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Semigroups, multisemigroups and representations

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Abstract

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This thesis consists of four papers about the intersection between semigroup theory, category theory and representation theory. We say that a representation of a semigroup by a matrix semigroup is effective if it is injective and define the effective dimension of a semigroup S as the minimal n such that S has an effective representation by square matrices of size n.

A multisemigroup is a generalization of a semigroup where the multiplication is set-valued, but still associative.

A 2-category consists of objects, 1-morphisms and 2-morphisms. A finitary 2-category has finite dimensional vector spaces as objects and linear maps as morphisms. This setting permits the notion of indecomposable 1-morphisms, which turn out to form a multisemigroup.

Paper I computes the effective dimension Hecke-Kiselman monoids of type A. Hecke-Kiselman monoids are defined by generators and relations, where the generators are vertices and the relations depend on arrows in a given quiver.

Paper II computes the effective dimension of path semigroups and truncated path semigroups. A path semigroup is defined as the set of all paths in a quiver, with concatenation as multiplication. It is said to be truncated if we introduce the relation that all paths of length N are zero.

Paper III defines the notion of a multisemigroup with multiplicities and discusses how it better captures the structure of a 2-category, compared to a multisemigroup (without multiplicities).

Paper IV gives an example of a family of 2-categories in which the multisemigroup with multiplicities is not a semigroup, but where the multiplicities are either 0 or 1. We describe these multisemigroups combinatorially.

Keywords: representation theory, semigroups, multisemigroups, category theory, 2-categories

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Effective Representations of Hecke-Kiselman Monoids of type A, manuscript

II Effective Representations of Path Semigroups, *Semigroup Forum 92 (2016), no. 2, 449–459*

III Multisemigroups with multiplicities and complete ordered semi-rings, *Beitr. Algebra Geom. 58 (2017), no. 2, 405–426.*

IV Sub-bimodules of the identity bimodule for cyclic quivers, manuscript

Contents

1. Introduction

1.1 Semigroups

One of the most important notions in mathematics is that of a *function*. For simplicity, let us for the moment consider functions from some set X to itself. Such functions are also known as *transformations* of X. Then a function from X to X is a subset $f \subset X \times X$ which has the property that, for any $x \in X$, there is a unique $y \in X$ such that $(x,y) \in f$. The latter is usually written $f(x) = y$. The set of all functions from X to X is denoted by $\mathscr{T}_X$. To specify that f is a function from X to X, one usually writes $f \in \mathscr{T}_X$ or $f : X \to X$. An easy example of a function is the *identity function* $\mathrm{id}_X : X \to X$ defined via $\mathrm{id}_X(x) = x$, for all $x \in X$.

Functions can be *composed*, that is, given two functions $f, g \in \mathscr{T}_X$, one defines their *composition* $f \circ g$ as the function in $\mathscr{T}_X$ given by

$$(f \circ g)(x) := f(g(x)).$$

This composition $\circ$ is itself a function, $\circ : \mathscr{T}_X \times \mathscr{T}_X \to \mathscr{T}_X$, and it has the following two properties:

- $\circ$ is *associative*, that is

$$f \circ (g \circ h) = (f \circ g) \circ h, \quad \text{for all} \quad f, g, h \in \mathscr{T}_X;$$

- id_X is a *unit element* with respect to $\circ$, that is

$$f \circ \mathrm{id}_X = \mathrm{id}_X \circ f = f, \quad \text{for all} \quad f \in \mathscr{T}_X.$$

The above observations suggest to study an axiomatic framework which is behind the general notions of semigroups and monoids. A *semigroup* is a pair $(S, *)$, where

- S is a non-empty set;
- $* : S \times S \to S$ is an associative binary operation.

Thus, $(\mathscr{T}_X, \circ)$ is a natural example of a semigroup.

A *monoid* is a triple $(S, *, e)$, where

- $(S, *)$ is a semigroup;
- e is an identity element in S.

Thus, $(\mathscr{T}_X, \circ, \mathrm{id}_X)$ is a natural example of a monoid.

Semigroups and monoids are abundant in mathematics and many other examples are familiar from junior years in high school. Various sets of numbers, in particular: positive integers, non-negative integers, integers, rational numbers, real numbers, complex numbers, quaternions and many other sets of numbers have natural structures of semigroups with respect to the arithmetic operations of addition or multiplication. In most of the cases, there is also a unit element for this operation, namely, 0 for addition and 1 for multiplication, and hence the semigroup structure extends naturally to that of a monoid.

Arithmetic operations are not the only one which naturally equip sets of various numbers with the structure of a semigroup. Subsets of real numbers can be given the structure of a semigroup with respect to the binary operation max of taking the maximal of two given numbers, alternatively, with respect to the binary operation min of taking the minimal of two given numbers. Some other natural examples of semigroups are

- the Boolean of a set with respect to union;
- the Boolean of a set with respect to intersection;
- any non-empty set S with $*$ given by $s*t := s$, for all $s,t \in S$;
- any non-empty set S with $*$ given by $s*t := t$, for all $s,t \in S$;
- any non-empty set S with a fixed element x and with $*$ given by $s*t := x$, for all $s,t \in S$;
- the set of all $n \times n$ matrices over a fixed ring with respect to matrix multiplication.
- the set of all invertible $n \times n$ matrices over a fixed unital ring with respect to matrix multiplication.

A *group* is a monoid $(G,*,e)$ in which every element is *invertible*, that is, for each $x \in G$, there exists $y \in G$ such that $y*x = x*y = e$. It can be show that, for a given x, such y is unique. It is denoted x^{-1}. Many of the examples mentioned above are, in fact, groups. For instance, $(\mathbb{Z},+)$ or the Boolean of a set with respect to union. Group theory is nowadays an important and intensively developed branch of modern mathematics.

