Abstract

In this project we review the method of using c-extremization and computing anomalies to obtain AdS/CFT theories. We start with a quick introduction to CFT’s and AdS/CFT correspondence which gives us the tools to later understand the 2D $\mathcal{N} = (2,0)$ SCFT and its gravity duals in particular $AdS_5 \times S^5$ and $AdS_7 \times S^4$ compactified on Riemann surfaces.

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Contents

1 Introduction 1

2 Conformal field theory 2
  2.1 The conformal group .................................................. 2
  2.2 Operator product expansions and correlation functions ......... 5
  2.3 $\mathcal{N} = 2$ superconformal algebra ............................. 7
  2.4 Anomaly polynomials .................................................... 8
  2.5 Superconformal R-symmetry in two dimensions ..................... 12

3 AdS/CFT correspondence 15
  3.1 $\mathcal{N} = 4$ 4D SYM ................................................... 15
  3.2 M5 ................................................................. 22

4 Discussion 32

Appendices 34

A BPS equations $AdS_5 \times S^5$ STU model 34

B BPS equations for $AdS_7 \times S^4$ 34
1 Introduction

One of the biggest achievements in String theory is the \((AdS/CFT)^1\) correspondence proposed by \([1]\), this correspondence tells us that there is a relation between gravity theories and field theories. We can use this duality to study strongly coupled Quantum field theories by using classical gravitational theories. The classical gravity theories we will study for the correspondence is Supergravity, this theory is the combination of supersymmetry and General relativity. It is also known as the low energy effective field theory of String theory and M-theory. The \(AdS/CFT\) correspondence has applications outside of String theory such as how to explain confinement and chiral-symmetry breaking in QCD, it also plays a role in understanding superconductors this duality is known as the \(AdS/CMT\).

The first example of the correspondence was the \(N = 4\) \(D = 4\) SYM and the \(AdS_5 \times S^5\) solution where we have a stack of \(N\) D3 branes in type-IIB string theory. Taking the large \(N\) limit of the field theory such that the brane decouples from the bulk and looking at the near horizon geometry where we now observe a black hole of \(AdS_{D+1}\), we can trust the supergravity solution as long as \(N\) is large. The near horizon limit plays a crucial role in studying the correspondence an example of this is when one studies thebrane solutions of \(M\)-theory \(i.e\) \(M2\) and \(M5\) membranes. One takes two limits of the membrane solutions: one where \(r\) goes to infinity which gives a Minkowski geometry this is what we observe on the boundary where the field theory lives. The near horizon limit where \(r\) goes to zero reveals the geometry of the black hole in the case for \(M2\) the black hole is described by \(AdS_4 \times S^7\) and the field theory dual to this is the \(ABJM\) theory \([2]\) which is a three dimensional Chern-Simons matter theory. The near horizon limit of \(M5\) gives the black hole geometry \(AdS_7 \times S^4\) and the field theory dual to this is the \((2,0)\) Super Conformal field theory (SCFT), this field theory has no Lagrangian description and makes it very difficult to study. The \((2,0)\) theory is the grand emperor of SCFT’s one can compactify this theory on several geometries \(e.g\) Riemann surfaces to obtain lower dimensional SCFT and study their gravity duals. A very interesting case is to study two dimensional CFT where the conformal group is infinite-dimensional, these theories are highly constraint and in some cases exactly solvable. In this project we will mainly focus on compactifying the \((2,0)\) theory on Riemann surfaces describes by \(\mathbb{R}^d \times \Sigma_g\) these theories describe twisted field theories at low energies. The so called twisted theories are obtained by making the choice such that the external gauge fields are equal to the spin connection, this choice is made so that we can consider a constant spinor. One can effectively see this as the coupling to the external gauge fields changing the spin of all fields \([3]\).

In order to compute the central charges of these two dimensional SCFT one needs to obtain the exact R-symmetry, in a non-conformal supersymmetric theory with a R-symmetry \(U(1)_R\) and Abelian flavor symmetries, the R-symmetry is not uniquely defined since any linear combination of these symmetries produce a equally good R-symmetry. If a theory flows to an IR fixed point the R-symmetry is singled out and computing the exact R-symmetry is a non-trivial task. It was shown in \([4]\) that using c-extremization where one does not have any mixing of symmetries, one can compute the exact R-symmetry in a unitary SCFT with normalizable vacuum. We will consider all possible Abelian currents where we single out the R-symmetry current and in doing so one can construct a quadratic function that is proportional to the ‘t Hooft anomalies. The quadratic function is then extremized and from this one obtains the exact R-symmetry current and moreover the function at the critical point is equal to the central charge at the IR fixed point.

The supergravity theories we consider describe some flow from \(d + 2\) dimensional

\(^1\)Anti de Sitter/Conformal field theory
field theory to a \( d \) dimensional field theory. The twisted theories we will consider is \( \mathcal{N} = 4 \ D = 4 \) SYM which we compatify on a Riemann surface \( \Sigma_g \), this gives us a two dimensional SCFT where the gravity dual is \( AdS_3 \) in the IR limit and at the UV limit we describe some \( AdS_5 \) geometry. The amount of supersymmetry one preserves in these solutions depends on the twisting parameter \( i.e \) it depends on how one embeds the surface \( \Sigma_g \) in a higher dimensional theory. The scalar fields in the gravity description are completely described by the twisting parameters this allows one to investigate where the good \( AdS \) vacua is for the different choices of Riemann surfaces. Using holographic tools one can then compute the large \( N \) limit of the supergravity theory and for all the cases we consider in this project they are indeed in agreement with the central charge computed from the field theory. The other case we will consider is the six dimensional \((2, 0)\) SCFT with the gravity dual describe by the gauged supergravity in seven dimensions \([5]\) with the twisted compactification of four manifolds, \( i.e \) the product of two Riemann surfaces. The low energy theory is described as a \( M5 \) membrane wrapping the product of the two Riemann surfaces in a non-trivial construction. The gravity solution in this construction also admits a flow between \( AdS_3 \) and \( AdS_7 \) which will be computed numerically.

The outline of this project is as follows: in section one we start off with the general CFT where we present the necessary tools, we also discuss the anomaly polynomials and how the c-extremization technique is constructed. We then move on to section three where we go through our two example and compute the central charges in the field theory and by using holographic tools we obtain the large \( N \) limit of the gravity theory. We also plot the good \( AdS \) vacua and numerically compute the flows for the given theories.

## 2 Conformal field theory

In this section we will briefly go through some concepts about Conformal field theories (CFT) which in this project will be crucial to later explain the \( AdS/CFT \) correspondence. We will start with the conformal group and then move on to discuss operator product expansions and mention the map between states and local operators.

Then write down the \( \mathcal{N} = 2 \) superconformal algebra and discuss the superconformal R-symmetry and also discuss anomalies in two dimensions. The key references will be \([6–9]\)

### 2.1 The conformal group

We will start with considering a \( D \) dimensional manifold that is conformally flat, where the conformal symmetry is the symmetry that preserves angles. A conformal transformation in a \( D \) dimensional spacetime is a change of coordinates that rescale the line element \( i.e \) there should be a transformation that changes the metric up to a conformal factor

\[
\text{dilatation: } \quad x_\mu \to \lambda x_\mu \quad dx^2 \to \lambda^2 dx^2 , \\
\text{conformal transformation: } \quad x_\mu \to x'_\mu \quad dx^2 \to dx'^2 = \Omega^2 dx^2 ,
\]

(2.1)

where \( \Omega \) is the conformal factor that depends on the spacetime coordinates and the conformal transformations implies that there is an inversion element. Taking an infinitesimal transformation \( x'_\mu \to x_\mu + \xi_\mu \) implies that the symmetry is determined by the solution of the conformal Killing equation

\[
\partial_{(\mu} \xi_{\nu)} - \frac{2}{D} \eta_{\mu\nu} \partial_{\rho} \xi_{\rho} = 0 .
\]

(2.2)

notice in \( D = 2 \) with a non-zero metric elements \( \eta_{zz} = 1 \) the Killing equation is reduced to \( \partial_z \xi_z = \partial_{\bar{z}} \xi_{\bar{z}} = 0 \) which implies that there is infinite dimensional conformal algebra,
this is however not true for conformal algebras in $D > 2$. The general solution for the infinitesimal transformation is given as
\begin{equation}
\xi^\mu = a^\mu + \lambda_\mu^{\mu\nu}x_\nu + \lambda_D x_\mu + (x^2 \lambda_K^\mu - 2x^\mu x \cdot \lambda_K) ,
\end{equation}
where $a^\mu$ are the translations $P_\mu$, the Lorentz rotations $M_{\mu\nu}$, corresponds to $\lambda_\mu^{\mu\nu}$, the dilatation $\lambda_D$ is generated by $D$ and $\lambda_K^\mu$ are the parameters of the special conformal transformation $K_\mu$. This can be expressed as the full set of conformal transformations $\delta_C$
\begin{equation}
\delta_C = a^\mu P_\mu + \lambda_\mu^{\mu\nu}M_{\mu\nu} + \lambda_D D + \lambda_K^\mu K_\mu ,
\end{equation}
the vacuum of a conformal theory is annihilated by all these generators and the number of generators is given as $\frac{1}{2}(D+2)(D+1)$ which is isomorphic to $SO(2, D)$. The conformal algebra for these generators is given as
\begin{equation}
[M_{\mu\nu}, M_{\rho\sigma}] = -2\eta_{[\mu, \rho]}^{\ [\sigma]}, \ [P_\mu, M_{\nu\rho}] = \eta_{\mu}[\nu] P_\rho, \ [M_{\mu\nu}, D] = 0, [D, K_\mu] = -K_\mu , \ [M_{\mu\nu}, K_\rho] = \eta_{\mu}[\nu] K_\rho, \ [P_\mu, K_\nu] = 2\eta_{\mu\nu} D + 2 M_{\mu\nu}, [D, P_\mu] = P_\mu .
\end{equation}
The first commutator is the algebra of the Lorentz group $SO(1, D - 1)$, the commutators between $M_{\mu\nu}$ and $P_\mu, K_\mu, D$ state that $D$ is a scalar and $P_\mu, K_\mu$ are vectors, the commutators between $D$ and $P_\mu, K_\mu$ are the ladder operators that increases and decreases its eigenvalues respectively and the commutator between $K_\mu$ and $P_\mu$ state that the $P$ and $K$ close on a Lorentz transformation and a dilatation. All the generators can be assembled in the following way
\begin{equation}
M_{MN} = \begin{pmatrix}
M_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\
-\frac{K_\mu - P_\mu}{2} & 0 & D \\
\frac{K_\mu + P_\mu}{2} & -D & 0
\end{pmatrix},
\end{equation}
one thing to notice is that this is the same as the algebra for $AdS_{D+1}$ which implies that there is a correspondence between conformal algebra and AdS algebra which is a crucial property of AdS/CFT. A scale invariant theory is also conformally invariant and one can construct currents that are associated with the conformal transformations $J_\mu = T_{\mu\nu}\xi^\nu$. The conservation of the current corresponds to translations of the current which requires the conservation of the stress energy tensor $\partial^\mu T_{\mu\nu} = 0$. If the stress energy tensor is symmetric then this implies that the conservation of the current corresponds to Lorentz transformations and the current for dilation $J_\mu = T_{\mu\nu}x^\nu$ is conserved if
\begin{equation}
\partial^\mu (T_{\mu\nu} x^\nu) = T^\nu_\nu = 0 .
\end{equation}
The condition of scale invariance is precisely the tracelessness of the stress energy tensor, this implies that a scale invariant theory the conformal currents are automatically conserved if the stress energy tensor is symmetric i.e
\begin{equation}
\partial^\mu (T_{\mu\nu} \xi^\nu) = 0 .
\end{equation}
One thing to stress here is that this is on a classical level, once we consider a quantum theory the stress energy tensor might not vanish as one will see when considering curved backgrounds where trace anomalies appear. Requiring that the left moving central charge is equal to the right moving central charge $c_R = c_L$ tells us that the theory we consider has no gravitational anomalies this will be clear when we consider four dimensional $\mathcal{N} = 4$ Super Yang-Mills.
Theories without scales and dimensionful parameters are classically scale invariant, recall the $\phi^4$ theory in QFT the action is scale invariant if we rescale the spacetime coordinates and the field with a specific weight

$$\phi(x) \to \lambda^\Delta \phi(\lambda x),$$

where $\Delta$ is the scaling dimension and $\lambda$ is the coupling constant. When a theory is conformally invariant the mass operator $P_\mu P^\mu$ does not commute anymore with other generators e.g. the dilatation $D$ due to this the S-matrix formulation does not make sense. The mass and energy can be rescaled by a conformal transformation this implies that the states corresponding to the S-matrix has energy values going from zero to infinity and there is no good way of labeling the states. In a conformal theory we want good conformal transformations properties for dilatations where we set $\lambda = e^{\alpha}$ such that $e^{i\alpha D}$ generates a dilatation.

The quantum version of equation (2.9) is given as $[D,\phi(x)] = i(\Delta + x_\mu \partial^\mu u)\phi(x)$ which identifies fields of conformal dimensions $\Delta$. We will also restrict to fields or operators that annihilated at $x = 0$ by a lowering operator $K_\mu$ these are called primary operators and $P_\mu$ is called descendants. In unitary field theories there is a lower bound on the dimension of fields and the representation of each conformal group must have some operator of lowest dimension which must be annihilated by $K_\mu$ i.e primary operators are classified according to the dimension $\Delta$ and the Lorentz quantum numbers. Before we continue with operator product expansions and the constraints that appear in CFT's let us further discuss the quantum numbers associated with the primary operators.

Consider states that are labeled by $(\Delta,j_R,j_L)$ where we define a primary conformal field in the $(j_R,j_L)$ representation of the Lorentz group by

$$[D,O_{(j_R,j_L)}(0)] = i\Delta O_{(j_R,j_L)}(0),$$

$$[K_\mu,O_{(j_R,j_L)}(0)] = 0,$$

the descendents $\partial \ldots \partial O_{(j_R,j_L)}(0)$ reconstruct the operator by Taylor expansions and the representations correspond to the conformal dimensions and the Lorentz quantum numbers of the primary operators. Using the compact subgroup $SO(2) \times SO(4) \subset SO(2,4)$ allows us to study and classify unitary representations of the conformal group using states with finite norm. We classify the states by the eigenvalues of $H = (P_0 + K_0)/2$ and $SO(4) = SU(2) \times SU(2)$ identified with $(\Delta,j_R,j_L)$. Unitary imposes bounds on the representations where one demands that all the states in the representation have positive norm [7]. The saturation of the bounds correspond to representations with null states which can be removed by a shorter representation due to differential constraints on the primary field, the constraints are given as

$$\Delta \geq 1 + j_R \ j_L = 0, \ \text{or} (j_R \to j_L),$$

$$\Delta \geq 2 + j_R + j_L \ (j_R,j_L \neq 0).$$

The two unitary bounds are satisfied by free massless fields and conserved tensor fields, the saturation of the bounds correspond to

$$\partial^2 \Phi_{(0,j_L)} = 0, \ \partial^{\alpha_1 \dot{\alpha}_1} O_{\alpha_1 \ldots \alpha_2 j_R \dot{\alpha}_1 \ldots \dot{\alpha}_2 j_L} = 0,$$

the first bound for $j_R = 0$ say that the dimensions of the scalar primary is $\Delta \geq 1$ and if the field is free this is one. The second bound for $j_L = j_R = 1/2$ says that the spin one operator $J_\mu$ has dimension greater than 3 and equal to 3 iff it is a conserved current $\partial^\mu J_\mu = 0$, this bound can be generalised and is given as

$$\Delta \geq \frac{d-2}{2}.$$
This can be extended to superconformal group and this will be what we consider from now on, the superconformal group $SU(2,2|N)$ correspond to a theory with $N$ supersymmetries by adding $N$ supercharges $Q^a$ and $N$ superconformal charges $S^a$ and the generators of a $U(\mathcal{N})$ global symmetry $R^a$. The commutation relations are given as
\[
[D, Q^a] = i\frac{1}{2} Q^a, \quad [D, S^a] = i\frac{1}{2} S^a, \quad [K_\mu, Q^a] = -i(\sigma_\mu)_\alpha^\beta S^\alpha, \\
[P_\mu, S_{\alpha}^a] = (\sigma_\mu)^\alpha^\beta Q_{\beta a}, \quad [Q^a_{\alpha}, Q_{\beta b}] = 2\delta^b_a (\sigma^\mu)_{\alpha\beta} P_\mu, \quad [S_{\alpha}^a, S^b_\beta] = 2\delta^b_a (\sigma^\mu)^\alpha^\beta K_\mu \\
\{Q^a_\alpha, S_{\beta}^b\} = -\delta^b_a (\sigma^\mu)^\alpha^\beta J_{\mu\nu} - 2i\delta^a_{\alpha} \delta^\beta_{a} D - 4\delta^b_a R^a_{\mu},
\] (2.14)
where $(Q^a)^\dagger = \bar{Q}_{\alpha a}$ and $(S^a)^\dagger = \bar{S}_{\alpha\dot{a}}$. The first two commutators specify the dimensions of the charges where $Q$ and $S$ are the raising and lowering operator respectively for $D$, we also see that we needed to introduce the charge $S$ in order to close the algebra. The last one is the commutation relation for the conformal partner. There are also commutators for the $R$-symmetry quantum numbers which are given as
\[
[R^a_{\mu}, Q^a_\alpha] = \delta^b_a Q^a_\alpha - \frac{1}{4} \delta^b_a Q^c_\alpha, \quad [R^a_{\mu}, S^b_\alpha] = -\delta^b_a S^a_\beta + \frac{1}{4} \delta^b_a S^c_\alpha,
\] (2.15)
the $R^a_{\mu}$ are the generators of $U(\mathcal{N})$ and close the corresponding algebra. Superconformal representations have a lowest state which is annihilated by both $K$ and $S$ which is identified by $\Delta, j_L, j_R$ under the conformal group $R, a_1 \ldots a_{N-1}$ under the $R$-symmetry $U(1) \times SU(\mathcal{N})$.

