Dyonic supersymmetric solutions in supergravity

Lukas Rødland
Supervisor: Giuseppe Dibitetto
Subject reader: Joseph Minahan
Master of Science Degree in Physics
Uppsala University

December 15, 2017
Abstract

In this thesis we reproduce the result of [1] and [2]. We consider dyonic solutions that preserve some supersymmetry in theories with a negative cosmological constant. More specifically, we look at a dyonic torus solution in gauged $N = 2$ supergravity and show that it preserves some residual supersymmetry. We do this by using the integrability conditions of the Killing spinor equations to find the Killing spinors. We find that the solution preserves $\frac{1}{2}$ of the original supersymmetry in two special cases, which are closely related to extremal black hole solutions. Starting from an ansatz of a six-dimensional dyonic string in gauged $N = (1, 0)$ supergravity we use the integrability condition to explicitly find a $\frac{1}{4}$-BPS solutions. We do this by solving the Killing spinor equation, and then show that the integrability condition implies that the solution solves the field equations.

Sammanfattning


Supersymmetrisk lösningar är viktiga att studera eftersom de har många speciella egenskaper. Generellt sett är lösningar till gravitationsteorier svåra att finna. Supersymmetrisk lösningar kan ofta vara mycket lättare att hitta än lösningar som inte är supersymmetriska.

I den sista delen av uppsatsen undersöker vi en supergravitations-teori i sex dimensioner och visar att det är mycket lättare att finna supersymmetriska lösningar än lösningar utan supersymmetri. Vi använder detta resultat för att finna en dyonisk stränglösning av teorin. En dyonisk sträng är ett endimensionellt objekt som både har elektrisk och magnetisk laddning.
1 Introduction 4

2 Supergravity 9
2.1 Vielbein formalism .................................................. 9
2.2 Covariant derivatives .................................................. 11
2.3 p-form gauge fields and dual tensors ............................... 15
2.4 How to couple gravity to fermions, 1. and 2. order formalism 16
2.5 Supersymmetry ......................................................... 17
2.6 Supergravity ............................................................. 19
2.7 Killing spinors and BPS solutions ................................... 21
2.7.1 Integrability conditions .......................................... 23

3 AdS Torus black hole is supersymmetric? 24
3.1 AdS black holes ....................................................... 25
3.2 N=2 gauged supergravity ............................................. 26
3.3 Find Killing spinors ................................................... 27
3.3.1 Case 1, \( q_m = \eta = 0 \) ........................................... 33
3.3.2 Case 2, \( l = \infty, \eta = 0 \) ........................................ 35
3.4 A short discussion of the solutions .................................. 37

4 Dyonic string solution in 6D, N=(1,0) gauged supergravity 38
4.1 N=(1,0), D=6 gauged supergravity ................................ 38
4.2 Integrability of the Killing spinor equation .......................... 40
4.3 Dyonic string solution .................................................. 49
4.3.1 Field equation and Bianchi identities for F and H ....... 51
4.3.2 Killing spinor conditions ....................................... 52

5 Conclusions 55

A Differential equation with projectors 56

B Spin connection 57

C Spin connection and Ricci tensor components for the dyonic string ansatz 59

D Spinors in different dimensions 59

References 60
1 Introduction

At the beginning of the last century, theoretical physics went through two revolutions. One was the realization that Newton’s classical mechanics did not work for objects with a high velocity and the other that it did not work at small scales. The two theories that emerged was General Relativity and quantum mechanics. Quantum mechanics was later merged with Einstein’s special relativity to give a theory that worked at high velocities.

Einstein’s theory of General Relativity is a theory of gravity, and hence it tells about physics at large scales. Quantum field theory is a theory that describes physics at very small scales. Symmetries play an essential role both in quantum field theory and in General Relativity, and the similarities in how they appear are striking. The basis of General Relativity is the idea of a local Lorentz symmetry, which for instance tells us that the speed of light is finite, and general coordinate transformations, which tells us that it does not matter what set of coordinates we want to use since the physics will look the same in all coordinates.

Quantum field theory is the merger of quantum mechanics with Special relativity, that is, quantum mechanics with global Poincaré symmetry. The Standard Model of particle physics, which is often called the most successful theory ever made, is a quantum field theory where, in addition to the global Poincaré symmetry, we have a local symmetry, given by the group $U(1) \times SU(2) \times SU(3)$. This local symmetry is called a gauge symmetry, and it decides what kinds of particles the theory contains, and how the particles interact with each other.

It is also possible to look at General Relativity as a gauge theory, where the gauge group is the general coordinate transformations. This makes it possible to think of General Relativity as a theory with a spin-2 particle, in the same way as the gauge group of a quantum field theory gives the particles of the theory. This is however not a quantum field theory since General Relativity is a non-renormalizable theory, which means that it is not well behaved at high energies.

The Standard Model together with General Relativity describes all the known forces of nature, but we know that this is not the full picture, one of the reasons is that we do not have a framework that explains both quantum field theory and General Relativity, such a theory is called quantum gravity.

Coleman and Mandula showed in 1967 that we could only combine the spacetime symmetries with the internal symmetries in a trivial way. This severely limits the possibility to add different kinds of symmetry to new theories since it means that a quantum field theory must have symmetries on the same form as in the standard model, i.e., the Poincaré group times
the gauge group. It was later shown that one could get around this theorem by considering symmetries that relate the two different types of particles we have in nature which are called, bosons and fermions.

Supersymmetry is a symmetry that relates fermionic and bosonic particles. The development of supersymmetric theories for particle physics started in the early 1970s. If a theory has unbroken supersymmetry, it means that every particle has a superpartner with the same mass. We do not observe this in nature, but it is still possible that supersymmetry exists as a broken symmetry. There are many reasons for believing this from a phenomenological standpoint. One is the unification of the coupling constants in the gauge group of the standard model at high energies. From a theoretical point of view, supersymmetry has many appealing properties. When studying particle physics one consider a global supersymmetry, but if one wants to have a theory of gravity we need local supersymmetry.

One of the reasons for studying supergravity is that if we want a theory with supersymmetry and General Relativity, supergravity is the only option. This is one of the reasons for the extensive work on supergravity that started in 1975 when Pran Nath and Richard Arnowitt considered gauged supersymmetry in [3]. After this countless different supergravities has been studied, and generalizations were made to higher dimensions and by adding more matter terms or more supersymmetry. Another reason to study supergravity is that the low energy limit of M-theory, a potential theory of everything, is the unique 11-dimensional supergravity. Since we live in 4 spacetime dimensions the 11-dimensional supergravity must be compactified in some way, this makes low dimensional supergravities interesting to study.

The simplest supergravity one can construct is the minimal four-dimensional \( N = 1 \) supergravity. One can construct that theory by gauging the super-Poincaré algebra. The gauging of the Poincaré part gives you a spin 2-particle called the graviton. The graviton must then have a spin \( \frac{3}{2} \) superpartner, which is called the gravitino. One can also consider extended supergravity theories, i.e., \( N \geq 2 \). The implications of looking at extended supergravities are that you will get more fields in the supermultiplets. Extended supergravities also have a symmetry acting on the supercharges, called R-symmetry.

If we make this symmetry into a local symmetry, we get so-called gauged supergravities.

Solutions to supergravity with some unbroken supersymmetry are particularly interesting to study. These solutions admit Killing spinors, which is the analog to Killing vectors for supersymmetries. Solutions that preserve some supersymmetry is called BPS solutions. BPS solutions have a lot of nice properties, one of them is that they are often easier to find than non-supersymmetric solutions. This is because of a close relationship between

5
the Killing spinor equations and the equations of motion for the system. It is often enough to find solutions of the Killing spinor equations since that often enough to make it a solution of the equation of motion as well. In general, it is hard to find solutions of theories of gravity, and it can be easier to look at a corresponding supergravity theory and try to find solutions to that theory since that solution will also be a solution of the non-supersymmetric gravity theory. Another reason why we study BPS-solutions is that they are important for counting of microstates of black holes.

BPS-solutions are closely related to extremal black holes in the sense that most extremal black holes are BPS solutions. Extremal black holes are solutions with the minimal allowed mass for a given charge and angular momentum, e.g., for zero angular momentum you have the extremal Reisner-Nordström solutions. That is, a spherically symmetric solution with electric and magnetic charge, with mass $G M^2 = q_e^2 + q_m^2$. This solution is a BPS solution in an ungauged $N = 2$ supergravity.

Objects that carry both electric and magnetic charge are called dyonic. The existence of dyonic objects is closely related to both the spacetime dimension and the dimension of the object. In four dimensions we have dyonic point particles since the electromagnetic duality tells us that two form field strengths are dual to two-form field strengths. Two-form field strengths are defined from one-form gauge fields, and objects that are charged under one form gauge fields are zero-dimensional objects. In six dimensions we can consider two-form gauge fields, where it’s corresponding field strength is a tree-form. We can have self-dual tree-form field strengths, which gives us the possibility of dyonic one-dimensional objects, i.e., dyonic strings.

The AdS/CFT correspondence, conjectured by Maldacena in [4] made the study of solutions to AdS gravity relevant. It is, in general, hard to find solutions of General Relativity. It is often easier to look for solutions with some supersymmetry, i.e., supersymmetric solutions of supergravity. Such solutions are called BPS-solutions. BPS black hole solutions are particularly interesting to study, because of their importance to the problem of counting microstates of a black hole.

AdS gravity, unlike General Relativity, admits topological black hole solutions, e.g., a black hole with event horizon with the topology of some manifold. Here we will review the case of a non-rotating, dyonically charged black hole solution with the topology of a torus in four-dimensional AdS gravity following the work of Caldarelli and Klemm in [1]. Caldarelli and Klemm show that this black hole solution is a BPS solution for two special cases when we look at the solution from the perspective of the $N = 2$ gauged AdS supergravity in four dimensions. We do this by assuming that it is a BPS solution and study the integrability conditions for Killing spinors. Using
the integrability conditions we explicitly compute the Killing spinors, which shows us that the solutions are BPS. We can compare this with a similar result from Romans, in [5], where he studies the case with a spherical event horizon. Both the spherical and the toroidal case admits two independent BPS solution with very similar conditions for the solutions to be BPS. In both cases, the two solutions are naked singularities. Caldarelli and Klemm extended the results to higher genus and rotating black hole solutions, where they found the conditions for those solutions to be BPS.

Similar results can be found for the case with zero cosmological constant. There the extreme Reisner-Nordström solutions are BPS solutions in the context of ungauged $N = 2$ supergravity in four-dimensions, as shown in [6–9]. The difference to what Romans showed is that, for the case with a negative cosmological constant, we have a BPS solution for a gauged $N = 2$ supergravity, while, for the one with zero cosmological constant, the solution is BPS for the ungauged $N = 2$ supergravity.

Solutions with unbroken supersymmetry are also important in higher dimensions. One supergravity theory that is particularly interesting is the gauged $N = (1, 0)$ theory in 6-dimensions. The minimal model of this theory was first found in [10] and is called Salam-Sezgin model. The theory is a chiral theory, and it has 8 real supercharges. The M-theory origin of the Salam-Sezgin model was first found in [11] by Cverič, Gibbons, and Pope.

The Salam-Sezgin model has a unique maximal vacuum solution with the geometry of four-dimensional Minkowski space times a two-sphere. This solution was found by Salam in 1984 [12].

The uniqueness of the vacuum was shown in [13]. The fact that this theory has a unique maximally symmetric vacuum makes the theory very interesting since this is in great contrast to other theories, which may have many such solutions.

It has been shown in [14–19] that the $N = (1, 0)$ theories can naturally be compactified to a four-dimensional $N = 1$ theory with a small cosmological constant and chiral matter. This has, of course, made the study of this theory important. Gibbons and Pope found a consistent Pauli reduction to four dimensions in [20] that gave a theory with gauge group SU(2) and zero cosmological constant.

The $\mathbb{R}^{1,3} \times S^2$ vacuum solution can be extended to a family of $\frac{1}{2}$-BPS vacuum solution with the geometry of $AdS_3 \times S^3$, where $S^3$ is a squashed three sphere [2]. This is a generalization of the maximally symmetric vacuum in the sense that it has the $\mathbb{R}^{1,3} \times S^2$ solution as a limit case when the Hopf fiber of the squashed sphere goes to zero. Güven, Liu, Pope, and Sezgin also show in [2] that this solution can be found as a near horizon limit of a dyonic string solution. This solution is the study of the last part of this thesis. The
dyonic string solution preserves $\frac{1}{4}$ of the supersymmetries. One can, in the corresponding ungauged supergravity, also find dyonic string solutions, such solutions were found in [21, 22].

Generalizations of the dyonic string studied here can be found in theories with more matter multiplets. In [23] it was found a $\frac{1}{4}$-BPS dyonic string solution in a theory coupled to a $E_6 \times E_7 \times U_R(1)$. The first generalization to a theory with hyperscalars was found in 2006 by Jong, Kaya, and Sezgin in [24], but the dyonic string solution only had one unbroken supersymmetry in this case, i.e., a $\frac{1}{8}$-BPS solution.

In section 2 we start with a review the necessary supergravity that we need for the rest of the thesis. We start with a review of the vielbein formalism of gravity section 2.1 since it is needed for coupling of fermions to gravity. We then review some other concepts from differential geometry, mainly about covariant derivatives and curvature tensors. In section 2.3 we introduce p-form gauge fields and discuss dyonic objects in various dimensions. The next few sections are about giving an overview supergravity. We do this in an informal way so that we can justify the different kinds of supergravities used later in the book. We do this through examples and some discussion about some more advanced topics, like gauged supergravities, and extended supergravities. In the last section in 2 we study BPS solutions of supergravity, that is, solutions that have Killing spinors. There we introduce the concept of Killing spinors, and finally, we talk about the integrability conditions of the Killing spinor equations, which we will use extensively throughout this book.

In section 3 we follow [1], there we calculate the Killing spinors of a dyonically charged black hole in four-dimensional AdS spacetime with the topology of a torus. In subsection 3.1 we review the charged topological black hole solution in four-dimensional AdS gravity with the topology of a torus. Here we give the metric corresponding to the solution and calculate the spin connection using the Christoffel symbols. In section 3.2 we introduce the gauged $N = 2$ AdS supergravity. We give the bosonic part of the action and the local supersymmetry transformations of the fields. In section 3.3 we show that the solution from section 3.1 is a BPS solution in the supergravity theory introduced in 3.2. We do this by explicitly solving the Killing spinor equations, given by the supersymmetry transformations. Using the integrability conditions for the Killing spinor equations, found in section 2, is used to show that the solution is a BPS solution in two different cases, which resembles the extreme black hole solutions for charged black holes in a theory without a cosmological constant. The integrability condition also shows that the solution is $\frac{1}{2}$-BPS.

In section 4 we review the work of Güven, Liu, Pope and Sezgin in [2]. In
subsection 4.1 we review the 6-dimensional N=(1,0) gauged supergravity. We will only consider the theory with a gravity multiplet, a tensor multiplet and a U(1) multiplet, where the U(1) multiplet comes from the gauging of the R-symmetry. It is possible to consider the theory with more matter fields, as done in [23]. Just as for the four-dimensional case, we give the Killing spinor equations, which is equivalent to the supersymmetry transformations of the fermionic fields. Here we also give the equations of motion of the system. We need the equations of motion in section 4.2, where we write the integrability conditions of the Killing spinor equations in terms of the field equations and the supersymmetry transformations. By rewriting the integrability conditions in this form, we show that a solution that admits Killing spinors, i.e., a BPS solution, also solves some of the fields equations. We use this fact in subsection 4.3 to find a dyonic string solution. We start with a general ansatz for a dyonic string and solve the Killing spinor equations. Then, since the ansatz solves the Killing spinor equation, it also solves the equation of motion and is hence a solution of the supergravity theory.

2 Supergravity

In this section, we will start with an introduction to the Vielbein formalism of gravity which is needed to be able to couple fermions to gravity. We then give a short introduction to the basics supersymmetry and supergravity. We end this section by discussing a certain class of solutions of supergravity that conserve some of the original supersymmetry of the theory, so-called BPS solutions.

We will assume some basic knowledge of differential geometry, but no knowledge of the Vielbein formalism is required. We will follow [25] and [26].

2.1 Vielbein formalism

In this section we consider a pseudo-Riemannian manifold, (M, g), where the metric g with signature (n, m), which means that it is invariant under SO(n, m) transformations.

When we want to couple gravity to fermions, we have to use a non-coordinate basis of the tangent space called Vielbein in d dimensions, or Vierbein in 4 dimensions. This method is different from the usual formulation of General Relativity where the primary object is the metric. We define the components of the metric as

\[ g_{\mu\nu} := g \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right) , \]
where \( x^\mu \) is our choice of local coordinates. In this formulation, we use \( \frac{\partial}{\partial x^\mu} \) as a basis for the tangent space. This is not the case for the vielbein formalism where the main object is the vielbein, which is a set of vector fields that form a basis for the tangent space at every point and is hence different than the coordinate basis in general. Where the coordinate basis depends entirely on our choice in coordinates, the Vielbein are instead closely related to the metric. To get a rigorous definition of a vielbein, we first have to define what we mean by a frame.

A frame on a pseudo-Riemannian manifold is a set of vector fields that form a basis of the tangent space at every point. A Vielbein is a special kind of orthonormal frame, \( e_a(x) = e^\mu_a \partial_\mu \), with the property
\[
g^{\mu\nu}(x) = e^\mu_a(x)\eta_{ab} e^\nu_b.
\]
Equation (1) makes it possible to define an inverse of the vielbein, \( e^a_{\mu} \), with \( e^a_{\mu} := e^\mu_a dx^\mu \) a set of one-forms which is a basis of the cotangent space. The choice of \( e^\mu_a \) is not unique, but all choices are related by a SO\((n, m)\) transformation, where \( n \) and \( m \) comes from the signature of the metric. We will consider all the different choices as equivalent. In the special case where the signature of the metric is diag\((-1, 1, 1, 1)\), the transformations are Lorentz transformations, and the transformations is given by
\[
e^a_{\mu}(x) = \Lambda^{-1a}_{b}\ e^b_{\mu}(x).
\]
Because of this the Latin indices, \( a,b,c, \ldots \) are often called Lorentz indices, and we can raise and lower them by using the flat metric \( \eta_{ab} \). The Greek indices can be raised and lowered by using the curved metric \( g_{\mu\nu} \) since the Vielbein transforms as a covariant vector under general coordinate transformations with respect to the Greek indices, that is
\[
e'^a_{\mu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} e^a_{\rho}(x).
\]
Since \( e_a \) forms a basis of the tangent space, we can write vectors in terms of this basis as
\[
V = V^a e_a = V^a e^\mu_a \partial_\mu = V^\mu \partial_\mu.
\]
We see that \( V^\mu = V^a e^{a\mu}_\mu \) relates the components of the vectors written in the coordinate basis and the components written in the Vielbein basis, and hence \( V^{\mu} e^a_{\mu} = V^a \). We can, of course, do the same thing for covectors, since \( e^a \) is a basis of the cotangent space, the components are related as \( V_\mu = V_a e^a_{\mu} \). The relation between indices in the coordinate basis and the vielbein basis can be generalized to all tensors in an obvious way.
We can think about the Vielbein as a matrix of dimension $\text{Dim} M \times \text{Dim} M$. From equation (1) we can see that the determinant of the Vielbein is

$$e := \text{Det} e^a_{\mu} = \sqrt{-\text{Det} g_{\mu\nu}}.$$  

It is convenient to introduce the so called anholonomy coefficients $\Omega^{c}_{ab}$ defined via the Lie derivative as follows

$$[e_{a}, e_{b}] := \mathcal{L}_{e_{a}} e_{b} = -\Omega^{c}_{ab} e_{c}.$$  

By expanding the commutator and using $e_{a} = e_{a}^{\mu} \partial_{\mu}$ we find

$$[e_{a}, e_{b}] = e_{a}^{\mu} (\partial_{\mu} e_{b}) \partial_{\nu} - e_{b}^{\nu} (\partial_{\nu} e_{a}) \partial_{\mu} = e_{a}^{\mu} (\partial_{\mu} e_{b}) e^{c}_{c} e_{c} - e_{b}^{\nu} (\partial_{\nu} e_{a}) e^{c}_{c} e_{c},$$

then by using the property $e^{c}_{c} e_{a}^{\nu} \partial_{\nu} e_{b}^{\mu} = - e_{a}^{\nu} e_{b}^{\mu} \partial_{\nu} e^{c}_{c}$, we find a explicit formula for the anholonomy coefficients which is

$$\Omega^{c}_{ab} = e_{a}^{\mu} e_{b}^{\nu} (\partial_{\mu} e^{c}_{c} - \partial_{\nu} e^{c}_{c}). \quad (2)$$

The reason why the Vielbein formalism makes it possible to couple fermions to gravity is that we consider $e_{a}^{\mu}$ and $e_{a}^{\mu} \Lambda^{-1 b}_{a}$, where $\Lambda^{-1 b}_{a}$ is a $\text{SO}(n, m)$ transformation, to be equivalent. We can extend this to the group $\text{SPIN}(n, m)$ which is a double covering of $\text{SO}(n, m)$, we can in this way include “spinor-representations” of the Lorentz group (or more generally of $\text{SO}(n, m)$), which means that it is meaningful to speak about spinors.

