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Hodge Decomposition for Manifolds with Boundary and Vector Calculus

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Abstract

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This thesis describes the Hodge decomposition of the space of differential forms on a compact Riemannian manifold with boundary, and explores how, for subdomains of 3-space, it can be translated into the language of vector calculus. In the former, more general, setting, we prove orthogonality of the decomposition. In the latter setting, we sketch the full proof, based on results from algebraic topology and about the solvability of boundary value problems for certain PDEs.

Contents

1 Introduction

Vector calculus, also known as vector analysis, is a branch of mathematics that extends the elements of integral and differential calculus to vector fields defined on subsets of three-dimensional space (or some other suitable space). Its importance is indicated by its many applications in physics and engineering, where, among other things, it is used to desribe electromagnetic and gravitational fields and various flow fields. The classical operators known as gradient, curl and divergence, typically denoted by ∇, $\nabla\times$ and $\nabla\cdot$, and the Laplacian, which we denote by ∇^2, are important objects of study, and are related to each other by various vector calculus identities that can be used to facilitate computations.

Since a smooth three-dimensional manifold is, in a sense, nothing more than bits and pieces of three-dimensional space that are glued together in a smooth and seamless way, vector calculus can be naturally extended to this setting. But on a manifold, it is oftentimes more convenient to work with differential forms than with vector fields. In particular, on a Riemannian manifold, the Riemannian metric provides a natural (that is, basis-independent) isomorphism between vector fields and differential 1-forms. In the case of a three-dimensional Riemannian manifold, the so-called Hodge star operator, denoted by $\star$, lets us extends this isomorphism to differential 2-forms as well, and also lets us construct the codifferential d^*, which is, in a sense, the adjoint operator to the exterior derivative d on differential forms. In this setting, the gradient, curl and divergence of vector analysis find a natural generalization in the exterior derivative, and the Laplacian is generalized by the Laplace-de Rham operator Δ. In Chapters 2 and 3 we set up the machinery of differential forms on Riemannian manifolds and look at how the classical language of vector calculus can be translated into, and is generalized by, the modern language of differential geometry.

Hodge theory, named after William Vallance Douglas Hodge (1903–1975), puts the theory of partial differential equations to work to study the cohomology of smooth manifolds. That is, it studies certain topological properties of a manifold by means of PDEs. Central to this study is the exterior derivative, the codifferential and the Laplace-de Rham operator. The Hodge decomposition theorem, which lies at the heart of Hodge theory, uses these operators to decompose the space of differential k-forms into a direct sum of

L^2-orthogonal subspaces. In Chapter 4 we introduce the notions of singular homology and de Rham cohomology, and state some results that will prove to be useful later when, in Chapter 5, we state the Hodge decomposition theorem (Theorem 5.5) as well as a special case of this theorem that applies to vector fields on certain domains in three-space (Theorem 5.12) and that lets us put our results from the previous chapters to the test.

Chapter 6 introduces various tools that are then used in Chapter 7 to sketch a full proof of Theorem 5.12.

Conventions

This might be a good time and place to say a few words about the conventions used throughout this thesis. Everything in this section applies everywhere in this thesis unless otherwise stated.

Smooth is taken to mean C^∞. Manifolds are assumed to be smooth, as are differential forms and vector fields. The letter M is used to denote a manifold, and g is used to denote a Riemannian metric. The dimension of a manifold is denoted by n. Three dimensional manifolds are sometimes denoted by Y (this is because the letter Y resembles a three-way intersection) For domains in $\mathbb{R}^n$ we write D. Vector spaces are assumed to be real, and rings are assumed to be unital. The letter Γ denotes the space of smooth sections of a fiber bundle, e.g. $\Gamma(TM)$ denotes the field of smooth sections of the tangent bundle or, in other words, the space of smooth vector fields on M.

2 Differential forms

This chapter introduces the major players in what is to come: Riemannian manifolds with boundary, differential forms, the Hodge star operator and the codifferential. The aim is to introduce those parts of manifold theory that are relevant to the topics that lie ahead, i.e. a discussion of the Hodge decomposition theorem and how it can be translated into the language of vector calculus in Euclidean 3-space.

For an introduction to smooth manifold theory and Riemannian manifolds, Lee's books [8] and [7] are useful. A brief refresher that aims to present the parts of manifold theory necessary to introduce and prove the Hodge decomposition is found in Schwarz's book [12].

2.1 Riemannian manifolds

Let M be a smooth manifold with boundary (note that M does not have to have a boundary). We write TM for the tangent bundle of M, and T_pM for the tangent space to M at the point $p \in M$. Similarly, T^*M denotes the ctangent bundle of M, and we write T_p^*M for the cotangent space to M at p. A Riemannian metric on M is a family of positive definite inner products

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}, \quad p \in M$$

such that for all (smooth) vector fields V and W the map

$$p \longmapsto g_p(V(p), W(p))$$

is smooth. A Riemannian manifold (M, g) is a smooth manifold M quipped with a Riemannian metric.

Example 2.1. The pair $(\mathbb{R}^n, \cdot)$, where $\mathbb{R}^n$ is understood as a smooth manifold in the usual way and $\cdot$ is the dot product, is a Riemannian manifold. Whenever we talk about $\mathbb{R}^n$ as a Riemannian manifold, it is implied, unless otherwise stated, that the Riemannian metric is given by the dot product.

A useful property of Riemannian manifolds is the existence of local orthonormal frames. Given an open subset $U \in M$, a local orthonormal frame on U is a set of (not necessarily smooth) vector fields $\{E_1, \ldots, E_n\}$

defined on U that are orthonormal with respect to the Riemannian metric at each point $p \in U$, that is, $g_p(E_i(p), E_j(p)) = \delta_{ij}$. It is convenient to know that at every pont p of a Riemannian manifold there exists a local orthonormal frame on an open set containing p.

If M is oriented, the orientation of M induces an orientation of T_pM for each $p \in M$. A local orthonormal frame $\{E_1, \ldots, E_n\}$ defined on $U \in M$ is said to be (positively) oriented if the ordered basis $(E_1(p), \ldots, E_n(p))$ of T_pM is an ordered basis at each $p \in U$.

Of particular interest to us is the class of smooth manifolds called *regular domains in* $\mathbb{R}^n$. These are properly embedded codimension 0 submanifolds with boundary. In addition to this, we consider them as Riemannian submanifolds of $\mathbb{R}^n$ equipped with the Euclidean metric. Note that regular domains in $\mathbb{R}^n$ are orientable.

2.2 Differential forms

Let M be a smooth n-manifold. We let $\Lambda^k(T^*M)$ denote the k-th exterior power of the cotangent space T^*M. A smooth *differential k-form* is a smooth section of $\Lambda^k(T^*M)$, i.e. a smooth map

$$\eta : M \longrightarrow \Lambda^k(T^*M),$$

so that

$$(\pi \circ \eta)(x) = x$$

for all points $x \in M$. Throughout this text, we will often write "differential form", "differential k-form", "k-form" or just "form" when referring to a smooth section of $\Lambda^k(T^*M)$, omitting *smooth*, as smoothness of forms is always understood. A form of degree n is sometimes called a *top level form*, or just a top form.

The space of k-forms on M is

$$\Omega^k(M) := \Gamma(\Lambda^k(T^*M))$$

and the space of all differential forms on M is

$$\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M).$$

The wedge product on k-covectors, i.e. elements of $\Lambda^k(T_p^*M)$ at some point $p \in M$, extends pointwise to define a product on $\Omega^k(M)$, also known as the wedge product and denoted by $\wedge$. The following proposition states some of its properties.

Proposition 2.2. *Let M be a smooth manifold. Then the following statements are true:*

1. *The wedge product is associative, bilinear and satisfies*

$$\eta \wedge \zeta = (-1)^{kl} \zeta \wedge \eta$$

for all $\eta \in \Omega^k(M)$ *and* $\zeta \in \Omega^l(M)$.

2. *Let* $\eta_1, \ldots, \eta_k \in \Omega^1(*M)$ *and let* $v_1, \ldots, v_k \in T_pM$ *for some point* $p \in M$*. Then*

$$(\eta_1 \wedge \ldots \wedge \eta_k)_p(v_1, \ldots, v_k) = \det(\eta_i(v_j)).$$

The wedge product might be easiest to grasp by looking at an example.

Example 2.3. Let Y be a smooth 3-manifold and consider the differential 2-form η and 1-form ζ, which in some local coordinates are given by $\eta = x\, dx \wedge dy$ and $\zeta = 5\, dx + y\, dz$. Then

$$\begin{aligned} \eta \wedge \zeta &= (x\, dx \wedge dy) \wedge (5\, dx + y\, dz) \\ &= 5x\, dx \wedge dy \wedge dx + xy\, dx \wedge dy \wedge dz \\ &= xy\, dx \wedge dy \wedge dz, \end{aligned}$$

and since $\eta \wedge \zeta = (-1)^2 \eta \wedge \zeta$, we get the same result if we swap η and ζ in the above computation.

A very important map that takes differential k-forms to $(k+1)$-forms is the exterior derivative, denoted by d. It is defined to be the unique map that satisfies the following properties:

1. It is $\mathbb{R}$-linear.

2. Let $\eta \in \Omega^k(M)$ and $\zeta \in \Omega^l(M)$. Then

$$d(\eta \wedge \zeta) = d\eta \wedge \zeta + (-1)^k \zeta \wedge d\eta.$$

3. $dd = 0$.

4. It is the differential for 0-forms, i.e. for smooth functions.

As with the wedge product, it is probably easiest to get a grasp of the exterior derivative by looking at an example.

Example 2.4. A 2-form η is expressible in local coordinates as $\eta = f\, dx_1 \wedge dx_2$ over 1-form basis $dx_1, \ldots, dx_n$. We calculate

$$d\eta = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right) dx_1 \wedge dx_2 = \left(\sum_{i=3}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_1 \wedge dx_2 \right).$$

A useful property of differential forms is that they can be *pulled back* from one smooth manifold to another via smooth maps. Let M_1 and M_2 be smooth manifolds with boundary, let $F : M \longrightarrow N$ be a smooth map and let η be a k-form on N. Then the *pullback of* η *by* F is the k-form on M given by

$$(F^*\eta)_p(V_1, \dots, V_k) = \eta_p(dF_p(V_1), \dots, dF_p(V_k)),$$

for $V_1, \dots .V_k \in T_pM$ and for every point $p \in M$. We state the following important property of the pullback and the exterior derivative without proof.

Proposition 2.5. *Let M and N be smooth manifolds with boundary and let $F : M \longrightarrow N$ be a smooth map. Then*

$$F^*(d\eta) = d(F^*\eta).$$

As we shall see in the next chapter, the exterior derivative generalizes many important notions from vector calculus, such as the gradient and curl of a vector field in $\mathbb{R}^3$.

2.3 Orientations and integration on manifolds

Let (M, g) be an oriented Riemannian manifold with boundary. An *orientation form* on M is a nowhere vanishing top form η such that

$$\eta(E_1, \dots, E_n) > 0$$

whenever $\{E_1, \dots, E_n\}$ is a local oriented orthonormal frame at point $x \in M$. An argument using partitions of unity shows that the existence of a non-vanishing top form on M is equivalent to to M being orientable. The choice of such a form up to multiplication by a positive function is equivalent to choosing an orientation on M.

Whenever M is orientable, then so is ∂M. Moreover, by another partitions of unity argument, it can be shown that there exists a smooth outward pointing vector field N along ∂M. Let $p \in \partial M$, and let $(E_1, \dots, E_{n-1})$ be a local oriented frame of ∂M at p. We define the orientation of $T_p\partial M$ by declaring the orientation of $(E_1(p), \dots, E_{n-1}(p))$ to be positive if $(N(p), E_1(p), \dots, E_{n-1}(p))$ is positively oriented as a frame of T_pM. This defines an orientation for the boundary ∂M in a consistent way.

There exists a unique orientation form called the *Riemannian volume form* (or simply the volume form), on M, which we denote by ω_g, and which has the defining property that

$$\omega_g(E_1, \dots, E_n) = 1$$

for every local oriented orthonormal frame $(E_1, \dots, E_n)$.

Remark 2.6. Specifying a volume form does not determine a unique Riemannian metric on a smooth manifold. On the other hand, the Riemannian metric together with an orientation is enough to determine the volume form uniquely.

We define the integral of a top form $\eta = f\ dx_1 \wedge \ldots \wedge dx_n$ over a domain of integration D in $\mathbb{R}^n$ as

$$\int_D f\ dx_1 \wedge \ldots \wedge dx_n := \int_D f\ dx_1 \ldots dx_n.$$

Via pullbacks to local coordinates and partitions of unity this definition can be extended to allow for the integration of top forms over compact orientable smooth manifolds. We will not dwell on the details here, as that is better left to any textbook on differential geometry, e.g. [8].

One of the most important and elegant theorems concerning the integration of differential forms is Stokes' theorem. We state it here without proof.

Theorem 2.7 (Stokes' theorem)**.** *Let M be a compact orientable smooth manifold with boundary and let $\eta \in \Omega^{k-1}(M)$. Then*

$$\int_M d\eta = \int_{\partial M} \eta.$$

Remark 2.8. In Stokes' theorem, the integral over the boundary is to be interpreted in the following way. Let $\imath : \partial M \longrightarrow M$ denote the natural inclusion map. Then

$$\int_{\partial M} \eta := \int_{\partial M} \imath^* \eta,$$

where $\imath^*\eta$ denotes the pullback of η to ∂M by $\imath$. Also, ∂M is assumed to have the induced boundary orientation.

On an compact oriented Riemannian manifold M, the existence of the volume form ω_g lets us integrate real-valued funtions as follows. Let $f \in C^\infty(M)$. Then the integral of f over M is given by

$$\int_M f\, \omega_g.$$

In particular, by integrating $f = 1$ over M, the volume form lets us define "the volume" of M as

$$\mathrm{Vol}(M) := \int_M \omega_g.$$

2.4 An inner product on $\Lambda^k(T_p^*M)$

Let (M,g) be an oriented Riemannian manifold, let $p \in M$ be a point and let $\{E_1,\dots,E_n\}$ be an orthonormal frame at p and $\{e_1,\dots,e_n\}$ the corresponding dual frame. A basis for $\Lambda^k(T_p^*M)$ is then given by the set

$$\mathcal{B} = \{e_\alpha \mid \alpha \text{ a } k\text{-dimensional multi-index with } \alpha_1 < \ldots < \alpha_k\}.$$

We now define an inner product $\langle\cdot,\cdot\rangle_g$ on $\Lambda^k(T_p^*M)$ as follows:

$$\begin{aligned}\langle\cdot,\cdot\rangle_g : \Lambda^k(T_p^*M) \times \Lambda^k(T_p^*M) &\to \mathbb{R}\\ (\eta,\zeta) &\mapsto \frac{1}{k!}\sum_{\substack{1\le\\ i_1,\dots,i_k\\ \le n}} \eta(E_{i_1},\dots,E_{i_k})\zeta(E_{i_1},\dots,E_{i_k}).\end{aligned}$$

This inner product is independent of the choice of orthonormal frame and is hence well defined. It has the property that it makes $\mathcal{B}$ orthonormal, which we will state as Proposition 2.11. But first we give an alternative definition of $\langle\cdot,\cdot\rangle_g$.

Proposition 2.9. *Let $\eta,\zeta \in \Lambda^k(T_p^*M)$. Then*

$$\langle\eta,\zeta\rangle_g = \sum_{\sigma\in S(k,n)} \eta(E_{\sigma(1)},\dots,E_{\sigma(k)})\zeta(E_{\sigma(1)},\dots,E_{\sigma(k)}),$$

where $S(k,n)$ denotes the subset of S_n consisting of all permutations σ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(n)$.

Proof. Let $\eta,\zeta \in \Lambda^k(T_p^*M)$. Then, by definition,

$$\langle\eta,\zeta\rangle_g = \frac{1}{k!}\sum_{\substack{1\le\\ i_1,\dots,i_k\\ \le n}} \eta(E_{i_1},\dots,E_{i_k})\zeta(E_{i_1},\dots,E_{i_k}).$$

But since differential forms are alternating, any summand with a repeated index, i.e. with $E_{i_j} = E_{i_l}$ for some $j \neq l$, contributes zero to the above sum. Hence

$$\begin{aligned}&\sum_{\substack{1\le\\ i_1,\dots,i_k\\ \le n}} \eta(E_{i_1},\dots,E_{i_k})\zeta(E_{i_1},\dots,E_{i_k})\\ &\qquad = \sum_{\substack{1\le i_1,\dots,i_k\le n\\ i_j\neq i_l}} \eta(E_{i_1},\dots,E_{i_k})\zeta(E_{i_1},\dots,E_{i_k})\\ &\qquad = \frac{1}{(n-k)!}\sum_{\sigma\in S_n} \eta(E_{\sigma(1)},\dots,E_{\sigma(k)})\zeta(E_{\sigma(1)},\dots,E_{\sigma(k)}). \qquad (2.1)\end{aligned}$$

Also, since each summand is the product of two alternating forms, evaluated at the same vectors, swapping any two arguments changes the sign of both forms, and hence the sign of a summand is unchanged under such swaps. This implies that each summand in (2.1) is invariant under permutations of the k vectors $E_{\sigma(1)}, \dots, E_{\sigma(k)}$, and since there are $k!$ such permutations for each $\sigma \in S_k$, of which precisely one is expressible as $E_{\tau(1)}, \dots, E_{\tau(k)}$ with $\tau \in S(k,n)$, and since we can permute the $n-k$ "complementary indices" arbitrarily, we get

$$\begin{aligned}\langle \eta, \zeta \rangle_g &= \frac{1}{k!} \sum_{\substack{1\leq \\ i_1,\dots,i_k \\ \leq n}} \eta(E_{i_1}, \dots, E_{i_k})\zeta(E_{i_1}, \dots, E_{i_k}) \\ &= \frac{1}{k!}(n-k)! \cdot k! \sum_{\sigma \in S(n)} \eta(E_{\sigma(1)}, \dots, E_{\sigma(k)})\zeta(E_{\sigma(1)}, \dots, E_{\sigma(k)}) \\ &= \sum_{\sigma \in S(k,n)} \eta(E_{\sigma(1)}, \dots, E_{\sigma(k)})\zeta(E_{\sigma(1)}, \dots, E_{\sigma(k)}),\end{aligned}$$

which is what we wanted to show. □

Remark 2.10. The elements of $S(k,n)$ are called (k,n)-shuffles.

Proposition 2.11. *The basis*

$$\mathcal{B} = \{e_\alpha \mid \alpha \text{ a } k\text{-dimensional multi-index with } \alpha_1 < \ldots < \alpha_k\}$$

of $\Lambda^k(T_p^*M)$ *is orthonormal with respect to* $\langle \cdot, \cdot \rangle_g$.

Proof. Let $e_\alpha, e_\beta \in \mathcal{B}$. Since $e_\alpha(E_{\sigma(1)}, \dots, E_{\sigma(k)}) = 1$ if only if $\sigma(i) = \alpha_i$, $1 \leq i \leq k$, and otherwise equals zero, we have

$$\begin{aligned}\langle e_\alpha, e_\beta \rangle_g &= \sum_{\sigma \in S(k,n)} e_\alpha(E_{\alpha_{\sigma(1)}}, \dots, E_{\alpha_{\sigma(k)}}) e_\beta(E_{\alpha_{\sigma(1)}}, \dots, E_{\alpha_{\sigma(k)}}) \\ &= e_\alpha(E_{\alpha_1}, \dots, E_{\alpha_k}) e_\beta(E_{\alpha_1}, \dots, E_{\alpha_k}).\end{aligned}$$

But $e_\beta(E_{\sigma(1)}, \dots, E_{\sigma(k)}) = 1$ if only if $\sigma(i) = \beta_i$, and otherwise equals zero, it must be the case that

$$\langle e_\alpha, e_\beta \rangle_g = \begin{cases} 1 & \text{if } \alpha = \beta \text{, that is, if } e_\alpha = e_\beta \\ 0 & \text{otherwise} \end{cases},$$

which is what we set out to prove. □

Remark 2.12. Applying $\langle , \rangle_g$ pointwise to differential forms yields the map

$$\langle \cdot, \cdot \rangle_g : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M).$$

Remark 2.13. In Section 3.2 we introduce an isomorphism $\sharp : T_p^*M \longrightarrow T_pM$ that lets us define $\langle\cdot,\cdot\rangle_g$ for covectors as

$$\langle \eta, \zeta\rangle_g := \langle \eta^\sharp, \zeta^\sharp\rangle.$$

For decomposable elements $\eta = \eta_1 \wedge \ldots \wedge \eta_k$ and $\zeta = \zeta_1 \wedge \ldots \wedge \zeta_k$ in $\Lambda^k(T_p^*M)$, we can then define $\langle\cdot,\cdot\rangle_g$ as

$$\langle \eta_1 \wedge \ldots \wedge \eta_k, \zeta_1 \wedge \ldots \wedge \zeta_k\rangle_g := \det(\langle \eta_i, \zeta_j\rangle_g). \tag{2.2}$$

It is clear that the above definition coincides with our original definition for all basis vectors $e_\alpha, e_\beta \in \mathcal{B}$, since we have

$$\det(\langle e_{\alpha_i}, e_{\beta_j}\rangle_g) = \det \begin{pmatrix} \delta_{\alpha_1\beta_1} & & 0 \\ & \ddots & \\ 0 & & \delta_{\alpha_k\beta_k} \end{pmatrix} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}.$$

A multilinearity argument then shows that (2.2) also coincides with the original definition for arbitrary decomposable η and ζ, and from there it extends linearly to allow for indecomposable elements.

