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U.U.D.M. Project Report 2017:37

Model theory of algebraically closed fields

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Examensarbete i matematik, 15 hp
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Oktober 2017

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Abstract

We construct a first order theory for algebraically closed fields and show that it is decidable in a given characteristic and that it eliminates both quantifiers and imaginaries. We also illuminate the connections to algebraic interpretations of the results.

Contents

1 Introduction

The joint history of model theory and algebra stretches back to the first half of the 20th century when logicians gradually started to realise that model theory was surprisingly capable of dealing with algebraic structures. Since then, model-theoretic algebra has earned its own name by proving on several occasions that taking a model-theoretic perspective on algebra is at least illuminating, if not more. From the early, purely model-theoretic, results, model-theoretic algebra has been used to prove far more advanced results well outside of the purely model-theoretic realm (see [Bou98] for such an example).

Led by Tarski in the mid-20th century, the study of fields from a model-theoretic perspective proved to be particularly interesting. Since the beginning, the theory of algebraically closed to fields, alongside that of real closed ordered fields, has been used as an archetypical example of a theory admitting a certain property. We will show some of these well-known properties for algebraically closed fields, all of which can be found (often together with analogue results for real closed ordered fields) in almost any introductory text on the subject (see, for example [Mar02] or [MMP96]).

Suggestion for further reading (treating, among other things, results concerning saturation, stability and further applications to algebraic geometry) : [MMP96], [HPS00] and [Bou98].

1.1 Prerequisites and Notation

This paper, though on an introductory level seen as a paper on the model theory of fields, does require previous knowledge of mathematical logic usually covered in an undergraduate course on the subject. [Mar02] or [Hod96] will provide all this knowledge and much more. Familiarity with abstract algebra is helpful but not necessary to follow this paper, as the approach will throughout be model-theoretic as opposed to algebraic.

$\mathbb{N}$ will denote the natural numbers including 0 and $\mathbb{N}_+$ will denote the natural numbers excluding 0.

In contrast to some authors, we will make two distinctions worth pointing out; The distinction between a vocabulary (a set of constant-, function-, and relation symbols) and its language (the set of first order formulas which can be formed from the vocabulary); The distinction between a structure (denoted calligraphically) and its underlying set - e.g if $\mathcal{M}$ is a structure, M is its underlying set.

$\mathcal{V}_r$ will denote the vocabulary $\langle +, -, \cdot, 0, 1\rangle$, where $+$ and $\cdot$ are binary function symbols, - is a unary function symbol and 0 and 1 are constant symbols. $\mathcal{L}_r$ denotes the language of $\mathcal{V}_r$.

Frequently, we will be so informal as to use "formula", "sentence", "theory" or "structure" to mean $\mathcal{V}$-formula etc. for an understood vocabulary $\mathcal{V}$.

Sometimes we will use abbreviations such as $\bar{x}$ to denote a vector $x_1, ..., x_n$ of some arbitrary length $n \in \mathbb{N}$. Consequential abbreviations (e.g $\forall\bar{x}$) are to be interpreted in the natural way ($\forall x_1 ... \forall x_n$).

Finally, we use $\equiv$ to denote equivalence between formulas.

1.2 ACF and ACF$_p$

Informally speaking, a *field* is a set, together with two associative and commutative operators, commonly denoted $+$ (addition) and $\cdot$ (multiplication), such that $\cdot$ distributes over $+$, all elements in the set have additive and multiplicative inverses, and there is an additive- and a multiplicative identity element, commonly denoted 0 and 1, respectively. It is conventional to leave out the $\cdot$ in expressions.

Before we start studying these structures model-theoretically, we need to go over some core concepts from field theory.

Definition 1. A field $\mathcal{K}$ is *algebraically closed* if it contains a root to every non-constant polynomial in the ring K[X] of polynomials in one variable with coefficients in K.

Definition 2. The *algebraic closure* of a field $\mathcal{K}$, denoted $\overline{\mathcal{K}}$, is the smallest algebraically closed field containing $\mathcal{K}$ as a subfield (a subset which is a field under the same operations).

It can be shown that every field has an algebraic closure, and that it is unique up to isomorphism. See [Lan02]V§2 for a full proof of this.

Definition 3. A field has *characteristic k* if $k = min\,\{n \mid \sum_1^n 1 = 0\}$. If there is no such k, the field is said to have *characteristic 0.*

Let T_{fields} denote the (first-order) theory containing the axioms

(i) $\forall x \forall y \forall z (x + (y + z) = (x + y) + z \wedge x(yz) = (xy)z)$ (Associativity)

(ii) $\forall x \forall y (x + y = y + x \wedge xy = yx)$ (Commutativity)

(iii) $\forall x \forall y \forall z (x(y + z) = xy + xz)$ (Distributivity)

(iv) $\forall x (x + (-x) = 0 \wedge (\neg(x = 0) \rightarrow \exists y (xy = 1)))$ (Inverses)

(v) $\forall x (x + 0 = x \wedge x \cdot 1 = x)$ (Identities)

These are the field axioms, and thus T_{fields} is the theory of fields, and every $\mathcal{K} \models T_{fields}$ is a field. Note that the way the axioms are stated forces $\mathcal{K}$ to interpret the constants 0 and 1 as the additive and multiplicative identity, respectively.

For every $n \in \mathbb{N}_+$, let

$$\varrho_n \equiv \forall a_0 \forall a_1 ... \forall a_n \exists x \left(\sum_{i=0}^{n} a_i x^i = 0 \right),$$

which expresses that every non-constant polynomial has a root.

For every $n \in \mathbb{N}_+$, let

$$\chi_n \equiv \forall x \left(\sum_{i=1}^{n} x = 0 \right).$$

Let ACF denote $T_{fields} \cup \{\varrho_n | n \in \mathbb{N}_+\}$, and for p prime, let ACF_p denote ACF $\cup \{\chi_p\}$ while ACF_0 denotes ACF $\cup \{\neg\chi_n | n \in \mathbb{N}_+\}$. Thus, ACF is the theory of algebraically closed fields and for $p = 0$ or p prime, ACF_p is the theory of algebraically closed fields of characteristic p. Henceforth, whenever ACF_p is mentioned without specification, p is assumed to be 0 or prime. (Indeed, there are no fields of any other characteristic).

2 Decidability and Quantifier Elimination

2.1 Decidability

Recall that a theory T is said to be *decidable* if there is an algorithm determining for every sentence ϕ, whether $T \vdash \phi$.