In the next section we will talk about $\text{SO}(n, m)$ gauge theories, but what we implicitly mean is $\text{SPIN}(n, m)$ gauge theories, since we want to include spinors. The corresponding connection will be precisely the one that objects that transform as tensors with respect to the Lorentz indices will transform as tensors when acted upon by the covariant derivative.

### 2.2 Covariant derivatives

We want to define a covariant derivative that sends tangent-space tensors to tangent-space tensors, that is objects with latin indices that transforms as

$$V^{a} = \Lambda^{-1 b}_{a} V^{b}.$$  

We call this covariant derivative $D$, and it should be on the form $d + \omega$, where $\omega$ is the $\text{SO}(n, m)$ connection usually called the spin connection. The covariant derivative will act on a one-form, $V^{a}$ as follows,

$$DV^{a} = dV^{a} + \omega^{a}_{b} \wedge V^{b}.$$  

11
$DV^a$ will transforms as a tensor if $\omega^a_b$ transforms as a SO$(n, m)$ connection, that is as

$$\omega^a_b = \Lambda^{-1}a_c d\Lambda^c_b + \Lambda^{-1} c_d \omega^c_d \Lambda^d_b.$$ 

In this formalism the torsion tensor of the connection can be defined as

$$De^a := T^a. \quad (3)$$

In most cases related to gravity the torsion vanishes, which makes (3) a good way of finding the components of the spin connection. In supergravity one will usually have non-zero torsion since we have fermions coupled to gravity. We can set all the fermionic fields to zero in most of the calculations. This will make the torsion tensor vanish as well, since the torsion is usually a product of the fermionic fields. The reason why we can consider only theories with vanishing fermions is that we can always get a theory with non-vanishing fermions by performing a supersymmetry transformation Equation (3) is often called the first Cartan structure formula.

In local coordinates we can write the spin connection as $\omega^a_{\mu b}$, which transforms as a vector under coordinate transformations. One important property of the spin connection is that it is antisymmetric in $a$ and $b$. The way the covariant derivative acts on vector fields, one-forms and tensors is

$$\begin{align*}
D_{\mu} V_a &= \partial_{\mu} V_a - V_b \omega^b_{\mu a} \\
D_{\mu} V^a &= \partial_{\mu} V^a + \omega^a_{\mu b} V^b \\
D_{\mu} V_{ab} &= \partial_{\mu} V_{ab} - V_{cb} \omega^c_{\mu a} - V_{ac} \omega^c_{\mu b}.
\end{align*} \quad (4)$$

More generally the covariant derivative act on a covariant field, $\psi$, in the following way

$$D_{\mu} \psi = \partial_{\mu} \psi + \frac{1}{2} \omega^a_{\mu b} \Gamma(M_{ab}) \psi,$$

where $M_{ab}$ are elements in the Lie algebra $\mathfrak{so}(n, m)$, and $\Gamma(M_{ab})$ is the representation under which $\psi$ transforms. The spinor representation can be chosen as the second rank Clifford matrices $\frac{1}{2} \gamma_{ab}$, where we define $\gamma_a$ from the property $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, and $\gamma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b]$. This tells us how the covariant derivative acts on spinors (we supress spinor indices)

$$D_{\mu} \psi = \partial_{\mu} \psi + \frac{1}{4} \omega^a_{\mu b} \gamma_{ab} \psi. \quad (5)$$

If we act with the covariant derivative on the flat metric $\eta_{ab}$ we find

$$D_{\mu} \eta_{ab} = -\eta_{cb} \omega^c_{\mu a} - \eta_{ac} \omega^c_{\mu b} = -\omega_{\mu ba} - -\omega_{\mu ab} = 0, \quad (6)$$

12
i.e. the connection is metric compatible with respect to $\eta$.

If we have a torsion free spin connection, then we can find the components of the spin connection just by using the Vielbein $e_a^\mu$. Starting from equation (3) we will find the spin connection in terms of the Vielbein and the torsion tensor. In local coordinates, $x^\mu$, the torsion tensor can be written as $T^a = T^a_{\mu\nu}dx^\mu \wedge dx^\nu$, and we define $T_{\mu\rho} := T^a_{\mu\nu}e_a^\rho$ which is antisymmetric in the first two indices. In terms of the components, equation (3) can be written as

$$T_{\mu\rho} = e_{a\rho}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) + \omega_{\mu\rho\nu} - \omega_{\nu\rho\mu}$$

We can rewrite this in terms of the anholonomy coefficients from equation (2), which in local coordinates is

$$\Omega_{\mu\rho\nu} = \Omega_{\mu}^{ab}e_{a\rho}e_{b\nu} = e_{a\rho}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu),$$

which is antisymmetric in $\mu$ and $\nu$. This gives

$$T_{\mu\rho} = \Omega_{\mu\rho\nu} + \omega_{\mu\rho\nu} - \omega_{\nu\rho\mu},$$

and hence

$$-\omega_{\mu\rho\nu} + \omega_{\nu\rho\mu} = \Omega_{\mu\rho\nu} - T_{\mu\rho\nu}.\tag{7}$$

Since $\omega_{\mu\nu}$ is antisymmetric in $\rho$ and $\nu$ it is easy to see that

$$2\omega_{\mu\rho} = (\omega_{\mu\rho} + \omega_{\rho\mu}) + (\omega_{\rho\mu} + \omega_{\mu\rho}) + (\omega_{\rho\nu} + \omega_{\nu\rho}) + (\omega_{\mu\nu} + \omega_{\nu\mu}) = (\Omega_{\mu\rho\nu} - T_{\mu\rho\nu}) + (\Omega_{\rho\mu\nu} - T_{\rho\mu\nu}) + (\Omega_{\rho\nu\mu} - T_{\rho\nu\mu}).$$

We will denote the spin connection with zero torsion $\omega(e)$ since it is completely determines from the Vielbein. It follows directly from (7) that

$$\omega_{\mu\rho\nu}(e) = \omega_{\mu}^{ab}e_{a\rho}e_{b\nu} = \frac{1}{2}(\Omega_{\mu\rho\nu} + \Omega_{\rho\mu\nu} + \Omega_{\rho\nu\mu}).$$

Another useful quantity to define is the total covariant derivative, which transforms tangent-space tensors to tangent space tensors and world tensors to world tensors. That is, it acts differently on the Latin indices and the Greek indices, for example,

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^a_{\mu\rho}e^a_\rho + \omega_{\mu}^{ab}e^b_\nu.\tag{8}$$

Here $\Gamma^a_{\mu\nu}$ is the affine connection. It is worth noting that the affine connection will, in general, have non-zero torsion.

The total covariant derivative does not add any new structure to the theory, and it only re-expresses the information of the spin connection to the
coordinate basis where it contains the affine connection. By that, we mean
that since the total covariant derivative acts differently on different indices,
it should give the same result as if we rewrote the expression in terms of only
Latin indices, which we can do using the Vielbein. To make this work we
need the so called “Vielbein postulate”, \( \nabla e^a_\mu \rho = 0 \), then we get

\[
\nabla e^a_\mu V^\nu = \nabla e^a_\mu (V^a e^a_\mu) = \nabla e^a_\mu e^a_\nu + V^a e^a_\mu e^a_\nu \\
= e^a_\nu D \mu V^a + V^a e^a_\mu e^a_\nu = e^a_\nu D \mu V^a.
\]

It follows directly from the Vielbein postulate and equation (8) that the
relation between the affine connection and the spin connection is

\[
\Gamma^\rho_{\mu \nu} = e^\rho_a (\partial e^a_\mu + \omega^a_{\mu b} e^b_\nu).
\]

(9)

When we have zero torsion, then the affine connection is just the Christof-
fel symbols. Since the Christoffel symbols are easy to calculate, this gives us
a good way of calculating the spin connection by using equation (9).

Another consequence of the Vielbein postulate is metric compatability of
the total covariant derivative

\[
\nabla g_{\mu \nu} = \partial g_{\mu \nu} - \Gamma^\sigma_{\mu \nu} g_{\sigma \rho} - \Gamma^\sigma_{\mu \rho} g_{\nu \sigma} = 0.
\]

Later, when we speak about supergravity, we will interpret the Vielbein
as a spin-2 particle which we call the graviton. Since supergravity is a super-
symmetric theory, the graviton must have a fermionic superpartner, which
we call the gravitino. The gravitino will have spin \( \frac{3}{2} \), and it is a mixed quan-
ty in the sense that it has one vector index and one spinor index (which we
will not write explicitly), that is \( \psi_\mu \). The way the total covariant derivative
acts upon the gravitino is

\[
\nabla \psi_\nu = \left( \partial_\mu + \frac{1}{4} \omega^a_{\mu b} \gamma_{a b} \right) \psi_\nu - \Gamma^\rho_{\mu \nu} \psi_\rho.
\]

This implies

\[
D_\mu \psi_\nu = \left( \partial_\mu + \frac{1}{4} \omega^a_{\mu b} \gamma_{a b} \right) \psi_\nu. \quad (10)
\]

We can define the curvature 2-form \( \rho^{ab} := \frac{1}{2} R_{\mu \nu}^{ab} dx^\mu \wedge dx^\nu \) for the covari-
ant derivative \( D_\mu \) as

\[
\rho^{ab} := d \omega^{ab} + \omega^a_{c b} \wedge \omega^c_{ab}. \quad (11)
\]
Equation (11) is called the second Cartan structure equation, and later we will use it to calculate the Ricci tensor. We can define the Ricci tensor as $R_{\mu\nu} := R_{\mu}^{\sigma \nu \sigma}$.

It is worth noting that $R_{\mu\nu ab}$ has the properties of a field strength for a $SO(n, m)$ gauge theory.

We can generalize the notion of curvature to any covariant derivative, $\hat{D}_\mu$, as the commutator between the covariant derivatives.

$$\hat{R}_{\mu\nu} := [\hat{D}_\mu, \hat{D}_\nu].$$ (12)

When acting on a spinor $\psi$ we have the relation

$$[D_\mu, D_\nu] \psi = \frac{1}{4} R_{\mu\nu ab} \gamma^{ab} \psi.$$ (13)

When considering AdS gravity we have to modify the covariant derivative. When acting on spinors the appropriate covariant derivative for AdS gravity is

$$\hat{D}_\mu \psi := \left( D_\mu - \frac{1}{2L} \gamma_\mu \right) \psi := \left( \partial_\mu + \frac{1}{4} \omega_\mu^{\ ab} \gamma_{ab} - \frac{1}{2L} \gamma_\mu \right) \psi.$$ (14)

The corresponding curvature tensor $\hat{R}_{\mu\nu} := [\hat{D}_\mu, \hat{D}_\nu]$ is related to the curvature tensor from the Lorentz covariant derivative, $R_{\mu\nu} := [D_\mu, D_\nu]$, as

$$\left( R_{\mu\nu ab} + \frac{1}{L^2} (e_{a\mu} e_{b\nu} - e_{b\mu} e_{a\nu}) \right) \gamma^{ab} \psi = \hat{R}_{\mu\nu ab} \gamma^{ab} \psi.$$ (15)

### 2.3 p-form gauge fields and dual tensors

Later, we will look at a six-dimensional supergravity that has a 2-form gauge field as well as the normal 1-form gauge field $A_\mu$. Here we will give a short review of (Abelian) p-form gauge fields.

If $A_{(1)}$ is a 1-form gauge field, then we define the field strength $F_{(2)} := dA_{(1)}$. The field strength satisfies the following equation of motion and Bianchi identity

$$d F_{(2)} = 0$$
$$d \star F_{(2)} = 0$$

The corresponding action is

$$S_1 := -\frac{1}{2} \int \star F_{(2)} \wedge F_{(2)}$$
Where ∗ is the Hodge star. The Hodge star sends p-forms to $D - p$-forms. For $D = 4$ this implies that the dual field strength $\star F_{(2)}$ is a 2-form as well. This relation is the electromagnetic duality. $A_{(1)}$ describes a $U(1)$-charged particle. The reason why it is a particle and not some higher dimensional object is that $A_{(1)}$ is locally the trajectory of the worldline for the charged particle. Similarly, we will have that a 2-form gauge field describes a string.

For a p-form gauge field, $A_{(p)}$, we define the field strength as $F_{(p+1)} := dA_{(p)}$, and the action as

$$S_p := -\frac{1}{2} \int \star F_{(p+1)} \wedge F_{(p+1)}.$$  

The Hodge duality still works, and in general we have that $p$-form gauge fields are equivalent to a $(D - p - 2)$-form gauge field. This implies self-dual 2-forms for $D = 6$, which means that we have the possibility for dyonic strings, i.e., strings with magnetic and electric charge.

### 2.4 How to couple gravity to fermions, 1. and 2. order formalism

If we want to couple gravity to fermions, we have to describe gravity using the vielbein formalism instead of the standard formalism using the metric and the coordinate basis of the tangent space. Here we will look at two different ways of doing this called the first order formalism and the second order formalism of gravity. It is possible to express both formalisms using the metric and the coordinate basis, but here we are interested in using the vielbein and the spin connection as the main objects. The difference between first and second order is that the first order formalism is described by first order differential equations, while the second order formalism is described by second order differential equations. This difference comes from that we in the first order formalism consider the vielbein and the spin connection to be independent of each other, or equivalently if we consider the metric and the affine connection to be independent. The second order formalism is the usual formalism where one considers the connection as a function of the vielbein (or metric). In this section, we will only consider the 4-dimensional case for simplicity, but this can easily be generalized.

In the second order formalism of gravity, we consider the vielbein as the dynamical variable describing gravity. Starting from the action, we get a second order differential equation from the variation with respect to $e_a^\mu$, e.g.

$$S = \int (R(e) + L)|e|d^4x,$$  

16
where \( R(e) \) is the Ricci scalar derived from the unique torsion free spin connection, \( \omega_{\mu ab}(e) \), derived from the vielbein. And \( L \) is a Lagrangian describing some matter field, e.g. \( L = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} \).

In the first order formalism of gravity we consider the vielbein, \( e_a^\mu \), and the spin connection \( \omega_{\mu ab} \), as two independent fields. The implications of this are that we get equations of motion from the variation of both fields. The equation of motions from the spin connection and the vielbein will the be first order differential equations which are easier to solve than the second order equations that we get from the second order formulation.

If we consider the action without any matter fields,

\[
S = \int (e_a^\mu e_b^\nu R_{\mu \nu}^{\ ab}) |e|,
\]

the solution of the spin connection equation of motion is the torsion free spin connection \( \omega_{\mu ab} = \omega_{\mu ab}(e) \). This gives the same results as the for the second order formalism. In general, the solution will be on the form

\[
\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab},
\]

where \( K_{\mu ab} \) is called the cotorsion tensor and is related to the torsion tensor as

\[
K_{\mu ab} = -\frac{1}{2} (T_{\mu \nu \rho} - T_{\nu \rho \mu} + T_{\rho \mu \nu}).
\]

If we insert the solution \( \omega_{\mu ab} \) into the first order action, we will get the equivalent second order action. This equivalence implies that the theory with torsion in the first order formalism is equivalent to the torsion free theory in the second order formalism some matter fields.

The main reason why the first order formalism is interesting is that it is easier to show that the supergravity actions is invariant under supersymmetry transformations.

### 2.5 Supersymmetry

Supersymmetry transforms bosonic particles into fermionic particles and vice versa. The global supersymmetry transformations looks schematically like

\[
\begin{align*}
\delta(\epsilon) B &= \epsilon F \\
\delta(\epsilon) F &= \epsilon B,
\end{align*}
\]

where \( B, F \) are bosonic and fermionic fields respectively and \( \epsilon \) is an infinitesimal supersymmetry parameter that carries spin indices.
When we are studying particle physics, we consider the supersymmetry as a global symmetry, but if we want to include supersymmetry in a theory of gravity, it must be as a local symmetry. Such a theory is what we call supergravity. If we make the supersymmetry parameter spacetime dependent, i.e., $\epsilon = \epsilon(x)$, then, if we want a consistent theory, it must be a supergravity theory.

We will look at global $N = 1$ supersymmetry as an extension of the Poincaré symmetries in 4 dimensions, and the super-Poincaré algebra is given by

$$\{Q_\alpha, Q^\beta\} = -\frac{1}{2}(\gamma_\mu)\beta P^\mu$$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})\alpha Q_\beta$$

$$[P_\mu, Q_\alpha] = 0.$$  \hspace{1cm} (18)

Here $P^\mu$ are the generators of translation, $M_{\mu\nu}$ of Lorentz symmetries and $Q_\alpha$ are the supercharges.

We can add more supercharges to the theory and get what we call extended supersymmetry. Then we will get an extra index on the supercharges, $Q^i_\alpha$, where $i$ runs from 1 to $N$, where $N$ is the amount of supercharges. There are limits to how many supercharges one can add to the theory, and this changes for different dimensions, for example in 4 dimensions we can have up to $N = 8$ which means that the theory has 32 real super charges.

Since supersymmetries transform bosonic fields to fermionic and vice versa, it makes it possible to group the particles into supermultiplets, which are the particles that transform into each other under supersymmetry transformations. For $N = 1$ this means that every bosonic field has a fermionic superpartner. This is still true for supergravity theories, where the main objects are the graviton i.e., the vielbein, and its superpartner, the gravitino, which is a spin $\frac{3}{2}$ particle, $\psi_\mu$.

The gravity supermultiplets for a $N = 2$ supergravity has the following content

$$(e^i_\mu, \psi^i_\mu, A_\mu), \text{ for } i = 1, 2.$$  \hspace{1cm} (19)

The fields that are not in the gravity multiplet is called matter fields in supergravity. If we want to add matter to the theory, it must come in a complete supermultiplet. Some important supermultiplets that we will encounter later are vector multiplet, and the tensor multiplet.

The vector multiplet only contains states with spin up to 1. The vector can be a Yang-Mills field for some gauged symmetry, so it is also called gauge multiplet.
The tensor multiplet consists of anti-symmetric tensors, $T_{\mu\nu}$. In six dimensions this tensor can have self-dual properties under Hodge duality.

2.6 Supergravity

Here we will give a short overview of supergravity theories, starting from the minimal $N = 1$ four-dimensional theory and then discuss extensions to that theory, like extended supergravities, gauged supergravities and AdS supergravities.

