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The infinity-Laplacian and its properties

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1 Introduction

The aim is to show that from the $p-$Laplace equation it is possible to obtain the $\infty-$Laplace equation by letting $p \to \infty$. G. Aronsson deduced the $\infty-$Laplace equation in 1967. There are several properties and applications that can be used when $p \to \infty$, we will concentrate on the stochastic game tug-of-war.

Another example, that is important but will not be focused on here is image processing. For image processing the $\infty-$Laplace equation is better to use than the ordinary Laplace equation, the most advantageous property is that at the isolated boundary points the boundary values can be measurable. Since the boundary can be arbitrary irregular and the non-existence of any flow along the contours i.e. in the directions of the stream lines the $\infty-$Laplacian is just the second derivative. Hence, the equation can be written as

$$\frac{\Delta_\infty u}{|\nabla u|^2} = \frac{\partial^2 u}{\partial \nu^2},$$

where $\nu = \frac{\nabla u}{|\nabla u|}$.

The modern theory of viscosity solutions is constructed by the $\infty-$Laplace equation and partial differential equations. It is the modern theory of viscosity solutions that enables the definition of solutions, since these circumvent the problem that the second derivative does not always exist for the $\infty-$Laplace equation.

One example of a viscosity solution when $\Delta_\infty = 0$ (in $\mathbb{R}^2$) is the function, $u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$. An example of a viscosity subsolution in $\mathbb{R}^n$ is the function $|x - x_0|$. If $\int_{\mathbb{R}^n} |y|\rho(y)\, dy < \infty$ then the function defined by the superposition is

$$V(x) = \int_{\mathbb{R}^n} |x - y|\rho(y)\, dy,$$

where $\rho \geq 0$ is a viscosity subsolution.[1]

The connection between probability calculations and the $\infty-$Laplace equation was discovered by Y.Peres, O.Schramm, S.Sheffield and D.Wilson in 2009 and thereafter the stochastic zero-sum game Tug-of-War. A brief summary of the game follows:

A token is placed at some point x_0 in a domain Ω and moves randomly within the ball $B_\varepsilon(x_n)$. The first player moves the token to a new position x_{n+1}, with probability $\frac{\alpha}{2}$, then the other player moves the token to a different position, with the same probability $\frac{\alpha}{2}$. The token moves with probability $\beta = 1-\alpha$, in the interval I $\in$ [0,1] where β is the probability that the token moves according to a Brownian motion, which does not affect the game. The game ends when the token leaves the domain Ω and the second player pays the first player the amount equal to a value of a given boundary pay-off function. [2]

2 The p-Laplace Equation

In order to show existence and uniqueness for the Dirichlet boundary value problem, the Sobolev space needs to be introduced.

Definition 1. ***(Sobolev space)***
The Sobolev space is denoted by $W^{1,p}(\Omega)$ and consists of functions u, such that $u, \nabla u \in L^p(\Omega)$,

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}). \tag{1}$$

Provided with the following norm,

$$\| u \|_{W^{1,p}(\Omega)} = \| u \|_{L^p(\Omega)} + \| \nabla u \|_{L^p(\Omega)} \tag{2}$$

which is a Banach space [1]*.* [1]

Hence, all Lipschitz continuous functions defined in the domain Ω belongs to the space $W^{1,\infty}(\Omega)$ and $W_0^{1,p}(\Omega)$ is the closure $C_0^{\infty}(\overline{\Omega})$ with respect to the Sobolev norm.

It is necessary for p to have a large value and if $p > n$, then the boundary values are considered in the classical sense and the Sobolev space only consists of continuous functions ($n =$ the number of coordinates in $\mathbb{R}^n$). Consequently, when $p > n$ all domains Ω are regular for the Dirichlet problem.

A variant of Morrey's inequality from (Lindqvist 2015, 11) with the important property that the constant remains bounded when p is large follows,

Lemma 2. *Let $p > n$ and assume that Ω is an arbitrary bounded domain in $\mathbb{R}^n$. If $v \in W_0^{1,p}(\Omega)$, then*

$$|v(x) - v(y)| \leq \frac{2pn}{p-n} |x-y|^{1-\frac{n}{p}} \| \nabla v \|_{L^p(\Omega)} \tag{3}$$

for a.e. $x, y \in \Omega$. One can redefine v in a set of measure zero and extend it to the boundary so that $v \in C^{1-\frac{n}{p}}(\overline{\Omega})$ and $v|_{\partial\Omega} = 0$.

The inequality is valid for many domains even if the boundary values are not required to be zero. For instance, the inequality is valid for $v \in W^{1,p}(\Omega)$, if the domain Ω is a cube Q.

[1] "A *Banach space* is a normed vector space which is complete under the metric associated with the norm." [3]

Consider a smooth curve $v \in C^1(Q) \cap W^{1,p}(Q)$, when $p \to \infty$ the behavior of the constant is crucial, this will be shown by first taking the integral of the following,

$$\begin{aligned} v(x) - v(y) &= \int_0^1 \frac{d}{dx} v(x + t(y-x))\, dt \\ &= \int_0^1 \langle y - x, \nabla v(x + t(y-x)) \rangle \, dt \end{aligned}$$

with respect to y over Q, thus

$$\begin{aligned} |v(x) - v_Q| &= \left| \fint_Q \int_0^1 \langle y - x, \nabla v(x + t(y-x)) \rangle \, dt\, dy \right| \\ &\leq diam(Q) \fint_Q \int_0^1 |\nabla v(x + t(y-x))| \, dt\, dy \\ &\leq diam(Q) \int_0^1 \left(\fint_Q |\nabla v(x + t(y-x))|^p \, dy \right)^{\frac{1}{p}} dt \end{aligned}$$

Now, let $\xi = x + t(y - x)$, $d\xi = t^n dy$, then we obtain the inner integral as,

$$\begin{aligned} \fint_Q |\nabla v(x + t(y-x))|^p \, dy =& \frac{1}{t^n} \fint_{Q_t} |\nabla v(\xi)|^p \, d\xi \\ \leq& \frac{1}{t^n} \fint_Q |\nabla v(\xi)|^p \, d\xi, \end{aligned}$$

where $Q_t \subset Q$.

Then, recall that $p > n$,

$$\begin{aligned} |v(x) - v_Q| &\leq \frac{diam(Q)}{|Q|^{\frac{1}{p}}} \int_0^1 t^{-\frac{n}{p}} \parallel \nabla v \parallel_{L^p(Q)} \, dt \\ &= \frac{1}{1 - \frac{n}{p}} \frac{diam(Q)}{|Q|^{\frac{1}{p}}} \parallel \nabla v \parallel_{L^p(Q)} . \end{aligned}$$

By the triangle inequality we obtain,

$$\begin{aligned} |v(x) - v(y)| &\leq |v(x) - v_Q| + |v(y) - v_Q| \\ &\leq \frac{2p}{p-n} \frac{diam(Q)}{|Q|^{\frac{1}{p}}} \parallel \nabla v \parallel_{L^p(Q)} . \end{aligned}$$

To sum up, one can pick a help cube $Q' \subset Q$ such that $|x-y| \leq diam(Q')$.

Theorem 3. *This theorem follows from (Lindqvist 2015, 13-14)*
Pick p>n and consider an arbitrary bounded domain $\Omega \in \mathbb{R}^n$. Assume that $g \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is given. Then there exists a unique function $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ with boundary values g which minimizes the variational integral,

$$I(v) = \int_{\Omega} |\nabla v|^p \, dx \tag{4}$$

among all similar functions.

The minimizer is a weak solution to the $p-$Laplace equation, i.e.

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle \, dx = 0, \tag{5}$$

when $\eta \in C_0^{\infty}(\Omega)$.

On the other hand, a weak solution in $C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is always a minimizer (among functions with its own boundary values).

Proof. The following is based on the proof in (Lindqvist 2015, 14-16).
From this,

$$\left| \frac{\nabla u_1 + \nabla u_2}{2} \right|^p < \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2},$$

when $\nabla u_1 \neq \nabla u_2$, upon integration, the uniqueness of the minimizer follows.

Namely, $\frac{u_1+u_2}{2}$ would be admissible and

$$I(u_1) \leq I\left(\frac{u_1+u_2}{2}\right) < \frac{I(u_1)+I(u_2)}{2} = I(u_1),$$

if u_1 and u_2 were two minimizers, unless $\nabla u_1 = \nabla u_2$ almost everywhere (a.e.). Let $u_1 = u_2$, to avoid contradiction.

From the minimizing property one can deduce the Euler-Lagrange Equation,

$$I(u) \leq I(u + \varepsilon\eta).$$

Note that, the function $v(x) = u(x) + \varepsilon\eta(x)$ is admissible. Then, by infinitesimal calculus we have the following,

$$\frac{d}{d\varepsilon} I(u + \varepsilon\eta) = 0,$$

for $\varepsilon = 0$. Therefore the first variation vanishes, i.e. equation (5) holds.

By using the Dirichlet Method in the calculus of variation, due to Lebesgue, one can show that the minimizer exists,

$$I_0 = \inf_v \int_\Omega |\nabla v|^p \, dx,$$

the infimum is taken over the class of admissible functions.
Consider a minimizing sequence of admissible functions v_j,

$$\lim_{j \to \infty} I(v_j) = I_0,$$

where $0 \leq I_0 \leq I(g) < \infty$.
Suppose that $I(v_j) < I_0 + 1$, $j = 1, 2, ...$ and that

$$\min g \leq v_j(x) \leq \max g,$$

in Ω, considering the functions to be cut at the constant heights $\min g$ and $\max g$. Observe that the integral decreases by this procedure.
The Sobolev norms $\parallel v_j \parallel_{W^{1,p}(\Omega)}$ are uniformly bounded, this means that the conventional way to use the Sobolev inequality is not to cut the functions, instead using the Sobolev inequality,

$$\parallel v_j - g \parallel_{L^p(\Omega)} \leq \; C \parallel \nabla(v_j - g) \parallel_{L^p(\Omega)}$$

bound the norms uniformly,

$$\begin{aligned}
\parallel v_j \parallel_{L^p(\Omega)} &\leq \; \parallel v_j - g \parallel_{L^p(\Omega)} + \parallel g \parallel_{L^p(\Omega)} \\
&\leq C \parallel \nabla(v_j - g) \parallel_{L^p(\Omega)} + \parallel g \parallel_{L^p(\Omega)} \\
&\leq C[\parallel \nabla v_j \parallel_{L^p(\Omega)} + \parallel \nabla g \parallel_{L^p(\Omega)}] + \parallel g \parallel_{L^p(\Omega)} \\
&\leq C[(I_0 + 1)^p + \parallel \nabla g \parallel_{L^p(\Omega)}] + \parallel g \parallel_{L^p(\Omega)} \\
&\equiv M < \infty,
\end{aligned}$$

where $j = 1, 2, 3, ...$

From weak compactness we can conclude that, there exists a function $u \in W^{1,p}(\Omega)$ and a subsequence such that,

$$\nabla v_{j_k} \rightharpoonup \nabla u \qquad \text{weakly in} \quad L^p(\Omega).$$

From lemma (2) we distinguished that $u \in C(\overline{\Omega})$, since $p > n$. Thus, $u = g$ on $\partial\Omega$ and the continuity is established!
From weak lower semi-continuity of convex integrals we obtain the following,

$$I(u) \leq \lim_{k \to \infty} \inf I(v_{j_k}) = I_0.$$

$I(u) \geq I_0$, since u is admissible.
Hence, we have that u is a minimizer and the existence is established!

It is time to show that, the weak solutions of the Euler-Lagrange equation are minimizers. Note that, there are variational integrals for which this is not the true. Take the integral of the following inequality[2],

$$|\nabla(u+\eta)|^p \ \geq \ |\nabla u|^p + p\langle |\nabla u|^{p-2}\nabla u, \nabla \eta\rangle,$$

we get

$$\int_\Omega |\nabla(u+\eta)|^p \, dx \ \geq \ \int_\Omega |\nabla u|^p \, dx + 0 \ = \ \int_\Omega |\nabla u|^p \, dx.$$

Thus u is a minimizer. So, the weak solutions of the Euler-Lagrange equation are minimizers! □

Remark : There exist a unique $p-$harmonic function $u \in C(\overline{\Omega})$ with boundary values g, such that the given boundary values g are merely continuous and $g \in C(\overline{\Omega})$ but $g \notin W^{1,p}(\Omega)$. It may happen that $\int_\Omega |\nabla u|^p dx = \infty$. (For the ordinary Dirichlet integral (p = 2), Hadamard gave a counter example).

2.1 Letting $p \to \infty$

To obtain the perfect passage to the limit in the $p-$Laplace equation $\Delta_p u = 0$ as $p \to \infty$, the connection with the problem of extending the Lipschitz boundary values to the whole domain Ω has to be made.

The Lipschitz function is defined as follows,

Definition 4. *A function $f : \Omega \to \mathbb{R}$ is Lipschitz continuous for some constant L if:*

$$|f(x) - f(y)| \leq L|x-y|,$$

where $x, y \in \Omega$.

Let $g : \partial\Omega \to \mathbb{R}$ be Lipschitz continuous,

$$|g(\xi_1) - g(\xi_2)| \leq L|\xi_1 - \xi_2|,$$

where $\xi_1, \xi_2 \in \partial\Omega$.

The following bounds were found by McShane and Whitney. For a Lipschitz continuous function $u : \overline{\Omega} \to \mathbb{R}$ with boundary values $g = u|_{\partial\Omega}$, the following holds,

$$\max_{\xi\in\partial\Omega}\{g(\xi) - L|x-\xi|\} \leq u(x) \leq \min_{\xi\in\partial\Omega}\{g(\xi) + L|x-\xi|\},$$

since u depends on x the two bounds are Lipschitz extensions of g and L is the the Lipschitz constant. Note that uniqueness fails in several variables, when the two bounds

[2]Observe that, since $|w|^p$ is convex, the inequality $|b|^p \geq |a|^p + p\langle |a|^{p-2}a, b-a\rangle$ holds for vectors.

does not coincide with the Lipschitz continuous extension and the Lipschitz constant.

The Rademacher theorem state that a Lipschitz function is a.e. differentiable.

Theorem 5. ***(Rademacher's Theorem)***
Let a function f be Lipschitz continuous, then f is totally differentiable a.e. in the domain Ω. Hence, the expansion holds at almost every point,

$$f(y) = f(x) + \langle \nabla f(x), y - x \rangle + o(|y - x|)$$

as $y \to x$, when $x \in \Omega$

The following is a special case of the Ascoli Theorem from (Lindqvist 2015, 8)

Theorem 6. ***(Ascoli's Theorem)***
Let the sequence $f_k : \Omega \to \mathbb{R}$ be equibounded:

$$\sup_{\Omega} |f_k(x)| \ \leq \ M < \infty,$$

where $k = 1, 2, ...$ and equicontinuous:

$$|f_k(x) - f_k(y)| \ \leq \ C|x - y|^{\alpha},$$

where $k = 1, 2, ...$
Then, there exists a continuous function f and a subsequence f_{k_j} such that $f_{k_j} \to f$ locally uniformly in Ω.
All functions can be extended continuously to the boundary and the convergence is uniform in the closure $\overline{\Omega}$, if the domain Ω is bounded.

Proof. The proof follows from (Lindqvist 2015,9-10).
Construct a subsequence which converges at the rational points and let $q_1, q_2, q_3, ...$ be a numbering of the rational points in Ω. By the assumptions the sequence $f_1(q_1), f_2(q_2), f_3(q_3), ...$ is bounded and according to Weierstrass theorem [3] it has a convergent subsequence, $f_{1j}(q_1),\ j = 1, 2, 3, ...$ Now, consider the point q_2. Hence the sequence $f_{11}(q_2), f_{12}(q_2), f_{13}(q_2), ...,$ is bounded. By extracting a convergent subsequence, $f_{21}(q_2), f_{22}(q_2), f_{23}(q_2), ...$ proceeding to extract subsequences of subsequences, the following scheme is obtained,

$$\begin{array}{lll}
f_{11}, f_{12}, f_{13}, f_{14} ... \textit{converges} & \textit{at} & q_1 \\
f_{21}, f_{22}, f_{23}, f_{24} ... \textit{converges} & \textit{at} & q_1, q_2 \\
f_{31}, f_{32}, f_{33}, f_{34} ... \textit{converges} & \textit{at} & q_1, q_2, q_3 \\
f_{41}, f_{42}, f_{43}, f_{44} ... \textit{converges} & \textit{at} & q_1, q_2, q_3, q_4 \\
\text{............................} & &
\end{array}$$

At every rational point the diagonal sequence $f_{11}, f_{22}, f_{33}, f_{44}, ...,$ converges.
Simplify by denoting $f_{k_j} = f_{jj}$.

