UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2017:32

The Ratliff-Rush Operation for Certain Monomial Ideals in $K[x, y]$

Petter Restadh

Examensarbete i matematik, 15 hp
Handledare: Veronica Crispin Quinonez
Examinator: Jörgen Östensson
Augusti 2017

Department of Mathematics
Uppsala University

THE RATLIFF-RUSH OPERATION FOR CERTAIN MONOMIAL IDEALS IN $K[x, y]$

PETTER RESTADH

ABSTRACT. Given a regular ideal $\mathfrak{a}$ in a Noetherian ring we can define the Ratliff-Rush ideal associated to $\mathfrak{a}$ as the union $\widetilde{\mathfrak{a}} = \bigcup \mathfrak{a}^{n+1} : \mathfrak{a}^n$. Moreover, this is the largest ideal with the same high powers as $\mathfrak{a}$. We explain why that is so and some of its consequenses. Then, with the use of results about numerical semigroups we will see how we can compute $\widetilde{\mathfrak{a}}$ for some ideals in $K[x, y]$ and give examples of some properties that $\widetilde{\mathfrak{a}}$ possesses in this case.

Contents

1. Introduction 3
2. Basic Concepts 3
2.1. Basic Definitions 3
2.2. Arithmetics on Ideals 5
2.3. Monomial Ideals 7
2.4. Noetherian Rings 9
3. Numerical Semigroups 10
4. Ratliff-Rush Ideal 12
4.1. Ratliff-Rush Ideals in Noetherian Rings 13
4.2. Ideals in $K[x, y]$ Generated by Monomials of the Same Degree 14
5. Examples and Various Properties 17
5.1. Examples 17
5.2. Some Interesting Properties 18
References 24

1. Introduction

Let R be a ring. On the set of ideals in R we can define the four operations addition, $\mathfrak{a}+\mathfrak{b}$, multiplication, $\mathfrak{a}\cdot\mathfrak{b}$, intersection, $\mathfrak{a}\cap\mathfrak{b}$, and ideal quotient, $\mathfrak{a}:\mathfrak{b}$. Using the multiplication we can define powers of ideals and using the other operations we can study how these powers behave.

Ratliff and Rush [5] were the first ones who studied the union $\widetilde{\mathfrak{a}} = \bigcup_{k\geq 0}(\mathfrak{a}^{k+1} : \mathfrak{a}^k)$ for regular ideals, that is containing a non-zero-divisor, in Noetherian rings. The ideal $\widetilde{\mathfrak{a}}$ is called the Ratliff-Rush ideal associated to $\mathfrak{a}$ and we will see that, for regular ideals, $\widetilde{\mathfrak{a}}$ has the same high powers as $\mathfrak{a}$. Moreover it is the largest ideal, with respect to inclution, with this property.

In some literature the associated Ratliff-Rush ideals are called Ratliff-Rush closures. This is not surprising. We have both $\mathfrak{a} \subseteq \widetilde{\mathfrak{a}}$ and $\widetilde{\widetilde{\mathfrak{a}}} = \widetilde{\mathfrak{a}}$. Although in general $\mathfrak{a} \subseteq \mathfrak{b}$ does not imply $\widetilde{\mathfrak{a}} \subseteq \widetilde{\mathfrak{b}}$, so we will not name it as such.

The Ratliff Rush operation behaves, in general, quite unpredictble and erradically. For example, $\widetilde{\mathfrak{a}}\widetilde{\mathfrak{b}}$ is not always equal to $\widetilde{\mathfrak{a}\mathfrak{b}}$, and as just mentioned, $\mathfrak{a} \subseteq \mathfrak{b}$ does not imply $\widetilde{\mathfrak{a}} \subseteq \widetilde{\mathfrak{b}}$ and $\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{b}}$ is not always equal to $\widetilde{\mathfrak{a}+\mathfrak{b}}$. This often makes the computations hard.

In this paper we will begin with some basic concepts about algebra in general, working our way up to arithmetic on ideals and Noetherian rings. We will after that work our way through some relevant results from [5]. Then we will present some results about numerical semigroups from [2]. We will need this work to prove that for some special cases the associated Ratliff-Rush ideals can be described in a much easier way. This will allow us to use a computer to assist us with the computations to produce examples so we can look at some properties of the associated Ratliff-Rush ideal.

2. Basic Concepts

Something many of the later results will revolve around is *numerical semigroups*. In fact, the study of the associated Ratliffe-Rush ideals on monomial ideals in $K[x, y]$ boils down to just that, as we will see when we work our way through results from [2]. All results from this section can be found in [3]. We will sometimes use sligtly different definitions than the ones used in [3] but even so they are equivalent.

Definition 2.1 (Numerical semigroups)**.** A numerical semigroup is a set $S \subseteq \mathbb{N}$ such that:

i) $0 \in S$
ii) $\mathbb{N}\backslash S$ is finite
iii) $a, b \in S \Rightarrow a + b \in S$

Notice that the operation, "+", is the usual addition on the natural numbers, so we know it is commutative and associative. To be more precise, any numerical semigroup is in fact a commutative monoid.

More about this later on. For now we will begin with the basics.

2.1. Basic Definitions.

Here we will recall the most fundamental concepts. We will not discuss them in detail, but when we move on to the later chapters it is expected of the reader to have a understanding of them.

Definition 2.2 (Binary Operation)**.** A binary operation on a set A is a function $* : A \times A \to A$. We will write $a * b$ instead of $*(a, b)$.

Definition 2.3 (Monoid)**.** A monoid is a set S and a binary operation $*$ such that:

i) The operation is associative, $(a * b) * c = a * (b * c)$
ii) There exist a unit element $e \in S$ such that $a = a * e = e * a$

A *semigroup* is a monoid that does not necessarily fulfill ii).

A *group* is a monoid such that for each a there exits b such that $a * b = b * a = e$. This b is usually written as a^{-1} and called a's inverse.

If the operation is commutative, that is $a * b = b * a$, the group or monoid is called commutative or abelian. Then we usally use "+" instead of "*" and write the inverses as "$-a$".

Definition 2.4 (Ring)**.** A ring is a set R and two binary operations, $+$ and $\cdot$ such that:

i) The pair R and $+$ is an abelian group
ii) The pair R and $\cdot$ is a monoid
iii) The multiplication is distributive over the multiplication, that is $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

Any ring can be seen as a quintuple $(R, +, \cdot, 0, 1)$, where 0 is the additive unit and 1 is the multiplicative unit.

If R and $\cdot$ are a commutative monoid the ring is called commutative.

Worth noting is that sometimes rings are defined without a mutiplicative unit and rings in the sense defined above is called rings with unit element, but we will only work with rings of the second kind. Moreover, the following holds in all rings:

- For all $r \in R$ we have $r \cdot 0 = 0$.
- The multiplicative and additive unit are unique.
- For all $r \in R$ we have $(-1) \cdot r = -a$.

Example 2.5. $\mathbb{Z}$ with the usual "$+$" and "$\cdot$" is a commutative ring.

Example 2.6. $R = \{0\}$ with $0 + 0 = 0$ and $0 \cdot 0 = 0$ is called the *trivial ring* or the *zero ring*.

Definition 2.7 (Subring)**.** A subring to a ring R is any subset $R' \subseteq R$ such that $(R', +, \cdot, 0, 1)$ is a ring.

In other words it is a ring within a ring.

Definition 2.8 (Field)**.** A field is a non-trivial ring such that $R \backslash \{0\}$ and $\cdot$ is an abelian group. The multiplicative inverse for a is denoted a^{-1} and we often write ab instead of $a \cdot b$.

Definition 2.9 (Polynomial ring)**.** Let R be a commutative ring. The polynomial ring over R, written as $R[x]$, is defined with the underlying set

$$R[x] = \{p = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n \mid n \in \mathbb{N}\}$$

with $p_i \in R$. Then we have the addition $p + q = r$ where $r_i = p_i + q_i$ and multiplication $p \cdot q = r$ where $r_i = \sum_{k+l=i} p_k \cdot q_l$.

Notice that with $R[x, y]$ we mean $R[x][y]$. Sometimes $R[x, y]$ can be defined in other ways, but they will behave in exactly the same way, or be *isomorphic*. We will not define being "*isomorphic*", but we can think of it as "being the same, but with different names".

An important number associated with every element in a polynomial ring is the *degree*. It is the same as in most other places, namely:

Remark 2.10 (Degree). The degree of a non-zero element p in $R[x]$ is the greatest n such that $p_n \neq 0$. The degree of 0 is $-\infty$. We will denote the degree of p by $\deg(p)$.

Definition 2.11 (Ideal)**.** An ideal $\mathfrak{a}$ in a ring R is a subset of R that is closed with respect to the addition with any element in $\mathfrak{a}$ and multiplication with any ring element. That is

i) if $a,\ b \in \mathfrak{a}$ then $a + b \in \mathfrak{a}$
ii) if $a \in \mathfrak{a}, r \in R$ then $r \cdot a \in \mathfrak{a}$
iii) if $a \in \mathfrak{a}, r \in R$ then $a \cdot r \in \mathfrak{a}$

Noteworthy is that ii) and iii) are equivalent if R is commutative.

Example 2.12. In any ring R, both R, the ring itself, and $\{0\}$, the *zero ideal*, are ideals. These are called the *trivial ideals*. We will denote the zero-ideal with $\mathfrak{o}$.

Remark 2.13. An ideal $\mathfrak{a}$ contains the multiplicative unit if and only if $\mathfrak{a}$ is the ring. If you are not used to ideals, showing this is a good exercise.

In fact, these two ideals are special in such a way that they are the only two that always exist in a ring. Studying ideals is interesting because they tell us how the ring behaves, as the following result will show.

Theorem 2.14. *For a commutative ring R the following two are equivalent:*

(1) R is a field.
(2) R has only two ideals, namely trivial ones.

Proof. Let R be a field. Take any non-zero ideal, $\mathfrak{a}$, then there exist a non-zero element in $\mathfrak{a}$, call it a. Since R is a field a^{-1} exists in R. Then by iii) in the definition 2.11 $a \cdot a^{-1} = 1 \in \mathfrak{a}$. So by remark 2.13 $\mathfrak{a} = R$.

The other way around, assume R only has two ideals. We only need to verify that $\mathfrak{a}_a = \{r \cdot a : r \in R\}$ is indeed a non-zero ideal for all non-zero $a \in R$. Since then it follows that $\mathfrak{a}_a = R \ni 1$ so there must exist $x \in R$ such that $x \cdot a = 1$ and therefore a has an inverse. Since a was chosen arbitrary, it follows that R is indeed a field.

Firstly, $1 \cdot a = a \in \mathfrak{a}_a$ so $\mathfrak{a}_a$ is non-zero. Left to do is verifying the conditions in definition 2.11 and we are done.

□

2.2. **Arithmetics on Ideals.** In this section we will continue with some more definitions and core concepts. We will talk about generators for ideals and define some operators that will be useful later on.

Definition 2.15 (Generating set)**.** Given any set A in a ring R. We call the smallest, ordered by inclusion, ideal containing A for the ideal generated by A. We denote this as $\langle A \rangle$. We say that a set A generates an ideal I, or I is generated by A, if $\langle A \rangle = I$.

The definition of a set generating a subgroup or semigroup is analogous.