The importance of $\langle\cdot,\cdot\rangle_g$ for our purposes lies in the fact that it lets us define the Hodge star operator on differential forms.

2.5 The Hodge star operator

Let (M, g) be a Riemannian manifold with boundary. We now introduce an automorphism

$$\star : \Omega^*(M) \longrightarrow \Omega^*(M),$$

known as the Hodge star operator, defined by requiring that for a k-form η, the identity

$$\zeta \wedge \eta = \langle \zeta, \star\eta\rangle_{\Lambda^k}\, \omega_g$$

holds for all $\zeta \in \Omega^{n-k}(M)$. It is linear over $C^\infty(M)$ and has the property that its restriction to $\Omega^k(M)$ is an isomorphism from the space of k-forms to the space of $(n-k)$-forms, that is

$$\star|_{\Omega^k(M)} : \Omega^k(M) \longrightarrow \Omega^{n-k}(M).$$

Note that this makes sense since

$$\dim \Lambda^k(T^*M) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda^{n-k}(T^*M).$$

We use the notation $\star_k$ when referring to the restriction of $\star$ to $\Omega^k(M)$, i.e.

$$\star_k := \star|_{\Omega^k(M)}.$$

Proposition 2.14. *Let (M, g) be a Riemannian manifold with boundary. The Hodge star operator is the unique automorphism on $\Omega^*(M)$ that maps the k-form η to the $(n-k)$-form $\star\eta$. Moreover, for each $k \in \{0, \dots, n\}$, the map $\star_k$ is an isomorphism from the space of k-forms to the space of $n-k$-forms on M.*

We use the following lemma, which is a finite-dimensional version of the Riesz representation theorem, in the proof of Proposition 2.14.

Lemma 2.15. *Let V be a finite-dimensional vector space endowed with a nondegenerate inner product g, and let f be a linear functional on V. Then there exists a unique vector $v \in V$ such that*

$$f(v) = g(v, w) \quad \text{for all } w \in V.$$

Proof. To show uniqueness, assume that v exists. Let $\{u_1, \dots, u_n\}$ be an orthonormal basis for V, and write $w = v_1 u_1 + \dots v_n u_n$. We must have $f(u_i) = g(u_i, v)$, and hence

$$v = \sum_{i=1}^{n} g(u_i, u_i) f(u_i) u_i = \sum_{i=1}^{n} f(u_i) u_i.$$

To show existence, we can easily check that v, defined as above, produces the desired result. □

Remark 2.16. In the proof of Lemma 2.15, we see that if we vary f smoothly in V^*, then v varies smoothly in V, and vice versa.

Proof of Proposition 2.14. Every top differential form on M can be written as $f\,\omega_g$ for some smooth function f on M. Fix $\eta \in \Omega^k(M)$. Then $\zeta \wedge \eta$ is a top form for all $\zeta \in \Omega^{n-k}(M)$, and thus

$$\zeta \wedge \eta = f_\eta(\zeta)\,\omega_g, \tag{2.3}$$

where $f_\eta(\zeta)$ is smooth and

$$f_\eta : \Omega^{n-k}(M) \longrightarrow C^\infty(M)$$

is linear over $C^\infty(M)$. Moreover, f_η is uniquely defined by (2.3), and the restriction of f_η to a point is linear over $\mathbb{R}$, i.e. at each point $p \in M$ the map

$$f_\eta|_{\Lambda^{n-k}(T_p^* M)} : \Lambda^{n-k}(T_p^* M) \longrightarrow \mathbb{R}$$

is a linear functional uniquely determined by (2.3). This observation lets us use Lemma 2.15 to deduce that there exists a unique form $\theta \in \Omega^{n-k}(M)$ such that

$$f_\eta(\zeta) = \langle \zeta, \theta \rangle_{\Lambda^k} \quad \text{for all } \zeta \in \Omega^k(M).$$

Take $\star\eta := \theta$. Then $\star$ is linear over $C^\infty(M)$ and $\ker \star = 0$, so it is indeed an automorphism, and since $\star\eta \in \Omega^{n-k}(M)$, its restriction to k-forms $\star_k$ is an isomorphism from $\Omega^k(M)$ to $\Omega^{n-k}(M)$. By Remark 2.16, θ is smooth. □

Our definition of $\star$ is not very practical when it comes to actual computations. Luckily for us, there exist equivalent definitions that lend themselves more easily to this task. The next proposition establishes several equivalent definitions of the Hodge star. We state it without proof.

Proposition 2.17. *Let (M, g) be a Riemannian manifold with boundary. The following definitions of $\star$ are equivalent.*

1. *Let $\eta \in \Omega^k(M)$. Then $\star\eta$ is defined by demanding that*

$$\zeta \wedge \eta = \langle \zeta, \star\eta \rangle_{\Lambda^k} \, \omega_g \quad \text{for all } \zeta \in \Omega^{n-k}(M).$$

2. *Let $\eta \in \Omega^k(M)$. Then $\star\eta$ is defined by demanding that*

$$\zeta \wedge \star\eta = \langle \zeta, \eta \rangle \, \omega_g \quad \text{for all } \zeta \in \Omega^k(M). \tag{2.4}$$

3. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal coframe defined on some open subset $U \in M$ and let $\sigma \in S_n$. Then*

$$\star(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}) := sgn(\sigma) \, e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}. \tag{2.5}$$

   Since $\star$ is linear, and since an orthonormal basis for $\Lambda^k(T^*M)$ on U can be found among the members of the set $\{e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)} \mid \sigma \in S_n\}$, this suffices to compute $\star\eta$ for all $\eta \in \Omega^k(M)$.

4. *Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame defined on some open subset $U \in M$ and let $\eta \in \Omega^k(M)$. Then $\star\eta$ is defined on U to be the $(n-k)$-form for which*

$$(\star\eta)(E_{\sigma(k+1)}, \ldots, E_{\sigma(n)}) := sgn(\sigma) \, \eta(E_{\sigma(1)}, \ldots, E_{\sigma(k)}) \tag{2.6}$$

   for all $\sigma \in S_n$.

Remark 2.18. Definitions 3 and 4 in Proposition 2.17 work equally well with $S(k, n)$ substituted for S_n.

Corollary 2.19. *Let (M, g) be a Riemannian manifold with boundary. Then $\star 1 = \omega_g$.*

Proof. This follows directly from definition 4 of Proposition 2.17, □

Proposition 2.20. *Applying $\star$ twice yields*

$$\star \star \eta = (-1)^{k(n-k)} \eta$$

for all $\eta \in \Omega^k(M)$.

Proof. From definition 3 of Proposition 2.17 we know that for an orthonormal coframe $\{e_1, \ldots, e_n\}$ defined on some open subset $U \in M$ and for $\sigma \in S_n$, we have

$$\star(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}) = \mathrm{sgn}(\sigma)\, e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}.$$

Applying the same definition of $\star$ twice yields

$$\begin{aligned}\star\star(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}) &= \mathrm{sgn}(\sigma) \star (e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}) \\ &= \mathrm{sgn}(\sigma)^2(-1)^x(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}) \\ &= (-1)^x(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}).\end{aligned}$$

where $x = k(n-k)$ since we can get from

$$(\sigma(k+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(k))$$

to

$$(\sigma(1), \ldots, \sigma(n))$$

by $k(n-k)$ adjacent transpositions (first move $\sigma(1)$ $n-k$ steps to the left by adjacent transpositions, then $\sigma(2)$ $n-k$ steps to the left etc. until we have moved $\sigma(k)$; this adds up to $k(n-k)$ adjacent transpositions). □

Corollary 2.21. *The inverse map of* $\star$ *is given by*

$$\begin{aligned}\star^{-1} : \Omega^*(M) &\longrightarrow \Omega^*(M) \\ \eta &\longmapsto (-1)^{k(n-k)} \star \eta,\end{aligned} \tag{2.7}$$

i.e.

$$\begin{cases} \star^{-1} = \star & \textit{in odd dimensions} \\ \star^{-1} = (-1)^k \star & \textit{in even dimensions.} \end{cases}$$

Proof. The result follows easily by working backwards from (2.7) and using Proposition 2.20. First, we have

$$(-1)^{k(n-k)} \star \eta = (-1)^{k(n-k)} \star^{-1} \star\star\, \eta = (-1)^{2k(n-k)} \star^{-1} \eta = \star^{-1}\eta.$$

Then it is straightforward to verify that

$$k(n-k) = nk - k^2 \equiv \begin{cases} 0 \pmod 2 & \text{if } n \text{ is odd} \\ k \pmod 2 & \text{if } n \text{ is even} \end{cases}.$$

□

Proposition 2.22. *Consider* $\mathbb{R}^3$ *with the standard metric, with* x, y *and* z *the standard coordinates. Then the Hodge star operator on* $\Omega^*(\mathbb{R}^3)$ *is given by*

$$\begin{aligned} &\star 1 = dx \wedge dy \wedge dz && \star(dx \wedge dy \wedge dz) = 1 \\ &\star dx = dy \wedge dz && \star(dy \wedge dz) = dx \\ &\star dy = -dx \wedge dz && \star(dx \wedge dz) = -dy \\ &\star dz = dx \wedge dy && \star(dx \wedge dy) = dz. \end{aligned}$$

Proof. Note that since $\star$ is linear, it is sufficient to determine where it takes the (standard) basis vectors of $\Omega^k(\mathbb{R}^3)$ for $0 \leq k \leq 3$. The basis vectors are

$$\begin{gathered} 1 \text{ for } \Omega^0(\mathbb{R}^3), \\ dx, dy \text{ and } dz \text{ for } \Omega^0(\mathbb{R}^3), \\ dx \wedge dy, dx \wedge dz \text{ and } dy \wedge dz \text{ for } \Omega^0(\mathbb{R}^3), \\ dx \wedge dy \wedge dz \text{ for } \Omega^0(\mathbb{R}^3). \end{gathered}$$

As we have already seen, these bases are orthonormal with respect to to the inner product on k-forms $\langle \cdot, \cdot \rangle_g$. The result thus follows immediately from Proposition 2.17. For example, let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$. Using cycle notation we can write $\sigma = (1\ 2)(3)$, hence $\operatorname{sgn}(\sigma) = (1\ 2)(3)$. Let (e_1, e_2, e_3) denote the standard coframe, i.e. $e_1 = dx, e_2 = dy$ and $e_3 = dz$. Then

$$\begin{aligned} \star dy = \star e_2 = \star e_{\sigma(1)} \\ &= \operatorname{sgn}(\sigma)\ e_{\sigma(2)} \wedge e_{\sigma(3)} \\ &= -e_1 \wedge e_3 = -dx \wedge dz \end{aligned}$$

and

$$\begin{aligned} \star(dx \wedge dy) &= \star(-dy \wedge dx) \\ &= - \star (dy \wedge dx) \\ &= - \star (e_2 \wedge e_1) \\ &= - \star (e_{\sigma(1)}, e_{\sigma(2)}) \\ &= -\operatorname{sgn}(\sigma)\ e_{\sigma(3)} = e_3 = dz. \end{aligned}$$

The rest of the proof follows from similar calculations. □

2.6 An L^2-inner product on differential forms

Let M be a compact smooth orientable manifold. We define an L^2-inner product on the space differential k-forms on M as

$$\begin{aligned} \langle \cdot, \cdot \rangle : \Omega^k(M) \times \Omega^k(M) &\longrightarrow \mathbb{R} \\ (\eta, \zeta) &\longmapsto \int_M \eta \wedge \star\zeta. \end{aligned}$$

That this defines an inner product on $\Omega^k(M)$ is easy to verify: since we can write $\eta \wedge \star\zeta = \langle \eta, \zeta \rangle_g, \omega_g$, the required properties of symmetry, linearity and positive-definiteness all follow from the corresponding properties of $\langle \cdot, \cdot \rangle_g$. along with the linearity of the integral.

Equipped with this inner product, the vector space $\Omega^k(M)$ becomes an infinite-dimensional inner product space.

Proposition 2.23. *Let (M, g) be a compact Riemannian manifold with boundary. The Hodge star operator preserves the L^2-inner product in the sense that*

$$\langle \star\eta, \star\zeta \rangle = \langle \eta, \zeta \rangle$$

for all $\eta, \zeta \in \Omega(M)^p$.

Proof. The result is obtained by a straighforward computation:

$$\langle \star\eta, \star\zeta \rangle = \int_M \star\eta \wedge \star\star\,\zeta = (-1)^{2p(n-p)} \int_M \zeta \wedge \star\eta = \langle \zeta, \eta \rangle = \langle \eta, \zeta \rangle.$$

□

2.7 The codifferential

Let (M, g) be a Riemannian manifold with boundary. The codifferential, denoted d^*, is the map defined by

$$\begin{aligned} d^* : \Omega^k(M) &\longrightarrow \Omega^{k-1}(M) \\ \eta &\longmapsto (-1)^{n(k-1)+1} \star d \star \eta \end{aligned} \tag{2.8}$$

for each k. Since $\star$ is used twice in the definition, the codifferential does not depend on the orientation of M.

Proposition 2.24. *Let (M, g) be a Riemannian manifold with boundary. An alternative expression for the codifferential is given by*

$$d^*\eta = (-1)^k \star^{-1} d \star \eta$$

for all $\eta \in \Omega^k(M)$.

Proof. Let $\eta \in \Omega^k(M)$. Then $d \star \eta \in \Omega^{n-k+1}(M)$. Using Corollary 2.21 we see that

$$\begin{aligned} \star(d \star \eta) &= (-1)^{(n-k+1)(n-(n-k+1))} \star^{-1} (d \star \eta) \\ &= (-1)^{kn-k-n-1} \star^{-1} (d \star \eta). \end{aligned}$$

Substituting the above expression into (2.8) yields

$$\begin{aligned} d^*\eta &= (-1)^{n(k-1)+1} \star d \star \eta \\ &= (-1)^{n(k-1)+1}(-1)^{kn-k-n-1} \star^{-1} d \star \eta \\ &= (-1)^k \star^{-1} d \star \eta, \end{aligned}$$

which is what we wanted to prove. □

Proposition 2.25. *Let (M, g) be a Riemannian manifold with boundary. The codifferential is nilpotent of degree 2, that is*

$$d^*d^*\eta = 0$$

for all $\eta \in \Omega^*(M)$.

Proof. Let $\eta \in \Omega^k(M)$. Using Proposition 2.24 along with the fact that $dd = 0$, we get

$$d^*d^*\eta = (-1)^{k-1} \star^{-1} d \star (-1)^k \star^{-1} d \star \eta = - \star^{-1} dd \star \eta = 0.$$

□

The following proposition gives an important property of the codifferential that is true only for Riemannian manifolds without boundary.

Proposition 2.26. *Let (M, g) be a compact Riemannian manifold without boundary. Then*

$$\langle d\eta, \zeta \rangle = \langle \eta, d^*\zeta \rangle$$

for all $\eta \in \Omega^k(M)$ and $\zeta \in \Omega^{k+1}(M)$.

Proof. Let $\eta \in \Omega^k(M)$ and $\zeta \in \Omega^{k+1}(M)$. Using the identity

$$d\eta \wedge \star\zeta = d(\eta \wedge \star\zeta) - (-1)^k(\eta \wedge d \star \zeta)$$

(recall that this is one of the defining properties of the exterior derivative) and the alternative characterization of d^* in Proposition 2.24, we compute

$$\begin{aligned} \langle d\eta, \zeta \rangle &= \int_M d\eta \wedge \star\zeta \\ &= \int_M \Big(d(\eta \wedge \star\zeta) - (-1)^k(\eta \wedge d \star \zeta) \Big) \\ &= \int_M d(\eta \wedge \star\zeta) + \int_M \eta \wedge (\star(-1)^{k+1} \star^{-1} d \star \zeta) \\ &= \int_{\partial M} \eta \wedge \star\zeta + \int_M \eta \wedge \star d^*\zeta = \langle \eta, d^*\zeta \rangle, \end{aligned}$$

where the last equality holds because ∂M is empty. □

Remark 2.27. In light of Proposition 2.26, the notation d^* makes perfect sense whenever M is compact without boundary, since then d^* is the L^2-adjoint of d. In this case, the condition that $\langle d\eta, \zeta \rangle = \langle \eta, d^*\zeta \rangle$, can be taken as the definition of d^*, and the fact that d^* can be expressed in a neat way using the Hodge star will then come as a nice surprise.

When M has nonempty boundary it is no longer the case that d and d^* are adjoint operators, since the leftmost integral in

$$\int_{\partial M} \eta \wedge \star\zeta = \int_M d(\eta \wedge \star\zeta) = \langle d\eta, \zeta\rangle - \langle \eta, d^*\zeta\rangle$$

is in general non-zero, but is zero when $\imath^*\zeta = 0$ or $\imath^* \star \eta = 0$, where $\imath^*$ is the pullback by the inclusion map $\imath : \partial M \longrightarrow M$. For this reason it might be sensible to use a different symbol than d^* when referring to the codifferential on a manifold with nonempty boundary. It is common to let δ denote the codifferential instead of d^*, but this alternative notation does not always seem to be motivated by the above considerations.

In any case, throughout this text we will use d^* to denote the codifferential, even when it is not the L^2-adjoint of d.

3 Vector calculus and differential forms

In this section we investigate how differential forms and vector calculus are related. In particular, we see how the exterior derivative d and the codifferential d^* extends the classical gradient, curl and divergence of vector calculus. As an application, we derive the well known Green's theorem, divergence theorem and Kelvin-Stokes theorem encountered in vector calculus, from Stokes' theorem (Theorem 2.7).

3.1 The Laplace-de Rham operator

Let (M, g) be a Riemannian manifold with boundary. The Laplace-de Rham operator Δ on differential k-forms on M is defined by

$$\begin{aligned} \Delta : \Omega^k(M) &\longrightarrow \Omega^k(M) \\ \eta &\longmapsto (d + d^*)^2(\eta). \end{aligned}$$

Hence, for a k-form η we have

$$\Delta\eta = (d + d^*)^2(\eta) = (dd^* + dd^*d)(\eta) = dd^*\eta + d^*d\eta$$

Proposition 3.1. *The Laplace-de Rham operator Δ commutes with $\star$, d and d^**

Proof. Let (M, g) be a Riemannian manifold with boundary and let $\eta \in \Omega^k(M)$.

We first show that Δ commutes with $\star$. By expanding the definitions and rearranging, we get

$$\begin{aligned} \star(dd^*\eta) &= \star d(-1)^{n(k-1)+1} \star d \star \eta \\ &= (-1)^{n(k-1)+1} \star d \star (d \star \eta) \\ &= (-1)^{n((n-k+1)-1)+1} \star d \star (d \star \eta) \\ &= d^*d(\star\eta) \end{aligned}$$

and

$$
\begin{aligned}
\star(d^* d\eta) &= \star(-1)^{n((k+1)-1)+1} \star d \star d\eta \\
&= \star \star d(-1)^{nk+1} \star d(-1)^{k(n-k)} \star (\star\eta) \\
&= (-1)^{(n-k)(n-(n-k))} d(-1)^{k+1} \star d \star (\star\eta) \\
&= d(-1)^{nk+1} \star d \star (\star\eta) \\
&= d(-1)^{n((n-k)-1)+1} \star d \star (\star\eta) \\
&= dd^*(\star\eta),
\end{aligned}
$$

so that

$$
\star\Delta\eta = \star(dd^*\eta + d^* d\eta) = \star(dd^*\eta) + \star(d^* d\eta) = \Delta \star \eta,
$$

i.e. Δ commutes with $\star$.

To show that $d\Delta\eta = \Delta d\eta$, we compute

$$
\begin{aligned}
d\Delta\eta &= d(dd^*\eta + d^* d\eta) \\
&= ddd^*\eta + dd^* d\eta \\
&= dd^* d\eta + d^* d^* d\eta = \Delta d\eta.
\end{aligned}
$$

The proof that $d^*\Delta\eta = \Delta d^*\eta$ is analogous. □

When M is compact, the Laplace-de Rham operator combines the two conditions of closedness ($d\eta = 0$) and coclosedness ($d^*\eta = 0$) into a single condition. The following proposition spells out the details.