[3] **Theorem (Weierstrass)** Every bounded infinite subset of R^k has a limit point in R^k. [4]

Claim that the constructed diagonal sequence converges at each point in the domain Ω, rational or not. Let $x \in \Omega$ be an arbitrary point and let q be a rational point near x. Thus,

$$\begin{aligned} &|f_{k_j}(x) - f_{k_i}(x)| \\ &\qquad \leq |f_{k_j}(x) - f_{k_j}(q)| + |f_{k_j}(q) - f_{k_i}(q)| + |f_{k_i}(q) - f_{k_i}(x)| \\ &\qquad \leq 2C|x-q|^\alpha + |f_{k_j}(q) - f_{k_i}(q)|. \end{aligned}$$

Fix q so close to x such that $2C|x-q|^\alpha < \frac{\varepsilon}{2}$, given that $\varepsilon > 0$. This is possible since the rational points are dense[4]. At the rational points the sequence converges, so denote that,

$$|f_{k_j}(x) - f_{k_i}(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for sufficiently large indices i, j. The sequence converge at the point x, by Cauchy's general convergence criterion[5]. The existence of the pointwise limit function has been stated,

$$f(x) = \lim_{j\to\infty} f_{k_j}(x).$$

It is time to show that the convergence is locally uniform. Assume $\overline{\Omega}$ to be compact, covered by balls $B(x,r)$ with diameter $2r = \varepsilon^{\frac{1}{\alpha}}$. Then, $\overline{\Omega}$ is covered by a finite number of these balls i.e.

$$\overline{\Omega} \subset \bigcup_{m=1}^{N} B(x_m, r).$$

Pick a rational point from each ball, $q'_m \in B(x_m, r)$. An index N_ε can be fixed such that,

$$\max_m |f_{k_j}(q'_m) - f_{k_i}(q'_m)| < \varepsilon,$$

where $i, j > N_\varepsilon$, since only a finite number of the chosen rational points q'_m are involved. Let $x \in \overline{\Omega}$ be arbitrary, then it must belong to some ball, $B(x_m, r)$. Hence we can write,

$$\begin{aligned} |f_{k_j}(x) - f_{k_i}(x)| &\leq 2C|x - q'_m|^\alpha + |f_{k_j}(q'_m) - f_{k_i}(q'_m)| \\ &\leq 2C(2r)^\alpha + |f_{k_j}(q'_m) - f_{k_i}(q'_m)| \\ &\leq 2C\varepsilon + \varepsilon, \end{aligned} \tag{6}$$

where $i, j > N_\varepsilon$.
So, the convergence is uniform in $\overline{\Omega}$, since the index N_ε is independent of how the point x is chosen. This shows that the convergence is uniform in .

□

[4]**Definition:** Let X be a metric space. All points and sets are understood to be elements of X. E is dense in X if every point of X is a limit point of E, or a point of E (or both). [4]

[5]**Cauchy convergence criterion:** A sequence converge in R^k if and only if it is a Cauchy sequence [4]

So, recall that we want to expand the Lipschitz boundary values over the domain Ω. By extending g from definition (4), we have for $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ that,

$$||\nabla g||_{L^\infty(\Omega)} \leq L.$$

To be clear, the above notation holds for the extended functions.

We want to minimize this function, among all functions in $W^{1,p}(\Omega)$,

$$I_p(v) = \int_\Omega |\nabla v|^p \, dx,$$

with boundary values, $v - g \in W_0^{1,p}(\Omega)$, then we let $p \to \infty$.
Recall $p > n$.

By theorem (3), a unique minimizer u_p exists, and especially on the boundary $\partial\Omega$, $u_p = g$ and $u_p \in C(\overline{\Omega} \cap W^{1,p}(\Omega))$. From the minimization property we obtain the following,

$$\| \nabla u_p \|_{L^p(\Omega)} \leq \| \nabla g \|_{L^p(\Omega)} \leq \; L|\Omega|^{1/p}.$$

By using both Ascoli's theorem and Morrey's inequality (lemma (2)) (Morrey's inequality is useful to $= u_p - g$) it follows that,

$$\begin{aligned}
|u_p(x) - u_p(y)| &\leq |g(x) - g(y)| + |v_p(x) - v_p(y)| \\
&\leq L|x-y| + \frac{2pn}{p-n}|x-y|^{1-\frac{n}{p}} \| \nabla v_p \|_{L^p(\Omega)} \\
&\leq L|x-y| + \frac{2pn}{p-n}|x-y|^{1-\frac{n}{p}} (\| \nabla u_p \|_{L^p(\Omega)} + \| \nabla g \|_{L^p(\Omega)}) \\
&\leq L|x-y| + LC_\Omega |x-y|^{1-\frac{n}{p}}
\end{aligned}$$

when $p > n+1$. The bound is $\| \, u_p \, \|_{L^\infty(\Omega)} \leq \| \, g \, \|_{L^p(\Omega)} + LC'_\Omega$. Since the family $\{u_p\}$, $p > n+1$ is equibounded and equicontinuous the Ascoli's theorem (6) is applicable, it is possible to extract a subsequence $p_j \to \infty$ such that,

$$u_{p_j} \to u_\infty \text{ unifromly in } \overline{\Omega},$$

where $u_\infty \in C(\overline{\Omega})$, with boundary values $u_\infty|_{\partial\Omega} = g$.

Hence,

$$|u_\infty(x) - u_\infty(y)| \leq C|x-y|.$$

where C depends on Ω and L.

For $p_j > s$, Hölder's inequality yields,

$$\begin{aligned}\left\{ \fint_\Omega |\nabla u_{p_j}|^s \, dx \right\}^{\frac{1}{s}} &\leq \left\{ \fint_\Omega |\nabla u_{p_j}|^{p_j} \, dx \right\}^{\frac{1}{p_j}} \\ &\leq \left\{ \fint_\Omega |\nabla g|^{p_j} \, dx \right\}^{\frac{1}{p_j}} \\ &\leq L.\end{aligned}$$

Hence,

$$\nabla u_{p_j} \rightharpoonup \nabla u_\infty \text{ weakly in } L^s(\Omega),$$

for any subsequence of p_j.
From weak lower semi-continuity we have,

$$\int_\Omega |\nabla u_\infty|^s \, dx \leq \lim_{j\to\infty} \inf \int_\Omega |\nabla u_{p_j}|^s \, dx.$$

Since weak convergence in L^p leads to weak convergence in L^s, for $s < p$, a single subsequence can be extracted through a diagonalization procedure, such that,

$$\nabla u_{p_j} \rightharpoonup \nabla u_\infty \;\text{ weakly in each }\; L^s(\Omega) \;\text{ simultaneously,}$$

when $n + 1 < s < \infty$.
Hence,

$$\left\{ \fint_\Omega |\nabla u_\infty|^s \, dx \right\}^{\frac{1}{s}} \leq L.$$

Since s is large enough and by letting $s \to \infty$ it follows that,

$$\| \nabla u_\infty \|_{L^\infty(\Omega)} \leq L,$$

where L is the Lipschitz constant of u_∞.

Theorem 7. ***(The Existence Theorem)***
There exists a function $u_\infty \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with boundary values $u_\infty = g$ on $\partial\Omega$, given $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. Which have the following minimizing property in each subdomain $D \subset \Omega$:
If $v \in C(\overline{D}) \cap W^{1,\infty}(D)$, and $v = u_\infty$ on $\partial\Omega$, then

$$||\nabla u_\infty||_{L^\infty(D)} \leq ||\nabla v||_{L^\infty(D)},$$

u_∞ can be obtained as the uniform limit $\lim u_{p_j} \in \Omega$, where u_{p_j} denotes the solution of the p_j-Laplace equation with boundary values g.

Proof. The following is based on the proof in (Lindqvist 2015, 19-21).
Show that, the minimizing property yields for the constructed function

$$u_\infty = \lim_{j\to\infty} u_{p_j} \qquad \text{in } \ D.$$

Observe that the uniqueness is not proved yet. Let v_{p_j} denote the solution to $\Delta_{p_j} v_{p_j} = 0$ in D with boundary values $v_{p_j}|_{\partial D} = u_\infty|_{\partial D}$ where v is a given function in $\overline{D}$.
Hence, the minimization property for v_{p_j} is,

$$\int_\Omega |\nabla v_{p_j}|^{p_j}\, dx \ \leq \ \int_\Omega |\nabla v|^{p_j}\, dx.$$

Claim that $v_{p_j} \to u_\infty$ uniformly in $\overline{D}$ and note that,

$$v_{p_j} - u_\infty = (v_{p_j} - u_{p_j}) + (u_{p_j} - u_\infty)$$

it is enough to show that

$$\| \, v_{p_j} - u_{p_j} \, \|_{L^\infty(D)} \ \to \ 0.$$

So,

$$\begin{aligned} \max_{\overline{D}}(v_{p_j} - u_{p_j}) \ &= \ \max_{\partial D}(v_{p_j} - u_{p_j}) \\ &= \ \max_{\partial D}(u_\infty - u_{p_j}) \\ &\leq \ \| \, u_\infty - u_{p_j} \, \|_{L^\infty(D)} \ \to \ 0, \end{aligned}$$

the same yields for $u_{p_j} - v_{p_j}$. Thus, the claim has been proved. Now, use the fact that the maximum is attained at the boundary for difference of two solutions for the $p-$Laplace equation and suppose that,

$$\alpha = \max_{\overline{D}}(v_p - u_p) > \max_{\partial D}(v_p - u_p) = \beta.$$

The open set is defined as,

$$G = \left\{ x \in D | \ v_p(x) - u_p(x) > \frac{\alpha+\beta}{2} \right\}.$$

Then, on the boundary ∂G, $G \subset\subset D$ and $v_p = u_p + \frac{\alpha+\beta}{2}$. Note that, the $p-$Laplace equation in G, have solutions v_p and $u_p + \frac{\alpha+\beta}{2}$ such that both have the same boundary values. Thus, by uniqueness the boundary values coincide in G. This is a contradiction at the maximum point(s). Consequently, as we claimed the maximum was at the boundary.

Recall that, $v_{p_j} \to u_\infty$ in D. Now, fix $s >> 1$. Then,

$$\begin{aligned} \left\{ \fint_D |\nabla v_{p_j}|^s\, dx \right\}^{\frac{1}{s}} \ &\leq \left\{ \fint_D |\nabla v_{p_j}|^{p_j}\, dx \right\}^{\frac{1}{p_j}} \\ &\leq \left\{ \fint_D |\nabla v|^{p_j}\, dx \right\}^{\frac{1}{p_j}} \\ &\leq \ \| \, \nabla v \, \|_{L^\infty(D)} \end{aligned}$$

for $p_j > s$. By weak lower semi-continuity it is possible to extract a weakly convergent subsequence $\nabla v_{p_j} \rightharpoonup \nabla u_\infty$ in $L^s(D)$ the following is obtained,

$$\left\{ \fint_D |\nabla u_\infty|^s \, dx \right\}^{\frac{1}{s}} \leq \| \nabla v \|_{L^\infty(D)} .$$

Thus, let $s \to \infty$,

$$\| \nabla u_\infty \|_{L^\infty(D)} \leq \| \nabla v \|_{L^\infty(D)}$$

□

If the classical solutions does not have critical points, the uniqueness can be shown by proving,

$$\min_{\Omega}(v-u) \;\geq\; \min_{\partial\Omega}(v-u), \tag{7}$$

where $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$. The following has to be satisfied $\Delta_\infty u \geq 0$ and $\Delta_\infty v \leq 0$, pointwise in the bounded domain Ω under extra the assumption $\nabla v \neq 0$.

Proof. Use the antithesis

$$\min_{\Omega}(v-u) < \min_{\partial\Omega}(v-u).$$

Note that, $\Delta_\infty v \geq \Delta_\infty u$ holds at the interior minimal point, but this is not a contradiction. So let us consider

$$w = \frac{1-e^{-\alpha v}}{\alpha} = v - \frac{\alpha v^2}{2} + ...,$$

where $\alpha > 0$ is sufficiently small such that,

$$\min_{\Omega}(w-u) < \min_{\partial\Omega}(w-u).$$

At the minimal point $\Delta_\infty w < 0$ and $0 \;\leq\; \Delta_\infty u \;\leq\; \Delta_\infty w$. Thus,

$$\Delta_\infty w = e^{-3\alpha v}(\Delta_\infty v - \alpha |\nabla v|^4) \leq -\alpha e^{-3\alpha v} |\nabla v|^4.$$

This is a contradiction and we have uniqueness for the Dirichlet boundary value problem.

□

Remark : There is a problem of unique continuation and therefore some qualities are lost as $p \to \infty$. When p is finite the principle of unique continuation holds in the plane, though in space the problem appear to be open. However, for the ∞-Laplace equation this is not true! There is an example that shows this:

$$u(x,y) = \begin{cases} 1 - \sqrt{x^2+y^2} & \text{if} \quad x \le 0,\ y \ge 0 \\ 1 - y & \text{if} \quad x \ge 0,\ y \ge 0 \end{cases} \tag{8}$$

set in the half-plane $y \ge 0$. When drawing the surface $z = u(x,y)$, we notice a cone and its tangent. Hence, this a viscosity solution of the $\infty-$Laplace equation and in some subdomain the above equation (8) provide ∞-harmonic functions that coincide with u.

The real analyticity is lost, since the $p-$harmonic functions are of class $C^{1,\alpha}_{loc}$ i.e. the gradients are locally Hölder continuous. In the open set, the functions are real analytic, where their gradients are not zero. This does not hold for the $\infty-$harmonic functions, it can be explained for $\nabla u = 0$ by adding an extra variable x_{n+1}, such that $\nabla v \neq 0$, but observe that both $v(x_1, x_2, ..., x_n, x_{n+1}) = u(x_1, x_2, ..., x_n) + x_{n+1}$ are $\infty-$harmonic functions. So, in some way the local situation with entirely no critical points is always reached. [1]

3 The Infinity-Laplace Equation

The $p-$Laplace equation,

$$\Delta_p u \equiv |\nabla u|^{p-4}(|\nabla u|^2 \Delta u + (p-2)\Delta_\infty u) = 0 \tag{9}$$

is the limit equation as $p \to \infty$. By dividing with $|\nabla u|^{p-4}$ and $p-2$ we obtain,

$$\frac{|\nabla u|^2 \Delta u}{p-2} + \Delta_\infty u = 0,$$

letting $p \to \infty$,

$$\Delta_\infty u = 0,$$

the solutions are called the $\infty-$harmonic functions and the $\infty-$Laplace equation is denoted as,

$$\Delta_\infty \equiv \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j},$$

where $u = u(x_1, x_2, ..., x_n)$ for n variables.

Classical solutions

The classical solutions satisfies a pointwise two-continuous function u, $u \in C^2(\Omega)$, where the domain Ω lies in the $n-$dimensional Euclidean space $\mathbb{R}^n$. Examples of classical solutions in domains where they are of class C^2 are:

$$\langle \mathbf{a}, x\rangle + b, \qquad b|x - \mathbf{a}| + c, \qquad a\sqrt{x_1^2 + x_2^2 + ... + x_n^2} + b,$$
$$x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}, \qquad tan^{-1}\left(\frac{x_2}{x_1}\right), \qquad tan^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2}}\right),$$
$$a_1 x_1^{\frac{4}{3}} + a_2 x_2^{\frac{4}{3}} + ... + a_n x_n^{\frac{4}{3}}, \qquad \text{where } \ a_1^3 + a_2^3 + ... + a_n^3 = 0.$$

All solutions $u \in C$ of the eikonal equation $|\nabla|^2 = C$ can be covered because,

$$\Delta_\infty u = \frac{\langle \nabla u, \nabla |\nabla u|^2\rangle}{2}.$$

Solutions in disjoint variables can be superposed, for instance

$$\sqrt{x_1^2 + x_2^2} - 7\sqrt{x_3^2 + x_4^2} + x_5 + x_6^{\frac{4}{3}} + x_7^{\frac{4}{3}}.$$

The classical solutions cannot solve the Dirichlet problem, since they are too few.