Remark 2.16. If it were not for the fact that for any two ideals their intersection is also an ideal, this would not be well-defined.

Remark 2.17. If at any point we speak about generators for anything other than the structures explicitly mentioned in definition 2.15 without defining generators on that object, the definition is analogous to 2.15. That is, the smallest such structure which contains the generators.

Generating sets is a core concept. Many proofs, theorems and definitions use generators and it is often easier to work with a generating set than the structure as a whole.

Example 2.18. In a ring R, the ring generated by $\langle 1 \rangle = R$.

Example 2.19. In the group $\mathbb{Z}$ the subgroup generated by 2 is $\langle 2 \rangle = \{\dots, -2, 0, 2, 4, \dots\}$. But $\mathbb{Z}$ seen as a monoid the monoid generated by 2 is $\langle 2 \rangle = \{0, 2, 4, \dots\}$.

Let us now use what we have discussed by studying some operations on the set of ideals in some ring R.

Definition 2.20. Define the following operations on some pair of ideals $\mathfrak{a}$ and $\mathfrak{b}$ as:

i) $\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$
ii) $\mathfrak{a} \cdot \mathfrak{b} = \{\sum_{i=0}^{k} a_i b_i \mid a_i \in \mathfrak{a}, b_i \in \mathfrak{b}\}$
iii) $\mathfrak{a} \cap \mathfrak{b} = \{r \in R \mid r \in \mathfrak{a} \text{ and } r \in \mathfrak{b}\}$

If the ring is commutative we can also define:

iv) $\mathfrak{a} : \mathfrak{b} = \{r \in R \mid r \cdot b \in \mathfrak{a} \ \forall b \in \mathfrak{b}\}$

The third operation is recognizible as the usual intersection between two sets.

Left to show is that these are in fact well defined operations on the set of ideals. Or, in other words, that $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cdot \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} : \mathfrak{b}$ are all ideals. We will show this for $\mathfrak{a} : \mathfrak{b}$ since that is the hardest one. The others will be left as an exercise for the reader.

Proof. We will go through the conditions from definition 2.11.

For any two elements r_1 and r_2 in $\mathfrak{a} : \mathfrak{b}$, any $k \in R$ and any element b in $\mathfrak{b}$ we have:

i) $(r_1 + r_2)b = r_1 b + r_2 b$ lies in $\mathfrak{a}$ since both $r_1 b$ and $r_2 b$ do.
ii) $(kr_1)b = k(r_1 b)$ lies in $\mathfrak{a}$ since $r_1 b$ does.
iii) Follows commutativity of the ring and by *ii*).

So $\mathfrak{a} : \mathfrak{b}$ is indeed an ideal.

□

First we will look at how these operations interact with eachother. Firstly some things that come directly from the ring. For any ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$

- $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$
- $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$
- $\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{c}$
- $(\mathfrak{b} + \mathfrak{c}) \cdot \mathfrak{a} = \mathfrak{b} \cdot \mathfrak{a} + \mathfrak{b} \cdot \mathfrak{a}$

And clearly if the ring is abelian:

- $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$

If any of the above seems unclear it is recommend proving it. It is really more of an exercise in writing than dealing with rings and ideals.

This area is full of small theorems and different ways these operations interact. We will later use some but far from all. Therefore the ones we will use, apart from the ones mentioned above, are summed up in this next theorem. Most of them can be found in chapter 1.3.5 in [3].

Theorem 2.21. *The following holds for any ideals in a ring R:*

i) If $\mathfrak{a} = \langle S \rangle$ and $\mathfrak{b} = \langle T \rangle$ then we have:
- $\mathfrak{a} + \mathfrak{b} = \langle S \cup T \rangle$
- $\mathfrak{a} \cdot \mathfrak{b} = \langle st \mid s \in S,\ t \in T \rangle$

Moreover if R is commutative

ii) If $\mathfrak{a} \subseteq \mathfrak{b}$ then $\mathfrak{a} : \mathfrak{c} \subseteq \mathfrak{b} : \mathfrak{c}$ and $\mathfrak{c} : \mathfrak{b} \subseteq \mathfrak{c} : \mathfrak{a}$

iii) $(\mathfrak{a} : \mathfrak{b}) : \mathfrak{c} = \mathfrak{a} : \mathfrak{c}\mathfrak{b}$

iv) $(\mathfrak{a} : \mathfrak{b})(\mathfrak{c} : \mathfrak{d}) \subseteq \mathfrak{a}\mathfrak{c} : \mathfrak{b}\mathfrak{d}$

v) $\mathfrak{a} : \mathfrak{b} \subseteq \mathfrak{a}\mathfrak{c} : \mathfrak{b}\mathfrak{c}$

Proof. We will show them one by one.

i) We will show that $\mathfrak{a} + \mathfrak{b} \subseteq \langle S \cup T \rangle$ and $\mathfrak{a}\mathfrak{b} \subseteq \langle st \mid s \in S,\ t \in T \rangle$. The other inclusion follows from the minimality condition in the definition 2.15 and the fact that $\mathfrak{a} \subseteq \mathfrak{a} + \mathfrak{b}$ since $0 \in \mathfrak{b}$.
- Take any element $a + b \in \mathfrak{a} + \mathfrak{b}$. By definition $a \in \langle S \cup T \rangle$ and $b \in \langle S \cup T \rangle$. Then, since $\langle S \cup T \rangle$ is an ideal $a + b \in \langle S \cup T \rangle$.
- Take any element $\sum_{i=0}^{k} a_i b_i \in \mathfrak{a}\mathfrak{b}$. Clearly $a_i b_i \in \langle st \mid s \in S,\ t \in T \rangle$. Then, since $\langle st \mid s \in S,\ t \in T \rangle$ is an ideal $\sum_{i=0}^{k} a_i b_i \in \langle st \mid s \in S,\ t \in T \rangle$.

ii) We have $r \in \mathfrak{a} : \mathfrak{c} \Leftrightarrow r\mathfrak{c} \subseteq \mathfrak{a} \Rightarrow r\mathfrak{c} \subseteq \mathfrak{b} \Leftrightarrow r \in \mathfrak{b} : \mathfrak{c}$ and $r \in \mathfrak{c} : \mathfrak{b} \Leftrightarrow r\mathfrak{b} \subseteq \mathfrak{c} \Rightarrow r\mathfrak{a} \subseteq \mathfrak{c} \Leftrightarrow r \in \mathfrak{c} : \mathfrak{a}$.

iii) $r \in (\mathfrak{a} : \mathfrak{b}) : \mathfrak{c} \Leftrightarrow r\mathfrak{c} \subseteq \mathfrak{a} : \mathfrak{b} \Leftrightarrow r\mathfrak{c}\mathfrak{b} \subseteq \mathfrak{a} \Leftrightarrow r \in \mathfrak{a} : \mathfrak{c}\mathfrak{b}$

iv) Take any element $\sum a_i b_i$ in $(\mathfrak{a} : \mathfrak{b})(\mathfrak{c} : \mathfrak{d})$. Then we see that for any $a_i b_i$ we have $a_i b_i \mathfrak{d}\mathfrak{b} \subseteq a_i \mathfrak{c}\mathfrak{b} = a_i \mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{c}$. The result follows.

v) $\mathfrak{a} : \mathfrak{b} = (\mathfrak{a} : \mathfrak{b})R = (\mathfrak{a} : \mathfrak{b})(\mathfrak{c} : \mathfrak{c}) \subseteq \mathfrak{a}\mathfrak{c} : \mathfrak{b}\mathfrak{c}$

□

2.3. **Monomial Ideals.** Now that we have talked about ideals in general we will focus on *monomial ideals*.

Definition 2.22 (Monomial)**.** A monomial in $R[x_0, \ldots, x_n]$ is any element on the form $kx_0^{\alpha_0} \ldots x_n^{\alpha_n}$ with $k \in R$. We usually write kx^{α} where $\alpha = (\alpha_0, \ldots, \alpha_n)$. If there exists a set of monomials that generates an ideal we will call such an ideal for a *monomial*.

Remark 2.23 (Degree)*.* On monomials we will use a slightly different defenition of a degree, namely $\sum_{i=0}^{n} \alpha_i$.

Why we use this degree will become more obvious later on.

So why do we look at these ideals? Why do they have their own subsection? The simple answer is "because they behave nicely", as we will see now. Let us first see how they behave under our operations defined in 2.20.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be generated by S and T respectively. Here we let S and T be some sets of monomials. Then by 2.21 we have

$$\mathfrak{a} + \mathfrak{b} = \langle S \cup T \rangle$$
$$\mathfrak{a} \cdot \mathfrak{b} = \langle st | s \in S,\ t \in T \rangle.$$

Since the right hand sides are sets of monomials, these ideals must be monomial ideals too.

Example 2.24. Let $\mathfrak{a} = \langle x^2,\ xy^2,\ y^3\rangle$ and $\mathfrak{b} = \langle xy^3,\ y^5\rangle$ in $K[x, y]$ for some field K. Then

$$\mathfrak{a} + \mathfrak{b} = \langle x^2,\ xy^2,\ y^3,\ xy^3,\ y^5\rangle.$$

But since y^3 divides both xy^3 and y^5, the right hand side is in fact $\langle x^2,\ xy^2,\ y^3\rangle = \mathfrak{a}$. We also have

$$\mathfrak{a} \cdot \mathfrak{b} = \langle x^3y^3,\ x^2y^5,\ x^2y^5,\ xy^7,\ xy^6,\ y^8\rangle.$$

But as before we see that xy^7 is not needed. So $\mathfrak{a} \cdot \mathfrak{b} = \langle xy^6,\ x^2y^5,\ x^3y^3,\ y^8\rangle$.

In fact, both $\mathfrak{a} : \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are also monomial ideals given that $\mathfrak{a}$ and $\mathfrak{b}$ are. The following theorem is a good starting point to understanding how monomial ideals work. It is theorem 1 in the second chapter in [3].

Theorem 2.25. *Let $\mathfrak{a}$ be a monomial ideal. A element $\sum a_i x^{\alpha_i}$ is in $\mathfrak{a}$ if and only if $a_i x^{\alpha_i}$ is in $\mathfrak{a}$ for all i.*

We refer to [3] for the proof, we will instead discuss a little about monomial ideals in $K[x, y]$. As we will see later, all ideals in $K[x, y]$ are finitely generated and therefore we will only discuss these ideals.

Firstly, an element kx^ay^b is in a monomial ideal $\mathfrak{a}$ if and only if x^ay^b is in $\mathfrak{a}$. This follows since K is a field and therefore all elements $k \in K$ are invertible. From this and theorem 2.25 the following can be proved:

Theorem 2.26. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be monomial ideals in $K[x, y]$. If a monomial m lies in their sum, it lies in one of them. That is if $m \in \mathfrak{a} + \mathfrak{b}$ then $m \in \mathfrak{a}$ or $m \in \mathfrak{b}$.*

Secondly, the generating elements can be ordered as $m_i = x^{a_i}y^{b_i}$ where $a_i > a_{i+1}$ and $b_i < b_{i+1}$. This can be done because otherwise some of the generating elements would be superfluous as in example 2.24. This is something unique to polynomial rings in two variables. How to order them, for example in $K[x, y, z]$, to get quicker calculations is the beginning of the study of *Gröbner bases.* This is something we will not mention any more in this paper, but more informaion about this can be found in [3].