We can think about theories of gravity as a theory where we have made the global Poincaré symmetry into a local symmetry. When we get a theory with a vielbein, which we interpret as the graviton. We want to find an action that is invariant under Poincaré transformations, without any matter the solution is the Einstein-Hilbert action, (16). We can add matter terms to this theory, e.g., Einstein-Maxwell action, where we add a kinetic term for the U(1) field strength.

For supergravity we want to gauge the whole super-Poincaré algebra, i.e., (18). For the $N = 1$ case we end up with a graviton and the spin $\frac{3}{2}$ gravitino. The action that is invariant under the super-Poincaré transformations is

$$S = \frac{1}{2\kappa^2} \int \left( R - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) |e| d^Dx.$$  \hspace{1cm} (20)

Where $D_\mu = \partial_\mu + \frac{i}{4} \omega_{\nu ab} \gamma^{ab}$, when acting on spinors. The last term is a Rarita-Schwinger term, which is the correct way to write down a Lorentz invariant term for spin $\frac{3}{2}$ particles.

In the previous section we briefly looked at supersymmetry transformations, in equation (17) we saw that global supersymmetry transformations are parametrized by constant spinors $\epsilon$. When we go to local supersymmetry transformations, $\epsilon$ becomes a function of spacetime, i.e., $\epsilon := \epsilon(x)$. The supersymmetry transformations that the action in equation (20) is invariant under is given by

$$\begin{cases} 
\delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \psi_\mu \\
\delta \psi_\mu = D_\mu \epsilon.
\end{cases}$$

A gauged supergravity is a supergravity that contains vector fields that gauge some subgroup of the R-symmetry, i.e., we promote some of the R-symmetry to local symmetries. This gives new terms in the Lagrangian containing kinetic terms of the field strengths. Gauged supergravities are closely related to supersymmetric AdS vacuum solutions, which makes them useful for finding supersymmetric solutions.
The number of supercharges is $N$ times the number of real components of an irreducible spinor in that spacetime dimension. E.g., in four dimensions we have irreducible Majorana spinors, which have 4 independent components in four dimensions, this implies that the total number of supercharges is $4N$. We denote them $Q^i$, for $i = 1, \ldots, N$ where we suppressed the spinor index. The supercharges relates particles with a difference $\frac{1}{2}$ in spin. Because of this, and the fact that we cannot have particles with spin larger than 2, we have a maximal $N$ for any given dimension. We do not want particles with spin higher than 2 because of the problems of writing down a Lagrangian with higher spins in a consistent way. The maximal amount of supercharges we can have in a supergravity theory is 32, and this corresponds to $N = 8$ in the four-dimensional case.

We have discussed the possibility of extended global supersymmetry, i.e., $N$ sets of supersymmetry generators $Q^i, i = 1, \ldots, N$. We can do this for local supersymmetry as well, i.e., for supergravity. For this to work consistently, we need a symmetry between the supercharges, $Q^i$, which is called R-symmetry. We can gauge the R-symmetry to make it into a local symmetry, that is, we will get a gauged supergravity theory.

When one adds more supercharges or gauge a symmetry, this changes the supercovariant derivative. In the next section, we will see how this is relevant for the so-called Killing spinor equations that determine the residual supersymmetries of a solution. The Killing spinor equation that comes from the supersymmetry variation of the gravitino field is by definition $\hat{D}_\mu \epsilon$, where $\hat{D}_\mu$ is the supercovariant derivative. The Killing spinor equations turn out to be the supersymmetry variations of the fields set to zero.

If we want a theory with a negative cosmological constant, we get Anti-de Sitter supergravity. The difference from Poincaré supergravity is that we now gauge the super-AdS group instead of the super-Poincaré group. It is worth noting that, when we make the spacetime symmetry into a local symmetry, it is a lot easier to do it with the Anti-de Sitter symmetry, and then get the Poincaré gravity from a so-called Wigner-inönü contraction, i.e., set the cosmological constant to zero. The main difference is that our covariant derivative, $D_\mu$, is modified to $\hat{D}_\mu \epsilon := (D_\mu - \frac{1}{2} \gamma_\mu) \epsilon$. The $N = 1$ action is then modified to

$$\frac{1}{2\kappa^2} \int d^Dx e(\hat{R} - \bar{\psi}_\mu \gamma^{\mu\nu} \hat{D}_\nu \psi_\mu).$$

Where $\hat{R}$ is the curvature defined by $\hat{D}_\mu$, i.e.,

$$[\hat{D}_\mu, \hat{D}_\nu] \epsilon = \frac{1}{4} \hat{R}_{\mu\nuab} \gamma^{ab}.$$
It is then easy to show that the modified Ricci scalar is related to the Poincaré Ricci scalar by

$$\hat{R} = R + \frac{D(D - 1)}{L^2}.$$

When we put the Ricci scalar back into the action we get a constant term that we can interpret as a cosmological constant, just as for (non-supersymmetric) AdS gravity.

This implies that we can write the action as the Poincaré action with some additional terms, this also holds for both gauged and extended supergravities.

### 2.7 Killing spinors and BPS solutions

Supergravity comes from gauging the super-Poincaré algebra, or some other global super algebra. That means that we have local supersymmetry transformations of our field which look schematically like

$$\begin{cases}
\delta(\epsilon)B = \epsilon F \\
\delta(\epsilon)F = \epsilon B
\end{cases}$$

where $F$ and $B$ are fermionic and bosonic field respectively, and $\epsilon = \epsilon(x)$ is a spinor that parametrize the local supersymmetry transformation.

For example in the $N = 1, D = 4$ Poincaré supergravity we have

$$\begin{cases}
\delta(\epsilon)e^a_{\mu} = \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_\mu \\
\delta(\epsilon)\Psi_\mu = D_\mu \epsilon.
\end{cases}$$

Solutions of supergravity that has some unbroken (global) supersymmetry are particularly interesting to study. This is because they have some special properties such as a bound on their mass which is called BPS-bound. These properties are why solutions with unbroken supersymmetry are called BPS-solutions.

BPS-solutions are often easier to find than non-BPS-solutions, which means that finding BPS-solutions can be a good way of finding new classical solutions of gravity. The reason for this is that BPS-solutions have what is called Killing spinors, and the existence of Killing spinors are closely related to the equations of motion of the theory which we will see later.

Killing spinors, just as Killing vectors are closely related to symmetries, but Killing spinors are related to unbroken supersymmetries. What we mean by unbroken supersymmetries of a solution of supergravity is that the solution is invariant under transformations of a subset of local supersymmetries of
the supergravity theory. Let’s say one of those unbroken supersymmetries is parametrized by a spinor, $\epsilon$, then that is equivalent to

\[
\begin{cases}
\delta(\epsilon)B = \epsilon F = 0 \\
\delta(\epsilon)F = \epsilon B = 0,
\end{cases}
\]

(21)

i.e., all the variations of the fields vanish under that transformation. The set of equations (21) is called the Killing spinor equations, and they are in general coupled differential equations. Solutions, $\epsilon(x)$, of the Killing spinor equations are called Killing spinors.

If our solution has $Q_0$ independent unbroken supersymmetries or equivalently independent Killing spinors, and the supergravity theory has $Q$ local real components supersymmetry, then we call the solutions $\frac{Q_0}{Q}$-BPS and say that the solution preserves $\frac{Q_0}{Q}$ of the supersymmetries.

$\frac{Q_0}{Q}$ is usually $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, this is because of projections operators of the type $\frac{1}{2}(1 - \gamma_{\mu})$ which halves the degree of freedom of the Killing spinors.

We will now look at a basic example of Killing spinors in the simplest supergravity, namely $N = 1, D = 4$ Poincaré supergravity. The supergravity solution we will consider is Minkowski spacetime, which means that the metric is $g_{\mu\nu} = \eta_{\mu\nu}$ and the spin $\frac{3}{2}$ gravitino $\Psi_{\mu}$ vanishes

\[
\begin{cases}
\Psi_{\mu} = 0 \\
e_{\mu}^{a} = \delta_{\mu}^{a}.
\end{cases}
\]

This gives supersymmetry transformations of the vielbein and the gravitino as

\[
\begin{cases}
\delta e_{\mu}^{a} = \frac{1}{2} \bar{\epsilon} \gamma_{\mu}^{a} \Psi_{\mu} = 0 \\
\delta \Psi_{\mu} = D_{\mu} \epsilon = \partial_{\mu} \epsilon
\end{cases}
\]

where $\delta e_{\mu}^{a} = 0$ since $\Psi_{\mu} = 0$ and $D_{\mu} = \partial_{\mu}$ since we are in flat space. The Killing spinor equation we have to solve is $\partial_{\mu} \epsilon = 0$ which gives 4 independent constant Killing spinors, which means that Minkowski space preserves all the supersymmetries and hence can be considered a background solution.

Usually one sets all the fermionic fields of the theory to 0 and only look at bosonic supergravity solutions. These solutions will also be solutions of General Relativity coupled to some matter fields. It is possible to work with solutions with non-vanishing fermionic fields, one of the reasons why we do not is that we can generate the fermionic fields from a solution without fermions by performing a supersymmetry transformation. It is possible that
there exist some supergravity solutions that we cannot get from supersymmetry transformations of solutions with vanishing fermions, but they are not as interesting since we do not observe macroscopic fermionic fields in nature.

If we set all the fermionic fields to 0 we can see that the supersymmetry transformations of the bosonic fields vanish as well, which means that the Killing spinor equations (21) simplify to

$$\delta(\epsilon) F = 0.$$  \hspace{1cm} (22)

As we have seen before, the Killing spinor equation for $N = 1, D = 4$ Poincaré supergravity is

$$D_\mu \epsilon = 0.$$

For $N = 1, D = 4$ AdS supergravity the transformation of the gravitino is given by

$$\delta \Psi_\mu = \hat{D}_\mu \epsilon = \left( D_\mu - \frac{1}{2l} \gamma_\mu \right) \epsilon,$$

which gives Killing spinor equation $\hat{D}_\mu \epsilon = 0$.

For more general supergravity theories the Killing spinor equation that comes from the gravitino can be written as

$$\tilde{D}_\mu \epsilon = 0,$$

where $\tilde{D}_\mu$ is often called the supercovariant derivative and will get additional terms depending on the theory as we will see later.

### 2.7.1 Integrability conditions

Let’s say that we have some supergravity theory with some fermionic field, for example, the spin $\frac{3}{2}$ gravitino, $\Psi_\mu$ that we always have, with Killing spinor equation $\hat{D}_\mu \epsilon = 0$. Then, if $\epsilon$ is a Killing spinor, then, trivially, $[\hat{D}_\mu, \hat{D}_\nu] \epsilon = 0$ as well, which means that it is an integrability condition of the differential equation. Moreover, it is an algebraic condition on the Killing spinors.

Since the Killing spinor equations can be quite hard to solve in general, it is convenient to have an integrability condition that can help to solve the differential equations. For the gravitino, the integrability condition can be written as

$$[\hat{D}_\mu, \hat{D}_\nu] \epsilon = \frac{1}{4} \tilde{R}_{\mu\nuab} \gamma^{ab} = 0,$$
where $\tilde{R}_{\mu\nu ab}$ is a generalization of the curvature tensor for the supercovariant derivative.

If we have more than one fermionic field, then have more integrability conditions by taking the commutator between the variations of different fields as we will see in section 4.

As an example we will again look at $N = 1, D = 4$ Poincaré supergravity, where the integrability condition on the Killing spinors is

$$\frac{1}{4} R_{\mu\nu ab} \gamma^{ab} \epsilon = 0.$$

We already know that the Minkowski spacetime solution have the maximal amount of unbroken supersymmetry, and since the curvature tensor in 0 the integrability condition is trivially satisfied.

Solutions with the maximum amount of unbroken symmetries are important for the study of vacuum solution in supergravity. Here we are interested in black hole solutions which imply that we want to break some of the supersymmetry. In section 3 we look at a black hole solution to $N = 2, D = 4$ gauged supergravity, and use the integrability condition to find the Killing spinors explicitly.

In section 4 we study $N = (1,0), D = 6$ gauged supergravity and starting from an ansatz of a dyonic string solution we find conditions for this ansatz to have Killing spinors. Then we use the integrability condition to show that this implied that the equation of motion of the system is satisfied when Killing spinors exist.

## 3 AdS Torus black hole is supersymmetric?

The AdS/CFT correspondence makes the study of asymptotically AdS solutions of supergravity theories important. BPS black hole solutions are particularly interesting to study because of their importance when it comes to the counting of microstates of black holes.

In this section, we will consider the gauged $N = 2, d = 4$ supergravity theory. In [5] they showed that the spherical Reissner-Nordström is a BPS solution in two different cases. The first case the magnetic charge vanishes and the electric charge is $q_e = m^2$, where $m$ is the mass parameter. In the second case $m = 0, q_m = \pm \frac{l}{2}$, where $l$ is related to the cosmological constant as $\Lambda = -\frac{3}{l^2}$. In supergravity with a cosmological constant there exist topological black holes that are asymptotically AdS, for example, a torus or other higher genus manifolds. The generalization to the torus and other higher genus cases was studied in [1]. Here we will reproduce the
results for the Reissner-Nordström black hole with topology of a torus. We want to find the configurations that preserve some supersymmetry, i.e., when it is a BPS solution. We will find that this happens in two different cases just as for the spherical space considered in [5].

In section 3.1 we will start by introducing the non-rotating charged torus solution in four dimensions. Then in section 3.2 we will introduce the gauged $N = 2, d = 4$ supergravity theory. In the rest of section 3 we use the integrability conditions to find the configurations of the solution that preserves some supersymmetry, then we will explicitly solve the Killing spinor equation and hence find the Killing spinors for those two cases.

3.1 AdS black holes

A four-dimensional asymptotically Anti-de Sitter non-rotating dyonic black hole with the topology of a torus has a metric on the form

\[ ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2dx^2 + r^2dy^2, \]

with $x, y \in [0, 1]$, where 0 and 1 are identified, and

We have a cosmological constant $\Lambda = -\frac{3}{l^2}$, and an electromagnetic potential

where $q_m$ and $q_e$ are magnetic and electric charges respectively.

Since the metric is diagonal an obvious choice of Vierbein, $e^a_\mu$, is

\[
\begin{align*}
    e^t_t &= \sqrt{V(r)} \\
    e^r_r &= \frac{1}{\sqrt{V(r)}} \\
    e^x_x &= r \\
    e^y_y &= r,
\end{align*}
\]

where the rest of the components is zero.

Since we are working in the second order formalism of gravity the spin connection is the unique torsion free one and is hence given by

\[
\omega_{\mu}^{ab} = \eta^{bb'} (e^a_\nu e^b'_{\lambda} \Gamma^{\nu}_{\mu\lambda} - e^b'_{\nu} \partial_{\mu} e^a_{\lambda}),
\]

where $\Gamma^{\nu}_{\mu\lambda}$ is the Christoffel symbols. The non-zero Christoffel symbols are

\[
\begin{align*}
    \Gamma^0_{01} &= \frac{V'(r)}{2V(r)}, & \Gamma^1_{00} &= \frac{V'(r)V(r)}{2}, & \Gamma^1_{11} &= -\frac{V'(r)}{2V(r)}, \\
    \Gamma^1_{22} &= -rV(r) = \Gamma^1_{33}, & \Gamma^2_{12} &= \frac{1}{r} = \Gamma^3_{31}.
\end{align*}
\]
This gives the following non-zero components of the spin connection:

\begin{align*}
\omega_{01} &= \frac{V'}{2} \\
\omega_{12} &= -\sqrt{V} \\
\omega_{31} &= \sqrt{V}.
\end{align*}

In the Killing spinor equation we will have terms where we contract the electromagnetic field strength with the gamma matrices, so we introduce the notation \( \tilde{F} := F_{ab} \gamma^{ab} \), where

\[ F_{ab} = F_{\mu\nu} e_a^\mu e_b^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu) e_a^\mu e_b^\nu. \]

The non-zero components of \( F_{ab} \) is \( F_{rt} = a'(r) \) and \( F_{xy} = \frac{b'(x)}{r^2} \), which gives

\[ \tilde{F} = 2a' \gamma^{10} + \frac{2b'}{r^2} \gamma^{23} = \frac{2d_e}{r^2} \gamma^{10} + \frac{2q_m}{r^2} \gamma^{23}. \]

\section{N=2 gauged supergravity}

Here we review the basic ingredients of \( N = 2 \) gauged supergravity which was first found in [27]. We look at the particles that are present in the theory and then write down the super covariant derivative of the theory which gives the Killing spinor equation which we will have to solve in the next sections for our problem.

In the ungauged \( N = 2 \) supergravity there is a SO(2) R-symmetry that is rotating the two independent Majorana supersymmetries. When we make this into a local symmetry by introducing a minimal gauge coupling between the photons and the gravitino we get the gauged theory.

The theory contains a graviton \( e^{a}_\mu \), a Maxwell gauge field \( A_\mu \) and two real gravitino \( \psi^i_\mu \), which can be combined into a complex spinor \( \psi_\mu = \psi^1_\mu + i \psi^2_\mu \).

The Lagrangian of the theory can be written as

\[ \mathcal{L} = e \left( -\frac{1}{4} R + \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu} \partial_\nu \psi_\mu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{8} (F^{\mu\nu} + \tilde{F}^{\mu\nu}) \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_5 \gamma_\epsilon \psi_\epsilon - \frac{1}{2l} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu - \frac{3}{2l^2} \right). \]

\( D_\mu \) denotes the gauge- and Lorentz covariant derivative and is defined by

\[ D_\mu := \partial_\mu + \frac{1}{4} \omega_\mu^{\ ab} \gamma_{ab} - \frac{i}{l} A_\mu. \]

\( \hat{F}_{\mu\nu} \) is the so-called super covariant field strength and is given by \( \hat{F}_{\mu\nu} = F_{\mu\nu} - \text{Im}(\bar{\psi}_\mu \psi_\nu) \). We can see that we have a cosmological constant \( \Lambda = -\frac{3}{l^2} \).
The action is invariant under the following local supertransformations
\[ \delta e^a_{\mu} = \text{Re}(\bar{\epsilon} \gamma^a \psi_{\mu}) \]
\[ \delta A_{\mu} = \text{Im}(\bar{\epsilon} \psi_{\mu}) \]
\[ \delta \psi_{\mu} = \hat{\nabla}_{\mu} \epsilon \]
where \( \epsilon \) is an infinitesimal Dirac spinor, and \( \hat{\nabla}_{\mu} \) is the supercovariant derivative defined by
\[ \hat{\nabla}_{\mu} = D_{\mu} + \frac{1}{2l} \gamma_{\mu} + \frac{i}{4} F_{ab} \gamma^{ab} \gamma_{\mu}. \] (27)

Since we are only interested in the bosonic part, we set \( \psi_{\mu} = 0 \), so that the only field that has non-zero variation is the gravitino \( \psi_{\mu} \). This means that a Dirac spinor \( \epsilon \) is a Killing spinor if it solves the Killing spinor equation
\[ \hat{\nabla}_{\mu} \epsilon = 0. \]

If the Killing spinor equation holds for \( \epsilon \), then it is also true that
\[ R_{\mu\nu} \epsilon := (\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} - \hat{\nabla}_{\nu} \hat{\nabla}_{\mu}) \epsilon = 0, \]
this is an integrability condition of the system, so it can be used to simplify the Killing spinor equation. The Killing spinor equation is a differential equation which can be hard to solve, therefore using the integrability condition to simplify it can help a lot. The supercovariant curvature tensor, \( \hat{R}_{\mu\nu} \), is given by an algebraic expression, and it is straightforward to calculate.