Proposition 3.2. *Let η be a differential form on a closed Riemannian manifold. Then $\Delta\eta = 0$ if and only if $d\eta = 0$ and $d^*\eta = 0$.*

Proof. Let M denote our manifold. Since M is compact, every differential form has compact support. Hence the inner product of two k-forms is always defined. Moreover, M has no boundary, which implies that d^* is the L^2-adjoint of d. Using these facts we compute

$$
\begin{aligned}
\langle \Delta\eta, \eta \rangle = \langle dd^*\eta + d^* d\eta, \eta \rangle &= \langle dd^*\eta, \eta \rangle + \langle d^* d\eta, \eta \rangle \\
&= \langle d^*\eta, d^*\eta \rangle + \langle d\eta, d\eta \rangle = ||d^*\eta||^2 + ||d^*\eta||^2,
\end{aligned}
$$

which shows that $\Delta\eta = 0$ implies that $d\eta = 0$ and $d^*\eta = 0$.

The other direction is trivial. □

3.2 Musical isomorphisms

Let M be a smooth oriented manifold equipped with a Riemannian metric g. The isomorphism

$$\flat : T_pM \longrightarrow T_p^*M$$

defined by requiring that

$$v^\flat(w) = g_p(v,w) \quad \forall v, w \in T_pM,$$

and its inverse

$$\sharp : T_p^*M \longrightarrow T_pM,$$

defined by

$$g_p(\omega^\sharp, v) = \omega(v) \quad \forall \omega \in T_p^*M, v \in T_pM,$$

take smooth vector fields to smooth covector fields and vice versa when applied pointwise to TM and T^*M. Hence, by applying $\flat$ to smooth vector fields and $\sharp$ to smooth covector fields, we get the inverse isomorphisms

$$\flat : \Gamma(TM) \longrightarrow \Omega^1(M)$$
$$\sharp : \Omega^1(M) \longrightarrow \Gamma(TM)$$

defined pointwise by

$$V^\flat(W) = g_p(V,W) \quad \forall V, W \in \Gamma(TM)$$
$$g_p(\omega^\sharp, V) = \omega(V) \qquad \forall \omega \in \Omega^1(M), V \in \Gamma(TM).$$

(See Lemma 2.15 and Remark 2.16.)

Next, we want to establish an isomorphism β from the tangent bundle to the bundle of alternating covariant $(n-1)$-tensors on M. This is achieved by composing $\star$ and $\flat$ as follows.

$$\beta : TM \longrightarrow \Lambda^{n-1}T^*M$$
$$v \longmapsto \star v^\flat$$

The inverse of β is given by

$$\beta^{-1} = (-1)^{n+1}\sharp\star\,.$$

Applying β to smooth vector fields and β^{-1} to smooth $n-1$-forms, we get the inverse isomorphisms

$$\beta : \Gamma(TM) \longrightarrow \Omega^{n-1}(M)$$
$$\beta^{-1} : \Omega^{n-1}(M) \longrightarrow \Gamma(TM)$$

defined by

$$\beta(V) = \star V^\flat \quad \text{and} \quad \beta^{-1}(\eta) = (-1)^{n+1}(\star\eta)^\sharp.$$

The interior product on differential forms provides an alternative way of expressing β, which follows from the next proposition and is stated as Corollary 3.4.

Proposition 3.3. *Let (M, g) be an oriented Riemannian n-manifold, let $V \in \Gamma(TM)$ and let $\eta \in \Omega^k(M)$. Then*

$$\star i_V(\eta) = (-1)^{k-1}(V^\flat \wedge \star\eta). \tag{3.1}$$

Proof. Let $p \in M$ and let $\{E_1, \ldots, E_n\}$ be a local orthonormal at p, with $\{e_1, \ldots, e_n\}$ the corresponding coframe, so that

$$V(p) = \sum_{i=1}^{n} v_i E_i \quad \text{and} \quad \eta(p) = f(p)\, e_{\pi(1)} \wedge \ldots \wedge e_{\pi(k)},$$

where $\pi \in S_n$ and because both sides are linear, without loss of generality, η has been chosen to be decomposable into a wedge product of basis 1-forms. As all the operations involved in the identity (3.1) are defined pointwise, we will work exclusively in T_pM and T_p^*M throughout the rest of this proof, and so we will omit writing p in calculations in order to improve legibility, e.g. we will write f instead of $f(p)$ and V instead of $V(p)$ as the reference to p is understood.

Expanding the left hand side of (3.1) yields

$$\begin{aligned} i_V(\eta) &= f \sum_{i=1}^{k} (-1)^{i-1}\, e_{\pi(i)}(V)\, e_{\pi(1)} \wedge \ldots \wedge \widehat{e_{\pi(i)}} \wedge \ldots \wedge e_{\pi(k)} \\ &= f \sum_{i=1}^{k} (-1)^{i-1}\, v_{\pi(i)}\, e_{\pi(1)} \wedge \ldots \wedge \widehat{e_{\pi(i)}} \wedge \ldots \wedge e_{\pi(k)}. \end{aligned}$$

Next, from the definition of $\star$ we see that

$$\star\eta = \operatorname{sgn}(\pi) f\, e_{\pi(k+1)} \wedge \ldots \wedge e_{\pi(n)}.$$

Since we are working in an orthonormal frame, we have $V^\flat = \sum_{i=1}^{n} v_i e_i$. A straightforward computation then shows that

$$\begin{aligned} V^\flat \wedge \star\eta &= \operatorname{sgn}(\pi) f \sum_{i=1}^{n} v_i e_i \wedge e_{\pi(k+1)} \wedge \ldots \wedge e_{\pi(n)} \\ &= \operatorname{sgn}(\pi) f \sum_{i=1}^{k} v_{\pi(i)} e_{\pi(i)} \wedge e_{\pi(k+1)} \wedge \ldots \wedge e_{\pi(n)}. \end{aligned}$$

Applying $\star$ to $i_V(\eta)$ introduces a factor $\operatorname{sgn}(\tau_{ik}\circ\pi) = (-1)^{i-k}\operatorname{sgn}(\pi)$, so that

$$\begin{aligned}\star i_V(\eta) &= (-1)^{k-1}\operatorname{sgn}(\pi) f \sum_{i=1}^{k} v_{\pi(i)} e_{\pi(i)} \wedge e_{\pi(k+1)} \wedge \ldots \wedge e_{\pi(n)} \\ &= (-1)^{k-1}(V^\flat \wedge \star\eta),\end{aligned}$$

which is the sought identity. □

Corollary 3.4. *Let (M, g) be an oriented Riemannian n-manifold and let $V \in \Gamma(TM)$. Then*

$$\beta(V) = i_V(\omega_g).$$

Proof. Take $\eta = \omega_g$ in Proposition 3.3. Then $k = n$, $\star\eta = 1$ and $V^\flat \wedge \star\eta = V^\flat$. Applying $\star$ to (3.1) and noting that $\star\star = (-1)^{k(n-k)}$, we get the left hand side

$$\star \star i_V(\omega_g) = (-1)^{(k-1)(n-(k-1))} i_V(\omega_g) = (-1)^{n-1} i_V(\omega_g),$$

which, when put together with the right hand side, yields

$$(-1)^{n-1} i_V(\omega_g) = (-1)^{n-1} \star V^\flat \quad \Longleftrightarrow \quad i_V(\omega_g) = \star V^\flat.$$

Since $\beta(V) = \star V^\flat$, we are done with the proof. □

3.3 Gradient and divergence

Let (M, g) be a Riemannian manifold. We define a map grad by

$$\begin{aligned}\operatorname{grad} : C^\infty(M) &\longrightarrow \Gamma(TM) \\ f &\longmapsto (df)^\sharp.\end{aligned} \tag{3.2}$$

For $f \in C^\infty(M)$, we call the vector field $\operatorname{grad}(f)$ the gradient vector field (or just the gradient) of f. By definition, for $V \in \Gamma(TM)$ we have $\langle \operatorname{grad} f, V\rangle = df(V) = Vf$. Thus $\langle \operatorname{grad} f, V\rangle$ gives the directional derivative of f along V, and $\operatorname{grad} f$ is the unique vector field with this property.

Moving on, we define another map, div, by

$$\begin{aligned}\operatorname{div} : \Gamma(TM) &\longrightarrow C^\infty(M) \\ V &\longmapsto \star^{-1}(d\beta(V)).\end{aligned} \tag{3.3}$$

The div map does not depend on the orientation of the manifold. Indeed, if $\{e_1, \ldots e_n\}$ is an orthonormal coframe on M, then any odd permutation $\pi \in S_n$ yields a differently oriented orthonormal coframe $\{e_{\pi(1)}, \ldots, e_{\pi(n)}\}$ for which $\star(e_{\pi(1)} \wedge \ldots \wedge e_{\pi(n)}) = -\star(e_1 \wedge \ldots \wedge e_n)$. This follows from Proposition 2.17. From Corollary 3.4 we see that β changes sign in the same way under this change of coframe. In the definition of div, these two sign changes cancel each other, implying that div is defined invariantly on all Riemannian manifolds, regardless of orientation.

The codifferential, eager not to be left behind, can also be used to define grad and div. This will prove to be very useful later on, and the following proposition shows how it can be done.

Proposition 3.5. *Let (M, g) be a Riemannian manifold and let $f \in C^\infty(M)$ and $V \in \Gamma(TM)$. Then the following statements are true:*

1. $\operatorname{grad} f = -\beta^{-1}(d^* \star f)$.
2. $\operatorname{div} V = -d^* V^\flat$.

Proof.

1. Since $\star f$ is an n-form and $d^* \star f$ is an $(n-1)$-form we have

$$\begin{aligned} -\beta^{-1}(d^* \star f) &= (-1)^n [\star (-1)^n \star^{-1} d \star (\star f)]^\sharp \\ &= [df]^\sharp = \operatorname{grad} f. \end{aligned}$$

2. Since $V^\flat$ is a 1-form we have

$$\begin{aligned} -d^* V^\flat &= (-1)^{1+1} \star^{-1} d \star V^\flat \\ &= \star^{-1} d\beta(V) = \operatorname{div} V. \end{aligned}$$

□

3.4 Euclidean space

We will use the term Euclidean n-space, or just n-space, to denote $\mathbb{R}^n$ equipped with the Euclidean metric. If the dimension is unimportant or clear from context, we might drop the n and write Euclidean space instead.

Proposition 3.6. *The following statements are true in Euclidean n-space with standard coordinates $x_1, \ldots, x_n$:*

1. *Let* $V = \sum_{i=1}^n V_i \frac{\partial}{\partial x_i} \in \Gamma(T\mathbb{R}^n)$ *and* $\eta = \sum_{i=1}^n \eta_i dx_i \in \Omega^1(\mathbb{R}^n)$. *Then*

$$V^\flat = \sum_{i=1}^n V_i dx_i \quad \text{and} \quad \eta^\sharp = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i}. \tag{3.4}$$

2. *Let* $f \in C^\infty(\mathbb{R}^n)$. *Then*

$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

3. *Let* $V = \sum_{i=1}^{n} V_i \frac{\partial}{\partial x_i} \in \Gamma(T\mathbb{R}^n)$. *Then*

$$\operatorname{div} V = \sum_{i=1}^{n} \frac{\partial V_i}{\partial x_i}.$$

4. *Let* $V, W \in \Gamma(T\mathbb{R}^n)$. *Then* $V \cdot W = \star(V^\flat \wedge \star W^\flat)$, *where* $\cdot$ *denotes the usual dot product.*

Proof.

1. In standard coordinates on $\mathbb{R}^n$ we have $g = \delta_{ij} dx_i dx_j$. Hence

$$V^\flat(W) = g_p(V, W) = V \cdot W,$$

which implies that $V^\flat = \sum_{i=1}^{n} V_i dx_i$. From this we also conclude that $\eta^\sharp = \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x_i}$.

2. It suffices to observe that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

The result then follows from (3.4).

3. From the definitions we see that $(\operatorname{div} V)\, \omega_g = d \star V^\flat$. From there we compute

$$\begin{aligned} d(\star V^\flat) &= d \star \sum_{i=1}^{n} (V_i\, dx_i) = d \sum_{i=1}^{n} \star(V_i\, dx_i) \\ &= d \sum_{i=1}^{n} \Big((-1)^{i-1} V_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge x_n \Big) \\ &= \sum_{i=1}^{n} \left(\frac{dV_i}{dx_i}\, dx_1 \wedge \ldots \wedge x_n \right) = \left(\sum_{i=1}^{n} \frac{dV_i}{dx_i} \right) dx_1 \wedge \ldots \wedge x_n, \end{aligned}$$

from which the desired conclusion follows.

4. We compute

$$\begin{aligned} V^\flat \wedge \star W^\flat &= \sum_{i=1}^n V_i\, dx_i \wedge \star \sum_{i=1}^n W_i\, dx_i \\ &= \sum_{i=1}^n V_i\, dx_i \wedge \sum_{i=1}^n (-1)^{i-1} W_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\ &= \sum_{i=1}^n (-1)^{i-1} V_i W_i\, dx_i \wedge (dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n) \\ &= \left(\sum_{i=1}^n V_i W_i \right) \omega_g, \end{aligned}$$

and thus $\star(V^\flat \wedge \star W^\flat) = V \cdot W$.

□

From the above proposition we see that our definitions of grad and div generalize the classical gradient and divergence functions. We will denote these latter two functions by ∇ and $\nabla\cdot$ respectively, in order to distinguish them from grad and div. Thus, in Euclidean space we have

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad \nabla \cdot V = \sum_{i=1}^n \frac{\partial V_i}{\partial x_i},$$

so that the diagrams

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \xrightarrow{\nabla} & \Gamma(T\mathbb{R}^n) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \flat} \\ \Omega^0(\mathbb{R}^n) & \xrightarrow{d} & \Omega^1(\mathbb{R}^n) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma(T\mathbb{R}^n) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^n) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \star} \\ \Omega^{n-1}(\mathbb{R}^n) & \xrightarrow{d} & \Omega^n(\mathbb{R}^n) \end{array} \tag{3.5}$$

commute. In the special case that $n = 2$ we have $\Omega^1(\mathbb{R}^2) = \Omega^{2-1}(\mathbb{R}^2)$, so that $\flat$ and β have the same domain and codomain, which suggests that we may combine the two diagrams into one. But $\flat \neq \beta$ in the two dimensional case, so in order to combine the diagrams we need to choose either $\flat$ or β as our preferred map and then adjust either ∇ or $\nabla\cdot$ accordingly. Let's say we choose $\flat$. We then get the commutative diagram

$$\begin{array}{ccccc} C^\infty(\mathbb{R}^2) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^2) & \xrightarrow{F} & C^\infty(\mathbb{R}^2) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \flat} & & \downarrow{\scriptstyle \star} \\ \Omega^0(\mathbb{R}^2) & \xrightarrow{d} & \Omega^1(\mathbb{R}^2) & \xrightarrow{d} & \Omega^2(\mathbb{R}^2), \end{array}$$

where $F = (\nabla\cdot) \circ \text{rot}$, and rot is the linear map that rotates vector fields by sending $\dfrac{\partial}{\partial x}$ to $\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial y}$ to $-\dfrac{\partial}{\partial x}$. In 3-space, on the other hand, we need the classical curl operator on vector fields to glue the two diagrams together.

Similarly, in light of Proposition 3.5 we see that the diagrams

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \xrightarrow{\nabla} & \Gamma(T\mathbb{R}^n) \\ {\scriptstyle\star}\downarrow & & \downarrow{\scriptstyle\beta} \\ \Omega^n(\mathbb{R}^n) & \xrightarrow{-d^*} & \Omega^{n-1}(\mathbb{R}^n) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma(T\mathbb{R}^n) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^n) \\ {\scriptstyle\flat}\downarrow & & \downarrow{\scriptstyle id} \\ \Omega^1(\mathbb{R}^n) & \xrightarrow{-d^*} & \Omega^0(\mathbb{R}^n) \end{array} \tag{3.6}$$

commute. Again, when $n = 2$ it is possible adjust them so that they combine, and when $n = 3$ the curl operator lets us join them together.

3.5 Three-space

We now turn our attention to Euclidean 3-space, where the standard coordinates will be denoted x, y and z.

In 3-space, we define the curl operator on vector fields by

$$\begin{aligned} \text{curl} : \Gamma(TM) &\longrightarrow \Gamma(TM) \\ V &\longmapsto \beta^{-1} dV^\flat. \end{aligned}$$

The definition of curl only works for 3-dimensional manifolds, since $dV^\flat$ is a 2-form and β^{-1} takes $(n-1)$-forms to vector fields, so that $\beta^{-1}dV^\flat$ only makes sense when $n = 3$.

Remark 3.7. In fact, our definition of curl works well on any oriented Riemannian 3-manifold. We will use curl in this more general setting in Theorem 3.14.

Let $\nabla\times$ and $\times$ denote the classical curl operator and cross product on $\mathbb{R}^3$. The following proposition shows that curl coincides with $\nabla\times$ in $\mathbb{R}^3$.

Proposition 3.8. *Let V and W be smooth vector fields on $\mathbb{R}^3$ and let $f \in C^\infty(\mathbb{R}^3)$. Then the following statements are true:*

1. $\nabla \times V = \text{curl}\, V = (d^*\beta V)^\sharp$.
2. $V \times W = [\star(V^\flat \wedge W^\flat)]^\sharp$.
3. $\text{curl grad}\, f = 0$ *and* $\text{div curl}\, V = 0$.

Proof.

1. We begin by showing that $\nabla \times V = \beta^{-1}(dV^\flat)$. First we compute

$$\begin{aligned} dV^\flat &= \frac{\partial V_1}{\partial y} dy \wedge dx + \frac{\partial V_1}{\partial z} dz \wedge dx + \frac{\partial V_2}{\partial x} dx \wedge dy + \frac{\partial V_2}{\partial z} dz \wedge dy \\ &+ \frac{\partial V_3}{\partial x} dx \wedge dz + \frac{\partial V_3}{\partial y} dy \wedge dz \\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) dx \wedge dz \\ &+ \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) dx \wedge dy. \end{aligned}$$

Since $n = 3$ we have $\beta^{-1} = (-1)^{n+1} \sharp \star = \sharp \star$, where $\star(dy \wedge dz) = dx$, $\star(dx \wedge dz) = -dy$ and $\star(dx \wedge dy) = dz$. From this we conclude that $\nabla \times V = \beta^{-1} dV^\flat$.

Next we show that $\nabla \times V = (d^* \beta V)^\sharp$. Using the same identities as above, we have

$$\begin{aligned} d^* \beta V &= (-1)^{3(2+1)+1} \star d \star (V_1\, dy \wedge dz - V_2\, dx \wedge dz + V_3\, dx \wedge dy) \\ &= \star d(V_1\, dx + V_2\, d_y + V_3 d_z) \\ &= \star \left[\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) dx \wedge dz \right. \\ &\quad \left. + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) dx \wedge dy \right]. \end{aligned}$$

After applying $\sharp$ to the above expression we end up in the exact same situation as above. Again, this is because $\beta^{-1} = \sharp \star$. And so we are done.

2. We first compute

$$\begin{aligned} V^\flat \wedge W^\flat &= (V_1 dx + V_2 dy + V_3 dz) \wedge (W_1 dx + W_2 dy + W_3 dz) \\ &= V_1 W_2\ dx \wedge dy + V_1 W_3\ dx \wedge dz + V_2 W_1\ dy \wedge dx \\ &\quad + V_2 W_3\ dy \wedge dz + V_3 W_1 dz \wedge dx + V_3 W_2 dz \wedge dy \\ &= (V_2 W_3 - V_3 W_2)\ dy \wedge dz + (V_1 W_3 - V_3 W_1)\ dx \wedge dz \\ &\quad + (V_1 W_2 - V_2 W_1)\ dx \wedge dy. \end{aligned}$$

Applying $\star$ to the above expression yields

$$\begin{aligned}
[\star(V^\flat \wedge W^\flat)]^\sharp &= [(V_2W_3 - V_3W_2)\,dx - (V_1W_3 - V_3W_1)\,dy \\
&\qquad + (V_1W_2 - V_2W_1)\,dz]^\sharp \\
&= (V_2W_3 - V_3W_2)\,\frac{\partial}{\partial x} - (V_1W_3 - V_3W_1)\,\frac{\partial}{\partial y} \\
&\qquad + (V_1W_2 - V_2W_1)\,\frac{\partial}{\partial z} \\
&= V \times W,
\end{aligned}$$

which is what we set out to prove.