In contrast to group theory, modern theory of semigroups makes a stronger emphasis on properties of semigroups related to existence of *non-invertible* elements. In particular, this makes meaningful the theory of *ideals*. A *left* ideal in a semigroup S is a non-empty subset I of S which is closed with respect to left multiplication of elements in S, that is $S*I \subset I$. A *right* ideal in a semigroup S is a non-empty subset I of S which is closed with respect to right multiplication of elements in S, that is $I*S \subset I$. A *two-sided* ideal in a semigroup S is a non-empty subset I of S which is, at the same time, a left ideal and a right ideal. Natural examples of ideals are *principal ideals*. For $x \in S$,

- the principal left ideal $S^1 * x$ is defined as $\{x\} \cup \{s*x : s \in S\}$;
- the principal right ideal $x*S^1$ is defined as $\{x\} \cup \{x*s : s \in S\}$;
- the principal two-sided ideal $S^1 * x * S^1$ is defined as

$$\{x\} \cup \{x*s : s \in S\} \cup \{s*x : s \in S\} \cup \{s*x*t : s,t \in S\}.$$

Comparison of principal ideals equips S with three natural pre-orders, called *Green's pre-orders*:

1. $s \preceq_{\mathscr{L}} t$ if $S^1 * s \subset S^1 * t$
2. $s \preceq_{\mathscr{R}} t$ if $s * S^1 \subset t * S^1$
3. $s \preceq_{\mathscr{J}} t$ if $S^1 * s * S^1 \subset S^1 * t * S^1$

Equivalence classes with respect to these preorders are *Green's relations* $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$, see [4]. If one wants to understand some semigroup S, the first question to ask is how Green's relations for S look like. For all semigroups mentioned above, description of Green's relations is a standard result in the classical semigroup theory. We refer the reader to [2] for further details.

1.2 Categories

Similarly to how semigroups arise as a natural axiomatic framework to study generalizations of functions from some set to itself, categories arise as a natural axiomatic framework to study generalizations of functions between possibly different sets. A *category* $\mathscr{C}$ consists of

- a class of *objects*, denoted $\mathscr{C}$ or $\mathrm{Ob}(\mathscr{C})$,
- for each pair (i,j) of objects in $\mathscr{C}$, a (possibly empty) set $\mathscr{C}(i,j)$ of *morphisms* from i to j,
- for each triple (i,j,k) of objects in $\mathscr{C}$, a *composition map*

$$\circ = \circ_{i,j,k} : \mathscr{C}(j,k) \times \mathscr{C}(i,j) \to \mathscr{C}(i,k),$$

- for each object i in $\mathscr{C}$, an *identity morphism* $\varepsilon_i \in \mathscr{C}(i,i)$;

and all these data are supposed to satisfy the conditions that

- composition is associative, whenever both sides of the associativity axiom make sense;
- identity morphisms are (local) identity elements with respect to composition, whenever both sides of the identity axiom make sense.

The motivating example of a category is the category **Set** of finite sets, where

- objects are all sets;
- given sets X and Y, morphisms from X to Y are functions from X to Y;
- composition in **Set** is composition of functions;
- given a set X, the corresponding identity morphism is id_X.

Other examples of categories include:

- vector spaces over a fixed field and linear maps;
- topological spaces and continuous maps;
- fixed type of algebraic structure (for instance, semigroups, groups, rings or modules) and homomorphisms between them.

Another interesting family of examples of categories are *path categories* of quivers. Let $\Gamma = (\Gamma_v, \Gamma_a, \varphi_s, \varphi_t)$ be a non-empty *quiver*, a.k.a. a *directed graph*, where

- Γ_v is the set of *vertices*;
- Γ_a is the set of *arrows*;
- $\varphi_s : \Gamma_a \to \Gamma_v$ is the function which assigns to an arrow its *source*;
- $\varphi_t : \Gamma_a \to \Gamma_v$ is the function which assigned to an arrow its *target*.

With such Γ we can associate a category $\mathscr{P}_\Gamma$ defined as follows:

- objects of $\mathscr{P}_\Gamma$ are vertices of Γ;
- morphism from i to j in $\mathscr{P}_\Gamma$ are all *directed paths* from i to j in Γ;
- composition in $\mathscr{P}_\Gamma$ is concatenation of paths;
- for $x \in \Gamma_v$, the corresponding identity morphisms is the *trivial path* (i.e. a path which contains no arrows) starting and ending at point i, denoted ε_i.

One can also further define the *path semigroup* S_Γ associated with Γ in the following way:

- elements of S_Γ are all directed paths over Γ, together with an additional special element $\mathbf{z}$.
- given $x, y \in S_\Gamma$, the product $x * y$ is defined as concatenation of y followed by x provided that this makes sense, and $\mathbf{z}$ otherwise.

The special element $\mathbf{z}$ is a *zero* element of S_Γ. Unless $|\Gamma_v| = 1$, the semigroup S_Γ does not have any identity element.

1.3 Effective representations

A *representation* of some mathematical objects is a "realization" of this object using some other, hopefully easier to handle object. Classical examples of such "easier" objects are permutations, functions (a.k.a. transformations) or linear transformations (or matrices). For example, one classically studies

- representations of groups using permutations;
- representations of semigroups using transformations;
- representations of algebras using linear transformations (or matrices).

Given a semigroup S and a field $\Bbbk$, one can define the so called *semigroup algebra* $\Bbbk[S]$ of S over $\Bbbk$ and hence also naturally study representations of semigroups (or groups) using linear transformations or matrices. Formally, a representation is defined as a *homomorphism* from our object of study (for instance, group, semigroup or algebra) to our classical object (the group of all permutations of a set, the semigroup of all transformations of a set or the algebra of all linear operators on a vector space).

Of course, representations may be different, in particular, it is possible that we can lose some information during the process of representation. One obvious case when such a things can happen is when two *different* elements of the original object are represented by the *same* element in the representation. If this degeneration does not occur, we follow [20] and call the corresponding representation *effective*. Obviously, to understand the original object completely, one needs to look for effective representations of this object. This makes study of effective representations interesting and important question.