### 2.2 Operator product expansions and correlation functions

We will now define local operators for the CFT’s these objects are also called fields, fields in the CFT term refer to any local expression we can write which includes $\phi$ and its derivatives $\partial^n \phi$ or operators such as $e^{i\phi}$. All of these are fields in the CFT and we will now define the operator product expansion (OPE) which tells us what happens as local operators approach each other. Two local operators inserted at a points can be closely approximated by a string of operator at one of these points where we denote the local operators as $O_i$ and the OPE is defined as
\[
O_i(z, \bar{z})O_j(w, \bar{w}) = \sum_k C^k_{ij}(z - w, \bar{z} - \bar{w})O_k(w, \bar{w}).
\] (2.16)
The $C^k_{ij}(z - w, \bar{z} - \bar{w})$ are a set of functions which only depend on the separation of two operators. The correlation functions are always assumed to be time-ordered which means that everything commutes since the ordering is inside the correlation function. The OPEs have a singular behaviour as $z \to w$ and this in general is all we will need to know.

Since the conformal group is much larger than the Poincaré group this sets restrictions on the correlation functions of primary fields which must be invariant under conformal transformation. The Ward identities for the conformal group gives constraints on the Green functions and one will always have primary operators $O_i$ with fixed scale dimension $\Delta_i$. The set of $(O_i, \Delta_i)$ gives the spectrum of the CFT and the two point functions and three point functions are completely fixed by conformal invariance which are equal to
\[
\langle O_i(z)O_j(w) \rangle = \frac{A_{\Delta_i}}{|z - w|^{2\Delta_i}},
\] (2.17)
and the three-point function is given as
\[
\langle O_i(z)O_j(w)O_k(u) \rangle = \frac{\lambda_{ijk}}{|z - w|^{\Delta_i + \Delta_j - \Delta_k}|w - u|^{\Delta_j + \Delta_k}|u - z|^{\Delta_i + \Delta_k - \Delta_k}}.
\] (2.18)
where \(\lambda_{ijk}\) is a constant. The field algebra of any conformal field theory includes the energy momentum tensor \(T_{\mu\nu}\) which is an operator of dimensions \(\Delta = d\) and the Ward identities of the conformal algebra relate correlation functions with \(T\) to correlation functions without \(T\). When there is global symmetries the conserved currents \(J_\mu\) are necessarily operators of dimension \(\Delta = d - 1\), the scaling dimension of other operators are not determined by the conformal group. The leading order of singularities for a primary operator whose OPE with the stress energy tensor is of order \((z-w)^{-2}\) and the OPE of this form tells us exactly the conformal dimension and the OPE of the stress energy tensor with itself gives us the central charge of the CFT.

One of the nice properties of CFTs is the one-to-one correspondence between local operators \(\mathcal{O}\) and the states \(|\mathcal{O}\rangle\) in the radial quantization of the theory, such that the Virasoro generators \(L_0, L_0^2\) which are bounded from below satisfy \(L_0|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle\) and \(L_n|\mathcal{O}\rangle = 0\) for \(n > 0\). These states are called highest weight state. In radial quantization the time coordinate is chosen to be the radial direction in \(\mathbb{R}\) and \(L\) the Virasoro generators \(L\) operators \(O\) energy tensor with itself gives us the central charge of the CFT.

For simplicity we only consider the holomorphic current and the zero modes \(J\) primary operator whose OPE with the stress energy tensor is of order \((z-w)\) and the states \(|O\rangle\) can be mapped to the state \(|\mathcal{O}\rangle = \lim_{z\to 0}\mathcal{O}(z)|0\rangle\). We can also define the action of the conformal group generators on a primary field given by

\[
[D, \mathcal{O}(z)] = i(-\Delta + x^\mu \partial_\mu)\mathcal{O}(z), \quad [M_{\mu\nu}, \mathcal{O}(z)] = i(x_{[\mu} \partial_{\nu]} - x_{\nu} \partial_\mu + \Sigma_{\mu\nu})\mathcal{O}(z), \\
[P_\mu, \mathcal{O}(z)] = i\partial_\mu \mathcal{O}(z), \quad [K_\mu, \mathcal{O}(z)] = \left(i\left(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta\right) - 2x^\nu \Sigma_{\mu\nu}\right)\mathcal{O}(z)
\]

(2.19)

where \(\Sigma_{\mu\nu}\) are the matrices of a finite dimensional representation of the Lorentz group acting on the indices of the primary field \(\mathcal{O}\). While we are still on the discussion on primary operators, let us also introduce current algebras these are two dimensional denoted by \(J^A(z, \bar{z})\) which takes its values in a compact Lie group symmetry in a CFT. For simplicity we only consider the holomorphic current and the zero modes \(J^0_A\) are the generators of the Lie algebra of \(G\) given as

\[
[J^A_0, J^B_0] = if^{AB}_C J^C_0,
\]

(2.20)

the algebra of these currents is an infinite-dimensional extension of this and is known as the Kac-Moody algebra \(\hat{G}\). These currents have conformal dimension \(\Delta = 1\) and the mode expansion is given as

\[
J^A(z) = \sum_{n=-\infty}^{\infty} J^A_n z^{-n-1}, \quad A = 1, 2, \ldots \dim \hat{G},
\]

(2.21)

and the OPE for these currents gives us the Kac-Moody algebra defined as

\[
J^A(z)J^B(w) \sim \frac{k\delta^{AB}}{2(z-w)^2} + \frac{if^{AB}_C J^C}{z-w} + \ldots .
\]

(2.22)

The parameter \(k\) is the Kac-Moody algebra called the level which is related to the parameter \(c\) in the Virasoro algebra. The energy momentum tensor associated with an arbitrary Kac-Moody algebra is

\[
T(z) = \frac{1}{k + h_G} \sum_{A=1}^{\dim G} J^A(z) J^A(z),
\]

(2.23)

\(^{\text{Only valid in two dimensional CFT}}\)
where we have the dual Coxeter number $\tilde{h}_G$ we can define the central charge of the CFT defined by a current algebra w.r.t to the energy momentum tensor defined above to obtain
\[
c = \frac{k \dim G}{k + \tilde{h}_G}.
\]
(2.24)
The Kac-Moody currents will play an important role later on when we discuss c-extremization where we will see that it determines the exact dimension of the chiral primary operators and the Virasoro right moving central charge of the theory.

2.3 $\mathcal{N} = 2$ superconformal algebra

Let us now discuss the $\mathcal{N} = 2$ superconformal algebra. We will consider algebra with the conformal weight $h \leq 2$ and we will also assume that there is only one (2,0) constraint current which is the overall energy momentum tensor, as we saw in the previous section the OPE of two currents is proportional to $(z - w)^{2h}$ more details on the superconformal algebras can be found in [9]. We will consider $\mathcal{N} = 2$ which admits two supercurrents but we will write them into one complex supercurrent given as
\[
T_F^\pm = \frac{1}{\sqrt{2}} (T_{F1} \pm iT_{F2}) ,
\]
(2.25)
one thing to remember is that $T_F^\pm$ and $\omega$ are primary fields and that $T_F^\pm$ has charge $\pm 1$ under the $U(1)$ generated by $\omega^4$. The constant $c$ in $T_F^\pm T_F^\pm$ and $\omega(z)\omega(w)$ must be the central charges. Let us now write down the superconformal and Abelian current algebras in terms of modes. A holomorphic operator $\mathcal{O}(z)$ of conformal weight $(0, h)$ is decomposed in modes $\mathcal{O}_m$ given as
\[
\mathcal{O}(z) = \sum_{m \in \mathbb{Z} + \alpha} \frac{1}{z^{m+h}} \mathcal{O}_m , \quad \mathcal{O}_m = \frac{1}{2\pi i} \oint dz z^{m+h+1} \mathcal{O}(z) ,
\]
(2.27)
where $\alpha \in [0, 1)$ depends on the boundary condition in radial quantization. We denote the supercharges as the modes $G^\pm_{-\frac{1}{2}}$ of the supercurrents $T_F^\pm(z)$ and $\omega_0$ is the R-charge,
the $N = 2$ algebra reads

\[
[L_m, L_n] = \frac{c}{12} (m^3 - m) \delta_{m+n,0} + (m - n) L_{m+n}
\]

\[
[L_m, G^\pm_r] = \left( \frac{m}{2} - r \right) G^\pm_{m+r}, \quad \{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} = 0
\]

\[
[L_m, \omega_n] = -n \omega_{m+n}, \quad [\omega_n, G^+_r] = \pm G^+_r, \quad [\omega_m, \omega_n] = \frac{c}{3} m \delta_{m+n,0}
\]  

\[
\{G^+_r, G^-_s\} = \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0} + 2 L_{r+s} + (r - s) \omega_{r+s}
\]

\[2.28\]

We will also write down the Abelian current algebra. We have not yet defined what it is but will be explained further in. The Abelian current algebra is given as

\[
[j^A_{a,m}, j^B_{b,n}] = \delta_{ab} k^{AB} m \delta_{m+n,0}, \quad [j^A_{a,m}, \psi^B_{b,r}] = 0, \quad \{\psi^A_{a,r}, \psi^B_{b,s}\} = \delta_{ab} k^{AB} \delta_{m+n,0}.
\]  

\[2.29\]

We also have the superconformal algebra on the current multiplet $J = (\phi, \psi^\pm, j)$ which gives us the following relations

\[
[L_m, \psi^A_{a,r}] = - \left( \frac{m}{2} + r \right) \psi^A_{a,m+r}, \quad [\omega_m, \psi^A_{a,r}] = i \epsilon_{ab} \psi^A_{b,m+r}
\]

\[
[L_m, j^A_{a,m}] = \frac{i}{2} q^A_m (m^2 + m) \delta_{m+n,0} - n j^A_{a,m+n}, \quad [\omega_m, j^A_{a,n}] = \epsilon_{ab} q^A_b m \delta_{m+n,0}
\]  

\[2.30\]

\[
\{G^+_r, \psi^A_{a,s}\} = \frac{\delta_{ab} + i \epsilon_{ab}}{\sqrt{2}} \left( iq^A_b \left( r + \frac{1}{2} \right) \delta_{r+s,0} + j^A_{b,r+s} \right).
\]

We have now defined the superconformal algebra we will be using now on.

### 2.4 Anomaly polynomials

Anomalies are connected to the topology of the configuration space of gauge theories [10], we will see how one can use descent formalism to obtain gauge anomalies from $2n$ dimensional which is related to characteristic classes in $2n + 2$ dimensions. The characteristic classes are constructed out of a fiber bundle and the tangent bundle which encodes the anomalies of the theory. The anomaly polynomial of a $2n$ dimensional right moving Weyl fermion in the representation $r$ of a gauge group $G$ takes the elegant form

\[
I_{2n+2} = \text{ch}_r (F) \hat{A}(R) \bigg|_{2n+2},
\]  

\[2.31\]

which is the same as the Dirac index density. The $\text{ch}_r$ is the Chern character and $\hat{A}$ is the Dirac genus and they are given as

\[
\text{ch}_r (F) = \text{dim} r + c_1 + \frac{c_1^2 - 2 c_2}{2} + \ldots, \quad \hat{A}(R) = 1 - \frac{p_1}{24} + \ldots.
\]  

\[2.32\]

We will focus on two dimensional theories and for a two dimensional Weyl fermion with a $U(1)^N$ bundle where the polynomials above are being used can be written as

\[
I_4 = \frac{1}{2} \sum_{I, M} k^{I M} c_1 (F^I) \wedge c_1 (F^M) - \frac{k}{24} p_1 (R) = - \sum_{I, M} \frac{k^{I M}}{8 \pi^2} F^I \wedge F^M + \frac{k}{192 \pi^2} \text{Tr} R^2
\]  

\[2.33\]

where $c_1$ is the first Chern class and $p_1$ is the first Pontryagin class. The two form field strength is given as $F^I = dA^I$ which takes the values in $U(1)^N$, $R = d\Gamma + \Gamma \wedge \Gamma$ which is a matrix valued curvature form constructed out of the one form $\Gamma^\mu_\nu \equiv \Gamma^\mu_\nu dx^\nu$. Using the descent formalism we can extract the anomalies, so let us begin by writing

\[
I_4 = dI_3, \quad I_3 = - \sum_{I, M} \frac{k^{I M}}{8 \pi^2} A^I \wedge F^M + \frac{k}{192 \pi^2} \text{Tr} \Gamma \wedge R.
\]  

\[2.34\]
The variation of the Chern-Simons form $I_3$ is locally exact and can be written as

$$\delta I_3 = dI_2^{(1)}, \quad I_2^{(1)} = -\sum_{I,M} \frac{k^{I M}}{8\pi^2} \lambda^I F^{M} + \frac{k}{192\pi^2} \text{Tr}(v d\Gamma)$$  \tag{2.35}

where we have the gauge variation given as $\delta A^I = d\lambda^I$ and the coordinate transformation $x^\mu \rightarrow x^\mu - \xi^\mu(x)$, the gauge connection one form transforms as $\delta A^I = \nabla v$ with $v^\alpha_\beta = \partial_\beta \xi^\alpha$. Using the anomaly inflow mechanism which states that the the anomalous gauge variation given by the bulk action flows into boundary and cancels the anomalous gauge variation localized on the boundary [11]. Hence the variation of the quantum action $S$ is equal to $\delta_S = 2\pi \int d^2 x I_2^{(1)}$

$$\delta S = \int d^2 x \left( \partial_\mu \frac{\delta S}{\delta A^I_\mu} + 2\nabla_\mu \xi^\nu \frac{\delta S}{\delta g^I_{\mu \nu}} \right) = \int d^2 x \sqrt{-g} \left( -\frac{k^{I M}}{8\pi} \lambda^I F^{M}_{\mu \nu} - \frac{k}{96\pi} \xi^\alpha \epsilon^{\mu \nu} \partial_\mu \partial_\nu \Gamma^\beta_{\alpha \rho} \right),$$  \tag{2.36}

and the anomalies are then given as

$$\nabla^\mu J^I_\mu = \sum_M \frac{k^{I M}}{8\pi} F^M_{\mu \nu} \epsilon^{\mu \nu}, \quad \nabla^\mu T^{\mu \nu} = \frac{k}{96\pi} g^{\nu \alpha} \epsilon^{\mu \rho} \partial_\mu \partial_\nu \Gamma^\beta_{\alpha \rho}.$$  \tag{2.37}

Let us now review [12] to get a feeling of a more physical picture of anomalies. We want to computes the two dimensional anomalies that will be a useful when we discuss c-extremization. We consider Lorentz signature $(+, -)$ and take the gamma matrices to satisfy $\{\gamma^a, \gamma^b\} = 2\epsilon^{a b}$. The chirality is $\gamma^3$ which satisfy $\gamma_a \gamma^3 = -\epsilon_{a b} \gamma^b$ and the vielbein is given as $g_{\mu \nu} = \epsilon_{\mu \nu} \gamma^a \gamma^b$. For convenience we will introduce light cone coordinates

$$x^\pm = \frac{x^1 \pm x^0}{\sqrt{2}}, \quad x^0 = \frac{x^+ - x^-}{\sqrt{2}}, \quad x^1 = \frac{x^+ + x^-}{\sqrt{2}}.$$  \tag{2.38}

Let us discuss a simpler model first to get the general idea of how to compute the anomalies, we will start with a spin-$1/2$ anomaly. The action for this field is given as

$$S = \int d^2 x \bar{\psi} \gamma^\mu (\partial_\mu - i A_\mu) \psi,$$  \tag{2.39}

where the equations of motion is given as $\partial_+ \psi = 0$ hence the fermion in two dimensions with negative chirality is an object that travels at the speed of light and this is an anomaly. Using the equation of motion we can find the current for this action and it is simple given as $J^\mu = \bar{\psi} \gamma^\mu \psi$ and the only non-vanishing component is $J_+$, using this we can compute the two point function using the fermion propagator which gives us

$$U_{++} = \frac{1}{4\pi^2} \int dk_+ dk_- \frac{1}{(k_- + p_- + \frac{i\epsilon}{p_+ + k_+}) (k_- + \frac{i\epsilon}{k_+})}.$$  \tag{2.40}