Proposition 1. ACF_p *is κ-categorical for all $\kappa > \aleph_0$.*

That is, all algebraically closed fields of a given characteristic are isomorphic if they are of the same uncountable cardinality.

Proof. If we admit the following two well-known and purely algebraic facts about fields, and if we recall that for a given p, all fields that model ACF_p have the same characteristic, the result follows immediately.

Fact 1 Any algebraically closed field of an uncountable cardinality κ has transcendence degree κ.

Fact 2 Algebraically closed fields are determined up to isomorphism by their characteristic and transcendence degree.

□

The proofs of the two facts above contain too much theory from abstract algebra to be included in a paper of this kind. See [Lan02] for a proof of the first fact. The second fact is a classic theorem due to Steinitz, who first proved it in the 1910 article [Ste10].

Proposition 2. ACF_p *is complete.*

Proof. First, consider the polynomial

$$f(x) = 1 + \prod_{a \in F} (x - a)$$

for a finite field $\mathcal{F}$. Then $f(a) = 1$ for all $a \in F$, making it impossible for $\mathcal{F}$ to be algebraically closed. Thus any algebraically closed field must be infinite.

Next, recall the Łoś-Vaught test: If $\mathcal{L}$ is the language of some vocabulary $\mathcal{V}$, then any satisfiable, κ-categorical, where $\kappa \geq |\mathcal{L}|$, $\mathcal{V}$-theory with no finite models is complete.

The result then follows from Proposition 1.

□

Theorem 1. ACF_p *is decidable.*

Proof. Fix p to be prime or 0 and let

$S = \{\phi \ \mid \mathrm{ACF}_p \models \phi\}$,

$F = \{\phi \ \mid \mathrm{ACF}_p \models \neg\phi\}$.

Since ACF_p is satisfiable and complete, we have that $S \cap F = \emptyset$ and $S \cup F = \mathcal{L}$, respectively. By the completeness theorem, $S = \{\phi \mid \mathrm{ACF}_p \vdash \phi\}$ and $F = \{\phi \mid \mathrm{ACF}_p \vdash \neg\phi\}$. It can easily be shown that both $\mathcal{L}$ and ACF_p are recursive, making both S and F recursively enumerable, which in turn makes them recursive as they are each others complement. Thus, for any sentence ϕ, there is an algorithm deciding whether $\phi \in S$, which is equivalent to it deciding whether $\mathrm{ACF}_p \vdash \phi$.

□

2.2 Quantifier Elimination

Recall that a theory T is said to have *quantifier elimination* if for every formula $\phi(\bar{x})$ there is a formula $\psi(\bar{x})$ containing no quantifiers such that

$$T \models \forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

Lemma 1. *Suppose* $\mathcal{M} \subseteq \mathcal{N}$, $\bar{a} \in M^k$ *and that* $\phi(\bar{x})$ *is quantifier-free. Then* $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a})$.

That is, quantifier-free formulas are preserved under substructure and extension.

Proof. First, we prove by induction on terms that $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$ for any term $t(\bar{x})$ and $\bar{b} \in M^k$.

If t is the constant symbol c, then $t^{\mathcal{M}} \equiv c^{\mathcal{M}} \equiv c^{\mathcal{N}} \equiv t^{\mathcal{N}}$.

If t is the variable x_i, then $t^{\mathcal{M}}(\bar{b}) \equiv b_i \equiv t^{\mathcal{N}}(\bar{b})$.

If t is the n-ary function f with the terms $t_1, ...t_n$ such that $t_i^{\mathcal{M}}(\bar{b}) \equiv t_i^{\mathcal{N}}(\bar{b})$ for $i \in 1, ..., n$, as entries, then

$$\begin{aligned} t^{\mathcal{M}}(\bar{b}) &\equiv f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), ..., t_n^{\mathcal{M}}(\bar{b})) \\ &\equiv f^{\mathcal{N}}(t_1^{\mathcal{M}}(\bar{b}), ..., t_n^{\mathcal{M}}(\bar{b})) \\ &\equiv f^{\mathcal{N}}(t_1^{\mathcal{N}}(\bar{b}), ..., t_n^{\mathcal{N}}(\bar{b})) \\ &\equiv t^{\mathcal{N}}(\bar{b}). \end{aligned}$$

Now that this has been proven, we can start proving the lemma itself by induction on formulas.

If ϕ is $t_1 = t_2$ for terms t_1 and t_2, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a}) \\ &\iff t_1^{\mathcal{N}}(\bar{a}) = t_2^{\mathcal{N}}(\bar{a}) \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

If ϕ is the n-ary relation R with terms $t_1, ..., t_n$ as entries, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff (t_1^{\mathcal{M}}(\bar{a}), ..., t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}} \\ &\iff (t_1^{\mathcal{M}}(\bar{a}), ..., t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{N}} \\ &\iff (t_1^{\mathcal{N}}(\bar{a}), ..., t_n^{\mathcal{N}}(\bar{a})) \in R^{\mathcal{N}} \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

Thus, the lemma is true for all atomic formulas. To complete the proof and show that the lemma holds for *all* quantifier-free formulas, we use the fact that the set of quantifier-free formulas is the smallest set that contains the atomic formulas and is closed under negation and conjunction, so that we only have to consider two more cases.

If ϕ is $\neg\psi$ and the lemma is true for ψ, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \not\models \psi(\bar{a}) \\ &\iff \mathcal{N} \not\models \psi(\bar{a}) \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

If ϕ is $\psi_1 \wedge \psi_2$ and the lemma is true for ψ_1 and ψ_2, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \models \psi_1(\bar{a}) \text{ and } \mathcal{M} \models \psi_2(\bar{a}) \\ &\iff \mathcal{N} \models \psi_1(\bar{a}) \text{ and } \mathcal{N} \models \psi_2(\bar{a}) \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

□

Proposition 3. *Assuming $\mathcal{V}$ is a vocabulary containing at least one constant symbol, let T be a $\mathcal{V}$-theory and let $\phi(x_1, ..., x_n)$ be a $\mathcal{V}$-formula (note that n=0 is a possibility). Then the following are equivalent:*

(i) There is a quantifier-free $\mathcal{V}$-formula $\psi(x_1, ..., x_n)$ such that

$$T \models \forall x_1 ... \forall x_n \left(\phi(x_1, ..., x_n) \leftrightarrow \psi(x_1, ..., x_n)\right).$$

(ii) For any two $\mathcal{V}$-structures $\mathcal{M}_1, \mathcal{M}_2 \models T$ such that $\mathcal{N} \subseteq \mathcal{M}_1$ and $\mathcal{N} \subseteq \mathcal{M}_2$, and any $a_1, ..., a_n \in N$ we have that

$$\mathcal{M}_1 \models \phi(a_1, ..., a_n) \Leftrightarrow \mathcal{M}_2 \models \phi(a_1, ..., a_n).$$

In words, *(ii)* states that any two models of T that share a common substructure satisfy the same sentences as long as any free variables are replaced by elements from the underlying set of that common substructure.