### 3.3 Find Killing spinors

In this section, we will obtain the Killing spinor equation for the dyonic AdS genus one black hole using equation (27). Then we are going to calculate the curvature tensor \( \hat{R}_{\mu\nu} \), which gives two special cases with non-trivial solutions and then use it to simplify the Killing spinor equations, which we solve explicitly for both cases.

We start by calculating \( D_{\mu} \) using equation (26). The non-zero components of the spin connection is given in (24), and the electromagnetic potential is \( A_t = a(r), A_y = b(x) \). Putting it all together we get
\[ D_t = \partial_t + \frac{V'}{4} \gamma_{01} - \frac{i}{\ell} a(r) \]
\[ D_r = \partial_r \]
\[ D_x = \partial_x + \frac{\sqrt{V}}{2} \gamma_{21} \]
\[ D_y = \partial_y + \frac{\sqrt{V}}{2} \gamma_{31} - \frac{i}{\ell} b(x). \]
The last two terms in the Killing spinor equation, (27), both have a \( \gamma_\mu \) term, and it is important to remember that \( \gamma_\mu = \gamma_a e_\mu^a \) where \( \gamma_a \) is the flat gamma matrices that satisfy \( \{ \gamma_a, \gamma_b \} = 2 \eta_{ab} \). We also have a \( \hat{F} \) term in the super covariant derivative, which is equal to \( \hat{F} \), which we calculated in (25), since \( \psi_\mu = 0 \).

Putting it all together we get

\[
\hat{\nabla}_t = D_t + \frac{1}{2l} \sqrt{V} \gamma_0 + \frac{i}{4} \sqrt{V} \hat{F} \gamma_0 \\
\hat{\nabla}_r = D_r + \frac{1}{2l} \sqrt{V} \gamma_1 + \frac{i}{4} \sqrt{V} \hat{F} \gamma_1 \\
\hat{\nabla}_x = D_x + \frac{r}{2l} \gamma_2 + \frac{i}{4} r \hat{F} \gamma_2 \\
\hat{\nabla}_y = D_y + \frac{r}{2l} \gamma_3 + \frac{i}{4} r \hat{F} \gamma_3
\]

written out more explicitly:

\[
\hat{\nabla}_t = \partial_t + \frac{1}{2r} \left( \frac{q}{r} - \frac{z^2}{r^2} + \frac{r^2}{l^2} \right) \gamma_{01} + \frac{i}{4} \frac{q_r}{r} + \frac{1}{2l} \sqrt{V} \gamma_0 + \frac{i}{4} \sqrt{V} \hat{F} \gamma_0 \\
\hat{\nabla}_r = \partial_r + \frac{1}{2l} \sqrt{V} \gamma_1 + \frac{i}{4} \sqrt{V} \hat{F} \gamma_1 \\
\hat{\nabla}_x = \partial_x + \frac{\sqrt{V}}{2} \gamma_{21} + \frac{r}{2l} \gamma_2 + \frac{i}{4} r \hat{F} \gamma_2 \\
\hat{\nabla}_y = \partial_y + \frac{\sqrt{V}}{2} \gamma_{31} - \frac{i}{4} q_m x + \frac{r}{2l} \gamma_3 + \frac{i}{4} r \hat{F} \gamma_3.
\]

Next, we want to simplify the Killing spinor equation \( \hat{\nabla}_\mu \epsilon \) by using the integrability condition \( \hat{R}_{\mu
u} \epsilon = 0 \). That means that we have to calculate the commutator between the different supercovariant derivatives. We note that \( \hat{R}_{tx} \) and \( \hat{R}_{ty} \) is going to have the same form only with \( \gamma_3 \) replacing \( \gamma_2 \). The same is true for \( \hat{R}_{tx} \) and \( \hat{R}_{ty} \).

It is convenient to introduce an operator \( \mathcal{P} := \frac{r^2}{2l} i \hat{F} \gamma_1 \) since it has the property \( \mathcal{P}^2 = 1 \). We will show that it is possible to write the curvature tensor on the form

\[
\hat{R}_{\mu\nu} = \mathcal{P} G_{\mu\nu} \mathcal{O}
\]

where \( G_{\mu\nu} \) is a function times \( \gamma_{\mu\nu} \) and \( \mathcal{O} \) is some operator.

When we have a derivative in the commutator we must remember that the commutator acts on the spinor \( \epsilon \), and that

\[
[\partial_x, f(x)]\epsilon(x) = \partial_x (f(x)\epsilon(x)) - f(x)\partial_x \epsilon(x) = f'(x)\epsilon(x)
\]
We start by calculating $\hat{R}_{tr}$. Nothing in $\hat{\nabla}_r$ depends on time, so $[\partial_t, \hat{\nabla}_r] = 0$, this gives

$$\hat{R}_{tr} = - \partial_r \left( \frac{1}{2r} \left( \frac{\eta}{r} - \frac{z^2}{r^2} + \frac{r^2}{l^2} \right) \gamma_{01} + \frac{i q_e}{l r} + \frac{1}{2l} \sqrt{V} \gamma_0 + \frac{i}{4} \sqrt{V} \gamma_0 \right) +$$

$$+ \frac{1}{2l^2} \gamma_{01} - \frac{i}{8l} [\gamma_{10}, \gamma_{01}] + \frac{i}{8l} [\gamma_{01}, \gamma_{10}] - \frac{1}{16} [\gamma_{01}, \gamma_{10}]$$

$$+ \frac{V'}{8l \sqrt{V}} [\gamma_{01}, \gamma_1] + \frac{i V'}{16 \sqrt{V}} [\gamma_{01}, \gamma_1],$$

where

$$[\gamma_{01}, \gamma_1] = -2 \gamma_1 \gamma_{01}, \quad [\gamma_{10}, \gamma_{01}] = -4 \frac{b'}{r^2} \gamma_{23} \gamma_{01}$$

$$[\gamma_{01}, \gamma_{10}] = 4 \frac{b'}{r^2} \gamma_{23} \gamma_{01}, \quad [\gamma_{01}, \gamma_{10}] = -8 \frac{z^2}{r^4} \gamma_{01}$$

Now we want to factorize out $\gamma_{01}$, we start with the derivative part

$$\partial_r (\nabla_r - \partial_r) = - \frac{i q_e}{l r^2} + \frac{i V'}{4l \sqrt{V}} \gamma_0 + \frac{i V'}{8 \sqrt{V}} \gamma_{01} - \frac{i}{2r} \sqrt{V} \gamma_{01} - \frac{\eta}{r^3} \gamma_{01} + 3 \frac{z^2}{r^4} \gamma_{01} + \frac{1}{2l^2} \gamma_{01}$$

$$= \left[ \frac{i q_e}{l r^2} \gamma_{01} - \frac{V'}{4l \sqrt{V}} \gamma_{01} - \frac{i V'}{8 \sqrt{V}} \gamma_{01} + \frac{i \sqrt{V}}{2r} \gamma_{01} - \frac{\eta}{r^3} \gamma_{01} + 3 \frac{z^2}{r^4} \gamma_{01} + \frac{1}{2l^2} \gamma_{01} \right] \gamma_{01},$$

where we in the second step multiplied by 1 several places in such a way that we could factorize out $\gamma_{01}$. The other part of the curvature, the one without derivatives, can also be factorized as something times $\gamma_{01}$ in the following way:

$$\frac{1}{2l^2} \gamma_{01} - \frac{i}{8l} [\gamma_{10}, \gamma_{01}] + \frac{i}{8l} [\gamma_{01}, \gamma_{10}] - \frac{1}{16} [\gamma_{01}, \gamma_{10}]$$

$$+ \frac{V'}{8l \sqrt{V}} [\gamma_{01}, \gamma_1] + \frac{i V'}{16 \sqrt{V}} [\gamma_{01}, \gamma_1]$$

$$= \frac{1}{2l^2} \gamma_{01} + \frac{i}{8l} \gamma_{23} \gamma_{01} + \frac{z^2}{2r^4} \gamma_{01} - \frac{V'}{4l \sqrt{V}} \gamma_{10} \gamma_{01} - \frac{i V'}{8 \sqrt{V}} \gamma_{10} \gamma_{01}.$$
Putting it all together we get

\[
\hat{R}_{tx} = -\left[-\frac{i q_e}{l r^2} \gamma_{10} - \frac{V'}{4l \sqrt{V}} \gamma_1 - \frac{i V'}{8 \sqrt{V}} \hat{F} \gamma_1 + \frac{i \sqrt{V}}{2r} \hat{F} \gamma_1 - \frac{\eta}{r^3} + \frac{3 z^2}{2 r^4} + \frac{1}{2 l^2}\right] \gamma_{01} +
\]

\[
+ \left[\frac{1}{2 l^2} + i \frac{2 b'}{2 l r^2} \gamma_{23} + \frac{z^2}{2 r^4} - \frac{V'}{4l \sqrt{V}} \gamma_1 - \frac{i V'}{8 \sqrt{V}} \hat{F} \gamma_1\right] \gamma_{01} = \]

\[
= \left(i \frac{2 l}{2} \left(2 a' \gamma_{10} + \frac{2 b'}{2 l r^2} \gamma_{23}\right) - i \frac{\sqrt{V}}{2r} \hat{F} \gamma_1 + \left(\frac{\eta}{r^3} - \frac{z^2}{r^4}\right) \mathcal{P} \mathcal{P} \right) \gamma_{01} = \]

\[
= \mathcal{P} \left(\frac{z}{l r^2} \gamma_1 - \frac{z \sqrt{V}}{r^3} + \frac{z}{r^3} \left(\frac{\eta}{z} - \frac{z}{r}\right) \mathcal{P} \right) \gamma_{01} = -\mathcal{P} \frac{\gamma_{01}}{r^3} \left(\frac{r}{l} \gamma_1 + \sqrt{V} + \left(\frac{\eta}{z} - \frac{z}{r}\right) \mathcal{P}\right),
\]

which we can see is on the form (28).

Next we calculate \(\hat{R}_{tx}\), which also gives us the result for \(\hat{R}_{ty}\). There is no \(t\)-dependence in \(\nabla_x\) and no \(x\)-dependence in \(\nabla_t\), so the derivative terms will be zero. Some of the terms will be zero since

\[
[\gamma_{01}, \gamma_{12}] = [\gamma_{02}, \hat{F} \gamma_2] = [\hat{F} \gamma_0, \hat{F} \gamma_2] = [\gamma_{01}, \gamma_{2}] = 0.
\]

The surviving terms is

\[
\hat{R}_{tx} = \frac{\sqrt{V}}{2l^2} r \gamma_{02} + \frac{i \sqrt{V} r}{8 l} \gamma_{10} - \frac{i V}{8} \hat{F} \gamma_{12} + \frac{i r \sqrt{V}}{8 l} \hat{F} \gamma_{02} -
\]

\[
- \frac{\sqrt{V}}{4 r} \left(\frac{\eta}{r} - \frac{z^2}{r^2} + \frac{r^2}{l^2}\right) \left[\gamma_{01}, \gamma_{12}\right].
\]

The commutators between the gamma can be written as

\[
[\gamma_{01}, \hat{F} \gamma_2] = \frac{4 b'}{r^2} \gamma_{23} \gamma_{02}, \quad [\hat{F} \gamma_0, \gamma_{12}] = -2 \hat{F} \gamma_{1} \gamma_{02},
\]

\[
[\hat{F} \gamma_0, \gamma_{2}] = 4 a' \gamma_{10} \gamma_{02}, \quad [\gamma_{01}, \gamma_{12}] = 2 \gamma_{02},
\]

30
putting this into equation (29) gives

\[
\hat{R}_{tx} = \left( \frac{\sqrt{V}r}{2l^2} + \frac{i\sqrt{V}r}{4l} f' + \frac{iV}{4} f' \gamma_1 - \frac{\sqrt{V}}{2r} \left( \frac{\eta}{r} - \frac{z^2}{r^2} + \frac{r^2}{l^2} \right) \right) \gamma_0^2 =
\]

\[
= \sqrt{V} \left( \frac{i}{4l} f' \gamma_1 \gamma_1 + \frac{i\sqrt{V}}{4} f' \gamma_1 - \frac{1}{2r} \left( \frac{\eta}{r} - \frac{z^2}{r^2} \right) \mathcal{P} \mathcal{P} \right) \gamma_0^2 =
\]

\[
= \sqrt{V} \mathcal{P} \left( \frac{z}{2lr} \gamma_1 + \frac{z\sqrt{V}}{2r^2} - \frac{z}{2r^2} \left( \frac{\eta}{r} - \frac{z}{r} \right) \mathcal{P} \right) \gamma_0^2 =
\]

\[
= \mathcal{P} \gamma_0^2 \frac{z}{2r^2} \sqrt{V} \left( \frac{r}{l} \gamma_1 + \sqrt{V} + \left( \frac{\eta}{z} - \frac{z}{r} \right) \mathcal{P} \right).
\]

In the second line we factorized out the square root of the potential and multiplied two of the terms by one in such a way that we can factorize out \( \mathcal{P} \) in the third line to try to get \( \hat{R}_{tx} \) into the same general form as \( \hat{R}_{tr} \).

To get the \( \hat{R}_{ty} \) component of the curvature tensor we just have to change \( \gamma_0^2 \) to \( \gamma_0^3 \) in the expression for \( \hat{R}_{tx} \), this gives \( \hat{R}_{ty} = \mathcal{P} \gamma_0^3 \frac{z}{2r^2} \sqrt{V} \left( \frac{r}{l} \gamma_1 + \sqrt{V} + \left( \frac{\eta}{z} - \frac{z}{r} \right) \mathcal{P} \right) \).

Since we want to write all the components of the curvature tensor on the same form, the strategy for the following computations is the same as for the previous ones, that is to factorize out \( \gamma_{\mu\nu} \) and \( \mathcal{P} \) in such a way that \( \hat{R}_{\mu\nu} = \mathcal{P} \gamma_{\mu\nu} f(r) \left( \frac{r}{l} \gamma_1 + \sqrt{V} + \left( \frac{\eta}{z} - \frac{z}{r} \right) \mathcal{P} \right) \) for some function \( f(r) \).

Now we want to calculate \( \hat{R}_{rx} \) and \( \hat{R}_{ry} \), which will be on the same form since \( \nabla_x \) and \( \nabla_y \) has the same \( r \) dependence and the terms with gamma matrices are the same when you switch \( \gamma_2 \) to \( \gamma_3 \).

\[
\hat{R}_{rx} = [\hat{\nabla}_r, \hat{\nabla}_x] = \partial_r \left( -\frac{1}{2} \sqrt{V} \gamma_{12} + \frac{1}{2l} r \gamma_2 + \frac{i}{4} r f' \gamma_2 \right) - \frac{1}{4l} [\gamma_1, \gamma_{12}] + \frac{r}{2l^2 \sqrt{V}} \gamma_{12} +
\]

\[
+ \frac{ir}{8l \sqrt{V}} [\gamma_1, f' \gamma_2] - \frac{i}{8} [f' \gamma_1, \gamma_{12}] + \frac{ir}{8 \sqrt{V}} [f' \gamma_1, \gamma_2] - \frac{r}{16 \sqrt{V}} [f' \gamma_1, f' \gamma_2],
\]

where

\[
[f' \gamma_1, f' \gamma_2] = [f' \gamma_1, \gamma_{12}] = 0, \quad [\gamma_1, \gamma_{12}] = 2 \gamma_1 \gamma_{12}
\]

\[
[\gamma_1, f' \gamma_2] = \frac{4b'}{r^2} \gamma^{23} \gamma_{12}, \quad [f' \gamma_1, \gamma_2] = 4a' \gamma^{10} \gamma_{12}.
\]
Inserting this into the expression for $\hat{R}_{rx}$ gives,

$$\hat{R}_{rx} = \left[ \left( -\frac{V'}{4\sqrt{V}} + \frac{1}{2l} \gamma_1 - \frac{i}{4} F \gamma_1 \right) - \frac{1}{2l} \gamma_1 + \frac{r}{2l^2 \sqrt{V}} + \frac{ir}{4l \sqrt{V}} \right] \gamma_{12}$$

$$= \frac{1}{\sqrt{V}} \left[ -\frac{V'}{4} P \frac{i}{4} F \gamma_1 \sqrt{V} + \frac{r}{2l^2} P \frac{4l}{\sqrt{V}} \gamma_1 \gamma_{12} \right]$$

$$= P \frac{1}{\sqrt{V}} \left[ \left( -\frac{1}{4} \left( \frac{2\eta}{r^2} + \frac{2r}{l^2} - \frac{2z^2}{r^3} \right) + \frac{r}{2l^2} \right) P - \frac{z}{2r^2 \sqrt{V}} + \frac{z}{2r^2} \right] \gamma_{12}$$

$$= P \frac{z}{2r^2 \sqrt{V}} \left[ \sqrt{V} + \frac{r}{l} \gamma_1 - \left( \frac{\eta}{z} - \frac{z}{r} \right) \right] \gamma_{12}$$

$$= - P \frac{z}{2r^2 \sqrt{V}} \left[ \gamma_{12} \left[ \sqrt{V} + \frac{r}{l} \gamma_1 + \left( \frac{\eta}{z} - \frac{z}{r} \right) \right] \right]$$

which implies

$$\hat{R}_{ry} = - P \frac{z}{2r^2 \sqrt{V}} \left[ \sqrt{V} + \frac{r}{l} \gamma_1 + \left( \frac{\eta}{z} - \frac{z}{r} \right) \right] \gamma_{12}.$$

The last component is computed in the same way. We get a term $-\frac{i}{l} q_m$ from the derivatives since $\nabla_y$ has a term $-\frac{i}{l} q_m x$, the rest of the terms comes from the commutator between terms with gamma matrices.

$$\hat{R}_{xy} = - \frac{i}{l} q_m + \frac{V}{4} \left[ \gamma_{12}, \gamma_{13} \right] - \frac{ir \sqrt{V}}{8} \left[ \gamma_{12}, F \gamma_{33} \right] + \frac{r^2}{2l^2} \gamma_{23} + \frac{ir^2}{8l} \left[ \gamma_{22}, F \gamma_{33} \right] - \frac{ir \sqrt{V}}{8} \left[ F \gamma_{22}, \gamma_{33} \right] - \frac{r^2}{16} \left[ F \gamma_{22}, F \gamma_{33} \right]$$

with

$$\left[ \gamma_{12}, \gamma_{13} \right] = -2 \gamma_{23}, \quad \left[ \gamma_{12}, F \gamma_{33} \right] = -2 F \gamma_{11} \gamma_{23}$$

$$\left[ \gamma_{22}, \gamma_{33} \right] = 4a' \gamma_{10} \gamma_{23}, \quad \left[ F \gamma_{22}, \gamma_{33} \right] = -2 F \gamma_{11} \gamma_{23}$$

$$\left[ F \gamma_{22}, \gamma_{33} \right] = 4a' \gamma_{10} \gamma_{23}, \quad \left[ F \gamma_{22}, F \gamma_{33} \right] = \frac{8z^2}{r^4} \gamma_{23}$$

32
inserting this gives
\[
\hat{R}_{rx} = -i\frac{q_m}{l} - \frac{V}{2}\gamma_{23} + \frac{r^2}{2l^2}\gamma_{23} + \frac{ir^2}{2l}2d'\gamma_{10}\gamma_{23} + \frac{ir\sqrt{V}}{2}F\gamma_{3}\gamma_{23} - \frac{z^2}{2r^2}\gamma_{23}
\]
\[
= \left[ \frac{ir^2}{2l}2d\gamma_{23} + \frac{ir^2}{2l}2d'\gamma_{10} - \frac{V}{2} + \frac{r^2}{2l^2} - \frac{ir\sqrt{V}}{2}F\gamma_{3}\gamma_{23} - \frac{z^2}{2r^2}\right]\gamma_{23}
\]
\[
= \frac{ir^2}{2l}F\gamma_{1}\gamma_1 + \frac{ir\sqrt{V}}{2}F\gamma_{1} + \left( \frac{\eta}{r} - \frac{z^2}{r^2} \right) P P
\]
\[
= \gamma_{23}
\]
\[
= \gamma_{23}
\]
\[
= \gamma_{23}
\]
\[
= \gamma_{23}
\]
\[
= \gamma_{23}
\]
\[
= \gamma_{23}
\]
If we compare with equation (28) we can see that the operator \( O \) is
\[
O = \left[ \sqrt{V} + \frac{r}{l}\gamma_1 + \left( \frac{\eta}{z} - \frac{z}{r} \right) P \right].
\]
In conclusion the integrability condition is
\[
\hat{R}_{\mu\nu}\epsilon = \gamma_{23} = 0.
\]
(30)
Now we need to find a condition for when \( \hat{R}_{\mu\nu} \) is singular because that give us non-trivial solutions to (30). Since \( P \) has the property \( P^2 = 1 \) we get that \( \text{Det} P \neq 0 \) and hence it is not singular. \( G_{\mu\nu} \) is some scalar function times \( \gamma_{\mu\nu} \) which means that it is non-singular. This only leaves \( O \), which determinant is given by
\[
\text{Det} O = \frac{l^2\eta^4 - 4q_m^2(z^2 - r\eta)^2}{l^2z^4}.
\]
(31)
This gives us two separate cases when \( \text{Det} O = 0 \), the first case is when \( q_m = \eta = 0 \) which gives \( V(r) = \frac{q^2}{r^2} + \frac{r^2}{l^2} \), and the second case is when \( l = \infty, \eta = 0 \) which gives \( V(r) = \frac{z^2}{l^2} \).

