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Computational dynamics – real and complex

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Abstract

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The PhD thesis considers four topics in dynamical systems and is based on one paper and three manuscripts.

In Paper I we apply methods of interval analysis in order to compute the rigorous enclosure of rotation number. The described algorithm is supplemented with a method of proving the existence of periodic points which is used to check rationality of the rotation number.

In Manuscript II we provide a numerical algorithm for computing critical points of the multiplier map for the quadratic family (i.e., points where the derivative of the multiplier with respect to the complex parameter vanishes).

Manuscript III concerns continued fractions of quadratic irrationals. We show that the generating function corresponding to the sequence of denominators of the best rational approximants of a quadratic irrational is a rational function with integer coefficients. As a corollary we can compute the Lévy constant of any quadratic irrational explicitly in terms of its partial quotients.

Finally, in Manuscript IV we develop a method for finding rigorous enclosures of all odd periodic solutions of the stationary Kuramoto-Sivashinsky equation. The problem is reduced to a bounded, finite-dimensional constraint satisfaction problem whose solution gives the desired information about the original problem. Developed approach allows us to exclude the regions in L^2, where no solution can exist.

Keywords: Continued fractions, Generating functions, Rotation numbers, Rigorous computations, Interval analysis, Interval arithmetic, Multipliers, Quadratic map, Kuramoto-Sivashinsky equation

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To my parents,
Irina and Andrey

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Anna Belova,
Rigorous enclosures of rotation numbers by interval methods.
Journal of Computational Dynamics, vol.3, n.1, (2016).

II Anna Belova, Igors Gorbovickis,
Critical points of the multiplier map for the quadratic family.
Manuscript.

III Anna Belova, Peter Hazard,
Quadratic irrationals, generating functions and Lévy constants.
Manuscript, arXiv:1710.08990

IV Arnold Neumaier, Warwick Tucker, Anna Belova,
Finding all solutions of stationary Kuramoto-Sivashinsky equations.
Manuscript.

Contents

1. Introduction

This PhD thesis considers several topics in dynamical systems and includes four articles, the present introduction and a short summary of the papers.

A large part of the thesis is based upon computations. Due to the limited precision and the way numbers are stored and manipulated in a computer, the standard numerical methods never give mathematically precise results but rather approximate values, that might be far from the true ones. Additional inaccuracies in computations are also caused by the discretization of the mathematical models. In order to produce mathematically rigorous results we turn to computational methods that provide guaranteed error bounds. One such method is *interval analysis*, developed by Moore [14], that replaces numbers with intervals and operates with set-valued arithmetic. A short description of the foundations of the interval analysis is presented later in the introduction.

Methods of *rigorous* (or *validated*) numerics, based on interval analysis, aim to compute tight enclosures of the sought values rather than approximations so that the results of the computations are proved to be mathematically correct. Rigorous numerics play a fundamental role in study of problems involving non-linearities influencing the global behaviour of the system. Examples of several non-trivial results, proved using computer-assisted methods based on interval methods, include computer-assisted proof of chaos in the Lorenz equations developed by Mischaikow and Mrozek in [13], existence of Lorenz attractor proved by Tucker in [17], hyperbolicity of homotopy hyperbolic 3-manifolds by Gabai et al. in [4].

The introduction gives a short overview of the interval analysis and some important results from the field of dynamical systems that are used further in the manuscripts.

All projects included in the thesis concern questions about dynamical systems. Paper I, II and IV use numerical and computer-assisted methods. In the Paper I we apply interval methods in order to get a rigorous estimate of the rotation numbers. Manuscript II uses traditional numerical methods to find all critical points of the multiplier map. Topics treated in Manuscript III concern continued fractions of quadratic irrationals and evaluation of corresponding Lévy constants. Finally, in Manuscript IV we develop a method for finding rigorous enclosures of all odd periodic solutions of the stationary Kuramoto-Sivashinsky equation.

The introduction is organized in the following way. First, we recall some foundations of interval analysis. Then we describe the interval Newton method. The next section covers some basics about continued fractions and quadratic irrationals. After that we give a brief review of the rotation numbers, using some facts from the previous section. Finally, we recall some ideas from the field of complex dynamical systems.

1.1 Interval analysis

1.1.1 Interval arithmetic and interval functions

A good tool that can be used in order to guarantee the mathematical rigour in the numerical work is set-valued mathematics. This approach is based on interval arithmetic [14, 18, 15] and allows to control the propagation of numerical errors throughout long iterative process.