Moreover, for monomials m the following holds:

$$(m\mathfrak{a} : m\mathfrak{b}) = \mathfrak{a} : \mathfrak{b}$$

and

$$(m\mathfrak{a} : \mathfrak{b}) = m(\mathfrak{a} : \mathfrak{b})$$

This follows from the injectivity of the map given by $x \mapsto mx$. This gives us:

$$(m\mathfrak{a})^{n+1} : (m\mathfrak{a})^n = m^{n+1}\mathfrak{a}^{n+1} : m^n\mathfrak{a}^n = m(\mathfrak{a}^{n+1} : \mathfrak{a}^n) \tag{2.1}$$

This equality can be found in [2] as (1.1). With this we will only need to consider some specific cases later on. But for now we will leave it as it is.

2.4. **Noetherian Rings.** Noetherian rings have been studied extensively and it is still a big research area within algebra. Indeed, this paper could very well only have been about Noetherian rings. The contents of this chapter and ideas for the proofs can be found in [1]. Since we will not be using much of the properties that follows from a ring being Noetherian we will only show a few of them.

Definition 2.27 (Noetherian ring)**.** A Noetherian ring is a commutative ring R which satisfies one of the following conditions:

i) Any infinite chain of increasing ideals has a maximal element. In other words, if $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ then there exists a natural number N such that $\mathfrak{a}_n = \mathfrak{a}_N$ for all $n \geq N$. This is known as the ascending chain condition.
ii) All ideals in R are finitely generated.
iii) Any family of ideals in R ordered by inclusion has a maximal element.

It is also possible to define what it means for a non-commutative ring to be Noetherian, but we will only work with commutative Noetherian rings. In this case the definitions coincide.

Theorem 2.28. *The above three properties are equivalent.*

Proof. We will, in order, show that iii) $\Rightarrow$ i) $\Rightarrow$ ii) $\Rightarrow$ iii).

'iii)$\Rightarrow$ i)'

Take any increasing chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ and create $\{\mathfrak{a}_n\}_{n=1}^{\infty}$. This set has a maximal element by iii), say $\mathfrak{a}_N$. Then for $n \geq N$ $\mathfrak{a}_N \subseteq \mathfrak{a}_n$, but since $\mathfrak{a}_N$ maximal, $\mathfrak{a}_n = \mathfrak{a}_N$.

'i)$\Rightarrow$ ii)'

Assume R has an ideal $\mathfrak{a}$ which is not generated by some finite set. Take any element $a_1 \in \mathfrak{a}$, then let $\mathfrak{a}_1 = \langle a_1 \rangle \subsetneq \mathfrak{a}$. Define $\mathfrak{a}_{n+1}$ recursively by taking $a_{n+1} \in \mathfrak{a} \backslash \mathfrak{a}_n$ and defining $\mathfrak{a}_{n+1} = \mathfrak{a}_n + \langle a_{n+1} \rangle = \langle a_1, \dots, a_{n+1} \rangle$. Note that a_{n+1} must exist since no finite set generates $\mathfrak{a}$ but $\mathfrak{a}_n = \langle a_1, \dots, a_n \rangle$.

We have $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$ by construction. Furthermore $a_{n+1} \in \mathfrak{a}_{n+1}$ but $a_{n+1} \notin \mathfrak{a}_n$, again from construction, so $\mathfrak{a}_n \subsetneq \mathfrak{a}_{n+1}$. Then we have a counterexample to i) and thus the assumption is wrong.

'ii) $\Rightarrow$ iii)'

We will show that for any set of ideals all chains have a maximal element, the result then follows by Zorn's lemma. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be a chain of ideals. Define $\mathfrak{a} = \bigcup \mathfrak{a}_i$. Then $\mathfrak{a}$ will be an ideal, this can be verified by checking the conditions in definition 2.11. By assumption $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$. For each x_i there must exist n_i such that $x_i \in \mathfrak{a}_{n_i}$. Then define $n = \max\{n_i\}$. It follows that $\mathfrak{a} = \mathfrak{a}_n$ and since $\mathfrak{a}_i \subseteq \mathfrak{a}$ for all i it follows that the chain has a maximal element, namely $\mathfrak{a}_n$. We are done.

□

Example 2.29. All fields are Noetherian, simply because the two ideals are $\langle 1 \rangle$ and $\langle 0 \rangle$ and so they indeed fulfill the second condition in definition 2.27.

In this paper we will, for the most parts, only need the ascending chain condition. Although we do not yet know which rings are Noetherian. The next theorem will tell us something about that.

Theorem 2.30. *If R is Noetherian then $R[x]$ is Noetherian.*

This theorem is very useful. For example it tells us that $\mathbb{Z}[x_1, \dots, x_n]$ is Noetherian, simply because $\mathbb{Z}$ is. Though we will only use it to conclude that $K[x, y]$ is Noetherian for any field K.

Proof. Take any ideal $\mathfrak{a} \subseteq R[x]$ and consider the set $\mathfrak{b} = \{a_n \mid \sum_{i=0}^{n} a_i x^i \in \mathfrak{a}\}$, that is the coefficients of the highest degree terms of all polynomials in $\mathfrak{a}$. This set will be an ideal in R. Since R is Noetherian there exists a finite set generating this ideal, call it $\{p_i\} \subseteq R$.

Clearly it must exist a set of polynomials in $\mathfrak{a}$ on the form $f_i = p_{r_i} x^{r_i} + \sum_{k=0}^{r_i - 1} a_k x^k$. Set $r = \max\{r_i\}$.

We can now see that in the same way $\mathfrak{b}_k = \{a_n \mid n \leq k \text{ and } \sum_{i=0}^{n} a_i x^i \in \mathfrak{a}\}$, or the leading coefficients of all polynomials with degree k or less, will be ideals. Then create $\{f_{k,j}\}$ from each $\mathfrak{b}_k$ in the same way as we made $\{f_i\}$ from $\mathfrak{b}$. Define

$$\mathfrak{a}' = \left\langle \{f_i\} \cup \{f_{k,j}\}_{k \leq r} \right\rangle .$$

Clearly, since $\{f_i\}$ and each of $\{f_{k,j}\}$ are finite, $\mathfrak{a}'$ is finitely generated. Moreover since $\{f_i\} \subseteq \mathfrak{a}$ and $\{f_{k,j}\} \subseteq \mathfrak{a}$ we have $\mathfrak{a}' \subseteq \mathfrak{a}$.

Left to show is $\mathfrak{a} \subseteq \mathfrak{a}'$. Take any element $a = \sum_{i=0}^{n} a_i x^i \in \mathfrak{a}$. If $n \geq r$ then there exits $u_i \in R[x]$ such that $\sum u_i f_i$ has leading coefficient a_n and degree n. So then $a \in \mathfrak{a}$ if and only if $a - \sum u_i f_i \in \mathfrak{a}$. But $a - \sum u_i f_i$ must by construction have degree smaller or equal to $n - 1$. So it is enough to show that all elements of degree $\leq r$ in $\mathfrak{a}$ lie within $\mathfrak{a}'$.

In the same way using elements from $\mathfrak{b}_k$ instead of $\mathfrak{b}$ we conclude that $\mathfrak{a}$ lie in $\mathfrak{a}'$ if and only if all ring elements in $\mathfrak{a}$ lies in $\mathfrak{a}'$. But using $\mathfrak{b}_0$ and the same method this is the same as saying that the additive inverse lies in both ideals which holds for all ideals and we are done.

□

This is all we need to know about Noetherian rings for now.

3. Numerical Semigroups

Most of the results in this section are taken from [2]. Recall definition 2.1 of a numerical semigroup from before. We will later apply these results to conclude some very nice results about the Ratliff-Rush ideals.

Throughout this chapter, unless specified otherwise, all numbers are natural numbers and all semigroups $S \subseteq \mathbb{N}$.

We will often assume $a_0 < a_1 < \cdots < a_n$ when we speak of a generating set $\{a_0, \dots, a_n\}$. We can do this without loss of generality since we only change the index.

Remark 3.1. The following is noteworthy before we begin:

(1) Any set $\{a_0, \dots, a_n\} \subseteq \mathbb{N}$ generates a numerical subgroup if and only if $\gcd(a_i) = 1$.
(2) Any semigroup generated by $\{a_0, \dots, a_n\} \subseteq \mathbb{N}$ is isomorphic to the numerical semigroup generated by $\{\frac{a_0}{\gcd(a_i)}, \dots, \frac{a_n}{\gcd(a_i)}\} \subseteq \mathbb{N}$.
(3) The semigroup generated by $\{a_0, \dots, a_n\}$ is the set of linear combinations $\sum_{i=0}^{n} \lambda_i a_i$, with $\lambda_i \in \mathbb{N}$. Notice though that the coefficients, the λ_i's, are not necessarily unique.
(4) All semigroups are finitely generated.

Definition 3.2 (Frobenius number)**.** Let $S = \langle a_0, \dots, a_n \rangle$ be a semigroup and let $h = \gcd(a_i)$. Define the Frobenious number $g(S)$ as the greatest multiple of h not in S.

If S is a numerical semigroup then $g(S)$ is simply the largest number in $\mathbb{N}$ not in S.

There does not exist a closed formula for the Frobenius number yet. Although there does exist a formula for the special case when $S = \langle a,\ b \rangle$ and $\gcd(a,b) = 1$, which is $g(s) = (a-1)(b-1) - 1$. This was shown in [7]. This can be generalized by using (2) in remark 3.1. Then we get:

$$g(\langle a,\ b \rangle) = \gcd(a,b)((\frac{a}{\gcd(a,\ b)} - 1)(\frac{b}{\gcd(a,\ b)} - 1) - 1).$$

It is worth noting that we do not have $g(\langle A \rangle) \geq g(\langle A \cup \{a\})$ unless $\gcd(A) = \gcd(A \cup \{a\})$. This can be seen in the next example.

Example 3.3. Let $A = \{4,\ 6\}$. Then $\gcd(4,\ 6) = 2$ and $\langle A \rangle = \{0\} \cup \{4,\ 6,\ 8, \dots\}$ so $g(\langle A \rangle) = 2$. But set $a = 9$. Then $\gcd(A \cup \{a\}) = \gcd(4,\ 6,\ 9) = 1$ and $\langle A \cup \{a\} \rangle = \{0,\ 4,\ 6\} \cup \{8,\ 9,\ 10, \dots\}$ so $g(\langle A \cup \{a\} \rangle) = 7$.

Now to something else.

Definition 3.4. Let $S = \langle a_0, \dots, a_n \rangle$. Then define $\lambda\ : S \to \mathbb{N}$ as

$$\lambda(s) = \min\{\sum \lambda_i \mid s = \sum \lambda_i a_i\}.$$

This λ can be seen as a measurement of how far from the generators s is.

Definition 3.5. For each semigroup $S = \langle a_0, \dots, a_n \rangle$ define

$$\Lambda(S) = \max\{\lambda(s) |\ s \in S, s \leq a_n + g(S)\}.$$

We will use this Λ as a "worst case scenario", but more to that later. Notice that both λ and therefore Λ depends on which generators we choose. This next example will show this.