3. Unwinding the definitions and using that $d^2 = 0$ we see that

$$\operatorname{curl}\operatorname{grad} f = \beta^{-1}d((df)^\sharp)^\flat = \beta^{-1}ddf = 0$$

and

$$\operatorname{div}\operatorname{curl} V = \star^{-1}d\beta(\beta^{-1}dV^\flat) = \star^{-1}ddV^\flat = 0.$$

□

Proposition 3.8 tells us that the diagrams

$$\begin{array}{ccc}
\Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^3) \\
\flat\downarrow & & \downarrow\beta \\
\Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3)
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^3) \\
\beta\downarrow & & \downarrow\flat \\
\Omega^2(\mathbb{R}^3) & \xrightarrow{d^*} & \Omega^1(\mathbb{R}^3)
\end{array} \tag{3.7}$$

commute. Thus, in 3-space we can combine the diagrams (3.5) with (3.7), which together with the fact that in 3-space we have

$$\star^{-1} = \star \quad \text{and} \quad d^* = (-1)^k \star d\star \tag{3.8}$$

lets us draw the following commutative diagram with exact rows and isomorphisms in the columns, which nicely sums up much of our work so far.

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^3) \\
id\downarrow & & \flat\downarrow & & \beta\downarrow & & \downarrow\star \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\star\downarrow & & \star\downarrow & & \star\downarrow & & \downarrow\star \\
\Omega^3(\mathbb{R}^3) & \xrightarrow{-d^*} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d^*} & \Omega^1(\mathbb{R}^3) & \xrightarrow{-d^*} & \Omega^0(\mathbb{R}^3) \\
\star\downarrow & & \beta^{-1}\downarrow & & \sharp\downarrow & & \downarrow id \\
C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^3)
\end{array} \tag{3.9}$$

3.6 Vector calculus identities

Diagram (3.9) shows us how the d operator generalizes vector calculus once we figure out the appropriate ways to translate between vector fields and differential forms. In this section, we use the results of the previous sections to showcase how the theory of differential forms lets us deduce and reinterpret some familiar vector calculus identities. Recall that we already proved that $\nabla \times \nabla = 0$ and $\operatorname{div} \nabla \times = 0$ in Proposition 3.8. Throughout the rest of this section we work in Euclidean space. For a self-contained treatment of this material, including a discussion about applications to electodynamics, see Schleifer's article [11].

We start off with a few first order identities, i.e. identities involving only first derivatives.

Proposition 3.9 (First order identities)**.** *Let $f \in C^\infty(\mathbb{R}^3)$ and $V, W \in \Gamma(T\mathbb{R}^3)$. Then the following identities hold:*

1. $\nabla(fg) = f\nabla g + g\nabla f$
2. $\nabla \times (fV) = f(\nabla \times V) + (\nabla f) \times V$
3. $\nabla \cdot (fV) = f(\nabla \cdot V) + V \cdot (\nabla f)$

Proof.

1. Since d is the differential for 0-forms, we have
$$\nabla(fg) = (d(fg))^\sharp = (fdg + gdf)^\sharp.$$

2. Using that $d(fV)^\sharp = d(fV^\sharp) = df \wedge V^\sharp + fdV^\sharp$, we compute
$$\begin{aligned}\nabla \times (fV) = \beta^{-1}(d(fV)^\sharp) &= [\star(d(fV)^\sharp)]^\sharp \\ &= [\star(df \wedge V^\sharp)]^\sharp + [f \star (dV^\sharp))]^\sharp \\ &= (\nabla f) \times V + f(\nabla \times V)).\end{aligned}$$

3. In the following computation we use that $\star = \star^{-1}$ in 3-space.
$$\begin{aligned}\nabla \cdot (fV) = \star^{-1} d(\beta(fV)) &= \star^{-1} d(f\beta(V)) \\ &= \star^{-1}(df \wedge \star V^\flat + fd\beta(V)) \\ &= \star(df \wedge \star V^\flat) + \star^{-1}(fd\beta(V)) \\ &= V \cdot (\nabla f) + f(\nabla \cdot V).\end{aligned}$$

□

Second order identities, i.e. identities involving second derivatives, are keen not to be left out, so we include one such identity here. Let ∇^2 denote the classical Laplacian, defined by

$$\begin{aligned}\nabla^2 : C^2(\mathbb{R}^n) &\longrightarrow C^0\mathbb{R}^n) \\ f &\longmapsto \nabla \cdot \nabla f.\end{aligned}$$

The following proposition shows how the Laplace-de Rham operator generalizes the classical Laplacian.

Proposition 3.10. *Let $f \in C^\infty(\mathbb{R}^n)$. Then $\Delta f = -\nabla^2 f$.*

Proof. We compute

$$\begin{aligned}\Delta f = dd^* f + d^* df &= d(-1)^{3(0-1)+1} \star d \star f + d^* df \\ &= d^* \star df \\ &= d^*((df)^\sharp)^\flat \\ &= -\nabla \cdot (\nabla f) = -\nabla^2 f.\end{aligned}$$

□

3.7 Classical theorems

Stokes' theorem (Theorem 2.7) generalizes the well known Green's theorem, divergence theorem and Kelvin-Stokes theorem. We state and prove them in this section. Whereas Green's theorem is a more or less immediate consequence of Stoke's theorem, we rely on a lemma to prove the other two.

Theorem 3.11 (Green's theorem)**.** *Let $D \subset \mathbb{R}^2$ be a compact regular domain, and let $\eta = P\,dx + Q\,dy$ be a 1-form on D. Then*

$$\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy. \tag{3.10}$$

Proof. We have

$$d\eta = \frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial Q}{\partial x} dy \wedge dx = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx \wedge dy.$$

Applying Stokes' theorem yields

$$\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx \wedge dy = \int_D d\eta = \int_{\partial D} \eta.$$

From here the theorem follows from the definition of integration of differential forms over domains of integration in $\mathbb{R}^n$. □

Lemma 3.12. *Let (M, g) be an oriented Riemannian n-manifold with boundary, let $\tilde{g}$ be the induced metric on ∂M and let N be the outward-pointing unit normal vector field along ∂M. Then, for any vector field V along ∂M, we have*

$$\imath^*(\beta(V)) = g(V, N)\omega_{\tilde{g}},$$

where $\imath : \partial M \longrightarrow M$ is the inclusion map of the boundary.

Proof. Let $V = V_T + V_N$, where $V_N = g(V, N)\, N$ and $V_T = V - V_N$. That is, V_T and V_N are the components of V that are respectively tangent and normal to ∂M. Thus we get

$$\imath^*(\beta(V)) = \imath^*(\beta V_T + \beta V_N) = \imath^*(\beta(V_T)) + \imath^*(\beta(V_N)).$$

Recall that by Corollary 3.4 we have $\beta(V) = i_V(\omega_g)$. Hence, for vectors $V_1, \ldots, V_{n-1}$ tangent to ∂M we have

$$\begin{aligned}\beta(V_T)(V_1, \ldots, V_{n-1}) &= (i_{V_T}\omega_g)(V_1, \ldots, V_{n-1}) \\ &= \omega_g(V_T, V_1, \ldots, V_{n-1}) = 0,\end{aligned}$$

where the last equality holds because the tangent space of ∂M is $(n-1)$-dimensional, and thus a set of n vectors that are tangent to ∂M must be linearly dependent. It follows that

$$\imath^*(\beta(V_T)) = 0.$$

Consider now $\imath^*(\beta(V_N))$. Writing $V_N = g(V, N)N$ and using the linearity of the pullback operator, we get

$$\imath^*(\beta(g(V, N)N)) = g(V, N)\imath^*(\beta(N)).$$

Now $\imath^*(\beta(N)) = \imath^*(i_N\omega_g)$ is an orientation form on ∂M corresponding to the orientation induced by N. Let $p \in \partial M$ and let $(E_1, \ldots, E_{n-1})$ be an orthonormal frame of ∂M at p. Then we get a positively oriented orthonormal frame $(N(p), E_1(p), \ldots, E_{n-1}(p))$ for M at p, and thus

$$\begin{aligned}\imath^*(\beta(N))(E_1, \ldots, E_{n-1}) &= \imath^*(i_N\omega_g)(E_1, \ldots, E_{n-1}) \\ &= \omega_g(N, E_1, \ldots, E_{n-1}) = 1.\end{aligned}$$

It follows that $\imath^*(\beta(N)) = \omega_{\tilde{g}}$, and hence

$$\imath^*(\beta(V_N)) = g(V, N)\imath^*(\beta(N)) = g(V, N)\, \omega_{\tilde{g}},$$

which is what we wanted to prove. □

Theorem 3.13 (The divergence theorem)**.** *Let (M, g) be an oriented Riemannian n-manifold with boundary, let $\tilde{g}$ be the induced metric on ∂M, let N be the outward-pointing unit normal vector field along ∂M and let $V \in \Gamma(TM)$ be compactly supported. Then*

$$\int_M (\operatorname{div} V)\, \omega_g = \int_{\partial M} g(V, N)\, \omega_{\tilde{g}}.$$

Proof. By definition we have $\star \operatorname{div} V = d\beta(V)$, and since $d\beta(V)$ is an n-form, we have

$$d\beta(V)\omega_g = \star d\beta(V) = \star\star \operatorname{div} V = \operatorname{div} V.$$

Applying Stokes' theorem yields

$$\int_M (\operatorname{div} V)\, \omega_g = \int_M d\beta(V) = \int_{\partial M} \imath^* \beta(V), \tag{3.11}$$

where $\imath : \partial M \longrightarrow M$ is the inclusion map, and from Lemma 3.12 we have

$$\imath^*(\beta(V)) = g(V, N)\, \omega_{\tilde{g}}. \tag{3.12}$$

The divergence theorem is now obtained by combining (3.11) and (3.12). □

Theorem 3.14 (Kelvin-Stokes theorem)**.** *Let (Y, g) be an oriented Riemannian 3-manifold with boundary and let $S \subset Y$ be a compact oriented smooth 2-dimensional submanifold with boundary. Let N be the unit normal vector field along S that determines its orientation and let T be the unique positively oriented unit tangent vector field along ∂S. Let $V \in \Gamma(TY)$. Then*

$$\int_S g(\operatorname{curl} V, N)\, \omega_{\tilde{g}} = \int_{\partial S} g(V, T)\, \omega_{\hat{g}},$$

where $\tilde{g}$ and $\hat{g}$ are the metrics induced by g on S and ∂S respectively.

Proof. By slightly rearranging our definition of curl, we get

$$\beta(\operatorname{curl} V) = dV^\flat,$$

and by Stokes' theorem we then have

$$\int_S dV^\flat = \int_S \imath^*(dV^\flat) = \int_{\partial S} \jmath^* V^\flat = \int_{\partial S} V^\flat, \tag{3.13}$$

where $\imath : S \longrightarrow M$ and $\jmath : \partial S \longrightarrow S$ are the inclusion maps. From Lemma 3.12 we see that

$$\imath^* dV^\flat = \imath^*(\beta(\operatorname{curl} V)) = g(\operatorname{curl} V, N)\, \omega_{\tilde{g}}. \tag{3.14}$$

We now turn our attention to $\jmath^* V^\flat$. It is a top-degree form on ∂S, so it must be of the form $f\,\omega_{\hat{g}}$ for some $f \in C^\infty(\partial S)$. As T is the unique positively oriented unit tangent vector field on ∂S, we know that $\omega_{\hat{g}}(T) = 1$. Hence

$$f = f\,\omega_{\hat{g}}(T) = (\jmath^*\,V^\flat)(T) = V^\flat(T) = g(V, T),$$

and therefore

$$\jmath^* V^\flat = g(V, T)\,\omega_{\hat{g}}. \tag{3.15}$$

Lastly, substituting (3.14) and (3.15) into (3.13) completes the proof. □

4 Algebraic topology

In this section, we briefely introduce some notions and results from algebraic topology that will prove useful later on. In particular, we introduce singular homology and cohomology, as well as reduced and relative homology, and de Rham cohomology. I refer the reader to Hatcher, [6], and Bott and Tu, [2], for a tourough treatment of this material.

4.1 Singular homology and cohomology

The *standard n-simplex*, denoted Δ^n, is the convex combination of the standard basis vectors $e_0, \ldots, e_n$ in $\mathbb{R}^{n+1}$, i.e.

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_0, \ldots, x_n \geq 0 \text{ and } \sum_{i=0}^{n} x_i = 1 \right\}.$$

Let X be a topological space. A continuous mapping $\sigma^k : \Delta^k \longrightarrow X$ is called a *singular k-simplex*. Let R be a unital ring. (From this point onwards, we assume all rings to be unital.) We let $C_k(X; R)$ denote the group of formal sums of k-simplexes with coefficients in R, and call the elements of $C_k(X; R)$ *singular n-chains*. Define a (linear) homomorphism

$$\partial_k : C_k(X; R) \longrightarrow C_{k-1}(X; R),$$

called the boundary operator, as follows: whenever $\sigma^k = (\sigma^k_0, \ldots, \sigma^k_k)$ is a k-simplex, i.e. a basis vector of $C_k(X)$, let

$$\partial_k \sigma^k := \sum_i (-1)^i (\sigma^k_0, \ldots, \widehat{\sigma^k_i}, \ldots, \sigma^k_k).$$

The boundary operator has the property that $\partial_{k-1}\partial_k = 0$, and hence the diagram

$$\ldots \longrightarrow C_2(X; R) \xrightarrow{\partial_2} C_1(X; R) \xrightarrow{\partial_1} C_0(X; R) \xrightarrow{\partial_0} 0, \tag{4.1}$$

is a chain complex, called the *singular chain complex*. The *k-th singular homology group of X with coefficients in R*, denoted $H_k(X; R)$, is defined to be the k-th homology group of this complex, i.e.

$$H_k(X; R) := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}.$$

Remark 4.1. Since we take our coefficients from a ring R, the homology groups that we just defined are actually R-modules. If we take the coefficients from an Abelian group instead of a ring, the homology groups become groups.

Consider now the singular chain complex

$$\ldots \longrightarrow C_2(X;\mathbb{Z}) \xrightarrow{\partial_2} C_1(X;\mathbb{Z}) \xrightarrow{\partial_1} C_0(X;\mathbb{Z}) \xrightarrow{\partial_0} 0. \tag{4.2}$$

Here, the coefficients are integers, and thus the group $C_k(X;\mathbb{Z})$ can be thought of as being the free abelian group on the set of singular k-simplices. Take R to be a ring, and dualize (4.2) by applying the functor $\mathrm{Hom}(\cdot, R)$ to obtain the cochain complex

$$0 \longrightarrow C^0(X;R) \xrightarrow{\partial^0} C^1(X;R) \xrightarrow{\partial^1} C^2(X;R) \longrightarrow \ldots,$$

where $C^k(X;R) := \mathrm{Hom}(C_k, R)$ and $\partial^k := \mathrm{Hom}(\partial_k, R)$. The *$k$-th singular cohomology group of X with coefficients in R*, denoted $H^k(X;R)$, is defined to be the quotient

$$H^k(X;R) := \frac{\ker \partial^k}{\operatorname{im} \partial^{k-1}}.$$

Remark 4.2. We will oftentimes be lazy and refer to singular homology as just homology and singular cohomology as just cohomology. When $R = \mathbb{Z}$, one oftentimes simply writes $H_k(X)$ instead of $H_k(X;R)$ and $H^k(X)$ instead of $H^k(X;R)$.

Remark 4.3. The elements of $\ker \partial_0 = C_0(X;R)$ are formal sums of continuous maps from point $x_0 = 1 \in \mathbb{R}^1$ to X, whereas $\operatorname{im} \partial_1$ is generated by elements of the form $a - b$, where $a, b \in \Delta^0 \to X_i$ and X_i denotes a path component of X. From this it can be seen that the quotient $H_0(X;R) = \ker \partial_0 / \operatorname{im} \partial_1$ is isomorphic the free R-module over the path components of X. If the number of path components is $n < \infty$, then $H_0(X;R) \simeq R^n$.

We also need to consider reduced homology and relative homology. Let $\epsilon : C_0(X;R) \longrightarrow R$ be the map given by

$$\sigma\left(\sum_i n_i \sigma_i\right) := \sum_i n_i$$

and define the *k-th reduced homology of X with coefficient in R*, denoted by $\widetilde{H}_k(X)$, to be the k-th homology group of the chain complex

$$\ldots \longrightarrow C_2(X;R) \xrightarrow{\partial_2} C_1(X;R) \xrightarrow{\partial_1} C_0(X;R) \xrightarrow{\epsilon} R \longrightarrow 0.$$

The *k-th homology group of X relative to a subspace* $Y \subset X$, denoted by $H_k(X, Y; R)$ is defined to be the k-th homology group of the quotient chain complex

$$\cdots \longrightarrow \frac{C_2(X;R)}{C_2(Y;R)} \xrightarrow{\partial_2} \frac{C_1(X;R)}{C_1(Y;R)} \xrightarrow{\partial_1} \frac{C_0(X;R)}{C_0(Y;R)} \xrightarrow{\partial_0} 0,$$

where the boundary map ∂_i is the usual boundary map that has descended to a map on the quotient $C_i(X;R)/C_i(Y;R)$.

4.2 Poincaré-Lefschetz duality

In this section we prove that for any compact topological manifold M with boundary and for any field $\mathbb{F}$, we have $H_k(M;\mathbb{F}) \simeq H_{n-k}(M, \partial M;\mathbb{F})$ and $H_k(M, \partial M;\mathbb{F}) \simeq H_{n-k}(M;\mathbb{F})$, This follows from the Poincaré-Lefschetz duality theorem and the Universal coefficient theorem (Theorems 4.6 and 4.4), both of which we state without proof.

Theorem 4.4 (Universal coefficient theorem for cohomology)**.** *Let X be a topological space, let A a principal ideal domain and let N be a module over A. Then the sequence*

$$0 \to \mathrm{Ext}^1_A(H_{k-1}(X;A), N) \to H^k(X;N) \to \mathrm{Hom}_A(H_k(X;A), N) \to 0 \quad (4.3)$$

is a short exact sequence that splits.

Corollary 4.5. *Let M be a compact topological topological manifold with boundary and let $\mathbb{F}$ be a field. Then $H_k(M, \mathbb{F}) \simeq H^k(M, \mathbb{F})$.*

Proof. Take M to be a compact topological manifold. Then M is certainly also a topological space, so that Theorem 4.4 is applicable. Let $\mathbb{F}$ be a field and take $N := \mathbb{F}$ in Theorem 4.4 (this works since $\mathbb{F}$ is trivially a module over itself). Now, since (4.3) splits, we have

$$H^k(M;\mathbb{F}) \simeq \mathrm{Hom}_{\mathbb{F}}(H_k(M;\mathbb{F}), \mathbb{F}) \oplus \mathrm{Ext}^1_{\mathbb{F}}(H_{k-1}(M;\mathbb{F}), \mathbb{F})$$

Moreover, since $\mathbb{F}$ is a field, we have

$$\mathrm{Ext}^1_A(H_{k-1}(M;\mathbb{F}), \mathbb{F}) = 0,$$

and since M is a compact topological manifold, $H_k(M;\mathbb{F})$ is finite-dimensional (we will not prove this well known fact, but for smooth manifolds it follows easily from the Hodge decomposition theorem, as we will see in Chapter 5 and is thus isomorphic to its dual space, that is

$$\mathrm{Hom}_{\mathbb{F}}(H_k(M;\mathbb{F}), \mathbb{F}) \simeq H_k(M, \mathbb{F}).$$

We thus conclude that

$$H^k(M;\mathbb{F}) \simeq \mathrm{Hom}_{\mathbb{F}}(H_k(M;\mathbb{F}), \mathbb{F}) \simeq H_k(M;\mathbb{F}),$$

which is what we set out to prove. □

Theorem 4.6 (Poincaré-Lefschetz duality theorem)**.** *Let M be a compact, oriented n-dimensional topological manifold with boundary. Then, for any ring R,*

$$H_k(M;R) \simeq H^{n-k}(M,\partial M;R) \quad \textit{and} \quad H_k(M,\partial M;R) \simeq H^{n-k}(M;R).$$

The Poincaré-Lefschetz duality theorem is a generalization of the Poincaré duality theorem, with the latter following easily from the former.