We use the following notation for a closed interval and its endpoints

$$\mathbf{x} := [\underline{x}, \overline{x}] := [x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}], \tag{1.1}$$

and let $\mathbb{IR}$ be the set of all such intervals. We call $\mathbf{x}$ a *thin* interval when $\underline{x} = \overline{x}$.

If $\star$ denotes one of the arithmetic operators $+, -, \times, \div$, then the arithmetic of elements $\mathbf{a}, \mathbf{b}$ of the set $\mathbb{IR}$ is defined as follows

$$\mathbf{a} \star \mathbf{b} = \{a \star b : a \in \mathbf{a}, b \in \mathbf{b}\}, \tag{1.2}$$

with the exception that $\mathbf{a} \div \mathbf{b}$ is undefined if $0 \in \mathbf{b}$. Observe that, working with the closed intervals, the resulting interval can be expressed in terms of the endpoints of the arithmetic operands and will be the closed interval. Note that the definition (1.2) is equivalent to the real arithmetic when the intervals are thin.

The important property of the interval arithmetic is the *inclusion isotonicity*. Thus, if $\mathbf{a} \subseteq \mathbf{a}'$ and $\mathbf{b} \subseteq \mathbf{b}'$, then $\mathbf{a} \star \mathbf{b} \subseteq \mathbf{a}' \star \mathbf{b}'$.

One of the main reasons for using the interval arithmetic is that this approach provides a tool for enclosing the range $R(f;D)$ of a given function f over a domain D. Indeed, one can extend the real functions to interval functions, i.e. functions, that take and return intervals rather than real numbers. The extension of the rational functions to the interval versions can be easily done by substituting all occurrences of the real variable x by its interval version $\mathbf{x}$ and using the corresponding interval arithmetic operands instead of the real ones. We get then a rational interval functions $F(\mathbf{x})$, called a *natural* interval extension of f.

The key feature of the interval extension of the real function is the inclusion isotonicity, which means that if $\mathbf{a} \subseteq \mathbf{a}' \subseteq \mathbf{x}$, then $F(\mathbf{a}) \subseteq F(\mathbf{a}')$, provided that the inherited extension $F(\mathbf{x})$ is well-defined for some $\mathbf{x} \in \mathbb{IR}$. This property follows from the inclusion isotonicity of the interval arithmetic operands. Moreover, we have the following range enclosure

$$R(f;\mathbf{x}) \subseteq F(\mathbf{x}). \tag{1.3}$$

In fact, similar interval extension can be defined for any function from the set of *standard* functions, consisting of trigonometric and exponential functions. Thus, one can define similar range enclosure for any reasonable function, constructed as a finite number of compositions of arithmetic operators and standard functions.

Implementing interval arithmetic on a computer, one should take into account the rounding error. Since computers work with floating point numbers rather than with real numbers, in order to guarantee inclusion of the true result we must round the resulting intervals outward, i.e. the lower bound is rounded down and the upper bound is rounded up.

1.1.2 Interval Newton method

Now we describe a method based on the use of the interval arithmetic and interval functions.

The interval Newton method is a constructive implementation of the bounds required in Kantorovic's theorem in order to guarantee the convergence of Newton's method. As such, it can be used to prove the existence (or non-existence) of zeros of general n-dimensional maps.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function, and suppose that we have an interval extension of its derivative Df. Given an n-dimensional interval variable $\mathbf{z} \subset \mathbb{R}^n$, we can compute the interval image $Df(\mathbf{z}) \supseteq \{Df(z) : z \in \mathbf{z}\}$.

Define the interval Newton operator

$$N_f(\mathbf{z}) = \check{z} - [Df(\mathbf{z})]^{-1} f(\check{z}), \tag{1.4}$$

where $[Df(\mathbf{z})]$ is an interval matrix containing all Jacobian matrices of f of the form $Df(z)$ for $z \in \mathbf{z}$. Here $\check{z}$ is an arbitrary point from the interval vector $\mathbf{z}$ usually chosen to be the midpoint of $\mathbf{z}$. The operator (1.4) possesses the following key properties:

Theorem 1.1.1 *[14] Given the assumptions above,*

1) *If $N_f(\mathbf{z}) \subset \mathbf{z}$ then there exists exactly one point $z^\star \in \mathbf{z}$ such that $f(z^\star) = 0$.*
2) *If $N_f(\mathbf{z}) \cap \mathbf{z} = \emptyset$ then there are no zeros of f in $\mathbf{z}$.*

Let $\mathbf{z}_0 = \mathbf{z}$ be the initial enclosure of a possible zero of f, and define the sequence of intervals $\mathbf{z}_{k+1} = N_f(\mathbf{z}_k) \cap \mathbf{z}_k$, $k = 0, 1, 2, \ldots$. If a true zero $z^\star$ is contained in $\mathbf{z}_0$, and if the interval Newton operator is well-defined on this domain, then the operator remains well-defined for all iterations, we have $z^\star \in \mathbf{z}_k$, and the intervals $\mathbf{z}_k$ form a nested sequence converging to the zero of f.