The notion of effective representation has several aliases, notably that of *injective representation* and that of *faithful representation*. Here "injectivity" refers to injectivity of the representation map (as a map between two sets) but can be confused with the notion of injective object of a category (as each representation is an object of the category of representations). Further, the term "faithfulness" is often used in $\Bbbk$-linear representation theory (meaning effective representations of $\Bbbk[S]$) and hence can also lead to confusion as effective representations of a semigroup S do not have to correspond to effective representations of $\Bbbk[S]$. Because of these reservations, we have chosen to follow the terminology proposed in [20].

A classical result in group theory says that each finite group G has an effective representation by permutations of a set with $|G|$ elements. This is achieved, for example, by the *left regular* representation of G, where an element $g \in G$ is represented by a permutation of other elements in G given by $x \mapsto g * x$, where $x \in G$. In many cases (for example, if $|G|$ is a prime number), this bound cannot be improved. However, in some cases it can be improved. A trivial example is the permutation group S_n of an n-element set, where $n > 1$, which has $n!$ elements, but which has the obvious (*defining*) effective representation by permutations of an n-element set.

For semigroups, the situation is very similar. A classical result in semigroup theory says that each finite semigroup S has an effective representation by permutations of a set with $|S|+1$ elements. This is achieved as follows: We first consider the semigroup S^1 which coincides with S if the latter has an identity element, and, otherwise, is obtained from S by adding to S an extra identity element. In particular, $|S^1|$ either equals $|S|$ or $|S|+1$. The original semigroup S is naturally a subsemigroup of S^1. And the *left regular* representation of S^1 defined as in the previous paragraph turns out to always be effective. In particular, it also gives an effective representation of S, by restriction. Exactly as above, sometimes the bound $|S|$ or $|S|+1$ is sharp and sometimes it is not. However, it feels natural to ask, for a given finite (semi)group S, what is the minimal possible k such that S admits an effective representation by permutations (or transformations) of a k-element set. One could call this the *effective rank* of S.

The paper [20] studies a linearized version of this problem: given a semigroup S and a field $\Bbbk$, what is the minimal possible k such that S admits an effective representation by $k \times k$ matrices over $\Bbbk$. This minimal k is called the *effective dimension of S over* $\Bbbk$ and denoted $\mathrm{eff.dim}_{\Bbbk}(S)$. Here are some of the results obtained in [20]:

Theorem 1. Let $\Bbbk$ be a field. If S is a finite monoid, then we always have $\mathrm{eff.dim}_{\Bbbk}(S) \leq |S|-1$. If S is a finite semigroup, then $\mathrm{eff.dim}_{\Bbbk}(S) \leq |S|$.

Theorem 2. For $\Bbbk = \mathbb{C}$, the following holds:

- The effective dimension of the symmetric group S_n is $n-1$.
- The effective dimension of the semigroup $\mathscr{T}_n$ is n.
- The effective dimension of the semigroup $\mathscr{P}\mathscr{T}_n$ of all partial transformation of $\{1,2,\dots,n\}$ is n.
- The effective dimension of the symmetric inverse semigroup $\mathscr{I}\mathscr{S}_n$ is n.
- The effective dimension of the semigroup $\mathscr{B}_n$ of all binary relations on $\{1,2,\dots,n\}$ is $2^n - 1$.
- The effective dimension of the path semigroup of an acyclic quiver on n vertices is n.
- The effective dimension of the free left regular band with n generators is $\binom{n}{2} + n + 1$.

The first two papers in this thesis are concerned with computing the effective dimension of various classes of semigroups.

1.4 2-categories

Some categories have notable additional structure. Consider, for example, the category **Ab** of abelian groups. Here

- objects are abelian groups;
- morphisms are group homomorphisms;
- composition is composition of functions;
- identity morphisms are identity functions (homomorphisms).

One can note that

- for any fixed two abelian groups G and H, the set of all morphisms from G to H forms an abelian group;
- composition of homomorphisms is biadditive.

Similarly we can consider the category $\mathbf{Vect}_\Bbbk$ of vector spaces over a field $\Bbbk$. Here

- objects are vector spaces over $\Bbbk$;
- morphisms are $\Bbbk$-linear maps;
- composition is composition of functions;
- identity morphisms are identity functions.

Again, one can note that

- for any fixed two vector spaces V and W over $\Bbbk$, the set of all morphisms from V to W forms a vector space over $\Bbbk$;
- composition of homomorphisms is $\Bbbk$-bilinear.

These observations again admit a common generalization in the axiomatic framework of *enriched* categories. We will not define this notion in full generality here. We just mention an interesting example of enriched categories, called 2-*categories*, which is defined, in some sense, recursively. Recall that a category is called *small*, if both its objects and all morphisms are sets.

A 2-*category* is a category enriched over the category of small categories. In other words, a 2-category $\mathscr{C}$ consists of

- objects,
- small categories of morphisms,
- identity objects in the appropriate morphisms categories,
- bifunctorial composition,

and all these data are supposed to satisfy all the obvious axioms. The canonical example of a 2-category is the category **Cat** of small categories where

- objects are small categories,
- morphisms are categories where objects are functors and morphisms are natural transformations of functors,
- identities are identity functors,
- composition is composition of functors (resp. natural transformations).

For more details on 2-categories we refer the reader to [11], [12].

Recent interest to 2-categories comes from their appearance in the categorification approach to various problem in algebra and low-dimensional topology, see [7], [1] and [13]. The series [15], [14], [16], [18], [19], [17] of papers initiated a systematic study of a special class of 2-categories called *finitary 2-categories*. A 2-category $\mathscr{C}$ is called *finitary* (over a field $\Bbbk$) provided that

- it has finitely many objects;
- each morphism category $\mathscr{C}(i,j)$ is equivalent to the category of projective modules over some finite dimensional associative $\Bbbk$-algebra (which depends on both, i and j);
- composition is biadditive and $\Bbbk$-bilinear;
- all identity 1-morphisms are indecomposable.