Corollary 4.7 (Poincaré duality theorem)**.** *Let M be a closed topological manifold and let R be a ring. Then $H_k(M;R) \simeq H^{n-k}(M;R)$.*

Proof. Take M to be a closed topological manifold. Since M has no boundary we have $H^k(M) \simeq H^k(M,\partial M)$, and then the corollary follows from applying Theorem 4.6 to M. □

Proposition 4.8. *Let M be a compact topological manifold with boundary. Then, for any field $\mathbb{F}$,*

$$H_k(M;\mathbb{F}) \simeq H_{n-k}(M,\partial M;\mathbb{F}) \quad \textit{and} \quad H_k(M,\partial M;\mathbb{F}) \simeq H_{n-k}(M;\mathbb{F}).$$

Proof. The result is a direct consequence of Corollary 4.5 and Corollary 4.6. □

4.3 Alexander duality

In what lies ahead, we need to be able to compare the homology of a compact domain in $\mathbb{R}^n$ with the homology of its relative complement with respect to $\mathbb{R}^n$. Our main tool for this is the Alexander duality theorem, which we state here without proof.

Theorem 4.9 (Alexander duality theorem)**.** *Let X be nonempty, compact and locally contractible proper topological subspace of S^n. Then, for any ring R,*

$$\widetilde{H}_k(S^n \setminus X;R) \simeq \widetilde{H}^{n-k-1}(X;R).$$

Corollary 4.10. *Let X be nonempty, compact and locally contractible proper topological subspace of $\mathbb{R}^n$. Then, for any ring R,*

$$\widetilde{H}_k(\mathbb{R}^n \setminus X;R) \simeq \begin{cases} H^{n-k-1}(X;R) & \textit{if } k = n-1 \\ \widetilde{H}^{n-k-1}(X;R) & \textit{otherwise} \end{cases}$$

Proof. Since $S^n \simeq \mathbb{R}^n \cup \{\infty\}$, take $X' = X \cup \{\infty\}$ and note that $\widetilde{H}_i(X;R) \simeq \widetilde{H}_i(X';R)$ except for $i = 0$, since X' has one more connected componend than X. This implies that $H_0(X;R) \simeq \widetilde{H}_0(X';R)$. Applying Theorem 4.9 to X' thus yields the desired result. □

In particular, the above corollary, as well as Proposition 4.8, applies to smooth manifolds, since any smooth n-dimensional manifold can be embedded into $\mathbb{R}^{2n}$. The important special case that we are concerned about is when $X = D$ is a compact regular domain in $\mathbb{R}^3$.

4.4 Homology in $\mathbb{R}^3$

Let D be a compact regular domain in $\mathbb{R}^3$. Elements in $H_0(D;\mathbb{R})$ can be thought of as equivalence classes of points in D. Two points are then equivalent in $H_0(D;\mathbb{R})$ if they lie in the same connected component of D. Similarly, elements in $H_1(D;\mathbb{R})$ can be thought of as equivalence classes of oriented loops in D, and elements in $H_2(D;\mathbb{R})$ can be thought of as equivalence classes of closed oriented surfaces in D. We adopt this point of view in this section, in which we state some properties of singular homology for compact regular domains in $\mathbb{R}^3$. We do not aim for general statements here; we only state what we need and leave it to the reader to look up further details in eg. [6].

Proposition 4.11. *Let D be a compact regular domain in $\mathbb{R}^3$. Then the following statements are true:*

1. *Every element in $H_1(D;\mathbb{R})$ has a representative that is a closed curve in ∂D. Similarly, every element in $H_1(\mathbb{R}^3 \setminus D;\mathbb{R})$ has a representative that is a loop in ∂D.*

2. *Every element in $H_1(D;\mathbb{R})$ is expressible as the sum two classes, one of which that can be representated by a closed curve that bounds a surface in $H_2(D;\mathbb{R})$ and one of which that can be representated by a closed curve that bounds a surface in $H_2(\mathbb{R} \setminus D;\mathbb{R})$*

Proof. Since $H_i(\mathbb{R}^n;\mathbb{R}) = 0$ for $i > 0$, the result is given by looking at the Mayer-Vietoris sequence

$$\ldots \longrightarrow H_2(\mathbb{R}^n;\mathbb{R}) \longrightarrow H_1(\partial D;\mathbb{R}) \\ \longrightarrow H_1(D;\mathbb{R}) \oplus H_1(\mathbb{R}^n \setminus D;\mathbb{R}) \longrightarrow H_1(\mathbb{R}^n;\mathbb{R}) \longrightarrow \ldots$$

For more details, see the proof of Proposition 5.14. □

Proposition 4.12. *Let D be a compact regular domain in $\mathbb{R}^3$. Then the following statements are true:*

1. $\dim H_0(D;\mathbb{R}) = \dim H_3(D,\partial D;\mathbb{R}) =$ *the number of connected components of D.*

2. $\dim H_1(D;\mathbb{R}) = \dim H_2(D,\partial D;\mathbb{R}) = \sum_{i\in I} g_i$, *where I indexes the connected components of ∂D and g_i denotes the genus of the i-th such component.*

3. $\dim H_2(D;\mathbb{R}) = \dim H_1(D,\partial D;\mathbb{R}) =$ *the number of connected components of ∂D $-$ the number of connected components of D.*

Proof. Let D be as in the proposition.

1. Since D is a manifold, its path components and connected components coincide. The result then follows from Remark 4.3 and Proposition 4.8.

2. Proposition 4.11 tells us that $H_1(\partial D;\mathbb{R}) = H_1(D;\mathbb{R}) \oplus H_1(\mathbb{R}^3 \setminus D;\mathbb{R})$ and from Corollary 4.10 we know that $H_1(\mathbb{R}^3 \setminus D;\mathbb{R}) = H^1(D;\mathbb{R})$. From Corollary 4.5 we then see that $\dim H^1(\partial D;\mathbb{R}) = \dim H_2(D;\mathbb{R})$, so that $\dim H_1(\partial D;\mathbb{R}) = 2\dim H_1(D;\mathbb{R})$. But is

$$\dim H_1(\partial D;\mathbb{R}) = 2 \cdot \sum_{i\in I} g_i, \tag{4.4}$$

and hence, by Proposition 4.8, we have

$$\dim H_1(D;\mathbb{R}) = \dim H_2(D,\partial D;\mathbb{R}) = \sum_{i\in I} g_i. \tag{4.5}$$

3. This must follow from the Euler characteristic, since we already computed all $H_i(D;\mathbb{R})$ dimensions except $H_2(D;\mathbb{R})$ (and all of these are zero except for $i = 0, 1, 2$, by e.g. Alexander duality). By definition we have

$$\chi(D) = \dim H_0(D;\mathbb{R}) - \dim H_1(D;\mathbb{R}) + \dim H_2(D;\mathbb{R}),$$

and hence

$$\dim H_2(D;\mathbb{R}) = \chi(D) - \dim H_0(D;\mathbb{R}) + \dim H_1(D;\mathbb{R}).$$

Thus, if we knew $\chi(D)$ we would be done.

The key is that $\chi(Y) = 0$ for any *closed* 3-manifold Y, since by Poincaré duality we have

$$\chi(Y) = \dim H_0(Y;\mathbb{R}) - \dim H_1(Y;\mathbb{R}) + \dim H_2(Y;\mathbb{R}) - \dim H_3(Y;\mathbb{R}),$$

but

$$\dim H_1(Y;\mathbb{R}) = \dim H^2(Y;\mathbb{R}) = \dim H_2(Y;\mathbb{R})$$

and

$$\dim H_0(Y;\mathbb{R}) = \dim H^3(Y;\mathbb{R}) = \dim H_3(Y;\mathbb{R}),$$

so everything cancels.

To make a closed manifold from D, we simply glue D to itself (with reversed orientation) along the boundary (this is called *doubling*) to get

$$0 = \chi(D) + \chi(D) - \chi(\partial D).$$

Since $\chi(\partial D) = \sum_{i\in I}(2 - 2g_i)$, we thus have $\chi(D) = \sum_{i\in I}(1 - g_i)$.

To see that this works, we compute

$$\begin{aligned}\dim H_2(D;\mathbb{R}) &= \chi(D) - \dim H_0(D;\mathbb{R}) + \dim H_1(D;\mathbb{R}) \\ &= \sum_{i\in I}(1-g_i) - (\#\text{ con. comp. of } D) + \sum_{i\in I} g_i \\ &= (\#\text{ con. comp. of } \partial D) \\ &\quad - (\#\text{ con. comp. of } D),\end{aligned}$$

and the proof is done (we just need to apply Proposition 4.8).

□

We also need the following theorem, which we state without proof.

Theorem 4.13. *Let D be a compact regular domain in $\mathbb{R}^3$. Then we can pick bases $\{\Sigma_i\}_{i=1}^m$ for $H_2(D, \partial D; \mathbb{R})$, $\{C_i\}_{i=1}^m$ for $H_1(D;\mathbb{R})$, and $\{C_i'\}_{i=1}^m$ for $H_1(\mathbb{R}^3 \setminus D; \mathbb{R})$, such that the following properties hold true:*

1. *$\{\Sigma_i\}_{i=1}^m$ consists of smooth oriented cross sectional surfaces in D such that $\Sigma_i \subset \partial D$, for $1 \leq i \leq m$.*
2. *$\{C_i\}_{i=1}^m$ consists of smooth oriented loops in $D \setminus \partial D$.*
3. *$\{C_i'\}_{i=1}^m$ consists of smooth oriented loops in $\mathbb{R}^3 \setminus D$.*
4. *The intersection number of Σ_i and C_j is δ_{ij} and the linking number of C_i and C_j is δ_{ij}*

4.5 De Rham cohomology

Let M be a smooth manifold. A differential form $\eta \in \Omega^*(M)$ is said to be *closed* if $d\eta = 0$. If there exists another form ζ such that $\eta = d\zeta$, then η is said to be *exact*. The linearity of the exterior derivative tells us that the closed and exact k-forms are linear subspaces of $\Omega^k(M)$. Since $dd = 0$, all exact forms are closed (that is, $\operatorname{im} d \subset \ker d$) and we get the cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \dots,$$

where d_i denotes the restriction of d to $\Omega^i(M)$. This complex is known as the *de Rham complex*. The k-th *de Rham cohomology group*, denoted H^k_{dR}, is defined as the k-th cohomology group of (4.5), i.e.

$$H^k_{\mathrm{dR}}(M) := \frac{\ker d_k}{\operatorname{im} d_{k-1}}$$

Thus, for two closed k-forms η and ζ the assertion that $[\eta] = [\zeta]$ in $H^k_{\mathrm{dR}}(M)$ is equivalent to $\eta - \zeta$ being exact.

Example 4.14. In $\mathbb{R}^n$, all closed forms are exact. This follows from the Poincaré lemma, which states that every closed form is exact on a contractible manifold. As a result of this all the de Rham cohomology groups of $\mathbb{R}^n$ of degree greater that zero are trivial.

4.6 De Rham's theorem

De Rham's theorem tells us that for a smooth manifold M, the de Rham cohomology group $H^k_{dR}(M)$ is isomorphic to the singular cohomology group $H^k(M;\mathbb{R})$. We state it here without proof.

Theorem 4.15 (De Rham's theorem)**.** *Let M be a smooth manifold. Then the map*

$$I : H^k_{dR}(M) \longrightarrow H^k(M;\mathbb{R})$$
$$[\eta] \longmapsto \left(H_k(M;\mathbb{R}) \ni [c] \longrightarrow \int_c \eta \right)$$

is an isomorphism.

We state the following nice and simple consequence of Theorem 4.15 as a corollary.

Corollary 4.16. *A closed manifold (i.e. compact without boundary) is not contractible.*

Proof. Let M be a compact manifold without boundary. Let g be any Riemannian metric on M, and consider the associated volume form ω_g. As ω_g is a top form, it is automatically closed. However, it is not exact, as the following argument shows.

Suppose, in order to derive a contradiction, that ω_g is exact. Then $\omega_g = d\eta$ for some $(n-1)$-form η, and by Stokes' theorem the volume of M is

$$\int_M \omega_g = \int_M d\eta = \int_{\partial M} \eta = 0,$$

which is a contradiction, and therefore ω_g cannot be exact.

Since ω_g is closed but not exact, the cohomology of M is nontrivial. This, in turn, implies that M is not contractible, since cohomology is invariant under homotopy equivalence, and a contractible space is homotopy equivalent to a point. □

5 Hodge theory

In this chapter we use the tools that we have developed so far to take a look at Hodge theory. In particular, we will look at how the Hodge decomposition theorem (Theorem 5.5) provides a decomposition of the space of k-forms on a Riemannian manifold with boundary into an orthogonal direct sum of subspaces with respect to the L^2-metric introduced in Section 2.6, and how a special case of this decomposition theorem for vector fields on compact regular domains in $\mathbb{R}^3$ can be obtained (Theorem 5.12). For a more in-depth treatment of the Hodge decomposition theorem for manifolds with boundary, see Schwarz's book [12]. A more elementary introduction in the context of vector fields on compact domains in $\mathbb{R}^3$ is provided by Cantarella et al. in [4]. A survey that also introduces various applications of the Hodge decomposition theorem is given by Bhatia et al. in [1].

We begin by familiarizing ourselves with some new notation and definitions that will be useful in what lies ahead.

5.1 Notation and definitions

Throughout this section, unless otherwise specified, let (M, g) be a smooth Riemannian manifold with boundary. We write $C^k(M)$ and $cC^k(M)$ for the subspaces of $\Omega^k(M)$ of closed and co-closed k-forms, i.e.

$$C^k(M) := \{\eta \in \Omega^k(M) \mid d\eta = 0\}$$
$$cC^k(M) := \{\eta \in \Omega^k(M) \mid d^*\eta = 0\},$$

and we write $E^k(M)$ and $cE^k(M)$ for the subspaces of exact and co-exact k-forms, i.e.

$$E^k(M) := d(\Omega^{k-1}(M)) = \{\eta \in \Omega^k(M) \mid \eta = d\zeta\}$$
$$cE^k(M) := d^*(\Omega^{k+1}(M)) = \{\eta \in \Omega^k(M) \mid \eta = d^*\zeta\}.$$

Intersections of spaces are denoted by juxtaposition of letters, so that, for example, the space $CcC^k(M) = C^k(M) \cap cC^k(M)$ is the subspace of $\Omega^k(M)$ consisting of forms that are both closed and co-closed. This notation is borrowed from Cappell et al., who use it in [5].

We also need to consider forms that satisfy certain boundary conditions. With this in mind, we define the map

$$\mathbf{t} : \Gamma(\Lambda^k(T^*M)|_{\partial M}) \longrightarrow \Gamma(\Lambda^k(T^*M)|_{\partial M})$$

by demanding that

$$\mathbf{t}\eta(X_1, \dots, X_k) = \eta(X_1^{\|}, \dots, X_k^{\|}) \quad \forall\, X_1, \dots X_k \in \Gamma(TM|_{\partial M}), \tag{5.1}$$

where $X = X^{\perp} + X^{\|}$ is the decomposition of the vector field X along ∂M into its tangential and normal components. We then define the map $\mathbf{n}$ defined by

$$\begin{aligned} \mathbf{n} : \Gamma(\Lambda^k(T^*M)|_{\partial M}) &\longrightarrow \Gamma(\Lambda^k(T^*M)|_{\partial M}) \\ \mathbf{n}\eta &\longmapsto \eta|_{\partial M} - \mathbf{t}\eta. \end{aligned}$$

For any k-form η on M, we thus have $\eta = \mathbf{t}\eta + \mathbf{n}\eta$, and we say that $\mathbf{t}\eta$ is the *tangential component* of η and that $\mathbf{n}\eta$ is the *normal component* of η.

Let $\imath : \partial M \to M$ be the inclusion map of the boundary. By abuse of notation, $\mathbf{t}\eta$ is sometimes identified with the pullback form $\imath^*\eta$. But $\mathbf{t}\eta$ lives in the space $\Gamma(\Lambda^k(T^*M)|_{\partial M})$ whereas $\imath^*\eta$ lives in the space $\Omega^k(\partial M) = \Gamma(\Lambda^k(T^*\partial M))$, and since these are two different spaces we cannot have $\mathbf{t}\eta$ equal to $\imath^*\eta$. The reason why this identification still "makes sense" is clarified in by the following proposition and its proof.

Proposition 5.1. *Let (M, g) be a Riemannian manifold with boundary, let $\imath : \partial M \to M$ be the inclusion map of the boundary, and let $\eta \in \Omega^k(M)$. Then*

$$\mathbf{t}\eta = 0 \quad \textit{if and only if} \quad \imath^*\eta = 0.$$

Proof. Let $X_1, \dots, X_k \in \Gamma(TM|_{\partial M})$. From the definition of $\mathbf{t}\eta$, we see that $(\mathbf{t}\eta)(X_1, \dots, X_k) = 0$ whenever $X_i = X_i^{\perp}$ for some $1 \le i \le k$, and hence

$$(\mathbf{t}\eta)(X_1, \dots X_k) = (\mathbf{t}\eta)(X_1 - X_1^{\perp}, \dots X_k - X_k^{\perp}).$$

Therefore, we only need to consider the case $X_i = X_i^{\|}$ for all $1 \le i \le k$. Then $X_i, \dots, X_k \in \Gamma(T\partial M)$ and we have

$$(\mathbf{t}\eta)(X_1, \dots X_k) = (\imath^*\eta)(X_1, \dots, X_k),$$

and we are done with the proof. □

Proposition 5.2. *Let (M, g) be a Riemannian manifold with boundary and let $\eta \in \Omega^k(M)$. Then*

$$\star(\mathbf{n}\eta) = \mathbf{t}(\star\eta) \quad \textit{and} \quad \star(\mathbf{t}\eta) = \mathbf{n}(\star\eta). \tag{5.2}$$

Proof. Let N denote the outward-pointing unit normal vector field alon ∂M and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame such that $E_1|_{\partial M} = N$ and $E_2|_{\partial M}, \ldots, E_n|_{\partial M} \in T\partial M$. The above definitions, together with Proposition 2.17 and Remark 2.18, imply that for any (k,n)-shuffle $\sigma \in S(k,n) \subset S_n$ (as defined in Proposition 2.9), we have

$$\begin{aligned}(\star \mathbf{n}\eta)(E_{\sigma(k+1)}, \ldots, E_{\sigma(n)}) &= (\text{sgn } \sigma) \cdot (\eta|_{\partial M} - \mathbf{t}\eta)(N, E_{\sigma(2)}, \ldots, E_{\sigma(k)}) \\ &= (\text{sgn } \sigma) \cdot (\eta|_{\partial M})(N, E_{\sigma(2)}, \ldots, E_{\sigma(k)})\end{aligned}$$

and

$$\begin{aligned}(\mathbf{t} \star \eta)(E_{\sigma(k+1)}, \ldots, E_{\sigma(n)}) &= (\text{sgn } \sigma) \cdot (\eta|_{\partial M} - \mathbf{n}\eta)(N, E_{\sigma(2)}, \ldots, E_{\sigma(k)}) \\ &= (\text{sgn } \sigma) \cdot (\eta|_{\partial M})(N, E_{\sigma(2)}, \ldots, E_{\sigma(k)})\end{aligned}$$

whenever $\sigma(1) = 1$, and we have

$$\begin{aligned}&(\star \mathbf{n}\eta)(N, E_{\sigma(k+2)}, \ldots, E_{\sigma(n)}) \\ &\qquad = (\text{sgn } \sigma) \cdot (\mathbf{n}\eta)(E_{\sigma(1)}, \ldots, E_{\sigma(k)}) \\ &\qquad = (\text{sgn } \sigma) \cdot (\eta|_{\partial M} - \mathbf{t}\eta)(E_{\sigma(1)}, \ldots, E_{\sigma(k)}) = 0\end{aligned}$$

and

$$(\mathbf{t} \star \eta)(N, E_{\sigma(k+2)}, \ldots, E_{\sigma(n)}) = 0.$$

whenever $\sigma(k+1) = N$ (these are the only possible cases). This proves the first identity. Applying this identity to $\star\eta$ and taking the Hodge star of each side shows that

$$\star \star [\mathbf{n}(\star\eta)] = \star\mathbf{t}(\star \star \eta) \Longrightarrow \mathbf{n}(\star\eta) = \star\mathbf{t}(\eta),$$

where we have used that $\star\star = \pm 1$, and the proof is done. □

We denote by $\Omega^k_D(M)$ the space of all k-forms that satisfy the *Dirichlet boundary condition*, that is

$$\Omega^k_D(M) := \{\eta \in \Omega^k(M) \mid \mathbf{t}\eta = 0\}. \tag{5.3}$$

Similarly, we denote by $\Omega^k_N(M)$ the space of all k-forms that satisfy the *Neumann boundary condition*, that is

$$\Omega^k_N(M) := \{\eta \in \Omega^k(M) \mid \mathbf{n}\eta = 0\}. \tag{5.4}$$

The space $\Omega^k_N(M)$ can be characterized in an alternative way, as the following proposition shows.