Performing the contour integral with the poles given as $k_- = -i\epsilon/k_+$ and $k_- = -p_- - i\epsilon/(p_+ + k_+)$ we obtain that the two point function gives us the anomaly

$$U_{++} = \frac{i}{2\pi} \frac{p_+}{p_-}.$$  \tag{2.41}

and coupling each vertex to $A_+$ the two point function gives the effective action defined as

$$S_{eff} = \frac{1}{4\pi} \int d^2 p \frac{p_+}{p_-} A_+(p) A_-(-p).$$  \tag{2.42}

The same computation for the left moving fermion gives the same result, let us now consider a more general theory with right-moving spinors $\psi_R$ and left moving spinors
\[ \psi \] instead. We couple each vertex of the same expression can be obtained for the left moving Weyl fermion but with the symmetric matrix \( k^I_J = \sum_i Q_i^I Q_i^J \) and \( k^I_J = \sum_a Q_a^I Q_a^J \) and using the same procedure as above we can compute the one point function of the current gives us

\[ \mathcal{L} = \sum_{j,l} i \bar{\psi}_{Rj} \gamma^\mu (\partial_\mu - i Q_j^I A_\mu^I) \psi_{Rj} + \sum_{a,l} i \bar{\psi}_{La} \gamma^\mu (\partial_\mu - i Q_a^I A_\mu^I) \psi_{La} . \] (2.43)

Let us also define symmetric positive definite matrices \( k_R^I_J = \sum_i Q_i^I Q_i^J \) and \( k_L^I_J = \sum_a Q_a^I Q_a^J \) and computing the energy momentum tensor by considering a weak gravitational background such that the coupling to gravity is given as

\[ \mathcal{L} = \sum_{j,l} i \bar{\psi}_{Rj} \gamma^\mu (\partial_\mu - i Q_j^I A_\mu^I) \psi_{Rj} + \sum_{a,l} i \bar{\psi}_{La} \gamma^\mu (\partial_\mu - i Q_a^I A_\mu^I) \psi_{La} . \] (2.44)

Computing the one point function of the current gives us

\[ \langle \partial_\mu J^{I\mu} \rangle_A = \frac{i}{4\pi} \left( (2k_R^I_J + B^{IJ}) p_+ A_+^I - (2k_L^I_J + B^{IJ}) p_- A_-^I \right) , \] (2.45)

we see that we cannot set this to zero unless \( k_R^I_J = k_L^I_J \) and we see that an anomaly has appeared. We can impose \( B^{IJ} = -k_L^I_J - k_R^I_J \) this leaves the antisymmetric part free, however we require that the anomalies are symmetric and we end up with

\[ \partial_\mu J^{I\mu} = \sum_M \frac{k^{I|M}}{8\pi} \tau^M_\mu e^{\mu\nu} , \] (2.46)

the symmetric matrix \( k^{I|M} \) is defined as

\[ k^{I|M} = k_R^{I|M} - k_L^{I|M} = \text{Tr}_{\text{Weyl fermions}} \gamma^3 Q_I Q^M . \] (2.47)

Let us now consider the gravitational anomalies and the conformal anomalies, we start with a Weyl fermion of positive chirality coupled to a gravitational background and for the remaining of the project we will follow [4] which is the key reference. The action is thus given as

\[ S = \frac{i}{2} \int d^2x (\det e) e^{\mu a} \bar{\psi} \gamma_a \overset{\rightarrow}{\partial_\mu} \psi , \] (2.48)

computing the energy momentum tensor by considering a weak gravitational field such that the coupling to gravity is given as \( \mathcal{L} = -\frac{1}{2} h^{\mu\nu} T_{\mu\nu} \) yields us

\[ T_{\mu\nu} = \frac{i}{4} \bar{\psi} (\gamma_\mu \overset{\rightarrow}{\partial_\nu} + \gamma_\nu \overset{\rightarrow}{\partial_\mu}) \psi . \] (2.49)

We impose the chirality constraint \( \gamma^3 \psi = \psi \) where the only non-vanishing component is \( T_{++} \) and we compute the two point function to be

\[ U_{++}(p) = \int d^2x e^{-ipx} (T_{++}(x) T_{++}(0))_T , \] (2.50)

performing the contour integral in \( k_- \) yields us the anomaly \( U_{++}(p) = \frac{i}{2\pi} \frac{p^3}{p_-} \) and the same expression can be obtained for the left moving Weyl fermion but with \( p_- \) instead. We couple each vertex of \( U_{\mu\nu\rho\sigma} \) to \( -\frac{1}{2} h^{\alpha\beta} \) and also include the Bose symmetry we find the quartic effective action \( S_{\text{eff}}(h) \). We consider \( c_R \) right moving Weyl fermions.
and $c_L$ for left moving and after adding the local counterterms we obtain the following effective action

$$S_{eff} = \frac{1}{192\pi} \int d^2p \left( c_R \frac{p_+^3}{p_-} h_{--}(p) h_{--}(-p) + c_L \frac{p_-^3}{p_+} h_{++}(p) h_{++}(-p) + A p_+^2 h_{--}(p) h_{--}(-p) + B p_- h_{--}(p) h_{--}(-p) + C p_- h_{++}(p) h_{++}(-p) + D p_-^2 h_{++}(p) h_{++}(-p) \right).$$

(2.51)

We can now compute the one point functions where we consider the first order in the background and the divergences are given as

$$\langle \partial_\mu T^\mu_{++} \rangle_h = -\frac{i p_+}{192\pi} \left( (4c_R + A)p_+^2 h_{--} + 2(A + B)p_+ h_{--} + (2C + D)p_- h_{++} \right),$$

$$\langle \partial_\mu T^\mu_{--} \rangle_h = -\frac{i p_-}{192\pi} \left( (2c + A)p_-^2 h_{--} + 2(B + D)p_- h_{--} + (4c_L + D)p_+ h_{++} \right).$$

(2.52)

We now have two options we can either set the central charges to be equal $c_R = c_L = c$ by doing so we impose that $A = -B = -2C = D = 4c$ and the stress tensor becomes conserved $\nabla_\mu T^{\mu\nu} = 0$ and one indeed finds a anomaly given as

$$T^\mu_\mu = 2T_{++} = -\frac{c}{24} R.$$

(2.53)

The other choice is $c_R \neq c_L$ and the conservation of stress tensor cannot be achieved using the following renormalization scheme $A = -3c_R - c_L, B = 2C = 2(c_R + c_L), D = -c_R - 3c_L$ we find that the at the linearized order the non-conservation of the stress tensor is given as

$$\partial_\mu T^{\mu\nu} \simeq -\frac{c_R - c_L}{192\pi} \epsilon^{\mu\rho} \partial_\rho R.$$

(2.54)

The full consistent anomaly takes the form of

$$\nabla_\mu T^{\mu\nu} = \frac{c_R - c_L}{96\pi} g^{\nu\alpha} \epsilon^{\mu\rho} \partial_\alpha \partial_\beta T^{\beta\rho}_{\alpha\rho},$$

(2.55)

where we consider local Lorentz rotations which are non-anomalous and the stress energy tensor is symmetric. We see that the gravitational anomaly and the gauge anomaly is in agreement with (2.37).

The anomaly coefficients $k^{JM}$ and $k$ are well defined by the equations (2.37) as long as the symmetries are not broken and are invariant under the RG flow. We saw that the anomaly coefficients can be obtained by the poles at zero momentum in the two point functions. If the theory is conformal the anomaly coefficients are related to the terms in the conformal and current algebra in flat space. We will use Euclidean signature for convinience when working in two dimensional CFT’s and also radial quantization using complex coordinates $z, \bar{z}$, hence we define

$$z = x^1 + i x^0_E, \quad \bar{z} = x^1 - i x^0_E, \quad \partial_1 \equiv \frac{\partial_1 - i \partial_0^E}{2}, \quad \bar{\partial}_2 = \frac{\partial + i \partial_0^E}{2}.$$

(2.56)

We also define

$$T(z) = -2\pi T_{zz}(x), \quad \bar{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}}, \quad j^I(z) = -i\pi J^I(x), \quad \bar{j}^I(\bar{z}) = -i\pi J^I(x).$$

(2.57)

We will consider CFT that have the following properties: the theory is unitary and the Virasoro generators $L_0, \bar{L}_0$ are bounded below and also that the vacuum is normalizable. The primary operators whose conformal weights are $(h, \bar{h})$ are non-negative where an
operator $\mathcal{A}$ is holomorphic $\partial \mathcal{A}$ if and only of the weight is $\bar{h} = 0$ and antiholomorphic when $h = 0$. This primary operator that takes these values is the identity. The conserved currents have $h + \bar{h} = 1$ dimensions and the spin $|h - \bar{h}| = 1$ and we consider the current algebra OPE’s to be

$$T(z)T(0) \sim \frac{c_R}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \quad j^I(z)j^J(0) \sim \frac{k_{IJ}^R}{z^2},$$

$$\bar{T}(\bar{z})\bar{T}(0) \sim \frac{c_L}{2\bar{z}^4} + \frac{2\bar{T}(0)}{\bar{z}^2} + \frac{\overline{\partial \bar{T}}(0)}{\bar{z}}, \quad \bar{j}^I(\bar{z})\bar{j}^J(0) \sim \frac{k_{IJ}^L}{\bar{z}^2}. \quad (2.58)$$

Unitary constrains $k_{IJ}^R$ and $k_{IJ}^L$ to be semi positive definite and to vanish when global symmetries are trivial. If $(IJ)$ are both right moving then we have $k_{IJ}^R$ and if they are both left moving we have $k_{IJ}^L$ where we also have that $k = c_R - c_L$. If $k$ is zero then there is no gravitational anomaly since the left moving central charge is equal to the left moving charge.

### 2.5 Superconformal R-symmetry in two dimensions

In non conformal $\mathcal{N} = (2, 0)$ supersymmetric theory with a $U(1)_R$ R-symmetry the R-current is not uniquely defined. Any mixing of $U(1)_R$ with flavor symmetries i.e any linear combination of those symmetries under the charge $Q = 1$ produces equally good R-symmetries. In [13] they showed that any supersymmetric multiplet contains the energy-momentum tensor and a supersymmetric current which are not uniquely defined due to the improvement transformation. The R-current is in the same multiplet as stress tensor and the supercurrent and the mixing corresponds to linear improvement. Determining the R-symmetry makes is difficult due to the mixing of symmetries.

We will now determine the R-symmetry exactly in a unitary superconformal field theory (SCFT) with normalizable vacuum. We consider all the possible Abelian currents assigned with charge 1 to $Q$ and R-symmetry current, this is called the trial R-current and is given as

$$\Omega^r_\mu(t) = J^R_\mu + \sum_{I(\neq R)} t_I J^I_\mu. \quad (2.59)$$

The first term is the R-symmetry current and the second term is the Abelian flavor symmetry currents from the expression above one can construct a quadratic function $c^r_R$ which is proportional to the ’t Hooft anomalies of $\Omega^r_\mu$ given as

$$c^r_R = 3 \left( k^{RR} + 2 \sum_{I(\neq R)} t_I k^{RI} + \sum_{I,J(\neq R)} t_I t_J k^{IJ} \right). \quad (2.60)$$

In order to compute the exact R-symmetry we need to show that there is no mixed symmetries in the function $c^r_R$ i.e we want the term $k^{RI}$ to be zero. We want to characterize the exact R-symmetry in terms of anomalies such that it is invariant under renormalization group (RG) flow and independent of the physics at the IR fix point. If $J^I_\mu$ is right moving flavor current then it is a part of a supermultiplet if $J^I_\mu$ is left moving then $k^{RI}$ is zero due to $J^R_\mu$ being right moving and since the R-symmetry current is holomorphic it implies $\omega(z) = -i\pi J^R_\mu$. The multiplet of $\mathcal{N} = 2$ Kac-Moody currents $J^A_\mu = (\psi^{\dagger}_{1,2} J^{A}_{1,2})$ is what we will consider and it is made out of two $\mathcal{N} = 1$ current multiplets where $(A, a)$ runs over all right moving flavor currents. The Abelian current algebra is described by the OPEs which are the same as (2.29)

$$j^A_a(z)j^B_b(0) \sim \delta_{ab} \frac{k^{AB}}{z^2}, \quad j^A_a(z)\psi^B_b(0) \sim 0, \quad \psi^A_a(z)\psi^B_b(0) \sim \delta_{ab} \frac{k^{AB}}{z^2}. \quad (2.61)$$

12
One thing to note is that the regular $N = (2,0)$ current multiplet is given as $J = (\phi, \psi^\pm, j)$ this multiplet is however not the one we want to consider since normalizable vacuum forces the scalar fields to take their conformal weights to be $(0,0)$. Thus we have to use slightly modified OPEs that contain central terms $q_a^A$ which are called background charges and preserve superconformal symmetry, the OPEs are given as

$$
T_F(z)\psi^A_a(0) \sim \frac{iq^A_a}{z^2} + \frac{j^A_a(0)}{z}, \quad T_F(z)\psi^A_a(0) \sim -\epsilon_{ab} \left( \frac{iq^A_b}{z^2} + \frac{j^A_b(0)}{z} \right),
$$

$$
T_F(z)j^A_a(0) \sim \frac{\psi^A_a(0)}{z^3} + \frac{\partial \psi^A_a(0)}{z}, \quad T_F(z)j^A_a(0) \sim \epsilon_{ab} \left( \frac{\psi^A_b(0)}{z^2} + \frac{\partial \psi^A_b(0)}{z} \right). \tag{2.62}
$$

Writing the mode expansions given in (2.30) as OPEs takes the following forms

$$
T(z)\psi^A_a(0) \sim \frac{A^0_a(0)}{z^2} + \frac{\partial \psi^A_a(0)}{z}, \quad T(z)j^A_a(0) \sim \frac{iq^A_a}{z^3} + \frac{j^A_a(0)}{z^2} + \frac{\partial j^A_a(0)}{z},
$$

$$
\omega(z)\psi^A_a(0) \sim \epsilon_{ab} \frac{A^0_b(0)}{z}, \quad \omega(z)j^A_a(0) \sim \epsilon_{ab} \frac{q^A_b}{z^2},
$$

$$
T(z)\psi^A_a(0) \sim \frac{\delta_{ab} \pm i\epsilon_{ab}}{\sqrt{2}} \left( \frac{iq^A_a}{z^2} + \frac{j^A_a(0)}{z} \right), \quad T(z)j^A_a(0) \sim \frac{\delta_{ab} \pm i\epsilon_{ab}}{\sqrt{2}} \left( \frac{\psi^A_b(0)}{z^2} + \frac{\partial \psi^A_b(0)}{z} \right). \tag{2.63}
$$

unitary requires that $q^A_a$ are real note also that the OPE $T(z)j^A_a(0)$ contains the central term $q^A_a$ which makes the currents $j^A_a$ not primary operators. Taking the expectation value of the two point function $\langle T(p)j^A_a(-p)\rangle_T \sim p^2_+/p_-$ leads to anomalous violations of current conservation on a gravitational background and of covariance on a gauge background. We can solve this problem by adding a local counterterm in the action of current conservation on a gravitational background and of covariance on a gauge background. At the conformal fixed points the vanishing current conservation on a gravitational background and of covariance on a gauge background. We can solve this problem by adding a local counterterm in the action of current conservation on a gravitational background and of covariance on a gauge background. Hence this shows that at the IR fixed point the flavor currents $j^A_a$ are primary operators. The central charge at the point $c_R = c_R - 3q^A_a q^B_b (k^{-1})^{AB}$ is constrained by the unitary and hence forbids any mixing of gauge anomalies between the superconformal R-current and the right moving flavor currents at that particular point. Hence this shows that at the IR fixed point the flavor currents $k^{RI}$ vanish for all values of $I \neq R$. This statement is true if there is no violation of current conservation on a gravitational background and of covariance on a gauge background. At the conformal fixed point the vanishing of background charges makes all the currents primary fields which means that the two point functions of the stress tensor with flavor symmetries have no singular terms. Hence

$$
q^A_a = q^A_a - 2k^{AB}a^B_a, \quad c_R = c_R - 12\alpha^A_a q^A_a + 12\alpha^A_a \epsilon_{ab}j^A_b(z) \tag{2.65}
$$

Since $k^{AB}$ is positive definite we can cancel all the central terms in (2.63) by taking $\alpha^A_a = \frac{1}{2}(k^{-1})^{AB}_a q^B_a$ and hence all the right moving currents are primary operators. The central charge at the point $c_R = c_R - 3q^A_a q^B_b (k^{-1})^{AB}$ is constrained by the unitary and hence forbids any mixing of gauge anomalies between the superconformal R-current and the right moving flavor currents at that particular point. Hence this shows that at the IR fixed point the flavor currents $k^{RI}$ vanish for all values of $I \neq R$. This statement is true if there is no violation of current conservation on a gravitational background and of covariance on a gauge background. At the conformal fixed point the vanishing of background charges makes all the currents primary fields which means that the two point functions of the stress tensor with flavor symmetries have no singular terms. Hence
we can impose the constraint on the trial current (2.59) using the fact that there are no mixed anomalies which gives us all the anomalies of a given theory by

\[ \frac{\partial c^{tr}}{\partial t_{I}}(t^{*}) = 0, \quad \forall I \neq R. \]  

(2.66)

Since \( c^{tr}_{R}(t) \) is quadratic this gives us a unique solution of \( t_{I}^{*} \) and because of the unitarity the function is maximized along the directions \( t_{I} \) that corresponds to left moving currents \( J^{I}_{\mu} \) and minimizes along the right moving ones. This is what we will call c-extremization and using this we can compute exact central charges and compute the large \( N \) limit using holographic techniques.
3 AdS/CFT correspondence

In this section we will start with the original proposal of AdS/CFT by Maldacena [1], where he showed that there is a duality between gravity theories and field theories once a certain limit called near horizon has been taken into account. The near horizon geometry will be a key argument for trusting supergravity solutions at large \( N \).