1.2 Continued fractions and quadratic irrationals

In this section we recall some definitions and notation from the number theory, that will be used in Papers I and III.

Given an arbitrary number $\theta \in \mathbb{R}$, the simple continued fraction expansion of θ is denoted by

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots], \tag{1.5}$$

where a_0 is an integer and $a_1, a_2, \dots$ are positive integers called the *partial quotients* of the continued fraction.

We use continued fraction expansions both in Papers I and III. In Paper I we construct rotation number of a circle map via its continued fraction expansion. By definition rotation number lies in interval $[0,1)$. More details on rotation numbers follow in the next section. In Manuscript III we consider quadratic irrationals. For simplicity we also restrict ourselves to the interval $[0,1)$. Thus, to shorten notation for the rest of this chapter we assume that $\theta \in [0,1)$ and therefore the first coefficient of its continued fraction expansion $a_0 = 0$ and can be omitted from the notation.

Define the *nth convergent* of θ to be a rational number

$$\theta_n := [a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}, \tag{1.6}$$

where p_n and q_n are positive integers having no common factors. The following property is satisfied for all $n \in \mathbb{N}$:

$$\left| \theta - \frac{p_n}{q_n} \right| \le \inf_{\frac{p}{q} \in \mathbb{Q}: q \le q_n} \left| \theta - \frac{p}{q} \right|. \tag{1.7}$$

For this reason p_n/q_n is also called the *nth best rational approximant* of θ.

Observe that the numerators and denominators of the convergents satisfy the following recursion relation

$$p_n = a_n p_{n-1} + p_{n-2}; \qquad p_0 = 0, \qquad p_1 = 1, \tag{1.8}$$

$$q_n = a_n q_{n-1} + q_{n-2}; \qquad q_0 = 1, \qquad q_1 = a_1. \tag{1.9}$$

Finally, in order to state later the results of Manuscript III, we introduce the following definitions. We call $\theta \in [0,1] \backslash \mathbb{Q}$ a *quadratic irrational* if θ is an algebraic number with minimal polynomial of (strict) degree two. A theorem of Lagrange [8, p. 56] implies that θ has a pre-periodic simple continued fraction expansion, i.e.

$$\theta = [a_1, a_2, \dots, a_m, \overline{a_{m+1}, \dots, a_{m+\ell}}], \tag{1.10}$$

where the minimal such ℓ is called the *period* and any such m is called a *preperiod* of the simple continued fraction expansion.

1.3 Rotation numbers

Here we give a brief overview of the rotation number.

Suppose $f : S^1 \to S^1$ is an orientation-preserving homeomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The rotation number gives an idea of the average amount points get rotated along S^1 when iterated by the map f many times.

Let $\pi : \mathbb{R} \to S^1$ be a natural projection of the real line to the circle defined by $\pi(x) := x(\text{mod } 1)$. Then the homeomorphism $F : \mathbb{R} \to \mathbb{R}$ of the real line is called a *lift* of f if it satisfies $f \circ \pi(x) = \pi \circ F(x)$. Define

$$\rho(F) := \lim_{n\to\infty} \frac{F^n(x) - x}{n}. \tag{1.11}$$

By the classical result of Poincaré [16] this limit exists, is well-defined up to an integer and independent of the choice of x.

The *rotation number* ρ of f is the fractional part of the limit $\rho(F)$ for any lift F of f, thus

$$\rho := \rho(f) = \pi(\rho(F)). \tag{1.12}$$

Note that the rotation number $\rho(f)$ is invariant under *topological conjugacy*. Recall that two maps $g_1 : A \to A$ and $g_2 : B \to B$ are topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \circ g_1 = g_2 \circ h$:

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A \\ h\downarrow & & \downarrow h \\ B & \xrightarrow{g_2} & B \end{array}.$$

The homeomorphism h is called a *topological conjugacy*. If the map h is not a homeomorphism but a surjection, then g_1 and g_2 are said to be *topologically semi-conjugate*.

The circle map f has a periodic point if and only if its rotation number is rational. Moreover, if $\rho(f) = p/q$, where p and q are co-prime integers, then f has a periodic point x of prime period q, i.e. q is the least positive integer such that $f^q(x) = x$.