Proposition 5.3. *Let (M,g) be a Riemannian manifold with boundary. Then $\Omega^k_N(M)$ is given by*

$$\Omega^k_N(M) = \{\eta \in \Omega^k \mid \mathbf{t}(\star\eta) = 0\}.$$

Proof. Let (M,g) be a Riemannian manifold with boundary. From (5.4) we know that a k-form η is in $\Omega^k_N(M)$ if and only if $\mathbf{n}\eta = 0$, which is the case if and only if $\star(\mathbf{n}\eta) = 0$. But from Proposition 5.2 we know that $\star(\mathbf{n}\eta) = \mathbf{t}(\star\eta)$. Thus, η is in $\Omega^k_N(M)$ if and only if $\mathbf{t}(\star\eta) = 0$. □

Boundary conditions are applied to $E^k(M)$ and $cE^k(M)$ as follows:

$$E^k_D(M) := d(\Omega^{k-1}_D(M)) = \{\eta \in \Omega^k(M) \mid \eta = d\zeta, \zeta \in \Omega^{k-1}_D(M)\}$$
$$E^k_N(M) := d(\Omega^{k-1}_N(M)) = \{\eta \in \Omega^k(M) \mid \eta = d\zeta, \zeta \in \Omega^{k-1}_N(M)\}$$
$$cE^k_N(M) := d^*(\Omega^{k+1}_N(M)) = \{\eta \in \Omega^k(M) \mid \eta = d^*\zeta, \zeta \in \Omega^{k+1}_N(M)\}$$
$$cE^k_D(M) := d^*(\Omega^{k+1}_D(M)) = \{\eta \in \Omega^k(M) \mid \eta = d^*\zeta, \zeta \in \Omega^{k+1}_D(M)\}.$$

For the other subspaces the situation looks as follows:

$$C^k_D(M) = \Omega^k_D(M) \cap C^k(M) \qquad C^k_N(M) = \Omega^k_N(M) \cap C^k(M)$$
$$cC^k_D(M) = \Omega^k_D(M) \cap cC^k(M) \qquad cC^k_N(M) = \Omega^k_N(M) \cap cC^k(M)$$
$$CcC^k_D(M) = \Omega^k_D(M) \cap CcC^k(M) \quad CcC^k_N(M) = \Omega^k_N(M) \cap CcC^k(M).$$

Proposition 5.4. *Let (M,g) be a Riemannian manifold with boundary.*

1. *The differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ preserves Dirichlet boundary conditions, whereas the codifferential $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ preserves Neumann boundary conditions.*

2. *The Hodge star operator $\star$ provides the following vector space isomorphisms:*

$$C^k(M) \simeq cC^{n-k}(M)$$
$$E^k(M) \simeq cE^{n-k}(M) \qquad E^k_D(M) \simeq E^{n-k}_N(M)$$
$$CcC^k(M) \simeq CcC^{n-k}(M) \qquad CcC^k_D(M) \simeq CcC^{n-k}_N(M).$$

Proof. Let (M,g) be a Riemannian manifold with boundary.

1. Let $\eta \in \Omega^k_D(M)$. Then $\mathbf{t}\eta = 0$, and by Proposition 5.1 we also have $\imath^*\eta = 0$. Since d commutes with pullbacks we have $d(\imath^*\eta) = \imath^*(d\eta) = 0$, and so $\mathbf{t}(d\eta) = 0$. That is, $d\eta$ lies in $\Omega^{k+1}_D(M)$.

   Next, let $\eta \in \Omega^k_N(M)$. Then $\mathbf{t}(\star\eta) = 0$, and again, by Proposition 5.1, we have $\imath^* \star \eta = 0$. Using that $\star\star = \pm 1$, we compute

   $$\imath^*(\star d^*\eta) = \pm\imath^*(\star \star d \star \eta) = \pm\imath^*(d \star \eta) = 0,$$

   It then follows from Proposition 5.2 that $\mathbf{t}(\star d^*\eta) = 0$, which, using Proposition 5.4, proves that $d^*\eta \in \Omega^{k-1}_N(M)$.

2. Let $\eta \in \Omega^k(M)$. From the definition of the codifferential it is clear that $d\eta = 0 \Leftrightarrow d^*(\star\eta) = 0$ and that $d^*\eta = 0 \Leftrightarrow d(\star\eta) = 0$. This implies that $\star C^k(M) = cC^{n-k}(M)$ and that $\star CcC^k(M) \simeq CcC^{n-k}(M)$. Taking into account part 1 of this proposition, it is also clear that $\star E^k(M) = cE^{n-k}(M)$. Lastly, using Proposition 5.2 we get

$$\begin{aligned}\star\Omega^k_D(M) &= \{\star\eta \mid \eta \in \Omega^k(M), \mathbf{t}\eta = 0\} \\ &= \{\eta \mid \star\eta \in \Omega^{n-k}(M), \mathbf{t}(\star\eta) = 0\} \\ &= \{\eta \in \Omega^{n-k}(M) \mid \mathbf{n}\eta = 0\} \\ &= \Omega^{n-k}_N(M).\end{aligned}$$

Finally, since $\star\star = \pm 1$, we can apply $\star$ again to get isomorphisms in the reverse directions.

□

5.2 The Hodge decomposition theorem

In this section, the reference to M is understood, and will therefore be omitted. We thus write Ω^k instead of $\Omega^k(M)$ etc. Also, throughout the remainder of the text, we will write $+$ for direct sums, and reserve the use of $\oplus$ for orthogonal direct sums.

Theorem 5.5 (Hodge decomposition theorem)**.** *Let (M, g) be a compact, oriented, smooth Riemannian n-manifold with boundary. The space Ω^k of smooth differential k-forms on M decomposes as the orthogonal (with respect to the L^2-inner product on k-forms) direct sum*

$$\Omega^k = cE^k_N \oplus CcC^k \oplus E^k_D. \tag{5.5}$$

Moreover, CcC^k decomposes as

$$CcC^k = CcC^k_N \oplus EcC^k = CcE^k \oplus CcC^k_D. \tag{5.6}$$

The Hodge decomposition theorem can be split into two statements, one concerning span and the other concerning orthogonality. What the *span statement* tells us is that the various subspaces that appear as summands in the theorem actually span the space of differential k-forms, i.e. that

$$\Omega^k = \mathrm{span}(cE^k_N \cup CcC^k \cup E^k_D) \tag{5.7}$$

and

$$CcC^k = \mathrm{span}(CcC^k_N \cup EcC^k) = \mathrm{span}(CcE^k \cup CcC^k_D). \tag{5.8}$$

The *orthogonality statement* then tells us that these subspaces are orthogonal. That is, that the spaces cE^k_N, CcC^k and E^k_D are mutually orthogonal, as are CcC^k_N and EcC^k, and also CcE^k and CcC^k_D. Whereas the proof of the span statement is somewhat involved and requires analysis, the proof of the orthogonality statement is rather straightforward. We prove the orthogonality statement here. A proof of the span statement, which relies on results about the solvability of boundary value problems for certain partial differential equations, will be given only for Theorem 5.12, which is a special case of the Hodge decomposition theorem dealing with the decomposition of vector fields on compact domains in $\mathbb{R}^3$. This proof is the subject of Chapter 7.

Remark 5.6. For a closed manifold, the boundary conditions in Theorem 5.5 are vacuous, and the decomposition (5.5) becomes

$$\Omega^k = cE^k \oplus CcC^k \oplus E^k.$$

Also, $EcC^k = 0$ and $CcE^k = 0$.

Proof of orthogonality statement. We need to show that the summands in (5.5) and (5.6) are orthogonal.

We begin by showing that E^k_D is orthogonal to cC^k. Take $\eta = d\zeta \in E^k_D$ and $\theta \in cC^k$. Then, using Stokes' theorem, we compute

$$\begin{aligned}\langle d\zeta, \theta\rangle &= \int_M d\zeta \wedge \star\theta \\ &= \int_M d(\zeta \wedge \star\theta) \pm \int_M \zeta \wedge (d \star \theta) \\ &= \int_{\partial M} \zeta \wedge \star\theta = 0,\end{aligned}$$

where the last equality holds because ζ lives in Ω^{k-1}_D, and hence the restriction of ζ to ∂M is zero.

To show that cE^k_N is orthogonal to C^k, take $\eta = d^*\zeta \in cE^k_N$ and $\theta \in C^k$. Again, using Stokes' theorem, we see that

$$\begin{aligned}\langle d^*\zeta, \theta\rangle &= \int_M \theta \wedge \star(d^*\zeta) \\ &= \pm \int_M \theta \wedge d(\star\zeta) \\ &= \pm \int_M d(\theta \wedge \star\zeta) = \pm \int_{\partial M} \theta \wedge \star\zeta = 0,\end{aligned}$$

where the last equality holds because ζ lives in Ω^{k+1}_N and $\star$ takes Ω^k_N to Ω^{n-k}_D, and thus the restriction of $\star\zeta$ to ∂M is zero.

Since $E^k_D \subset C^k$, it now follows that E^k_D is orthogonal to cE^k_N.

Next, we show that EcC^k is orthogonal to CcC^k_N. Let $\eta = d\zeta \in EcC^k$ and $\theta \in CcC^k_N$. Since $d^*\theta = 0$, it must be the case that $d \star \theta = 0$. Moreover, since $\star\theta$ lives in Ω^k_D, the restriction of $\star\theta$ to ∂M is zero. Using these facts, we compute

$$\begin{aligned}\langle d\zeta, \theta\rangle &= \int_M d\zeta \wedge \star\theta \\ &= \int_M d(\zeta \wedge \star\theta) \pm \int_M \beta \wedge (d \star \theta) \\ &= \int_{\partial M} \zeta \wedge \star\theta = 0,\end{aligned}$$

i.e. η and γ are orthogonal.

Lastly, to see that CcE^k is orthogonal to CcC^k_D, we observe that $\star$ maps CcE^k to EcC^{n-k} and CcC^k_D to CcC^{n-k}_N, and that $\langle \eta, \zeta\rangle_g = \langle \star\eta, \star\zeta\rangle$ whenever $\eta, \zeta \in \Omega^k$ (see propositions 2.23 and 5.4). □

5.3 Hodge isomorphism theorem

Closely related to the Hodge decomposition theorem is the Hodge isomorphism theorem (also known as Hodge's theorem). We discuss it briefly here.

Let (M, g) be a compact, oriented Riemannian manifold without boundary, and consider a k-form η on M. By the Hodge decomposition theorem we know that there exists an orthogonal decomposition of η with respect to the L^2-inner product on differential forms as

$$\eta = d\alpha + d^*\beta + \gamma,$$

where $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k+1}(M)$ and $\gamma \in CcC^k(M)$. This decomposition is unique in the sense that $d\alpha$, $d^*\beta$ and γ are uniquely determined by η, although α and β are not.

A k-form ζ represents a class in $H^k_{\text{dR}}(M)$ if it is closed. Going back to η, we see that the closed forms in its decomposition are $d\alpha$ and γ, whereas $d^*\beta$ is not closed, and hence not exact (unless it is the zero form). From this we conclude that η represents a class in $H^k_{\text{dR}}(M)$ if and only if $d^*\beta = 0$, in which case we get

$$\eta = d\alpha + \gamma.$$

Moreover, it follows from the uniqueness of the Hodge decomposition that γ is the only harmonic representative of its cohomology class. This result, which we recovered with ease from the Hodge decomposition theorem, is known as the Hodge isomorphism theorem.

Theorem 5.7 (Hodge isomorphism theorem)**.** *Let M be a compact, oriented Riemannian manifold without boundary. Then every cohomology class in $H^k_{\text{dR}}(M)$ has a unique harmonic representative.*

It is well known that the kernel of an elliptic operator is finite dimensional. Since on a Riemannian manifold the Laplace-de Rham operator Δ is an elliptic operator, we get the following corollary to Theorem 5.7.

Corollary 5.8. *Let M be as in Theorem 5.7. Then the cohomology of M is finite dimensional.*

Let γ be the harmonic. Then every representative of the cohomology class of γ is of the form $\gamma + d\alpha$. The norm of a such element is

$$||\gamma + d\alpha||^2 = ||\gamma||^2 + 2\langle \gamma, d\alpha \rangle + ||d\alpha||^2 = ||\gamma||^2 + ||d\alpha||^2,$$

whence it follows that the element that minimizes the norm in a given class is its harmonic representative. We state this result as a corollary.

Corollary 5.9. *Let M be as in Theorem 5.7. Then the unique element that minimizes the norm in a given cohomology class in $H^k_{\mathrm{dR}}(M)$ is its harmonic representative.*

5.4 Hodge decomposition in three-space

Throughout this section, unless otherwise stated, let D be a compact regular domain in $\mathbb{R}^3$. As before, $\Gamma(TD)$ denotes the vector space of smooth vector fields on D. We define an L^2-inner product, denoted $\langle \cdot, \cdot \rangle$, on $\Gamma(TD)$ by

$$\begin{aligned} \langle \cdot, \cdot \rangle : \Gamma(TD) \times \Gamma(TD) &\longrightarrow \mathbb{R} \\ (V, W) &\longmapsto \int_D V \cdot W \, dV. \end{aligned} \tag{5.9}$$

A fact that will be useful is that this metric is related to the L^2-metric on differential 1-forms via the musical isomorphisms $\flat$ and $\sharp$. The following proposition makes this statement precise.

Proposition 5.10. *Let $\eta, \zeta \in \Omega^1(D)$ and $V, W \in \Gamma(T\mathbb{R}^n)$. Then*

$$\langle \eta, \zeta \rangle_g = \langle \eta^\sharp, \zeta^\sharp \rangle \quad \textit{and} \quad \langle V, W \rangle = \langle V^\flat, W^\flat \rangle_g.$$

Proof. Let D be a compact regular domain in $\mathbb{R}^n$ with smooth boundary and let $\eta, \zeta \in \Omega^1(D)$. Since by Proposition 3.6 we have $*(\eta^\sharp \wedge *\zeta^\sharp) = \eta^\sharp \cdot \zeta^\sharp$, we get

$$\begin{aligned} \langle \eta, \zeta \rangle_g &= \int_D \eta \wedge *\zeta \\ &= \int_D \star(\eta \wedge *\zeta) \, \omega_g \\ &= \int_D \eta^\sharp \cdot \zeta^\sharp \, dV = \langle \eta^\sharp, \zeta^\sharp \rangle, \end{aligned}$$

which shows that $\sharp$ preserves the L^2-inner product. It follows that $\flat$ preserves the inner product too, since it is the inverse isomorphism to $\sharp$. □

Remark 5.11. Note that Proposition 5.10 implies that orthogonality is preserved by $\flat$ and $\sharp$.

Using the L^2-inner product on vector fields on D, we obtain a special case of the Hodge decomposition theorem that deals with vector fields on D. Before we state the theorem, we need to introduce a five subspaces of $\Gamma(TD)$. The names of these subspaces may appear cryptic at this point, but later on we will come up with alternative characterizations that explain our notation in Chapter 7. The notation is originally found in [4]. The subspaces are

$$\begin{aligned}
FK(D) &= (\ker(\nabla\times))^{\perp} \\
HK(D) &= (\,\mathrm{im}\,(\nabla))^{\perp} \cap (\ker(\nabla\times)) \\
CG(D) &= (\,\mathrm{im}\,(\nabla)) \cap (\,\mathrm{im}\,(\nabla\times)) \\
HG(D) &= (\,\mathrm{im}\,(\nabla\times))^{\perp} \cap (\ker(\nabla\cdot)) \\
GG(D) &= (\ker(\nabla\cdot))^{\perp},
\end{aligned}$$

where ∇ is understood to be the map $\nabla : C^{\infty}(D) \to \Gamma(TD)$, and $\nabla\times$ and $\nabla\cdot$ are understood to be the maps $\nabla\times : \Gamma(TD) \to \Gamma(TD)$ and $\nabla\cdot : \Gamma(TD) \to \mathbb{R}$.

We now turn our attention to the Hodge decomposition of $\Gamma(TD)$. In what follows, unless otherwise stated, the reference to the manifold D will be understood and omitted. For example, we write FK instead of $FK(D)$ when the reference to D is understood.

Theorem 5.12. *Let D be a compact regular domain in $\mathbb{R}^3$. The space $\Gamma(TD)$ of smooth vector fields on D, endowed with the L^2-inner product $\langle\cdot,\cdot\rangle$, can be expressed as the orthogonal direct sum*

$$\Gamma(TD) = FK \oplus HK \oplus CG \oplus HG \oplus GG, \tag{5.10}$$

where

$$\begin{aligned}
\mathrm{im}\,(\nabla) &= CG \oplus HG \oplus GG \\
\mathrm{im}\,(\nabla\times) &= FK \oplus HK \oplus CG \\
\ker(\nabla\times) &= HK \oplus CG \oplus HG \oplus GG \\
\ker(\nabla\cdot) &= FK \oplus HK \oplus CG \oplus HG,
\end{aligned}$$

The following lemma will turn out to be useful when proving Theorem 5.12. Note that this lemma holds for more general manifolds than those to which Theorem 5.12 is applicable.

Lemma 5.13. *Let (M, g) be a compact, oriented, smooth Riemannian n-manifold with boundary. Then $E^k(M) = EcC^k(M) \oplus E^k_D(M)$ and $cE^k(M) = CcE^k(M) \oplus cE^k_N(M)$.*

Proof. First, we show that $E^k(M) = EcC^k(M) \oplus E^k_D(M)$.

Let $\eta \in E^k(M)$. Since η is exact we have $\eta = d\alpha$ for $\alpha \in \Omega^{k-1}(M)$. By the Hodge decomposition theorem we have

$$\Omega^k(M) = cE^k_N(M) \oplus CcC^k_N(M) \oplus EcC^k(M) \oplus E^k_C(M). \tag{5.11}$$

It is clear that $EcC^k(M) \oplus E^k_D(M) \subset E^k(M)$. Thus, in order to prove that $E^k(M) = EcC^k(M) \oplus E^k_D(M)$, it suffices to show that $E^k(M)$ is orthogonal to $cE^k_N(M)$ and $CcC^k_N(M)$, which implies, as is immediate from (5.11), that $E^k(M) \subset EcC^k(M) \oplus E^k_D(M)$.

To this end, suppose first that $\zeta \in cE^k_N(M)$. Using Stoke's theorem and that $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi$ for $\varphi \in \Omega^k(M)$ and $\psi \in \Omega^*(M)$, we compute

$$\begin{aligned}\langle \eta, \zeta \rangle = \int_M d\alpha \wedge \star\zeta &= \int_M d(\alpha \wedge \star\zeta) - (-1)^k \int_M \alpha \wedge \star d^*\zeta \\ &= \int_{\partial M} \alpha \wedge \star\zeta - (-1)^k \int_M \alpha \wedge \star d^*\zeta. \end{aligned} \tag{5.12}$$

Note that since $\zeta \in \Omega^k_N(M)$ we have $\star\zeta \in \Omega^{n-k}_D(M)$ and hence $\star\zeta = 0$ on ∂M. We also have $d^*\zeta = d^*d^*\beta = 0$. Thus both integrals in (5.12) are zero, i.e. $\langle \eta, \zeta \rangle = 0$.

Suppose now that $\zeta \in CcC^k_N(M)$. Again we have $\star\zeta \in \Omega^{n-k}_D(M)$ and again, since ζ is coclosed, $d^*\zeta = 0$, so that the integrals in (5.12) are zero and $\langle \eta, \zeta \rangle = 0$. We thus conclude that $E^k(M) = EcC^k(M) \oplus E^k_D(M)$.

Next, we show that $cE^k(M) = CcE^k(M) \oplus cE^k_N(M)$. Using the result just obtained together with the fact that the Hodge star preserves inner products (Proposition 2.23) as well as provides the isomorphisms $\star cE^k(M) = E^k(M)$, $\star EcC^k(M) = CcE^k(M)$ and $\star E^k_D(M) = cE^k_N(M)$, we see that $cE^k(M) = CcE^k(M) \oplus cE^k_N(M)$ emerges as the dual result to $E^k(M) = EcC^k(M) \oplus E^k_D(M)$ under $\star$. □

We need another lemma in the proof of Theorem 5.12, which is an easy Corollary to the following proposition.

Proposition 5.14. *Let D be a compact regular domain in $\mathbb{R}^n$. Then $C^k_D \subset E^k$.*

Proof. The boundary ∂D has a *bi-collar* (meaning that it has a neighbourhood in $\mathbb{R}^n$ that looks like $\partial D \times (-\epsilon, \epsilon)$, with ∂D sitting inside as $\partial D \times 0$, with *negative part* $U_- = \partial D \times (-\epsilon, 0)$ in D and *positive part* $U_+ = \partial M \times (0, \epsilon)$ in $\mathbb{R}^n \backslash D$. This lets us cover $\mathbb{R}^n$ by two open sets $A = D \cap U$ and $B = (\mathbb{R}^n \backslash D) \cap U$, with intersection $A \cup B = U$.