Once we established the correspondence we will use c-extremization to compute the central charge and then take the large \( N \) limit where we will see that indeed it gives the correct degree’s of freedom. We start with super Yang-Mills (SYM) where we see that at large \( N \) one uncovers the gravity theory living on \( AdS \times S^5 \) and then we also consider M5 branes wrapped on four manifolds.

3.1 \( \mathcal{N} = 4 \) 4D SYM

We start with type IIB string theory with string coupling \( g \) and in Minkowski space. Consider \( N \) parallel \( D3 \) branes separated by some distance \( r \) and for some low energies the theory on \( D3 \) brane decouples from the bulk. We the the energies to be fixed and take

\[
\alpha' \to 0, \quad U \equiv \frac{r}{\alpha'} = \text{fixed}.
\]

The second condition is saying that we keep the mass of the stretched strings fixed and the first condition is the limit where the interaction Lagrangian relating the bulk to the brane vanishes. The resulting theory on the brane is the \( D = 4 \) \( \mathcal{N} = 4 \) SYM, we will consider the theory on the superconformal point where \( r = 0 \) and the conformal group is given as \( SO(2, 4) \). The R-symmetry is given as \( SO(6) \sim SU(4)_R \) which rotates the six scalar fields into each other. In the bulk we have a supergravity theory so let us now consider the supergravity solution carrying \( D3 \) brane charge. The metric is given as

\[
ds^2 = f^{-1/2}dx_1^2 + f^{1/2}(dr^2 + r^2d\Omega_5^2), \quad f = 1 + \frac{4\pi gN\alpha'^2}{r^4},
\]

where the self dual five form field is given as \( F_5 = (1 + \ast)dt dx_1 dx_2 dx_3 df^{-1} \), \( x_1 \) denotes the transverse direction and \( d\Omega_5^2 \) is the metric on the unit five-sphere. Using the defined variable in (3.1) and rewriting the metric in terms of \( U \) we obtain the following metric

\[
ds^2 = \alpha' \left( \frac{U^2}{\sqrt{4\pi gN}} dx_1^2 + \sqrt{4\pi gN} dU^2 + \sqrt{4\pi gN} d\Omega_5^2 \right).
\]

This metric describes the five dimensional \( AdS_5 \times S^5 \) and the metric remains constant in \( \alpha' \) units. The radius of the fivesphere is \( R_5^2/\alpha' = \sqrt{4\pi gN} \) and is the same as the radius of \( AdS_5 \). In the large \( N \) limit we can trust the supergravity solution when \( gN \gg 1 \).

Let us now consider the two dimensional theories arising at low energy from the twisted compactification of four manifolds \( \mathcal{N} = 4 \) SYM with a gauge group \( G \) on a Riemann surface \( \Sigma_g \). We will use c-extremization to determine the central charge of the IR two-dimensional CFT for a gauge group \( U(N) \) these theories arise from the \( N \) \( D3 \)-branes wrapped on \( \Sigma_g \). In order to preserve some supersymmetry one generally has to twist the theory i.e turn on some background gauge field \( A_\mu \) coupled to the \( SO(6) \) R-symmetry. The supercharges transform in the representation \( 2 \otimes 4 \) of the product of the Lorentz and R-symmetry groups \( SO(3, 1) \times SO(6) \) and we can at least preserve \( \mathcal{N} = (0, 2) \) supersymmetry by choosing a background gauge field \( A_\mu \). Let us analyze the decomposition of the representations we split \( SO(3, 1) \times SO(6) \) and decompose into

\[
\left( \begin{pmatrix} i & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} -i & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right) \otimes \left( \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} -1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right)
\]

(3.4)
under $SO(1,1) \times SO(2)_g \times SO(2)_1 \times SO(2)_2 \times SO(2)_3$. In terms of the spin connection \( \tilde{\omega}_\mu = \frac{1}{2} \omega^a_{\mu} \epsilon_{ab} \) on the Riemann surface \( \Sigma_g \) and the Gauss-Bonnet theorem gives us

\[
\int d\tilde{\omega} = \frac{1}{2} \int \sqrt{g} R = 4\pi (1 - \mathfrak{g}) .
\]  

(3.5)

The covariant derivative of a spin-\( s \) field is \( D_\mu = \partial_\mu + i s \tilde{\omega}_\mu - i A_\mu \). For \( \mathfrak{g} = 0 \) we choose the 4D R-symmetry background \( A_\mu \) such that the field strength is given as \( F = - T d\tilde{\omega} \), for \( \mathfrak{g} = 1 \) we set \( F = - T \frac{2\pi}{\text{vol}(\Sigma)} \omega_{a\overline{b}} \), and for \( \mathfrak{g} > 1 \) we set \( F = T \tilde{\omega} \), the background is taken along the generators

\[
T = a_1 T_1 + a_2 T_2 + a_3 T_3 .
\]  

(3.6)

The generators \( T_{1,2,3} \) are the generators of an \( SO(2)^3 \) embedded block-diagonally into \( SO(6) \) and \( a_{1,2,3} \) are the constants parameterizing the twist and to preserve \( 2d \mathcal{N} = (0,2) \) supersymmetry we take

\[
a_1 + a_2 + a_3 = - \kappa .
\]  

(3.7)

When \( \kappa = 1 \) we have \( \mathfrak{g} = 0 \), \( \kappa = 0 \) for \( \mathfrak{g} = 1 \) and \( \kappa = -1 \) for \( \mathfrak{g} > 1 \) and we also choose the metric on \( \Sigma_g \) to be constant curvature \( R = 2\kappa \) where we define

\[
ds^2 = e^{2h} (dx^2 + dy^2), \quad \text{with} \quad h = \begin{cases} - \log \frac{1 + y^2}{2} & \text{for} \quad \mathfrak{g} = 0 \\ \frac{1}{2} \log 2\pi & \text{for} \quad \mathfrak{g} = 1 \\ - \log y & \text{for} \quad \mathfrak{g} > 1 \end{cases}
\]  

(3.8)

and the background flux is given as \( F = dA = \sum_I F^I T_I \) where \( F^I = -a_I e^{2h} dx \wedge dy \) and \( I = 1, 2, 3 \). When we have arbitrary \( a_I \) we have \( \mathcal{N} = (0,2) \) supersymmetries when one \( a_I \) is zero we have \( \mathcal{N} = (2,2) \) when two are zero and \( \mathfrak{g} \neq 1 \) we have \( \mathcal{N} = (4,4) \) and when all \( a_I \) are zero we have \( \mathcal{N} = (8,8) \) the two middle cases have been studied in [14] further analysis can be found in [4] on the other cases. From a geometric point of view one can see that for a gauge group \( U(N) \) the twisted theories describe low energy dynamics of \( N \) D3 branes on holomorphic two cycle \( \Sigma_g \) in a local Calabi-Yau fourfold \( X \). We define the line bundle over \( \Sigma_g \) as \( \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \) such that

\[
\mathbb{C}^{(1)} \times \mathbb{C}^{(2)} \times \mathbb{C}^{(3)} \rightarrow X \rightarrow \Sigma_g ,
\]  

(3.9)

where the degree of the line bundle \( \mathcal{L}_I \) is \(-2\kappa(\mathfrak{g} - 1)a_I \) for \( \mathfrak{g} \neq 1 \) and \( a_I \) for \( \mathfrak{g} = 1 \). The condition that \( X \) is a Calabi-Yau is given by (3.7) and when one of the \( a_I \) vanish we have \( X = CY_2 \times \mathbb{C} \) where the supersymmetry is enhanced to \( \mathcal{N} = (2,2) \) and so on. The low energy 2d theory inherits \( SO(2)^3 \) global symmetries which contains the 2d superconformal R-symmetry and the trial R-symmetry is a linear combination of the generators of \( SO(2)^3 \) given as

\[
T_R = \epsilon_1 T_1 + \epsilon_2 T_2 + (2 - \epsilon_1 - \epsilon_2) T_3 .
\]  

(3.10)

We have fixed the R-charge to be 1 and \( \epsilon_{1,2} \) parameterizes the mixing and we want to determine the exact R-symmetry by using c-extremization. The four dimensional theory has gaugini in the representation \( 2 \otimes 4 \) of \( SO(3,1) \times SO(6) \) which can be read of from the decomposition and the number of 2d massless chiral fermions follows from Kaluza-Klein reduction on \( \Sigma_g \) which come from the 4d gauginos. The gaugino decompose under \( SO(2)^3 \) with the following charges \( A : \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \), \( B : \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \), \( C : \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \), \( D : \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \) and by using the index theorem the number of right moving minus the left moving fermions is

\[
n^{(r)}_R - n^{(r)}_L = \frac{1}{2\pi} \int \text{Tr}_\sigma F = - t_\sigma \eta_{\Sigma} .
\]  

(3.11)
Where $\sigma$ takes the values of the chiral spinors charges $(A, B, C, D)$ we can define the charges of the fermions in the background field $A_\mu$ such that $t_A = \frac{a}{2}$, $t_B = \frac{a}{2} + a_1 + a_2$, $t_C = \frac{a}{2} - a_2$ and $t_D = \frac{a}{2} - a_1$ we also define

$$\eta^\Sigma = \begin{cases} 2|g - 1| & \text{for } g \neq 1 \\ 1 & \text{for } g = 1 \end{cases} .$$

(3.12)

Let us also point out the Dirac quantization notice that the quantization of the fluxes where the index was given above gives us certain constraints on the R-symmetry bundle when $g \neq 1$ we have the following quantization $2(g - 1)(\pm a_1 \pm a_2 \pm a_3) \in 2\mathbb{Z}$ which can be written as $2(g - 1)a_I \in 2\mathbb{Z}$ by using the twist condition (3.7) and for $g = 1$ we simply have $a_I \in \mathbb{Z}$. We can write this for both conditions in the following way

$$a_I \eta^\Sigma \in \mathbb{Z} .$$

(3.13)

We can now compute the trial right moving charge using c-extremization taking into account that the fermions are in the adjoint representation of the gauge group

$$c^{tr}_R(\epsilon_i) = 3d_G \sum_\sigma (n^{(\sigma)}_R - n^{(\sigma)}_L)(q^{(\sigma)}_R q^{(\sigma)}_L)^2 .$$

(3.14)

Using the R-symmetry generators of the $SO(2)^3$ and the index theorem together with the spinor charges one finds that the trial charge can be written as

$$c^{tr}_R(\epsilon_i) = -3d_G \eta^\Sigma \left(t_A \left(-\frac{1}{2} \epsilon_1 - \frac{1}{2} \epsilon_2 - \frac{1}{2}(2 - \epsilon_1 - \epsilon_2) \right)^2 + t_B \left(\frac{1}{2} \epsilon_1 + \frac{1}{2} \epsilon_2 - \frac{1}{2}(2 - \epsilon_1 - \epsilon_2) \right)^2 \\
+ t_C \left(\frac{1}{2} \epsilon_1 - \frac{1}{2} \epsilon_2 + \frac{1}{2}(2 - \epsilon_1 - \epsilon_2) \right)^2 + t_D \left(-\frac{1}{2} \epsilon_1 + \frac{1}{2} \epsilon_2 + \frac{1}{2}(2 - \epsilon_1 - \epsilon_2) \right)^2 \right) \\
= -3d_G \eta^\Sigma \left(t_A + t_B(\epsilon_1 + \epsilon_2 - 1)^2 + t_C(1 - \epsilon_2)^2 + t_D(1 - \epsilon_1)^2 \right) .$$

(3.15)

We can now find the extreme values for this trial central charge by taking the derivative w.r.t to the $\epsilon_{1,2}$ and we find that the extreme values are given as

$$\epsilon_i = \frac{2a_i(2a_i + \kappa)}{a_1^2 + a_2^2 + a_3^2 - 2(a_1a_2 + a_1a_3 + a_2a_3)} .$$

(3.16)

for simplicity we will introduce two variables that will be used and they are given as

$$\Theta = a_1^2 + a_2^2 + a_3^2 - 2(a_1a_2 + a_1a_3 + a_2a_3) ,$$

$$\Pi = (-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)(a_1 + a_2 - a_3) .$$

(3.17)

Plugging in the extreme values into the trial central charge and assuming that $\Theta \neq 0$ we obtain after some computation at the critical point the central charge takes the following value

$$c_R = -12\eta^\Sigma d_G \frac{a_1a_2a_3}{\Theta} .$$

(3.18)

The signature of the ‘t Hooft anomaly matrix determines the chirality of the currents at the IR fixed point, where positive eigenvalues correspond to right moving currents and negative eigenvalues to left moving currents. The signature of the Hessian determines the chirality of the flavor current. The second derivative of $c_R^{tr}$ is given as

$$\partial^2_c c_R^{tr} = -6\eta^\Sigma d_G a_1a_2/a_i, \quad \partial_c \partial_{c'} c_R^{tr} = -3\eta^\Sigma d_G (a_1 + a_2 - a_3) .$$

(3.19)
the Hessian determinant is given as \( \det_{ij} \partial_{\epsilon_i} \partial_{\epsilon_j} c^R = -9n_2^2 d_G^2 \Theta \). Since both currents are right moving we have that
\[
c_R - c_L = k = d_G \sum_{\sigma} (n_R^{(\sigma)} - n_L^{(\sigma)}) = 0 ,
\]
(3.20)
hence we have no gravitational anomalies. The full matrix of anomalies is given as
\[
k^{ij} = d_G \sum_{\sigma} (n_R^{(\sigma)} - n_L^{(\sigma)}) q_i^{(\sigma)} q_j^{(\sigma)} = \frac{n_2 d_G}{2} \begin{pmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix} .
\]
(3.21)
Let us now consider several cases where we twist the theory the critial point is not valid for \( g > 1 \) and \( a_1 = a_2 = \frac{1}{2} \), \( a_3 = 0 \) or any permutations of the same sort since the locus is zero i.e \( \Theta = 0 \). In [14] this was the case they considered where the trial current \( c^R \) is maximized for any \( \epsilon_1 + \epsilon_2 = 1 \) where we have that \( a_1 = a_3 = \frac{1}{2} \) gives a maximization along \( \epsilon_2 = 1 \) direction and \( a_2 = a_3 = \frac{1}{2} \) is maximized along \( \epsilon_1 = 1 \) direction. If we now consider the generator that is maximized along the \( \epsilon_1 \) direction we obtain \( T = \frac{1}{2} T_2 + \frac{1}{2} T_3 \) this breaks the \( SO(6) \rightarrow SO(2) \times SO(2) \times SU(2) \) where we consider that \( SO(2)^3 \subset SO(6) \). Using index theorem to compute the number of zero modes for the positive and negative chirality one can conclude that the left and right movers will also take the same index namely \( n_R - n_L \sim (g - 1) \). The symmetries at the IR fixed point for the left and right movers will thus be \( U(1)_R \) and \( U(1)_L \) and we see the UV global symmetry is enhanced to \( SU(2) \times U(1)_R \times U(1)_L \) and the supersymmetry is enhanced to \( \mathcal{N} = (2,2) \). The gaugino charges are given as \( A : (++) \) and \( B : (-+) \) where \( A \) is for right moving and \( B \) for the left moving, using this the generators for the R-symmetry are given as \( T_R = \frac{1}{2} (T_1 + T_2) + T_3 \) and the central charge can be computed to be
\[
c_R = c_L = 3d_G (g - 1) .
\]
(3.22)
There is also the case when we take \( a_1 = a_2 = 0, a_3 = 1 \) and this is the case where the supersymmetry is enhanced to \( \mathcal{N} = (4,4) \) however we will not consider this case since the vacuum is non-normalizable which tells us that the moduli space is non-compact and there is no \( AdS_3 \) solutions [3]. For a fixed gauge group and \( g > 1 \) there is a twist that minimizes the central charge where we choose \( a_1 = \frac{1}{3} \) this gives us the central charge \( c_R = \frac{8}{3} (g - 1) d_G \) and this twist preserves \( SU(3) \times U(1) \) global symmetry and this corresponds to the degrees of the three line bundles is equal.