Gradient flow

For the classical solutions there are an interesting explanation of the equation $\Delta_\infty u = 0$

Let the curve be smooth enough and denoted $x(t) = (x_1(t), x_2(t), ..., x_n(t))$. Then, by differentiating $|\nabla u(x(t))|^2$ along the curve we obtain,

$$\frac{d}{dt}|\nabla u(x(t))|^2 = 2\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{dx_i}{dt}.$$

Hence let the curve be a solution to,

$$\frac{dx}{dt} = \nabla u(x(t)) \tag{10}$$

We obtain the stream lines,

$$\frac{d}{dt}|\nabla u(x(t))|^2 = 2\Delta_\infty u(x(t))$$

along the gradient flow.

Note that, $|\nabla u|$ denotes the speed and it is constant along the stream lines. In the direction of the gradient, the fastest growth occurs and the surface level of a function u have the normal Δu.

Variational solutions

Let g ba a Lipschitz continuous function, that is defined on the boundary of a bounded domain Ω. Again start by minimizing the variational integral,

$$I(u) = \int_\Omega |\nabla u(x)|^p \, dx,$$

among all functions in $C(\overline{\Omega}) \cap C^1(\Omega)$ with boundary values g. Hence, a solution to the $p-$Laplace equation (9) exists, u_p, which is a unique minimizer.

Thus, a variational solution of the ∞-Laplace equation is achieved as a uniform limit

$$u_\infty = \lim_{p_j \to \infty} u_{p_j}$$

through suitable sequences and the function is Lipschitz continuous. Because,

$$\lim_{p\to\infty}\left\{\int_D |\nabla u(x)|^p \, dx\right\}^{\frac{1}{p}} = ess \sup_{x \in D} |\nabla u(x)| = \parallel \nabla u \parallel_{L^\infty(D)},$$

a conjecture is that u_∞ is a minimizer for $\parallel \nabla u \parallel_{L^\infty}$. For each subdomain D it must be true that,

$$\parallel \nabla u_\infty \parallel_{L^\infty(D)} \leq \parallel \nabla v \parallel_{L^\infty(D)},$$

for all v that is Lipschitz continuous in $\overline{D}$ and for $v = u_\infty$ on the boundary ∂D. Considering,

$$\sup_{D\times D} \frac{|u(x) - u(y)|}{|x - y|} = \parallel \nabla u \parallel_{L^\infty(D)}$$

interpret this as having an optimal Lipschitz extension, with boundary value $g \in \Omega$. Later on, we will see that all solutions are variational!

3.1 Mean Value Formula

The $\infty-$harmonic functions have some kind of mean-value property and formula is used in game theory, as we will show further on. There are some properties that one should bear in mind. From Gauss we know that a continuous function u, is harmonic if and only if it comply with the mean value formula.
So, $\Delta u = 0$ in Ω if and only if

$$u(x) = \fint_{B(x,\varepsilon)} u(y)\, dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{11}$$

where every point $x \in \Omega$. There exist a mean value formula for $p-$harmonic functions that is:

$$u(x) = \frac{p-2}{p+n} \frac{\max\limits_{\overline{B}(x,\varepsilon)} \{u\} + \min\limits_{\overline{B}(x,\varepsilon)} \{u\}}{2} + \frac{2+n}{p+n} \fint_{B(x,\varepsilon)} u(y)\, dy + o(\varepsilon^2), \text{ as } \varepsilon \to \infty,$$

where the first term is non-linear and the second term is linear, note that the larger the p, the "more" non-linear is the first term. This formula is used in game theory where the coefficient adds up to 1 (coefficients = probabilities),

$$\frac{p-2}{p+n} + \frac{2+n}{p+n} = 1$$

Note that when $p = 2$ the formula reduces to equation (11) and when $p = \infty$ the asymptotic formula is,

$$u(x) = \frac{\max\limits_{\overline{B}(x,\varepsilon)} \{u\} + \min\limits_{\overline{B}(x,\varepsilon)} \{u\}}{2} + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0. \tag{12}$$

This happens to be a viscosity solution of $\Delta_\infty u = 0$ in Ω where $u \in C(\Omega)$ if and only if equation (12) is true for every point in Ω. This can be interpreted in a specific way described in the following lemma, from (Lindqvist 2015,34).

Lemma 8. *Assume that $\phi \in C^2(\Omega)$ and that $\nabla\phi(x_0) \neq 0$ at the point $x_0 \in \Omega$. Then the asymptotic formula*

$$\phi(x_0) = \frac{\max\limits_{\overline{B}(x_0,\varepsilon)} \{\phi\} + \min\limits_{\overline{B}(x_0,\varepsilon)} \{\phi\}}{2} + \frac{\Delta_\infty(x_0)}{|\nabla\phi(x_0)|^2} + o(\varepsilon^2), \; \textit{as } \varepsilon \to 0 \tag{13}$$

holds.

Proof. The proof follows from the one in (Lindqvist 2015, 34-35).
To verify this, Taylor's formula and some infinitesimal calculus is used. Because $\nabla\phi(x_0) \neq$

0, it is possible to pick ε sufficiently small such that $|\nabla\phi(x_0)| > 0$ when $|x - x_0| \le \varepsilon$. At points for which,

$$x = x_0 \pm \varepsilon \frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \qquad |x - x_0| = \varepsilon, \tag{14}$$

the extremal values in $\overline{B}(x_0, \varepsilon)$ are achieved.
This can be discovered by using a Lagrangian multiplier. Further, for ε small the +-sign corresponds to the maximum. The endpoints of some diameter, is the maximum and minimum points which are about the opposite. Then,

$$\frac{\nabla\phi(x)}{|\nabla\phi(x)|} = \frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|} + \mathcal{O}(\varepsilon), \qquad \text{as} \quad \varepsilon \to 0. \tag{15}$$

For $x \in \partial B(x_0, \varepsilon)$ let x^* denote exactly opposite endpoints of a diameter, i.e., $x + x^* = 2x_0$.
By Taylor expansion,

$$\phi(y) = \phi(x_0) + \langle \nabla\phi(x_0), y - x_0\rangle + \frac{1}{2}\langle \mathbb{D}^2\phi(x_0)(y - x_0), y - x_0\rangle + o(|y - x_0|^2),$$

when $y = x$ and $y = x^*$, we have

$$\phi(x) + \phi(x^*) = 2\phi(x_0) + \langle \mathbb{D}^2\phi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2).$$

Observe that the first order terms vanishes.
Pick x to be the maximal point,

$$\phi(x) = \max_{|y - x_0| = \varepsilon} \{\phi(y)\}.$$

The following inequality is obtained by, inserting formula (14) with equation (15),

$$\begin{aligned}
\max_{\overline{B}(x_0,\varepsilon)} \{\phi\} + \min_{\overline{B}(x_0,\varepsilon)} \{\phi\} &\le \phi(x) + \phi(x^*) \\
&= 2\phi(x_0) + \langle \mathbb{D}^2\phi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \\
&= 2\phi(x_0) + \langle \mathbb{D}^2\phi(x_0)\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}, \frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}\rangle + o(\varepsilon^2) \\
&= 2\phi(x_0) + \frac{\Delta_\infty\phi(x_0)}{|\nabla\phi(x_0)|^2} + o(\varepsilon^2).
\end{aligned}$$

Now, pick x to be a minimal point,

$$\phi(x) = \min_{|y - x_0| = \varepsilon} \{\phi(y)\}$$

then the opposite inequality, can be deduced since,

$$\max_{\overline{B}(x_0,\varepsilon)} \{\phi\} + \min_{\overline{B}(x_0,\varepsilon)} \{\phi\} \ge \phi(x^*) + \phi(x).$$

Hence, the asymptotic formula has been proved (13). $\square$

Note that, when $\nabla\phi(x_0) = 0$, there is a bit of an issue with formula (13). Hence the following definition from (Lindqvist 2015, 36) contains changes for this situation.

Definition 9. *A function $u \in C(\Omega)$ satisfies the formula*

$$u(x) = \frac{\max\limits_{\overline{B}(x,\varepsilon)}\{u\} + \min\limits_{\overline{B}(x,\varepsilon)}\{u\}}{2} + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0 \tag{16}$$

in the viscosity sense, if the two following conditions are true:

- *if $x_0 \in \Omega$ and if $\phi \in C^2(\Omega)$ touches u from below at x_0, then*

$$\phi(x_0) \geq \frac{1}{2}\left(\max_{|y-x_0|\leq\varepsilon}\{\phi(y)\} + \min_{|y-x_0|\leq\varepsilon}\{\phi(y)\}\right) + o(\varepsilon^2), \qquad \text{as } \varepsilon \to 0.$$

  Besides, if it so happens that $\nabla\phi(x_0) = 0$, we require that the test function satisfies $\mathbb{D}^2\phi(x_0) \leq 0$.

- *if $x_0 \in \Omega$ and if $\psi \in C^2(\Omega)$ touches u from above at x_0, then*

$$\psi(x_0) \leq \frac{1}{2}\left(\max_{|y-x_0|\leq\varepsilon}\{\psi(y)\} + \min_{|y-x_0|\leq\varepsilon}\{\psi(y)\}\right) + o(\varepsilon^2), \qquad \text{as } \varepsilon \to 0.$$

  Besides, if it so happens that $\nabla\psi(x_0) = 0$, we require that the test function satisfies $\mathbb{D}^2\psi(x_0) \geq 0$.

Remark : When $\phi(x_0) = 0$ or $\psi(x_0) = 0$ the following form is used

$$\lim_{y\to x_0}\frac{\phi(y)-\phi(x_0)}{|y-x_0|^2} \leq 0, \qquad\qquad \lim_{y\to x_0}\frac{\psi(y)-\psi(x_0)}{|y-x_0|^2} \geq 0.$$

The main result for the mean value formula is the following characterization theorem from (Lindqvist 2015,36).

Theorem 10. *Let $u \in C(\Omega)$. Then $\Delta_\infty u = 0$ in the viscosity sense if and only if the mean value formula (16) holds in the viscosity sense.*

Proof. The proof follows from (Lindqvist 2015, 36-37).
Consider subsolutions. Assume that ψ touches u from above at the point $x_0 \in \Omega$. If $\nabla\psi(x_0) \neq 0$ then, according to formula (13), the condition $\Delta_\infty\psi(x_0) \geq 0$ is equivalent to the inequality for ψ in definition (9). Hence, the case is clear.
Suppose that $\nabla\psi(x_0) = 0$. If u satisfies formula (13) in the viscosity sense, the extra assumption

$$\lim_{y\to x_0}\frac{\psi(y)-\psi(x_0)}{|y-x_0|^2} \geq 0$$

can be used. If

$$\psi(x_\varepsilon) = \min_{|x-x_0|\leq\varepsilon}\psi(x)$$

conclude that,

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left\{\frac{1}{2}\left(\max_{\overline{B}(x_0,\varepsilon)} \psi + \min_{\overline{B}(x_0,\varepsilon)} \psi\right) - \psi(x_0)\right\} \\
&\quad= \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left\{\frac{1}{2}\left(\max_{\overline{B}(x_0,\varepsilon)} \psi - \psi(x_0)\right) + \frac{1}{2}\left(\min_{\overline{B}(x_0,\varepsilon)} \psi - \psi(x_0)\right)\right\} \\
&\quad\geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\frac{1}{2}\left(\min_{\overline{B}(x_0,\varepsilon)} \psi - \psi(x_0)\right) \\
&\quad= \frac{1}{2}\liminf_{\varepsilon\to 0}\left(\frac{\psi(x_\varepsilon) - \psi(x_0)}{|x_\varepsilon - x_0|^2}\right)\frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} \\
&\quad\geq 0
\end{aligned}
$$

where $\frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} \leq 1$. Thus, the inequality in definition (9) has been proved.
Conversely, the condition $\Delta_\infty\psi(x_0) \geq 0$ is satisfied, when $\nabla\psi(x_0) = 0$. Hence, the mean value property in the viscosity sense, guarantees that viscosity subsolution of the ∞-Laplace equation is obtained. □

.

3.2 Viscosity Solutions

The use of viscosity solutions circumvents the problem that the second derivative does not always exist in the $\infty-$Laplace equation. In the Dirichlet problem smooth boundary values can be prescribed such that no C^2- solution can achieve them. Smoothness is not allowed at the critical points, $\nabla u = 0$!

Theorem 11. ***(Aronsson's Theorem)***
Assume that $u \in C^2(\Omega)$ and that $\Delta_\infty u = 0$ in Ω.
Thus, either u reduces to a constant or $\nabla u \neq 0$ in Ω.

Proof. The case when $n = 2$ was proved with methods in complex analysis by Aronsson in [5] and the case for $n \geq 3$ is in [6].

□

Aronsson's theorem has a major significance for the Dirichlet problem.

$$
\begin{cases} \Delta_\infty u = 0, & \text{in } \Omega \\ \quad\;\; u = g, & \text{on } \partial\Omega \end{cases} \tag{17}
$$

if $g \in C(\overline{\Omega}) \cap C^2(\Omega)$, g is chosen such that for every function $f \in C(\overline{\Omega}) \cap C^2(\Omega)$, with boundary values $f = g$ on $\partial\Omega$ there must be at least one critical point in Ω.

The following is a topological phenomenon related to index theory: Pick Ω as the unit disk $x^2 + y^2 < 1$ and $g(x, y) = xy$. Thus, for every function f with the boundary values $f(\xi, \eta) = \xi\eta$ where $\xi^2 + \eta^2 < 1$, a solution or not, there must exist a critical point in the open unit disk.

Aronsson's theorem (11) would be contradicted if there are boundaries of class C^2, a solution to the Dirichlet problem cannot be of class C^2. This can also be seen by using symmetry, suppose that $\nabla u \neq 0$ in the disc. It can be conclude, that $u(x, y) = u(-x, -y)$ but when differentiating note that $\nabla u(0, 0) = 0$. Thus, there exists a critical point in any case!
Therefore we must introduce the concept of viscosity solutions. Crandall, Evans, Lions, Ishii and Jensen among others developed the modern theory of viscosity solutions. At first it was only for first order equations, then later on it was developed to second order equations.
Burgers' equation is one example of how the label viscosity was established for first order equations,

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0 & (-\infty < x < \infty,\ t > 0) \\ u(x, 0) = g(x) \end{cases} \tag{18}$$

an extremely small term is added representing viscosity.
To this Cauchy problem even smooth initial values $g(x)$ can oblige the solution $u = u(x, t)$ to develop shocks. Besides, the uniqueness fails frequently.
By adding a term $\varepsilon\Delta u$, that representing artificial viscosity the Burgers' Equation will be

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + \frac{1}{2}\frac{\partial u_\varepsilon^2}{\partial x} = \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2} \\ u_\varepsilon(x, 0) = g(x). \end{cases}$$

So, now $u_\varepsilon = u_\varepsilon(x, t)$ is smooth and unique. Additionally, $u_\varepsilon(x, t) \to u(x, t)$, this u is the desired viscosity solution of the original equation (18). Note that this is the method of vanishing viscosity.
Although to identifying the viscosity solution, the concept can be formulated directly from the original equation without the limit procedure with ε.
Return to the ∞-Laplace equation and instead of considering the limit procedure,

$$\Delta_\infty u_\varepsilon + \varepsilon\Delta u_\varepsilon = 0 \qquad \text{as } \varepsilon \to 0, \tag{19}$$

a concept of the viscosity solution directly from the original equation will be formulated. Equation (19) is the Euler-Lagrange equation for the following function,

$$J(v) = \int_\Omega e^{\frac{|\nabla v|^2}{2\varepsilon}}\, dx.$$

Start with infinitesimal calculus. Assume $v : \Omega \to \mathbb{R}$ be given, suppose that $\phi \in C^2(\Omega)$ is touching v from below at some point $x_0 \in \Omega$:

$$\begin{cases} v(x_0) &= \phi(x_0) \\ v(x) &> \phi(x), \quad \text{where } x \neq x_0 \end{cases} \tag{20}$$

If v is smooth, then by infinitesimal calculus,

$$\begin{cases} \nabla v(x_0) &= \nabla \phi(x_0), \\ D^2 v(x_0) &\geq D^2 \phi(x_0). \end{cases}$$

Here the Hessian matrix evaluated at the touching point x_0 is,

$$D^2 v(x_0) = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}(x_0) \right)_{n \times n}.$$

When looking at a symmetric matrix $A = (a_{ij})$ the following ordering is used,

$$A \geq 0 \iff \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq 0 \qquad \forall \xi = (\xi_1, \xi_2, ..., \xi_n),$$

where $A = D^2(v - \phi)(x_0)$.