Proof. [$(i) \Rightarrow (ii)$]: Suppose that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ for a quantifier-free $\psi(\bar{x})$ and let $\bar{a} \in N$ where $\mathcal{N}$ is a common substructure of $\mathcal{M}_1$ and $\mathcal{M}_2$, with the latter two being models of T. By Lemma 1,

$$\begin{aligned} \mathcal{M}_1 \models \phi(\bar{a}) &\iff \mathcal{M}_1 \models \psi(\bar{a}) \\ &\iff \mathcal{N} \models \psi(\bar{a}) \\ &\iff \mathcal{M}_2 \models \psi(\bar{a}) \\ &\iff \mathcal{M}_2 \models \phi(\bar{a}). \end{aligned}$$

[$(ii) \Rightarrow (i)$]: Let *(ii)* hold and let c be a constant symbol. In the case that $T \models \forall \bar{x}\phi(\bar{x})$, we will also have that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow c = c)$, and if $T \models \forall \bar{x}\neg\phi(\bar{x})$, then $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \neg(c = c))$, immediately making *(i)* hold. Thus we assume that both $T \cup \{\phi(\bar{x})\}$ and $T \cup \{\neg\phi(\bar{x})\}$ are consistent (or, equivalently, satisfiable by a structure).

Let $\Gamma(\bar{x}) = \{\psi(\bar{x}) \mid \psi \text{ is quantifier-free and } T \models \forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))\}$, and let $d_1, ..., d_m$ be *new* constant symbols.

Claim $T \cup \Gamma(\bar{d}) \models \phi(\bar{d})$.

Suppose, toward a contradiction, that this is not true and let $\mathcal{M}_1 \models T \cup \Gamma(\bar{d}) \cup \{\neg\phi(\bar{d})\}$. Also, let $\mathcal{N}$ be the substructure $\langle\{d_1, ..., d_m\}\rangle_{\mathcal{M}_1}$ generated by $\bar{d}$. Next, let Σ be $T \cup \mathcal{D}(\mathcal{N}) \cup \{\phi(\bar{d})\}$, where $\mathcal{D}(\mathcal{N})$ denotes the *literal*, or *atomic*, diagram of $\mathcal{N}$, i.e the set $\left\{\theta(\bar{d_\alpha}) \mid \theta \text{ is a literal and } \mathcal{N} \models \theta(\bar{d_\alpha}) \text{ for } \bar{d_\alpha} \in \{d_1, ..., d_m\}^k, k \in \mathbb{N}\right\}$. If Σ is inconsistent, there will be quantifier-free $\psi_1(\bar{d}), ..., \psi_n(\bar{d}) \in \mathcal{D}(\mathcal{N})$ such that

$$T \models \forall \bar{x} \left(\bigwedge_{i=1}^{n} \psi_i(\bar{x}) \rightarrow \neg\phi(\bar{x}) \right).$$

But then we will have

$$T \models \forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^{n} \neg\psi_i(\bar{x}) \right),$$

leading to $\bigvee_{i=1}^{n} \neg\psi_i(\bar{x}) \in \Gamma(\bar{x})$ and then $\mathcal{N} \models \bigvee_{i=1}^{n} \neg\psi_i(\bar{d})$ - a contradiction. However if Σ is consistent, any of its models must contain $\mathcal{N}$ as an embedded substructure, as $\mathcal{D}(\mathcal{N}) \subseteq \Sigma$. Thus, since *(ii)* is assumed to hold and $\mathcal{M}_1 \models \neg\phi(\bar{d})$, we have that $\mathcal{M}_2 \models \neg\phi(\bar{d})$ whenever $\mathcal{M}_2 \models \Sigma$, contradicting the fact that $\phi(\bar{d}) \in \Sigma$.

Now with the claim proven, by compactness, there are $\psi_1, ..., \psi_n \in \Gamma$ such that

$$T \models \forall \bar{x} \left(\bigwedge_{i=1}^{n} \psi_i(\bar{x}) \to \phi(\bar{x}) \right), \text{ and thus } T \models \forall \bar{x} \left(\bigwedge_{i=1}^{n} \psi_i(\bar{x}) \leftrightarrow \phi(\bar{x}) \right),$$

and $\bigwedge_{i=1}^{n} \psi_i(\bar{x})$ is quantifier-free as a conjunction of quantifier-free formulas.

□

Although not necessary for our purposes, it might be worth noting that the above proposition, and its proof thereafter, can be modified to include the case where the vocabulary contains no constant symbols. There will be no quantifier-free sentences in that case, but for every sentence there is an equivalent (in the theory) quantifier-free formula with one free variable.

Proposition 4. *Suppose that for every quantifier-free formula $\theta(\bar{x}, y)$ there is a quantifier-free formula $\psi(\bar{x})$ such that $T \models \forall \bar{x}(\exists y \theta(\bar{x}, y) \leftrightarrow \psi(\bar{x}))$. Then for every formula $\phi(\bar{x})$, there is a quantifier-free $\sigma(\bar{x})$ such that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \sigma(\bar{x}))$.*

Proof. Let $\phi(\bar{x})$ be a formula. We prove the proposition by induction on the complexity of $\phi(\bar{x})$.

If ϕ is quantifier-free, it is equivalent to itself. Otherwise, suppose, for $i = 0, 1$, that ψ_i is quantifier-free and that $T \models \forall \bar{x}(\theta_i(\bar{x}) \leftrightarrow \psi_i(\bar{x}))$.

If $\phi(\bar{x}) \equiv \neg\theta_0(\bar{x})$, then $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \neg\psi_0(\bar{x}))$.