Let us now study the gravity side of all this, we will construct the \( AdS_3 \) type \( IIB \) supergravity solution for generic \( a_I \) dual to the twisted SYM \( \mathcal{N} = 4 \) at large \( N \). We will use the notation of [15], the supergravity theory we consider here is the \( D = 5 \mathcal{N} = 8 \) maximal supergravity where the minimal truncation gives \( \mathcal{N} = 2 \) which is a consistent truncation. This theory contains two vector multiples coupled and three \( U(1) \) gauge fields and two scalars this is known as the STU model. We will work in the simplest consistent truncation which is the metric, three Abelian gauge fields \( A_I \) in the Cartan of \( SO(6) \) and two neutral scalars \( \phi_1 \) and \( \phi_2 \). The Lagrangian is given as
\[
\mathcal{L} = R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \phi_2)^2 + 4g^2 \sum_{I} X_I^{-2} - \frac{1}{4} \sum_{I} X_I^{-2} (F_I^1)^2 + \frac{1}{4} \epsilon^{\mu \nu \rho \sigma \lambda} F_{\mu \nu}^1 F_{\rho \sigma}^2 A_3^3 ,
\]
(3.23)
where the scalars are defined as
\[
X^1 = e^{-\frac{\phi_1}{\sqrt{2}}} \phi_3, \quad X^2 = e^{-\frac{\phi_1}{\sqrt{2}}} + \phi_3, \quad X^3 = e^{\frac{\phi_1}{\sqrt{2}}}, \quad X^1 X^2 X^3 = 1 .
\]
(3.24)
The five dimensional metric and the field strength is given as
\[
ds_5^2 = e^{f(r)} (-dt^2 + dz^2 + dr^2) + e^{2g(r) + 2h(x,y)} (dx^2 + dy^2) ,
\]

\[
F^I = -a_I e^{2h(x,y)} dx \wedge dy , \quad I = 1, 2, 3 .
\]
(3.25)
We consider closed Riemann surfaces $\Sigma_g$ with constant curvature metrics and the function $h(x, y)$ is defined as

$$h(x, y) = \begin{cases} -\log \frac{1+x^2+y^2}{2}, & \text{for } S^2 \\ \frac{1}{2} \log 2\pi, & \text{for } T^2 \\ -\log y & \text{for } \mathbb{H}^2 \end{cases},$$

(3.26)

where for $g = 1$ the range of coordinates is $x, y \in [0, 1]$ also note that $F^I$ takes its values in the $SO(2)^3$ group. We want to find the BPS equations in order to compute scalars in terms of the $a^I_i$'s, by doing so we evaluate the supersymmetric transformations of fermionic fields which are given as

$$\delta \psi_\mu = \left( \partial_\mu + \frac{1}{4} \omega_\mu \gamma ab + \frac{i}{8} X_f (\gamma_\mu \nu \rho - 4 \delta_\mu \nu \rho) F^I_{\nu \rho} + \frac{1}{2} X^I V_I \gamma_\mu - \frac{3i}{2} V_I A^I_\mu \right) \epsilon,$$

(3.27)

$$\delta \chi_{(j)} = \left( \frac{3}{8} (\partial_{\phi_j} X_I) F^I_{\mu \nu} \gamma^\mu \nu + \frac{3i}{2} V_I \partial_{\phi_j} X^I - \frac{1}{4} \delta_{jk} \partial_\mu \phi_k \gamma^\mu \right) \epsilon \quad j = 1, 2,$$

we also define $V_I = \frac{1}{3}$ and $X_I = \frac{1}{2} (X^I)^{-1}$. The spinors obey the following constraints

$$\gamma_\mu \epsilon = \epsilon, \quad \gamma_{\overline{z}} \epsilon = i \epsilon, \quad \partial \epsilon = \partial_x \epsilon = \partial_y \epsilon = \partial_y \epsilon = 0$$

(3.28)

the derivations for the BPS equations can be found in the appendix. The BPS equations are given as

$$0 = g' + \frac{1}{3} e^f (X^1 + X^2 + X^3) - e^{f - 2g} a_I X_I,$$

(3.29)

$$0 = f' + \frac{1}{3} e^f (X^1 + X^2 + X^3) + \frac{1}{2} e^{f - 2g} a_I X_I,$$

(3.30)

$$0 = \phi_1' + \frac{\sqrt{6}}{3} e^f (X^1 + X^2 - 2X^3) + \frac{\sqrt{6}}{2} (a_1 X_1 + a_2 X_2 - 2a_3 X_3),$$

(3.31)

$$0 = \phi_2' + \frac{\sqrt{6}}{3} e^f (X^1 - X^2) + \frac{3\sqrt{2}}{2} e^f (a_1 X_1 - a_2 X_2),$$

(3.32)

$$0 = a_1 + a_2 + a_3 + \kappa.$$  

(3.33)

We want to find the $AdS_3$ fixed points where we take $f(r) = f_0 - \log r$ and $g, \phi_1, \phi_2$ to be constants, this results in the BPS equations becoming simpler. Taking the difference of the first equation third equation gives us the following algebraic equation

$$2 - e^{-2g} (a_1 X^2 + a_2 X^1) / X^1 X^2 = 0,$$

this implies that $a_1 X^2 + a_2 X^1 \neq 0$ and we find

$$e^{-2g} = \frac{2}{a_1 X^2 + a_2 X^1}.$$  

(3.34)

We can now use this and plug the first equation into the second one and we find the following relation

$$e^{f_0} = \frac{2 X^1 X^2}{1 + (X^1)^2 X^2 + (X^2)^2}.$$  

(3.35)

This solves the functions for the metric and we are left to solve the functions for the scalars. Hence we need to solve the first and the forth equation to obtain the solutions for $X^1$ and $X^2$, for simplicity we define the following variables

$$Y = (X^1)^2 X^2, \quad Z = X^1 (X^2)^2.$$  

(3.36)

Taking the sum and the difference of first and forth equation gives us the following expressions

$$(a_2 - a_3) Y + a_1 (Z - 1) = 0 \quad \text{and} \quad (a_1 - a_3) Z + a_2 (Y - 1) = 0$$

and solving this set of equations gives us the following solutions

$$Y = \frac{a_1 (-a_1 + a_2 + a_3)}{a_3 (a_1 + a_2 - a_3)}, \quad Z = \frac{a_2 (a_1 - a_2 + a_3)}{a_3 (a_1 + a_2 - a_3)}.$$  

(3.37)
also notice that we obtain
\[ e^{2g+f_0} = \frac{a_1Z + a_2Y}{1 + Y + Z}. \]  

(3.38)

We can now use equation (3.37) together with the defined scalars \( X^I \) to obtain the value for the scalars in terms of the twist parameters \( a_I \)

\[
Y = e^{-3\phi_1/\sqrt{\pi} - \phi_2/\sqrt{\pi}} = \frac{a_1(-a_1 + a_2 + a_3)}{a_3(a_1 + a_2 - a_3)} \quad \Leftrightarrow \quad e^{\phi_2/\sqrt{\pi}} = \frac{a_3(a_1 + a_2 - a_3)}{a_1(-a_1 + a_2 + a_3)} e^{-3\phi_1/\sqrt{\pi}},
\]

\[
Z = e^{-3\phi_1/\sqrt{\pi} + \phi_2/\sqrt{\pi}} = \frac{a_2(a_1 - a_2 + a_3)}{a_3(a_1 + a_2 - a_3)} \quad \Leftrightarrow \quad e^{3\phi_1/\sqrt{\pi}} = \frac{a_3(a_1 + a_2 - a_3)}{a_2(a_1 - a_2 + a_3)} e^{\phi_2/\sqrt{\pi}},
\]

(3.39)

using this and after some computation we obtain the following values for the scalars

\[ e^{\sqrt{\pi} \phi_1} = \frac{a_3^2(a_1 + a_2 - a_3)^2}{a_1a_2(-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)}, \quad e^{\sqrt{\pi} \phi_2} = \frac{a_2(a_1 - a_2 + a_3)}{a_1(-a_1 + a_2 + a_3)}. \]

(3.40)

The warp factors for the \( AdS_3 \) solution is given in terms of the solutions we find above namely \( e^g \) and \( e^{f_0} \) they are given as

\[
e^g = \frac{a_1^2a_2^2a_3^2}{\Pi}, \quad e^{f_0} = -\frac{8a_1a_2a_3\Pi}{\Theta},
\]

(3.41)

where \( \Pi \) and \( \Theta \) was define above. We can now do an analysis of the positivity of the \( AdS_3 \) solution, the conditions for good supersymmetric vacua is given as \( X^1 > 0, X^2 > 0, e^{2g} > 0 \) and \( e^{f_0} > 0 \) this is equivalent in requiring that

\[ Y > 0, \quad Z > 0, \quad e^{2g+f_0} > 0. \]

(3.42)

This implies that for certain regions the parameter space \( (a_1, a_2, a_3) \) where the compactified \( N = 4 \) SYM theory flows to an IR fixed point, \( i.e \)

\[
g = 0: \quad \{ a_1 > 0, a_2 > 0 \} \cup \{ a_1 > 0, a_1 + a_2 < -1 \} \cup \{ a_2 > 0, a_1 + a_2 < -1 \},
\]

\[
g = 1: \quad \{ a_1 > 0, a_2 > 0 \} \cup \{ a_1 > 0, a_1 + a_2 < 0 \} \cup \{ a_2 > 0, a_1 + a_2 < 0 \},
\]

\[
g > 1: \quad \left\{ a_1 > \frac{1}{2}, a_2 > \frac{1}{2} \right\} \cup \left\{ a_1 > \frac{1}{2}, a_1 + a_2 < \frac{1}{2} \right\} \cup \left\{ a_2 > \frac{1}{2}, a_1 + a_2 < \frac{1}{2} \right\}
\]

\[ \cup \left\{ a_1 < \frac{1}{2}, a_2 < \frac{1}{2}, a_1 + a_2 > \frac{1}{2} \right\} \cup \left\{ a_1 = a_2 = \frac{1}{2}, a_3 = 0 \right\}. \]

(3.43)

For the higher genus Riemann surface \( g > 1 \) we find four open regions and where they also connect, for \( g = 1 \) we have a torus and one finds three regions and for the sphere \( g = 0 \) we have three open regions we also require that at least two \( a'_I \)s are positive, the plots are given below for the corresponding Riemann surfaces:

Figure 1: \( g > 1 \) left, \( g = 1 \) middle, \( g = 0 \) right

20
The region in the middle is the locus $\Theta = 0$ for $g = 0, 1$ and for the torus we have that it collapses to one point in the region. The large $N$ is computed as

$$c_R = \frac{3R_{AdS_3}}{2G_N^{(3)}} = \frac{2N^2 \text{vol}_S e^\Lambda}{\pi} = -12\eta_S N^2 a_1 a_2 a_3 \frac{1}{\Theta}, \quad (3.44)$$

where the volume form is given by integration on the Riemann surface of constant curvature. For the chosen values of $a'_I$s and at large $N$ we see that $N = 4$ SYM flows to a 2d IR fixed point and $AdS_3$ is the holographic dual to this theory. Let us also mention the signature of the eigenvalues of the Hessian to determine the chirality of the flavor currents. One finds for $g = 0, 1$ in all allowed regions the Hessian has one positive and one negative eigenvalues. For $g > 1$ the eigenvalues have opposite sign in the three infinite regions and are both negative in the finite region in the middle. We can also see that the Chern-Simons couplings are proportional to the matrix of 't Hooft anomalies (3.21) by considering the Chern-Simons term in the five dimensional supergravity theory given by

$$S_{5d} \supset \frac{1}{G_N^{(5)}} \int F^{(1)} \wedge F^{(2)} \wedge A^{(3)}.$$ \quad (3.45)$$

Using the definition of the field strengths and the integration over the Riemann surface $\Sigma$ and writing $G_N^{(5)} = \pi/2N^2$ we obtain the following three dimensional effective action

$$S_{3d} \supset -4\eta_S N^2 \int (a_1 A^{(2)} \wedge F^{(3)} + a_2 A^{(3)} \wedge F^{(1)} + a_3 A^{(1)} \wedge F^{(2)}) = k^{IJ} \int A^{(I)} \wedge F^{(J)}. \quad (3.46)$$

The signature of the 't Hooft anomaly matrix determines the chirality of the currents at the IR fixed point and in the good supergravity regions: the Hessian has signature (1, 1) which are the blue regions in the plots and the anomaly matrix has the signature (2, 1). When the Hessian has the signature (0, 2) which corresponds to the middle region in the plot for $g > 1$ then $k^{IJ}$ has signature (1, 2) this is also consistent with the fact that the R-symmetry is right moving.

We can compute the holographic RG flows numerically to see how they flow in the interpolating solution between $AdS_5 \times S^5$ in the UV and $AdS_3$ in the IR fixed point. We will introduce the following new radial variable given as

$$\rho = f + \frac{1}{2\sqrt{6}} \phi_1 + \frac{1}{2\sqrt{2}} \phi_2, \quad (3.47)$$

where we have that $\frac{d\rho}{d\rho} = -e^f D$ with $D \equiv X^1 + \frac{3a_1}{2} e^{-2g} X_1$. Using this the system of equations simplify to the following set of differential equations

$$0 = \frac{dg}{d\rho} - \frac{1}{D} \left( \frac{X^1 + X^2 + X^3}{3} - e^{-2g} a_1 X_1 \right),$$

$$0 = \frac{d\phi_1}{d\rho} \sqrt{6} D \left( \frac{X^1 + X^2 - 2X^3}{3} + e^{-2g} a_1 X_1 + a_2 X_2 - 2a_3 X_3 \right),$$

$$0 = \frac{d\phi_2}{d\rho} \sqrt{2} D \left( X^1 - X^2 + 3e^{-2g} a_1 X_1 - a_2 X_2 \right), \quad (3.48)$$

using this choice of the variable makes our BPS equations independent of the function $f(r)$ and solving the BPS equations one can then solve $f(\rho)$ directly from (3.47). The numerical solutions to these equations are shown in the figures below.

One last thing to mention is the uplift to the origin theory we compactified on, this can be done using the techniques developed in [15], this results in the type $IIB$
supergravity theory. The non vanishing fields are the ten dimensional metric and the self dual five-form flux. The metric is given as

\[ ds^2_{10} = \Delta^{1/2} ds^2_5 + \Delta^{-1/2} \sum_{I=1,2,3} X^I m^2_I \left( d\mu^2_I + \mu^2_I (d\phi_I + A^I)^2 \right), \quad (3.49) \]

where we have the following constraints on the coordinates of \( S^5 \) namely \( \sum_{I=1,2,3} \mu^2_I = 1 \) and their parametrization is given in the regular Euler angles:

\[ \mu_1 = \cos \theta \sin \xi, \quad \mu_2 = \cos \theta \cos \xi, \quad \mu_3 = \sin \theta, \quad \theta \in [0,\pi], \xi \in [0,2\pi) . \quad (3.50) \]

The warp factor \( \Delta \) and the one-form \( A^I \) is defined as

\[ \Delta = \sum_{I=1,2,3} X^I \mu^2_I, \quad F^I = -a_I e^{2h(x,y)} dx \wedge dy = dA^I, \quad (3.51) \]

the self dual five form flux is given as

\[ G^{(5)} = \sum_{I=1}^3 \left( 2X^I(X^I \mu^2_I - \Delta)\epsilon(5) + \frac{1}{2(X^I)^2} d(\mu^2_I) \wedge \left( (d\phi_I + A^I) \wedge \ast_5 F^I + X^I \ast_5 dX^I \right) \right). \quad (3.52) \]

This concludes the \( AdS_3 \) supergravity theory and its corresponding 2d \( \mathcal{N} = (0,2) \) SCFT, one thing we will not mention is the special properties of the bulk vector fields. The subtleties are due to having gauge fields that are both Yang-Mills and Chern-Simons kinetic terms in three dimensional gauge fields the details can be found in section 4 of [4].

### 3.2 M5

Let us now consider the six dimensional \( \mathcal{N} = (2,0) \) theory on \( \Sigma_1 \times \Sigma_2 \) the resulting theory is a two dimensional theory that appears as a low energy compactification of the six dimensional \( \mathcal{N} = (2,0) \) theory on a four manifold. The four manifolds of attention will be the product of two Riemann surfaces, where the main focus will be on preserving \( \mathcal{N} = (0,2) \) supersymmetries in two dimensions.

The six dimensional theory is a family of theories classified by the ADE Lie algebras plus the \( \mathcal{N} = (2,0) \) free tensor multiplet. The family of \( \mathcal{N} = (0,2) \) partial topological twist we will study corresponds to wrapping the \( M5 \) branes on \( \Sigma_1 \times \Sigma_2 \) as a Kähler 4-cycle in a Calabi-Yau fourfold. At large \( N \) limit we will construct \( AdS_3 \) vacua preserving at least \( \mathcal{N} = (0,2) \) supersymmetries.