Especially, it follows that $\phi_{x_j x_j}(x_0) \leq v_{x_j x_j}(x_0)$, where $j = 1, 2, ..., n$, and thus

$$\begin{aligned} \Delta \phi(x_0) &\leq \Delta v(x_0) \\ \Delta_\infty \phi(x_0) &\leq \Delta_\infty v(x_0) \\ \Delta_p \phi(x_0) &\leq \Delta_p v(x_0) \end{aligned}$$

for $n < p \leq \infty$. If v is a supersolution, i.e., $\Delta_p v \leq 0$, then

$$\Delta_p \phi(x_0) \leq 0.$$

If v does not have any derivatives the last inequality still holds.

Definition 12. *Let $n < p \leq \infty$.*

- *If $\Delta_p \phi(x_0) \leq 0$, when $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$, are such that ϕ touches v from below at x_0, then $v \in C(\Omega)$ is a viscosity supersolution of the equation, $\Delta_p v = 0$, in Ω.*
- *If $\Delta_p \psi(x_0) \geq 0$, when $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$, are such that ψ touches u from above at x_0, then $u \in C(\Omega)$ is a viscosity subsolution of the equation, $\Delta_p u = 0$, in Ω.*
- *If $h \in C(\Omega)$ is both a viscosity supersolution and a viscosity subsolution then, $h \in C(\Omega)$ is a viscosity solution.*

Proposition 13. ***(Consistency)***
This proposition is from (Lindqvist 2015, 27).
A function $u \in C^2(\Omega)$ is a viscosity solution of $\Delta_p u = 0$ if and only if $\Delta_p u(x) = 0$ holds pointwise in Ω.

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Proof. This proof follows from (Lindqvist 2015,27). Consider the case of subsolutions. Let u be a viscosity subsolution, then at point x_0 u itself will do as a test function, such that $\Delta_p u(x_0) \geq 0$.

In agreement to formula (20) use the following test function,

$$\psi(x) = u(x) + |x - x_0|^4,$$

such that when $x \neq x_0$ the touching from above is strict. Then $0 \leq \Delta_p \psi(x_0) = \Delta_p u(x_0)$.

If u is a classical subsolution, i.e. $\Delta_p u(x) \geq 0$ at every point, then definition (12) must be verified. Hence, for a test function touching from above at x_0 the inequality

$$\Delta_p \psi(x_0) \geq \Delta_p u(x_0)$$

is true by the infinitesimal calculus. Thus, since $\Delta_p u(x_0) \geq 0$ it follows that $\Delta_p \psi(x_0) \geq 0$. $\square$

An important example of a viscosity solution of $\Delta_\infty u = 0$ in $\mathbb{R}^2$ is the function

$$u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}.$$

Note that, at the coordinate axes the second derivative blows up and there is no appropriate test function at the origin. Indeed,

$$u \in C_{loc}^{1,\frac{1}{3}}(\mathbb{R}^2) \cap W_{loc}^{2,\frac{2}{3}-\varepsilon}(\mathbb{R}^2), \qquad \varepsilon > 0.$$

For every viscosity solution in the plane, it is assumed that it belongs to $C_{loc}^{1,\frac{1}{3}}(\mathbb{R}^2)$, i.e. the Hölder continuity with exponent $\frac{1}{3}$ yields locally for its gradient.
An example of a viscosity subsolution in $\mathbb{R}^n$ is the function $|x - x_0|$. In fact, another viscosity subsolution is the function defined by superposition

$$V(x) = \int_{\mathbb{R}^n} |x - y| \rho(y)\, dy,$$

where $\rho \geq 0$, if $\int_{\mathbb{R}^n} |y|\rho(y)\, dy < \infty$.

The biharmonic equation $\Delta\Delta u = -\delta$ in three dimensional space, have the fundamental solution $\frac{|x-x_0|}{8\pi}$, such that $\Delta\Delta V(x) = -8\pi\rho(x)$. If V is given then we can find an appropriate ρ!

Before proving that u_∞ is a viscosity solution, one must prove that u_{p_j} is a viscosity solution of the p_j-Laplace equation, recall the variational solution

$$u_\infty = \lim_{j \to \infty} u_{p_j}.$$

This was constructed through a sequence of solutions to $p-$Laplace equations in the proof of the Existence theorem (7).

Lemma 14. *Assume that $v \in C(\Omega) \cap W^{1,p}(\Omega)$ fulfill $\Delta_p v \leq 0$ in the weak sense,*

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \ \geq 0, \quad \forall \, \eta \geq 0, \ \eta \in C_0^\infty(\Omega).$$

Thus, v is also a viscosity supersolution in Ω.

Proof. The proof follows from (Lindqvist 2015, 28-29).
If the claim does not hold we have, at some point $x_0 \in \Omega$ a test function ϕ touching from below such that,

$$\Delta_p \phi(x_0) > 0$$

holds.
By continuity we have,

$$\Delta_p \phi(x) > 0 \ \text{ when } \ |x - x_0| < 2\rho,$$

for some small radius $\rho > 0$. Hence, ϕ is a classical solution in the ball $B(x_0, 2\rho)$.
Set,

$$\psi(x) \ = \ \phi(x) + \frac{1}{2} \min_{\partial B(x_0,\rho)} \{v - \phi\} \ = \ \phi(x) + \frac{m}{2}.$$

It is clear that, $\psi < v$ on $\partial B(x_0, \rho)$ and $\psi(x_0) > v(x_0)$.
Now, let the component of the open set $\{\psi > v\} \cap B(x_0, \rho)$, which include the point x_0, be denoted as D_p. Hence, conclude that $\psi = v$ on ∂D_ρ.
By the Comparison Principle [6], we have that $\psi \leq v \in D_\rho$, since ψ is a weak subsolution and v a weak supersolution of the same equation. This implies a contradiction, $\psi(x_0) \leq v(x_0) \leq \psi(x_0)$ and thus the lemma follows. □

Lemma 15. *Assume that $f_j \to f$ uniformly in $\overline{\Omega}$, where $f_j, f \in C(\overline{\Omega})$. If ϕ touches f from below at $x_0 \in \Omega$, then there are points $x_j \to x_0$ so that,*

$$f_j(x_j) - \phi(x_j) \ = \ \min_\Omega \{f_j - \phi\}$$

for some subsequence.

[6] Put the test function $\eta = [\psi - v]_+$ into

$$\int_{D_\rho} \langle |\nabla \psi|^{p-2} \nabla \psi, \nabla \eta \rangle \, dx \leq 0, \qquad \int_{D_\rho} \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx \geq 0$$

subtract to get,

$$\int_{D_\rho} \langle |\nabla \psi|^{p-2} \nabla \psi - |\nabla v|^{p-2} \nabla v, \nabla (\psi - v)_+ \rangle \, dx \leq 0.$$

By an elementary inequality,

$$2^{p-2} \int_{D_\rho} |\nabla (\psi - v)_+|^p \, dx.$$

Thus, $\nabla (\psi - v)_+ = 0$, the result follows.

Proof. The proof is follows from (Lindqvist 2015, 29-30).
Since $f_j - \phi = (f - \phi) + (f_j - f)$ and $f_j \to f$,

$$\inf_{\Omega \setminus B_r} \{f_j - \phi\} \geq \frac{1}{2} \inf_{\Omega \setminus B_r} \{f - \phi\} > 0$$

where j is sufficiently large. Let $B_r = B(x_0, r)$ be a small ball.
Thus,

$$\inf_{\Omega \setminus B_r} \{f_j - \phi\} > f_j(x_0) - \phi(x_0),$$

where $j > j_r$. The left-hand side has a positive limit whereas the right-hand side approaches zero. Hence, the infimum over $\Omega \setminus B_r$ is bigger then the value at the center. So there is a point $x_j \in \overline{B}(x_0, r)$ so that,

$$\min_{\Omega} \{f_j - \phi\} = f_j(x_j) - \phi(x_j),$$

where $j > j_r$. Then let $r \to 0$ through a sequence $r = 1, \frac{1}{2}, \frac{1}{3}, ...$ to finish the proof. □

Theorem 16. *The function v_∞ is a viscosity solution of the equation*

$$\max\{\varepsilon - |\nabla v_\infty|, \Delta_\infty v_\infty\} = 0.$$

Conversely,

$$\varepsilon \leq |\nabla \phi(x_0)| \;\; \text{and} \;\; \Delta_\infty \phi(x_0) \leq 0,$$

is true for a test function touching from below and

$$\varepsilon \geq |\nabla \psi(x_0)| \;\; \text{or} \;\; \Delta_\infty \psi(x_0) \geq 0.$$

is true for a test function touching from above.
In the special case, $\varepsilon = 0$, the equation is $\Delta_\infty v_\infty = 0$.

Proof. The proof follows from (Lindquist 2015, 32-33).
Verify that v_p is a viscosity solution of its own Euler-Lagrange equation

$$\Delta_p v = -\varepsilon^{p-1}. \tag{21}$$

In weak form,

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle \, dx = \varepsilon^{p-1} \int_\Omega \eta \, dx,$$

where $\eta \in C_0^\infty(\Omega)$.
This would be equal to the pointwise equation (21) if v were of class $C^2(\Omega)$ and $p \geq 2$.
The proof of this is the same as in lemma (14).
Hence, let $p_j \to \infty$. The case with supersolutions is,

$$\Delta_{p_j} \varphi(x_0) \leq -\varepsilon^{p-1}$$

with φ touching v_{p_j} from below at point x_0.
Now, assume that ϕ touches v_∞ from below at x_0, i.e. $\phi(x_0) = v_\infty(x_0)$ and $\phi(x) < v_\infty(x)$ where $x \neq x_0$. There are points $x_k \to x_0$ such that $v_{p_{j_k}} - \phi$ achieve its minimum at x_k, by lemma (15).
Hence,

$$\Delta_{p_{j_k}} \phi(x_k) \leq -\varepsilon^{p-1}$$

explicitly,

$$|\nabla\phi(x_k)|^{p_{j_k}-4}\Big\{|\nabla\phi(x_k)|^2\Delta\phi(x_k) + (p_{j_k}-2)\Delta_\infty\phi(x_k)\Big\} \ \leq \ -\varepsilon^{p_{j_k}-1}.$$

It is impossible that $\nabla\phi(x_k) = 0$, if $\varepsilon \neq 0$. So, by dividing with $\frac{|\nabla\phi(x_k)|^{p_{j_k}-4}}{p_{j_k}-2}$ we obtain,

$$|\nabla\phi(x_k)|^2\,\frac{\Delta\phi(x_k)}{p_{j_k}-2} + \Delta_\infty\phi(x_k) \leq -\frac{\varepsilon^3}{p_{j_k}-2}\left(\frac{\varepsilon}{|\nabla\phi(x_k)|}\right)^{p_{j_k}-4}. \tag{22}$$

The left-hand side has the limit $\Delta_\infty\phi(x_0)$, by continuity.
If $|\nabla\phi(x_0)| < \varepsilon$, a contradiction occurs, $\Delta_\infty\phi(x_0) = -\infty$. Hence the first established required inequality is,

$$\varepsilon - |\nabla\phi(x_0)| \leq 0,$$

this implies that the last term in equation (22) tends to zero.
The second established required inequality is,

$$\Delta_\infty\phi(x_0) \leq 0,$$

this concludes the case when $\varepsilon > 0$.

There is nothing to show if $\varepsilon = 0$, thus $\nabla\phi(x_0) = 0$. But if $\nabla\phi(x_0) \neq 0$ then,

$$\frac{\Delta\phi(x_k)}{p_{j_k}-2} + \frac{\Delta_\infty\phi(x_k)}{|\nabla\phi(x_k)|^2} \leq 0,$$

when k is large. It follows that $\Delta_\infty\phi(x_0) \leq 0$. This concludes the case with supersolutions. (To show the case with subsolution is easier.) $\square$

Now, we want to prove uniqueness of viscosity solutions, for the Dirichlet problem in a bounded domain, this was done by Jensen.

Theorem 17. ***(Jensen's Theorem)***
Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$. Given a Lipschitz continuous function $f : \partial\Omega \to \mathbb{R}$, there exists a unique viscosity solution $u \in C(\overline{\Omega})$ with boundary values f. Thus, $u \in W^{1,\infty}(\Omega)$, and $\| \nabla u \|_{L^\infty}$ has a bound depending only on the Lipschitz constant of f.

Let us present two help equations with $\varepsilon > 0$, the situation is,

$$\begin{cases} \max\{\varepsilon - |\nabla u^+|, \Delta_\infty u^+\} = 0, & \text{Upper equation} \\ \Delta_\infty u = 0, & \text{Equation} \\ \min\{|\nabla u^-| - \varepsilon, \Delta_\infty u^-\} = 0, & \text{Lower equation.} \end{cases}$$

When the equations have the same boundary values the order is $u^- \leq u \leq u^+$. For the Lower equation the subsolutions are needed and for the Upper equation supersolutions are needed.

Lemma 18. *A variational solution of the Upper Equation satisfies*

$$\| \nabla v \|_{L^\infty(\Omega)} \leq \ L + \varepsilon.$$

Note that for the Lower Equation, the stages are

$$\begin{aligned} \Delta_p u \ &= \ + \varepsilon^{p-1}, \\ \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle \, dx \ &= \ - \varepsilon^{p-1} \int_\Omega \eta \, dx, \\ \int_\Omega \left(\frac{1}{p} |\nabla u|^p + \varepsilon^{p-1} u \right) dx \ &= \ \min . \end{aligned}$$

Further, let u_p^- , u_p , u_p^+ be solutions to the Upper equation, the equation and the Lower equation, with the same boundary values f, then by comparison $u_p^- \leq u_p \leq u_p^+$. The equations have weak solutions which are viscosity solutions. Pick a sequence $p \to \infty$ such that,

$$u_p^- \to u^-, \quad u_p \to h, \quad u_p^+ \to u^+.$$

Then it yields,

$$\begin{aligned} div(|\nabla u_p^+|^{p-2} \nabla u_p^+) &= -\varepsilon^{p-1}, \\ div(|\nabla u_p^-|^{p-2} \nabla u_p^-) &= +\varepsilon^{p-1}. \end{aligned}$$

Now take $u_p^+ - u_p^-$ as a test function in the weak formulation of the equation,

$$\int_\Omega \langle |\nabla u_p^+|^{p-2} \nabla u_p^+ - |\nabla u_p^-|^{p-2} \nabla u_p^-, \nabla u_p^+ - \nabla u_p^- \rangle \, dx \ = \ \varepsilon^{p-1} \int_\Omega (u_p^+ - u_p^-) \, dx.$$

The elementary inequality for vectors (with $p > 2$),

$$\langle |b|^{p-2} b - |a|^{p-2} a, b - a \rangle \geq 2^{p-2} |b - a|^p$$

gives,

$$4 \int_\Omega \left| \frac{\nabla u_p^+ - \nabla u_p^-}{2} \right|^p dx \ \leq \ \varepsilon^{p-1} \int_\Omega (u_p^+ - u_p^-) \, dx.$$

By extracting the p^{th} roots we get,

$$\| \nabla u^+ - \nabla u^- \|_{L^\infty(\Omega)} \leq 2\,\varepsilon.$$

Integration gives,

$$\| u^+ - u^- \|_{L^\infty(\Omega)} \leq \varepsilon \;\; diam(\Omega).$$

The result $u^- \leq h \leq u^+ \leq u^- + C\varepsilon$ is needed for the constructed functions.

The following lemma from (Lindquist 2015, 49) implies uniqueness.

Lemma 19. *If $u \in C(\overline{\Omega})$ is an arbitrary viscosity solution of the equation $\Delta_\infty u = 0$ with $u = f$ on $\partial\Omega$, then*

$$u^- \leq u \leq u^+,$$

where u^-, u^+ are constructed variational solutions of the help equations.