If $\phi(\bar{x}) \equiv \theta_0(\bar{x}) \wedge \theta_1(\bar{x})$, then $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow (\psi_0(\bar{x}) \wedge \psi_1(\bar{x})))$.

If we put $\sigma(\bar{x}) \equiv \neg\psi_0(\bar{x})$ in the first case, and $\sigma(\bar{x}) \equiv \psi_0(\bar{x}) \wedge \psi_1(\bar{x})$ in the second, then σ is a quantifier-free formula equivalent to ϕ. This shows that whenever ϕ is a negation or conjunction of a formula or formulas equivalent to a quantifier-free formula or formulas, then ϕ is as well. By the reasoning in the proof of Lemma 1, negation and conjunction are the only two connectives we need to consider in the induction.

Now, with the above in mind, we will show that we can "remove" (existential) quantifiers one-by-one. Recall that "$\forall$" is just an abbreviation of "$\neg\exists\neg$".

Suppose that $\phi(\bar{x}) \equiv \exists y \theta(\bar{x}, y)$, that ψ is a quantifier-free formula, and that $T \models \forall \bar{x} \forall y(\theta(\bar{x}, y) \leftrightarrow \psi(\bar{x}, y))$. Then we have that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \exists y \psi(\bar{x}, y))$, which in turn - by assumption - implies $T \models \forall \bar{x}(\exists y \psi(\bar{x}, y) \leftrightarrow \sigma(\bar{x}))$ for some quantifier-free σ. And then, finally, $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \sigma(\bar{x}))$.

□

A combination of the above two propositions may serve as a test for quantifier elimination of a theory.

Corollary 1. *Suppose that for $\mathcal{M}_1, \mathcal{M}_2 \models T$, such that $\mathcal{N} \subseteq \mathcal{M}_1$ and $\mathcal{N} \subseteq \mathcal{M}_2$ for some $\mathcal{N}$, we have that for any $b \in M_1$, there is $c \in M_2$ such that $\mathcal{M}_1 \models \theta(\bar{a}, b) \iff \mathcal{M}_2 \models \theta(\bar{a}, c)$ for any quantifier-free θ and any $\bar{a} \in N^k, k \in \mathbb{N}$.*

Then T has quantifier elimination.

Proposition 4 tells us that to test a theory for quantifier elimination, we only have to make sure that every formula on the form $\exists y \theta(\bar{x}, y)$, where $\theta(\bar{x}, y)$ is quantifier-free, is equivalent to a quantifier-free formula $\sigma(\bar{x})$. Given the form of these formulas, the hypothesis of the corollary fulfils condition (ii) in Proposition 3, assuring us that there indeed is an equivalent quantifier-free formula.

Theorem 2. ACF *has quantifier elimination.*

Proof. We will use the above corollary to prove the theorem.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two algebraically closed fields containing $\mathcal{F}$ as a subfield and note that the algebraic closure $\overline{\mathcal{F}}$ is then a subfield of both $\mathcal{K}_1$ and $\mathcal{K}_2$. Let $\bar{a} \in F$, $b \in K_1$ and let $\phi(\bar{x}, y)$ be a quantifier-free formula such that $\mathcal{K}_1 \models \phi(\bar{a}, b)$. If we can show that there is a $c \in K_2$ such that $\mathcal{K}_2 \models \phi(\bar{a}, c)$, we will have shown that ACF has quantifier elimination.

Since ϕ is quantifier-free, it can be written in disjunctive normal form, i.e disjunctions of conjunctions of literals. Given the contents of our vocabulary $\mathcal{V}$, any atomic formula $\psi(x_1, ...x_n)$ in our language $\mathcal{L}$ can be written on the form $p(x_1, ..., x_n) = 0$ (and a negated atomic formula would be on the form $p(x_1, ..., x_n) \neq 0$), where $p \in \mathbb{Z}[X_1, ..., X_n]$, and furthermore, we can view $p(\bar{a}, y)$ as a polynomial in $F[Y]$.

These facts together mean that there are polynomials $f_{i,j}, g_{i,j} \in F[Y]$ such that

$$\mathcal{F} \models \phi(\bar{a}, y) \leftrightarrow \bigvee_{i=1}^{k} \left(\bigwedge_{j=1}^{m_i} f_{i,j}(y) = 0 \wedge \bigwedge_{j=1}^{n_i} g_{i,j}(y) \neq 0 \right).$$

Then, since $\mathcal{K}_1 \models \phi(\bar{a}, b)$, we have that

$$\mathcal{K}_1 \models \bigwedge_{j=1}^{m_i} f_{i,j}(b) = 0 \wedge \bigwedge_{j=1}^{n_i} g_{i,j}(b) \neq 0$$

for some $i \leq k$.

In the case that $f_{i,j}$ is not the zero-polynomial for some $j \leq m_i$, b is algebraic in $\mathcal{F}$, i.e $b \in \overline{F} \subseteq K_2$ and then $\mathcal{K}_2 \models \phi(\bar{a}, b)$.

In the case that $f_{i,j}$ is the zero-polynomial for all $j \leq m_i$, we can turn our attention to the $g_{i,j}$-polynomials, which, for each j, must have finitely many roots due to the fact that neither of them are the zero-polynomial. Letting $D = \{d \mid g_{i,j}(d) = 0 \text{ for some } j \leq n_i\}$, we can see that $K_2 \setminus D \neq \emptyset$ since all algebraically closed fields are infinite, and then if $c \in K_2 \setminus D$, we have that $\mathcal{K}_2 \models \phi(\bar{a}, c)$.

□

2.3 Remarks

Quantifier elimination and decidability are two properties closely connected for many theories. Completeness of ACF_p follows quite directly from quantifier elimination of ACF and Tarski, who first proved decidability for $\mathbb{C}$ in [Tar57], did so by means of quantifier elimination. He even constructed an explicit algorithm eliminating quantifiers, and indeed, there is such an algorithm for any decidable theory admitting quantifier elimination.

In a more concrete sense, what it means for ACF to eliminate quantifiers is something that was hinted at in the proof of Theorem 2. Namely, that any formula with n free variables expressed in the language $\mathcal{V}_r$, since it is equivalent to a formula written in disjunctive normal form, is equivalent to a disjunction of systems of equalities and inequalities between polynomials in n variables over $\mathbb{Z}$. In particular, any *sentence* will in the same way be equivalent to a disjunction of systems of equalities and inequalities between integers.