A typical example of a finitary 2-category is the 2-category of projective endofunctors of A-mod, where A is a finite dimensional associative $\Bbbk$-algebra, see [15]. This 2-category is defined as follows:

- it has one object, which has to be thought of as a small category $\mathscr{A}$ equivalent to A-mod;
- 1-morphisms are endofunctors of $\mathscr{A}$ from the additive closure of functors given by tensoring with projective A-A-bimodules and the identity A-A-bimodule ${}_AA_A$;
- 2-morphisms are natural transformations of functors.

Some other examples of finitary 2-categories can be found in [5] and [6].

Each finitary 2-category has two layers of structure:

- the combinatorial (discrete) structure of 1-morphisms,
- the $\Bbbk$-linear (continuous) structure of 2-morphism.

The discrete structure of 1-morphisms has some similarities with semigroups (and sometimes can be described by a semigroup). Unfortunately, this does not work in general as, for example, composition of indecomposable functors is, in general, decomposable. Therefore, in order to understand combinatorics of 1-morphisms, the paper [14] proposed to look instead at a generalization of semigroups, called *multisemigroups*, which will be discussed in the next section.

1.5 Multisemigroups

Let $(S,*)$ be a semigroup. Then the operation $*$ is a map from $S\times S$ to S, that is, given $s,t\in S$, their product $s*t$ is a new element of S. The notion of a *multisemigroup* generalizes this concept by asking $*$ to be *multivalued*, that is to have values in *subsets* of S.

In more detail, a *multisemigroup* is a pair $(S,*)$ where

- S is a non-empty set,
- $*$ is a *multivalued* operation on S, that is a map from $S\times S$ to the set of all subsets of S;

such that the following associativity axiom is satisfied, for all $a,b,c\in S$:

$$\bigcup_{r\in b*c} a*r=\bigcup_{s\in a*b} s*c.$$

If every $s*t$ is a singleton, one recovers, as a special case, the usual notion of a semigroup. Multisemigroups have various aliases, in particular, *hypergroups*, *hypersemigroups*, *semi-hypergroups* and others, see [10] for details.

An interesting example of a multisemigroup is the following one: let G be a group and H a subgroup of G. Consider the set G/H of right H-cosets in G. Then, for any $x,y\in G$, the set $xHyH$ is a union of right H-cosets. This equips G/H with the structure of a multisemigroup. If (and only if) H is normal, all $xHyH$ are single cosets and hence G/H is a genuine semigroup (the quotient of the group G by a normal subgroup H).

The notions of ideals, principal ideals, Green's preorders and Green's relations have natural (obvious) generalizations to the setup of multisemigroups, see [10].

1.6 Multisemigroups and finitary 2-categories

Let $\mathscr{C}$ be a finitary 2-category. Consider the (finite) set $\mathscr{S}[\mathscr{C}]$ of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$. In [14] it was observed that $\mathscr{S}[\mathscr{C}]$ has the natural structure of a multisemigroup given by the following, for $F,G\in\mathscr{S}[\mathscr{C}]$:

$$F*G:=\{H\in\mathscr{S}[\mathscr{C}] : H \text{ is isomorphic to a direct summand of } F\circ G\}.$$

Combinatorics of Green's relations for the multisemigroup $\mathscr{S}[\mathscr{C}]$ plays crucial role in representation theory of $\mathscr{C}$ developed in the series [15], [14], [16], [18], [19], [17].

The last paper of the thesis is devoted to an explicit description of the cell structure for a special class of finitary 2-categories.

1.7 Grothendieck decategorification

Recall that, for a skeletally small additive category $\mathscr{X}$, its *split Grothendieck group* $[\mathscr{X}]_\oplus$ is defined as the quotient of the free abelian group generated by isomorphism classes of objects in $\mathscr{X}$ modulo the relation $[X] = [Y] + [Z]$, whenever $X \cong Y \oplus Z$.

Let $\mathscr{C}$ be a finitary 2-category and $\mathscr{S}[\mathscr{C}]$ the multisemigroup of $\mathscr{C}$. There is an important connection between $\mathscr{C}$ and linear algebra, given by the so-called *Grothendieck decategorification* of $\mathscr{C}$, denoted $[\mathscr{C}]$. This Grothendieck decategorification is an ordinary category which has

- the same objects as $\mathscr{C}$;
- for each pair (i, j) of objects, the morphism set $[\mathscr{C}](i, j)$ is given by the split Grothendieck group $[\mathscr{C}(i, j)]_\oplus$ of the additive category $\mathscr{C}(i, j)$;
- composition and identities are induced from those in $\mathscr{C}$.

In some sense, the information contained in $[\mathscr{C}]$ is rather similar to that described by $\mathscr{S}[\mathscr{C}]$. However, there is one serious difference: $[\mathscr{C}]$ remembers not only which indecomposable 1-morphisms H appear as direct summands of the composition $F \circ G$, but it also remembers the multiplicity of H as a direct summand of $F \circ G$. The latter multiplicities are structure constants of the associative algebra defined by $[\mathscr{C}]$ and hence they cannot be ignored (and play very important role) in representation theory. The problem is which algebraic structure can be used to describe these multiplicities axiomatically.

The third paper of the thesis introduces and studies a generalization of multisemigroups, called *multi-multisemigroup* or *multisemigroups with multiplicities*, which is designed to deal with the problem described above.

2. Papers in this thesis

2.1 Description of Paper I

Let Γ be a fixed finite quiver without loops and with at most one arrow in each direction between any pair of vertices. Following [3], we define the corresponding *Hecke-Kiselman monoid* HK_Γ as the quotient of the free semigroup generated by Γ_v modulo the following relations:

- $x^2 = x$, for each $x \in \Gamma_v$ (*idempotency*),
- $xy = yx$, for all $x, y \in \Gamma_v$ which do not have any arrow between them (*commutativity*),
- $xyx = yxy = xy$, for all $x, y \in \Gamma_v$ such that there is an arrow $x \to y$ but no arrow $y \to x$,
- $xyx = yxy$, for all $x, y \in \Gamma_v$ connected by arrows in both directions.