On the other hand, if the rotation number $\rho(f)$ is irrational, then there are two possibilities. Either all orbits are dense in S^1 and f is topologically conjugate to a rigid rotation $R_\rho := x + \rho(\text{mod } 1)$. Otherwise there exists a Cantor set $\mathscr{C} \subset S^1$ which is invariant under the map f and both forward and backward orbits of all points converge to $\mathscr{C}$, and hence f is semi-conjugate to a rigid rotation R_ρ.

In [1] the rotation number ρ is constructed via the continued fraction expansion obtained from the topological behaviour of the circle map, thus the rotation number can be represented in the form

$$\rho = [a_1, a_2, \ldots].$$

The continued fraction coefficients a_i depend on increasingly high iterates of the circle map. Using this approach, it is possible to construct a sequence of intervals of rapidly decreasing widths, all containing the rotation number ρ.

In general the convergence is quadratic, and when the rotation number satisfies a Diophantine condition, the convergence can be made cubic, see [1]. Recall that real number ρ is Diophantine if there exist positive B and γ such that $|\rho - p/q| \geq b/q^{2+\gamma}$ for all rational numbers p/q. Note that the set of rotation numbers satisfying a Diophantine condition has full measure in $[0,1]$. On the other hand, the complement lies dense in $[0,1]$. In what follows, we will not assume any Diophantine properties of the rotation number.

Based on the algorithm for finding the coefficients a_i of the continued fraction expansion of the rotation number ρ described in details in [1] we recall the following theorem.

Theorem 1.3.1 *[1] Let N_i be the number of iterates needed to compute q_i. Then the following holds:*

(a) If ρ is irrational, then $\frac{p_i}{q_i}$ converges to ρ as $i \to \infty$. If ρ is rational, then the process terminates ($a_{i+1} = \infty$) and the last estimate $\frac{p_i}{q_i} = \rho$.

(b) If p_i and q_i are found, then ρ is contained in a closed interval A with end-points $\frac{p_i}{q_i}$ and $\frac{(a+1)p_i+p_{i-1}}{(a+1)q_i+q_{i-1}}$, where the integer a is a lower bound of a_{i+1}.

(c) $|A| \leq 4/N_i^2$. For any $N_i \leq N < N_{i+1}$, $|A| \leq 2/(q_i N_i)$. If $\{a_i\}_{i \geq 1}$ satisfies the Diophantine condition $a_{i+1} < Bq_i^{\gamma}$ for some B and $\gamma > 0$, then $|A| \leq 2(B+2)^{1/(1+\gamma)(1/N)^{1+1/(1+\gamma)}}$.

In Paper I we will use a version of case (b) to derive explicit, computable bounds on rotation number ρ for a given f.

1.4 Multiplier map for the quadratic family

In this section we recall some ideas from the field of complex dynamics used in Manuscript II.

Let $p_c : \mathbb{C} \to \mathbb{C}$ be the quadratic family $p_c(z) = z^2 + c$ with $c \in \mathbb{C}$.

We consider the space of quadratic polynomials with a marked periodic point of a given period n. In other words, given n, let the *period n curve* $\mathrm{Per}_n \subset \mathbb{C} \times \mathbb{C}$ be the closure of the locus of points (c,z) such that z is a periodic point of p_c of period n. Observe that each pair $(c,z) \in \mathrm{Per}_n$ determines a periodic orbit. Let $\mathbb{Z}_n$ be cyclic group of order n, acting on Per_n by cyclicly permuting points of the same periodic orbits for each fixed value of c. Thus, the factor space $\mathrm{Per}_n/\mathbb{Z}_n$ consists of pairs $(c,\mathcal{O})$, where $\mathcal{O}$ denotes a periodic orbit of p_c. The space $\mathrm{Per}_n/\mathbb{Z}_n$ has a structure of a smooth algebraic curve. For details see [11].