Examples The following two algebraically apprehensible examples showcase individually the two aspects "disjunction" and "system" from the above discussion.

(i) $\phi(a_0, ..., a_n) \equiv \exists x(a_0 + \sum_{i=1}^{n} a_i x^i = 0)$ expresses that the nth-degree polynomial with coefficients $a_0, ..., a_n$ has a root, which we for algebraically closed fields know is true for all non-constant polynomials and the zero-polynomial. Thus ϕ is equivalent to

$$a_0 = 0 \vee \bigvee_{i=1}^{n} a_i \neq 0.$$

(ii) $\psi(a_0, a_1, a_2) \equiv \exists x \exists y(a_2x^2 + a_1x + a_0 = 0 \wedge a_2y^2 + a_1y + a_0 = 0) \wedge \neg(x = y)$ is the formula expressing that the 2nd degree polynomial with coefficients a_0, a_1, a_2 has two distinct roots. In algebraically closed fields this is true if and only if both a_2 and the discriminant $a_1^2 - 4a_2a_0$ are not equal to 0. Thus ψ is equivalent to

$$\begin{cases} a_2 \neq 0 \\ a_1^2 - 4a_2a_0 \neq 0 \end{cases}$$

Before we move on to the next section, we have one more notable result:

Definition 4. A theory T is *model-complete* if for any $\mathcal{M}$, $\mathcal{N} \models T$, we have that $\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathcal{M} \preccurlyeq \mathcal{N}$.

In other words, all embeddings are elementary.

Proposition 5. ACF *is model-complete.*

Proof. Let $\mathcal{F} \subseteq \mathcal{K}$ be algebraically closed fields. We show that for any $\phi(\bar{x})$ and any $\bar{a} \in F^k$, $\mathcal{F} \models \phi(\bar{a}) \iff \mathcal{K} \models \phi(\bar{a})$:

By quantifier elimination, there is a quantifier-free $\psi(\bar{x})$ such that $\mathcal{F} \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. As seen earlier, quantifier-free formulas are preserved under substructures and extensions, so

$$\begin{aligned}\mathcal{F} \models \phi(\bar{a}) &\iff \mathcal{F} \models \psi(\bar{a}) \\ &\iff \mathcal{K} \models \psi(\bar{a}) \\ &\iff \mathcal{K} \models \phi(\bar{a}).\end{aligned}$$

And thus $\mathcal{F} \preccurlyeq \mathcal{K}$.

□

Note that the above proof is not specific to ACF, but holds for any theory admitting quantifier elimination.

For ACF, model-completeness has the following algebraic interpretation:

Corollary 2. *If $\mathcal{K}_1 \subseteq \mathcal{K}_2$ are algebraically closed, any (finite) system of equalities and inequalities over K_1 which has a solution in K_2 has a solution in K_1.*

3 Definability

In this section we investigate what is definable in an algebraically closed field. We will focus on sets, functions and equivalence relations - although one could expand the model-theoretic notion of definability to include structures, such as groups. But first we must define definability itself.

Definition 5. For a structure $\mathcal{M}$ and a subset $A \subseteq M$,

(i) a set $X \subseteq M^k$ is *A-definable* if there is a formula $\phi(x_1, ..., x_k, y_1, ..., y_l)$ and $\bar{a} \in A^l$ such that $X = \{\bar{m} \in M^k \mid \mathcal{M} \models \phi(\bar{m}, \bar{a})\}$,

(ii) a function f is *A-definable* if its graph $G_f = \{(\bar{x}, \bar{y}) \mid \bar{y} = f(\bar{x})\}$ is A-definable as a set,

(iii) a relation R is *A-definable* if it is A-definable as a set.

Note that if, in the above definition, $A = \emptyset$ then $l = 0$ and $X \subseteq M^k$ is defined by a formula with k free variables. $\emptyset$-definability is in fact the most useful kind of definability for our purposes, but when there is no need to specify a set A, we will simply use "definable" to mean that a set, function or relation is M-definable.

3.1 Strong Minimality via Algebraic Geometry

When studying definable sets in algebraically closed fields, we may benefit from involving some notions from algebraic geometry. The following example illustrates just how apt model theory is as a language for algebraic geometry.

For a field $\mathcal{K}$, a set which is a boolean combination (i.e a finite combination of unions, intersections and complements) of kernels (or zero sets) of polynomials over K is called *constructible*. The kernels of polynomials over

$\mathbb{Z}$ are exactly the sets definable by the atomic formulas of $\mathcal{V}_r$. As noted earlier, if $p \in \mathbb{Z}[X_1, ..., X_n, ..., X_{n+m}]$ and if $k_1, ..., k_m \in K$, then we can view $p(X_1, ..., X_n, \bar{k})$ as a polynomial in $K[X_1, ..., X_n]$. Now, considering that the set-theoretic boolean operations union, intersection and complement have first-order semantic analogues in the form of disjunction, conjunction and negation, it is not hard to see that the constructible sets of the field $\mathcal{K}$ are exactly the quantifier-free definable sets (i.e sets which are definable by a quantifier-free formula) of $\mathcal{K}$.

As a consequence of quantifier elimination it is clear that, the constructible sets of an algebraically closed field are exactly its definable sets.

Definition 6. A theory T is *strongly minimal* if for every $\mathcal{M} \models T$ and every definable set $X \subseteq M$, X is either finite or cofinite.

Theorem 3. ACF *is strongly minimal.*

Proof. As seen above, the definable sets of an algebraically closed field are exactly its constructible sets. Thus every definable set is a boolean combination of kernels of polynomials. Since any non-zero polynomial has a finite number of roots, these sets are finite (or the whole field). And since the property of being "finite or cofinite" is preserved under boolean combinations, any definable set is either finite or cofinite.

□

There is plenty more to be said about the connections between algebraic geometry and model theory, and a lot of it can be found in [Bou98] or [HPS00]. For a more detailed proof of the above, see [Mar02].

3.2 Definable equivalence relations

Quotient structures appear frequently in different areas of mathematics (not least in algebra). Consider for instance the case where we have a group G and a normal subgroup H. Then it might be interesting to study the coset space G/H, and the importance of definable equivalence relations becomes apparent - for, in fact, the relation $xEy \iff y \in Hx$ is a definable equivalence relation whenever G and H are definable. For this reason, we expand our definition of a structure to include *many-sorted* structures, and in particular Shelah's construction $\mathcal{M}^{eq}$.