We note that the first two relations imply that, for all $x, y \in \Gamma_v$ that are not connected by arrows, we automatically have

$$xyx = xxy = xy = xyy = yxy = yx.$$

If $\Gamma_1 = \{(x,y) : x, y \in \Gamma_v, x \neq y\}$, then the corresponding HK_Γ is the Kiselman monoid defined in [9] (inspired by [8]). If Γ is a uniform orientation of an A-D-E Dynkin quiver, then HK_Γ is isomorphic to the 0-Hecke monoid, that is the monoid underlying the Iwahori-Hecke algebra with parameter 0, see [21].

Paper I studies effective representations of certain Hecke-Kiselman monoids.

Let $\Bbbk$ be an integral domain and

$$W = \bigoplus_{v \in \Gamma_v} \Bbbk v$$

the formal vector space over $\Bbbk$ with basis Γ_v. Let $f : \Gamma_a \to \Bbbk \setminus \{0\}$ be a function (we will call it a *weight* function). For notational reasons we denote the weight on the arrow from x to y (assuming this arrow exists) by f_{xy}. Set

$$\theta_x^f(y) = \begin{cases} y, & x \neq y; \\ \sum_{z \to x} f_{zx} z, & x = y; \end{cases}$$

and extend this by linearity to an endomorphism of W. We use this to define the map

$$R_f : \Gamma_v \to End_\Bbbk(W) \quad \text{via} \quad \Gamma_v \ni x \mapsto \theta_x^f. \tag{2.1}$$

Let $(\Gamma_v)^*$ be the free monoid generated by Γ_v. By the universal property of the monoid $(\Gamma_v)^*$, the map (2.1) extends uniquely to a monoid homomorphism

$(\Gamma_v)^* \to End_{\Bbbk}(W)$, which we also denote by R_f. This construction generalizes the construction of what is called a *linear integral representations* in [3] and we have the following statement that generalizes [3, Proposition 7].

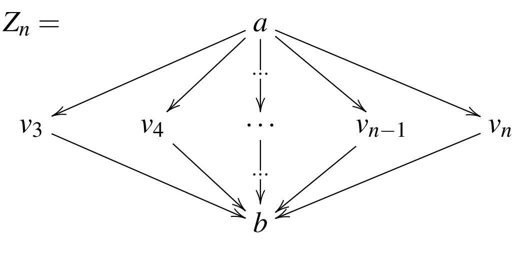
Theorem 3. R_f induces a well-defined homomorphism $HK_\Gamma \to End_{\Bbbk}(W)$ (which we will also denote by R_f).

The main results of Paper I are the following:

Theorem 4. Let Γ be any orientation of a type A Dynkin quiver. Then the representation $R_{\mathbf{1}}$ of HK_Γ, where $\mathbf{1}$ is the constant function with value $1 \in \Bbbk$, is effective.

Consider the following family of graphs:

$Z_n =$



Theorem 5. Let $\Gamma = Z_n$ and let f be defined by $f_{av_i} = 1, f_{v_ib} = 2^i$, for all i. Then R_f is an effective representation of HK_Γ.

For applications of Theorem 4, see [6]. Additionally to the results mentioned above, it is also shown that odd Fibonacci numbers appear as the cardinalities of certain bipartite Hecke-Kiselman monoids and they also count the number of multiplicity free elements in Hecke-Kiselman monoids associated with type A Dynkin quivers.

2.2 Description of Paper II

Paper II is motivated by some questions which remained unanswered in the paper [20]. As already mentioned above, in [20] it is shown that the effective dimension (over $\mathbb{C}$) of the path semigroup of a finite acyclic quiver Γ equals the cardinality of Γ_v. Acyclic quivers are the only quivers for which the associated path semigroups are finite.

The paper [20] also studies *truncated* path semigroups for arbitrary (finite) quivers. Let Γ be a finite quiver and N a positive integer. Then we can consider the corresponding path semigroup S_Γ and its Rees quotient $S_{\Gamma,N}$ by the ideal generated by all paths of length N. The semigroup $S_{\Gamma,N}$ is finite as Γ is finite. The following result is proved in [20].

Theorem 6. Assume that Γ is a finite quiver which contains a loop at every vertex and that $\Bbbk$ is an algebraically closed field. Then the effective dimension of $S_{\Gamma,N}$ over $\Bbbk$ equals $N|\Gamma_v|$.

The main idea behind Paper II is to consider the general case, that is the case which does not have the additional assumption that Γ contains a loop at every vertex. The first main result of Paper II is the following:

Theorem 7. Assume that Γ is a finite quiver and $\Bbbk$ an algebraically closed field. Then the effective dimension of S_Γ over $\Bbbk$ equals $|A|+|\Gamma_v|$, where A denotes the set of all $x \in \Gamma_v$ for which the subsemigroup of S_Γ consisting of all paths which start and terminate at x is not commutative.

The second main result of Paper II is the following:

Theorem 8. Let Q be a finite quiver and let $\Bbbk$ be an uncountable field or a field of characteristic zero. Then the effective dimension of $S_{\Gamma,N}$ over $\Bbbk$ equals

$$\sum_{x \in \Gamma_v} d_x,$$

where

$$d_x := \min\left\{l_x^- + 1, l_x^+ + 1, N, \max\{l_x^- + l_x^+ + 2 - N, 1\}\right\}$$

and

$$l_x^- := \sup\{l(\omega) \,|\, \omega \text{ is a path terminating at } x\}$$
$$l_x^+ := \sup\{l(\omega) \,|\, \omega \text{ is a path starting at } x\}$$

(here $l(\omega)$ denotes the length of the path ω).

The formula for truncated path semigroups is a bit messy, but, if we fix Γ and let N vary, then there are constants a and b, such that

$$\text{eff.dim}_{\Bbbk}(\mathscr{S}_{\Gamma,N}) = aN + b, \quad \text{for all} \quad N > |\Gamma_v|.$$

For countable fields, the same formulae are valid as lower bounds on effective dimension.

The idea of the proof behind both these results is as follows. On the one hand, we must prove that any representation of the semigroup into a matrix semigroup of lower degree necessarily fails to be effective. On the other hand, we must prove that the claimed effective dimension suffices. For the latter, we provide an explicit constructions and prove efficiency for this constructions. At the corresponding place in [20], some abstract results from algebraic geometry were used instead.