The R-symmetry of the six dimensional \( \mathcal{N} = (2,0) \) theory is \( Sp(2) \cong SO(5) \), the four manifold \( M_4 = \Sigma_1 \times \Sigma_2 \) with the genera \( g_1 \) and \( g_2 \) which will generate the holonomy
group $SO(2)_1 \times SO(2)_2$. The supercharges transform in the representation $4 \otimes 4$ with symplectic Majorana condition under $SO(5,1) \times SO(5)$, the decomposition is given as

\[
\left(\begin{array}{c}
\frac{i}{2} \frac{1}{2} \\
\frac{i}{2} \frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\frac{1}{2} \frac{1}{2} \\
\frac{1}{2} \frac{1}{2}
\end{array}\right) + \left(\begin{array}{c}
\frac{i}{2} \frac{1}{2} \\
\frac{i}{2} \frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\frac{1}{2} \frac{1}{2} \\
\frac{1}{2} \frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\frac{1}{2} \frac{1}{2} \\
\frac{1}{2} \frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\frac{1}{2} \frac{1}{2} \\
\frac{1}{2} \frac{1}{2}
\end{array}\right)
\right) ,
\]

(3.53)

under $SO(1,1) \times SO(2)_1 \times SO(2)_2 \times SO(2)_A \times SO(2)_B$. We can preserve 2d $\mathcal{N} = (0,2)$ supersymmetry by turning on an Abelian background $A_\mu$ coupled to an $SO(2)^2$ subgroup of $SO(5)$ embedded block-diagonally. The background gauge field with field strength $F = dA = \sum_{\sigma=1,2} F_\sigma$ which is the sum of two components that live on the Riemann surfaces. We again follow the same method and reasoning as previous. The subgroup of $SO(2)$ strength $(0)$ $SO(2)$ conditions in a more compact form given as

\[
\text{g}
\]

The field strength in terms of the two spin connections $\tilde{\Omega}_\sigma = \frac{1}{2} \tilde{\omega}_\mu^{(\sigma)ab} \epsilon_{ab}$ such that the Gauss-Bonnet theorem gives us $\frac{1}{2} \int \sqrt{g} R_\sigma = 4 \pi (1 - g_\sigma)$. The field strength components are given as $F_\sigma = -T_\sigma d\tilde{\omega}(\sigma)$ for $g_\sigma = 0$, $F_\sigma = -T_\sigma \frac{2 \pi}{e\sigma(\Sigma)} d\text{vol}_\Sigma$ for $g = 1$ and $F_\sigma = T_\sigma d\tilde{\omega}(\sigma)$ for $g_\sigma > 1$. The generators for the Riemann surfaces is taken to be

\[
T_\sigma = a_\sigma T_A + b_\sigma T_B , \quad \sigma = 1, 2 ,
\]

(3.54)

where $a_\sigma, b_\sigma$ are real constants that parameterize the twist and are the generators of $SO(2)^2 \subset SO(5)$. The preservation of supersymmetries is given in the same way as the previous construction namely

\[
a_\sigma + b_\sigma = -\kappa_\sigma .
\]

(3.55)

We can parameterize the solutions to these constraints in terms of two real numbers $z_\sigma$,

\[
a_\sigma = -\frac{\kappa_\sigma + z_\sigma}{2} , \quad b_\sigma = -\frac{\kappa_\sigma - z_\sigma}{2} .
\]

(3.56)

The R-symmetry bundle requires that $2(g_\sigma - 1)(\pm a_\sigma, \pm b_\sigma) \in 2\mathbb{Z}$ for $g \neq 1$ the well definiteness over $\Sigma_d$ where the fermions of the 6d theory transform under the representation $4 \otimes 4$ of $SO(5,1) \times SO(5)$ and under $SO(2)_A \times SO(2)_B \subset SO(5)$ with the charge $(\pm \frac{1}{2}, \pm \frac{1}{2})$. For $g_\sigma = 1$ we have that $a_\sigma, b_\sigma \in \mathbb{Z}$, we can write these Dirac quantization conditions in a more compact form given as

\[
\eta_\sigma a_\sigma , \eta_\sigma b_\sigma \in \mathbb{Z} \quad \sigma = 1, 2 .
\]

(3.57)

Let us discuss the amount of supersymmetries one preserves from the twisting: the non flat case $g_{1,2} \neq 1$ which satisfy $0 = a_1 + b_1 + \kappa_1 = a_2 + b_2 + \kappa_2$ the only covariantly constants components of the supercharges is $\left(\begin{array}{c}
\frac{i}{2} \\
\frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right)$ and we preserve 2d $\mathcal{N} = (0,2)$ supersymmetry. If $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$ then the two supercharges $\left(\begin{array}{c}
\frac{i}{2} \\
\frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right)$ are covariantly constant and the global symmetry is enhanced to $SO(3)_A \times SO(2)_B$ and we preserve $\mathcal{N} = (0,4)$ supersymmetry. When one $a_\sigma$ and one $b_\sigma$ vanishes the covariantly constant supercharges are $\left(\begin{array}{c}
\pm \frac{1}{2} \\
\frac{1}{2}
\end{array}\right) \otimes \left(\begin{array}{c}
\pm \frac{1}{2} \\
\frac{1}{2}
\end{array}\right)$ $\otimes \left(\begin{array}{c}
\pm \frac{1}{2} \\
\frac{1}{2}
\end{array}\right)$ we have $\mathcal{N} = (2,2)$. When three of the parameters vanish we preserve $\mathcal{N} = (4,4)$ and when all the parameters vanish and $g_{1,2} = 1$ we preserve maximally supersymmetry $\mathcal{N} = (8,8)$. The low energy theory has a global symmetry $SO(2)^2$ which contains the 2d superconformal R-symmetry. The trial R-symmetry is a linear combination of the generators of $SO(2)^2$:

\[
T_R = (1 + \epsilon) T_A + (1 - \epsilon) T_B ,
\]

(3.58)

where the real number $\epsilon$ parameterizes the mixing and we find the R-charge of the complex supercharge to 1. The reason for this is because for charge $Q = 1$ the M5 brane has zero modes which form the tensor multiplet this leads to the contributions from the
fermions and the self dual 2-form for the gravitational anomaly [10]. Let us mention a few details for the M5 membrane anomaly before we continue with finding the exact R-symmetry.

The chiral world-volume fields of he M5 membrane lead to contributions \( I_8^{\text{inf}}(Q) \) and the second contribution arises as the anomaly inflow \( f_{M11} C_3 \wedge I_8^{\text{inf}} \) which leads to a contribution \( I_8^{\text{inf}} \). These zero modes and the anomaly inflow of the bulk will indeed cancel the tangent bundle anomaly. Following the [10] we have that the contribution of the fermions to the tangent bundle as well as the 2-form contribution gives the total tangent bundle anomaly:

\[
I_8^{\text{total}} = I_8^{\text{ferm}} + I_8^{B^+} = \frac{1}{192} (p_1^2(TW^6) - 4p_2(TW^6)) .
\] (3.59)

This anomaly is canceled if the inflow of the bulk coupling given by \( f_{M11} C_3 \wedge I_8^{\text{inf}} \) where \( I_8^{\text{inf}} \) is given by

\[
X_8 = -\frac{1}{192} (p_1(TM^{11}) - 4p_2(TW^{11})) .
\] (3.60)

The descendant formula in this case is not so trivial since the 2-form is not exact since it allows \( 2\pi \) periods, this tells us that the function is not a smooth function [16] since it supports a dirac-delta function at the origin. The \( M5 \) membrane acts as a magnetic source of \( C_3 \) via \( dG_4 = 2\pi\delta_3(W^6 \mapsto M^{11}) \) and this is how one generally computes the anomaly inflow for tangent bundles.

Let us now discuss the normal bundle anomaly which corresponds to the gravitational anomaly where the theory along the tangent bundle on the world-volume couples to a \( SO(5) \) gauge theory. These anomalies arise when we consider diffeomorphism of \( M^{11} \) where we have \( SO(5) \) gauge transformations on the normal bundle. The \( M5 \) membrane background breaks the Lorentz symmetry \( SO(10,1) \rightarrow Spin(5,1) \otimes Spin(5) \) where the tangent bundle of \( M^{11} \) is decomposed as

\[
TM^{11}|_{W^6} = TW^6 \oplus N ,
\] (3.61)

where \( N \) is the normal bundle. The normal bundle indeed has the gauge transformations that transform the anti symmetric tensor field as a singlet under \( SO(5) \) and does not contribute to the normal bundle anomaly. The world-volume fermions are chiral spinors with values in the rank four bundle \( S(N) \) and the total anomaly due to the fermion fields is given as

\[
I_8^{\text{ferm}} = \frac{1}{2} \text{ch}(S(N), \hat{A}(TW^6))|_8 .
\] (3.62)

We can represent the Chern classes in terms of Pontryagin classes, to compute \( \text{ch}(S(N)) \) we notice that the Chern roots of the \( SO(5) \) bundle \( N \) are \( \pm \lambda_1, \pm \lambda_2 \) and the eigenvalues of the curvature of \( S(N) \) are \( \pm (\lambda_1 \pm \lambda_2) \) which are also the Chern roots. Using this we can compute the Dirac genus and the Chern class of \( S(N) \) to be

\[
\hat{A}(TW^6) = 1 - \frac{p_1(TW^6)}{24} + \frac{7p_1(TW^6)^2 - 4p_2(TW^6)}{5760}, \quad \text{ch}(S(N)) = 4 + \frac{p_1(N)^2}{2} + \frac{p_1(N)^2}{96} + \frac{p_2(N)}{24} .
\] (3.63)

The inflow from the bulk can be computed using the properties of the Pontryagin classes such that we have \( p_1(TM^{11}|_{W^6}) = p_1(TW^6) + p_1(N) \) and \( p_2(TM^{11}|_{W^6}) = p_2(TW^6) + p_2(N) + p_1(TW^6)p_1(N) \). This gives us the following anomaly inflow of the bulk to be given as

\[
I_8^{\text{inf}} = \frac{1}{48} \left(p_2(TW^6) + p_2(N) - \frac{1}{4} \left(p_1(TW^6) - p_1(N) \right)^2 \right) ,
\] (3.64)
summing all the equations above one finds the normal bundle anomaly given as

\[ I_{\text{total}}^8 = \frac{p_2(N)}{24}. \]  

(3.65)

This anomaly is however canceled if one carefully studies the Chern-Simons term in the 11D supergravity in the presence of M5 membranes [?]. The contribution of the Chern-Simons term indeed cancels the normal bundle anomaly where the anomaly of the Chern-Simons term is given as \( I_{CS}^8 = -\frac{p_2(N)}{24} \). The results of the M5 membrane anomaly computations have applications in black hole entropy [?]. We can now move on to compute the exact R-symmetry of our twisted theory on \( \Sigma_1 \times \Sigma_2 \) using these techniques in the spirit of [17–20].

We will integrate the 6d anomaly polynomial on \( \Sigma_1 \times \Sigma_2 \), where we denote the Chern roots \( t_\sigma \) of the tangent bundles on \( \Sigma_\sigma \) and the Chern roots by \( n_{A,B} \) of the R-symmetry bundle. The condition for the supersymmetry is then given as

\[ t_1 + t_2 + n_A + n_B = 0. \]  

(3.66)

To compute the 8-form we couple the symmetry U(1) to a non trivial bundle \( F_R \), the U(1) symmetry can be seen from the decomposition of \( SO(4) \cong SU(2)_1 \times SU(2)_2 \). This induces a shift in the Chern roots for the R-symmetry which will be what detects the anomalies in the 2d trial R-symmetry

\[ n_A = a_1f_1 + a_2f_2 + (1 + \epsilon)c_1(F_R), \quad n_B = b_1f_1 + b_2f_2 + (1 - \epsilon)c_1(F_R) \]  

(3.67)

We also introduced the two classes \( f_\sigma \) on the surfaces \( \Sigma_\sigma \) along the field strength components \( F_\sigma \) which will be define as \( f_\sigma = n_\sigma \) which for \( g = 1 \) is equal to one and \( g \neq 1 \) is equal to \( 2|g - 1| \) and \( \int t_\sigma = 2(1 - g) \). We have seen the form of the M5 membrane anomaly (3.64) for a Abelian 6d \( \mathcal{N} = (2,0) \) also known as a free M5 membrane, from now one we will denote the R-symmetry as NW which is the same as the normal bundle on the M5 membrane. We can write the anomaly polynomial of \( G \)-type (\( G = A_N, D_N, E_N \)) in the following way

\[ I_8[G] = r_G I_{\text{inflow}}^8 + \frac{d_G h_G}{24} p_2(NW), \]  

(3.68)

where \( r_G, d_G \) and \( h_G \) are the rank, dimensions and Coxeter number of \( G \) given in the table below:

<table>
<thead>
<tr>
<th>( G )</th>
<th>( r_G )</th>
<th>( d_G )</th>
<th>( h_G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{N-1} )</td>
<td>( N )</td>
<td>( N^2 - 1 )</td>
<td>( N )</td>
</tr>
<tr>
<td>( D_N )</td>
<td>( N )</td>
<td>( N(2N - 1) )</td>
<td>( 2N - 2 )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>6</td>
<td>78</td>
<td>12</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>7</td>
<td>133</td>
<td>18</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>8</td>
<td>248</td>
<td>30</td>
</tr>
</tbody>
</table>

notice that \( d_G = r_G(h_G + 1) \) and \( d_G h_G \) is a multiple of 6. In terms of the Chern roots \( n_i \) the Pontryagin classes are

\[ p_1 = \sum_i n_i^2, \quad p_2 = \sum_{i<j} n_i^2 n_j^2 \]  

(3.69)

with this we can compute the anomaly polynomial of the \( G \) type by integrating over \( \Sigma_1 \times \Sigma_2 \). The only terms that survive in the integration are terms of linear order i.e. anything of degree higher than \( TW_2 \) vanishes automatically\(^5\), then following anomaly

\(^{5}\)The normal bundle is taken to be the product of two vector bundles where the Whitney sum is \( NW = TW_2 \oplus NW_4 \).
theory. We will consider the seven dimensional maximal gauged supergravity \[5, 21\]

\[
\int_{\Sigma_1 \times \Sigma_2} I_8[G] = \frac{\eta \eta_2}{24} \left( (d_G h_G(\kappa_1 \kappa_2 (3 - \epsilon^2) + 2(\kappa_1 z_2 + \kappa_2 z_1) - z_1 z_2 (1 - 3\epsilon^2))
+ 3r_G(\kappa_1 \kappa_2 + z_1 z_2 \epsilon^2))c_1(F_R)^2 - \frac{r_G}{4}(\kappa_1 \kappa_2 + z_1 z_2)p_1(TW_2) \right),
\]

we can compare this to the anomaly polynomial in the two dimensional theory

\[
I_4 = \frac{c_R R}{6} c_1(F_R)^2 - \frac{c_R - c_L}{24} p_1(TW_2).
\]

We can thus extract the trial right moving central charge and the gravitational anomaly

\[
c^{tr}_R(\epsilon) = \frac{\eta \eta_2}{24} \left( d_G h_G(\kappa_1 \kappa_2 (3 - \epsilon^2) + 2(\kappa_1 z_2 + \kappa_2 z_1) - z_1 z_2 (1 - 3\epsilon^2)) + 3r_G(\kappa_1 \kappa_2 + z_1 z_2 \epsilon^2) \right),
k = c_R - c_L = \frac{\eta \eta_2}{4} r_G(\kappa_1 \kappa_2 + z_1 z_2).
\]

We can extremize this trial current function and we obtain the following point

\[
\epsilon = \frac{d_G h_G(\kappa_1 z_2 + \kappa_2 z_1)}{d_G h_G(\kappa_1 \kappa_2 - 3z_1 z_2) - 3r_G z_1 z_2},
\]

plugging this into the trial current we obtain the following right and left moving currents

\[
c_R = \frac{\eta \eta_2 d_G^2 h_G^2 P + 3d_G h_G r_G(z_1^2 z_2^2 + \kappa_1 \kappa_2 z_1 z_2 + \kappa_1^2 \kappa_2^2) - 9r_G^2 \kappa_1 \kappa_2 z_1 z_2}{4 d_G h_G(\kappa_1 \kappa_2 - 3z_1 z_2) - 3r_G z_1 z_2},
\]

\[
c_L = \frac{\eta \eta_2 d_G^2 h_G^2 P + 2d_G h_G r_G(3z_1^2 z_2^2 - \kappa_1 \kappa_2 z_1 z_2 + \kappa_1^2 \kappa_2^2) - 3r_G^2 z_1 z_2 (z_1 z_2 - \kappa_1 \kappa_2)}{4 d_G h_G(\kappa_1 \kappa_2 - 3z_1 z_2) - 3r_G z_1 z_2},
\]

where \( c_L = c_R - k \) and \( P \) is defined as

\[
P = 3z_1^2 z_2^2 + \kappa_1^2 z_2 + \kappa_2^2 z_1 - \kappa_1 \kappa_2 z_1 z_2 + 3\kappa_1^2 \kappa_2^2.
\]

Taking the second derivative of the trial current to see the sign of the flavor current, gives the following expression

\[
\frac{\partial c^{tr}_R}{\partial \epsilon^2} = \frac{\eta \eta_2}{2} r_G(h_G(h_G + 1)(3z_1 z_2 - \kappa_1 \kappa_2) + 3z_1 z_2),
\]

the sign is positive and later we will see when the parameters give us good \( AdS_3 \) vacua by analyzing the regions in more detailed. The central charges above are valid for all \( ADE \) Lie algebras an example is by plugging in \( A_N \) theory for large \( N \) the central charge is then given as

\[
c_N^{tr} \simeq c_N \simeq \frac{\eta \eta_2 N^3}{4} \frac{3z_1^2 z_2^2 + \kappa_1^2 z_2 + \kappa_2^2 z_1 - \kappa_1 \kappa_2 z_1 z_2 + 3\kappa_1^2 \kappa_2^2}{\kappa_1 \kappa_2 - 3z_1 z_2},
\]

and for \( D_N \) theories we have \( c_N^{tr} \simeq c_N^{DN} \simeq 4c_N^{tr} \). Notice that for the twisted theory we have a gravitational anomaly.