The multiplier map is defined on this space and maps a pair $(c,\mathcal{O})$ to the multiplier of the periodic orbit $\mathcal{O}$, i.e. to the derivative of the n-th iteration of the map at any point from the orbit $\mathcal{O}$. Particularly, let $\tilde{\lambda}_n : \mathrm{Per}_n \to \mathbb{C}$ be the map defined by

$$(c,z) \mapsto \frac{\partial p_c^n}{\partial z}(z) = 2^n z_1 \cdots z_n, \tag{1.13}$$

where $z_1, \ldots, z_n$ denote the points of the periodic orbit of period n. Observe that for all regular points of the projection $(c,z) \to c$, the value $\tilde{\lambda}_n$ is the multiplier of the periodic point z. Furthermore, if z_1 and z_2 belong to the same periodic orbit, then the values of the multiplier at these points are the same. Hence the map $\tilde{\lambda}_n$ projects to the well-defined map $\lambda_n : \mathrm{Per}_n/\mathbb{Z}_n \to \mathbb{C}$, called a *multiplier map*, that assigns to each pair $(c, \mathcal{O})$ the multiplier of the periodic orbit $\mathcal{O}$. Both λ_n and $\tilde{\lambda}_n$ are proper algebraic maps (c.f. [11]).

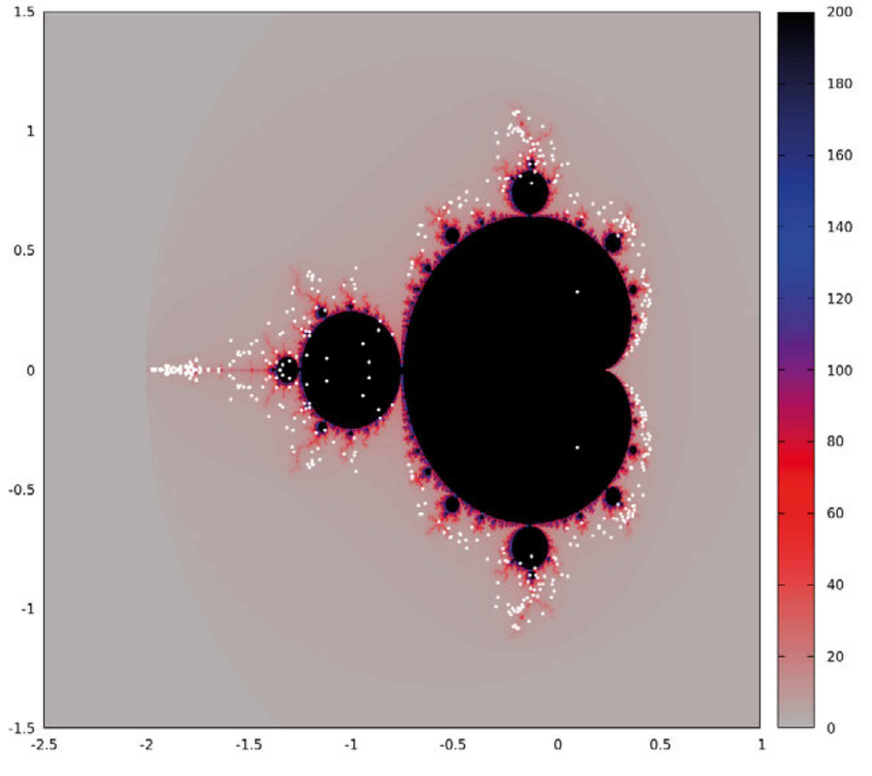

Figure 1.1. Mandelbrot set and critical points of the multiplier map for period $n = 10$.

Recall that the set of complex numbers c for which the orbit of 0 is bounded under iterates of p_c is known as the *Mandelbrot set M*. A connected component of the interior of the Mandelbrot set is hyperbolic if the corresponding

p_c has an attractive periodic point (i.e. the absolute value of the multiplier is less than 1). An inverse branch of the multiplier map is a Riemann mapping from the unit disk to the corresponding hyperbolic component [12]. Recall that a Riemann mapping of a simply connected domain is a conformal diffeomorphism of the unit disk onto that domain. It is important to know how far this Riemann mapping can be analytically extended outside of the unit disk: if the Riemann map has a large extension, then its behaviour on the unit disk is nicely controllable, which, in turn, provides control of the shape of the hyperbolic components of the Mandelbrot set - an important problem in complex dynamics. Critical points of the multiplier map are the only obstacles for such analytic extension. The aim of Manuscript II is to find all critical points of the multiplier map for different values of n. Figure 1.4 shows the critical points of the multiplier map for the period $n = 10$.

2. Results of the Thesis

2.1 On Paper I

In the first paper we realize the Bruin's [1] method for estimation of the rotation number ρ of a given circle map f. We derive explicit, computable bounds on ρ for given f and use rigorous computations. All computations are based on set-valued mathematics. This approach allows us to guarantee the correctness of the n first decimals of the computed estimate of the rotation number ρ.