It is also worth pointing out that most of our results do not depend on certain enlargements of Γ. For example, for truncated path semigroups, we can add as many arrows as we like from x to y provided that we already have an oriented path from x to y. For path semigroups, the same is true except when we create a new oriented cycle. In particular, it follows that the effective dimension of any path semigroup over an uncountable field is finite.

Our results in Paper II thus answer some open questions left in [20] and also extend and complement some of the results in [20].

2.3 Description of Paper III

The main motivation for Paper III is to define a proper abstract framework to study multisemigroups in which multiplicities of elements can be taken into account. The principal original challenge is to properly define the notion of *multiplicity*. To do that we interpret multiplicities as elements of certain complete ordered semirings. In more details it goes as follows:

Fix a cardinal κ which we understand as the *limit* of *infinity* multiplicity and consider the semiring Card_κ defined as follows:

- elements of Card_κ are all cardinals which are less than or equal to κ,
- addition in Card_κ is defined to be the usual addition of cardinals (i.e. cardinality of the disjoint union) if the outcome of this addition is in Card_κ and the result of this addition is defined to be κ otherwise.
- multiplication in Card_κ is defined to be the usual multiplication of cardinals (i.e. cardinality of the cartesian product) if the outcome of this multiplication is in Card_κ and the result of this multiplication is defined to be κ otherwise.

Some important examples are:

- When $\kappa = 0$, the set Card_0 consists of one element with the obvious (uniquely defined) addition and multiplication.
- When $\kappa = 1$, the set Card_1 consists of two elements $\{0,1\}$ and is isomorphic to the boolean semiring.
- When $\kappa = \omega$ (the first infinite cardinal), the set Card_ω is $\{0,1,2,\dots,\omega\}$. The semiring Card_ω contains, as a subsemiring, the semiring of non-negative integers with respect to the usual addition and multiplication.

Let now κ be a fixed cardinal. Given a base set X, define the *κ-multi-Boolean* $\mathscr{B}_\kappa(X)$ of X as the set of all functions from X to the complete semiring Card_κ. By construction, $\mathscr{B}_\kappa(X)$ is equipped with the natural structure of a complete semiring.

A *multisemigroup with multiplicities bounded by* κ is a pair (S,μ), where

- S is a non-empty set;

- $\mu : S \times S \to \mathscr{B}_\kappa(S)$, written $(s,t) \mapsto \mu_{s,t} : S \to \mathrm{Card}_\kappa$;

such that the following *distributivity* requirement is satisfied: for all $r,s,t \in R$, we have

$$\sum_{i\in S} \mu_{s,t}(i)\mu_{r,i} = \sum_{j\in S} \mu_{r,s}(j)\mu_{j,t}. \tag{2.2}$$

We note that here, for a cardinal λ and a function $\nu : S \to \mathrm{Card}_\kappa$, by $\lambda\nu$ we mean the function from S to Card_κ defined as

$$\lambda\nu = \sum_{i\in\lambda} \nu,$$

or, in other words, this is just adding up λ copies of ν.

For a complete unital semiring R and a non-empty set X, consider the complete R-module R^X. An *algebra* structure on R^X is a map $\bullet : R^X \times R^X \to R^X$ such that, for all $f_i, f, g, h \in R^X$, we have

$$\left(\sum_{i\in I} f_i\right) \bullet g = \sum_{(i,j)\in I\times J} f_i \bullet g;$$
$$g \bullet \left(\sum_{i\in I} f_i\right) = \sum_{(i,j)\in I\times J} g \bullet f_i;$$
$$f \bullet (g \bullet h) = (f \bullet g) \bullet h.$$

If R is a semiring and X a set, then, for $x \in X$, we denote by $\chi_x : X \to R$ the *indicator function* of x defined as follows:

$$\chi_x(y) = \begin{cases} 1, & x = y; \\ 0, & x \neq y. \end{cases}$$

The main result of Paper III is the following statement, inspired by [10], which describes multisemigroup with multiplicities in terms of algebra structures:

Theorem 9.

1. Let κ be a cardinal and (S,μ) be a multisemigroup with multiplicities bounded by κ. Then Card_κ^S has a unique structure of a complete Card_κ-algebra such that $\chi_s \bullet \chi_t = \mu_{s,t}$, for all $s,t \in S$.
2. Conversely, any complete Card_κ^S-algebra $(\mathrm{Card}_\kappa^S, \bullet)$ defines a unique structure of a multisemigroup with multiplicities bounded by κ on S via $\mu_{s,t} := \chi_s \bullet \chi_t$, for $s,t \in S$.

If we take $\kappa = 0$, we recover the usual definition of a semigroup. If we take $\kappa = 1$, we recover the usual definition of a multisemigroup as in [10]. For

$\kappa = \omega$, we obtain the notion of a multisemigroup with multiplicities (bounded by ω) which is suitable to the study of Grothendieck decategorifications of finitary 2-categories.

2.4 Description of Paper IV

Paper IV continues investigation which started in [5], [6] and [22]. The paper [6] introduces and studies certain finitary 2-categories of bimodules associated to subbimodules of the identity bimodule for tree quivers. One of the main results is an explicit description of the cell combinatorics for the multisemigroup of the corresponding finitary 2-category. It turns out that, in this particular case, this multisemigroup is, in fact, a semigroup. Furthermore, it is shown that tensor product of two indecomposable subbimodules of the identity bimodules for path algebras of tree quivers is either again an indecomposable subbimodule of the identity bimodule or zero. So, even taking multiplicities in account, all combinatorics of 1-morphisms is governed by a proper semigroup. Therefore these finitary 2-categories categorify semigroup algebras.