In fact, if there are two viscosity solutions u_1, u_2 then,

$$-\varepsilon \;\leq\; u^- - u^+ \;\leq\; u_1 - u_2 \;\leq\; u^+ - u^- \;\leq\; \varepsilon,$$

where ε is given. Thus $u_1 = u_2$. This implies that every viscosity solution is a variational one and that the variational solutions are unique!

3.2.1 The Comparison Principle

The inequalities,

$$\varepsilon \leq |\nabla u^{\pm}| \leq L + \varepsilon,$$

are satisfied by the variational solutions u^+ and u^- from the help equations. Hence, the upper bound in lemma (18) holds a.e. in Ω and the test functions the lower bound holds in the viscosity sense.

The following lemma from (Lindqvist 2015,46) will be used in further reading.

Lemma 20. ***(Ishii's Lemma)***
Let u and v belong to $C(\Omega)$. If there exists an interior point $(x_j, y_j) \in \Omega \times \Omega$ for which the maximum

$$\max_{x,y\in\Omega} \left\{ u(x) - v(y) - \frac{j}{2}|x-y|^2 \right\}$$

is attained, then there are symmetric matrices $\mathbb{X}_j$ and $\mathbb{Y}_j$ so that

$$\Big(j(x_j - y_j), \mathbb{X}_j \Big) \in \overline{J^{2,+}} u(x_j),$$
$$\Big(j(x_j - y_j), \mathbb{Y}_j \Big) \in \overline{J^{2,-}} u(y_j),$$

and $\mathbb{X}_j \leq \mathbb{Y}_j$

Lemma 21. *(Lindqvist 2015, 50) If u is a viscosity solution of the equation $\Delta_\infty u = 0$ and if $u \leq f = u^+$ on $\partial\Omega$, then $u \leq u^+$ in Ω. The analogous comparison holds for the viscosity supersolutions above u^-.*

Proof. The proof follows from (Lindquist 2015, 50-52).
Suppose that $u^+ > 0$, by adding a constant. Denote $v = u^+$ and claim that $v \geq u$. Use the antithesis,

$$\max_{\Omega}(u - v) > \max_{\partial\Omega}(u - v).$$

First construct a strict supersolution $w = g(v)$ of the Upper equation, so that,

$$\max_{\Omega}(u - w) > \max_{\partial\Omega}(u - w) \tag{23}$$

and

$$\Delta_\infty w \leq -\mu < 0.$$

Use the approximation,

$$g(t) = log(1 + A(e^t - 1)),$$

where $A > 1$. When $A = 1 \implies g(t) = t$.
Assume $t > 0$ then,

$$0 < g(t) - t < A - 1,$$
$$0 < g'(t) - 1 < A - 1.$$

We have,

$$g''(t) = -(A-1)\frac{g'(t)^2}{Ae^t},$$

in order to deduce the equation for $w = g(v)$.
By differentiation,

$$\begin{aligned} w &= g(v), \\ w_{x_i} &= g'(v)\, v_{x_i}, \\ w_{x_i x_j} &= g''(v)\, v_{x_i} v_{x_j} + g'(v)\, v_{x_i x_j}, \\ \Delta_\infty w &= g'(v)^3 \Delta_\infty v + g'(v)^2 g''(v) |\nabla v|^4. \end{aligned}$$

By multiplying the Upper equation $\max\{\varepsilon - |\nabla v|, \Delta_\infty v\} \leq 0$ for supersolutions by $g'(x)^3$ we have,

$$\Delta_\infty w \;\leq\; g'(v)^2 g''(v) |\nabla v|^4 \;=\; -(A-1)A^{-1}e^{-v} g'(v)^4 |\nabla v|^4.$$

Note that the right-hand side is negative.

The inequality $\varepsilon \leq |\nabla v|$ have the following estimate,

$$\Delta_\infty w \leq -\varepsilon^4 (A-1) A^{-1} e^{-v}.$$

Given $\varepsilon > 0$, fix A near 1 such that,

$$0 < w - v = g(v) - v < A - 1 < \delta,$$

where δ is so small such that equation (23) holds. By these adjustments, the tiny negative quantity,

$$-\mu = -\varepsilon^4 (A-1) A^{-1} e^{-\|v\|_\infty}$$

will make it in the strict equation.
Hence, the final equation is

$$\Delta_\infty w \leq -\mu.$$

Since $g'(v) > 1$, the bound, $\varepsilon \leq |\nabla w|$, will be needed.

A test function ϕ that touches v from below at some point x_0, should replace the function v and a test function φ, that touches w from below at the point $y_0 = g(\phi(x_0))$ should replace w.

The following has now been proved,

$$\Delta_\infty \varphi(y_0) \leq -\mu, \qquad \varepsilon \leq |\nabla \varphi(y_0)|,$$

when φ touches w from below at y_0.

Since we want to use Ishii's lemma (20), start by doubling the variables:

$$M = \sup_{\substack{x \in \Omega \\ y \in \Omega}} \left(u(x) - w(y) - \frac{j}{2}|x-y|^2 \right).$$

The maximum is attained at the interior points x_j, y_j, for large indices, hence $x_j \to \hat{x}$, $y_j \to \hat{x}$, when $\hat{x}$ is some interior point. Thus, due to equation (23) $\hat{x}$ cannot be on the boundary.
There exist a symmetric $n \times n-$ matrices $\mathbb{X}_j$ and $\mathbb{Y}_j$, due to Ishii's Lemma, such that $\mathbb{X}_j \leq \mathbb{Y}_j$ and,

$$(j(x_j - y_j), \mathbb{X}_j) \in \overline{J^{2,+}} u(x_j),$$
$$(j(x_j - y_j), \mathbb{Y}_j) \in \overline{J^{2,-}} w(y_j),$$

"where the closure of the subjets appear". The equations can be written as,

$$j^2 \langle \mathbb{Y}_j (x_j - y_j), (x_j - y_j) \rangle \leq -\mu,$$
$$j^2 \langle \mathbb{X}_j (x_j - y_j), (x_j - y_j) \rangle \geq 0.$$

Now, subtract to reach the contradiction,

$$j^2 \langle (\mathbb{Y}_j - \mathbb{X}_j)(x_j - y_j), (x_j - y_j) \rangle \leq -\mu.$$

The ordering that $\mathbb{X}_j \leq \mathbb{Y}_j$ is a contradiction, since

$$j^2 \langle (\mathbb{Y}_j - \mathbb{X}_j)(x_j - y_j), (x_j - y_j) \rangle \geq 0.$$

Therefore, the antithesis does not hold and we have $u \leq v$. $\square$

At last, we establish the general form of the comparison principle (Lindquist 2015, 52).

Theorem 22. ***(The Comparison Principle)***
Suppose that $\Delta_\infty u \geq 0$ and $\Delta_\infty v \leq 0$ in the viscosity sense in a bounded domain Ω. If

$$\liminf v \geq \limsup u \quad \textit{on} \quad \partial\Omega, \quad \textit{then} \quad v \geq u \quad \textit{in} \quad \Omega.$$

Proof. The proof follows from lemma (21) (Lindqvist 2015, 52).
For the indirect proof, let $\varepsilon > 0$ and consider a component D_ε of the domain $\{v+\varepsilon < u\}$. Clearly, $D_\varepsilon \subset\subset \Omega$. For the domain D_ε (not for Ω) construct the help functions u^+ and u^-, such that $u^+ = u^- = u = v + \varepsilon$ on ∂D_ε. Thereby

$$v + \varepsilon \geq u^- \geq u^+ - \varepsilon \geq u - \varepsilon$$

since ε is arbitrary the result follows. □

The existence and uniqueness of variational solutions of the Dirichlet boundary value problem have now been shown. Since the variational solutions are also viscosity solution they must also be unique. To sum up, there exists only one variational solution with given boundary values and the existence was determined in theorem (7).[1]

4 Tug-of-War

In 2009, O.Schramm, Y.Peres, D.Wilson and S.Sheffield discovered the relation between the Probability calculations and the ∞-Laplace equation. The equation was detected by using the mathematical game tug-of-war. [1]

In this section we will show the relation by introduce $p-$harmonious functions.

4.1 The game

Let us consider a two-player zero-sum game in a bounded domain $\Omega \subset \mathbb{R}^N$. Pick $\Gamma_N \equiv \partial\Omega \backslash \Gamma_D$ and $\Gamma_D \subset \partial\Omega$. Then let $F : \Gamma_D \to \mathbb{R}$ be Lipschitz continuous. The game begins when a player places a token at a point $x_0 \in \overline{\Omega} \backslash \Gamma_D$, then a coin is tossed and the player who wins can move the game position to any $x_1 \in \overline{B}_\varepsilon(x_0) \cap \overline{\Omega}$.
For every turn the coin is tossed and the winner choose the new game state $x_k \in \overline{B}_\varepsilon(x_{k-1}) \cap \overline{\Omega}$. The game stops when the token has reached some $x_\tau \in \Gamma_D$ and player I earns $F(x_\tau)$, therefore player II earns $-F(x_\tau)$ (loses $F(x_\tau)$). Hence F is called the final payoff function. The strategies adopted by the players and the game tosses affect the game states $x_0, x_1, x_2, ..., x_\tau$, where every x_k except x_0 are random variables.
We have that:

1. The initial state $x_0 \in \overline{\Omega} \backslash \Gamma_D$. (Public knowledge, both player 1 and player 2 knows about it).
2. Each player i chooses an action $a_0^i \in \overline{B}_\varepsilon(0)$ that is announced to the other player, this defines an action profile $a_0^i = \{a_0^1, a_0^2\} \in \overline{B}_\varepsilon(0) \times \overline{B}_\varepsilon(0)$
3. The new state $x_1 \in \overline{B}_\varepsilon(x_0)$

[10]

In the following sections suppose that $2 \leq p < \infty$. Games that approximate the $p-$Laplacian is given by

$$\Delta_p u = div(|\nabla u|^{p-2} \nabla u).$$

Intuitively, by summing all three cases with their corresponding probability:

1. Player I, moves the token from the current position x_n with probability $\frac{\alpha}{2}$ to a new position x_{n+1} within the ball $B_{\varepsilon(x_n)}$.
2. Player II, moves the token from the current position to a new position with probability $\frac{\alpha}{2}$, in the same ball $B_{\varepsilon(x_n)}$
3. Hence the token moves randomly in the ball $B_{\varepsilon(x_n)}$ with probability $\beta = 1 - \alpha$ in the interval I $\in$ [0,1]. When the token is no longer in the domain Ω the game

has come to an end. Then at the exit position for the token x_τ player II pays the amount equal to a value of a given boundary pay-off function to player I. [2]

The expected payoff at the point can be calculated. The noise is distributed uniformly on $B_\varepsilon(x)$, in this version of tug-of-war with noise. Hence we are now allowed to use the dynamic programming principle, in the following form

$$u_\varepsilon(x) = \frac{\alpha}{2}\{\sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy, \tag{24}$$

to draw the conclusion that the game has a value and that the value is $p-$harmonious.

4.1.1 p-harmonious functions

Hence the aim is to study the functions that satisfies formula (24), when $\varepsilon > 0$ is fixed and $\alpha + \beta = 1$ where α, β are non-negative. From now on these kind of functions will be called p-harmonious functions.
Now, let $\varepsilon > 0$ be fixed and consider a bounded domain $\Omega \subset \mathbb{R}^N$. For p-harmonious functions the boundary values can be imposed by indicate a compact boundary strip of width ε by,

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^N \backslash \Omega : d(x, \partial\Omega) \leq \varepsilon\}.$$

For $x \in \Omega$ the ball $\overline{B}_\varepsilon(x)$ is not necessarily included in Ω, therefor the aim is to use the boundary strip Γ_ε instead of the boundary $\partial\Omega$.

Definition 23. *In a bounded domain Ω, let u_ε be a p-harmonious function, $\varepsilon > 0$. The boundary values are a bounded Borel function $F : \Gamma_\varepsilon \to \mathbb{R}$ if*

$$u_\varepsilon(x) \;= \frac{\alpha}{2}\{\sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy, \qquad \forall x \in \Omega,$$

where $\alpha = \frac{p-2}{p+N}$ and $\beta = \frac{2+N}{p+N}$.
Then $u_\varepsilon(x) = F(x)$, for every x in Γ_ε,

Now let us formulate 'p-harmonious'. So we know from previous that if u is harmonic, the mean-value property is satisfied.
When $\alpha = 0$ and $\beta = 1$ equation (24) will be,

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy. \tag{25}$$

When $\alpha = 1$ and $\beta = 0$ equation (24) will be,

$$u_\varepsilon = \frac{1}{2}\{\sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon\} \tag{26}$$

these equations are called *harmonious* functions and are values of the tug-of-war games.

If we let $\varepsilon \to 0$ these functions approximates solutions to the ∞-Laplacian! Remember, the p-Laplacian is given by,

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\{(p-2)\Delta_\infty u + \Delta u\} \tag{27}$$

The goal is to prove that the p-harmonious functions converge uniformly to the p-harmonic function with given boundary data and are uniquely determined by their boundary values.

4.1.2 A heuristic argument

It follows from expansion and from equation (27), that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0. \tag{28}$$

This still holds in the viscosity sense, when $\nabla u = 0$, it has been shown in [9]. By taking the average over $B_\varepsilon(x)$ of the classical Taylor expansion[7] we get,

$$u(x) - \fint_{B_\varepsilon(x)} u\,dy = -\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + \mathcal{O}(\varepsilon^3), \tag{29}$$

where u is smooth. Using the notation

$$\fint_{B_\varepsilon(x)} udy = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u\,dy.$$

Hence, notice that the maximizing direction is nearly the same as the gradient direction. By summation of the two Taylor expansions we obtain

$$\begin{aligned} u(x) - \frac{1}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} &\approx u(x) - \frac{1}{2}\left\{u\left(x + \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + u\left(x - \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right\} \\ &= -\frac{1}{2}\varepsilon^2\Delta_\infty u(x) + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{30}$$

By multiplying equation (29) and equation (30) by the constants α and β, $(\alpha+\beta = 1)$, we get

$$u(x) - \frac{1}{2}\alpha\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta\fint_{B_\varepsilon(x)} u\,dy = -\alpha\frac{\varepsilon^2}{2}\Delta_\infty u(x) - \beta\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + \mathcal{O}(\varepsilon^3)$$

7

$$u(y) = u(x) + \nabla u(x)(y-x) + \frac{1}{2}\langle D^2u(x)(y-x), (y-x)\rangle + \mathcal{O}(|y-x|^3)$$

Hence, choose α and β such that the operator in formula (28) is obtained on the right-hand side. It is from this we obtain the constants α and β that we used previously in definition (23),

$$\alpha = \frac{p-2}{p+N} \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{31}$$

So, now we have that

$$u(x) \;=\; \frac{1}{2}\alpha\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u\, dy + \mathcal{O}(\varepsilon^3) \qquad \text{as} \quad \varepsilon \to 0.$$

4.1.3 Tug-of-war and p-harmonious functions

Now we want to find the link between tug-of-war games and p-harmonious functions. Let $\varepsilon > 0$ be fixed and again consider the two-player zero-sum game tug-of-war. Place the token at a point $x_0 \in \Omega$ and toss a biased coin with probabilities α and β where $\alpha + \beta = 1$. To get heads have probability α and to get tails have probability β, since the game is tug-of-war the winner of the toss moves the game position to a new position $x_1 \in \overline{B}_\varepsilon(x_0)$. If the coin that is tossed shows tails the game state is moved to a random point in the ball $B_\varepsilon(x_0)$, according to the uniform probability. Hence, the game continues by the same way from x_1.
Thus, this game gives an infinite sequence of game states $x_0, x_1, ...$, where every x_k is a random variable. Let $x_\tau \in \Gamma_\varepsilon$ be the first point in Γ_ε in the sequence, where τ implies a hit at the boundary Γ_ε. The payoff is then $F(x_\tau)$, the given payoff function is $F : \Gamma_\varepsilon \to \mathbb{R}$, such that player I wins $F(x_\tau)$ while player II will lose $F(x_\tau)$ (-$F(x_\tau)$). Hence, despite that $\beta > 0$, or equivalently $p < \infty$ the game will end almost surely (a.s.).

$$\mathbb{P}^{x_0}_{S_I,S_{II}}(\{\omega \in H^\infty : \tau(\omega) < \infty\}) = 1$$

for any strategy.
Hence the value of the game for player I and player II is then given by:

$$\begin{aligned} \text{Player I:} \qquad u_I^\varepsilon(x_0) \;&=\; \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)] \\ \text{Player II:} \qquad u_{II}^\varepsilon(x_0) \;&=\; \sup_{S_{II}} \inf_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)]. \end{aligned}$$

Note that when the game begins at x_0 the best expected outcome each player can ensure is $u_I^\varepsilon(x_0)$ and $u_{II}^\varepsilon(x_0)$.
Let us look at the dynamic programming principle applied to the game.