The practical use of the algorithms for estimating the rotation number ρ requires computations of long orbits. Typically, using interval arithmetic, one observes a quick deterioration of bounds due to rounding errors. In order to overcome this problem, we use a combination of the interval Newton method and a multiple shooting technique. This allows us to compute longer accurate trajectories without increasing the precision of the numerical computations. Therefore we can perform the computations with a reasonable speed.

Moreover, we add a sub-algorithm based on the interval Newton method that proves the existence (and uniqueness) of moderately long periodic orbits. This makes it possible to handle the case of rational rotation numbers.

We illustrate our method on two examples: the Arnold family and the delayed logistic map. We base our code on the CAPD [3] interval library.

2.2 On Manuscript II

The goal of the second manuscript is to study (compute) critical points of the multiplier map λ_n, defined in section 1.4.

In [10] the multipliers of the fixed points were used to parameterize the moduli space of degree 2 rational maps. Using this parameterization it was proved that this moduli space is isomorphic to $\mathbb{C}^2$. In an attempt to generalize this approach, it was observed in [5] that the multipliers of any $m-1$ distinct periodic orbits provide a local parameterization of the moduli space of degree m polynomials in a neighborhood of its generic point. It is then a natural question to describe the set of polynomials at which this local parameterization fails, that is, to describe the set of all critical points of the multiplier map.

In order to achieve this goal, we solve numerically the following system of algebraic equations

$$\left\{\begin{array}{lcl} p_c^{\circ n}(z) - z & = & 0 \\ z' - \frac{\partial p_c^{\circ n}}{\partial c}(z)\left(1 - \frac{\partial p_c^{\circ n}}{\partial z}(z)\right)^{-1} & = & 0 \\ \frac{d\lambda_n}{dc} & = & 0, \end{array}\right. \tag{2.1}$$

with three unknowns c, z, z', where $z' := \frac{dz}{dc}$. Any critical point of the multiplier map corresponds to a solution of the above system.

Furthermore, with help of the Riemann-Hurwitz formula and the results derived by Milnor in [11], we estimate the upper bound for the number of critical points of the multiplier map. This allows us to make sure that all solutions are found. The number N_{λ_n} of the critical points of the multiplier map λ_n satisfies the following inequality

$$N_{\lambda_n} \leq \nu(n) - \frac{\nu(n)}{n} - \frac{1}{2} \sum_{\substack{\forall r,p \text{ s.t.} \\ n=rp \\ p<n}} \nu(p) \cdot \varphi(r). \tag{2.2}$$

Here $\nu(n)$ denotes the number of periodic points of p_c of period n for a generic value of c, and $\varphi(r)$ is the Euler function that counts the positive integers up to r that are relatively prime to r.

In this second manuscript we describe the details of the numerical algorithm used to solve the system (2.1), and discuss the results of the numerical experiments. The algorithm is implemented in C++, and run for periods $n = 3, \dots, 10$.

2.3 On Manuscript III

Hong observed in [6] that the generating function of the Fibonacci sequence $\Phi(z) = \sum \phi_i z^i = z + z^2 + 2z^3 + 3z^4 + \dots$ is a rational function that attains an integer value if $z = \phi_i/(\phi_i + 1)$ for some even integer i. Bulawa and Lee in [2] showed that in the Fibonacci case these were the only rational points with integer values. The natural question is for which other continued fraction expansions the generating functions are rational. In the third manuscript we show that the generating function corresponding to the sequence of denominators of the best rational approximants of a quadratic irrational is a rational function with integer coefficients.

The main result of this work is the following theorem:

Theorem 2.3.1 *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be a quadratic irrational. For each n, let p_n/q_n denote the n-th best rational approximant to θ. Then the generating functions*

$$F(z) = \sum_{n \geq 0} p_n z^n, \qquad G(z) = \sum_{n \geq 0} q_n z^n, \tag{2.3}$$

are both rational with integer coefficients.

In fact, we can compute the generating functions F and G explicitly in terms of the partial convergents of θ. Namely, if the continued fraction expansion of θ is eventually periodic of period ℓ and we let m denote the second smallest pre-period, then

$$F(z) = \frac{\left(\sum_{1\le j\le \ell} p_{m+j} z^{j-1} + (-1)^{\ell-1} \sum_{\ell+1\le j\le 2\ell} p_{m+j-2\ell} z^{j-1}\right) z^{m+1}}{1-(-1)^m\left(q_m p_{m+\ell-1} - p_m q_{m+\ell-1} - q_{m-1} p_{m+\ell} + p_{m-1} q_{m+\ell}\right) z^{\ell} + (-1)^{\ell} z^{2\ell}} \tag{2.4}$$

and

$$G(z) = \frac{\left(\sum_{1\le j\le \ell} q_{m+j} z^{j-1} + (-1)^{\ell-1} \sum_{\ell+1\le j\le 2\ell} q_{m+j-2\ell} z^{j-1}\right) z^{m+1}}{1-(-1)^m\left(q_m p_{m+\ell-1} - p_m q_{m+\ell-1} - q_{m-1} p_{m+\ell} + p_{m-1} q_{m+\ell}\right) z^{\ell} + (-1)^{\ell} z^{2\ell}} \tag{2.5}$$