Definition 7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

If E is an $\emptyset$-definable equivalence relation on M^n for some $n \in \mathbb{N}_+$, we let the *sort* $S_E = M^n/E$ be the set of equivalence classes with respect to E, and we let $\pi_E \colon M^n \to S_E$ be the quotient map, i.e $\pi_E(\bar{m}) = \bar{m}/E$, where $\bar{m}/E$ denotes the equivalence class of $\bar{m}$ with respect to E.

$\mathcal{M}^{eq}$ is the many-sorted structure having as its underlying set the disjoint union of M and, for every $\emptyset$-definable relation E, the sorts S_E, and where, after a function symbol has been added to $\mathcal{L}$ for every E, the new function symbols are interpreted as the π_E. The new elements - the equivalence classes of the relations - are called *imaginary* elements, or *imaginaries.* The original elements are sometimes referred to as *real* elements.

For some theories we need not perform this construction on its models - a property called *elimination of imaginaries*.

Definition 8. A theory T *eliminates imaginaries* if for every $\mathcal{M} \models \mathrm{T}$ we have that for every $n \in \mathbb{N}_+$ and every $\emptyset$-definable equivalence relation E on M^n, there is an $\emptyset$-definable function $f\colon K^n \to K^l$ for some $l \in \mathbb{N}_+$ such that $\forall \bar{x}, \bar{y} \in K^n$, $\bar{x}E\bar{y} \iff f(\bar{x}) = f(\bar{y})$.

The name of the property comes from the fact that we can identify the quotient M^n/E with the image of f - every *imaginary* element can be represented by a tuple of *real* elements via f.

The more standard ways to prove elimination of imaginaries for ACF often provide a more general method than the proof we will give here. Our proof, though not quite as beautiful, is more specific to algebraically closed fields. The original, more algebraic, proof is found in [Poi89].

We start by examining a special case:

Lemma 2. *Let $\mathcal{K}$ be any field and let E be the equivalence relation*

$$\forall \bar{x} \forall \bar{y} \left(\bar{x}E\bar{y} \iff (\bar{x}_1, ..., \bar{x}_n) \text{ is a permutation of } (\bar{y}_1, ..., \bar{y}_n) \right),$$

where $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$, $\bar{y} = (\bar{y}_1, ..., \bar{y}_n) \in K^{mn}$.

Then, for some $l \in \mathbb{N}_+$, there is a definable function $f\colon K^{mn} \to K^l$ such that $\bar{a}E\bar{b} \iff f(\bar{a}) = f(\bar{b})$.

Proof. Let $\bar{a} \in K^{mn}$ such that $\bar{a} = (\bar{a}_1, ..., \bar{a}_n)$, where $\bar{a}_i = (a_{i,1}, ..., a_{i,m})$. Let $p_{\bar{a}_i} \in K[X_1, ..., X_m, Y]$ denote the polynomial $Y - \sum_{j=1}^{m} a_{i,j}X_j$, and let $p_{\bar{a}} = \prod_{i=1}^{n} p_{\bar{a}_i}$.

We know from algebra, that if $\mathcal{F}$ is a field, then $F[\bar{X}]$ is a unique factorization domain, i.e polynomials over F factor uniquely. Thus $p_{\bar{a}} = p_{\bar{b}}$ if and only if the sequence of factors $(p_{\bar{a}_1}, ..., p_{\bar{a}_n})$ is a permutation of $(p_{\bar{b}_1}, ..., p_{\bar{b}_n})$. And finally, if we let $f(\bar{a})$ be the sequence of coefficients of $p_{\bar{a}}$, we have that $\bar{a}E\bar{b} \iff f(\bar{a}) = f(\bar{b})$.

The reader should be easily convinced that both f and E above are definable. □

Before we continue, we introduce the model-theoretic notion of "algebraic". Worth noting is that it can be shown to coincide with the field-theoretic notion.

Definition 9. Let $\mathcal{M}$ be a structure, $A \subseteq M$ and E a definable equivalence relation on M^n. Then

(I) $a \in M$ is said to be *algebraic* over A if there is a formula $\phi(x, y_1, ..., y_k)$ and $\bar{b} \in A^k$ such that $\mathcal{M} \models \phi(a, \bar{b})$ and $|\{x \in M \mid \mathcal{M} \models \phi(x, \bar{b})\}| < \omega$.

(II) $a \in M$ is algebraic over $\{\bar{b}/E, \bar{c}\}$ for $\bar{b} \in M^n$ and $\bar{c} \in M^m$ if there is a formula $\psi(x, y_1, ..., y_n, z_1, ..., z_m)$ such that

   (i) $\mathcal{M} \models \psi(a, \bar{b}, \bar{c})$,

(ii) $|\{x \in M \mid \mathcal{M} \models \psi(x, \bar{b}, \bar{c})\}| < \omega$, and

(iii) $\bar{b}E\bar{b}' \Rightarrow \mathcal{M} \models \forall x(\psi(x, \bar{b}, \bar{c}) \leftrightarrow \psi(x, \bar{b}', \bar{c}))$.

(III) $\bar{a} = (a_1, ..., a_l)$ is algebraic over $\{\bar{b}/E, \bar{c}\}$ if each a_i is.

Lemma 3. *If $\bar{a}$ is algebraic over $\{\bar{b}/E, \bar{c}, \bar{d}\}$ and $\bar{d}$ is algebraic over $\{\bar{b}/E, \bar{c}\}$, then $\bar{a}$ is algebraic over $\{\bar{b}/E, \bar{c}\}$.*

Proof. Let each a_i in $\bar{a}$ be algebraic over $\{\bar{b}/E, \bar{c}, \bar{d}\}$ by the formula $\phi_i(x, \bar{u}, \bar{v}, \bar{w})$ and let each d_j in $\bar{d}$ be algebraic over $\{\bar{b}/E, \bar{c}\}$ by the formula $\psi_j(w, \bar{u}, \bar{v})$. By definition,

(1) $\mathcal{M} \models \phi_i(a_i, \bar{b}, \bar{c}, \bar{d})$

(2) $|\{x \in M \mid \mathcal{M} \models \phi_i(x, \bar{b}, \bar{c}, \bar{d})\}| < \omega$

(3) $\bar{b}E\bar{b}' \Rightarrow \mathcal{M} \models \forall x(\phi_i(x, \bar{b}, \bar{c}, \bar{d}) \leftrightarrow \phi_i(x, \bar{b}', \bar{c}, \bar{d}))$

and

(1') $\mathcal{M} \models \psi_j(d_j, \bar{b}, \bar{c})$

(2') $|\{w \in M \mid \mathcal{M} \models \psi_j(w, \bar{b}, \bar{c})\}| < \omega$

(3') $\bar{b}E\bar{b}' \Rightarrow \mathcal{M} \models \forall w(\psi_j(w, \bar{b}, \bar{c}) \leftrightarrow \psi_j(w, \bar{b}', \bar{c}))$.