Let us now discuss the supergravity solutions that describes the backreaction of \( N \) \( M5 \) membranes on \( \Sigma_1 \times \Sigma_2 \) at large \( N \), in order to be able to study the \( 6d \mathcal{N} = (2, 0) \) theory. We will consider the seven dimensional maximal gauged supergravity \[5, 21\]
and it was shown to be a consistent truncation of eleven dimensional supergravity on $S^4$ [22, 23]. The Lagrangian is given as

$$L = \frac{\sqrt{-g}}{2} \left( R + \frac{m}{2} (T^2 - 2 t_i T^{ij}) - P_{\mu ij} P^{\mu ij} - \frac{1}{2} (\Pi_A^i \Pi_B^j F_{\mu AB})^2 - m^2 (\Pi^{-1})_A^i S_A^{\mu \nu \rho} \right)$$

$$- 3 m \delta^{AB} S_A \wedge F_B + \sqrt{3} \epsilon_{ABCDE} \delta^{AG} S_C \wedge F^{BC} \wedge F^{DE} + \frac{1}{16 m^2} (2 \Omega_5[A] - \Omega_3[A]) .$$

(3.78)

The indices $A, B = 1, \ldots, 5$ are in the fundamental representation of the $SO(5)_g$ gauge group and $i, j = 1, \ldots, 5$ are in the $SO(5)_c$ local composite gauge group. We have 14 scalars that are parameterized by the scalar coset $SL(5, \mathbb{R})/SO(5)_c$ given by the matrix $\Pi_A^i$ which transforms as $5$ of both $SO(5)_g$ and $SO(5)_c$. The scalar kinetic term $P_{\mu ij}$ and the composite $SO(5)_c$ connection $Q_{\mu ij}$ which appaeers in the covariant derivatives, they are given as

$$P_{\mu ij} = \frac{1}{2} (M_{\mu ij} + M_{\mu ji}), \quad Q_{\mu ij} = \frac{1}{2} (M_{\mu ij} - M_{\mu ji}), \quad M_{\mu ij} = (\Pi^{-1})_A^i (\delta_A^B \partial_\mu + 2 m A_{\mu A} B) \Pi_B^j \delta_{kj} .$$

(3.79)

The gauge group has vectors $A^{AB}_\mu$ and the 3-form gauge potential $S_A$ whose 4-form field strength is given as $F_A = dS_A + 2 m A_A B S_B$. The Chern-Simons 7-forms $\Omega_3[A], \Omega_5[A]$ are given as

$$\Omega_3[A] = \epsilon^{\alpha \beta \gamma \delta \epsilon \eta \xi} \text{Tr} (A_\alpha \wedge F_{\beta \gamma} - \frac{2}{3} A_\alpha \wedge A_\beta \wedge A_\gamma) \text{Tr} (F_\delta \wedge F_\epsilon) ,$$

$$\Omega_5[A] = \epsilon^{\alpha \beta \gamma \delta \epsilon \eta \xi} \text{Tr} \left( A_\alpha \wedge F_{\beta \gamma} \wedge F_\delta \wedge F_\epsilon \wedge F_\eta \wedge F_\xi - \frac{4}{5} A_\alpha \wedge A_\beta \wedge A_\gamma \wedge F_\delta \wedge F_\epsilon \wedge F_\eta \wedge F_\xi \right)$$

$$- \frac{2}{5} A_\alpha \wedge A_\beta \wedge F_\gamma \wedge A_\epsilon \wedge F_\eta \wedge F_\xi + \frac{4}{5} A_\alpha \wedge A_\beta \wedge A_\gamma \wedge A_\delta \wedge A_\epsilon \wedge F_\eta \wedge F_\xi$$

$$- \frac{8}{35} A_\alpha \wedge A_\beta \wedge A_\gamma \wedge A_\delta \wedge A_\epsilon \wedge A_\eta \wedge A_\xi \right) \right)$$

(3.80)

and finally the matrix of scalars and the potential are given as

$$\Pi_A^i = \text{diag}(e^{\lambda_1}, e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_2}, e^{-2 \lambda_1 - 2 \lambda_2}), \quad T_{ij} = \delta_{AB} (\Pi^{-1})_A^i (\Pi^{-1})_B^j , \quad T = \delta_{ij} T_{ij} .$$

(3.81)

The non zero components of the $SO(5)$ gauge fields are in the Cartan of $SO(5)$ and their Abelian field strength will be denoted $F^A, F^B$, the full ansatz is given as

$$d\sigma^2 = e^{2 f(r)} (-dt^2 + dx^2 + dr^2) + \sum_{\sigma = 1, 2} e^{-2 g_\sigma (x)} (dx_\sigma^2 + dy_\sigma^2) ,$$

$$F^A = -\frac{1}{4} \sum_{\sigma = 1, 2} a_\sigma e^{2 h_\sigma (x)} dx_\sigma \wedge dy_\sigma , \quad F^B = -\frac{1}{4} \sum_{\sigma = 1, 2} b_\sigma e^{2 h_\sigma (x)} dx_\sigma \wedge dy_\sigma \right)$$

$$\lambda_1 = \lambda_1 (r), \quad \lambda_2 = \lambda_2 (r) ,$$

the metric functions $h_\sigma$ are taken to be the same as in the five dimensional supergravity. For $g = 0$ the coordinates $x_\sigma, y_\sigma$ span $\mathbb{R}^2$, for $g = 1$ they span $[0, 1]^2$, for $g > 1$ the range is $x_\sigma \in \mathbb{R}, y_\sigma > 0$ describes the hyperbolic plane $\mathbb{H}^2$ where we take the quotient of this space with a discrete subgroup of $PSL(2, \mathbb{R})$ to obtain a compact Riemann surface$^6$.

We set the gauge coupling $g = 4$ therefore the normalization of the fluxes is given by

$$\frac{1}{2 \pi} \int F^{A,B} \in \mathbb{Z}/4. \quad \text{We also have that the 3-form gauge potential for the given ansatz takes the form}$$

$$S_5 = -\frac{1}{32 \sqrt{3}} [(a_1 b_2 + a_2 b_1) e^{-4(\lambda_1 + \lambda_2) - 2(g_1 + g_2)} dt \wedge dz \wedge dr .$$

(3.83)

$^6$This is known as the Fuchsian and we need to take the quotient in order for the total area of the surface to be $4\pi(g - 1)$.
In order to derive the BPS equations to analyze the regions for good $AdS_3$ vacua we need the supersymmetry transformations and they are given as

\[ \delta \psi_\mu = \left( \partial_\mu + \frac{1}{4} \omega^{ab}_\mu + \frac{1}{4} Q_{\mu ij} \Gamma^{ij} + \frac{m}{20} T^\mu - \frac{1}{40} \left( \gamma_\mu^\rho - 8 \delta_\mu^\nu \rho \right) \Gamma_\nu \Pi_\nu \Pi F^{AB} \right) + \frac{m}{10 \sqrt{3}} \left( \gamma_\mu^\nu \rho \sigma - \frac{9}{2} \delta_\mu^\nu \rho \right) \Gamma^i (\Pi^{-1})^A_i S_{A \nu \rho \sigma} \epsilon , \]

\[ \delta \chi_i = \left( \frac{1}{2} P_{\mu ij} \gamma^\mu \Gamma^j + \frac{m}{2} \left( \Gamma_{ij} - \frac{1}{2} \delta_{ij} T \right) \Gamma^{ij} + \frac{1}{16} \left( \Gamma_{kt} \Gamma_i - \frac{1}{5} \Gamma_i \Gamma_{kt} \right) \gamma^{\mu \nu} \Pi_\mu \Pi F^{AB} \right) + \frac{m}{20 \sqrt{3}} \gamma^{\mu \nu} \left( \Gamma_i^j - 4 \delta_i^j \right) (\Pi^{-1})^A_i S_{A \nu \rho} \epsilon , \]

(3.84)

here the $\gamma^a$ are gamma matrices for seven dimensional spacetime and $\Gamma^i$ are $SO(5)_c$ gamma matrices, where we also have $\Gamma^i \chi_i = 0$. We also impose the following projections on the supergravity spinor $\epsilon$,

\[ \gamma_3 \epsilon = \epsilon , \quad \gamma_{45} \epsilon = \gamma_{67} \epsilon = \Gamma^{12} \epsilon = \Gamma^{34} \epsilon = \pm \epsilon . \]  

(3.85)

The BPS equations impose the same differential equations on the metric and hence preserves $1/8$ supersymmetry that is one complex supercharge, the BPS equations set

\[ a_\sigma + b_\sigma = -\kappa_\sigma . \]

(3.86)

The BPS equations are thus given as

\[ e^{-f} f' = -\frac{1}{5} \left( 2 e^{-2 \lambda_1} + 2 e^{-2 \lambda_2} + e^{4 \lambda_1 + 4 \lambda_2} \right) - \frac{1}{80} (a_1 b_2 + a_2 b_1) e^{-2 \lambda_1 - 2 \lambda_2 - 2 g_1 - 2 g_2} \]

\[ - \frac{1}{20} \left( a_1 e^{2 \lambda_1 - 2 g_1 + a_2 e^{2 \lambda_2 - 2 g_2} + b_1 e^{2 \lambda_2 - 2 g_1} + b_2 e^{2 \lambda_2 - 2 g_2} \right) , \]

(3.87)

\[ e^{-f} g_1' = -\frac{1}{5} \left( 2 e^{-2 \lambda_1} + 2 e^{-2 \lambda_2} + e^{4 \lambda_1 + 4 \lambda_2} \right) + \frac{1}{40} (a_1 b_2 + a_2 b_1) e^{-2 \lambda_1 - 2 \lambda_2 - 2 g_1 - 2 g_2} \]

\[ - \frac{1}{20} \left( 4 a_1 e^{2 \lambda_1 - 2 g_1} + 4 b_1 e^{2 \lambda_2 - 2 g_1} - 2 b_2 e^{2 \lambda_2 - 2 g_2} \right) , \]

(3.88)

\[ e^{-f} g_2' = -\frac{1}{5} \left( 2 e^{-2 \lambda_1} + 2 e^{-2 \lambda_2} + e^{4 \lambda_1 + 4 \lambda_2} \right) + \frac{1}{40} (a_1 b_2 + a_2 b_1) e^{-2 \lambda_1 - 2 \lambda_2 - 2 g_1 - 2 g_2} \]

\[ - \frac{1}{20} \left( 4 a_2 e^{2 \lambda_2 - 2 g_2} + 4 b_2 e^{2 \lambda_2 - 2 g_2} - 2 a_1 e^{2 \lambda_1 - 2 g_2} - b_1 e^{2 \lambda_2 - 2 g_1} \right) , \]

(3.89)

\[ e^{-f} \lambda_1' = -\frac{2}{5} \left( 3 e^{-2 \lambda_1} - 2 e^{-2 \lambda_2} - e^{4 \lambda_1 + 4 \lambda_2} \right) + \frac{1}{80} (a_1 b_2 + a_2 b_1) e^{-2 \lambda_1 - 2 \lambda_2 - 2 g_1 - 2 g_2} \]

\[ - \frac{1}{20} \left( 3 a_1 e^{2 \lambda_1 - 2 g_1} + 3 a_2 e^{2 \lambda_1 - 2 g_2} - 2 b_1 e^{2 \lambda_2 - 2 g_1} - 2 b_2 e^{2 \lambda_2 - 2 g_2} \right) , \]

(3.90)

\[ e^{-f} \lambda_2' = -\frac{2}{5} \left( 3 e^{-2 \lambda_2} - 2 e^{-2 \lambda_1} - e^{4 \lambda_1 + 4 \lambda_2} \right) + \frac{1}{80} (a_1 b_2 + a_2 b_1) e^{-2 \lambda_1 - 2 \lambda_2 - 2 g_1 - 2 g_2} \]

\[ - \frac{1}{20} \left( 3 b_1 e^{2 \lambda_2 - 2 g_2} + 3 b_2 e^{2 \lambda_2 - 2 g_2} - 2 a_1 e^{2 \lambda_1 - 2 g_1} - 2 a_2 e^{2 \lambda_1 - 2 g_2} \right) , \]

(3.91)

to get an $AdS_3$ vacua we further constraint the radial functions to be

\[ e^{f(r)} = e^{f_0} / r , \quad g_1, g_2 = const, \quad X_1 \equiv e^{2 \lambda_1} = const, \quad X_2 \equiv e^{2 \lambda_2} = const . \]

(3.92)
Using the constraints and simplifying the BPS equations we now can obtain metric functions and the scalars, we first consider the linear combination (3.88) – (3.90) – (3.91) and (3.89) – (3.90) – (3.91) this linear combinations reduce to the following quantities

\[
e^{2g_1} = \frac{a_1 X_1 + b_1 X_2}{4X_1^2 X_2^2}, \quad e^{2g_2} = \frac{a_2 X_1 + b_2 X_2}{4X_1^2 X_2^2}.
\]

(3.93)

We see that \(a_1 X_1 + b_1 X_2 \neq 0\) and \(a_2 X_1 + b_2 X_2 \neq 0\) and the non vanishing combination give the following variables

\[
Y \equiv X_1^2 X_2^2, \quad Z \equiv X_1^2 X_3^2.
\]

(3.94)

In order to obtain the scalar fields in terms of the twist parameters we have to rewrite (3.90), (3.91) in terms of \(Y\) and \(Z\) in doing so we obtain a linear equation in \(Y, Z\) and from here we solve it and find that they take the form

\[
Y = \frac{(a_1^2 b_2 + a_2 b_1^2)(a_2^2 b_1 + a_1 b_2^2)}{(a_1^2 b_2^2 + a_2^2 b_1^2 + a_1 a_2 b_1 b_2)(a_1 b_2 + a_2 b_1 - a_1 a_2)}, \quad Z = \frac{a_1 b_2 + a_2 b_1 - a_1 a_2}{a_1 b_2 + a_2 b_1 - b_1 b_2} Y.
\]

(3.95)

Writing these expression in terms of the scalar functions we obtain the following quantities

\[
e^{10\lambda_1} = \frac{Y^3}{Z^2} = \frac{(a_1^2 b_2 + a_2 b_1^2)(a_2^2 b_1 + a_1 b_2^2)(a_2 b_1 + a_1 b_2 - b_1 b_2)^2}{(a_1^2 b_2^2 + a_2^2 b_1^2 + a_1 a_2 b_1 b_2)(a_1 b_2 + a_2 b_1 - a_1 a_2)^3},
\]

(3.96)

\[
e^{10\lambda_2} = \frac{Z^3}{Y^2} = \frac{(a_2^2 b_1 + a_1 b_2^2)(a_2^2 b_1 + a_1 b_2^2)(a_2 b_1 - a_1 a_2 + a_1 b_2)^2}{(a_1^2 b_2^2 + a_2^2 b_1^2 + a_1 a_2 b_1 b_2)(a_1 b_2 + a_2 b_1 - b_1 b_2)^3}.
\]

(3.97)

We can finally solved the \(AdS_3\) warp factor using the known relations above, in doing so we obtain from equation (3.87) the following function

\[
e^{f_0} = \frac{b_1 b_2 - a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2 - 2a_2 b_1 - 2a_2 b_1} X_2,
\]

(3.98)

and for computing the central charge using again the holographic tools as we did in the previous case we obtain

\[
e^{f_0 + 2g_1 + 2g_2} = -\frac{a_1^2 b_2^2 + a_2^2 b_1^2 + a_1 a_2 b_1 b_2}{16(2a_1 b_2 + 2a_2 b_1 - a_1 a_2 - b_1 b_2)}.
\]

(3.99)

We can now analyze the positivity of the \(AdS_3\) solution of this type, the sufficient condition for good \(AdS_3\) vacua is the following conditions: \(Y > 0, Z > 0, a_1/Z + b_1/Y > 0, a_2/Z + b_2/Y > 0\) and \(e^{f_0} > 0\). The twisting parameters are given by the relation (3.55) and there is only good vacua if at least one of the Riemann surfaces is a hyperbolic plane \(\mathbb{H}^2\). The following regions are valid for the good vacua: for \(g_1 > 1, g_2 > 1\) we have

\[
\{ 1 - 3z_1 z_2 > |z_1 + z_2| \},
\]

(3.100)

the case for \(g_1 = 1, g_2 > 1\) we have

\[
\left\{ z_1 < 0, z_2 > \frac{1}{3} \right\} \cup \left\{ z_1 > 0, z_2 < -\frac{1}{3} \right\}
\]

(3.101)

we could also have \(g_1 > 1, g_2 = 1\) we just exchange \(z_1\) and \(z_2\). For the last case we have \(g_1 = 0, g_2 > 1\) we also obtain two regions namely

\[
\left\{ \frac{1 + z_2^2}{2z_1} < z_2 < \frac{z_1 + 1}{1 - 3z_1}, z_1 > 1 \right\} \cup \left\{ z_1 - 1 < z_2 < -\frac{1 + z_2^2}{2z_1}, z_1 < -1 \right\},
\]

(3.102)
for the other region we just exchange $z_1$ and $z_2$ again with the genera given as $g_1 > 1, g_2 = 0$. Hence we plot the good $AdS_3$ vacua which can be seen in figure 3.