As a corollary to this analysis, we can compute the *Lévy constant* of a quadratic irrational. Lévy constants give a way of bounding the error between a quadratic irrational and its n-th best rational approximant. Recall that, given a real number θ with nth best rational approximant p_n/q_n for each n, the Lévy constant of θ, when it exists, is given by the following expression

$$\beta(\theta) = \lim_{n\to\infty} \frac{1}{n} \log q_n. \tag{2.6}$$

Lévy showed [9], following earlier work by Khintchine [8], that

$$\beta(\theta) = \frac{\pi^2}{12\log 2}, \qquad \text{for Lebesgue-almost every } \theta. \tag{2.7}$$

It was shown by Jager and Liardet [7] that for every quadratic irrational, the Lévy constant exists. As an immediate corollary to Theorem (2.3.1) above we get a new proof of the following result, which was implicitly contained in [7].

Theorem 2.3.2 *Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$ be a quadratic irrational. Let ℓ denote the (eventual) period of the simple continued fraction expansion of θ. Let M_θ denote the element of* $\mathrm{PSL}(2,\mathbb{Z})$ *corresponding to the simple continued fraction expansion of θ. Then*

$$\beta(\theta) = \frac{1}{\ell} \log \mathrm{rad}(M_\theta), \tag{2.8}$$

where by $\mathrm{rad}(M_\theta)$ *we denote the spectral radius.*

The linear transformation M_θ from the above theorem is defined explicitly for the quadratic irrationals as $M_\theta := M_0 M_1 M_0^{-1}$, where M_1 corresponds to the periodic part of the continued fraction expansion of θ and has the following form:

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & a_{m+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_{m+2} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{m+\ell} \end{bmatrix}, \tag{2.9}$$

and M_0 corresponds to the pre-periodic part and is defined as follows:

$$M_0 = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_m \end{bmatrix}. \tag{2.10}$$

2.4 On Manuscript IV

In the forth manuscript we aim to accurate cover all odd, periodic solutions of the stationary Kuramoto-Sivashinsky (KS) equation in one dimension

$$au'''' + bu'' + cuu' = 0, \quad u \in L^2([0,2\pi]), \quad u(-x) = -u(x), \tag{2.11}$$

where $a,b,c > 0$ are fixed. We are focusing on excluding regions in L^2, where no solutions can exist.

The problem is reduced to a bounded, finite-dimensional constraint satisfaction problem (CSP) whose solution gives the desired information about the original problem.

We first derive an a priori bound for a Sobolev-like norm formulated in the following theorem

Theorem 2.4.1 *Let α,β,γ be real numbers, $\beta \geq 0$. Let L be a densely defined symmetric operator on $\mathbb{X} := C_0^1([0,\pi])$, and let $u \in \mathbb{X}$ be a solution of*

$$Lu = (u^2)', \qquad u(0) = u(\pi) = 0. \tag{2.12}$$

Then

$$(u, Lu) = 0, \tag{2.13}$$

and for every $w \in C^1([0,\pi])$ such that $f := w'$ satisfies

$$f(s) \geq \gamma - \frac{2\beta}{s(\pi - s)} \quad \text{for } 0 < s < \pi, \tag{2.14}$$

it follows that

$$\alpha(u, Lu) - \beta\|u'\|^2 + \gamma\|u\|^2 + (Lw, u) \leq 0. \tag{2.15}$$

Using the known transformation (e.g. [19]) of the stationary KS equation with periodic boundary condition we derive the equivalent quadratic equation $Ax = N(x,x)$ with $x \in \ell^2(\mathbb{N})$ in infinite dimensions, where the components x_k

of x are the coeffecients of the Fourier series $u(s) = \sum_{k=1}^{\infty} x_k \sin ks$. Here A is a densely defined operator with components of the form

$$A_k := (a/c)k^3 - (b/c)k,$$

and the components of $N(x,x)$ are symmetric bilinear forms given by

$$N_k(x,y) := \sum_{j=1}^{\infty} (x_j y_{k+j} + y_j x_{k+j}) - \sum_{j=1}^{k-1} x_j y_{k-j}.$$