Example 3.6. Let

$$S = \langle 3,\ 5,\ 6,\ 8,\ 9 \rangle = \{3\} \cup \{5,\ 6\} \cup \{8, 9, \dots\}$$

and

$$T = \langle 3,\ 5 \rangle = \{3\} \cup \{5,\ 6\} \cup \{8, 9, \dots\}.$$

Then $g(S) = g(T) = 7$ but $\Lambda(S) = 2$ and $\Lambda(T) = 4$. This despite that as semigroups S and T are the same.

Although this behavior might seem somewhat unorthodox it is a quite useful property. This way we can consider any set of generators instead of only some specific set. In some cases we are unable to choose generators and instead they are given to us as a necessity. Then a flexible definition, like this one, is easier to use.

Let us now see to some of the properties of λ.

Theorem 3.7. *Let $S = \langle a_0, \dots, a_n \rangle$. Moreover let $\alpha < 1$ and β be real nonnegative numbers. Then if $s \in S$ and $s \leq \alpha a_n l + \beta$ then $\lambda(s) \leq l$ if l large enough.*

Proof. For all $s > g(S)$ there exits a unique positive integer k such that $g(S) + a_n k + 1 \leq s \leq g(S) + a_n(k+1)$. From this we clearly have $\lambda(s) \leq \Lambda(S) + k \leq \Lambda(S) + \frac{s-g(S)-1}{a_n}$. Using $s \leq \alpha a_n l + \beta$ we get:

$$\lambda(s) \leq \Lambda(S) + \frac{s - g(S) - 1}{a_n} \leq \Lambda(S) + \alpha l + \frac{\beta - g(S) - 1}{a_n}.$$

If we now can find large enough l such that $\Lambda(S) + \alpha l + \frac{\beta - g(S) - 1}{a_n} \leq l$ we are done. This is true for all $l \geq \frac{a_n \Lambda + \beta - g(S) - 1}{a_n(1-\alpha)}$.

□

What this says is that $\lambda(s)$ cannot become too large in proportion to s. This is however in some sense the best one because the condition $\alpha < 1$ is necessary. This can be seen in remark 2.5 in [2], it is as follows:

Remark 3.8. Let $S = \langle 2,\ 5\rangle$. Choose $\alpha = 1$ and $\beta = 4$. Then for $s = 5l + 4 = 5l + 2 \cdot 2 \in S$ clearly $\lambda(s) = l + 2$ for all l. So $\alpha < 1$ is necessary.

Though the following result tells us that all $\alpha < 1$ are in some sense 'relevant'.

Theorem 3.9. *Let* $S = \langle a_0, \ldots, a_n\rangle$. *Then* $\lim_{s\to\infty} \frac{s}{\lambda(s)} = a_n$.

Proof. For all $s > g(S)$ there exits a unique non-negative integer k such that $g(S) + a_n k + 1 \leq s \leq g(S) + a_n(k+1)$. Then we have $k \leq \lambda(s) \leq \Lambda(S) + k$. Putting this together we get:

$$\frac{g(S) + a_n k + 1}{\Lambda(S) + k} \leq \frac{s}{\lambda(s)} \leq \frac{g(S) + a_n k + a_n}{k}.$$

From this the limit follows directly since $k \to \infty$ as $s \to \infty$.

□

Our work thus far leads up to the following theorem, which is why we have a chapter about numerical semigroups.

Theorem 3.10. *If* $S = \langle a_0, \ldots, a_n\rangle$ *and* $T = \langle b_0, \ldots, b_n\rangle$ *such that* $a_0 = b_n = 0$ *and* $a_i + b_i = d$ *for all* i. *Then for some fixed* β *and all large enough* l *the following holds:*

if $s \in S$ *and* $s \leq \alpha a_n l + \beta \leq dl$, *then there exist* $\lambda_0, \ldots, \lambda_n$ *such that* $s = \sum \lambda_i a_i \in S$ *and* $\sum \lambda_i = 0$. *Furthermore* $dl - s = \sum \lambda_i b_i \in T$.

Proof. By theorem 3.7, if l is large enough then, since $s \leq \alpha a_n l + \beta$, we have $\lambda(s) \leq l$. By definition of λ and the fact that $a_0 = 0$ there exist $\lambda_1, \ldots, \lambda_n$ such that $s = \sum \lambda_i a_i$ with $\sum \lambda_i \leq l$. Then let $\lambda_0 = l - \sum \lambda_i$. Then $s = \sum_{i=0}^{n} \lambda_i a_i$ with $\sum \lambda_i = l$. By assumptions we get $dl - s = \sum d\lambda_i - \sum \lambda_i a_i = \sum \lambda_i b_i \in T$.

□

4. Ratliff-Rush Ideal

The study of powers of ideals can be used in a couple of different areas within mathematics. How these powers behave in general is something we know very little about. Let $\mathfrak{a}$ be a regular ideal, that is contaning a non-zero divisor. The union $\widetilde{\mathfrak{a}} = \bigcup_{l\in\mathbb{Z}^+}(\mathfrak{a}^{l+1} : \mathfrak{a}^l)$ was studied by Ratliffe and Rush in [5] and was found to have some nice properties. All ideals in this chapter are assumed to be regular.

4.1. **Ratliff-Rush Ideals in Noetherian Rings.** We will now repeat the important results from [5].

Definition 4.1 (The Ratliff-Rush ideal associated to $\mathfrak{a}$)**.** Let $\mathfrak{a}$ be a regular ideal. Define the Ratliff-Rush ideal associated to $\mathfrak{a}$ as $\widetilde{\mathfrak{a}} = \bigcup \mathfrak{a}^{l+1} : \mathfrak{a}^l$.

Remark 4.2. An ideal is said to be Ratliff-Rush if $\mathfrak{a} = \widetilde{\mathfrak{a}}$.

Theorem 4.3. *In Noetherian rings,* $\widetilde{\mathfrak{a}} = \bigcup_{l \in \mathbb{Z}^+} (\mathfrak{a}^{l+1} : \mathfrak{a}^l)$ *is the largest ideal such that* $\mathfrak{a}^n = \widetilde{\mathfrak{a}}^n$ *for all large enough* n*.*

Theorem 4.3 can be found as Theorem (2.1) in [5]. Before we sketch a proof we will study some properties of $\widetilde{\mathfrak{a}} = \bigcup_{l \in \mathbb{Z}^+} (\mathfrak{a}^{l+1} : \mathfrak{a}^l)$. First of all

$$(\mathfrak{a}^2 : \mathfrak{a}^1) \subseteq (\mathfrak{a}^3 : \mathfrak{a}^2) \subseteq \cdots \subseteq (\mathfrak{a}^{l+1} : \mathfrak{a}^l) \subseteq (\mathfrak{a}^{l+2} : \mathfrak{a}^{l+1}) \subseteq \dots$$

This follows from $x\mathfrak{a}^l \subseteq \mathfrak{a}^{l+1} \Rightarrow x\mathfrak{a}^{l+1} \subseteq \mathfrak{a}^{l+2}$. Since the ring is Noetherian there exists n such that $\mathfrak{a}^{n+1} : \mathfrak{a}^n = \mathfrak{a}^{m+1} : \mathfrak{a}^m$ for all $m \geq n$. It follows that $\widetilde{\mathfrak{a}} = \mathfrak{a}^{n+1} : \mathfrak{a}^n$.

Definition 4.4 (Ratliff-Rush reduction number)**.** The Ratliff-Rush reduction number is defined as

$$r(\mathfrak{a}) = \min\{n \mid \widetilde{\mathfrak{a}} = \mathfrak{a}^{n+1} : \mathfrak{a}^n\}.$$

Furthermore $\mathfrak{a} \subseteq \widetilde{\mathfrak{a}}$. This follows trivially since $\mathfrak{a} \subseteq \mathfrak{a}^2 : \mathfrak{a}^1$. From this and theorem 2.21 ii) we get

$$\mathfrak{a}^k \subseteq \mathfrak{a}^{k-1}(\widetilde{\mathfrak{a}})^1 \subseteq \cdots \subseteq \mathfrak{a}^1(\widetilde{\mathfrak{a}})^{k-1} \subseteq (\widetilde{\mathfrak{a}})^k$$

for all k. We will be using this when showing 4.3. In fact, what we will be showing is that for large enough k

$$\mathfrak{a}^k \subseteq \mathfrak{a}^{k-1}(\widetilde{\mathfrak{a}})^1 \subseteq \cdots \subseteq \mathfrak{a}^1(\widetilde{\mathfrak{a}})^{k-1} \subseteq (\widetilde{\mathfrak{a}})^k \subseteq \mathfrak{a}^k. \tag{4.1}$$

From this $\mathfrak{a}^k = (\widetilde{\mathfrak{a}})^k$ follows. Now to proving equation 4.1. We will as earlier mentioned only sketch some parts of the proof. This because it contains some concepts we will not need for anything other than this proof and those concepts would deserve their own chapter. For more details about these parts of the theorem see proof of theorem (2.1) in [5] and, (22.1) and (22.2) in [4].

Proof. What really is left to show of equation 4.1 is $(\widetilde{\mathfrak{a}})^k \subseteq \mathfrak{a}^k$ for large enough k.

There will exist two positive natural numbers h and c and a non-zero divisor $a \in \mathfrak{a}^h$ such that $\mathfrak{a}^k : aR \cap \mathfrak{a}^c = \mathfrak{a}^{k-h}$ for all large enough k. Or put in another way, a superficial regular element a of $\mathfrak{a}$ of degree h. This is ensured by (22.1) and (22.2) in [4].

Moreover, by (3.12) in [4], for even larger n, say $n \geq n^*$, we will have $\mathfrak{a}^n : aR = \mathfrak{a}^{n-h}$. Using the fact that $a \in \mathfrak{a}^h$ we have $aR \subseteq \mathfrak{a}^h$ and therefore

$$\mathfrak{a}^{n^*-h+i} \subseteq \mathfrak{a}^{n^*-h+i+1} : \mathfrak{a} \subseteq \cdots \subseteq \mathfrak{a}^{n^*-h+i+h} : \mathfrak{a}^h \subseteq \mathfrak{a}^{n^*+i} : aR = \mathfrak{a}^{n^*-h+i}$$

for all $i \in \mathbb{N}$.

From this it follows that $\mathfrak{a}^{k+1} : \mathfrak{a} = \mathfrak{a}^k$ for all $k \geq n^* - h$. Using that $(\mathfrak{a} : \mathfrak{b}) : \mathfrak{c} = \mathfrak{a} : \mathfrak{b}\mathfrak{c}$ we get

$$\mathfrak{a}^{k+m} : \mathfrak{a}^m = (\mathfrak{a}^{k+m} : \mathfrak{a}) : \mathfrak{a}^{m-1} = \mathfrak{a}^{k+m-1} : \mathfrak{a}^{m-1} = \cdots = \mathfrak{a}^{k+1} : \mathfrak{a} = \mathfrak{a}^k$$

for all $m \geq 1$. Let l be a number $\geq k$ such that $\widetilde{\mathfrak{a}} = \mathfrak{a}^{l+1} : \mathfrak{a}^l$. Using the fact that $(\mathfrak{a} : \mathfrak{b})(\mathfrak{c} : \mathfrak{d}) \subseteq \mathfrak{ac} : \mathfrak{bd}$ we now get $\mathfrak{a}^l \subseteq \widetilde{\mathfrak{a}}^l \subseteq (\mathfrak{a}^{l+1} : \mathfrak{a}^l)^l \subseteq \mathfrak{a}^{l+l^2} : \mathfrak{a}^{l^2} = \mathfrak{a}^l$. So for large enough l we have $\mathfrak{a}^l = \widetilde{\mathfrak{a}}^l$.