Now we will study the two cases separately. where we first simplify the Killing spinor equation using the integrability condition and then we will get a differential equation which we can solve.

3.3.1 Case 1, \( q_m = \eta = 0 \)

We want to simplify \( \hat{\nabla}_\mu\epsilon = 0 \) by using the integrability condition \( \hat{R}_{\mu\nu}\epsilon = 0 \), which in this case gives \( O \epsilon = \left[ \sqrt{V} + \frac{r}{l}\gamma_2 - \frac{z\eta}{r}\gamma_1 \right] \epsilon = 0 \), where \( V(r) = \frac{q^2}{r^2} + \frac{r^2}{l^2} \). For the time component this gives
\[ \hat{\nabla}_t \epsilon = \left( \partial_t + \frac{iq_e}{lr} + \sqrt{\frac{V}{2l}} \gamma_0 + \frac{iq_e \sqrt{V}}{2r^2} \gamma_1 + \frac{r}{2l^2} \gamma_{01} - \frac{q_e^2}{2r^3} \gamma_{01} \right) \epsilon = \]

\[ = \left( \partial_t + \frac{\gamma_0}{2l} \left[ -2\frac{iq_e}{r} \gamma_0 + \sqrt{V} + \frac{r}{l} \gamma_1 - \left( \frac{ilq_e \sqrt{V}}{r^2} \gamma_0 + \frac{q_e^2 l}{r^3} \right) \gamma_1 \right] \right) \epsilon \]

where in the first line we just used \( \eta = q_m = 0 \), and in the second line we factorized out \( \gamma_0 \). Using \( \mathcal{O} \epsilon = 0 \) we can replace \( \sqrt{V} + \frac{r}{l} \gamma_1 \) by \( \frac{iq_e}{r} \gamma_0 \) this implies that

\[ -2 \frac{iq_e}{r} \gamma_0 + \sqrt{V} + \frac{r}{l} \gamma_1 = -\frac{iq_e}{r} \gamma_0. \]

The part in the parenthesis multiplying \( \gamma_1 \) can be simplified as well,

\[ - \left( \frac{ilq_e \sqrt{V}}{r^2} \gamma_0 + \frac{q_e^2 l}{r^3} \right) \gamma_1 = - \frac{ilq_e \gamma_0}{r^2} \left( \sqrt{V} - \frac{iq_e}{r} \gamma_1 \gamma_0 \right) \gamma_1 = \frac{iq_e}{r} \gamma_0. \]

Putting it all together we find that the time component of the Killing spinor equation is trivial and hence that the Killing spinor must be constant in time, i.e.,

\[ \hat{\nabla}_t \epsilon = \partial_t \epsilon = 0. \]

The r-component of the supercovariant derivative in this case is simplified to

\[ \hat{\nabla}_r = \partial_r + \frac{1}{2r \sqrt{V}} \left( \frac{r}{l} \gamma_1 + \frac{iq_e}{r} \gamma_0 \right), \]

this cannot be simplified further using \( \mathcal{O} \epsilon = 0 \).

We can treat the \( x \) and \( y \) component of the supercovariant derivative in the same way since \( q_m = 0 \), which means that the only difference in their respective Killing spinor equations is that \( \gamma_2 \) is replaced \( \gamma_3 \).

\[ \hat{\nabla}_x = \partial_x - \frac{1}{2} \sqrt{V} \gamma_{12} + \frac{r}{2l} \gamma_2 + \frac{irq_e}{4r^2} \gamma_{01} \gamma_2 = \]

\[ = \partial_x - \frac{1}{2} \left( \sqrt{V} \gamma_1 - \frac{r}{l} - \frac{iq_e}{r} \gamma_{01} \right) \gamma_2 = \]

\[ = \partial_x - \frac{1}{2} \gamma_1 \left( \sqrt{V} - \frac{r}{l} \gamma_1 + \frac{iq_e}{r} \gamma_0 \right) \gamma_2 = \partial_x - \frac{1}{2} \gamma_{12} \mathcal{O} = \partial_x, \]
which gives $\hat{\nabla}_y \epsilon = \partial_y \epsilon = 0$. So we have simplified the Killing spinor equations to

$$
\hat{\nabla}_t \epsilon = \partial_t \epsilon = 0 \\
\hat{\nabla}_r \epsilon = \left[ \partial_r + \frac{1}{2r \sqrt{V}} \left( \frac{r}{l} \gamma_1 + \frac{q_e i}{r} \gamma_0 \right) \right] \epsilon = 0 \\
\hat{\nabla}_x \epsilon = \partial_x \epsilon = 0 \\
\hat{\nabla}_y \epsilon = \partial_y \epsilon = 0.
$$

(32)

From this we can see that $\epsilon$ only depend on $r$ and the Killing spinor equation has been reduced to a differential equation with a constraint

$$
\left\{ \begin{array}{l}
\partial_r \epsilon(r) = -\frac{1}{2r \sqrt{V}} \left( \frac{r}{l} \gamma_1 + \frac{q_e i}{r} \gamma_0 \right) \\
\frac{1}{2\sqrt{V}} \mathcal{O} \epsilon(r) = \frac{1}{2} \left( 1 + \frac{r}{l \sqrt{V}} - \frac{q_e}{r \sqrt{V}} i \gamma_0 \right) \epsilon = 0.
\end{array} \right.
$$

(33)

In appendix A we show how to solve this kind of differential equation. In this case the solution is

$$
\epsilon(r) = \left( \left( -\frac{r \sqrt{V}}{q_e} - \frac{r^2}{l q_e} \right)^{1/2} - \left( \frac{r^2}{l q_e} - \frac{r \sqrt{V}}{q_e} \right)^{1/2} i \gamma_0 \right) P(-\gamma_1) \epsilon_0,
$$

where $\epsilon_0$ is a constant complex 4-dimensional spinor and $P(-\gamma_1)$ is the projection operator defined in appendix A which projects the constant 4-dimensional spinor $\epsilon_0$ onto a 2-dimensional subspace. This means that we have 2 independent solutions of the Killing spinor equation, and hence this solution is $\frac{1}{2}$-BPS.

### 3.3.2 Case 2, $l = \infty$, $\eta = 0$

In this case the integrability condition implies $\mathcal{O} \epsilon = \left( \sqrt{V} - \frac{z^2}{r} \right) \epsilon = 0$ and $V(r) = \frac{z^2}{r^2}$. Now we are going to use this to simplify the Killing spinor equation, starting with the time component

$$
\hat{\nabla}_t \epsilon = \left( \partial_t + \frac{i}{4} \sqrt{V} \gamma_0 - \frac{z^2}{2r^3} \gamma_0 \right) \epsilon = \left( \partial_t + \frac{\gamma_{01}}{2} \left( \frac{i}{2} \sqrt{V} \gamma_1 - \frac{V}{r} \right) \right) \epsilon = \left( \partial_t + \frac{\gamma_{01}}{2r} \left( V - V \right) \right) \epsilon = \partial_t \epsilon.
$$

35
In the last line we used
\[ 0 = \mathcal{O} \epsilon \Rightarrow V \epsilon = \sqrt{V} \frac{F}{2} \gamma_1 \epsilon. \]

Next we calculate the time component where we find that the Killing spinors must have some \( r \)-dependence
\[
\hat{\nabla}_r \epsilon = \left( \partial_r + \frac{i}{4\sqrt{V}} F \gamma_1 \right) \epsilon = \left( \partial_r + \frac{1}{2r\sqrt{V}} ( \frac{z}{r} \mathcal{P} ) \right) \epsilon = \left( \partial_r + \frac{\sqrt{V}}{2r\sqrt{V}} \right) \epsilon = \left( \partial_r + \frac{1}{2r} \right) \epsilon.
\]

Once again the \( x \)- and the \( y \)-component is the same since the term \(-\frac{i}{4}q_m x \) in \( \hat{\nabla}_y \) vanishes in the limit \( l \rightarrow \infty \).

\[
\hat{\nabla}_x \epsilon = \left( \partial_x - \frac{1}{2} \sqrt{V} \gamma_{12} + \frac{i}{4} r F \gamma_2 \right) \epsilon = \left( \partial_x + \left( -\sqrt{V} + \frac{i}{2} r F \gamma_1 \right) \gamma_{12} \right) \epsilon = \left( \partial_x + \left( -\sqrt{V} + \sqrt{V} \right) \gamma_{12} \right) \epsilon = \partial_x \epsilon.
\]

We have found that, just as in the first case, the Killing spinors only depend on the radius, and they are solutions of a differential equation with a constraint, that is
\[
\partial_r \epsilon(r) = -\frac{1}{2r} \epsilon(r)
\]
\[
\mathcal{O} \epsilon(r) = 0.
\]

It is easy to see that the \( r \)-dependence of \( \epsilon \) is \( r^{-1/2} \). The constraint \( \mathcal{O} \epsilon = 0 \) can be made into a projection operator
\[
\frac{1}{2\sqrt{V}} \mathcal{O} \epsilon = \frac{1}{2} \left( 1 - \frac{z}{r\sqrt{V}} \mathcal{P} \right) \epsilon = \frac{1}{2} \left( 1 - \frac{q_e z i \gamma_0 - q_m z i \gamma_{123}}{z} \right) \epsilon = 0.
\]

This is a projection operator since \( \left( -\frac{q_e z i \gamma_0 - q_m z i \gamma_{123}}{z} \right)^2 = 1 \), and hence the solution to (35) is
\[
\epsilon = r^{-1/2} \frac{1}{2} \left( 1 + \frac{q_e z i \gamma_0 + q_m z i \gamma_{123}}{z} \right) \epsilon_0
\]
for some arbitrary spinor \( \epsilon_0 \), this reduces the dimension of the solution space from 4 complex dimensions to 2 complex dimensions making this solution \( \frac{1}{2} \)-BPS.
3.4 A short discussion of the solutions

In the previous section, we found that we had two solutions that preserve some supersymmetry, both of them preserved $\frac{1}{2}$ of the supersymmetries. The first case we found that the potential is

$$V(r) = \frac{q_e^2}{r^2} + \frac{r^2}{l^2}.$$ 

This describes a naked singularity, since $V(r) \neq 0$ for non-zero radius. This solution describes a naked singularity with only electric charge, and it has the topology of $\mathbb{R}^2 \times \mathbb{T}$, where $\mathbb{T}$ is the torus. If we compare the solution with the Reisner-Nordström solution we can see that the solutions mass is proportional to the parameter $\eta$ which we defined in section 3.1. This implies that the mass of the solution is zero.

In the other case the solution is given by

$$V(r) = \frac{z^2}{r^2}.$$ 

This is a dyonic solution, but since $\eta = 0$ this is a naked singularity with vanishing mass. We also have that $l = \infty$, which implies that the cosmological constant, $\Lambda$, is zero.

We can compare these two solutions to the results for a spherically symmetric Reisner-Nordström solutions in AdS space found by Romans in [5]. Romans looked at the dyonic black hole with the topology of a sphere from the perspective of $N = 2$ gauged supergravity as well and found two independent BPS-solutions. Both solutions correspond to naked singularities, just as in the case for the torus. Another similarity is that one of the solutions has zero magnetic charge, but it differs in the sense that this solution has non-zero mass which is equal to $m^2 = q_e^2$, hence the electric charge is proportional to the mass. The other solution is also quite similar; this is a dyonic solution where the magnetic charge depends on the cosmological constant.

In the dyonic solution for the case of the torus, this was not the case, here $q_m^2$ could take on any values.

Caldarelli and Klemm also show that the rotating torus solution with electric and magnetic charge admits BPS solutions that are naked singularities. The same also holds for black holes with higher genus topology, except here there is a non-naked $\frac{1}{4}$-BPS solution for the rotating case. The only other case that they find that is not a naked singularity is the BPS solutions of the charged, spherical and rotating solution, which are non-naked for non-vanishing angular momentum.
4 Dyonic string solution in 6D, $N=(1,0)$ gauged supergravity

In this section we will follow [2], and use the integrability conditions to find a dyonic string solution of $N = (1,0), D = 6$ gauged supergravity, starting from a dyonic string ansatz.

4.1 $N=(1,0), D=6$ gauged supergravity

Here, we will review the basics of the $N = (1,0), D = 6$ gauged supergravity from [12]. It is a chiral theory with 8 real supersymmetries with the U(1)-R symmetry gauged. It has been given much attention because it naturally gives rise to a small cosmological constant when compactified to four dimensional Minkowski space via the vacuum solution $\mathbb{R}^{1,3} \times S^2$.

All the fermions of this theory are chiral, which means that $\gamma^* \lambda = \pm \lambda$, where $\gamma^* = \gamma_0 \ldots \gamma_5$. We can choose the plus sign, and hence only look at left handed spinors. In this section $M,N,P,\ldots$ will denote the coordinate basis, and $a,b,c,\ldots$ will denote the tangent space basis. We have the following supermultiplets

\[
\begin{align*}
(e^a_M, \psi_M, B^+_M N) & \quad \text{graviton} \\
(\phi, \chi, B^-_M N) & \quad \text{tensor/dilaton} \\
(A_M, \lambda) & \quad \text{U(1)-vector}.
\end{align*}
\]

Where $B^\pm$ gives rise to selfdual/anti-selfdual 3-form field strengths. $\lambda, \chi$ are spin $\frac{1}{2}$ particles, $\psi_M$ is the spin $\frac{3}{2}$ gravitino, $A_M$ is the vector gauge field from the U(1) symmetry and $\phi$ is a dilaton.

The full Lagrangian and supersymmetry transformations of this theory can be found in [12], but here we only need the bosonic part of the Lagrangian and the supersymmetry transformations of the fermionic fields. This is because we are interested in solutions where the fermions vanish. The bosonic part of the Lagrangian is given by

\[
\mathcal{L} = R \ast 1 - \frac{1}{4} \ast d \phi \wedge d \phi - \frac{1}{2} e^{\phi} \ast H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{\phi} \ast F_{(2)} \wedge F_{(2)} - 8 g^2 e^{-\frac{\phi}{2}} \ast 1.
\]

Where $\ast$ denotes the hodge star and $g$ is the gauge coupling. The field...
strengths $F_{(2)}$ and $H_{(3)}$ are defined by
\[ F_{(2)} := dA_{(1)} \]
\[ H_{(3)} := dB_{(2)} + \frac{1}{2} F_{(2)} \wedge A_{(1)}. \] (39)

The equations of motion for the bosonic fields is given by
\[ R_{MN} = J_{MN} := \frac{1}{4} \partial_M \phi \partial_N \phi + \frac{1}{2} e^\phi \left( F_{MN}^2 - \frac{1}{8} F_{MN}^2 g_{MN} \right) + \frac{1}{4} e^\phi \left( H_{MN}^2 - \frac{1}{6} H_{MN}^2 g_{MN} \right) + 2g^2 e^{-\frac{\phi}{2}} \]
\[ \Box \phi = J := \frac{1}{4} e^\frac{\phi}{2} F^2 + \frac{1}{6} e^\phi H^2 - 8g^2 e^{-\frac{\phi}{2}} \] (40)
\[ \Box \phi = J := \frac{1}{4} e^\frac{\phi}{2} \star F_{(2)} + \frac{1}{6} e^\phi \star H_{(3)} \wedge F_{(2)} \]
\[ d(e^\frac{\phi}{2} \star H_{(3)}) = 0 \]
where we have defined
\[ F_{MN}^2 := F_{MP} F_{PN} \]
\[ H_{MN}^2 := H_{MPQ} H_{PQN} \]

The supersymmetry transformations of the fermionic fields generated by a spinor $\epsilon$ is given by
\[ \delta \psi_M := \tilde{D}_M \epsilon := \left( D_M + \frac{1}{48} e^\frac{\phi}{2} H_{NPQ}^+ \gamma^{NPQ} \gamma_M \right) \epsilon \]
\[ \delta \chi := - \frac{1}{4} \Delta_\phi \epsilon := - \frac{1}{4} \left( \gamma^N \partial_N \phi - \frac{1}{6} e^\frac{\phi}{2} H_{NPQ}^+ \gamma^{NPQ} \right) \epsilon \] (41)
\[ \delta \lambda := \frac{1}{4 \sqrt{2}} \Delta_F \epsilon := \frac{1}{4 \sqrt{2}} \left( e^\frac{\phi}{2} \star F - 8ig e^{-\frac{\phi}{2}} \right) \epsilon. \]

Where the $D_M$ is the Lorentz and gauge covariant derivative and is given by
\[ D_M \epsilon := \left( \partial_M + \frac{\omega_{Mab} \gamma^{ab}}{4} - ig A_M \right) \epsilon. \] (42)

Hence the Killing spinor equations are
\[ \tilde{D}_M \epsilon = 0 \]
\[ \Delta_\phi \epsilon = 0 \]
\[ \Delta_F \epsilon = 0 \] (43)

It is worth noting that the $\pm$ on $H$ denoting (anti-)self-duality is redundant since $\epsilon$ is chiral and hence implies projections onto (anti-)self-dual parts.
4.2 Integrability of the Killing spinor equation

In this section, we will show, by using the integrability conditions of section 2.7.1, that the existence of Killing spinors and that the field equations for $F$ and $H$ are solved implies that Einstein field equation and the dilaton field equation is solved as well.

The integrability conditions we consider are
\[
\gamma^N[\tilde{D}_M, \tilde{D}_N]\epsilon = 0
\]
\[
\gamma^M[\tilde{D}_M, \Delta_\phi]\epsilon = 0
\]
\[
\gamma^M[\tilde{D}_M, \Delta_F]\epsilon = 0
\]
for a Killing spinor $\epsilon$. We will see that the contraction with the gamma matrices will make some of the calculations considerably easier. One relation we will use lot is
\[
\gamma^M\gamma^{A_1...A_n}\gamma_M = (-1)^n(d - 2n)\gamma^{A_1...A_n}.\tag{44}
\]
In six dimensions this implies $\gamma^M \tilde{H} \gamma_M = 0$, which is something that we will use several times.