Applying a priori bounds and the above trasformation we construct a positive definite, densely defined operator B with components $B_k := \alpha' k^4 - \beta' k^2 + \gamma$ with $\alpha', \beta', \gamma > 0$ such that every solution satisfies the constraint

$$\|x - x^0\|_B \leq \bar{r} \tag{2.16}$$

in the norm $\|x\|_B := \left(\sum_{k=1}^{\infty} B_k x_k^2 \right)^{1/2}$. The components of x^0 and $\bar{r}$ are defined explicitly for each k. Thus, the problem is transformed to the following fixed point problem

$$x = B^{-1}(Bx - Ax + N(x,x)) \tag{2.17}$$

in the ball (2.16).

Finally, in order to solve (2.17) we construct a sequence of CSPs of increasing dimension and hence increasing resolution. Thus, for each $n = 1,2,3,\ldots$ we derive a finite system (C_n) of inequalities relating the vector $z := x_{1:n}$ of leading coefficients and a residual error measure r. We derive an explicit error bound for the remaning components of any solution x.

One then first solves the lower-dimensional CSPs and refines the covering found there by moving to higher dimensions. Covering methods are able to rigorously prove the nonexistence of a solution and therefore we can exclude the regions where solutions cannot exists.

3. Sammanfattning på svenska

Denna avhandling bygger på fyra vetenskapliga artiklar som behandlar ett flertal ämnen inom dynamiska system och tillämpningen av numeriska och datorassisterade metoder.

För att underlätta förståelsen av artiklarna, inleds avhandlingen med en introduktion till området där grundläggande intervallanalys och viktiga idéer från både rella och komplexa dynamiska system beskrivs. Introduktionen åtföljs av en kort sammanfattning av forskningsresultaten.

I Artikel I tillämpar vi intervallanalys för att rigoröst kunna beräkna inneslutningar av rotationsnummer. Bruins metod som teoretiskt beskrivs i [1] implementeras i kod i form av ett rigoröst beräkningsprogram. Givet en cirkelavbildning erhåller vi explicita och beräkningsbara undre och övre gränser för dess rotationsnummer. Alla beräkningar baseras på mängdvärd matematik. Detta tillvägagångssätt garanterar att vi erhåller n korrekta värdesiffror av det sökta av rotationsnummret. När vi baserar beräkningarna på intervallaritmetik observerar vi snabbt en försämring av gränserna, vilket kan härledas till små, lokala avrundningfel. För att råda bot på detta implementerar vi en algoritm som baseras på en kombination av multipla randvärdesproblem och en mängdvärd Newtonmetod.

Slutmålet för Artikel II är att beräkna kritiska punkter av en multiplikatoravbildning för den komplexa kvadratiska familjen. För att uppnå detta mål omformulerar vi problemet som ett system av algebraiska ekvationer. Dessa löses numeriskt med Newtons metod. Dessutom tar vi fram en uppskattning för en övre gräns av antalet kritiska punkter. På så sätt kan vi kontrollera huruvida alla lösningar hittats. Även här presenteras resultat av numeriska experiment för olika perioder.

I Artikel III visar vi att den genererande funktionen, som motsvarar följden av nämnare av bästa rationella approximationen till kvadratiska irrationella tal, är en rationell funktion med heltalskoefficienter. Man kan dessutom effektivt beräkna den genererande funktionen genom att använda sig av konvergenter av kedjebråk. En konsekvens av detta resultat är att vi kan beräkna Lévys konstant, för givna kvadratiska irritionella tal, som en logaritm av spektralradien för ett element av $\mathrm{PSL}(2,\mathbb{Z})$ som motsvarar kedjebråket av talet.

I Artikel IV undersöker vi en ny metod för att (med validerad numerik) finna ett begränsat område i ett lämpligt funktionsrum som inkluderar alla udda periodiska lösningar av den stationära och endimensionella Kuramoto-Sivashinsky-ekvationen. Problemet reduceras till ett begränsat ändligtdimensionellt bivillkorsproblem tillsammans med feluppskattningar för de associerade diskretiseringsfelen. Tillsammans kan vi ta fram information om var lösningarna til det ursprungliga (oändligtdimensionella) problemet måste finnas. Den nya tekniken vi utvecklat går att använda på randvärdesproblem för kvasilinära elliptiska partiella differentialekvationer då á priori-estimat på lösningarna kan fastställas och egenfunktionerna för principaldelen av operatorn är känd.

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