For the maximality, assume $\mathfrak{a}^k = \mathfrak{b}^k$ for all $k \geq k'$. Then we must have $(\mathfrak{a} + \mathfrak{b})^{2k'} = \sum \mathfrak{a}^i \mathfrak{b}^{2k'-i} \subseteq \mathfrak{a}^{2k'} = \mathfrak{b}^{2k'}$ since each term must be either $\mathfrak{a}^{2k'}$ (the ones where $2k' - i \geq k'$) or $\mathfrak{b}^{2k'}$ (the ones where $i \geq k'$) and $\mathfrak{a}^k = \mathfrak{b}^k$ for all $k \geq k'$. So then for all $b \in \mathfrak{b}$ and $n \geq 2k'$ we must have $\mathfrak{a}^n \subseteq \langle \mathfrak{a},\ b \rangle^n \subseteq (\mathfrak{a} + \mathfrak{b})^n = \mathfrak{a}^n$. This gives us $b\mathfrak{a}^{n-1} \subseteq \mathfrak{a}^n$ and hence $b \in \mathfrak{a}^n : \mathfrak{a}^{n-1} \subseteq \widetilde{\mathfrak{a}}$ and by symmetry the result follows.

□

Corollary 4.5. *If we have $\mathfrak{a}^k = \widetilde{\mathfrak{a}}^k$ for some k, then $\mathfrak{a}^{k'} = \widetilde{\mathfrak{a}}^{k'}$ for all $k' \geq k$.*

Proof. By equation 4.1 we must then have $\mathfrak{a}^k = \mathfrak{a}^{k-l}\widetilde{\mathfrak{a}}^l$ for all $l \leq k$. So $\widetilde{\mathfrak{a}}^{k+1} = \widetilde{\mathfrak{a}}\mathfrak{a}^k = \widetilde{\mathfrak{a}}\widetilde{\mathfrak{a}}^{k-1}\mathfrak{a} = \widetilde{\mathfrak{a}}^k\mathfrak{a} = \mathfrak{a}^{k+1}$. The result follows by induction over k.

□

From this we get an upper limit to the Ratliff-Rush reduction number, namely as:

Corollary 4.6. *If we have $\mathfrak{a}^k = \widetilde{\mathfrak{a}}^k$ for some k, then $r(\mathfrak{a}) \leq k$.*

Proof. For all $l \geq k$ we have $\widetilde{\mathfrak{a}} \subseteq \widetilde{\mathfrak{a}}^{l+1} : \widetilde{\mathfrak{a}}^l = \mathfrak{a}^{l+1} : \mathfrak{a}^l \subseteq \widetilde{\mathfrak{a}}$.

□

Though this inequality does not tell us much, for how will we know when $\mathfrak{a}^k = \widetilde{\mathfrak{a}}^k$ without calculating $\widetilde{\mathfrak{a}}$ and thus in turn calculating $r(\widetilde{\mathfrak{a}})$? A couple of other things can also be proven from theorem 4.3, here are a couple. Most of them are from Remark (2.3) in [5].

Corollary 4.7. *The following holds:*

i) The associated Ratliff-Rush ideals are Ratliff-Rush, that is $\widetilde{\widetilde{\mathfrak{a}}} = \widetilde{\mathfrak{a}}$.
ii) Any high enough power of an ideal is Ratliff-Rush, that is $\widetilde{\mathfrak{a}^n} = \mathfrak{a}^n$ for large enough n.
iii) If $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \widetilde{\mathfrak{a}}$ then $\widetilde{\mathfrak{b}} = \widetilde{\mathfrak{a}}$.
iv) In general $\widetilde{\mathfrak{a}}\widetilde{\mathfrak{b}} \neq \widetilde{\mathfrak{a}\mathfrak{b}}$.

We will later see examples of iv) when we actually can calculate $\widetilde{\mathfrak{a}}$. Because even if we have theorem 4.3, $\mathfrak{a} : \mathfrak{b}$ is not always easily calculated. Even if it were, $\mathfrak{a}^{n+1} : \mathfrak{a}^n = \mathfrak{a}^{n+2} : \mathfrak{a}^{n+1}$ does not imply $\mathfrak{a}^{n+2} : \mathfrak{a}^{n+1} = \mathfrak{a}^{n+3} : \mathfrak{a}^{n+2}$, so calculating $\widetilde{\mathfrak{a}}$ is not something easily done in general. But with the right equipment it can sometimes be done.

4.2. **Ideals in $K[x, y]$ Generated by Monomials of the Same Degree.** Here we will continue to study results from [2] and also try to tie the knot together. We will see how we can calculate the Ratliff-Rush ideal associated to $\mathfrak{a}$ if $\mathfrak{a}$ is on a certain form.

Let $\mathfrak{a}$ be generated by monomials of the same degree. That is $\mathfrak{a} = \langle x^{a_i} y^{b_i} \rangle_{i=0}^n$ where $a_i + b_i = d$ for all i. As discussed previously we can assume $a_i > a_{i+1}$ and $b_i < b_{i+1}$. Moreover equation 2.1 shows that it suffices to study $\langle x, y \rangle$-primary ideals, or, equivalently, ideals such that $a_0 = b_n = 0$. With the *degree of the monomial* we will refer to d. We will only work with ideals on this form in this section.

For any ideal $\mathfrak{a} = \langle x^{a_i}y^{b_i}\rangle$ define the two semigroups $S = \langle a_i\rangle$ and $T = \langle b_i\rangle$. We can see that

$$\mathfrak{a}^l = \langle \prod_{\sum \lambda_i = l} (x^{a_i}y^{b_i})^{\lambda_i}\rangle = \langle \prod_{\sum \lambda_i = l} (x^{\sum \lambda_i a_i} y^{\sum \lambda_i b_i})\rangle$$

Or put in another way:

$$\mathfrak{a}^l = \langle x^s y^t \mid s \in S,\ t \in T \text{ such that } \lambda(s) \leq l,\ \lambda(t) \leq l \text{ and } s + t = dl\rangle$$

The next corollary will more or less say that for large enough l the conditions $\lambda(s) \leq l$ and $\lambda(t) \leq l$ are superfluous. This way, high enough powers of $\mathfrak{a}$ will reach a 'maximal state'.

Theorem 4.8. *For sufficienly large l the following holds:*

(1) If $s \in S$, $s \leq u$ and $s + u \geq dl$ for some $u \in \mathbb{N}$, then $x^s y^u \in \mathfrak{a}^l$
(2) If $t \in T$, $t \leq v$ and $t + v \geq dl$ for some $v \in \mathbb{N}$, then $x^v y^t \in \mathfrak{a}^l$

Proof. We will show them one at the time.

(1) Since $s+u \geq dl$ there exists $j,\ b \in \mathbb{N}$ such that $d(l+j) \leq s+u = d(l+j)+b \leq d(l+j+1)-1$. We want to show that $x^s y^u \in \mathfrak{a}^{l+j} \subseteq \mathfrak{a}^l$.
Since $s + u \leq d(l + j + 1) - 1$ and $s \leq u$ we have $s \leq \frac{d(l+j)}{2} + \frac{d-1}{2}$. So the prerequisites for theorem 3.10 are fulfilled. So if $l + j$ are large enough, or just l since $l \leq l + j$, we can write $s = \sum \lambda_i a_i$ and $u = b + \sum \lambda_i b_i$ with $\sum \lambda_i = l + j$. Therefore $x^s y^u = y^b \prod (x^{a_i} y^{b_i})^{\lambda_i} \in \mathfrak{a}^{l+j} \subseteq \mathfrak{a}^l$.
(2) This part is proved in the same way.

□

Corollary 4.9. *For large enough l we have $\mathfrak{a}^l = \langle x^s y^t \mid s \in S,\ t \in T$ such that $s + t = dl\rangle$.*

Proof. Clearly $\mathfrak{a}^l = \langle x^s y^t \mid s \in S,\ t \in T \text{ such that } \lambda(s) \leq l,\ \lambda(t) \leq l \text{ and } s + t = dl\rangle \subseteq \langle x^s y^t \mid s \in S,\ t \in T \text{ such that } s + t = dl\rangle$.

The other inclusion follows from the fact that if $s + t = dl$ then either $s \leq t$ or $t \leq s$. The rest follows from the previous theorem.

□

This would be the 'maximal state' we mentioned earlier. Because clearly no other elements than the ones on the form $x^s y^t$ such that $s \in S,\ t \in T$ and $s + t = dl$ will ever be a minimal generator to $\mathfrak{a}^l$ and with this all of them will.

Moreover, when the power of an ideal starts behaving this way it will, possibly a bit later, become Ratliff-Rush. Although we still need a few more theorems to prove this.

So as we can see, how powers of ideals behave depends on how their respective semigroups behave. With this in mind, let us define some ideals.

Definition 4.10. For an ideal $\mathfrak{a} = \langle x^{a_i}y^{b_i}\rangle$ let $S = \langle a_i\rangle$ and $T = \langle b_i\rangle$. Define $\mathfrak{a}_S = \langle x^s y^{d-s} \mid s \in S,\ s \leq d\rangle$ and $\mathfrak{a}_T = \langle x^{d-t} y^t \mid t \in T,\ t \leq d\rangle$ where d is the degree of $\mathfrak{a}$.

The next theorem will prove their relevance.

Theorem 4.11. *For sufficiently large l and some ideal $\mathfrak{a}_{M,l}$ we have*

$$\mathfrak{a}^l = (x^{d(l-1)})\mathfrak{a}_T + (y^{d(l-1)})\mathfrak{a}_S + (x^d y^d)\mathfrak{a}_{M,l}$$

Proof. First of, let $l \geq \max\{g(S),\ g(T)\}$. Then any element $y^{d(l-1)}(x^s y^{d-s}) = x^s y^{dl-s}$ with $s \leq S$ generates $\mathfrak{a}^l$ if and only if $s \in S$ and $\lambda(s) \leq l$. By our choice of l we have $\lambda(s) \leq l$ for all $s \leq d$. So by definition of $\mathfrak{a}_S$ we have that $x^s y^t$ with $s \leq d$ lies in $y^{d(l-1)}\mathfrak{a}_S$ if and only if it lies in $\mathfrak{a}^l$.

Using symmetry, $x^s y^t$ with $t \leq d$ lies in $x^{d(l-1)}\mathfrak{a}_T$ if and only if it lies in $\mathfrak{a}^l$.

The only elements left are $x^{d+s}y^{d+t}$. Choose $\mathfrak{a}_{M,l} = \mathfrak{a}^l : x^d y^d$ and the result follows.

□

Even though we now can describe powers of $\mathfrak{a}$ we still do not know how anything about how $\widetilde{\mathfrak{a}}$ looks like. The next theorem takes care of that for us. It allows us to actually compute the Ratliff-Rush associated ideal, and this relatively easy.