Applying the Meyer-Vietoris sequence for de Rham cohomology yields

$$\begin{aligned} H^k_{\mathrm{dR}}(R^n) = 0 \longrightarrow H^k_{\mathrm{dR}}(A) = H^k_{\mathrm{dR}}(D) \oplus H^k(B) = H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus D) \\ \longrightarrow H^k_{\mathrm{dR}}(U) = H^k_{\mathrm{dR}}(\partial D) \longrightarrow H^{k+1}_{\mathrm{dR}}(\mathbb{R}^n) = 0. \end{aligned}$$

When $k \neq 0$, we get the middle map

$$H^k_{\mathrm{dR}}(D) \oplus H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus D) \to H^k_{\mathrm{dR}}(\partial D)$$

which is $(\imath^* - \jmath^*)$ (that is, induced by inclusions), and which is an isomorphism. In particuar, $\imath^*$ is injective. When $k = 0$, $\imath^*$ sends locally constant functions on D to their restriction to ∂D. Also in this case $\imath^*$ is injective. In more human terms, if a form representing class $a \in H^k_{\mathrm{dR}}(D)$ becomes exact after restricting to the boundary, then it is already exact on D.

Certainly, a form which is closed and satisfies Dirichlet boundary condition becomes exact (in fact zero!) when restricted to the boundary. So it must be exact on D. □

Lemma 5.15. *Let D be a compact regular domain in $\mathbb{R}^n$. Then CcC^k_N and CcC^k_D are orthogonal.*

Proof. By Proposition 5.14, $CcC^k_D \subset C^k_D$ is exact. Applying Theorem 5.5 then yields the desired result. □

Remark 5.16. Theorem 5.5 does not state that CcC^k_N and CcC^k_D are orthogonal for the simple reason that, in general, they are not. The problem of determining for which manifolds the are orthogonal is discussed by Shonkwiler in [13]. The particular case for domains in $\mathbb{R}^3$ is then dealt with by Poelke, using Shonkwiler's results, in [10].

We are now in a position to derive the Hodge decomposition theorem for vector fields on compact regular domains in $\mathbb{R}^3$, i.e. 5.12, from the general Hodge decomposition theorem, i.e. 5.5.

Proof of Theorem 5.12. In this proof, we will only consider 1-forms on D, and will therefore omit superscripts in order to improve legibility. For example, we write E instead of E^1 when referring to exact 1-forms on D.

First, suppose that η is closed 1-form in cE_N. Then η is also coclosed (since $d^*d^* = 0$) and hence $\eta \in CcC$, which contradicts (5.5) unless $\eta = 0$. Hence the only closed 1-form in cE_N is the zero form, and since all forms in CcC and E_D are closed, it follows from Theorem 5.5 that $C = CcC \oplus E_D$ and that

$$cE_N = C^\perp.$$

For similar reasons we have $cC = CcC \oplus cE_N$, and hence

$$E_D = cC^\perp.$$

It follows that

$$C = CcC \oplus E_D = CcC_N \oplus EcC \oplus E_D = CcC_N \oplus E,$$

where the last equality comes from Lemma 5.13, and from which it follows that CcC_N is the orthogonal complement of E in C, that is

$$CcC_N = E^{\perp} \cap C$$

Similarly, again using Lemma 5.13, we have

$$cC = CcC \oplus cE_N = CcC_D \oplus CcE \oplus cE_N = CcC_D \oplus cE,$$

thus CcC_D is the orthogonal complement of cE in cC, that is

$$CcC_D = cE^{\perp} \cap cC$$

Lastly, since EcE is a subset of CcC that is orthogonal to both CcC_N and CcC_D, and since by Lemma 5.15 CcC_N is orthogonal to CcC_D, we have the decomposition

$$CcC = CcC_N \oplus EcE \oplus CcC_D,$$

which lets us express Ω as the direct sum of five mutually orthogonal subspaces as follows:

$$\begin{aligned}\Omega &= cE_N \oplus CcC \oplus E_D \\ &= cE_N \oplus CcC_N \oplus EcE \oplus CcC_D \oplus E_D \\ &= C^{\perp} \oplus (E^{\perp} \cap C) \oplus EcE \oplus (cE^{\perp} \cap cC) \oplus cC^{\perp}.\end{aligned}$$

This concludes the first part of the proof.

For the second part of the proof, we need to identify the five subspaces in the above decomposition with subspaces of $\Gamma(TY)$ via a suitable isomorphism. Recall diagram (3.9). From it we deduce that the two diagrams

$$\begin{array}{ccccc} C^{\infty}(\mathbb{R}^3) & \xrightarrow{\nabla} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^3) \\ {\scriptstyle id}\downarrow & & {\scriptstyle \flat}\downarrow & & {\scriptstyle \beta}\downarrow \\ \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) \end{array} \tag{5.13}$$

and

$$\begin{array}{ccccc} \Gamma(T\mathbb{R}^3) & \xrightarrow{\nabla\times} & \Gamma(T\mathbb{R}^3) & \xrightarrow{-\nabla\cdot} & C^{\infty}(\mathbb{R}^3) \\ {\scriptstyle \beta}\downarrow & & {\scriptstyle \flat}\downarrow & & {\scriptstyle id}\downarrow \\ \Omega^2(\mathbb{R}^3) & \xrightarrow{d^*} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d^*} & \Omega^0(\mathbb{R}^3) \end{array} \tag{5.14}$$

are commutative. Since D is a regular domain in $\mathbb{R}^3$, these diagrams work equally well with D substituted for $\mathbb{R}^3$. This suggest that $\flat^{-1} = \sharp$ is our isomorphism of choice. Indeed, it follows from the commutativity of the diagrams that $\sharp$ provides the following isomorphisms

$$E \xrightarrow{\sim} \text{im}\,(\nabla)$$
$$cE \xrightarrow{\sim} \text{im}\,(\nabla\times)$$
$$C \xrightarrow{\sim} \ker(\nabla\times)$$
$$cC \xrightarrow{\sim} \ker(\nabla\cdot),$$

that is to say $E^\sharp = \{\eta^\sharp \mid \eta \in \Omega\} = \text{im}\,(\nabla)$ and so forth.

Now, recall that by Proposition 5.10), orthogonality is preserved by $\sharp$. From the above discussion we thus conclude that summands in (5.10) are given by

$$\begin{aligned}(cE_N)^\sharp &= (C^\perp)^\sharp = (\ker(\nabla\times))^\perp = FK \\ (CcC_N)^\sharp &= (E^\perp \cap C)^\sharp = (E^\perp)^\sharp \cap C^\sharp = (\text{im}\,(\nabla))^\perp \cap (\ker(\nabla\times)) = HK \\ (EcE)^\sharp &= (\text{im}\,(\nabla)) \cap (\text{im}\,(\nabla\times)) = CG \\ (CcC_D)^\sharp &= (cE^\perp \cap cC)^\sharp = (cE^\perp)^\sharp \cap cC^\sharp = (\text{im}\,(\nabla\times))^\perp \cap (\ker(\nabla\cdot)) = HG \\ (E_D)^\sharp &= (cC^\perp)^\sharp = (\ker(\nabla\cdot))^\perp = GG.\end{aligned} \tag{5.15}$$

Finally, since $\sharp$ preserves orthogonality, these subspaces are orthogonal. □

Remark 5.17. The above proof could just as well have been carried out in the setting of 2-forms on D instead of 1-forms. Diagram (3.9) shows that β^{-1} is a suitable substitute for $\sharp$ in this case. The fact that $\beta^{-1} = \sharp\star$ on D together with the way $\star$ maps the various subspaces of Ω^1 in the Hodge decomposition isomorphically to those of Ω^2 suggests that a proof using 2-forms could be carried out more or less the same way as the proof we have presented here.

Remark 5.18. To summarize the contents of this chapter in terms of differential forms, we have the following diagram for compact regular domains in Euclidean n-space:

$$\begin{array}{rcccccccccc} \Omega^k &=& cE^k_N &\oplus& CcC^k_N &\oplus& EcE &\oplus& CcC^k_D &\oplus& E^k_D \\ \ker(d) &=& && CcC^k_N &\oplus& EcE &\oplus& CcC^k_D &\oplus& E^k_D \\ \text{im}\,(d) &=& && &\oplus& EcE &\oplus& CcC^k_D &\oplus& E^k_D \\ \text{im}\,(d^*) &=& cE^k_N &\oplus& CcC^k_N &\oplus& EcE &\oplus& && \\ \ker(d^*) &=& cE^k_N &\oplus& CcC^k_N &\oplus& EcE &\oplus& CcC^k_D && \end{array}$$

Since CcC^k_N and CcC^k_D are not orthogonal in general on Riemannian manifolds with boundary, we must rely on Proposition 5.14 to obtain this five-term version of the Hodge decomposition.

6 The Biot-Savart formula and boundary value problems

In order to prove the spanning statement of Theorem 5.12 in Capter 7, we need some results concerning the solutions to Dirichlet and Neumann boundary value problems for Poisson's equation on compact regular domains in $\mathbb{R}^3$, as well as some basic facts about the Biot-Savart formula, which is an equation that lets us compute the electric field generated by an electric current.

6.1 Dirichlet and Neumann problems

We need the following theorem, which we state without proof, about solutions to boundary value problems known as Dirichlet and Neumann problems for Poisson's equation.

Theorem 6.1. *Let D be a compact regular domain in $\mathbb{R}^3$, let $\{D_i\}_{i=1}^k$ denote the connected components of D, and let N be the outward-pointing unit normal vector field on ∂D. Then, for each $f \in C^\infty(D)$ and $u \in C^\infty(\partial D)$, the following two statements are true.*

1. *The boundary value problem*

$$\begin{cases} \nabla^2\varphi = f & \text{in } D \\ \varphi = u & \text{on } \partial D, \end{cases} \tag{6.1}$$

known as the Dirichlet problem, has a unique solution φ. Moreover, φ is smooth on D.

2. *The boundary value problem*

$$\begin{cases} \nabla^2\varphi = f & \text{in } D \\ \dfrac{\partial\varphi}{\partial N} = u & \text{on } \partial D, \end{cases} \tag{6.2}$$

known as the Neumann problem, has a solution φ if and only if the integrability condition

$$\int_{D_i} f\,dV = \int_{\partial D_i} u\,dS$$

holds for each $i \in \{1, 2, \ldots, k\}$. Solutions are unique modulo a function that is constant on each D_i, i.e. if φ_1 and φ_2 both solve (6.2) then $\psi = \varphi_1 - \varphi_2$ is constant on each connected component of D. Moreover, if φ is a solution to (6.2), then φ is smooth on D.

6.2 The Biot-Savart formula

In this section we state some of the properties of the Biot-Savart formula that we will use in Chapter 7. For more information on the Biot-Savart formula, the article [3] by Cantarella et al. is useful. A short background and history is provided by Parsley in [9].

Let D be a compact regular domain in $\mathbb{R}^3$ and let V be a vector field on D. Consider the map BS given by

$$BS(V)(x) := \int_D V(y) \times \nabla\Phi(x,y)\, dV(y), \tag{6.3}$$

where $\Phi(x,y) := -\dfrac{1}{4\pi|x-y|}$ is the fundamental solution to Laplace's equation in $\mathbb{R}^3$. We call (6.3) the Biot-Savart formula. Note that since

$$\nabla\Phi(x,y) = \nabla\left(-\frac{1}{4\pi|x-y|}\right) = \frac{x-y}{4\pi|x-y|^3},$$

we can expand the right-hand side of (6.3) to get

$$BS(V)(x) = \frac{1}{4\pi}\int_D V(y) \times \frac{x-y}{|x-y|^3}\, dV(y).$$

Proposition 6.2. *Let D be a compact regular domain in $\mathbb{R}^3$, let N be the outward-pointing unit normal vector field along ∂D, and let V be a smooth vector field on D. Then the following properties are true of BS:*

1. *$BS(V)$ is smooth in D and in $\mathbb{R}^3 \setminus D$.*
2. *$\nabla \cdot BS(V) = 0$ in D and in $\mathbb{R}^3 \setminus D$.*
3. *The curl of $BS(V)$ is given by*

$$\nabla_x \times BS(V)(x) = V'(x) + \frac{1}{4\pi}\nabla_x \int_D \frac{\nabla_y \cdot V(y)}{|x-y|^3}\, dV(y) - \frac{1}{4\pi}\int_{\partial D} \frac{V(y)\cdot N}{|x-y|^3}\, dS(y),$$

where $V'(x) = \begin{cases} V(x) & \text{for } x \in D \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus D \end{cases}$.

Remark 6.3. The above proposition makes clear that if we the restrict the vector field $BS(V)$ to D we get an operator

$$BS : \Gamma(TD) \to \Gamma(TD).$$

This operator is called the Biot-Savart operator.

Consider now an oriented loop C in $\mathbb{R}^3 \setminus D$. Let $I \in \mathbb{R}$ and, for $x \in C$, let $I(x)$ denote the vector field with constant length I along C that is tangent to and pointing in the direction of C at each point. We want to modify (6.3) so that it works in this new setting. The idea is to thicken C into a thin tube of radius r that $I(x)$ can be extended to. We then apply (6.3) and take the limit as r goes to zero. Thus we get

$$BS(I)(x) := \frac{1}{4\pi} \int_C I(y) \times \frac{x-y}{|x-y|^3}\, dy. \tag{6.4}$$

This expression naturally shares some of the properties of (6.3).

Proposition 6.4. *Let D be a compact regular domain in $\mathbb{R}^3$, let C be an oriented smooth loop in $\mathbb{R}^3 \setminus D$ and let $I \in \mathbb{R}$ with $I(x)$ the corresponding vector field along C. Then the following properties are true of BS:*

1. *$BS(I)$ is smooth in D.*
2. *$\nabla \cdot BS(I) = 0$ in D.*
3. *$\nabla \times BS(I) = 0$ in D.*
4. *If D is a loop in D, then*

$$\int_C BS(I) \cdot ds = I \cdot \mathit{link}(C, D),$$

where $\mathit{link}(C, D)$ denotes the linking number of C and D.

7 Proof of spanning statement for domains in $\mathbb{R}^3$

7.1 Introduction

As we have seen, the Hodge decomposition theorem for vector fields on domains in 3-space (Theorem 5.12) follows from the Hodge decomposition theorem for differential forms on compact Riemannian manifolds (Theorem 5.5). Recall that we only proved the *orthogonality statement* of Theorem 5.5. That is, we proved that the sums appearing in the theorem are L^2-orthogonal. A direct consequence of this is that the sums are direct. We did not prove the *spanning statement*, i.e. we did not prove that equations (5.7) and (5.8) hold true.

In this chapter we proceed to prove, step by step, that for a compact regular domain D in $\mathbb{R}^3$, we have

$$\Gamma(TD) = \text{span}(FK \cup HK \cup CG \cup HG \cup GG),$$

which we refer to as the *spanning statement* of Theorem 5.12. Together with the proof of the orthogonality statement of Theorem 5.5, this provides a proof of Theorem 5.12. The proof used here is, except some minor modifications, the same as the by Cantarella et al. in [4].

We remind the reader that we write $+$ for a direct sum and reserve the use of $\oplus$ for orthogonal direct sums.

7.2 Notation and definitions

We are concerned with compact regular domains in $\mathbb{R}^3$. Throughout the rest of this chapter D will denote such a domain, unless otherwise stated. That is, $D \subset \mathbb{R}^3$ will denote a properly embedded submanifold of codimension 0 with boundary. We let N denote the outward-pointing unit normal vector field along ∂D, and adopt the convention that any expression in which N appears is understood to apply only to ∂D.

In order to to establish the spanning statement of Theorem 5.12, we will proceed by writing $\Gamma(TD)$ as the span of smaller and smaller subspaces, eventually ending up with five such subspaces. This is done in four *splitting statements*, numbered 1 through 4. We also need to show that these five subspaces are precisely those that appear in Theorem 5.12. This is done in five *inclusion statements*, numbered 1 through 5.

We begin with Splitting statement 1, which states that $\Gamma(TD) = \text{span}(K(D) \cup G(D))$, where $K(D) =$ *knots* and $G(D) =$ *gradients* are defined as follows:

$$K(D) := \{V \in \Gamma(TD) \mid \nabla \cdot V = 0,\ V \cdot N = 0\}$$
$$G(D) := \{V \in \Gamma(TD) \mid \exists \varphi \in C^\infty(D) \text{ s.t. } V = \nabla\varphi\}.$$

Splitting statement 2 then states that $K(D) = \text{span}(FK'(D) \cup HK'(D))$, where $FK'(D) =$ *fluxless knots* and $HK'(D) =$ *harmonic knots* are given by

$$FK'(D) := \{V \in K(D) \mid \text{all interior fluxes are zero}\}$$
$$HK'(D) := \{V \in K(D) \mid \nabla \times V = 0\},$$

where the *interior flux* of a vector field $V \in \Gamma(TD)$ through a smooth orientable surface $\Sigma \subset M$ with a given orientation and with $\partial\Sigma \subset \partial M$, is the real number Φ given by

$$\Phi := \int_\Sigma V \cdot N\, dS.$$

That all interior fluxes of V are zero simply means that $\Phi = 0$ for any such surface. In the case that V is divergence free and tangent to ∂M (which, in particular, is the case for $V \in K(D) \supset FK'(D)$) the value of Φ depends only on the homology class of Σ in $H_2(D, \partial D)$. Hence, if $\{\Sigma_i\}_{i=1}^k$ is a set of smooth orientable surfaces that form a basis for $H_2(D, \partial D)$, then a vector field $V \in K(D)$ is also in $FK'(D)$ if and only if the interior flux through each Σ_i is zero.

Recall that for a vector field V, exactness corresponds, via the isomorphism $\flat$, to $V = \nabla\varphi$, coexactness to $V = \nabla \times U$, closedness to $\nabla \times V = 0$ and coclosedness to $\nabla \cdot V = 0$. This follows from the commutative diagram (3.9) and we extend this terminology to include vector fields in addition to differential forms (that is, we say $V \in \Gamma(TD)$ is closed whenever $V \in \text{im}\,(\nabla)$ etc.). Boundary conditions are translated similarly, i.e. via the musical isomorphisms, from the definitions in Chapter 5. This leads us to the first inclusion statement.

Inclusion statement 1. $HK' = HK$.

Proof. It is clear from the above definitions and from (5.15) that $HK'(D) = HK(D)$, since forms in $HK'(D)$ are closed, coclosed and satisfy the Neumann boundary condition. □

Next up is Splitting statement 3, which states that $G(D) = \mathrm{span}(DFG(D) \cup GG'(D))$, where $DFG(D)$ = *divergence free gradients* and $GG'(D)$ = *grounded gradients* are given by

$$DFG(D) := \{V \in G(D) \mid \nabla \cdot V = 0\}$$
$$GG'(D) := \{V \in G(D) \mid \varphi|_{\partial D} = 0\}.$$

From these definitions we immediately get the second inclusion statement.

Inclusion statement 2. $GG' = GG$.

Proof. Vector fields in $GG'(D)$ are precisely those that are exact with Dirichlet boundary conditions, and hence $GG'(D) = GG(D)$. □

Finally, Splitting statement 4 states that $DFG = \mathrm{span}(CG'(D) \cup HG'(D))$, where $CG'(D)$ = *curly gradients* and $HG'(D)$ = *harmonic gradients* are given by

$$CG'(D) := \{V \in DFG(D) \mid \text{all boundary fluxes are } 0\}$$
$$HG'(D) := \{V \in DFG(D) \mid \varphi \text{ is locally constant on } \partial D\}.$$

That all boundary fluxes of a vector field $V \in \Gamma(TD)$ are 0 means that the flux of V through any connected component of ∂D is zero, and that a function $\varphi \in C^\infty(D)$ is locally constant on ∂D is equivalent to φ being constant on each connected component of ∂D.

Inclusion statement 3. $HG' = HG$.

Proof. Every vector field $V = \nabla\varphi \in HG'(D)$ is exact, thus in particular closed, and coclosed. Since φ is locally constant on ∂D, it follows that V is normal to ∂D, and hence satisfies the Dirichlet boundary condition. Therefore $HG'(D) = HG(D)$. □

We have already seen in the first three inclusion statements that $HK'(D) = HK(D)$, $HG'(D) = HG(D)$ and $GG'(D) = GG(D)$, so we omit the prime symbols for these subspaces and write $HK(D)$ instead of $HK'(D)$ etc. By the end of this chapter it will become clear that $FK'(D) = FK(D)$ and $CG'(D) = CG(D)$ as well. Until that point, we keep the prime symbols in place for these two spaces to avoid confusion as to how each space is defined.