Lemma 24. ***(The dynamic programming principle)***
The value function for player I satisfies the following formulas:

$$\begin{cases} u_I^\varepsilon(x_0) \;=\; \frac{\alpha}{2}\{\sup_{\overline{B}_\varepsilon(x_0)} u_I^\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_I^\varepsilon\} + \beta \fint_{B_\varepsilon(x_0)} u_I^\varepsilon\, dy, & x_0 \in \Omega \\ u_I^\varepsilon(x_0) \;=\; F(x_0), & x_0 \in \Gamma_\varepsilon \end{cases} \tag{32}$$

and hence the value functions for player II satisfies:

$$\begin{cases} u_{II}^{\varepsilon}(x_0) = \frac{\alpha}{2}\{\sup_{\overline{B}_{\varepsilon}(x_0)} u_{II}^{\varepsilon} + \inf_{\overline{B}_{\varepsilon}(x_0)} u_{II}^{\varepsilon}\} + \beta \fint_{B_{\varepsilon}(x_0)} u_{II}^{\varepsilon}\, dy, & x_0 \in \Omega \\ u_{II}^{\varepsilon}(x_0) = F(x_0), & x_0 \in \Gamma_{\varepsilon} \end{cases}$$

Proof. The proof of this lemma can be found in [7]. □

A brief explanation why the programing principle holds follows, by taking into account the expectation of the payoff at x_0. The probability to get heads is α and the probability to get tails is β, the coin is tossed fair and the game begins. By this in mind if player I wins the toss (gets heads, with probability $\frac{1}{2}$) the player steps to a point maximizing the expectation, but if player II wins (gets heads, with probability $\frac{1}{2}$) the player moves the game state to a point minimizing the expectation. If any of the players get tails the game state moves to a random point according to a uniform probability on $B_{\varepsilon}(x_0)$. By summing these different outcomes the expectation at x_0 can be obtained. Note that in general for the tug-of-war game without noise the value functions are discontinuous.[10]

Before, further reading the following theorem need to be stated.

Theorem 25. ***(Fatou's theorem)***
This follows from the theorem in (Rudin, 320).
Assume that $E \in \mathbb{R}$. If $\{f_n\}$ is a sequence of nonnegative measurable functions and

$$f(x) = \lim_{n\to\infty} \inf f_n(x),$$

where $x \in E$, then

$$\int_E f\, d\mu \;\leq\; \lim_{n\to\infty} \inf \int_E f_n\, d\mu.$$

Proof. The proof can be found in [4]. □

Theorem 26. ***(Comparison principle)***
(Rossi 2010, 344) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. If v_{ε} is a p-harmonious function with boundary values $F_v \in \Gamma_{\varepsilon}$ such that $F_v \geq F_{u_I^{\varepsilon}}$, then $v \geq u_I^{\varepsilon}$.

Proof. The following is based on the proof in (Rossi 2010, 344-345).
We want to show that:

- Player II can make the process a supermartingale by choosing a strategy according to the minimal values of v.

- Under this strategy the expectation of the process is bounded by v, this is implied by the optional stopping theorem.

- An upper bound for u_I^{ε} is provided by this process.

Then let player I follows any strategy an let player II follows a strategy S_{II}^0 such that: At a point $x_{k-1} \in \Omega$ the player chooses to step to another point that almost minimizes v, to wit, to a point $x_k \in \overline{B}_\varepsilon(x_{k-1})$ so that,

$$v(x_k) \leq \inf_{\overline{B}_\varepsilon(x_{k-1})} v + \eta 2^{-k}$$

for any fixed $\eta > 0$.
Start from the point x_0, then

$$\begin{aligned}
&\mathbb{E}_{S_I, S_{II}^0}^{x_0}[v(x_k) + \eta 2^{-k} | x_0, ..., x_{k-1}] \\
&\qquad \leq \frac{\alpha}{2}\left\{ \inf_{\overline{B}_\varepsilon(x_{k-1})} v + \eta 2^{-k} + \sup_{\overline{B}_\varepsilon(x_{k-1})} v \right\} + \beta \fint_{B_\varepsilon(x_{k-1})} v \, dy + \eta 2^{-k} \\
&\qquad \leq v(x_{k-1}) + \eta 2^{-(k-1)},
\end{aligned}$$

the strategy for player I is estimated by sup and the fact that v is p-harmonious is also used. Hence,

$$M_k = v(x_k) + \eta 2^{-k}$$

is a supermartingale. Further since $F_v \geq F_{u_I^\varepsilon}$ at Γ_ε we derive with use of Fatou's theorem (25) and the optional stopping theorem for M_k,

$$\begin{aligned}
u_I^\varepsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_1, S_{II}}^{x_0}[F_{u_I^\varepsilon}(x_\tau)] \\
&\leq \sup_{S_I} \mathbb{E}_{S_1, S_{II}^0}^{x_0}[F_v(x_\tau) + \eta 2^{-\tau}] \\
&\leq \sup_{S_I} \lim_{k\to\infty} \inf \mathbb{E}_{S_1, S_{II}^0}^{x_0}[v(x_{\tau\wedge k})\eta 2^{-(\tau\wedge k)}] \\
&\leq \sup_{S_I} \mathbb{E}_{S_1, S_{II}^0}[M_0] = v(x_0) + \eta,
\end{aligned}$$

when $\tau \wedge k = min(\tau, k)$. Since η is arbitrary the claim is proved. □

Likewise, it can be proven that u_{II}^ε is the largest p-harmonious function. Now, let player I use the strategy that always step to the point where v is almost maximized and let player II choose any strategy. Hence this leads to that $v(x_k) - \eta 2^{-k}$ is a submartingale. Thus, Fatou's theorem (25) and the optional stopping theorem prove the claim.
It is time to show that the game has a value! The uniqueness of p-harmonious functions with given boundary values, is proven by the comparison principle and the fact that the game has a value proves.

Theorem 27. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let F be a given boundary datum in Γ_ε. Then $u_I^\varepsilon = u_{II}^\varepsilon$, hence the game has a value.*

Proof. The following is based on the proof in (Rossi 2010, 345-346).
It is always true that $u_I^\varepsilon \leq u_{II}^\varepsilon$. Thus, we have to show that $u_{II}^\varepsilon \leq u_I^\varepsilon$. By using

the same method as in the proof of theorem (26): Player II follows the strategy S_{II}^0 so that $x_{k-1} \in \Omega$, remember that player II always chooses to step to a point that almost minimizes u_I^ε, to wit, to a point x_k so that

$$u_I^\varepsilon(x_k) \leq \inf_{\overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon + \eta 2^{-k}$$

for any fixed $\eta > 0$. Start from the point x_0, then it follows from the dynamic programming principle for u_I^ε and the choice of strategies that

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_I,S_{II}^0}[u_I^\varepsilon(x_k)+\eta 2^{-k}|x_0,...,x_{k-1}] \\
&\qquad \leq \frac{\alpha}{2}\{\sup_{\overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon + \inf_{\overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon + \eta 2^{-k}\} + \beta \fint_{B_\varepsilon(x_{k-1})} u_I^\varepsilon \, dy + \eta 2^{-k} \\
&\qquad = u_I^\varepsilon(x_{k-1}) + \eta 2^{-(k-1)}.
\end{aligned}$$

Hence,

$$M_k = u_I^\varepsilon(x_k) + \eta 2^{-k}$$

is a supermartingale. By using the optional stopping theorem and Fatou's theorem (25) we get,

$$\begin{aligned}
u_{II}^\varepsilon(x_0) &= \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)] \\
&\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[F(x_\tau) + \eta 2^{-\tau}] \\
&\leq \sup_{S_I} \lim_{k\to\infty} \inf \mathbb{E}^{x_0}_{S_I,S_{II}^0}[u_I^\varepsilon(x_{\tau\wedge k}) + \eta 2^{-(\tau\wedge k)}] \\
&\leq \sup_{S_I} \mathbb{E}_{S_I,S_{II}^0}[u_I^\varepsilon(x_0) + \eta] \\
&= u_I^\varepsilon(x_0) + \eta.
\end{aligned}$$

Here we used the fact that the game ends a.s., and again since $\eta > 0$ is arbitrary the proof is complete. □

With a fixed boundary datum, theorem (26) and theorem (27) implies that there exist a unique p-harmonious function.

Theorem 28. *(Rossi 2010, 346) Let $\Omega \in \mathbb{R}^n$ be a bounded open set. Then there exists a unique $p-$harmonious function in Ω with given boundary values F.*

Proof. The existence part of this theorem is accomplished since the value functions of the game are $p-$harmonious functions due to the dynamic principle. The uniqueness part of this theorem is implied by theorem (26) and theorem (27).

□

Corollary 29. *This follows from (Rossi 2010, 346). The value of the game with payoff function F coincides with the p-harmonious function with boundary values F.*

4.1.4 Maximum principles

Now we want to show that the strong comparison principle and the strong maximum principle hold for p-harmonious functions. Next theorem states that p-harmonious functions satisfy the strong maximum principle.

Theorem 30. *(Rossi 2010, 346) Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected set. If u_ε is $p-$harmonious in Ω with boundary values F, then*

$$\sup_{\Gamma_\varepsilon} F \geq \sup_{\Omega} u_\varepsilon.$$

If there exist a point $x_0 \in \Omega$ so that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is a constant in Ω.

Proof. This follows from the proof in (Rossi 2010, 346-347).
Observe that, if the maximum is attained inside the domain, then all the quantities in the definition of a p-harmonious function must be equal to the maximum. This is only possible in the connected domain if the function is a constant. By this in mind we begin with the observation of a p-harmonious function u_ε with a boundary datum F satisfying

$$\sup_{\Omega} |u_\varepsilon| \leq \sup_{\Gamma_\varepsilon} |F|.$$

Then, suppose that there exists a point $x_0 \in \Omega$ so that

$$u_\varepsilon(x_0) = \sup_{\Omega} u_\varepsilon = \sup_{\Gamma_\varepsilon} F.$$

Using definition (23) of a $p-$harmonious function and obtain

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy.$$

Thus, since $u_\varepsilon(x_0)$ is the maximum, the terms on the right-hand side,

$$\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon, \qquad \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon, \qquad \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

must be smaller than or equal to $u_\varepsilon(x_0)$. Consequently,

$$u_\varepsilon(x) = u_\varepsilon(x_0) = \sup_{\Omega} u_\varepsilon, \tag{33}$$

where $x \in B_\varepsilon(x_0)$ when $p > 2$. For $p > 2$ we have that $\alpha, \beta > 0$ and thereby the terms must be equal to $u_\varepsilon(x_0)$. The argument can be repeated for each $x \in B_\varepsilon(x_0)$ and by continuing in this way, the result can be extended to the hole domain since Ω is a connected. Hence this implies that for every $p > 2$, u is a constant.
At last if $p = 2$ then formula (33) holds for nearly every $x \in B_\varepsilon(x_0)$ and therefore for nearly every x in the whole domain. Thus,

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy,$$

since this is true at every point in Ω and since u is a constant nearly everywhere, it follows that u is a constant everywhere. ∎

Before the next theorem is presented, note that $p-$harmonious functions with continuous boundary values satisfy the strong comparison principle. The validity of the strong comparison principle is not known for the p-harmonic functions in $\mathbb{R}^N, N \leq 3$.

Theorem 31. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected set. Let u_ε and v_ε be p-harmonious functions with continuous boundary values $F_u \geq F_v$ in Γ_ε. Then, if there exists a point $x_0 \in \Omega$ so that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$ it follows that*

$$u_\varepsilon = v_\varepsilon \quad in \ \ \Omega,$$

and, additionally, the boundary values satisfy

$$F_u = F_v \quad in \quad \Gamma_\varepsilon.$$

Proof. The following is based on the proof in (Rossi 2010, 348).
Note that the proof is based on the fact that $p < \infty$, since the strong comparison principle does not hold for infinity harmonic functions. By theorem (26) and corollary (29), $F_u \geq F_v \implies u_\varepsilon \geq v_\varepsilon$. By the definition of a p-harmonious function we obtain,

$$\begin{aligned} u_\varepsilon(x_0) &= \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy, \\ v_\varepsilon(x_0) &= \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy, \end{aligned}$$

Now compare the right-hand sides. Since $u_\varepsilon \geq v_\varepsilon$, we have

$$\begin{aligned} \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon &\leq \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon, \\ \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon &\leq \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon, \\ \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy &\leq \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy. \end{aligned} \tag{34}$$

Since $u_\varepsilon(x_0) = v_\varepsilon(x_0)$ there must be equalities in (34). Especially there is equality in the third inequality in (34), and hence,

$$u_\varepsilon = v_\varepsilon \quad \text{almost everywhere in} \quad B_\varepsilon(x_0).$$

The fact that the domain Ω is connected implies that

$$u_\varepsilon = v_\varepsilon \quad \text{almost everywhere in} \quad \Omega \cup \Gamma_\varepsilon.$$

Especially, $F_u = F_v$ everywhere in Γ_ε, since F_u and F_v are continuous. Since the boundary values coincide, the uniqueness result from theorem (28), shows that $u_\varepsilon = v_\varepsilon$ everywhere in Ω. ∎

4.1.5 Letting $\varepsilon \to 0$

It is time to show that p-harmonious functions with a fixed boundary datum converge to the unique $p-$harmonic function. Suppose that the bounded domain Ω satisfies the boundary regularity condition in the next theorem. This we need to define in order to prove that the p-harmonious functions converge to the unique solution of the Dirichlet problem for the $p-$Laplacian in Ω, with fixed continuous boundary values.

Theorem 32. ***(Boundary regularity condition)***
There exists $\delta' > 0$ and $\mu \in (0,1)$ so that, $\forall \delta \in (0, \delta']$ and $y \in \partial\Omega$, there exists a ball

$$B_{\mu\delta}(z) \subset B_\delta(y) \backslash \Omega.$$

When Ω is a Lipschitz domain, it satisfies the requirement.

Theorem 33. *Let F be a continuous function and let Ω be a bounded domain satisfying the condition in theorem (32). Then consider the unique viscosity solution u to*

$$\begin{aligned} div(|\nabla u|^{p-2}\nabla u)(x) &= 0, \quad\ \ x \ \ in \ \ \Omega, \\ u(x) &= F(x), \ x \ \ in \ \ \partial\Omega. \end{aligned} \tag{35}$$

Let u_ε be the unique p-harmonious function with boundary values F. Then,

$$u_\varepsilon \to u \qquad uniformly \ in \qquad \overline{\Omega} \qquad as \quad \varepsilon \to 0.$$

Since the limit in theorem (33) entirely depends on the values of F on $\partial\Omega$, the same limit is given for any continuous extension of $F|_{\partial\Omega}$ to Γ_{ε_0}.