Theorem 4.12. *Let $\mathfrak{a}$, $\mathfrak{a}_S$ and $\mathfrak{a}_T$ be as in definition 4.10. Then the Ratliff-Rush ideal associated to $\mathfrak{a}$ is*

$$\widetilde{\mathfrak{a}} = \mathfrak{a}_S \cap \mathfrak{a}_T.$$

Proof. Since $\widetilde{\mathfrak{a}} = \mathfrak{a}^{l+1} : \mathfrak{a}^l$ for large enough n it is enough to show that $\mathfrak{a}_S \cap \mathfrak{a}_T = \mathfrak{a}^{l+1} : \mathfrak{a}^l$ for large enough l. Since $\mathfrak{a}$ monomial $\mathfrak{a}^l$ is monomial and by theorem 2.25 any polynomial $\sum k_i x^{a_i} y^{b_i}$ lies in $\mathfrak{a}^l$ if and only if each term $kx^a y^b$ lies in $\mathfrak{a}^l$. So it enough to consider monomial elements m.

If $m \in \mathfrak{a}_S \cap \mathfrak{a}_T$ then by definition there exist $s' \in S$, $t'' \in T$ and $m',\ m'' \in K[x,y]$ such that $m = m' x^{s'} y^{d-s'} = m'' x^{d-t''} y^{t''}$. By corollary 4.9 $\mathfrak{a}^l$ is generated by $x^s y^t$ such that $s + t = dl$, if l is large enough. We want to show that for all of these generators $mx^s y^t \in \mathfrak{a}^{l+1}$ and thus $m \in \mathfrak{a}^{l+1} : \mathfrak{a}^l$. Since $s + t = dl$ either $s \leq \frac{dl}{2}$ or $t \leq \frac{dl}{2}$.

Assume $s \leq \frac{dl}{2}$. Then by the first equality $mx^s y^t = m' x^{s+s'} y^{d(l+1)-(s+s')}$. Since $s \leq \frac{dl}{2}$ and $s' \leq d$ we get $s+s' \leq \frac{dl}{2} + d = \frac{d(l+1)}{2} + \frac{d}{2}$. Using theorem 3.10 we get that if l is indeed large enough we have $s + s' = \sum \lambda_i a_i$ and $d(l+1) - (s+s') = \sum \lambda_i b_i$ where $\sum \lambda_i = l+1$. Using these equalities we obtain $mx^s y^t = \prod (x^{a_i} y^{b_i})^{\lambda_i} \in \mathfrak{a}^{l+1}$. If $t \leq \frac{dl}{2}$ we need only to use the second equality ($m = m'' x^{d-t''} y^{t''}$) and we obtain the same result. Hence $\mathfrak{a}_S \cap \mathfrak{a}_T \subseteq \mathfrak{a}^{l+1} : \mathfrak{a}^l$ for all large enough l.

The other way around assume $m \notin \mathfrak{a}_S \cap \mathfrak{a}_T$, that is $m \notin \mathfrak{a}_S$ or $m \notin \mathfrak{a}_T$. Begin with $m \notin \mathfrak{a}_S$. Then $my^{dl} \notin y^{dl}\mathfrak{a}_S$ and by theorem 4.11 $my^{dl} \notin \mathfrak{a}^{l+1}$ and hence $m \notin \mathfrak{a}^{l+1} : \mathfrak{a}^l$ since $y^{dl} \in \mathfrak{a}^l$. If $m \notin \mathfrak{a}_T$ then in the same way $mx^{dl} \notin \mathfrak{a}^{l+1}$ and $m \notin \mathfrak{a}^{l+1} : \mathfrak{a}^l$, which completes the proof.

□

Remark 4.13. No step in calculating $\mathfrak{a}_S \cap \mathfrak{a}_T$ is complicated. The author has in fact made a program available at [6]. This way, producing examples is very easy and it is even possible to check some specific property for all relevant monomial ideals with $d \leq 20$. Although it does take some time.

5. Examples and Various Properties

Now that we know how to calculate the associated Ratliff-Rush ideals for some ideals we can produce examples and see how they behave. We will begin with some examples of earlier mentioned properties and then look closer at some specific classes and state a few things about them.

All Ideals are presumed to have the same form as in the previus chapter. That is $\mathfrak{a}$ is generated by monomials $\{x^{s_i}y^{t_i}\}_{i=0}^{n}$ such that $s_n = t_0 = 0$, $s_i + t_i = d$ and $s_i > s_{i+1}$. Moreover we have the related semigroups $S = \langle s_i \rangle$ and $T = \langle t_i \rangle$.

5.1. Examples.

Here we will take examples of earlier mentioned properties.

Example 5.1. Let $\mathfrak{a} = \langle x^{10},\ x^8y^2,\ x^5y^5,\ x^3y^7,\ y^{10}\rangle$. Then $\mathfrak{a}^2 : \mathfrak{a}^1 = \mathfrak{a}^3 : \mathfrak{a}^2 = \langle x^{10},\ x^8y^2,\ x^7y^4,\ x^5y^5,\ x^3y^7,\ y^{10}\rangle$ but $\mathfrak{a}^4 : \mathfrak{a}^3 = \langle x^{10},\ x^8y^2,\ x^6y^4,\ x^5y^5,\ x^3y^7,\ y^{10}\rangle$. Thus $\mathfrak{a}^{n+1} : \mathfrak{a}^n = \mathfrak{a}^{n+2} : \mathfrak{a}^{n+1}$ does not imply $\mathfrak{a}^{n+2} : \mathfrak{a}^{n+1} = \mathfrak{a}^{n+3} : \mathfrak{a}^{n+2}$.

Example 5.2. Let $\mathfrak{a} = \langle x^5,\ x^4y,\ y^5\rangle$ and $\mathfrak{b} = \langle x^5,\ x^3y^2,\ y^5\rangle$.

We see that $\mathfrak{a} = \widetilde{\mathfrak{a}} = \langle x^5,\ x^4y,\ y^5\rangle$, $\mathfrak{b} = \widetilde{\mathfrak{b}} = \langle x^5,\ x^3y^2,\ y^5\rangle$ and so $\widetilde{\mathfrak{a}} \cdot \widetilde{\mathfrak{b}} = \mathfrak{a}\mathfrak{b} = \langle x^{10},\ x^9y^1,\ x^8y^2,\ x^7y^3,\ x^5y^5,\ x^4y^6,\ x^3y^7,\ y^{10}\rangle$. But on the other hand $\widetilde{\mathfrak{a}\mathfrak{b}} = \langle x^{10},\ x^9y,\ x^8y^2,\ x^7y^3,\ x^6y^4,\ x^5y^5,\ x^4y^6,\ x^3y^7,\ y^{10}\rangle$. So in general $\widetilde{\mathfrak{a}} \cdot \widetilde{\mathfrak{b}} \neq \widetilde{\mathfrak{a}\mathfrak{b}}$.

Example 5.3. Let $\mathfrak{a} = \langle x^5,\ y^5\rangle$, $\mathfrak{b} = \langle x^4y,\ xy^4\rangle = xy\langle x^3,\ y^3\rangle$. Then we have $\widetilde{\mathfrak{a}} = \mathfrak{a}$ and $\widetilde{\mathfrak{b}} = (\widetilde{xy\langle x^3,\ y^3\rangle}) = xy(\widetilde{\langle x^3,\ y^3\rangle}) = xy\langle x^3,\ y^3\rangle = \mathfrak{b}$. So $\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{b}} = \mathfrak{a} + \mathfrak{b} = \langle x^5,\ x^4y,\ xy^4,\ y^5\rangle$, but $\widetilde{\mathfrak{a} + \mathfrak{b}} = \langle x,\ y\rangle^5$. So in general $\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{b}} \neq \widetilde{\mathfrak{a} + \mathfrak{b}}$.

Example 5.4. If l is large enough $\mathfrak{a}^l$ is Ratliff-Rush.

This we already know from corollary 4.7 but we will show it using some of the new methods presented to us in the last chapter.

Proof. Let l be large enough such that $dl > g(S) + g(T)$ and $\mathfrak{a}^l$ is on the form in corollary 4.9. Then ${\mathfrak{a}^l}_S = \mathfrak{a}^l + x^{g(S)+1}\langle x,\ y\rangle^{dl-g(S)-1}$. Indeed it must be so since for all $x^sy^t \in \langle x,\ y\rangle^{dl-g(S)-1}$ we have $x^{g(S)+1}x^sy^t = x^{g(S)+s+1}y^t \in {\mathfrak{a}^l}_S$ because $g(S) + s + 1 \in S$.

The other way around we see that $\mathfrak{a}^l_S = \langle x^sy^{dl-s} \mid s \in S\rangle$. Therefore for any x^sy^t generating $\mathfrak{a}^l_S$, if $s \geq g(S) + 1$ $x^sy^t \in x^{g(S)+1}\langle x,\ y\rangle^{dl-g(S)-1}$ an if $s \leq g(s)$ then $t = dl - s \geq g(T) + 1$ and so by corollary 4.9 $x^sy^t \in \mathfrak{a}^l$ since we assumed l was large enough for corollary 4.9 to hold.

Using this consider $x^sy^t \in \widetilde{\mathfrak{a}^{\mathfrak{l}}} = {\mathfrak{a}^{\mathfrak{l}}}_S \cap {\mathfrak{a}^{\mathfrak{l}}}_T$ we get three cases,

i) $s \leq g(S)$,
ii) $t \leq g(T)$,
iii) $g(S) + 1 \leq s$ and $g(T) + 1 \leq t$.

Let us do them one by one.

i) We get $x^sy^t \in \mathfrak{a}^l_S = \mathfrak{a}^l + x^{g(S)+1}\langle x,\ y\rangle^{dl-g(S)-1}$. Since $s \leq g(S)$ we clearly have $x^sy^t \notin x^{g(S)+1}\langle x,\ y\rangle^{dl-g(S)-1}$ and theorem 2.26 we get $x^sy^t \in \mathfrak{a}^l$.
ii) Follows by symmetry.
iii) If $s + t \geq dl$ we have $s + t \geq dl > g(S) + g(T)$ and so there exists s' and t' in $\mathbb{N}$ such that $s' + t' = dl$ with $s \geq s' > g(S)$ and $t \geq t' > g(T)$. Clearly $x^{s'}y^{t'} \in \mathfrak{a}^l$ since $\mathfrak{a}^l$ is on the form of corollary 4.9. If $s + t \leq dl$ clearly $x^sy^t \notin \mathfrak{a}^l_S$ since $\mathfrak{a}^l_S$ is generated by monomials of degree dl.

It follows that $\widetilde{\mathfrak{a}^l} \subseteq \mathfrak{a}^l$ and we are done.

□

5.2. **Some Interesting Properties.** We will now look closer at some specific classes of ideals and prove some things about them. We still assume that all ideals are in $K[x,\ y]$, monomial and the generators are on the form $x^{s_i}y^{t_i}$ where $s_n = t_0 = d$, $s_i + t_i = d$ and $s_i > s_{i+1}$.

Theorem 5.5. *Let $\mathfrak{a} = \langle x^{s_i}y^{t_i}\rangle$ be an ideal of degree d. Let $t_{min} = \min\{t_i > 0\}$. If for all i and all j either $s_i+s_j > d-t_{min}$ or there exists k such that $s_i+s_j = s_k \in S$, then $\mathfrak{a}$ is Ratliff-Rush.*

This is a slightly stronger version of corollary 3.8 in [2].