The central charge for the dual theory can again be taken in the leading order of $N$ and we obtain

$$c_R \simeq c_L \simeq \frac{8N}{\pi^2} e^{f_0+2g_1+2g_2} \text{vol}(\Sigma_1 \times \Sigma_2) = 2\eta_1 \eta_2 N^3 \frac{a_1^2 b_1^2 + a_2^2 b_2^2 + a_1 a_2 b_1 b_2}{16(2a_1 b_2 + 2a_2 b_1 - a_1 a_2 - b_1 b_2)}.$$  

This is indeed the exact central charge we computed (3.77) if we express $a_\sigma, b_\sigma$ in terms of $z_\sigma, \kappa_\sigma$, which indeed shows that this gravity theory is in fact the dual of our 2d $\mathcal{N} = (0,2)$ theory.

We will now compute the matrix $k^{IJ}$ of the ’t Hooft anomalies in the field theory and taking the large $N$ limit we indeed obtain the Chern-Simons matrix of the effective three dimensional supergravity. Let us write down the Chern-Simons -forms and simplify it

$$S_{7D} \supset \frac{1}{16m} \int (2\Omega_5[A] - \Omega_3[A])$$

$$= \frac{1}{16m} \int d^7x \sqrt{-g} \epsilon^{\alpha\beta\gamma\delta\epsilon\eta\xi} (2 \text{Tr}(A_\alpha F_{\beta\gamma} F_{\delta\epsilon} F_{\eta\xi}) - \text{Tr}(A_\alpha F_{\beta\gamma}) \text{Tr}(F_{\delta\epsilon} F_{\eta\xi}))$$

$$= -\frac{1}{4m} \int d^7x \sqrt{-g} \left( (A^A \wedge F^A) \wedge F^B + (A^A \wedge F^B) \wedge F^A \wedge F^B \right.$$  

$$\left.+ (A^B \wedge F^A) \wedge F^A \wedge F^B + F^A \wedge F^A \wedge (A^B \wedge F^B) \right)$$

$$= -\frac{1}{2m} \int \left( b_1 b_2 A^A \wedge F^A + a_1 a_2 A^B \wedge F^B + 2(2a_1 b_2 + a_2 b_1) A^A \wedge F^B \right),$$

we have used the field strength quantization and the trace in the second line is over the $SO(5)_g$ indices of the gauge field $A$. The matrix $k^{IJ}$ of ’t Hooft anomalies is computed in [4] where one obtains this by integrating the anomaly polynomial of the 6d $\mathcal{N} = (2,0)$ over the product of two riemann surfaces. The matrix is given as

$$k^{IJ} = \frac{1}{48} \xi_1 \eta_2 \left( b_1 b_2 (4d_G h_G + 3r_G) + 3a_1 a_2 r_G b \right) (a_1 b_2 + a_2 b_1) (4d_G h_G + 3r_G)$$

$$+ \left( a_1 a_2 (4d_G h_G + 3r_G) + 3 b_1 b_2 r_G \right),$$

the matrix has one negative and one positive eigenvalue for all values of $(a_\sigma, b_\sigma)$ which lead to good $AdS_3$ vacua. The flavor current is left moving for this type of SCFT and this is in agreement with the fact that the second derivative of the trial current is negative in the allowed regions for $(a_\sigma, b_\sigma)$. If we consider the $A_N$ theory at large $N$ we indeed recover the matrix of the Chern-simons levels.

Like in the previous section we will now consider the holographic RG flows for this type of theory, we have a interpolating solution from an asymptotically local $AdS_7$ vacua...
to a $AdS_3$ vacua. We will solve the BPS equations numerically and in order to do so we need to introduce the new radial variable

\[ \rho = f(r) - \lambda_1(r) - \lambda_2(r), \]  

such that \( \frac{d\rho}{dr} = -e^f D \) with

\[ D \equiv e^{4(\lambda_1+\lambda_2)} + \frac{1}{16} (a_1 b_2 + a_2 b_1) e^{-2(\lambda_1+\lambda_2+g_1+g_2)}. \]  

The BPS equation are given as

\[ D g'_1(\rho) = \frac{1}{5} \left( 2e^{-2\lambda_1} + 2e^{-2\lambda_2} + e^{4\lambda_1+\lambda_2} \right) - \frac{1}{40} \left( a_1 b_2 + a_2 b_1 \right) e^{-2\lambda_1-2\lambda_2-2g_1-2g_2} \]
\[ - \frac{1}{20} \left( 4a_1 e^{-2\lambda_1-2g_1} + 4b_1 e^{-2\lambda_2-2g_1} - a_2 e^{-2\lambda_1-2g_2} - b_2 e^{-2\lambda_2-2g_2} \right), \]

\[ D g'_2(\rho) = \frac{1}{5} \left( 2e^{-2\lambda_1} + 2e^{-2\lambda_2} + e^{4\lambda_1+\lambda_2} \right) - \frac{1}{40} \left( a_1 b_2 + a_2 b_1 \right) e^{-2\lambda_1-2\lambda_2-2g_1-2g_2} \]
\[ - \frac{1}{20} \left( 4a_2 e^{-2\lambda_2-2g_2} + 4b_2 e^{-2\lambda_2-2g_2} - a_1 e^{-2\lambda_1-2g_1} - b_1 e^{-2\lambda_2-2g_1} \right), \]

\[ D \lambda'_1(\rho) = \frac{2}{5} \left( 3e^{-2\lambda_1} - 2e^{-2\lambda_2} - e^{4\lambda_1+\lambda_2} \right) - \frac{1}{80} \left( a_1 b_2 + a_2 b_1 \right) e^{-2\lambda_1-2\lambda_2-2g_1-2g_2} \]
\[ + \frac{1}{20} \left( 3a_1 e^{-2\lambda_1-2g_1} + 3a_2 e^{-2\lambda_1-2g_2} - 2b_1 e^{-2\lambda_2-2g_1} - 2b_2 e^{-2\lambda_2-2g_2} \right), \]

\[ D \lambda'_2(\rho) = \frac{2}{5} \left( 3e^{-2\lambda_2} - 2e^{-2\lambda_1} - e^{4\lambda_1+\lambda_2} \right) - \frac{1}{80} \left( a_1 b_2 + a_2 b_1 \right) e^{-2\lambda_1-2\lambda_2-2g_1-2g_2} \]
\[ + \frac{1}{20} \left( 3b_1 e^{-2\lambda_2-2g_1} + 3b_2 e^{-2\lambda_2-2g_2} - 2a_1 e^{-2\lambda_1-2g_1} - 2a_2 e^{-2\lambda_1-2g_2} \right). \]

Integrating this numerically we obtain the following graphs:

Figure 4: Numerical solution for \( g_1(\rho), g_2(\rho), \lambda_1(\rho) \) and \( \lambda_2(\rho) \) appearing in the same order above, with the numerical values \((z_1, z_2) = (11, -3), (15, -7), (23, -13)\) for red, yellow and green respectively. \( \kappa_1 = \kappa_2 = -1 \) and \( g_1 = g_2 = 2 \).

The uplift for the maximally supersymmetric gauged supergravity in 7D to the eleven dimensional theory is again given in the same way as in [15]. The eleven dimensional solution is presented in terms of the functions \( X_1 \equiv e^{2\lambda_1}, X_2 \equiv e^{2\lambda_2}, X_0 \equiv (X_1 X_2)^{-2} \).
the metric then takes the form
\[ ds_{11}^2 = \Delta^{1/3} ds_7^2 + \frac{1}{4} \Delta^{-2/3} \left( X_0^{-1} d\mu_0^2 + \sum_{i=1,2} X_i^{-1} \left( d\mu_i^2 + \mu_i^2 (d\phi_i + 4A^{(i)})^2 \right) \right) , \]  
\hspace{1cm} (3.109)

where the \( ds_7^2 \) is defined above and the \( A^{(i)} \) are the two dimensional gauge fields that we called \( A^A \) and \( A^B \). We also define the following relations
\[ \Delta = \sum_{\alpha=0,1,2} X_\alpha \mu_\alpha^2 , \quad \sum_{\alpha=0,1,2} \mu_\alpha^2 = 1 , \]  
\hspace{1cm} (3.110)

the periods of the angular coordinates \( \phi \) are \( 2\pi \) and for \( \mu_\alpha \) we use the Euler angles given as
\[ \mu_0 = \cos \alpha , \quad \mu_1 = \sin \alpha \cos \beta , \quad \mu_2 = \sin \alpha \sin \beta . \]  
\hspace{1cm} (3.111)

The truncation we used in this section for the two gauge potentials is \( U(1) \times U(1) \) Cartan subalgebra of \( SO(5) \) and the two scalars are retained. The four form flux of the eleven dimensional solution is
\[ *_{11}F^{(4)} = 4 \sum_{\alpha=0,1,2} (X_\alpha^2 \mu_\alpha^2 - \Delta X_\alpha \epsilon_{(7)} + 2\Delta X_0 \epsilon_{(7)} + \frac{1}{4} \sum_{\alpha=0,1,2} X_\alpha^{-1} (\ast_7 dX_\alpha) \wedge d(\mu_\alpha^2) + \frac{1}{4} \sum_{i=1,2} X_i^{-2} d(\mu_i^2) \wedge (d\phi_i + 4A^{(i)}) \wedge \ast_7 F^{(i)} , \]  
\hspace{1cm} (3.112)

where \( F^{(i)} = dA^{(i)} \), \( \epsilon_{(7)} \) is the volume form of \( ds_7^2 \) and \( *_7, *_{11} \) are the Hodge operators for \( ds_7^2 \) and \( ds_{11}^2 \). At the IR fixed points the eleven dimensional metric is warped product of \( AdS_3 \) with an eight dimensional compact manifold which is a squashed \( S^4 \) fibred over \( \Sigma_1 \times \Sigma_2 \).

### 4 Discussion

We have shown how using c-extremization proves to be very useful in understanding two dimensional CFT’s. It is a very powerful tool to find the exact R-symmetry and central charges for two dimensional SCFT’s. We have only discussed two of the solutions however there exist much more solutions using c-extremization. There is several solutions that exist for example: Kähler 4-cycles in Calabi-Yau four folds and also for \( G_2 \) holonomy manifolds [4]. A large class of seven dimensional supergravity solutions have also been constructed in [24] where they considered \( M5 \) membranes wrapping supersymmetric cycles such as special Lagrangian cycles. The two dimensional \( \mathcal{N} = (0,2) \) SCFT’s dual theory is described by \( AdS_3 \times \Sigma_1 \times \Sigma_2 \) or \( AdS_3 \times \Sigma_1 \) they correspond to the near horizon geometry of \( AdS_7 \times S^4 \) and \( AdS_3 \times S^5 \). The holographic RG flows indeed flow from the IR fixed point to the UV fixed point indicating that one could uplift the solutions to String/M-theory.

An interesting question would be to study the c-theorem for \( \mathcal{N} = (0,2) \) theories in order to understand the full RG flow as we did not solved the BPS equations analytically. Perhaps further studying c-extremization one could have an insight on how to show that RG flows in two dimensions is a gradient flow. So far one has exact computations of R-symmetry and central charges for two dimensions using c-extremization, for three dimensions one uses F-maximization [25, 26] and for four dimensions we have a-maximization [27]. For a-maximization there exist constructions of five dimensional supergravity related to four dimensional SCFT by attractor equation where the central charge is proportional to the inverse of the superpotential [28, 29].
Recently there has been constructions of superconformal quantum mechanics described by some Witten index with fugacities (chemical potentials) [30]. They showed that using supersymmetric localization one finds that the partition function for topologically twisted index $S^2 \times S^1$ gives the entropy of black holes in described by the geometry $AdS_2 \times S^2$. In their case they showed that the complex scalars are extremized at the black hole horizon and that the prepotential is proportional to the log $Z_{S^2 \times S^1}$ which indeed is the entropy. The constructions of new $AdS/CFT$ theories using these type of techniques have been shown to be very successful, however there is still problems to be addressed such as showing that the RG flows for various SCFT’s are gradient flows and would be interesting to further explore.
Appendices

A BPS equations $AdS_5 \times S^5$ STU model

Here we present the computation for the BPS equations using the supersymmetry transformations (3.27), the variation of $\delta \psi_t$ gives the function for $f$, $\delta \psi_x$ gives the BPS equation for $g$ and $\delta \chi(1), \delta \chi(2)$ gives the functions for the scalars $\phi_1, \phi_2$. For the variation of the gravitino transformation we need the spin connection. The spin connection components are given as

$$\omega^t_r = \omega^z_r = f', \quad \omega^z_r = \omega^y_r = e^{g-f}g'. \quad (A.1)$$

Let us write down all the supersymmetry transformations for the given coordinates and from there we can simplify them using the spin connection above and choosing the appropriate twist on the background fields $A_\mu$ in this case the only non vanishing background fields are $A^t_2$:

$$0 = \delta \psi_t = \left( \partial_t + \frac{1}{4} \omega^a_{tb} \gamma_{ab} + \frac{i}{8} X_I \left( \gamma_t^{\mu \rho} - 8 \delta_t^{\mu} \gamma^\rho \right) F_{\mu \rho}^I + \frac{1}{2} X^I V_I \gamma_t \right) \epsilon, \quad (A.2)$$

We also use the field strengths given in that section, plugging in everything one can easily see the BPS equations given in (3.33). The last equation there is obtained by taking the sum of $\delta \psi_x + \delta \psi_y$ this is indeed the twisting parameter given in the ansatz for the field theory approach.

B BPS equations for $AdS_7 \times S^4$

For the BPS equations in this supergravity theory we have to work slightly harder to obtain the BPS equations, the spin connection components are given as

$$\omega^t_r = f' e^{-f}, \quad \omega^{z_1} r = \omega^{z_2} r = g^t_1 e^{-f}, \quad \omega^{y_1} r = \omega^{y_2} r = g^t_1 e^{-f}. \quad (B.1)$$

recall that we are using the truncation $U(1) \times U(1)$ such that $A^{12} = A^1$ and $A^{34} = A^2$. Let be more explicit for the $\delta \psi_t$ component and the rest follows the same type of computation,

$$0 = \delta \psi_t = \left( \partial_t + \frac{1}{4} \omega^a_{tb} \gamma_{ab} + \frac{1}{4} Q_{tij} \Gamma^{ij} + \frac{m}{20} \delta^{ij} \delta_{AB} (\Pi^{-1} A^i) \Gamma (\Pi^{-1} B^j) \right) \epsilon \quad (B.2)$$

Notice that the term $Q_{tij}$ since it does not contain any derivatives of the coordinate $t$ and also we have no background fields chosen for this field, this automatically is zero,
and the first term for the 3-form gauge potential is also zero due to the index of the gamma matrix. We also set $m = 2$. The same computation can be done for $\delta \psi_1x_1$ and $\delta \psi - x_2$ however the computation for $\delta \chi_1$ requires some additional computations. We start by writing out the variation and take care of the terms that needs some extra attention:

$$0 = \delta \chi_1 = \left( \frac{1}{2} P_{\mu j} \gamma^\mu \Gamma_j + \frac{m}{2} \left( T_{ij} - \frac{1}{5} \delta_{ij} T \right) \Gamma_j \right) + \frac{1}{16} \left( \Gamma_{k\ell} \Gamma_1 - \frac{1}{5} \Gamma_1 \Gamma_{k\ell} \right) \gamma^{\mu\nu}\Pi^k_A \Pi^\ell_B F_{\mu\nu}^{AB} + \frac{m}{20\sqrt{3}} \gamma^{\mu\nu\rho} \left( \Gamma^j_1 - 4\delta^j_1 \right) \left( \Pi^{-1}_j A S_{A\mu\nu\rho} \right) \epsilon ,$$

(B.3)

the term we have to look at more carefully now is the $P_{\mu 1 j}$. We obtain the following expression

$$P_{\mu 1 j} = (\Pi^{-1}_1)^A \left( \delta^B_A \partial_\mu + 2mA_{\mu A} B \right) \Pi_B^k \delta_k j + (\Pi^{-1}_1)_j \left( \delta^B_A \partial_\mu + 2mA_{\mu A} B \right) \Pi_B^k \delta_k 1 = (\Pi^{-1}_1)^A \partial_\mu \Pi_A^j + (\Pi^{-1}_1)_j \partial_1 A^j = 2\lambda^j_1 \epsilon^- f .$$

(B.4)

The rest of the terms follow the same procedure as the previous case however one has to pay slightly more attention to the factors. The full expressions are given in (3.87)-(3.91).

References