Let $\psi(\bar{w}, \bar{u}, \bar{v}) \equiv \bigwedge \psi_j(w_j, \bar{u}, \bar{v})$, and let k_i be such that $|\{x \in M \mid \mathcal{M} \models \phi_i(x, \bar{b}, \bar{c}, \bar{d})\}| \leq k_i$. "$\exists_{\leq k}$" is an abbreviation and denotes the sentiment expressible in first-order logic that "there exists less than or equal to k". Consider

$$\theta_i(x, \bar{u}, \bar{v}) \equiv \exists \bar{w}(\phi_i(x, \bar{u}, \bar{v}, \bar{w}) \wedge \exists_{\leq k_i} x' \phi_i(x', \bar{u}, \bar{v}, \bar{w}) \wedge \psi(\bar{w}, \bar{u}, \bar{v})).$$

We will show that θ_i makes a_i algebraic over $\{\bar{b}/E, \bar{c}\}$.

Items (1), (2) and (1') together imply $\mathcal{M} \models \theta_i(a_i, \bar{b}, \bar{c})$.

Item (2') and the $\exists_{\leq k_i}$ part of θ_i ensures $|\{x \in M \mid \mathcal{M} \models \theta_i(x, \bar{b}, \bar{c})\}| < \omega$.

Suppose $\bar{b}E\bar{b}'$. Then it follows from items (3) and (3') that $\mathcal{M} \models \forall x(\theta_i(x, \bar{b}, \bar{c}) \leftrightarrow \theta_i(x, \bar{b}', \bar{c}))$.

Thus each a_i is algebraic over $\{\bar{b}/E, \bar{c}\}$ by the formula $\theta_i(x, \bar{u}, \bar{v})$.

□

Lemma 4. *Let $\mathcal{K} \models$ ACF and let E be an equivalence relation on K^n defined by $\phi(\bar{x}, \bar{y}, \bar{c})$. If $\bar{b} \in K^n$, then there is $\bar{a} \in K^n$ algebraic over $\{\bar{b}/E, \bar{c}\}$ such that $\bar{a}E\bar{b}$.*

Proof. Let $m \leq n$ be maximal non-negative integer for which there is $\bar{a} = (a_1, ..., a_m)$ algebraic over $\{\bar{b}/E, \bar{c}\}$ such that

$$\mathcal{K} \models \exists u_{m+1} ... \exists u_n \phi(\bar{a}, \bar{u}, \bar{b}, \bar{c}).$$

If we can show that $m = n$, then $\mathcal{K} \models \phi(\bar{a}, \bar{b}, \bar{c})$ and $\bar{a}$ is the tuple satisfying the lemma. Suppose, toward a contradiction, that $m < n$ and consider the set

$$X = \{x \in K \mid \mathcal{K} \models \exists v_{m+2} ... \exists v_n \phi(\bar{a}, x, \bar{v}, \bar{b}, \bar{c})\}.$$

If X is finite, then any $a_{m+1} \in X$ is algebraic over $\bar{b}/E, \bar{c}, \bar{a}$ and, by Lemma 3, also over $\bar{b}/E, \bar{c}$ contradicting the maximality of m.

If X is infinite, then, by strong minimality, $K \setminus X$ is finite and unable to contain all elements a_{m+1} algebraic over $\emptyset$ (and thus over $\bar{b}/E, \bar{c}$). Again, this contradicts the maximality of m.

Thus $m = n$.

□

The different ways to go about proving elimination of imaginaries for ACF all involve some degree of more advanced theory than what we have seen so far. We will keep further theory to a minimum, though the introduction of a few new concepts is seemingly unavoidable.

Definition 10. Let $\mathcal{M} \models$ T, where T is a complete theory with infinite models in a countable language $\mathcal{L}$. For $A \subseteq M$, let $\mathcal{L}_A$ be $\mathcal{L}$ with constant symbols added for every element in A, interpreted in the natural way. $\text{Th}_A(\mathcal{M})$ is then the set of $\mathcal{L}_A$-sentences true in $\mathcal{M}$.

(i) An *n-type* is a set p of $\mathcal{L}_A$-formulas in n free variables which is consistent with $\text{Th}_A(\mathcal{M})$.

(ii) An n-type p is *complete* if for all $\mathcal{L}_A$-formulas $\phi(x_1, ..., x_n)$, either $\phi \in p$ or $\neg\phi \in p$.

(iii) $S_n^{\mathcal{M}}(A)$ denotes the set of all complete n-types over A.

(iv) An n-type p is *realized* if there is a $\bar{m} \in M^n$ such that $\mathcal{M} \models \phi(\bar{m})$ for every $\phi \in p$.

Definition 11. Let $\mathcal{M}$ be an infinite model of a complete theory T in a countable language $\mathcal{L}$.

$\mathcal{M}$ is *saturated* if for every $n \in \mathbb{N}_+$ and every $A \subseteq M$ such that $|A| < |M|$, every $p \in S_n^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$.

Definition 12. Let T be a complete theory in a countable language and κ an infinite cardinal.

T is *κ-stable* if for all $n \in \mathbb{N}_+$, $|S_n^{\mathcal{M}}(A)| = \kappa$ whenever $\mathcal{M} \models$ T, $A \subseteq M$, and $|A| = \kappa$.

We admit the following two well-known results without proof.

Proposition 6. *(i) If a theory is uncountably categorical, it is ω-stable.*

(ii) If a theory is ω-stable, each of its models has an elementary extension which is saturated.

Thus any algebraically closed field has a saturated elementary extension, and this conclusion is extremely useful; Proving that a model of some theory admits a certain property can often be easier if the model is saturated. However, if the property is expressible with a first order sentence, it transfers to its elementary substructures.

At last, we are ready to prove our final theorem.