Proof. Clearly any element on the form $x^sy^{t+t_{min}}$ is in $\mathfrak{a}$ if and only if it is in $\mathfrak{a}_S$, that is $\mathfrak{a} : y^{t_{min}} = \mathfrak{a}_S : y^{t_{min}}$. But since $\mathfrak{a} \subseteq \mathfrak{a}_T$ we also have $\mathfrak{a} : y^{t_{min}} \subseteq \mathfrak{a}_T : y^{t_{min}}$. Thus we get

$$\begin{aligned}\widetilde{\mathfrak{a}} : y^{t_{min}} &= (\mathfrak{a}_S \cap \mathfrak{a}_T) : y^{t_{min}}\\ &= \mathfrak{a}_S : y^{t_{min}} \cap \mathfrak{a}_T : y^{t_{min}}\\ &= \mathfrak{a} : y^{t_{min}} \cap \mathfrak{a}_T : y^{t_{min}}\\ &= \mathfrak{a} : y^{t_{min}}.\end{aligned}$$

Therefore any element $x^sy^{t+t_{min}}$ is in $\mathfrak{a}$ if and only if it is in $\widetilde{\mathfrak{a}}$. All other elements x^sy^t must have $t < t_{min}$. Then they are in $\mathfrak{a}_T$ if and only if $s \geq d$, but all such elements must be in $\mathfrak{a}$ since $x^d \in \mathfrak{a}$ and therefore also in $\widetilde{\mathfrak{a}}$. So in conclusion any element x^sy^t is in $\mathfrak{a}$ if and only if they are in $\widetilde{\mathfrak{a}}$.

□

Example 5.6. Let $\mathfrak{a} = \langle x^7,\ x^4y^3,\ x^3y^4,\ x^2y^5,\ y^7\rangle$. Then we see that $\mathfrak{a}$ fufills the prerequsites of theorem 5.2.

We have $t_{min} = 3$, $\mathfrak{a}_S = \langle x^7,\ x^6y,\ x^5y^2,\ x^4y^3,\ x^3y^4,\ x^2y^5,\ y^7\rangle$ and $\mathfrak{a}_T = \langle x^7,\ x^4y^3,\ x^3y^4,\ x^2y^5,\ xy^6,\ y^7\rangle$.

In $\widetilde{\mathfrak{a}} = \mathfrak{a}_S \cap \mathfrak{a}_T$ we clearly do not have any elements on x^sy^t where $t \leq t_{min}$ and $s \leq 7$ since none of these elements are present in $\mathfrak{a}_T$. The rest of the elements are shared between $\mathfrak{a}$ and $\mathfrak{a}_S$ thus we get $\widetilde{\mathfrak{a}} = \mathfrak{a}_S \cap \mathfrak{a}_T \subseteq \mathfrak{a}$. So $\mathfrak{a}$ is Ratliff-Rush.

Remark 5.7. Since we assumed that s_i and therefore t_i where ordered $t_{min} = t_1$ and $s_{min} = s_{n-1}$. Hence we will start naming them as such.

Corollary 5.8. *Let $\mathfrak{a}$ be as in the previus theorem. If $2s_{n-1} + t_1 \geq d$ then $\mathfrak{a}$ is Ratliff-Rush.*

Proof. We have $s_i + s_j + t_1 \geq 2s_{n-1} + t_1 \geq d$ for all $s_i,\ s_j \in S\backslash\{0\}$. If $s_j = 0$ then $s_i + s_j = s_i \in S$. The rest follows from theorem 5.5.

□

Corollary 5.9. *If $\mathfrak{a} = \langle x^d,\ x^ay^{d-a},\ y^d\rangle$, then $\mathfrak{a}$ is Ratliff-Rush.*

Proof. We have $2a + (d - a) = d + a \geq d$ and thus corollary 5.8 gives us the result.

□

We will now look closer on the generators for $\widetilde{\mathfrak{a}}$ and on how the generators for $\mathfrak{a}$ affect them.

Example 5.10. Let $\mathfrak{a} = \langle x^7,\ x^5y^2,\ x^2y^5,\ y^7\rangle$. Then we get $S = T = \langle 2,\ 5,\ 7\rangle = \{0\} \cup \{2\} \cup \{4,\ 5, \dots\}$ and therefore $\mathfrak{a}_S = \langle x^7,\ x^6y,\ x^5y^2,\ x^4y^3,\ x^2y^5,\ y^7\rangle$ and $\mathfrak{a}_T = \langle x^7,\ x^5y^2,\ x^3y^4,\ x^2y^5,\ xy^6,\ y^7\rangle$. From theorem 4.12 we then obtain $\widetilde{\mathfrak{a}} = \mathfrak{a}_S \cap \mathfrak{a}_T = \langle x^7,\ x^5y^2,\ x^4y^4,\ x^2y^5,\ y^7\rangle = \mathfrak{a} + \langle x^4y^4\rangle$. We see that $\widetilde{\mathfrak{a}}$ is not necessarily generated by monomials of the same degree.

With this in mind, define $m(\mathfrak{a})$ as the difference between the maximal degree of a minimal generator for $\widetilde{\mathfrak{a}}$ and the degree of $\mathfrak{a}$. In the previous example we then have $m(\mathfrak{a}) = 1$.

In fact the previous example is the ideal of the smallest degree having $m(\mathfrak{a}) \geq 1$, that is $\widetilde{\mathfrak{a}}$ has minimal generators of greater than the degree of $\mathfrak{a}$. This is easily checked with a computer. Though something more interesting is that it is the first one in a class of ideals where the associated Ratliff-Rush ideals are not generated by monomials of the same degree, as we will see in the next example.

Example 5.11. Let $\mathfrak{a} = \langle x^d,\ x^{d-a}y^a,\ x^ay^{d-a},\ y^d\rangle$ be an ideal such that $\gcd(a,d) = 1$. Let n be the unique integer such that $an < d < a(n+1)$. Then if $n \geq 3$ then $\widetilde{\mathfrak{a}}$ has a minimal generator of degree $a(n+1)$ and thus $m(\mathfrak{a}) = a(n+1) - d$.

Proof. We see that $S = \langle a,\ d-a,\ d\rangle$ so $\{s \in S\ :\ s \leq d\} = \{0,\ a,\ 2a, \dots,\ a(n-1),\ d-a,\ an, d\}$ and therefore $\mathfrak{a}_S = \langle x^d,\ x^{an}y^{d-an},\ x^{d-a}y^a, \dots,\ x^ay^{d-a},\ y^d\rangle$. Symmetry then gives us $\mathfrak{a}_T = \langle x^d,\ x^{d-a}y^a, \dots,\ x^ay^{d-a},\ x^{d-na}y^{na},\ y^d\rangle$. So

$$\widetilde{\mathfrak{a}} = \mathfrak{a}_S \cap \mathfrak{a}_T =$$

$$= \langle\ \mathrm{lcm}(y^d, y^d),\ \mathrm{lcm}(y^d, x^{d-na}y^{na}),\ \mathrm{lcm}(x^{d-na}y^{na}, x^ay^{d-a}),\ \mathrm{lcm}(x^ay^{d-a}, x^ay^{d-a}),$$

$$\mathrm{lcm}(x^ay^{d-a}, x^{d-(n-1)a}y^{(n-1)a}), \dots, \mathrm{lcm}(x^d, x^d)\rangle =$$

$$= \langle y^d,\ x^{d-na}y^d,\ x^ay^{na},\ x^ay^{d-a},\ x^{d-(n-1)a}y^{d-a}, \dots, x^d\rangle$$

$$= \langle y^d, x^ay^{d-a}, x^{2a}y^{(n-1)a}, x^{d-(n-2)a}y^{d-2a}, \dots, x^{d-2a}y^{d-(n-2)a}, x^{(n-1)a}y^{2a}, x^{d-a}y^a, x^d\rangle =$$

$$\mathfrak{a} + \langle x^{ka}y^{(n-k+1)a} \mid 1 < k < n\rangle + \langle x^{d-ka}y^{d-(n-k)a} \mid 1 < k < n-1\rangle.$$

$x^{ka}y^{(n-k+1)a}$ has degree $(n+1)a$. So if there exists k such that $2 \leq k \leq n-1$ we are done. This is true for all $n \geq 3$.

□

Remark 5.12. Let $\mathfrak{a}$ be as in the previous example, and choose $d = 3a+1$. That is let $\mathfrak{a} = \langle x^{3a+1},\ x^{2a+1}y^a,\ x^ay^{2a+1},\ y^{3a+1}\rangle$. Clearly $\gcd(d,\ a) = 1$. Then we have $S = T = \{0,\ a,\ 2a,\ 2a+1,\ 3a,\ 3a+1,\ \dots\}$ so

$$\mathfrak{a}_S = \mathfrak{a} + \langle x^{3a}y,\ x^{2a}y^{a+1}\rangle =$$

$$\langle x^{3a+1},\ x^{3a}y,\ x^{2a+1}y^a,\ x^{2a}y^{a+1},\ x^ay^{2a+1},\ y^{3a+1}\rangle$$

and

$$\mathfrak{a}_T = \langle x^{3a+1},\ x^{2a+1}y^a,\ x^{a+1}y^{2a},\ x^ay^{2a+1},\ xy^{3a},\ y^{3a+1}\rangle.$$

Which gives us

$$\widetilde{\mathfrak{a}} = \mathfrak{a}_S \cap \mathfrak{a}_T =$$

$$= \langle x^{3a+1},\ x^{2a+1}y^a,\ x^{2a}y^{2a},\ x^ay^{2a+1},\ y^{3a+1}\rangle = \mathfrak{a} + \langle x^{2a}y^{2a}\rangle$$

		$m(\mathfrak{a})$							
		1	2	3	4	5	6	7	8
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	7	①	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0
	9	7	0	0	0	0	0	0	0
	10	1	①	0	0	0	0	0	0
	11	32	0	0	0	0	0	0	0
	12	4	0	0	0	0	0	0	0
	13	113	4	①	0	0	0	0	0
	14	57	12	0	0	0	0	0	0
	15	386	16	4	0	0	0	0	0
$\deg(\mathfrak{a})$	16	208	51	1	①	0	0	0	0
	17	1318	50	15	0	0	0	0	0
	18	720	132	4	4	0	0	0	0
	19	3 962	204	46	1	①	0	0	0
	20	2 717	395	19	11	0	0	0	0
	21	11 446	5 024	130	740	0	0	0	0
	22	8 687	1 195	55	33	1	①	0	0
	23	35 946	1 500	365	24	12	0	0	0
	24	24 151	3 109	173	89	4	4	0	0
	25	102 925	4 595	944	68	36	1	①	0
	26	80 609	8 079	514	234	11	11	0	0
	27	281 698	11 513	2 366	208	96	4	4	0

Figure 5.1. A table of how big the generators for $\widetilde{\mathfrak{a}}$ can become

This class of ideals are interesting in a couple of ways. Firstly, they maximize the $m(\mathfrak{a})$ in comparison to the degree of $\mathfrak{a}$, within the class of monomials in example 5.11. Moreover they do this while still not being isomorphic to some monomial ideal of smaller degree. We can, of course, get quite large differences if we only double or triple the degree of all monomials generating $\mathfrak{a}$ but, in some sense, they will be uninteresting.