Theorem 34. *Let Ω be a bounded open set and let $\Omega \cup \Gamma_\varepsilon \subset \Omega'$. Let u be p-harmonious with non-vanishing gradient $\nabla u \neq 0$ in Ω' and let u_ε be the p-harmonious function in Ω with boundary values $u \in \Gamma_\varepsilon$. Then*

$$u_\varepsilon \to u \qquad uniformly \ in \qquad \overline{\Omega} \qquad as \quad \varepsilon \to 0.$$

Proof. The following is based on the proof in (Rossi 2010, 349-351). Since the value of the game coincides with the p-harmonious function, we can use a game-theoretic approach. Remember that u satisfies,

$$u(x) = \frac{\alpha}{2}\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\} + \beta \fint_{B_\varepsilon(x)} u\, dy + \mathcal{O}(\varepsilon^3) \tag{36}$$

with a uniform error term for $x \in \overline{\Omega}$ as $\varepsilon \to 0$. Note that the error term is uniform due to the assumption on u.
Now, suppose that $p > 2$, leads to that $\alpha > 0$ such that the strategies are relevant. Let

player II follows the strategy S^0_{II} such that, at a point x_{k-1}, the player move to a point that minimizes u, to wit, to a point $x_k \in B_\varepsilon(x_{k-1})$ so that

$$u(x_k) \;=\; \inf_{\overline{B}_\varepsilon(x_{k-1})} u(y).$$

Pick $C_1 > 0$ so that $|\mathcal{O}(\varepsilon^3)| \leq C_1\varepsilon^3$. With strategy S^0_{II}, the supermartingale will be

$$M_k = u(x_k) - C_1 k\varepsilon^3.$$

In fact,

$$\begin{aligned}
&\mathbb{E}_{S_I,S^0_{II}}(u(x_k) - C_1k\varepsilon^3 | x_0, ..., x_{k-1}) \\
&\qquad\qquad \leq \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_{k-1})} u + \inf_{\overline{B}_\varepsilon(x_{k-1})} u\right\} + \beta \fint_{B_\varepsilon(x_{k-1})} u\,dy - C_1k\varepsilon^3 \qquad (37)\\
&\qquad\qquad \leq u(x_{k-1}) - C_1(k-1)\varepsilon^3.
\end{aligned}$$

Note that the first inequality follows from the choice of strategy and the second follows from equation (36). By using Fatou's theorem (25) and the optional stopping theorem for supermartingales we can estimate $u^\varepsilon_{II}(x_0)$.

$$\begin{aligned}
u^\varepsilon_{II}(x_0) =& \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)] \\
\leq& \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[u(x_\tau)] \\
\leq& \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[u(x_\tau) + C_1\tau\varepsilon^3 - C_1\tau\varepsilon^3] \\
\leq& \sup_{S_I}(\lim_{k\to\infty}\inf \mathbb{E}^{x_0}_{S_I,S^0_{II}}[u(x_{\tau\wedge k}) - C_1(\tau\wedge k)\varepsilon^3] + C_1\varepsilon^3\mathbb{E}^{x_0}_{S_I,S^0_{II}}[\tau]) \\
\leq& u(x_0) + C_1\varepsilon^3 \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[\tau].
\end{aligned}$$

For $u_\varepsilon = u^\varepsilon_{II} = u^\varepsilon_I$,the analogous argument for player I and the above inequality leads to,

$$u(x_0) - C_1\varepsilon^3 \inf_{S_{II}} \mathbb{E}^{x_0}_{S^0_I,S_{II}}[\tau] \quad\leq\quad u_\varepsilon(x_0) \quad\leq\quad u(x_0) + C_1\varepsilon^3 \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[\tau]. \qquad (38)$$

Hence, to finish the proof we need to prove that there exists a C, when letting $\varepsilon \to 0$ such that,

$$\mathbb{E}^{x_0}_{S_I,S^0_{II}}[\tau] \leq C\varepsilon^{-2}. \qquad (39)$$

To determine this, we show that for a $\varepsilon > 0$ small enough, the following is a supermartingale

$$\tilde{M}_k = -u(x_k)^2 + u(x_0)^2 + C_2\varepsilon^2 k.$$

If the toss is won by player II, we obtain,

$$u(x_k) - u(x_{k-1}) \leq -C_3\varepsilon,$$

since $\nabla u \neq 0$, where $C_3 = \inf_{x\in\Omega} |\nabla u|$.
Then

$$\begin{aligned}\mathbb{E}_{S_I,S_{II}^0}&[(u(x_k)-u(x_{k-1}))^2|x_0,...,x_{k-1}]\\ &\geq \frac{\alpha}{2}((-C_3\varepsilon)^2+0)+\beta\cdot 0\\ &=\frac{\alpha}{2}C_3^2\varepsilon^2.\end{aligned} \tag{40}$$

So we obtain,

$$\begin{aligned}&\mathbb{E}_{S_I,S_{II}^0}[\tilde{M}_k-\tilde{M}_{k-1}|x_0,...,x_{k-1}]\\ &\quad= \mathbb{E}_{S_I,S_{II}^0}[-u(x_k)^2+u(x_{k-1})^2+C_2\varepsilon^2|x_0,...,x_{k-1}]\\ &\quad= \mathbb{E}_{S_I,S_{II}^0}[-(u(x_k)-u(x_{k-1}))^2|x_0,...,x_{k-1}]\\ &\qquad- \mathbb{E}_{S_I,S_{II}^0}[2(u(x_k)-u(x_{k-1}))u(x_{k-1})|x_0,...,x_{k-1}]+C_2\varepsilon^2.\end{aligned} \tag{41}$$

Suppose that $u < 0$, since $u(x_{k-1})$ is determined by the point x_{k-1} the second term on the right-hand side can be estimated as,

$$\begin{aligned}-\mathbb{E}_{S_I,S_{II}^0}&[2(u(x_k)-u(x_{k-1}))u(x_{k-1})|x_0,...,x_{k-1}]\\ &= -2u(x_{k-1})(\mathbb{E}_{S_1,S_{II}^0}[u(x_k)|x_0,...,x_{k-1}]-u(x_{k-1}))\\ &\leq 2\parallel u\parallel_\infty C_1\varepsilon^3,\end{aligned}$$

where the inequality follows from equation (36). Equation (40) and equation (41) implies that,

$$\mathbb{E}_{S_I,S_{II}^0}[\tilde{M}_k-\tilde{M}_{k-1}|x_0,...,x_{k-1}]\leq 0,$$

when

$$-\frac{\varepsilon^2\alpha}{2}C_3^2+2\parallel u\parallel_\infty C_1\varepsilon^3+C_2\varepsilon^2\leq 0.$$

Choose C_2 so that $C_3 \geq 2\sqrt{\frac{C_2}{\alpha}}$ and pick $\varepsilon < \frac{C_2}{2\|u\|_\infty C_1}$ then the above equation holds. Hence, M_k is a supermartingale. Note that we assumed that $p > 2$, then $\alpha > 0$.
By the optimal stopping theorem for supermartingales

$$\mathbb{E}_{S_I,S_{II}^0}[\tilde{M}_{\tau\wedge k}]\leq \tilde{M}_0=0,$$

hence,

$$C_2\varepsilon^2\mathbb{E}^{x_0}_{S_I,S_{II}^0}[\tau\wedge k]\leq \mathbb{E}^{x_0}_{S_I,S_{II}^0}[u(x_{\tau\wedge k})^2-u(x_0)^2].$$

Since u is bounded in Ω, the result follows by passing to the limit with k.
At last, when $p = 2$ the mean-value property holds due to the classical mean-value property for harmonic functions. The claim follows by doing the proof ones again until equation (38), but without the correction term.

□

Until now we have shown the convergence result for p-harmonious functions when $\nabla u \neq 0$, now it is time to treat the Dirichlet problem immediately with a different proof for the uniform convergence.

Lemma 35. *The following lemma is a variant of the classical Arzela-Ascoli compactness lemma from (Rossi 2010, 351-352)*
Let $\{u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ *be a set of functions such that:*

1. *There exists a* $C > 0$ *such that* $|u_\varepsilon(x)| < C,\ \forall \varepsilon > 0$ *and* $\forall x \in \overline{\Omega}$.
2. *Given* $\eta > 0$, *there exists constants* r_0 *and* ε_0 *such that,* $\forall \varepsilon < \varepsilon_0$ *and any* $x, y \in \overline{\Omega}$ *with* $|x - y| < r_0$, *it holds that*

$$|u_\varepsilon(x) - u_\varepsilon(y)| < \eta.$$

Thus, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ *and a subsequence denoted by* $\{u_\varepsilon\}$ *such that*

$$u_\varepsilon \to u \quad \textit{uniformly} \quad \textit{in} \quad \overline{\Omega} \quad \textit{as} \quad \varepsilon \to 0.$$

Proof. The following is from (Rossi 2010, 352).
Let $X \subset \overline{\Omega}$ be a dense countable set and u be a candidate for the uniform limit. A diagonal procedure provides a subsequence $\{u_\varepsilon\}$ to converges for all $x \in X$, since the functions are uniformly bounded. Then limit is denoted by $u(x)$. Note that u is only defined for $x \in X$. Hence, let $\eta > 0$, then there is a r_0 such that, for any $x, y \in X$ where $|x - y| < r_0$, it holds that

$$|u(x) - u(y)| < \eta.$$

Thus, u can be extended continuously in the whole closed domain $\overline{\Omega}$ by letting,

$$u(z) = \lim_{\substack{x \to z, \\ x \in X,}} u(x).$$

Now pick a finite covering,

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} B_r(x_i)$$

with $\varepsilon_0 > 0$ to show that $\{u_\varepsilon\}$ to u uniformly we have

$$|u_\varepsilon(x) - u_\varepsilon(x_i)|, \qquad |u(x) - u(x_i)| < \frac{\eta}{3}, \qquad \forall x \in\ B_r(x_i) \ \text{ and } \ \varepsilon < \varepsilon_0$$

as well as,

$$|u_\varepsilon(x_i) - u(x_i)| < \frac{\eta}{3}, \qquad \forall x_i \quad \text{and} \quad \varepsilon < \varepsilon_0$$

The last inequality is due to $N < \infty$.
Hence, for any $x \in \overline{\Omega}$, one can find x_i such that $x \in B_r(x_i)$ and

$$|u_\varepsilon(x) - u(x)| \ \leq \ |u_\varepsilon(x) - u_\varepsilon(x_i)| + |u_\varepsilon(x_i) - u(x_i)| + |u(x_i) - u(x)| \ < \ \eta$$

for all $\varepsilon < \varepsilon_0$, where ε_0 is independent of x. □

Lemma 36. *A p-harmonious function u_ε with boundary values F satisfies*

$$\min_{y\in\Gamma_\varepsilon} F(y) \le u_\varepsilon(x) \le \max_{y\in\Gamma_\varepsilon} F(y). \tag{42}$$

Further, the goal is to show that $p-$harmonious functions are asymptotically uniformly continuous.

Lemma 37. *Let $\{u_\varepsilon\}$ be a family of p-harmonious functions in Ω with a fixed continuous boundary datum F. Then this family satisfies equation (36)*

$$u(x) = \frac{\alpha}{2}\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\} + \beta \fint_{B_\varepsilon(x)} u\, dy + \mathcal{O}(\varepsilon^3)$$

from lemma (35).

Proof. The following is based on the proof in (Rossi 2010, 353-354). By the continuity of F the case $x, y \in \Gamma_\varepsilon$ follows. Hence, let us study the case when $x \in \Omega$, $y \in \Gamma_\varepsilon$ and $x, y \in \Omega$.
The proof can be looked at in three steps.

1. Let $x \in \Omega$ and $y \in \Gamma_\varepsilon$. Close to a solution for the $p-$Dirichlet problem in an annular domain, use a comparison with a $p-$harmonious function. Then the $p-$harmonious function with the boundary datum F, is bounded by a slightly smaller constant then the maximum of the boundary values, close to $y \in \Gamma_\varepsilon$.
2. By iterating the above reasoning we can show that the p-harmonious function is close to the boundary values close to $y \in \Gamma_\varepsilon$, for $\varepsilon > 0$, small enough.
3. Extend the result to the case when $x, y \in \Omega$ by translation, take the boundary values from the strip.

Now, let $B_{\mu\delta}(z) \subset B_\delta(z)\backslash\Omega$, $\delta < \delta'$, by formula (33). Consider the problem,

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in B_{4\mu}(z)\backslash\overline{B}_{\mu\delta}(z), \\ u(x) = \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F & x \in \partial B_{\mu\delta}(z), \\ u(x) = \sup_{\Gamma_\varepsilon} F, & x \in \partial B_{4\delta}(z). \end{cases} \tag{43}$$

The explicit radially symmetric solution is

$$\begin{aligned} u(r) &= ar^{-\frac{(n-p)}{(p-1)}} + b, \quad \text{where } \ p \neq n \\ u(r) &= a\, log(r) + b, \quad \text{where } \ p = n, \end{aligned} \tag{44}$$

where $r = |x - z|$. Now extend the solutions to $B_{4\delta+2\varepsilon}(z)\backslash\overline{B}_{\mu\delta-2\varepsilon}(z)$. Since $\nabla u \neq 0$ by theorem (34) we have p-harmonious functions $\{u^\varepsilon_{fund}\} \in B_{4\delta+\varepsilon}(z)\backslash\overline{B}_{\mu\delta-\varepsilon}(z)$ with boundary values u, then

$$u^\varepsilon_{fund} \to u \quad \text{uniformly} \quad \text{in } \overline{B}_{4\delta+\varepsilon}(z)\backslash B_{\mu\delta-\varepsilon}(z) \quad \text{as } \varepsilon \to 0.$$

Thus,

$$|u^{\varepsilon}_{fund} - u| = o(1) \quad \text{in} \quad \overline{B}_{4\delta+\varepsilon}(z) \backslash B_{\mu\delta-\varepsilon}(z) \quad \text{where} \quad o(1) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

If $\varepsilon > 0$ is small enough, the comparison principle (theorem (26)) implies that in $B_\delta(y) \cap \Omega \subset B_{2\delta}(z) \cap \Omega$ there is a $\theta \in (0,1)$ so that,

$$u^{\varepsilon} \ \leq \ u^{\varepsilon}_{fund} + o(1) \ \leq \ u + o(1) \ \leq \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F + \theta\bigg(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F\bigg).$$

Note that $0 < \theta < 1$ does not depend on δ, this can be seen by determine a, b in equation (44).

Now, to the second step of the proof. We want to solve the $p-$harmonic function in $B_\delta(z) \backslash \overline{B}_{\mu\frac{\delta}{4}}(z)$ with boundary values $\sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F$ at $\partial B_{\mu\frac{\delta}{4}}(z)$, from the previous steps we have

$$\sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F + \theta\bigg(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F\bigg)$$

at $\partial B_\delta(z)$. The comparison principle and the explicit solution implies that, for a $\varepsilon > 0$ small enough, we obtain

$$u_\varepsilon \ \leq \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F + \theta^2\bigg(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F\bigg) \quad \text{in} \ \ B_{\frac{\delta}{4^k}}(y) \cap \Omega.$$

By iterating we see that for $\varepsilon > 0$ small enough, we have

$$u_\varepsilon \ \leq \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F + \theta^k\bigg(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta(y)\cap\Gamma_\varepsilon}} F\bigg) \quad \text{in} \ \ B_{\frac{\delta}{4^k}}(y) \cap \Omega.$$

Hence, now we have an upper bound for u_ε. To get the lower bound one uses an equal argument.
Since we have shown that for any $\eta > 0$, one can choose $\delta > 0$ small enough , a k large enough and $\varepsilon > 0$ small enough, thus for $x \in \Omega$, $y \in \Gamma_\varepsilon$ with $|x-y| < \frac{\delta}{4^k}$ it is true that

$$|u_\varepsilon(x) - F(y)| < \eta. \tag{45}$$

By this formula we see that, when $y \in \Gamma_\varepsilon$ the second condition in lemma (35) is satisfied. Now let the estimate be extended to the interior of the domain. Pick δ small enough and k large enough such that

$$|F(x') - F(y')| < \eta, \tag{46}$$

where $|x'-y'| < \frac{\delta}{4^k}$ and for $\varepsilon > 0$ small enough such that equation (45) is satisfied. Take a smaller domain

$$\tilde{\Omega} = \bigg\{ z \in \Omega : d(z, \partial\Omega) > \frac{\delta}{4^{k+2}} \bigg\}$$

with the boundary strip,

$$\tilde{\Gamma} = \left\{ z \in \overline{\Omega} : d(z, \partial\Omega) \leq \frac{\delta}{4^{k+2}} \right\}.$$

Assume that $x, y \in \Omega$ with $|x-y| < \frac{\delta}{4^{k+2}}$, if $x, y \in \tilde{\Gamma}$ we have the estimate

$$|u_\varepsilon(x) - u_\varepsilon(y)| \ \leq \ 3\eta, \tag{47}$$

by using equation (45) and by comparing the values of x, y to the closest boundary values. If we let $x, y \in \tilde{\Omega}$ we can define

$$\tilde{F}(z) = u_\varepsilon(z - x + y) + 3\eta \quad \text{in } \tilde{\Gamma}.$$

Thus, by equations (45), (46) and (47) we obtain

$$\tilde{F}(z) \geq u_\varepsilon(z) \ \text{ in } \ \tilde{\Gamma}.$$

We want to solve the $p-$harmonious function $\tilde{u_\varepsilon} \ \in \tilde{\Omega}$ with boundary values $\tilde{F} \ \in \tilde{\Gamma}$. At last, by the uniqueness and the comparison principle we derive that,

$$u_\varepsilon(x) \ \leq \ \tilde{u}_\varepsilon(x) \ = \ u_\varepsilon(x - x + y) + 3\eta = u_\varepsilon(y) + 3\eta \quad \text{in } \tilde{\Omega}.$$

This finish the proof, to get the lower bound one can use the argument in a similar way. □

To show that the process is a supermartingale, small corrections are made, at every step. The expectation of the stopping time τ need to be estimated in order to show that the effect of the correction is small enough, in the long run. In a large annular domain, where the outer boundary have reflecting conditions, there is a random walk for which τ is bounded by the exit time τ^*.