Theorem 4. ACF *eliminates imaginaries.*

That is, if we let $\mathcal{K} \models$ ACF and if E is an $\emptyset$-definable equivalence relation on K^n, then for some $l \in \mathbb{N}_+$ there is an $\emptyset$-definable function $f\colon K^n \to K^l$ such that $\bar{x}E\bar{y} \iff f(\bar{x}) = f(\bar{y})$.

Proof. Let $\mathcal{K} \models$ ACF be uncountable and saturated and let E be an $\emptyset$-definable equivalence relation on K^n.

For every formula $\phi(\bar{x}, \bar{y})$ and every $k \in \mathbb{N}_+$, let $\Theta_{\phi,k}(\bar{y})$ be the conjunction of

(i) $\forall \bar{x}(\phi(\bar{x}, \bar{y}) \to \bar{x}E\bar{y})$;

(ii) $\forall \bar{x}\forall \bar{z}(\bar{y}E\bar{z} \to (\phi(\bar{x}, \bar{y}) \leftrightarrow \phi(\bar{x}, \bar{z})))$;

(iii) $\exists_{=k}\bar{x}(\phi(\bar{x}, \bar{y}))$.

By Lemma 4, we can for all $\bar{a} \in K^n$ find ϕ and k such that $\Theta_{\phi,k}(\bar{a})$ holds. By (ii), $\Theta_{\phi,k}(\bar{a})$ and $\bar{b}E\bar{a}$ together imply $\Theta_{\phi,k}(\bar{b})$.

For all $\alpha \in I = \{(\phi, k) \mid \phi \text{ is a formula and } k \in \mathbb{N}_+\}$, let X_α be the set defined by $\Theta_\alpha(\bar{y})$ and let $p = \{\neg\Theta_\alpha(\bar{x}) \mid \alpha \in I\}$.

Suppose, toward a contradiction, that p is consistent. Then it is an n-type over K. Since $|p| < |K|$, it must even be an n-type over some subset $A \subset K$ with $|A| < |K|$. Let q be the complete type over A extending p in accordance with $\mathrm{Th}_A(\mathcal{K})$. By saturation, q is realized in $\mathcal{K}$, and with it, so is p. But a realization of p in $\mathcal{K}$ is a contradiction, since for all $\bar{a} \in K$, there is an $\alpha \in I$ such that $\Theta_\alpha(\bar{a})$ holds.

Thus p is inconsistent. By the compactness theorem, it is finitely inconsistent. Therefore there are formulas $\phi_1, ..., \phi_m$ and positive integers $k_1, ..., k_m$ such that, for all $\bar{a} \in K^n$, there is some $i \leq m$ such that $\Theta_{\phi_i,k_i}(\bar{a})$ holds.

If for every $\bar{a} \in X_i = \{\bar{y} \mid \Theta_{\phi_i,k_i}(\bar{y})\}$, we let $Y_i(\bar{a}) = \{\bar{b} \mid \phi_i(\bar{b}, \bar{a})\}$, then for every $\bar{a}, \bar{b} \in X_i$, $\bar{a}E\bar{b} \iff Y_i(\bar{a}) = Y_i(\bar{b})$. Since $Y_i(\bar{a})$ and $Y_i(\bar{b})$ are finite sets, $Y_i(\bar{a}) = Y_i(\bar{b})$ is equivalent to stating that any ordering of $Y_i(\bar{a})$ and $Y_i(\bar{b})$ will be permutations of one another. Then, by Lemma 2, there are $\emptyset$-definable functions $g_i : X_i^{k_i} \to K^{l_i}$ for some l_i which we can use to define functions $f_i : K^n \to K^{l_i}$ as $f_i(\bar{x}) = (0, \ldots, 0)$ if $\bar{x} \notin X_i$ and $f_i(\bar{x}) = g_i(\bar{y}_1, \ldots, \bar{y}_{k_i})$, where $\bar{y}_1, \ldots, \bar{y}_{k_i}$ is an enumeration of $Y_i(\bar{x})$, if $\bar{x} \in X_i$. Our f_i are then such that $Y_i(\bar{a}) = Y_i(\bar{b}) \iff f_i(\bar{a}) = f_i(\bar{b})$ or, equivalently, $\bar{a}E\bar{b} \iff f_i(\bar{a}) = f_i(\bar{b})$ for $\bar{a}, \bar{b} \in X_i$.

Finally, let $f\colon K^n \to K^{\sum l_i}$ be the function $\forall \bar{x} \in K^n, f(\bar{x}) = (f_1(\bar{x}), ..., f_m(\bar{x}))$. Then $\forall \bar{a}, \bar{b} \in K^n, \bar{a}E\bar{b} \iff f(\bar{a}) = f(\bar{b})$, and we have, for $l = \sum l_i$, found our $\emptyset$-definable function f.

The above asserts that the theorem is true for any uncountable saturated algebraically closed field $\mathcal{K}$. However, by Proposition 6 together with the Upward Löwenheim-Skolem Theorem, every algebraically closed field has an uncountable saturated elementary extension. Since the property of eliminating imaginaries is expressible with a first order sentence, it transfers to any elementary submodel of $\mathcal{K}$, making the theorem true for all algebraically closed fields.

□

References

[Bou98] Elisabeth Bouscaren (Ed.). *Model Theory and Algebraic Geometry: An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture* (Lecture Notes in Mathematics; 1696), Springer, 1998.

[Hod96] Wilfrid Hodges. *Model Theory*, Cambridge University Press, 1996.

[HPS00] Deirdre Haskell, Anand Pillay, Charles Steinhorn (Ed.). *Model Theory, Algebra, and Geometry*, Cambridge University Press, 2000.

[Lan02] Serge Lang. *Algebra*, (revised 3rd ed.) Springer, 2002.

[Mar02] David Marker. *Model Theory: An Introduction*, Springer, 2002.

[MMP96] David Marker, Margit Messmer, Anand Pillay. *Model Theory of Fields*, Springer, 1996.

[Poi89] Bruno Poizat. *An introduction to algebraically closed fields and varieties*, pp. 41-67 in: The model theory of groups, Ali Nesin and Anand Pillay (Eds.), Univ. Notre Dame Press, 1989.

[Ste10] Ernst Steinitz. *Algebraische Theorie der Körper*, pp. 167-309 in: Journal für die reine und angewandte Mathematik vol. 137, 1910.

[Tar57] Alfred Tarski. *A decision method for elementary algebra and geometry*, (2nd ed.) Rand Corporation Publication, 1957.