Secondly, they show that a linear relation between $m(\mathfrak{a})$ and the degree of the ideal is possible for certain ideals. Quite trivially the relation cannot be "greater" than linear. Otherwise, for large enough degrees, $m(\mathfrak{a})$ would grow faster than the degree of the ideal, and in time be greater. This clearly cannot happen. So the question instead becomes how "big" can this linear difference become?

In figure 5.1 the number in the i:th row column j represents how many ideals of degree i that have $m(\mathfrak{a}) \geq j$.

Notice that only ideals that are not isomorphic to some ideal of lower degree were tested. So for example, $\langle x^8,\ x^6y^2,\ y^8\rangle$ was not tested.

The encircled numbers represent the ideals in remark 5.12, and we see that within our table these ideals maximises this difference. We can also see that the numbers seem to increase with higher degree and fewer divisors.

Proposition 5.13. *Let* $\mathfrak{a} = \langle x^{s_i}y^{t_i}\rangle_{i=0}^{n}$ *such that* $s_n = t_0 = 0$, $s_i + t_i = d$ *and* $s_i > s_{i+1}$. *If* $m(\mathfrak{a}) \geq a-1$ *then* $\mathfrak{a}$ *must have degree greater or equal to* $3a+1$.

	1	2	3	4	5	6	7	8	9	10
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3	4	0	0	0	0	0	0	0	0	0
4	7	①	0	0	0	0	0	0	0	0
5	13	2	①	0	0	0	0	0	0	0
6	21	6	4	①	0	0	0	0	0	0
7	38	15	6	4	①	0	0	0	0	0
8	59	32	24	8	4	①	0	0	0	0
9	103	78	48	14	8	4	①	0	0	0
10	159	157	118	47	18	8	4	①	0	0
11	267	346	247	101	34	16	8	4	①	0
12	406	698	581	222	78	34	16	8	4	①

FIGURE 5.2. A table showing when $\mathfrak{a}^k$ and $\widetilde{\mathfrak{a}}^k$ coincide.

Proof. For $\widetilde{\mathfrak{a}}$ to have a minimal generator on the form $x^{s'}y^{t'}$ we must have $x^{s'}y^{d-s'}, x^s y^{d-s} \in \mathfrak{a}_S$, by symmetry $s < s'$, and $x^{d-t'}y^{t'}, x^{d-t}y^t \in \mathfrak{a}_S$, by symmetry $t < t'$, such that $\operatorname{lcm}(x^{s'}y^{d-s'}, x^{d-t'}y^{t'}) = x^{s'}y^{t'}$. This simply because $\langle a_i \rangle \cap \langle b_j \rangle = \langle \operatorname{lcm}(a_i, b_j) \rangle$. Moreover there cannot exist $x^{s''}y^{d-s''} \in \mathfrak{a}_s$ or $x^{d-t''}y^{t''} \in \mathfrak{a}_t$, such that $s < s'' < s'$ or $t < t'' < t'$ since then $x^{s'}y^{t'}$ would not be minimal. In terms of S and T this translates as there must exist $s,\ s' \in S$ and $t,\ t' \in T$ that fufill the following properties:

(1) $s < d - t' < s' < d - t$.
(2) There does not exist $s'' \in S$ or $t'' \in T$ such that $s < s'' < s'$ or $t < t'' < t'$.

Since we assumed $m(\mathfrak{a}) = a - 1$ we can let $s' + t' = d + a - 1$.

Clearly (2) gives us $s' - s \leq s_{n-1}$ and $t' - t \leq t_1$. Otherwise $s + s_{n-1}$ would be a counterexample. So then we have $a - 1 = s' + t' - d = s' - (d - t') < s' - s \leq s_{n-1}$, that is $a \leq s_{n-1}$. Symmetry gives us $a \leq t_1$.

Assume by symmetry $s_{n-1} \leq t_1$. Then corollary 5.8 states that if $2s_{n-1} + t_1 \geq d$ holds, $\mathfrak{a}$ is Ratliff-Rush. This clearly cannot hold. Therefore we have $3a \leq 3s_{n-1} \leq 2s_{n-1} + t_1 < d$. So $3a + 1 \leq d$.

□

Now we have looked at the degree of the generators, but a more striking question is when $\mathfrak{a}^k = \widetilde{\mathfrak{a}}^k$. Let $n(\mathfrak{a}) = \min\{k \in \mathbb{Z}^+ \mid \mathfrak{a}^k = \widetilde{\mathfrak{a}}^k\}$. Corollary 4.6 then simply states $r(\mathfrak{a}) \leq n(\mathfrak{a})$.

Example 5.14. Let $\mathfrak{a} = \langle x^d,\ x^{d-1}y,\ xy^{d-1},\ y^d \rangle$. Then $\widetilde{\mathfrak{a}} = \langle x,\ y \rangle^d$ and $n(\mathfrak{a}) = d - 2$.

In figure 5.2 we have encircled the ideals just mentioned in 5.14. The number in row d column n represents the number of ideals of degree d with $n(\mathfrak{a}) = n$.

As we can see, the numbers at the end of each row are powers of 2. For example, row 10 ends with 8 4 1.Let us examine these ideals closer. The ideal of degree 10 with $n(\mathfrak{a}) = 8$ is:

$$\mathfrak{a}_0 = \langle x^{10},\ x^9 y,\ xy^9,\ y^{10} \rangle.$$

The 4 ideals of degree 10 with $n(\mathfrak{a}) = 7$ are:

$$\begin{aligned}
\mathfrak{a}_1 =&\langle x^{10},\ x^9y,\ x^8y^2,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_2 =&\langle x^{10},\ x^9y,\ x^2y^8,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_3 =&\langle x^{10},\ x^9y,\ x^2y^8,\ y^{10}\rangle\\
\mathfrak{a}_4 =&\langle x^{10},\ x^8y^2,\ xy^9,\ y^{10}\rangle.
\end{aligned}$$

The 8 ideals of degree 10 with $n(\mathfrak{a}) = 6$ are:

$$\begin{aligned}
\mathfrak{a}_5 =&\langle x^{10},\ x^9y,\ x^8y^2,\ x^7y^3,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_6 =&\langle x^{10},\ x^9y,\ x^3y^7,\ x^2y^8,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_7 =&\langle x^{10},\ x^9y,\ x^7y^3,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_8 =&\langle x^{10},\ x^9y,\ x^3y^7,\ x^2y^8,\ y^{10}\rangle\\
\mathfrak{a}_9 =&\langle x^{10},\ x^9y,\ x^3y^7,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_{10} =&\langle x^{10},\ x^8y^2,\ x^7y^3,\ xy^9,\ y^{10}\rangle\\
\mathfrak{a}_{11} =&\langle x^{10},\ x^9y,\ x^3y^7,\ y^{10}\rangle\\
\mathfrak{a}_{12} =&\langle x^{10},\ x^7y^3,\ xy^9,\ y^{10}\rangle
\end{aligned}$$

And as far as we have checked, for large enough d, the ideal of degree d with $n(\mathfrak{a}) = d - 2$ is:

$$\mathfrak{a}_0 =\langle x^d,\ x^{d-1}y,\ xy^{d-1},\ y^d\rangle.$$

The 4 ideals of degree d with $n(\mathfrak{a}) = d - 3$ are:

$$\begin{aligned}
\mathfrak{a}_1 =&\langle x^d,\ x^{d-1}y,\ x^{d-2}y^2,\ xy^{d-1},\ y^d\rangle\\
\mathfrak{a}_2 =&\langle x^d,\ x^{d-1}y,\ x^2y^{d-2},\ xy^{d-1},\ y^d\rangle\\
\mathfrak{a}_3 =&\langle x^d,\ x^{d-1}y,\ x^2y^{d-2},\ y^d\rangle\\
\mathfrak{a}_4 =&\langle x^d,\ x^{d-2}y^2,\ xy^{d-1},\ y^d\rangle
\end{aligned}$$

and so on. What is interesting is how the generators are located. They seem to avoid numbers around $\frac{d}{2}$. This is in line with the upper limits presented in [2], since this leads to larger values of $\Lambda(S)$.

With 4.7 *iii)* in mind we can partially order the ideals as $\mathfrak{a} \sqsubset \mathfrak{b} \Leftrightarrow \mathfrak{a} \subseteq \mathfrak{b} \subseteq \widetilde{\mathfrak{a}}$. Then we get that if $\mathfrak{a} \sqsubset \mathfrak{b}$ then we have $n(\mathfrak{a}) \geq n(\mathfrak{b})$.

By theorem 4.11 we see that if $\mathfrak{a}$ and $\mathfrak{b}$ are of the same degree with the same Ratliff-Rush associated ideal, then $\mathfrak{a}_S = \mathfrak{b}_S$ and $\mathfrak{a}_T = \mathfrak{b}_T$, since they coinside after large enough powers. With this in mind it is possible to construct an ideal $\mathfrak{a}'$ from $\mathfrak{a}$ such that $\mathfrak{a}' \sqsubset \mathfrak{a}$ as following:

(1) Let d be the degree of $\mathfrak{a}$. Take the minimal generators to S, call them a_i. Create $\{x^{a_i}y^{d-a_i}\} \cup \{x^d\}$.
(2) Repeat with T and create $\{x^{d-b_i}y^{b_i}\} \cup \{y^d\}$.
(3) Let $\mathfrak{a}' = \langle \{x^{a_i}y^{d-a_i}\} \cup \{x^d\} \cup \{x^{d-b_i}y^{b_i}\} \cup \{y^d\} \rangle$.

Clearly $\mathfrak{a}' \subseteq \mathfrak{a}$, $S_{\mathfrak{a}} = S_{\mathfrak{a}'}$ and $T_{\mathfrak{a}} = T_{\mathfrak{a}'}$, and thus $\mathfrak{a}' \sqsubset \mathfrak{a}$. Moreover, by construction, $\mathfrak{a}'$ must be the smallest ideal having the same associated semigroups, and is thus the smallest ideal, within the class of ideals we are discussing here with the same high powers as $\mathfrak{a}$, keeping theorem 4.11 in mind, and is thus the smallest ideal, here with respect to inclution, with the same assosiated Ratliff-Rush ideal.

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[2] V. Crispin Quiñonez, *Ratliff-Rush monomial ideals*, Algebraic and geometric combinatorics, 43–50, Contemp. Math., **423**, Amer. Math. Soc., Providence, RI, (2006).

[3] R. Fröberg, *An Introduction to Gröbner Bases*, John Wiley Sons Ltd, Chichester, 1997.

[4] M. Nagata, *Local Rings*, Interscience Tracts 13, Interscience Publishers, New York, NY, 1962.

[5] L. J. Ratliff, Jr. and D. E. Rush, *Two Notes on Reductions of Ideals*, Indiana Univ. Math. J. **27** (1978), no. 6, 929–934.

[6] P. Restadh http://user.it.uu.se/∼pere4728/

[7] J. J. Sylvester, *Question 7382*, Mathematical Questions from the Educational Times **41**, 21, (1884).

E-mail address: `petter.restadh@gmail.com`