We start by calculating $\gamma^M[\tilde{D}_M, \Delta_F]$, where we implicitly act on a spinor $\epsilon$. We want to write it in terms of the field equations, Bianchi identities, $\Delta_\phi$ and $\Delta_F$. We start by writing out the expression explicitly.
\[
\gamma^M[\tilde{D}_M, \Delta_F] = \gamma^M[\partial_M + 1/4\omega_{MNP}\gamma^{NP} - igA_M + i\frac{e^\phi}{2} \tilde{H} \gamma_M, e^\frac{\phi}{2} \tilde{F} - 8ig e^{-\frac{\phi}{2}}] =
\]
\[
= e^{\frac{\phi}{2}} 2ig \partial \phi + e^{\frac{\phi}{2}} \left( \frac{(\partial \phi) \tilde{F}}{4} + \partial \tilde{F} + \frac{\partial \phi}{4} - \frac{1}{4} \gamma^M \tilde{F} \omega_{MNP}\gamma^{NP} \right) -
\]
\[
- \frac{e^{\frac{3\phi}{4}}}{48} \gamma^M \tilde{F} \tilde{H} \gamma_M.\tag{45}
\]
Since we have a term with derivatives on the field strength $F_{(2)}$ we should be able to rewrite it such that some terms become the field equation for $F_{(2)}$ or the Bianchi identity for $F_{(2)}$, since this is the only places we have derivatives on $F_{(2)}$. We start by rewriting the terms with $e^{\frac{\phi}{2}}$.
\[
\partial \tilde{F} = 2\partial^A F_{AB}\gamma^B + \partial_N F_{AB}\gamma^{NAB}
\]
\[
\frac{(\partial \phi) \tilde{F}}{4} = \frac{\tilde{F} \partial \phi}{4} + (\partial A \phi)F_{A}\gamma^B
\]
\[
- \frac{\gamma^M \tilde{F} \omega_{MNP}\gamma^{NP}}{4} = - \frac{\phi \tilde{F}}{4} + 2F_{A}^P \omega_{MPN}\gamma^{MN}\gamma^A + 2F_{NP}^M \omega_{MPN}\gamma^M
\]
If we then put all those terms together we get

\[
\begin{align*}
&2 \partial^A F_{AB} \gamma^B + \partial_N F_{AB} \gamma^{NAB} + \frac{F \phi}{4} + (\partial^A \phi) F_{AB} \gamma^B + \\
&+ 2 F_A^P \omega_{MPN} \gamma^{MN} \gamma^A + 2 F^NP \omega_{MPN} \gamma^M \right) e^\frac{\phi}{4} = \\
&2 - \frac{\phi}{4} \nabla^M \left( e^\frac{\phi}{2} F_{MP} \right) \gamma^P + e^\frac{\phi}{2} \partial_M F_{AB} \gamma^{MAB} + \frac{e^\frac{\phi}{4}}{4} F \phi.
\end{align*}
\]

The term with \( e^{\frac{3\phi}{4}} \) in \( (45) \) can be rewritten as

\[
\frac{1}{48} \gamma^M \nabla^M e^\frac{\phi}{4} \frac{1}{12} F_{AB} \gamma^B \nabla^A \gamma^A = \frac{1}{24} \left( \frac{\phi}{\nabla^4} + \frac{\phi}{\nabla^4} \right) + H_{NPQ} F^{NP} \gamma^Q.
\]

In the first step we moved \( \gamma^M \) to the right past \( \nabla^P \) and used \( \gamma^M \nabla^P \gamma^M = 0 \), then in the next step we split the expression into two equal parts and in one part we moved \( \gamma^B \) to the right, and in the other \( \gamma^A \) to the left and simplified the expression.

Putting it all together we get

\[
\begin{align*}
\gamma^M [\tilde{D}_M, \Delta_F] &= 2 e^{-\frac{\phi}{4}} \nabla^M \left( e^{\frac{\phi}{2}} F_{MP} \right) \gamma^P + e^{\frac{\phi}{4}} \partial_M F_{AB} \gamma^{MAB} + \frac{e^{\frac{\phi}{4}}}{4} F \phi + 2 e^{-\frac{\phi}{4}} \nabla^M \left( e^{\frac{\phi}{2}} F_{MP} \right) - e^{\frac{3\phi}{4}} \left( \frac{\phi}{\nabla^4} + \frac{\phi}{\nabla^4} \right) + H_{NPQ} F^{NP} \gamma^Q.
\end{align*}
\]

Now we add and subtract

\[
\frac{1}{4} \left[ \Delta_\phi, \Delta_F \right] = \frac{1}{4} \left( \frac{\phi}{\nabla^4} F e^{\frac{\phi}{4}} - F (\phi) e^{\frac{\phi}{4}} + e^{\frac{3\phi}{4}} \frac{\phi}{\nabla^4} \right).
\]

which gives

\[
\begin{align*}
\gamma^M [\tilde{D}_M, \Delta_F] &= e^{\frac{\phi}{4}} \partial_M F_{AB} \gamma^{MAB} + 2 e^{-\frac{\phi}{4}} \left( \nabla^M \left( e^{\frac{\phi}{2}} F_{MP} \right) - \frac{e^\phi}{2} H_{MNPF^M} \right) \gamma^P + \\
&+ \frac{1}{4} \left[ \Delta_\phi, \Delta_F \right] - \frac{1}{4} \phi \phi \left( F e^{\frac{\phi}{4}} + 8 e^{-\frac{\phi}{4}} \phi \right) + \\
&+ \frac{e^\phi}{2} \left( \gamma^N \partial_N \phi - \frac{1}{6} e^\phi H_{NPQ} \gamma^{NPQ} \right).
\end{align*}
\]

Then, by using the definition of \( \Delta_\phi \) and \( \Delta_F \) we get

\[
\begin{align*}
\gamma^M [\tilde{D}_M, \Delta_F] &= e^{\frac{\phi}{4}} \partial_M F_{AB} \gamma^{MAB} + 2 e^{-\frac{\phi}{4}} \left( \nabla^M \left( e^{\frac{\phi}{2}} F_{MP} \right) - \frac{e^\phi}{2} H_{MNPF^M} \right) \gamma^P + \\
&+ \frac{1}{4} \left[ \Delta_\phi, \Delta_F \right] + \frac{1}{4} \phi \phi (\Delta_F) + \frac{e^\phi}{2} F (\Delta_F)
\end{align*}
\]

(47)
If $\epsilon$ is a Killing spinor, then this integrability condition is
$$
\gamma^M [\hat{D}_M, \Delta_F] \epsilon = 0.
$$
The last row of the equation above is zero when acting on a Killing spinor by
definition of $\Delta_\phi$ and $\Delta_F$. This integrability condition is then equivalent to
the vanishing of the first row, i.e., the sum of the field equation and Bianchi
identity for $F^{(2)}$ is satisfied.

Next we want to calculate the integrability condition from the commuta-
tor between the supercovariant derivative and the supersymmetry transfor-
mation from $\chi$, i.e.

$$
0 = \gamma^M [\hat{D}_M, \Delta_\phi] = \gamma^M [\partial_M + \frac{\omega_{Mab}\gamma^{ab}}{4} - igA_M + \frac{e^\phi}{48} H \gamma_M, \partial \phi - \frac{1}{6} e^{\frac{\phi}{2}} H].
$$
If we write out the commutator, use equation (44) and $\partial \phi = \Box \phi := \partial^M \partial_M \phi$, then we get

$$
\gamma^M [\hat{D}_M, \Delta_\phi] = \Box \phi + \frac{e^\phi}{288} \gamma^M H H \gamma_M - \frac{e^{\frac{\phi}{2}}}{6} \left( \frac{1}{2} (\partial \phi) H + \partial H + \frac{1}{4} \gamma^M \omega_{MAB}[\gamma^{AB}, H] + \frac{1}{8} \gamma^M \partial \phi \partial H \gamma_M \right). \tag{48}
$$
Since we have the $\Box \phi$ term, we want to rewrite the expression such that we
get the whole dilaton field equation,

$$
\Box \phi - \frac{1}{4} e^{\frac{\phi}{2}} F^2 - \frac{1}{6} e^\phi H^2 + 8g^2 e^{-\frac{\phi}{4}}.
$$
The last term, $8g^2 e^{-\frac{\phi}{4}}$, has an exponential factor of $e^{-\frac{\phi}{4}}$ which is something
we are missing in equation (48). We are also missing the $F^2$ terms of the
dilaton equation in (48). We can get both terms from adding and subtracting
\( \frac{1}{8} \Delta_F \Delta_F = \frac{1}{8} \left( e^{\frac{\phi}{2}} \frac{F}{\gamma} + 64 g e^{-\frac{\phi}{4}} \right) \), where we defined $\Delta_F := e^{\frac{\phi}{2}} F + 8g e^{-\frac{\phi}{4}}$.
Notice that $F^2 = -\frac{1}{2} \frac{F}{\gamma} F + \frac{1}{2} F_{MN} F_{PQ} \gamma^{MNPQ}$, so we get exactly the $F^2$ terms
that we want.

The only thing missing from the dilaton field equation is now the $H^2$
term, which we will try to get from $\frac{e^\phi}{288} \gamma^M H H \gamma_M$ since that is the only
place where we have $H^2$ in (48). If we move $\gamma^M$ in between the two $H$
we only get the terms from commuting the gamma matrices since the term
$H \gamma^M H \gamma_M = 0$ by equation (44), i.e.,

$$
\gamma^M H H \gamma_M = 6 \gamma^{AM} \gamma^M g^{AM}.
$$
Now it is natural to move the new $\gamma^A$ from the right to $\gamma^{BC}$ to make a $\mathcal{H}$ term. If we do this we end up with
\[
\frac{\gamma^M}{6 \cdot 48} \mathcal{H} = - \frac{1}{48} \mathcal{H} + \frac{1}{8} H_{BC}^A H_{ARP} \gamma^{BC} \gamma^{RP} = - \frac{1}{48} \mathcal{H} + \frac{1}{8} H^2_{BCRP} \gamma^{BCRP} - \frac{1}{4} H^2.
\]
So now we have a term with $H^2$, the problem is that it has the wrong prefactor. We want to have $-\frac{1}{6} H^2$, not $-\frac{1}{2} H^2$. We can get this by using the fact that
\[
\mathcal{H} = -6 H^2 + 9 H^2_{BCEF} \gamma^{BCEF}.
\]
Using this we find
\[
\frac{\gamma^M}{6 \cdot 48} \mathcal{H} = - \frac{1}{3 \cdot 48} \mathcal{H} - \frac{2}{3 \cdot 48} \mathcal{H} + \frac{1}{8} H^2_{BCRP} \gamma^{BCRP} - \frac{1}{4} H^2 = - \frac{1}{3 \cdot 48} \mathcal{H} - \frac{1}{6} H^2,
\]
which is what we wanted.

The only terms in (48) that are left to consider is the $e^{\hat{\phi}}$ terms, i.e.
\[
-\frac{e^{\hat{\phi}}}{6} \left( \frac{1}{2} (\partial \phi) \mathcal{H} + \mathcal{H} + \frac{1}{4} \gamma^M \omega_{MAB} \gamma^{AB} + \frac{1}{8} \gamma^M \partial \phi \mathcal{H} \gamma_M \right).
\]  
(50)

We want this to be equivalent to some combination of the field equation for $H$, the Bianchi identity for $H$ and the supersymmetry variations. We get this if we use the following relations
\[
\begin{aligned}
-\frac{1}{48} \mathcal{H} \gamma_M &= - \frac{1}{24} \mathcal{H} \partial \phi \\
-\frac{1}{12} \mathcal{H} \partial \phi &= \frac{1}{12} \mathcal{H} \partial \phi - \frac{1}{2} \partial^M \partial^M H_{MBC} \gamma^{BC} \\
-\frac{\partial \mathcal{H}}{6} &= - \frac{1}{6} \partial^M H_{MBC} \gamma^{MABC} - \frac{1}{2} \partial^M H_{MAB} \gamma^{AB}.
\end{aligned}
\]

Then (50) becomes
\[
e^{\hat{\phi}} \left( -\frac{1}{6} \partial^M H_{MBC} \gamma^{MABC} + \frac{1}{24} \mathcal{H} \partial \phi - \frac{1}{2} (\nabla^M H_{MAB} \gamma^{AB}) - \frac{1}{2} (\partial^M \partial \phi) H_{MAB} \gamma^{AB} \right)
= e^{\hat{\phi}} \left( -\frac{1}{6} \partial^M H_{MBC} \gamma^{MABC} + \frac{1}{24} \mathcal{H} \partial \phi - \frac{1}{2} e^{-\hat{\phi}} \nabla^M (e^\phi H_{MAB}) \gamma^{AB} \right),
\]
(51)

where $\nabla^M$ is the Lorentz covariant derivative.
If we put all of it together we find

\[ \gamma^M [\tilde{D}_M, \Delta_\phi] = \left( \Box \phi - \frac{1}{4} e^{\frac{2}{\phi}} F^2 - \frac{1}{6} e^{\phi} H^2 + 8g^2 e^{-\frac{2}{\phi}} \right) - \]

\[ - \frac{1}{8} \Delta_F \Delta_F - \frac{1}{2} e^{-\frac{2}{\phi}} \nabla^M (e^{\phi} H_{MAB}) \gamma^{AB} - \]

\[ - \frac{1}{6} e^{\frac{2}{\phi}} \left( \partial_M H_{NPQ} - \frac{3}{4} F_{MN} F_{PQ} \right) \gamma^{MNPQ} + \frac{1}{24} e^{\frac{2}{\phi}} \hat{\mathcal{H}} \Delta_\phi. \]

(52)

Which is zero when acting on a Killing spinor \( \epsilon \). This tells us that when we have an ansatz that solves the Killing spinor conditions, (43), for some \( \epsilon \) and also solves the Bianchi identity and field equation for \( H \), it automatically satisfies the dilaton equation field equation as well. This is because for such an ansatz we would have \( \Delta_F \epsilon = \Delta_\phi \epsilon = 0 \) by definition. Then the only things that are left in (52) is the field equations for the dilaton and \( H \) and the Bianchi identity for \( H \). Since the Bianchi identity and field equation for \( H \) is satisfied the dilaton field equation must be satisfied as well for the integrability condition \( \gamma^M [\tilde{D}_M, \Delta_\phi] = 0 \) to hold. The Integrability condition must hold by definition since we have a Killing spinor \( \epsilon \), hence the dilaton field equation is satisfied by such an ansatz as well.

Now we want to calculate the last integrability condition;

\[ \gamma^M [\tilde{D}_N, \tilde{D}_M] = \gamma^M [\nabla_N - igA_N + \frac{1}{48} e^{\frac{2}{\phi}} \hat{\mathcal{H}} \gamma_N, \nabla_M - igA_M + \frac{1}{48} e^{\frac{2}{\phi}} \hat{\mathcal{H}} \gamma_M]. \]

(53)

The commutator between the Lorentz covariant derivatives gives

\[ \gamma^M [\nabla_N, \nabla_M] = -\frac{1}{2} R_{NM} \gamma^M. \]

(54)

This implies that we should write the integrability condition in such a way that it contains the equation of motion \( R_{MN} = J_{MN} \) since this is the only place in the field equations, Bianchi identities, and the supersymmetry transformations of the fermions where \( R_{MN} \) appears. We will start by writing the integrability conditions in such a way, and then try to take care of the other terms after that.

\( J_{MN} \) contains \( H^2 \) terms, so we start by calculating the part of the integrability condition that contains terms quadratic in the three-form field strength, i.e.,

\[ e^{\phi} \gamma^M \frac{1}{48} \hat{\mathcal{H}} \gamma_N, \hat{\mathcal{H}} \gamma_M = e^{\phi} \gamma^M \frac{1}{48} \hat{\mathcal{H}} \gamma_N \hat{\mathcal{H}} \gamma_M. \]
where we used $\gamma^M \nabla^N \gamma_M = 0$ from equation (44). We want to write this in a way such that the integrability condition implies the Einstein equation, $R_{MN} = J_{MN}$, and since we have a $-\frac{1}{2} R_{MN}$ term we need to get $-\frac{1}{48} H^2 \gamma_N + \frac{1}{8} H^2_{MN} \gamma^M$. As we will see, we get this and one extra term which we will take care of later. For now we will drop the factor for $e^\phi$.

$$\frac{1}{48^2} \gamma^M \nabla^N \gamma^N \nabla^M \gamma_M = \frac{1}{48} \gamma_N \nabla^N \gamma_M \nabla^M \gamma^M +$$

$$\frac{1}{8 \cdot 48} \gamma^M H^{ABC} \nabla_{ABC} \gamma^N \gamma_M - \frac{1}{24 \cdot 48} \nabla^M \gamma^N. \tag{55}$$

We rewrote this in such a way that we can use our previous result, equation (49), to rewrite $\frac{1}{48^2} \gamma_N \nabla^N \gamma^M$. We want to get a term with $\nabla^N \gamma^N$, so we use

$$\nabla^N \gamma^N = - \nabla^N \gamma^N + 6 \nabla^N H_{AB} \gamma^{AB}, \tag{56}$$

which follows from commuting the gamma matrix, for both the $\nabla^N \gamma^N$ from equation (49) and (55), which gives

$$\frac{1}{48^2} \gamma^M \nabla^N \gamma^N \nabla^M \gamma_M = \frac{1}{12 \cdot 48} \nabla^N \gamma^N \nabla^M \gamma^M - \frac{1}{48} H^2 \gamma_N -$$

$$- \frac{1}{4 \cdot 48} \left( \nabla^N H_{AB} \gamma^{AB} + H_{AB} \nabla^N \gamma^{AB} \nabla^M \gamma_M \right) + \frac{1}{8 \cdot 48} \gamma^M H_{AB} \gamma^{AB} \nabla^N \gamma^N. \tag{57}$$

The last line can be simplified, by moving $\gamma^{AB}$ in the first term in the parenthesis to the left, which gives

$$- \frac{1}{4 \cdot 48} \left( \nabla^N H_{AB} \gamma^{AB} + H_{AB} \nabla^N \gamma^{AB} \nabla^M \gamma_M \right) =$$

$$= - \frac{1}{2 \cdot 48} H_{AB} \gamma^{AB} \nabla^N + \frac{1}{16} H^2_{NPAB} \gamma^{PAB}. \tag{58}$$

The last term in (57) can be simplified to

$$\frac{1}{8 \cdot 48} \gamma^M H_{NPAB} \gamma^{PAB} \nabla^N \gamma^N \nabla^M \gamma_M =$$

$$= \frac{1}{8 \cdot 48} H_{NPAB} \gamma^{PAB} \nabla^N \gamma^N \nabla^M \gamma_M.$$  

If we combine (58) and (59) we see that most of the terms cancel, and we are left with

$$- \frac{1}{4 \cdot 48} \left( \nabla^N H_{AB} \gamma^{AB} + H_{AB} \nabla^N \gamma^{AB} \nabla^M \gamma_M \right) + \frac{1}{8 \cdot 48} \gamma^M H_{AB} \gamma^{AB} \nabla^N \gamma^N \nabla^M \gamma_M = \frac{1}{8} H^2_{MN} \gamma^M. \tag{60}$$
This implies that
\[ \frac{1}{48^2} \gamma^M \bar{\gamma} \gamma_N \bar{\gamma} \gamma_M = \frac{1}{12 \cdot 48} \bar{\gamma} \gamma_N \bar{\gamma} - \frac{1}{48} H^2 \gamma_N + \frac{1}{8} H^2_{MN} \gamma^M \] (61)

This is what we wanted for the $H^2$ part of $J_{MN}$ in the field equation, and one extra term, $\frac{1}{12 \cdot 48} \bar{\gamma} \gamma_N \bar{\gamma}$.

