one step since the normal degree shifts are on the top side of the complex and the bottom side is shifted with one from the above. This completes the proof. $\square$

With this proposition we can work out the Khovanov homology more efficiently. For the trefoil we have $k = 3$, thus the complex becomes

$$
0 \to C^{-3}\left(\asymp\right)\{-2\} \xrightarrow{2x} C^{-2}\left(\asymp\right)\{-1\} \xrightarrow{0} C^{-1}\left(\asymp\right) \xrightarrow{\Delta} C^{0}\left(\asymp\right) \to 0. \tag{65}
$$

The vector spaces here are one dimensional, thus we can derive the cohomology groups to be $H^{-3}(D) = (x)$, $H^{-2}(D) = (1)$, $H^{-1}(D) = 0$ and $H^{0}(D) = (1 \otimes 1, 1 \otimes x)$ where D is the diagram of the trefoil. The last one is achieved by taking the quotient

$$
H^{0}(D) = \frac{(1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x)}{x \otimes x, 1 \otimes x + x \otimes 1} = (1 \otimes 1, 1 \otimes x). \tag{66}
$$

The quantum grading for $x \in H^{-3}(D)$ is $j = -7$, for $1 \in H^{-2}(D)$ is $j = -4$ and for $1 \otimes 1, 1 \otimes x \in H^{0}(D)$ are $j = -1$ as well as $j = -3$ respectively. With the degree shifts in the complex we arrive at the same result as earlier.

7 Switching the Field into an Arbitrary Ring

In this section we make Khovanov homology stronger by exchanging the field $\mathbb{Q}$ into an arbitrary ring R. We compute the more generalized Khovanov homology for an arbitrary torus knot $(T_{2,k})$ [2].

Recall the definition of R-modules (definition 11). We can think of R-modules as a vector space where its coefficients are taken from a ring instead of a field. In the proof for proposition (2) we do not use the fact that we were working with the field $\mathbb{Q}$. Thus it also holds for rings. So why would we work with rings instead of fields? Consider we have the quotient $span(x)/span(2x)$. This is zero if we are working with $\mathbb{Q}$ since it has an inverse of 2, but if we look at the ring $\mathbb{Z}$ we get that this is equal to $\mathbb{Z}_2$. This makes the Khovanov invariant stronger since we get more information from the knots. With this cleared up we can begin by calculating the Khovanov homology for a general k-torus over an arbitrary ring R.

Proposition 3. *The cohomology groups of $T_{2,k}$ are*

$$\begin{aligned} H^i(T_{2,k}) &= 0 \quad \text{for } i < -k \text{ and } i > 0, && (67)\\ H^0(T_{2,k}) &= R\{-k\} \oplus R\{2-k\}, && (68)\\ H^{-1}(T_{2,k}) &= 0, && (69)\\ H^{-2j}(T_{2,k}) &= (R/2R)\{-4j-k\} \oplus R\{-4j+2-k\} \quad \text{for } 1 \le j \le \frac{k-1}{2}, j \in \mathbb{Z}, && (70)\\ H^{-2j-1}(T_{2,k}) &= R\{-4j-2-k\} \quad \text{for } 1 \le j \le \frac{k-1}{2}, j \in \mathbb{Z}, && (71)\\ H^{-k}(T_{2,k}) &= R\{-3k\} \oplus R\{2-3k\} \qquad \text{for even } k. && (72) \end{aligned}$$

Proof. By using proposition (2) our chain complex becomes

$$\begin{aligned} &0 \longrightarrow C(\rangle\langle)\{1-k\} \xrightarrow{x(-1)^k x} C(\rangle\langle)\{3-k\} \xrightarrow{x(-1)^{k-1}x} \dots \\ &\dots \xrightarrow{2x} C(\rangle\langle)\{k-3\} \xrightarrow{0} C(\rangle\langle)\{k-1\} \xrightarrow{\Delta} C(\asymp)\{k\} \longrightarrow 0, \end{aligned} \tag{73}$$

where $2x$ denotes the map that multiplies with $2x$ on the tensor factor. Note that every other map is zero since $m_{lx} = m_{rx} = x$. The knots only have k number of cohomology groups so (67) follows from that. The equality (68) follows from the fact that $Im\,\partial^{-1} = span(1 \otimes x + x \otimes 1, x \otimes x)$ and $\ker \partial^0 = V \otimes V$, so the cohomology group becomes $span(1 \otimes 1, 1 \otimes x)$. The quantum grading for these elements are $2-k$ and $-k$ respectively. The kernel of ∂^{-1} is trivial, thus (69) follows. The cohomology group at $-2j$ is $span(1, x)/span(2x)$ and their quantum gradings are $1-2j-k$ and $-1-2j-k$ respectively. The degree shift is $-2j+1$, if we add this degree shift to the quantum gradings we get (70). The cohomology group for $-2j-1$ is $span(x)$ since the differential is multiplication with $2x$. The quantum grading becomes $-1-2j-k$ and with the degree shift $-2j$ we get (71). Lastly we have to look at when k is even and the first entry of the complex. The differential is the zero function so the cohomology group is

$span(1, x)$. We get the quantum gradings to be $-1-2k$ and $1-2k$, this with the degree shift $1-k$ we arrive at (72). $\square$

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