The main idea of Paper IV is to look at the easiest possible example which lies outside the setup considered in [6]. This is an example of an oriented cyclic quiver. It turns out the situation here is different from the tree case and the multisemigroup of the corresponding finitary 2-category is now a proper multisemigroup and not a semigroup. Here are two examples of a quiver which we consider:

$$\begin{array}{ccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 \\ \downarrow & & & & \uparrow \\ 4 & \longleftarrow & 5 & \longleftarrow & 6 \end{array} \qquad \begin{array}{ccccc} 1 & \xrightarrow[\beta]{} & 2 & \xrightarrow[\gamma]{} & 3 \\ {\scriptstyle\alpha}\downarrow & & & & \uparrow{\scriptstyle\eta} \\ 4 & \xrightarrow[\delta]{} & 5 & \xrightarrow[\xi]{} & 6 \end{array} \tag{2.3}$$

So, we start with an arbitrary non-uniform orientation Γ of a cyclic quiver and consider the corresponding reduced path algebra $A := \Bbbk[S_\Gamma]/\Bbbk\mathbf{z}$, where $\Bbbk$ is an algebraically closed field. Non-uniformity of the orientation ensures that A is finite dimensional. Subbimodules of the identity A-A-bimodule ${}_AA_A$ are the same thing as (two-sided) ideals in A. There are two types of ideals in A:

- *Linearized semigroup ideals*, that is ideals which are obtained by linearizing an ideal of the finite path semigroup S_Γ.
- *Exceptional ideal* which exist only in the situation when Γ has exactly one sink. Such ideal is a one-dimensional subspace of A spanned by a linear combination (with non-zero coefficients) of two different paths of maximal length in Γ.

For the examples in (2.3), exceptional ideals exist only for the quiver to the right. An example of an exception ideal for this quiver is $\Bbbk(\gamma\beta + 2\eta\xi\delta\alpha)$.

Let $\tilde{\Gamma}$ be the unoriented graph defined as follows:

- vertices of $\tilde{\Gamma}$ are all maximal paths of Γ;
- edges of $\tilde{\Gamma}$ are all sinks and sources of Γ;
- each sink (or source) is adjacent to each maximal path which terminates (resp. starts) in it.

Clearly, $\tilde{\Gamma}$ is an unoriented cycle. For a linearized semigroup ideal I of A, we define its *graph* Γ_I as the subgraph $\tilde{\Gamma}$

- vertices of Γ_I are exactly the maximal paths over Γ which belong to I,
- edges in Γ_I are exactly the sinks and sources in Γ for which the corresponding trivial path ε_x is in I.

The first main result in Paper IV is the following:

Theorem 10. Let I be a non-zero linearized semigroup ideal. Then the ideal I is indecomposable (as an ideal) if, and only if, the graph Γ_I is connected.

We split all indecomposable linearized semigroup ideals into three types.

- Ideals of type I are those ideals whose graph is the full circle.
- Ideals of type II are those ideals whose graph is a chain with at least two vertices.
- Ideals of type III are those ideals whose graph consists of a single vertex.

We observe the following:

Lemma 11. Let I and J be two indecomposable linearized semigroup ideals in $\Bbbk Q$. If $I \cdot J$ is not indecomposable, then both I and J are of type II and $\Gamma_I \cup \Gamma_J$ is the full circle.

Linearized semigroup ideals can be described using combinatorics of *generalized Dyck paths*, for each linearized semigroup ideal I, we have the corresponding generalized Dyck path $\pi(I)$. For a generalized Dyck path π, define the *initial slope* $\mathbf{is}(\pi)$ of π as the number of the column from which π starts. Define the *terminal slope* $\mathbf{ts}(\pi)$ of π as the number of the row in which π terminates. The second main result in Paper IV is the following combinatorial characterization of decomposability of tensor products (over A) of linearized semigroup ideals.

Theorem 12. Let I and J be two indecomposable linearized semigroup ideals in $\Bbbk Q$ of type II. Then $I \cdot J$ decomposes into a direct sum of two non-zero components if and only if the following condition is satisfied: For each isolated vertex ω in $\Gamma_I \cup \Gamma_J$ such that the edge x in Γ_J adjacent to ω is a sink, we have

$$\mathbf{ts}(\pi(J)_\omega) \geq \mathbf{is}(\pi(I)_\omega),$$

where $\pi(I)_\omega$ denotes the intersection of $\pi(I)$ with the triangle corresponding to ω and similarly for $\pi(J)_\omega$ (our normalization is such that $\mathbf{is}(\pi(J)_\omega) = 1$ and $\mathbf{ts}(\pi(I)_\omega) = \mathfrak{l}(\omega) + 1$).

We also give explicit formulae for enumeration of indecomposable ideals in A. These formulae are rather complicated polynomial summation formulae in classical Catalan numbers.

3. Populärvetenskaplig sammanfattning på svenska

En av de mest fundamentala begreppen inom matematik är funtioner. Några exempel är $f(x) = e^x$ och $g(x) = \sin(x)$. Vi kan till exempel sätta samman funxtionerna ovan till $f \circ g(x) = e^(\sin x)$ eller $g \circ f(x) = sin(e^x)$. Exemplet belyser att $f \circ g$ och $g \circ f$ i allmänhet är olika funktioner. Däremot kommer funktionerna $f \circ (g \circ h)$ och $(f \circ g) \circ h$ alltid att vara samma, så länge de är definierade. Det beror på att bägge sidor tar x till $f(g(h(x)))$. Den här egenskapen, som gör att $f \circ g \circ h$ är en väldefinierad funktion, kallas *associativitet*.

De bägge funktionerna ovan har en egenskap gemensam. Deras värdemängder sammanfaller med respektive värdemängder. Den situationen är lite speciell, men mycket viktig.

Låt oss fixera en mängd X och studera mängden $\mathscr{T}_X$ av funktioner $f : X \to X$ vars definitionsmängd och värdemängd är X. Vi kallar dessa funktioner för *transformationer* av X. Alla dessa funktioner kan sättas samman och sammansättningen är associativ. In speciell funktion är identitesfunktionen id_X som definieras genom $\mathrm{id}_X(x) = x$ för alla x i X. Den uppfyller att $f \circ \mathrm{id}_X = f = \mathrm{id}_X \circ f$ för alla funktioner $f : X \to X$.