The last part from $H^2$ can be written as
\[ \frac{e^\phi}{12 \cdot 48} \bar{\gamma} \gamma_N \bar{\gamma} = -\frac{e^\phi}{2 \cdot 48} \gamma_N \bar{\gamma} \Delta \phi + \frac{e^\phi}{2 \cdot 48} \bar{\gamma} \gamma_N \bar{\gamma} \phi. \] (62)

It is convenient to rewrite the last term above by moving $\gamma_N$ to the right;
\[ \frac{e^\phi}{2 \cdot 48} \gamma_N \bar{\gamma} \phi = e^\phi \left( \frac{1}{48} \bar{\gamma} \gamma_N \bar{\gamma} \partial_N \phi - \frac{1}{2 \cdot 48} \bar{\gamma} \gamma_N \bar{\gamma} \phi \right). \] (63)

To get the $g^2$ term in the equation of motion, we have to add
\[ 0 = -\frac{1}{64} \gamma_N \bar{\gamma} F \bar{\gamma} + \frac{\gamma_N}{64} \frac{\gamma^M}{F} \bar{\gamma} F + g^2 e^{-\phi} \gamma_N. \] (64)

Next we look at the $F^2$ terms. $F_{MN}$ is the field strength from $A_M$, so the place they show up is in the commutator between the derivative and $A_M$. This is also the only non-zero commutator with $A_M$ since it is not multiplied with a gamma matrix, i.e., all terms with $F_{MN}$ comes from
\[ -ig \gamma^M ([\partial_N, A_M] + [A_N, \partial_M]) = -ig F_{NM} \gamma^M. \] (65)

But this is not any terms that are in the equation of motion that we want since it has a factor of $g$ and it is only one power of the field strength. The only place we have a factor of $g$ is in the supersymmetry transformation of $\lambda$, that is, $\Delta_F \epsilon$. We can add and subtract zero to get
\[ -ig F_{NM} \gamma^M = \frac{1}{8} e^\frac{\phi}{2} F_{NM} \gamma^M \Delta_F - \frac{1}{8} e^\frac{\phi}{2} F_{NM} \gamma^M \bar{F}. \] (66)

Now the integrability condition is
\begin{align*}
\gamma^M \frac{1}{48} \left( [\nabla_N, e^\frac{\phi}{2} \bar{\gamma} \gamma_M] - [\nabla_M, e^\frac{\phi}{2} \bar{\gamma} \gamma_N] \right) + \\
\left( -\frac{1}{2} R_{NM} + \frac{1}{8} e^\phi (H^2_{NM} - \frac{1}{6} H^2 g_{MN}) + g^2 e^{-\frac{\phi}{2}} g_{MN} \right) \gamma^M - \\
- \frac{1}{64} \gamma_N \bar{\gamma} F \bar{\gamma} + \frac{1}{8} e^\frac{\phi}{2} F_{NM} \gamma^M \Delta_F - \frac{1}{8} e^\frac{\phi}{2} F_{NM} \gamma^M \bar{F} - \frac{1}{8} e^\frac{\phi}{2} F_{NM} \gamma^M \bar{F} + \\
e^\frac{\phi}{2} \left( \frac{1}{48} \bar{\gamma} \gamma_N \bar{\gamma} \partial_N \phi - \frac{1}{2 \cdot 48} \bar{\gamma} \gamma_N \bar{\gamma} \phi \right),
\end{align*}
(67)
where all we are missing from $R_{MN} = J_{MN}$ in the second row is the $F^2$ and the $\partial \phi \partial N \phi$ terms. We get the field strength terms from

$$ e^{\frac{\phi}{2}} \left( \gamma_N \frac{1}{64} F F - \frac{1}{8} F_{NM} \gamma^M F \right) = $$

$$ e^{\frac{\phi}{2}} \left( - \frac{1}{32} F^2 \gamma_N + \frac{F_{AB} F_C D}{64} \frac{\gamma^{ABCD} \gamma_N}{\gamma^M} - \frac{1}{4} F_{NM} \gamma^M \right). \tag{69} $$

The get the dilaton term by adding and subtracting $\frac{1}{8} \partial_N \phi \Delta \phi = \frac{1}{8} (\partial_N \phi \phi - \frac{1}{6} \partial_N \phi \mathcal{H} e^{\frac{\phi}{2}}).$ This gives

$$ \gamma^M [\bar{D}_N, \bar{D}_M] = -\frac{1}{2} (R_{NM} - J_{NM}) \gamma^M - \frac{1}{8} \partial_N \phi \Delta \phi - $$

$$ - \frac{e^{\phi}}{2 \cdot 48} \Delta \phi - \frac{1}{64} \gamma_N \Delta F \Delta F + \frac{1}{8} e^{\phi} F_{NM} \gamma^M \Delta F + $$

$$ + \gamma^M \frac{1}{48} \left( [\nabla_N, e^{\frac{\phi}{2}} \mathcal{H} \gamma_M] - [\nabla_M, e^{\frac{\phi}{2}} \mathcal{H} \gamma_N] \right) + \frac{e^{\phi}}{48} (\mathcal{H} \partial_N \phi - \frac{1}{2} \mathcal{H} \phi \gamma_N) + $$

$$ + \frac{e^{\phi}}{64} F_{AB} F_C D \gamma^{ABCD} \gamma_N. \tag{70} $$

Now everything except the last two lines is on the form that we want, i.e., in terms of the field equations and the supersymmetry transformations. The first of the two commutators vanish since

$$ \gamma^M [\nabla_N, e^{\frac{\phi}{2}} \mathcal{H} \gamma_M] = \gamma^M \nabla_N e^{\frac{\phi}{2}} \mathcal{H} \gamma_M = 0, $$

where the first step follows from Leibniz rule and the second step is just $\gamma^M \mathcal{H} \gamma_M = 0$. The other commutator is not zero, but is given by

$$ - \frac{1}{48} \gamma^M [\nabla_M, e^{\frac{\phi}{2}} \mathcal{H} \gamma_N] = - \frac{1}{48} \gamma^M \nabla_M \left( e^{\frac{\phi}{2}} \mathcal{H} \right) \gamma_N = $$

$$ = - \frac{1}{48} \gamma^M e^{\frac{\phi}{2}} \nabla_M \mathcal{H} \gamma_N - \frac{1}{48} e^{\phi} (\phi \phi \mathcal{H}) \gamma_N = $$

$$ = - \frac{1}{48} e^{\frac{\phi}{2}} \partial A H_{BCD} \gamma^{ABCD} \gamma_N + \frac{1}{16} e^{\frac{\phi}{2}} \nabla^A H_{ABC} \gamma^{BC} \gamma_N + $$

$$ + \frac{e^{\phi}}{2 \cdot 48} H_{ABC} \partial D \phi \gamma^{ABCD} \gamma_N - \frac{e^{\phi}}{32} (\partial^A \phi) H_{ABC} \gamma^{BC} \gamma_N, $$

where in the last step we contracted the gamma matrices, with the first row corresponding to the first term in the second row and the last row to the second term.
The last two lines in equation (70) can then be rewritten as

\[
\frac{e^2}{48} \left( \frac{3}{4} F_{AB} F_{CD} - \partial_A H_{BCD} \right) \gamma^{ABCD} \gamma_N + \frac{e^2}{16} \left( \nabla^A H_{ABC} - (\partial^A \phi) H_{ABC} \right) \gamma^{BC} \gamma_N = \\
= \frac{e^2}{48} \left( \frac{3}{4} F_{AB} F_{CD} - \partial_A H_{BCD} \right) \gamma^{ABCD} \gamma_N - \frac{e^2}{16} \nabla^A (e^\phi H_{ABC}) \gamma^{BC} \gamma_N
\]

Which is what we want, since the integrability condition now reads:

\[
\gamma^M [\tilde{D}_N, \tilde{D}_M] = -\frac{1}{2} (R_{NM} - J_{NM}) \gamma^M - \frac{1}{8} \partial_N \phi \Delta - \\
- \frac{e^2}{2} \frac{H}{48} \Delta - \frac{1}{64} \gamma_N \Delta_F \Delta_F + \frac{1}{8} e^\phi F_{NM} \gamma^M \Delta_F + \\
+ \frac{e^2}{48} \left( \frac{3}{4} F_{AB} F_{CD} - \partial_A H_{BCD} \right) \gamma^{ABCD} \gamma_N - \frac{e^{-\phi}}{16} \nabla^A (e^\phi H_{ABC}) \gamma^{BC} \gamma_N.
\]

(71)

This implies that if the Killing spinor equations are satisfied and we act with the integrability condition on the Killing spinor we are left with

\[
\left( -\frac{1}{2} (R_{NM} - J_{NM}) \gamma^M + \frac{e^2}{48} \left( \frac{3}{4} F_{AB} F_{CD} - \partial_A H_{BCD} \right) \right) \gamma^{ABCD} \gamma_N - \frac{e^{-\phi}}{16} \nabla^A (e^\phi H_{ABC}) \gamma^{BC} \gamma_N
\]

Just as in the integrability condition from \( \Delta \), we have that \( \frac{3}{4} F_{AB} F_{CD} - \partial_A H_{BCD} \gamma^{ABCD} \) and \( \nabla^A (e^\phi H_{ABC}) \gamma^{BC} \) corresponds to the Bianchi identity and field equation for the field strength \( H^{(3)} \) respectively. This implies that if we have an ansatz that solves the Killing spinor equation, the field equation for \( H^{(3)} \) and the Bianchi identity for \( H^{(3)} \), then the Einstein field equation, \( R_{MN} = J_{MN} \), is satisfied as well.

To summarize this long section about the integrability conditions we give the three main results, and discuss how to use them. The three integrability conditions that we found was

\[
\gamma^M [\tilde{D}_M, \Delta_F] = e^\phi \partial_M F_{AB} \gamma^{MAB} + 2 e^{-\phi} \left( \nabla^M \left( e^\phi F_{MP} \right) - \frac{e^\phi}{2} H_{MNP} F^{MN} \right) \gamma^P + \\
+ \frac{1}{4} [\Delta_M, \Delta_F] + \frac{1}{4} \partial_M (\Delta_F) + e^\phi \frac{F}{2} (\Delta_F),
\]

(72)
\[
\gamma^M[D_M, \Delta_\phi] = \left( \Box \phi - \frac{1}{4} e^\frac{\phi}{2} F^2 - \frac{1}{6} e^{\phi} H^2 + 8g^2 e^{-\frac{\phi}{2}} \right) - \\
- \frac{1}{8} \Delta F \Delta F - \frac{1}{2} e^{-\frac{\phi}{2}} \nabla^M (e^{\phi} H_{MAB}) \gamma^{AB} - \\
- \frac{1}{6} e^{\frac{\phi}{2}} \left( \partial_M H_{NPQ} - \frac{3}{4} F_{MN} F_{PQ} \right) \gamma^{MNPQ} + \frac{1}{24} e^{\frac{\phi}{2}} \nabla^M \Delta_\phi
\]  
(73)

and

\[
\gamma^M[D_N, \hat{D}_M] = -\frac{1}{2} (R_{NM} - J_{NM}) \gamma^M - \frac{1}{8} \partial_N \phi \Delta_\phi - \\
- \frac{e^{\frac{\phi}{2}} \nabla^N}{2 \cdot 48} \Delta_\phi - \frac{1}{64} \gamma_N \Delta F \Delta F + \frac{1}{8} e^{\frac{\phi}{2}} F_{NM} \gamma^M \Delta F + \\
+ \frac{e^{\frac{\phi}{2}}}{48} \left( \frac{3}{4} F_{AB} F_{CD} - \partial_A H_{BCD} \right) \gamma^{ABCD} \gamma_N - \frac{e^{-\frac{\phi}{2}}}{16} \nabla^A \left( e^\phi H_{ABC} \right) \gamma^{BC} \gamma_N.
\]  
(74)

As mentioned in the text above this implies that if the Killing spinor equation is solved, i.e., \( \Delta_\phi \epsilon = \Delta F \epsilon = \hat{D}_\mu \epsilon = 0 \), then the three integrability conditions should hold, that is, when they act on \( \epsilon \) they vanish. We have written the integrability conditions in terms of \( \Delta_\phi, \Delta F \) and the field equations, which means that when the integrability conditions act on Killing spinors we are left with the field equations acting on the spinor, since the \( \Delta_\phi \) and \( \Delta F \) terms vanish trivially. This gives a relation between the equation of motion, i.e.,

\[
\gamma^M[D_M, \Delta_F] \epsilon = e^{\frac{\phi}{2}} \partial_M F_{AB} \gamma^{MAB} \epsilon + 2e^{-\frac{\phi}{2}} \left( \nabla^M \left( e^{\frac{\phi}{2}} F_{MP} \right) - \frac{e^\phi}{2} H_{MNP} F^{MN} \right) \gamma^P \epsilon.
\]

It is then trivial to see that if our ansatz solves the Bianchi identity for \( F_{(2)} \), i.e., \( \partial_M F_{AB} \gamma^{MAB} = 0 \), then the field equation, which is equivalent to \( \nabla^M \left( e^{\frac{\phi}{2}} F_{MP} \right) - \frac{e^\phi}{2} H_{MNP} F^{MN} \), must be satisfied as well.

This means that we can find solutions to our theory by solving the Killing spinor equations, the field equation for \( H_{(3)} \) and the Bianchi identity for \( F_{(2)} \) and \( H_{(3)} \), since then the rest of the equations of motion are satisfied as well.

### 4.3 Dyonic string solution

In 6 dimensions we have the possibility for one-dimensional dyonic objects, i.e., strings that carry both magnetic and electric charge. The reason for this is that a p-form gauge field is equivalent to a \((D - p - 2)\)-form gauge field.
In 6 dimensions this gives the possibility for self-dual 2-forms. The duality is between magnetic and electric charges (for a U(1) symmetry), and hence we can have objects that are dyonic. Since it is a 2-form, it implies that the charged objects are of dimension one, and it is a string.

We start with a general ansatz for a dyonic string in 6 dimensions. Here \(x^\mu = (t, x)\) is the coordinates of the string. \(\sigma_i\) for \(i = 1, 2, 3\) are the left invariant 1-forms on the three sphere which satisfy the exterior algebra

\[
d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k. \tag{75}\]

The ansatz that we start with is

\[
\begin{align*}
\mathrm{d} s_6^2 &= c^2 \mathrm{d} x^\mu \mathrm{d} x_\mu + a^2 (\sigma_1^2 + \sigma_2^2) + b^2 \sigma_3^2 + h^2 \mathrm{d} r^2 \\
H_{(3)} &= P \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + u \mathrm{d} x^h \wedge \mathrm{d} x^i \wedge \mathrm{d} r \epsilon_{\mu
u} \\
F_{(2)} &= k \sigma_1 \wedge \sigma_2. \tag{76}\end{align*}
\]

Here \(a, b, h, u, k, P, \phi\) are all functions of \(r\). \(a^2 (\sigma_1^2 + \sigma_2^2) + b^2 \sigma_3^2\) is the metric of the squashed 3-sphere. Since the metric is orthogonal there is an obvious choice of vielbein, that is

\[
e^\hat{0} = c dt, e^1 = c dx, e^1 = a \sigma_1, e^2 = a \sigma_2, e^3 = b \sigma_3, e^4 = h dr. \tag{77}\]

Since the torsion tensor is zero we can use equation (3) to find the spin connection, i.e.,

\[
de^a = - \omega^a_b \wedge e^b.
\]

the full computation can be found in appendix B, and this gives non-zero components

\[
\begin{align*}
\omega^{34} &= \frac{b'}{bh} e^3, \omega^{04} = \frac{c'}{hc} e^0, \omega^{14} = \frac{c'}{hc} e^1, \\
\omega^{24} &= \frac{a'}{ah} e^2, \omega^{14} = \frac{a'}{ah} e^1, \omega^{23} = - \frac{b}{2a^2} e^1, \\
\omega^{31} &= - \frac{b}{2a^2} e^2, \omega^{12} = \left( \frac{b}{2a^2} - \frac{1}{b} \right) e^3.
\end{align*}
\]

Then we can calculate the curvature tensor \(\rho^{ab} = \frac{1}{2} R_{MN}^{ab} dx^M \wedge dx^N\) using the definition of the curvature tensor \(\rho^{ab} := d\omega^{ab} + \omega^a_c \wedge \omega^{cb}\). Using this we can calculate the Ricci tensor \(R_{ab} := R_{abc} \wedge e^c\) where the result can be found in appendix C. Notice that the Ricci tensor is diagonal.
4.3.1 Field equation and Bianchi identities for F and H

The Bianchi identities for $F^{(2)}$ and $H^{(3)}$ is $d F^{(2)} = 0$ and $d H^{(3)} = 0$ respectively. For our ansatz this is equivalent to

\[
\begin{cases}
  k' dr \wedge \sigma_1 \wedge \sigma_2 = 0 \\
  P' dr \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 0
\end{cases} \Rightarrow \begin{cases}
  k' = 0 \\
  P' = 0
\end{cases}
\]  

(78)

i.e., $k$ and $P$ are constants.

The field equations for $F^{(2)}$ and $H^{(3)}$ in (40) gives more relations relations between the functions, so we need the Hodge duals of $F^{(2)}$ and $H^{(3)}$ which is is given by

\[
\begin{cases}
  \star F^{(2)} = \frac{kbhc_2}{\alpha} \sigma_3 \wedge dr \wedge dt \wedge dx \\
  \star H^{(3)} = -\frac{Pc_2h}{\alpha^2b} dt \wedge dx \wedge dr - \frac{ua^2b}{\hbar c^2} \sigma_1 \wedge \sigma_2 \wedge \sigma_3
\end{cases}
\]

The left hand side of the $F^{(2)}$ field equation can be rewritten as

\[
d(e^\phi \star F^{(2)}) = -e^\phi \frac{kb}{\alpha} \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 \wedge \tilde{e}_4,
\]

where we have used equation (75) and the definition of the vielbein. The right hand side of the field equation gives

\[
e^\phi \star H^{(3)} \wedge F^{(2)} = -e^\phi \frac{kP}{\alpha^2b} \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 \wedge \tilde{e}_4.
\]

This means that the field equation for $F^{(2)}$ implies

\[
b^2 = e^\phi P.
\]

(79)

The field equation for $H^{(3)}$, i.e. $d(e^\phi \star H^{(3)}) = 0$ gives

\[
0 = u' \left( e^\phi \frac{a^2b}{\hbar c^2} \right) + u \left( e^\phi \frac{a^2b}{\hbar c^2} \right) \Rightarrow \frac{u'}{u} = -\frac{\left( e^\phi \frac{a^2b}{\hbar c^2} \right)^{\prime}}{\left( e^\phi \frac{a^2b}{\hbar c^2} \right)}.
\]

Which gives

\[
u = Q e^{-\phi \frac{\hbar c^2}{a^2b}}
\]

(80)

where $Q$ is some integration constant.
4.3.2 Killing spinor conditions

Here we will solve the Killing spinor equation for the dyonic string ansatz. We showed earlier that if the ansatz solves the Killing spinor equation, it also solves $R_{MN} = J_{MN}$ and $\Box \phi = J$ and hence it will be a solution of the supergravity theory. We start with the Killing spinor equation that comes from $\delta \lambda = 0$.

$$\Delta \epsilon = 0 \Rightarrow e^\varphi \hat{F}^a \epsilon = 8 i g e^{-\varphi} \epsilon \Leftrightarrow \gamma^{12} \epsilon = \frac{4 i g a^2}{k} e^{-\varphi} \epsilon.$$

We can without loss of generality set

$$a^2 = \frac{k}{4g} e^{\frac{\varphi}{2}}$$

and get the projection condition

$$\gamma^{12} \epsilon = i \epsilon$$

for the Killing spinors. This reduces the amount of supersymmetry of the solution in half.