Lemma 38. *(Rossi 2010, 355) Let us consider an annular domain $B_R(y)\backslash\overline{B}_\delta(y)$ and a random walk such that, when at x_{k-1}, the next point x_k is chosen according to a uniform distribution at $B_\varepsilon(x_{k-1}) \cap B_R(y)$. Let*

$$\tau^* = \inf\{k : x_k \in \overline{B}_\delta(y)\}.$$

Then

$$\mathbb{E}^{x_0}(\tau^*) \ \leq \ \frac{C(\frac{R}{\delta})d(\partial B_\delta(y), x_0) + o(1)}{\varepsilon^2}, \qquad o(1) \ \ \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{48}$$

where $x_0 \in B_R(y)\backslash\overline{B}_\delta(y)$.

Proof. The following is based on the proof in (Rossi 2010, 355-356). Suppose that

$$g_\varepsilon(x) = \mathbb{E}^x(\tau^*),$$

where g_ε satisfies the dynamic programing principle

$$g_\varepsilon(x) = \fint_{B_\varepsilon(x)\cap B_R(y)} g_\varepsilon \, dz + 1,$$

since, when making a step to one of the neighboring points a random, the number of steps always increases by one. Then if we let $v_\varepsilon(x) = \varepsilon^2 g_\varepsilon(x)$ one get,

$$v_\varepsilon(x) = \fint_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon \, dz + \varepsilon^2.$$

By this equation we have a suggestion to the problem

$$\begin{cases} \Delta v(x) = -2(n+2), & x \in B_{R+\varepsilon}(y)\backslash\overline{B}_\delta(y), \\ v(x) = 0, & x \in \partial B_\delta(y,) \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial B_{R+\varepsilon}(y), \end{cases} \tag{49}$$

where the normal derivative is denoted $\frac{\partial v}{\partial \nu}$. In fact, when $B_\varepsilon(x) \subset B_{R+\varepsilon}(y)\backslash\overline{B}_\delta(y)$, the mean value property is satisfied by the solution to this problem by classical calculation,

$$v(x) = \fint_{B_\varepsilon(x)} v \, dz + \varepsilon^2. \tag{50}$$

Note that in problem (49) the solution in $r = |x-y|$ is strictly increasing, positive and radially symmetric. Hence, the solution is of the form:

$$\begin{cases} v(r) = -ar^2 - br^{2-n} + c, & \text{for } n > 2, \\ v(r) = -ar^2 - blog(r) + c, & \text{for } n = 2. \end{cases}$$

Let the function be extended as a solution to the same equation to $\overline{B}_\delta(y)\backslash\overline{B}_{\delta-\varepsilon}(y)$. Hence, v satisfies equation (50), for each $B_\varepsilon(x) \subset B_{R+\varepsilon}(x)\backslash\overline{B}_{\delta-\varepsilon}(y)$. It is true for each $x \in B_R(y\backslash\overline{B}_\delta(y))$ that

$$\fint_{B_\varepsilon(x)\cap B_R(y)} v \, dz \le \fint_{B_\varepsilon(x)} v \, dz = v(x) - \varepsilon^2,$$

since v increases in r. Hence,

$$\mathbb{E}[v(x_k) + k\varepsilon^2 | x_0, ..., x_{k-1}] = \fint_{B_\varepsilon(x_{k-1})} v \, dz + k\varepsilon^2 = v(x_{k-1}) + (k-1)\varepsilon^2$$

for $B_\varepsilon(x_{k-1}) \subset B_R(y)\backslash\overline{B}_{\delta-\varepsilon}(y)$. Hence if $B_{\varepsilon(x_{k-1})}\backslash B_R(y) \neq \emptyset$, then

$$\begin{aligned}\mathbb{E}[v(x_k) + k\varepsilon^2 | x_0, ..., x_{k-1}] &= \fint_{B_\varepsilon(x_{k-1})\cap B_R(y)} v\,dz + k\varepsilon^2 \\ &\leq \fint_{B_\varepsilon(x_{k-1})} v\,dz \\ &= v(x_{k-1}) + (k-1)\varepsilon^2,\end{aligned}$$

so, $v(x_k) + k\varepsilon^2$ is a supermartingale. The optional stopping theorem is,

$$\mathbb{E}^{x_0}[v(x_{\tau^*\wedge k}) + (\tau^* \wedge k)\varepsilon^2] \leq v(x_0). \tag{51}$$

Since $x_{\tau^*} \in \overline{B}_\delta(y)\backslash\overline{B}_{\delta-\varepsilon}(y)$, we obtain

$$0 \leq -\mathbb{E}^{x_0}[v(x_{\tau^*})] \leq o(1).$$

Moreover, the following estimate holds for the solution to the problem (49).

$$0 \leq v(x_0) \leq C\left(\frac{R}{\delta}\right) d(\partial B_\delta(y), x_0).$$

By passing to a limit with k in formula (51) we get,

$$\varepsilon^2\mathbb{E}^{x_0}[\tau^*] \leq v(x_0) - \mathbb{E}[u(x_{\tau^*})] \leq C\left(\frac{R}{\delta}\right) d(\partial B_\delta(y), x_0) + o(1).$$

□

Remark : The error term $o(1)$ can be improved to $Clog(1+\varepsilon)$, by estimating the dominating terms br^{-N+2} or $blog(r)$ of the explicit solution to problem (49) nearby $r = \delta$, the following obeys

$$|\mathbb{E}^{x_0}[v(x_{\tau^*})]| \leq Clog(1+\varepsilon).$$

Lemma 39. *Let Ω satisfies an exterior sphere condition such that, for each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n\backslash\Omega$, so that $y \in \partial B_\delta(z)$. Let F be Lipschitz continuous in Γ_ε. The p-harmonious function u_ε with boundary datum F satisfies*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq Lip(F)\delta + C\left(\frac{R}{\delta}\right)(|x-y|) + o(1), \tag{52}$$

$\forall\delta > 0$ small enough and $\forall x, y \in \Omega \cup \Gamma_\varepsilon$.

Proof. The following is based on the proof in (Rossi 2010, 357-358). There are two cases, the first is clear, $x, y \in \Gamma_\varepsilon$. So, the second case $x \in \Omega$ and $y \in \Gamma_\varepsilon$ additionally $x, y \in \Omega$, will be shown by using the relation to games.
Let $x \in \Omega$ and $y \in \Gamma_\varepsilon$, the exterior sphere condition tells us that $\exists B_\delta(z) \subset \mathbb{R}^n\backslash\Omega$ so that

$y \in \partial B_\delta(z)$. If player I choose the strategy S_I^z of pulling towards z. Then the following is a supermartingale

$$M_k = |x_k - z| - C\varepsilon^2 k$$

where C is a constant large enough independent of ε. In fact,

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_k - z|\ |x_0, ..., x_{k-1}] \\
&\qquad \leq \frac{\alpha}{2}\left\{|x_k - z| + \varepsilon + |x_k - z| - \varepsilon\right\} + \beta \fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \\
&\qquad \leq |x_{k-1} - z| + C\varepsilon^2,
\end{aligned}$$

where the strategy determines the first inequality and the second inequality is determined by the estimate

$$\fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \ \leq\ |x_{k-1} - z| + C\varepsilon^2.$$

This leads to,

$$\mathbb{E}^{x_0}_{S_1^z, S_{II}}[|x_\tau - z|] \ \leq\ |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{S_1^z, S_{II}}[\tau] \tag{53}$$

by the optional stopping theorem.
By lemma (38), we use the stopping time to compute $\mathbb{E}^{x_0}_{S_1^z, S_{II}}[\tau]$. Let player I pull towards z and let player II uses any strategy. Hence we obtain,

$$\mathbb{E}^{x_{k-1}}_{S_1^z, S_{II}} |x_k - z| \leq |x_{k-1} - z|$$

then we can show that the tug-of-war take place. Though, if the random walk happens, then

$$\mathbb{E}^{x_{k-1}}_{S_1^z, S_{II}} |x_k - z| \geq |x_{k-1} - z|.$$

Hence, the expectation can be bounded of the original process by considering an appropriate random walk in $B_R(z) \backslash B_\delta(z)$ for $B_R(z)$ so that $\Omega \subset B_{\frac{R}{2}}(z)$. The successor x_{k+1} is chosen after a uniform probability in $B_R(z) \cap B_\varepsilon(x)$, if $x_k \in B_R(z) \backslash \overline{B}_\delta(z)$.
Hence, by formula (48) we have

$$\varepsilon^2 \mathbb{E}^{x_0}_{S_I^z, S_{II}}[\tau] \ \leq\ \varepsilon^2 \mathbb{E}^{x_0}_{S_I^z, S_{II}}[\tau^*] \ \leq\ C\left(\frac{R}{\delta}\right)(d(\partial B_\delta(z), x_0) + o(1)).$$

Since we have that $y \in \partial B_\delta(z)$,

$$d(\partial B_\delta(z), x_0) \ \leq\ |y - x_0|.$$

Formula (53) implies that

$$\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_\tau - z|] \ \leq\ C\left(\frac{R}{\delta}\right)(|x_0 - y| + o(1)).$$

Then we obtain,

$$\begin{aligned} F(z) - C\left(\frac{R}{\delta}\right)(|x-y| + o(1)) &\leq \mathbb{E}^{x_0}_{S_I^z, S_{II}}[F(x_\tau)] \\ &\leq F(z) + C\left(\frac{R}{\delta}\right)(|x-y| + o(1)). \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{S_I}\inf_{S_{II}} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)] \;&\geq\; \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I^z, S_{II}}[F(x_\tau)] \\ &\geq\; F(z) - C\left(\frac{R}{\delta}\right)(|x_0-y| + o(1)) \\ &\geq\; F(y) - Lip(F)\delta - C\left(\frac{R}{\delta}\right)(|x_0-y| + o(1)). \end{aligned}$$

Note that, if player II chooses a strategy that points to z, one can achieve the upper bound and hence formula (52) follows.

At last, let the game starts at x and let the strategies for player I and player II be fixed. Let $x, y \in \Omega$. The virtual game that starts at point y is defined as:
Use the same coin tosses and random steps as in x. Then , player I chooses the strategy S_I^v and player II chooses strategy S_{II}^v , to wit, at game position y_{k-1}, on of the players chooses a step that would be taken at x_{k-1} in the game starting at x. The game goes one like this, until the first time $y_k \in \Gamma_\varepsilon$ or $x_k \in \Gamma_\varepsilon$, at that point we got $|x_k - y_k| = |x - y|$. Then we can apply the previous steps that worked for $x_k \in \Omega$, $y_k \in \Omega$ or $x_k, y_k \in \Gamma_\varepsilon$. □

Corollary 40. *Let $\{u_\varepsilon\}$ be a family of p-harmonious functions with a fixed continuous boundary datum F. Then there exists a uniformly continuous u and a subsequence $\{u_\varepsilon\}$ so that*

$$u_\varepsilon \;\to\; u \quad \textit{uniformly} \quad \textit{in} \quad \overline{\Omega}.$$

Now to the main result, we want to show that the limit u in corollary (40) is a solution to formula (35) in theorem (33). The plan is to work in the viscosity sense to show that the limit is a viscosity subsolution and supersolution. This ideas is from [8], where the p-harmonious functions are characterized in terms of asymptotic expansions.

Definition 41. *Let $1 < p < \infty$ then consider the following equation,*

$$-div(|\nabla u|^{p-2}\nabla u) = 0.$$

Thus,

1. A lower semicontinuous function u is a viscosity supersolution if, $\forall \phi \in C^2$ so that ϕ touches u at $x \in \Omega$ strictly from below with $\nabla\phi(x) \neq 0$, we obtain

$$-(p-2)\Delta_\infty\phi(x) - \Delta\phi(x) \;\geq\; 0.$$

2. An upper semi-continuous function u is a viscosity subsolution if, $\forall \phi \in C^2$ so that ϕ touches u at $x \in \Omega$ strictly from above with $\nabla \phi(x) \neq 0$, we obtain

$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \ \leq \ 0.$$

3. If u is both a subsolution and a supersolution, it is a viscosity solution.

Theorem 42. *Let F be a continuous function and let Ω be a bounded domain satisfying $u_\varepsilon(x) = u_{\varepsilon_0} = \sup_\Omega u_\varepsilon$. Then the uniform limit u of p-harmonious functions $\{u_\varepsilon\}$ is a viscosity solution to*

$$\begin{aligned} div(|\nabla u|^{p-2}\nabla u)(x) = 0, \qquad & x \in \Omega, \\ u(x) = F(x), \quad & x \in \partial\Omega. \end{aligned}$$

Proof. The following is based on the proof in (Rossi 2010, 359-360).
Due to lemma (37) we have that $u = F$ on $\partial\Omega$. Hence we want to show that u is p-harmonious in the viscosity sense in Ω.
Pick a point $x \in \Omega$ and a function $\phi \in C^2$ defined in a neighborhood of x. Let ϕ attains its minimum in $\overline{B}_\varepsilon(x_0)$ at a point x_1^ε.

$$\phi(x_1^\varepsilon) = \min_{y \in \overline{B}_\varepsilon(x)} \phi(y).$$

From the Taylor expansion in [8] we have that

$$\begin{aligned} &\frac{\alpha}{2}\left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi(y) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y) \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy - \phi(x) \\ &\geq \frac{\beta\varepsilon^2}{2(n+2)}\left((p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right) + o(\varepsilon^2). \end{aligned} \tag{54}$$

Hence, assume that ϕ touches u at x strictly from below and that $\nabla\phi(x) \neq 0$. Note that it is enough to test with these kind of functions, according to definition (41).
By uniform convergence, there exists a sequence $\{x_\varepsilon\}$ that converges to x, such that, $u_\varepsilon - \phi$ has its approximated minimum at x_ε. To wit, for $\eta_\varepsilon > 0$ $\exists x_\varepsilon$ so that,

$$u_\varepsilon(x) - \phi(x) \ \geq \ u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon.$$

Thus, if $\tilde{\phi} = \phi - u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$, suppose that $\phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon)$. Hence, recall that u_ε is p-harmonious, then we get

$$\eta_\varepsilon \geq -\phi(x_\varepsilon) + \frac{\alpha}{2}\left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi + \min_{y \in \overline{B}_\varepsilon(x)} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy,$$

now choose $\eta_\varepsilon = o(\varepsilon^2)$ and by equation (54) we obtain

$$0 \ \geq \frac{\beta\varepsilon^2}{2(n+2)}\left((p-2)\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle + \Delta\phi(x_\varepsilon)\right) + o(\varepsilon^2).$$

Since $\nabla\psi(x) \neq 0$, let $\varepsilon \to 0$, then,

$$0 \;\geq\; \frac{\beta}{2(n+2)}((p-2)\Delta_\infty\phi(x) + \Delta\phi(x)).$$

Consequently, u is a viscosity supersolution. Use a reverse inequality to formula (54) to show that u is a viscosity subsolution, choose ϕ to be a function such that touches u from above and consider the maximum point of the test function.

□

Conclusion: We have now observed that the viscosity solution of formula (35) is unique and that the whole family $\{u_\varepsilon\}$ converges as $\varepsilon \to 0$.

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