Denna situation generaliserar naturligt till halvgrupper, som är en mängd S tillsammans med en associativ operator $\circ : S \times S \to S$. Om vi dessutom har ett element 1 i S som uppfyller att $s \circ 1 = s = 1 \circ s$ för alla s i S så kallar vi S en *monoid*. Vi noterar att om S *inte* är en monoid så kan vi lägga till ett nytt element 1 med den önskade egenskapen utan att förstöra associativiteten. Vi låter S^1 betyda den minsta monoid som innehåller S.

Ett centralt tema i representationsteori är att jämföra ett objekt man vill förstå med någon form av favoritobjekt. För halvgrupper och monoider finns det två typer av favoritobjekt. Ett har vi redan mött; transformationshalvgruppen $\mathscr{T}_X$ (som trots sitt namn är en monoid). Varje halvgrupp kan bäddas in i en transformationshalvgrupp på följande sätt: Först bäddar vi in S i S^1. Därefter observerar vi att S^1 naturligt verkar på sig själv genom $s(t) = s \circ t$. Vidare ser vi att om s och s' är olika så är deras verkan på S^1 olika, eftersom $s(1) = s \circ 1 = s \neq s' = s' \circ 1 = s'(1)$. Därför kan vi bädda in S^1 i $\mathscr{T}_{S^1}$. Detta resonemang bör vara bekant för den som har studerat gruppteori. Skillnaden är att grupper per definition har ett neutralt element.

Den andra familjen av favoritobject vi har är matrishalvgruppen $Mat_n(\Bbbk)$, med avseende på multiplikation. Vi noterar att ändliga transformationshalvgrupper kan bäddas in i matrishalvgruppen genom att låta dem agera på $\Bbbk^X$, det vill säga vektorrummet som har X som bas. Därmed kan vilken ändlig halvgrupp som helst bäddas in i en matrishalvgrupp. Vi noterar dock att även om inbäddningarna ovan alltid fungerar, så är de oftast väldigt slösaktiga. Om vi fixerar en halvgrupp och en kropp $\Bbbk$ så undrar vi vad det minsta n är så att S kan bäddas in i $Mat_n(\Bbbk)$. Vi kallar det för *effektiv dimension*.

Artiklarna I och II handlar om att beräkna effektiva dimensioner för olika familjer av halvgrupper. Beräkningarna är uppdelade i två delar. Först hittar vi en undre begränsning genom att visa att injektivitet fallerar för mindre n, oavsett kropp. Sedan visar vi konstruktivt hur man kan bädda in halvgrupperna och varför det är just en inbäddning. Den andra halvan är något mer restriktiv med vilken kropp som fungerar för inbäddningen, men det fungerar för komplexa tal och alla tillräckligt stora kroppar.

Om man istället vill generalisera funktioner vars definitionsmängd och målmängd inte nödvändigtvis sammanfaller är det lämpligt att studera kategorier. En kategori består av objekt och morfismer mellan objekt. Objekten generaliserar definitions- och värdemängder och varje morfism går mellan två objekt (som kan vara samma). Morfismerna $f: Z \to U$ och $g: X \to Y$ kan sättas samman till en morfism $f \circ g: X \to U$ om och endast om $Y = Z$. Vi kräver att sammansättningen är associativ och att varje objekt X har en lokal identitetsmorfism 1_X så att för varje morfism $f: Y \to X$ och varje morfism $g: X \to Z$ har vi att $1_x \circ f = f$ och $g \circ 1_X = g$.

Ett aktivt område inom matematiken idag handlar om att översätta klassiska resultat inom matematiken i termer av kategorier. Den processen kallas för kategorifiering. Idén är snarlik den inom matematiken så klassiska generaliseringen, där man försöker samla liknande strukturer under samma tak och på så sätt slippa göra om samma arbeta gång på gång. För många strukturer och resultat går det utmärkt att kategorisera rakt av. För andra krävs det att vi betraktar kategorier med extra struktur.

En viktig typ av kategori med extra struktur som dyker upp i den här avhandlingen är 2-kategorier, där vi utöver morfismer (som vi kallar 1-morfismer) även har 2-morfismer, som går emellan 1-morfismer, samt ett par regler om hur dessa ska samspela. Ett annat exempel på viktiga kategorier med extra strukturer är kategorier där man kan addera morfismer som går mellan samma (ordnade) par av objekt. Då kan vi prata om odelbara morfismer, som är morfismer som inte kan uttryckas som summan av två andra morfismer.

Om vi lägger på lite extra struktur kan vi få kategorier där varje morfism kan beskrivas som en summa av odelbara morfismer på ett unikt sätt. Det finns ingen garanti att sammansättningen av två odelbara morfismer blir en ny odelbar morfism. Det man alltid kan göra är att skapa en så kallad multihalvgrupp, som är en generalisering av halvgrupper som tillåter att produkten av två element är en mängd av element. I det här fallet skapar vi en multihalvgrupp av odelbara morfismer i vår kategori och säger att produkten av två odelbara morfismer är mängden av summander till sammansättningen av morfismerna i kategorin.

När vi skapar en multihalvgrupp på det här sättet glömmer vi bort de multipliciteter som summanderna kommer med. För att behålla så mycket information som möjligt definierar vi i artikel III begreppet *multihalvgrupp med multipliciteter*.

En rad författare har upptäckt att flera familjer av besläktade kategorier har egenskapen att produkten av odelbara morfismer blir en ny odelbar morfism. I artikel IV studerar vi en liknande familj av kategorier, där vi avsiktligen bryter mot en gemensam egenskap hos de övriga kategorierna. Resultatet är en kategori där sammasättningen av odelbara morfismer inte nödvändigtvis är odelbar, men där varje summand dyker upp exakt en gång. Vidare ger vi en övre gräns på hur många summander som kan dyka upp då flera odelbara morfismer sätts samman, samt beskriver multihalgruppen (med multipliciteter) av odelbara morfismer.

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