From $\delta \chi = 0$ we have $(\partial \phi) \epsilon = \frac{1}{2} e^{\varphi} \not{H} \epsilon$. Since $\phi$ is a function of $r$ the left hand side is just $(\partial \phi) \epsilon = \gamma^{4} e^{\varphi} \epsilon$. On the other hand we have $\not{H} = 6 \frac{1}{a^2 b} \left( P \gamma^{123} + Q e^{-\phi} \gamma^{014} \right)$, where we used equation (80) to write it terms of $Q$.

Since $\epsilon$ is chiral, i.e. $\gamma^* \epsilon := \gamma^{01234} \epsilon = \epsilon$ we have

$$\gamma^{i_1 \ldots i_k} \epsilon = \gamma^{i_1 \ldots i_k} \gamma^* \epsilon.$$ (83)

This makes it possible to rewrite gamma matrix of rank $k$ acting on a spinor as a gamma matrix of rank $6 - k$ acting on the same spinor. For example $\gamma^{014} \epsilon = -\gamma^{123} \epsilon$. Using this relation we can rewrite the Killing spinor equation from $\chi$ as

$$\frac{\gamma^{4} \phi'}{h} \epsilon = \frac{1}{a^2 b} \left( P - Q e^{-\phi} \right) \gamma^{123} \epsilon.$$ (85)

If we now multiply by $\gamma^4$ on both sides and impose the projection condition

$$\gamma^{1234} \epsilon = \epsilon$$

we get the differential equation

$$\phi' = -\frac{h}{a^2 b} e^{\frac{\varphi}{2}} \left( P - Q e^{-\phi} \right).$$

52
The projection condition further cuts the amount of supersymmetry in half, so that the solution is $\frac{1}{4}$-BPS.

Next, we look at the Killing spinor equations from the gravitino. In this case, we have to look at the components separately. We will assume that $\epsilon$ only depend on $r$. The only non-zero component of the gauge field $A_{(1)}$ is $A_3 = -k$.

We start with the worldsheet components, $\mu = (\tilde{0}, \tilde{1})$, which gives the same differential equation.

$$\delta \psi_\mu = 0 \iff \hat{D}_\mu \epsilon = 0 \iff \frac{c'}{2h} \gamma^\mu \epsilon = \frac{c e^\frac{\phi}{2}}{8 a^2 b} \left( P + Q e^{-\phi} \right) \gamma^{123} \epsilon$$

In the last step we moved $\hat{H} \gamma^\mu$ over to the other side and used (83). Now we can multiply $\gamma^\mu$ on both sides of the equation and use the projection condition $\gamma^{1234} \epsilon = \epsilon$ to get

$$\frac{c'}{hc} = -\frac{e^\frac{\phi}{2}}{4 a^2 b} (P + Q e^{-\phi}).$$

(86)

The calculations for $\delta \psi_1 = 0$ and $\delta \psi_2 = 0$ are quite similar and give the same result, so we only give the calculation for $\delta \psi_1$ here.

$$0 = \frac{1}{4} \omega^1_{ab} \gamma^{ab} + \frac{1}{48} e^\frac{\phi}{2} \hat{H} \gamma^1 = \left( -\frac{b}{4a} \gamma^{23} + \frac{a'}{2h} \gamma^{14} \right) \epsilon + \frac{c e^\frac{\phi}{2}}{8 a b} \left( P \gamma^{23} - Q e^{-\phi} \gamma^{14} \right) \epsilon$$

$$= \left( -\frac{b}{4a} + \frac{a'}{2h} \right) \epsilon + \frac{c e^\frac{\phi}{2}}{8 a b} \left( P e^{-\phi} \right) \epsilon$$

where in the last step we multiplied by $\gamma^{14}$ and used (84) to eliminate the $\gamma$ matrices from the equation. This implies the relation

$$\frac{a'}{ha} = \frac{e^\frac{\phi}{2}}{4 a^2 b} \left( P + Q e^{-\phi} \right) - \frac{b}{2a^2}.$$  

(87)

This is the same relation we get from $\delta \psi_2 = 0$.

From $\delta \psi_3 = 0$ we also have a contribution from the U(1) gauge vector, other than that the calculation is basically the same as the previous one so we only give the result which is

$$\frac{b'}{hb} = \frac{e^\frac{\phi}{2}}{4 a^2 b} \left( P + Q e^{-\phi} \right) + \frac{b}{2a^2} - \frac{1}{b} + \frac{2kg}{b}. $$

(88)
The r-dependence of $\epsilon$ can be found from the differential equation $\delta \psi_4 = 0$.

$$\epsilon' = - \frac{e^{\frac{\varphi}{2}}}{a^2b} \left( P \gamma^{1234} + Q e^{-\phi} \gamma^{01} \right) \epsilon$$

$$= - \frac{e^{\frac{\varphi}{2}}}{a^2b} \left( P + Q e^{-\phi} \right) \gamma^{1234} \epsilon = - \frac{e^{\frac{\varphi}{2}}}{a^2b} \left( P + Q e^{-\phi} \right) \epsilon.$$  

This gives

$$\frac{\epsilon'}{\epsilon} = - \frac{e^{\frac{\varphi}{2}}}{a^2b} \left( P + Q e^{-\phi} \right).$$

If we combine this with equation (86) we get

$$\frac{\epsilon'}{\epsilon} = \frac{1}{2} c'. \quad (89)$$

This determines the r-dependence of the Killing spinors. To summarize we have Killing spinors of the form

$$\epsilon(r) = c^{1/2} \epsilon_0, \quad (90)$$

where $\epsilon_0$ is a constant spinor that satisfy

$$\left\{ \begin{array}{l} \gamma^{1234} \epsilon_0 = \epsilon_0 \\ \gamma^{12} \epsilon_0 = i \epsilon_0 \end{array} \right. \quad (91)$$

if all the following conditions are satisfied

$$k' = 0 \quad P' = 0 \quad b'^2 = e^{\frac{\phi}{2}} P \quad u = Q e^{-\phi} \frac{hc^2}{a^2b} \quad a'^2 = \frac{k}{4g} e^{\frac{\phi}{2}} \quad \phi' = - \frac{h}{a^2b} e^{\frac{\phi}{2}} \left( P - Q e^{-\phi} \right) \quad (92)$$

$$\frac{c'}{hc} = - \frac{e^{\frac{\phi}{2}}}{4a^2b} \left( P + Q e^{-\phi} \right) \quad \frac{b'}{hb} = \frac{e^{\frac{\phi}{2}}}{4a^2b} \left( P + Q e^{-\phi} \right) + \frac{b}{2a^2} - \frac{1}{b} + \frac{2kg}{b}$$

$$\frac{a'}{ha} = \frac{e^{\frac{\phi}{2}}}{4a^2b} \left( P + Q e^{-\phi} \right) - \frac{b}{2a^2}.$$
Then the Bianchi identity and field equation for $F(2)$ and $H(3)$ is satisfied, as well as the Killing spinor equations. Then, as discussed in section 4.2, the dilaton and Einstein field equation is solved as well, and the dyonic string ansatz is a solution of the gauged $(1,0)$ six dimensional supergravity.

The function $h$ is not determined by the conditions, so we are free to set it to any function of $r$. We can make an explicit choice for $h$ to make it easier to study the dyonic string solution.

In the near horizon limit, they show in [2] that this dyonic string solution approach the $AdS_3 \times S^3$ vacuum solution. This restores some of the supersymmetry and is a $\frac{1}{2}$-BPS solution.

5 Conclusions

We have used the integrability condition of the Killing spinor equations to both show that a solution of AdS gravity is a BPS solution, and also used it to find BPS solutions in 6 dimensions. This shows us how the integrability condition can be used in two different ways. Both as a way to find new solutions of supergravity and also to find out when existing solutions are BPS.

This makes the use of integrability conditions a very important tool. On the one side, it helps us to solve coupled equations of motions, which is in general quite hard to solve, and it helps us to find BPS solutions which might hide in some old classical solutions of gravity.

BPS solutions are in fact very important, not only because they are solutions to some gravity theory, but also because of all the nice properties they possess. One of these properties is that solutions with some unbroken supersymmetry ensure the stability of the solution under perturbation, this makes it possible to calculate the entropy of black holes by counting of microstates for BPS solutions. This was first done in 1996 by Stromminger and Vafa [28] for asymptotically flat black holes.

Generalizations to asymptotically AdS black holes were found in [29]. They calculated the microstates of static black hole solutions in four dimensions preserving two supercharges by using the AdS/CFT correspondence. The difference from the solution considered in this thesis is that they contain non-constant scalar fields. The first solutions of this kind were found in [30] and was found by constructed the recipe of [31]. The near horizon limit of solutions that are asymptotically $AdS_4$ are $AdS_2 \times S^2$. This is what makes the counting of microstates possible in this case since it makes it possible to do the calculations in the dual CFT theory, which will be a one-dimensional theory, i.e., a superconformal quantum mechanics theory. The correct CFT
to consider is the topologically twisted ABJM theory, which is the dual theory for asymptotically $\text{AdS}_4 \times S^7$ solutions of M-theory. There the dual to the $\text{AdS}_4$ part is a three-dimensional CFT which we compactify to the quantum mechanical system by going to the near horizon limit.

The entropy of the theory can then be identified as a generalization of the Witten index of the theory, which is related to the ground states of the theory and hence to the entropy. The Witten index can then be calculated using localization techniques. This method gives the correct entropy in the first order when compared with the supergravity calculations. There are still open problems regarding these calculations. For example, it is unclear why we have to extremize the Witten index with respect to the chemical potential, which characterizes the mixing of the symmetries, to get the entropy. It would also be interesting to expand the results to higher dimensions by studying, e.g., asymptotically $\text{AdS}_5 \times S^5$ solutions. This is technically more demanding since the dual CFT will be of a higher dimension than one. It would also be interesting to study solutions with less supersymmetry.

## A Differential equation with projectors

Here we show how to solve the kind of differential equation that we had in section 3.3.1. For more details see the appendix of [5].

Let $\Gamma_1, \Gamma_2$ be constant operators such that $\Gamma_1^2 = \Gamma_2^2 = 1$ and $\Gamma_1 \Gamma_2 = -\Gamma_2 \Gamma_1$ and let $\Pi(r) := \frac{1}{2}(1 + x(r)\Gamma_1 + y(r)\Gamma_2)$ be a projection operator where $x(r)$ and $y(r)$ are functions of $r$ with the property $x^2 + y^2 = 1$, and $y \neq 0$.

We want to solve a differential equation on the form

$$\epsilon' := \partial_r \epsilon(r) = (a(r) + b(r)\Gamma_1 + c(r)\Gamma_2)\epsilon(r) \quad (93)$$

with the constraint $\Pi \epsilon = 0$.

We can absorb $c(r)\Gamma_2$ into $b(r)\Gamma_1$ by using the constraint:

$$\Pi \epsilon = 0 \Rightarrow \Gamma_2 c = -\frac{1}{y}x\Gamma_1 c.$$

We have an integrability condition

$$x' + 2by^2 = 0.$$

When this is satisfied we have the solution

$$\epsilon(r) = (u(r) + v(r)\Gamma_2)P(-\Gamma_1)\epsilon_0 \quad (94)$$
where $\epsilon_0$ is an arbitrary constant spinor, $P(\Gamma) := \frac{1}{2}(1 + \Gamma)$ and

\[
u(r) = -\left(\frac{1 - x}{y}\right)^{1/2} e^{\int_0^r a(r')dr'}.
\]

Since $P(\Gamma)$ is a projection operator that acts on a constant spinor, it projects out some degrees of freedom of our space of solutions. This means that if one has several different, independent projections in your solution, you will have fewer degrees of freedom in the solution, which in the case of Killing spinors decides how much supersymmetry is left in your solution.

B Spin connection

Here we calculate the spin connection for the dyonic string ansatz by using the first Cartan structure formula, i.e. equation (3), which for zero torsion is

\[ de^a = \omega^a_{\ b} \wedge e^b. \] (95)

The exterior derivative of the different components is given by

\[
\begin{align*}
\text{de}^\mu &= \omega^\mu_\nu e^\nu \\
\text{de}^1 &= \frac{\alpha'}{\mathcal{h}_a} e^4 \wedge e^1 - \frac{1}{b} e^2 \wedge e^3 \\
\text{de}^2 &= \frac{\alpha'}{\mathcal{h}_a} e^4 \wedge e^2 + \frac{1}{b} e^1 \wedge e^3 \\
\text{de}^3 &= \frac{\kappa'}{\mathcal{h}_b} e^4 \wedge e^3 - \frac{b}{\mathcal{h}_a} e^1 \wedge e^2 \\
\text{de}^4 &= 0
\end{align*}
\] (96)

The Cartan structure equation for $a = \mu$ then implies that $\omega^{\mu4} = \frac{\alpha'}{\mathcal{h}_c} e^\mu + y^{4\mu} e^4$ for some constant $y^{4\mu}$. It also gives $\omega^a_{\ b} \wedge e^1 = 0$ for $i = 1, 2, 3$.

For $a = 1$ we get two equations

\[
\begin{align*}
\frac{\alpha'}{\mathcal{h}_a} e^4 \wedge e^1 - \frac{1}{b} e^2 \wedge e^3 &= -\omega^{12} \wedge e^2 - \omega^{13} \wedge e^3 - \omega^{14} \wedge e^4 \\
\omega^{01} \wedge e^0 &= \omega^{11} \wedge e^1
\end{align*}
\] (97)

We get the last equation since those terms do not contribute to the left hand side of the Cartan structure equation, but they can still cancel each other. From the first equation we can see that $\omega^{14} = \frac{\alpha'}{\mathcal{h}_a} e^1 + y^{14} e^4$ for some constant $y^{14}$. This gives the equations

\[
\begin{align*}
\omega^{14} &= \frac{\alpha'}{\mathcal{h}_a} e^1 + y^{14} e^4 \\
-\frac{1}{b} e^2 \wedge e^3 &= -\omega^{12} \wedge e^2 - \omega^{13} \wedge e^3 \\
\omega^{01} \wedge e^0 &= \omega^{11} \wedge e^1
\end{align*}
\] (98)
We get similar results from $a = 2, 3$. We get a $\omega^{i4} = x^{i4}e^i + y^{i4}e^4$ from all the equations. If we insert this into the Cartan structure equation for $a = 4$ we get that $y^{i4} = y^{i4} = 0$, and hence we have

$$\begin{cases} 
\omega^{i4} = \frac{c'}{hc} e^\mu \\
\omega^{14} = \frac{a'}{ha} e^1 \\
\omega^{24} = \frac{a'}{ha} e^2 \\
\omega^{34} = \frac{b'}{hb} e^3.
\end{cases} \quad (99)$$

We also get the equations $\omega^{0i} \land e^0 = \omega^{i4} \land e^4$ from $a' = 2, 3$. This implies $\omega^{i4} = x^{i4}e^0 + y^{i4}e^1$. If we plug this into $\omega^{i4} \land e^i = 0$ we get $\omega^{i4} = 0$.

The last set of equations we get from $a = i$ is

$$\begin{cases} 
-\frac{1}{b} e^2 \land e^3 = e^2 \land \omega^{12} - \omega^{13} \land e^3 \\
\frac{1}{b} e^1 \land e^3 = e^1 \land \omega^{12} - \omega^{23} \land e^3 \\
-\frac{1}{b} e^1 \land e^2 = -e^1 \land \omega^{13} + \omega^{23} \land e^2.
\end{cases} \quad (100)$$

It is now easy to see that the solution must be on the form

$$\begin{cases} 
\omega^{12} = x^{12} e^3 \\
\omega^{13} = x^{13} e^2 \\
\omega^{23} = x^{23} e^1.
\end{cases} \quad (101)$$

This gives a system of linear equations which is easy to solve. The solution is given by $x^{23} = -x^{13} = -\frac{b}{2\pi}$ and $x^{12} = -\frac{1}{b} + \frac{b}{2\pi} e^1$.

To summarize, all the non-zero components of the spin connection is

$$\begin{cases} 
\omega^{i4} = \frac{c'}{hc} e^\mu \\
\omega^{14} = \frac{a'}{ha} e^1 \\
\omega^{24} = \frac{a'}{ha} e^2 \\
\omega^{34} = \frac{b'}{hb} e^3 \\
\omega^{12} = -\frac{1}{b} + \frac{b}{2\pi} e^1 \\
\omega^{13} = \frac{b}{2\pi} e^2 \\
\omega^{23} = -\frac{b}{2\pi} e^1.
\end{cases} \quad (102)$$
C  Spin connection and Ricci tensor components for the dyonic string ansatz

The non-zero components of the Ricci tensor is

\[ R_{\bar{0}\bar{0}} = \left( \frac{c'}{hc} \right)^2 + 2 \frac{a'c'}{ach^2} \frac{b'c'}{bch^2} + \frac{c''}{ch^2} - \frac{c'h'}{ch^3} \]

\[ R_{\bar{1}\bar{1}} = -\left( \frac{c'}{hc} \right)^2 - 2 \frac{a'c'}{ach^2} \frac{b'c'}{bch^2} - \frac{c''}{ch^2} + \frac{c'h'}{ch^3} \]

\[ R_{11} = -2 \frac{a'c'}{ach^2} + \frac{1}{a^2} - \left( \frac{a'}{ah} \right)^2 - \frac{b^2}{2a^4} - \frac{a'b'}{ah^2} - \frac{ab'}{ah^3} + \frac{a''}{ah^2} + \frac{a'h'}{ah^3} \]

\[ R_{22} = R_{11} \]

\[ R_{33} = 2 \left( \frac{b}{2a^2} \right)^2 - 2 \frac{a'b'}{abh^2} + \frac{b'h'}{bh^3} - \frac{b''}{bh^2} - 2 \frac{b'c'}{ach^2} \]

\[ R_{44} = -2 \left( \frac{c''}{ch^2} - \frac{c'h'}{ch^3} \right) - 2 \left( \frac{a''}{ah^2} - \frac{a'h'}{ah^3} \right) - \left( \frac{b''}{bh^2} - \frac{b'h'}{bh^3} \right) \]

D  Spinors in different dimensions

The type of spinors relevant for supersymmetry is highly dependent of the spacetime dimension. We want to consider the most fundamental spinors of that dimensions. The most fundamental spinors are those corresponding to an irreducible representation of the spin group, and they are hence related to the symmetries of the gamma matrices.

In 3 + 1 dimensions we have Majorana spinors. Majorana spinors satisfy a reality condition

\[ \psi = \psi^c := B^{-1} \psi^* \]

where \( \psi^c \) is the charge conjugate. This cuts the degrees of freedom to half, which means that in 4 dimensions spinors have 4 independent components.

For majorana spinors there always exist a representation for the gamma matrices where they are all real, that is, all the components are real numbers. In this representation the matrix \( B \) is just the identity.

In 4 dimensions one could equally well consider Weyl spinors, i.e., \( \psi = P_L \psi \).

In 5 + 1 dimensions the fundamental spinors are symplectic Majorana Weyl spinors, that is, they are Weyl spinors that satisfy a reality condition

\[ \psi^i = \epsilon^{ij} (\psi^j)^c. \]
In 6 dimensions the spinors have 8 independent components.
Explicit representations are not needed for the calculations in this thesis.

References


[22] M. J. Duff, H. Lu, and C. N. Pope, “Heterotic phase transitions and


[hep-th].

http://www.cambridge.org/mw/academic/subjects/physics/
theoretical-physics-and-mathematical-physics/supergravity?
format=AR.

[26] T. Ortin, *Gravity and Strings*. Cambridge Monographs on
http://www.cambridge.org/mw/academic/subjects/physics/
theoretical-physics-and-mathematical-physics/


[28] A. Strominger and C. Vafa, “Microscopic origin of the

[29] F. Benini, K. Hristov, and A. Zaffaroni, “Black hole microstates in
AdS4 from supersymmetric localization,” *JHEP* 05 (2016) 054,


supersymmetric solutions of N=2, D=4 gauged supergravity coupled to
[hep-th].