In what follows, the reference to D is understood, so we omit it and write K instead of $K(D)$ etc.

7.3 Knots and gradients

Splitting statement 1. The span of $K \cup G$ is $\Gamma(TD)$.

Proof. Let V be a smooth vector field on D and define two smooth functions $f : D \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$ by $f := \nabla \cdot V$ and $g := V \cdot N$. It then follows from the divergence theorem (Theorem 3.13) that

$$\int_{D_i} f \, dV = \int_{\partial D_i} g \, dS,$$

where $\{M_i\}_{i \in I}$ is the set of connected components of D. Let φ solve the Neumann problem

$$\begin{cases} \nabla^2 \varphi = f & \text{in } D \\ \dfrac{\partial \varphi}{\partial N} = g & \text{on } \partial D \end{cases}$$

and define two vector fields V_G and V_K by $V_G := \nabla \varphi$ and $V_K := V - V_G$. By computing

$$V_G \cdot N = \nabla \varphi \cdot N = \frac{\partial \varphi}{\partial N} = g = V \cdot N,$$

we see that $V_K \cdot N = 0$ on ∂D. Also

$$\nabla \cdot V_G = \nabla^2 \varphi = f = \nabla \cdot V,$$

hence V_K is divergence free, that is $\nabla \cdot V_K = 0$. Thus V_K lives in K and V_G, being the gradient of φ, lives in G. This shows that $\Gamma(TD) = \operatorname{span}(K \cup G)$. □

7.4 Splitting knots

Splitting statement 2. The span of $FK' \cup HK$ is K.

Before we turn to the proof of Proposition 2, we state and prove inclusion statement four as well as a lemma.

Inclusion statement 4. $FK' = FK$.

Proof. Recall that

$$FK = \{\nabla \times U \mid \nabla \cdot U = 0, \; U \times N = 0\}.$$

That is, FK' consists of those forms that are closed, coclosed and satisfy the Neumann boundary condition.

We begin by showing that $FK' \subset \{\nabla \times U \mid \nabla \cdot U = 0, \; U \times N = 0\}$. To this end, let $V \in FK'$ and define $B := BS(V)|_D$. Then, by Proposition 6.4, $\nabla \cdot B = 0$ and $\nabla \times B = V$, so we just need to fix $B \times N$.

Let C be a closed curve on ∂D, and consider the circulation around C of the vector field $B' := B^{||}|_{\partial D}$, consisting of the component of $B|_{\partial D}$ that is parallel to ∂D. That is, consider

$$\int_C B' \cdot ds. \tag{7.1}$$

Notice that substituting B for B' in the above integral does not change its value. Hence, by Kelvin-Stokes theorem (Theorem 3.14), if C bounds a surface Σ that lies inside D we have

$$\int_C B' \cdot ds = \int_C B \cdot ds = \int_\Sigma (\nabla \times B) \cdot N \, dS = \int_\Sigma V \cdot N \, dS = 0, \qquad (7.2)$$

since V is fluxless. If, on the other hand, Σ lies outside D we get

$$\int_C B' \cdot ds = \int_C B \cdot ds = \int_\Sigma BS(V) \cdot ds = \int_\Sigma (\nabla \times BS(V)) \cdot N \, dS = 0, \quad (7.3)$$

because $\nabla \times BS(V) = 0$ outside D.

Now, since $\nabla \times BS(V)|_{\partial D} = V|_{\partial D}$ is tangent to ∂D, the value of (7.1) only depends on the homology class of the curve C in $H_1(\partial D)$. Also, every homology class in $H_1(D)$ can be expressed as the sum of two classes; one that bounds in D and one that bounds in $D^{\complement}$. Hence, from (7.2) and (7.3) we deduce that

$$\int_C B' \cdot ds = \int_C B \cdot ds = 0$$

for any closed curve C on ∂D. Thus, B' is a conservative vector field on ∂D, and can therefore be expressed as $B' = \nabla^{\|} f$, where $f : \partial D \to \mathbb{R}$ is a scalar potential of B' and $\nabla^{\|} f$ denotes the gradient of f along ∂D.

Take $\varphi : \partial D \to \mathbb{R}$ to be the solution of the Dirichlet problem

$$\begin{cases} \nabla^2 \varphi = 0 & \text{in } D \\ \varphi = f & \text{in } \partial D. \end{cases}$$

and define a vector field U by $U := B - \nabla\varphi$. Then $\nabla \cdot U = 0$, $\nabla \times U = \nabla \times B = V$ and $U \times N = 0$. To verify the last of these three expression, consider a vector v that is tangent to ∂D and calculate

$$B \cdot v = B' \cdot v = \nabla^{\|} f \cdot v = \nabla\varphi \cdot v,$$

whence it follows that $U \cdot v = (B - \nabla\varphi) \cdot v = 0$.

Next, we prove that $\{\nabla U \mid \nabla \cdot U = 0,\ U \times N = 0\} \subset FK'$. To this end, let $U \in \Gamma(TD)$ be such that $\nabla \cdot U = 0$ and $U \times N = 0$ and define V by $V := \nabla \times U$.

It is clear that $\nabla \cdot V = \nabla \cdot \nabla \times U = 0$, and since $U \times N = 0$ implies that U is orthogonal to ∂D we have, for any smooth surface Σ with $\partial\Sigma \subset \partial D$,

$$\int_\Sigma V \cdot N \, dS = \int_\Sigma (\nabla \times U) \cdot N \, dS = \int_{\partial\Sigma} U \cdot ds = 0,$$

which shows that all interior fluxes of V are zero.

To show that V is tangent to ∂D it is enough to recall Proposition 5.4, which states that the codifferential (in our case $\nabla\times$) preserves Neumann boundary conditions. This concludes the proof of the lemma. □

Lemma 7.1. *Let g denote the genus of D and let $\{\Sigma_i\}_{i=1}^g$ be a set of surfaces that form a basis for $H_2(D, \partial D; \mathbb{R})$. Then*

1. $HK \simeq H_1(D; \mathbb{R}) \simeq H_2(D, \partial D; \mathbb{R}) \simeq \mathbb{R}^g$.

2. *For each g-tuple of real numbers $(\Phi_1, \Phi_2, \ldots, \Phi_g)$, there exists a unique vector field $V \in HK$ such that Φ_i is the flux of V through Σ_i for $i \in \{1, 2, \ldots, g\}$.*

Proof. By Proposition 4.12, we have

$$H_1(D) \simeq H_1(\mathbb{R}^3 \setminus D; \mathbb{R}) \simeq H_2(D, \partial D; \mathbb{R}) \simeq \mathbb{R}^g.$$

Let $\{C_i\}_{i=1}^g$ and $\{C_i'\}_{i=1}^g$ be disjoint smooth loops that constitute a basis for $H_1(D; \mathbb{R})$ and $H_1(\mathbb{R}^3 \setminus D; \mathbb{R})$ respectively, and let $\{\Sigma_i\}_{i=1}^g$ be smooth orientable surfaces that constitute a basis for $H_2(D, \partial D; \mathbb{R})$. Additionally, these bases are to be chosen such that the intersection number of C_i with Σ_j is δ_{ij} and the linking number of C_i with C_i' is δ_{ij}. That this can be done is guaranteed by Theorem 4.13.

For each $1 \leq i \leq g$, let $I_i \in \mathbb{R}$ denote the magnitude of the vector field $I_i(x)$ along C_i' (as in Proposition 6.4), and let B denote the vector field on D given by

$$B(x) := \sum_{i=1}^{g} BS(I_i)(x) = \frac{1}{4\pi} \sum_{i=1}^{g} \int_{C_i} I_i(y) \times \frac{x-y}{|x-y|^3}\, dy.$$

It then follows from Proposition 6.4 that $\nabla \cdot B = 0$, $\nabla \times B = 0$ and that

$$\int_{C_i} B \cdot ds = I_i.$$

Let $g : \partial D \to \mathbb{R}$ be given by $g := B \cdot N$ and take φ to be the solution of the Neumann problem

$$\begin{cases} \nabla^2 \varphi = 0 & \text{in } D \\ \dfrac{\partial \varphi}{\partial N} = g & \text{on } \partial D. \end{cases}$$

Now, let $V = V(I_1, I_2, \ldots, I_g) := B - \nabla\varphi$. Then V has the following properties:

$$\nabla \cdot V = 0, \qquad \nabla \times V = 0, \qquad V \cdot N = 0, \qquad \int_{C_i} V \cdot ds = I_i. \tag{7.4}$$

From this it follows that $0 \neq V(I_1, I_2, \ldots, I_g) \in HK$ whenever at least one of the I_i is nonzero.

Now, for $i \in \{1, 2, \ldots, g\}$, let Φ_i denote the flux of V through Σ, i.e.

$$\Phi_i := \int_{\Sigma_i} V \cdot N_i \, dS, \tag{7.5}$$

where N_i denotes the unit normal vector field along Σ_i that defines its orientation. Note that the integral (7.5) depends only on the homology class of Σ_i. The fluxes Φ_i cannot all be zero unless $V = 0$, since then we would have $V \in FK \cap HK = 0$ as FK and HK are orthogonal (see the comment after Inclusion statement 4). It follows that the linear map

$$\mathbb{R}^g \longrightarrow \mathbb{R}^g$$
$$(I_1, I_2, \ldots, I_g) \longmapsto (\Phi_1, \Phi_2, \ldots, \Phi_g)$$

is an isomorphism.

Suppose now that $W \in HK$ also has flux Φ_i trough the surface Σ_i for $i \in \{1, 2, \ldots, g\}$. Then $V - W \in FK \cap HK = 0$, which implies that $V = W$. It follows that the vector fields $V(I_1, I_2, \ldots, I_g)$ are the only harmonic knots in $\Gamma(TD)$. □

We are now in a position to prove Proposition 2.

Proof of Splitting statement 2. Let $V \in FK$, let $\{\Sigma_i\}_{i=1}^k$ be a set of smooth orientable surfaces in D that form a basis for $H_2(D, \partial D)$. For $i \in \{1, 2, \ldots, k\}$, let

$$\Phi_i := \int_{\Sigma_i} V \cdot N \, dS,$$

that is, Φ_i is the flux of V through Σ_i.

By Lemma 7.1 there is a unique vector field $V_{HK} \in HK$ such that the flux of V_{HK} through Σ_i is Φ_i for $i \in \{1, 2, \ldots, k\}$. Let $V_{FK} := V - V_{HK} \in K$. Since all interior fluxes of $V_{FK'}$ are zero, it follows that $V_{FK} \in FK$, and hence $K = \text{span}(HK \cup FK')$. □

7.5 Splitting gradients

Splitting statement 3. The span of $DFG \cup GG$ is G.

Proof. Let $\varphi \in C^\infty(D)$ and let $V := \nabla\varphi$ be the corresponding gradient vector field. Let φ_1 solve the Dirichlet problem

$$\begin{cases} \nabla^2\varphi_1 = 0 & \text{in } D \\ \varphi_1 = \varphi & \text{on } \partial D \end{cases}$$

and define $\varphi_2 =: \varphi - \varphi_1$, so that $V_1 := \nabla\varphi_1$ and $V_2 := \nabla\varphi_2$ satisfy $V = V_1 + V_2$. Then $V_1 \in DFG$ since $\nabla \cdot V_1 = 0$ and $V_2 \in GG$ since $\varphi_2|_{\partial D} = 0$, and hence $G = \text{span}(DFG \cup GG)$. □

7.6 Splitting divergence free gradients

Splitting statement 4. The span of $CG' \cup HG$ is DFG.

Our proof of this proposition relies on three lemmas, which we state and prove first.

Lemma 7.2. *Let m denote the number of connected components of D and let n denote the number of connected components of ∂D. Then*

$$HG \simeq H_2(D;\mathbb{R}) \simeq H_1(D,\partial D;\mathbb{R}) \simeq \mathbb{R}^{n-m}.$$

Proof. Let $D_1, \ldots, D_m$ denote the connected components of D and let $\partial D_{i1}, \ldots, \partial D_{in_i}$ denote the connected components of ∂D_i for each $1 \leq i \leq m$ Then $n = \sum_{i=1}^{m} n_i$ is the number of connected components of ∂D.

For each boundary component ∂D_{ij}, let $c_{ij} \in \mathbb{R}$ be a constant and let $\varphi \in C^\infty(D)$ be the solution of the Dirichlet problem

$$\begin{cases} \nabla^2\varphi = 0 & \text{in } D \\ \varphi = c_{ij} & \text{on } \partial D_{ij}. \end{cases} \tag{7.6}$$

For each set of constants $\{c_{ij}\}$ this solution is unique.

Now, let $V = \nabla\varphi$. Then V is the gradient vector field of a function that is constant of each boundary component of D, and since $\nabla \cdot V = \nabla^2\varphi = 0$, we conclude that $V \in HG$. Moreover, every vector field HG is of the form $\nabla\psi$ with ψ solves (7.6), and is hence uniquely determined by the value of ψ on each boundary component of D. Thus, to every vector $v \in \mathbb{R}^n$ we can associate a vector field $\nabla\varphi \in HG$ by means of the linear map

$$\begin{aligned} F : \mathbb{R}^n &\longrightarrow HG \\ v &\longmapsto \nabla\varphi, \end{aligned}$$

where φ is the unique solution to (7.6) with

$$c_{11} = v_1, \ldots, c_{mn_m} = v_n. \tag{7.7}$$

The map F is not an isomorphism, as can be seen by considering the identity $\nabla\varphi = \nabla(\varphi + c)$, where c is a constant. Hence, if the vector $v - v' \in \mathbb{R}^n$ (here understood as an assignment of a constant to each boundary component of D via the correspondence (7.7)) is constant on each $\partial D_i = \bigcup_{j=1}^{n_i} \partial D_{ij}$, then $F(v) = F(v')$. If, on the other hand, $v - v'$ is not constant on each ∂D_i, then $F(v) \neq F(v')$.

Let $S \subset \mathbb{R}^n$ denote the subspace spanned by m vectors $v_1, \ldots, v_m$ such that v_i is constant on ∂D_i and 0 elswhere (again understood via the correspondence (7.7)). Then $\{v_1, \ldots, v_m\}$ forms a basis for S, and thus $S \simeq \mathbb{R}^m$. Let π denote the projection onto the quotient space $\mathbb{R}^n/S$. Because of the universal property of quotient vector spaces, the above discussion implies that there exists a unique map $G : \mathbb{R}^n/S \to HG$ such that the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{F} & HG \\ {\scriptstyle\pi}\downarrow & \nearrow_{G} & \\ \mathbb{R}^n/S & & \end{array} \tag{7.8}$$

commutes, and that G is in fact an isomorphism.

Since $\mathbb{R}^n/S \simeq \mathbb{R}^n/\mathbb{R}^m \simeq \mathbb{R}^{n-m}$, the existence of G proves that $HG \simeq \mathbb{R}^{n-m}$.

Finally, Proposition 4.12 tells us that $\dim H_2(D;\mathbb{R}) = \dim H_1(D, \partial D;\mathbb{R}) =$ *the number of connected components of ∂D minus the number of connected components of D*, i.e. $n - m$. □

Lemma 7.3. *CG' is a subset of CG.*

Proof. Recall that CG' lies in the image of ∇ and that $CG = (\operatorname{im}(\nabla)) \cap (\operatorname{im}(\nabla\times))$. To show that $CG' \subset CG$, we thus only need to show that CG' lies in the image of $\nabla\times$.

Let $V \in CG'$. By Proposition 6.2 we have

$$\nabla_x \times BS(V)(x) = V'(x) - \frac{1}{4\pi}\int_{\partial D} \frac{V(y)\cdot N}{|x-y|^3}\, dS(y), \tag{7.9}$$

where $V'(x) = \begin{cases} V(x) & \text{for } x \in D \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus D \end{cases}$, and where we have used the fact that $\operatorname{div} V = 0$. We want to show that $V = \nabla \times U$ for some $U \in \Gamma(TD)$, for which is is enough to show that the second term in (7.9) lies in $\operatorname{im}(\nabla\times)$.

To this end, let $A = B \setminus \operatorname{int}(D)$, where $B \subset \mathbb{R}^3$ is a ball containing D in its interior and $\operatorname{int}(D) = D \setminus \partial D$, and let N_B denote the outward-pointing unit normal vector field on B. Then take $W := \nabla\psi$, where ψ is a solution to the Neumann problem

$$\begin{cases} \nabla^2\psi = 0 & \text{in } A \\ \dfrac{\partial\psi}{\partial N} = -V\cdot N & \text{on } \partial D \\ \dfrac{\partial\psi}{\partial N_B} = 0 & \text{on } \partial B. \end{cases}$$

In $\mathbb{R}^3 \setminus \operatorname{int}(A)$ we then have

$$\begin{aligned} \nabla_x \times BS(W)(x) &= -\frac{1}{4\pi}\int_{\partial(\mathbb{R}^3\setminus A)} \frac{W(y)\cdot N}{|x-y|^3}\, dS(y) \\ &= \frac{1}{4\pi}\int_{\partial D} \frac{V(y)\cdot N}{|x-y|^3}\, dS(y). \end{aligned}$$

In particular, since $D \subset \mathbb{R}^3 \setminus \operatorname{int}(A)$, the above equation holds in D, and we thus have

$$\nabla \times (BS(V) + BS(W)) = V.$$

This implies that $V \in \operatorname{im}(\nabla\times)$, and we are done with the proof. □

Lemma 7.4. *The linear maps $\Phi_{ij} : \mathbb{R}^n \to \mathbb{R}$ given by*

$$\Phi_{ij}(v) = \int_{\partial D_{ij}} F(v) \cdot N \, dS$$

provide an automorphism $G \in Aut(\mathbb{R}^n/S)$ given by

$$\begin{aligned} G : \mathbb{R}^n/S &\longrightarrow \mathbb{R}^n/S \\ [v] &\longmapsto [(\Phi_{11}(v), \ldots, \Phi_{mn_m}(v))]. \end{aligned}$$

Proof. We first prove that G is well defined. Let $[v] = [v']$. Then, since $v - v'$ is constant on each ∂D_i, we have $F(v) = F(v')$, and thus $G([v]) = G([v'])$.

Note that for all $v \in \mathbb{R}^n$, we have

$$\sum_{j=1}^{n_i} \Phi_{ij}(v) = 0 \quad \text{for all } i \in \{1, \ldots, m\},$$

since $F(v) \in HG$ is divergence free.

Now, let $v \in \mathbb{R}^n$ be such that $G([v]) = 0$. Then $F(v) \in HG$ and, since all boundary fluxes are zero, $F(v) \in CG'$. But CG' and HG are orthogonal since $CG' \subset CG$ (this is a direct consequence of Lemma 7.3), whence it follows that $F(v) = 0$, so that v must be constant on each component of D. From this we conclude that $v \in S$. Thus $[v] = 0$, and hence G is an isomorphism. □

With the proofs of the lemmas behind us, we are now ready to give a proof of Proposition 4.

Proof of Splitting statement 4. Let $V = \nabla\varphi \in DFG$ and let $\{\Phi_{ij}\}$ denote the fluxes of V through the components ∂D_{ij} of D. Using the isomorphism provided by Lemma (7.4), we get a vector v of constant boundary values corresponding to these fluxes. Since V is divergence free, we know that the flux of V through each ∂D_i is zero, and hence v can be chosen so that

$$\sum_{j=1}^{n_i} c_{ij} = 0 \quad \text{for all } 1 \leq i \leq m,$$

where we have used the identifications provided by (7.7).

Now, let ψ solve the Dirichlet problem

$$\begin{cases} \nabla^2\psi = 0 & \text{in } D \\ \psi = c_{ij} & \text{on } \partial D_{ij}. \end{cases}$$

Then $V_1 = \nabla\psi \in HG$ has the same boundary fluxes as V, and $V_2 = V - V_1 = \nabla(\varphi - \psi) \in CG'$ since its boundary fluxes are all 0, and hence $DFG = \mathrm{span}(CG' \cup HG)$. □

7.7 Putting everything together

Inclusion statement 5. $CG' = CG$.

Proof. From Lemma 7.3 we have $CG' \subset CG$, and from Spanning statements 1 through 4 and Inclusion statements 1 through 4 we have

$$\Gamma(TM) = \mathrm{span}(FK \cup HK \cup CG' \cup HG \cup GG). \tag{7.10}$$

Hence, by Theorem 5.12, CG' must be equal CG. □

We are now finally in a position to prove the spanning statement in $\mathbb{R}^3$.

Proof of spanning statement in $\mathbb{R}^3$. The result follows immediately from Splitting statements 1 through 4 together with Inclusion statements 1 through 5. We get

$$\Gamma(TM) = \mathrm{span}(FK \cup HK \cup CG \cup HG \cup GG), \tag{7.11}$$

which is what we set out to prove. □

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