Reduction Mechanics of The Cosmological Constant

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Abstract

The quantum creation of membranes by a totally antisymmetric tensor and gravitational fields is considered in arbitrary space-time dimensions. The creation process is described by instanton tunneling. As membranes are created, the energy density associated with the antisymmetric field decreases, reducing the effective cosmological constant to a lower value. For a collection of parameters and initial conditions, the creation mechanism goes to a halt as soon as the cosmological constant is near zero. A brief exploration of a canonical gravity representation is also considered, where the system of ADM-composition in arbitrary space-time dimensions is introduced.

Sammanfattning

I detta arbete ger vi en teoretisk förklaring hur den kosmologiska konstanten kan reduceras med hjälp av en naturlig mekanism. Relaterat till denna naturliga mekanism, är möjligheten av att en skapelseprocess av membran existerar och det visar sig endast vara möjligt när den kosmologiska konstanten har ett positivt värde. Empiriska experiment visar att det observerbara universum vi lever i, har ett positivt värde på den kosmologiska konstanten och det möjliggör en teorerisk förklaring för skapelsen av membran.

Contents

1 Introduction

In general relativity the cosmological constant plays a roll of being imposed as a fixed parameter, to be determined via observations on the dynamical nature of space-time. It is when the introduction of a totally antisymmetric field, coupled via gravity, the cosmological constant gets shifted by an effective term proportional to the energy density of the antisymmetric field. Hence, the cosmological constant can be treated as a dynamical variable rather than being imposed as a fixed parameter. If a source is introduced for the antisymmetric field, the cosmological constant changes to being a dynamical quantity - it is no longer constant.

To be specific, consider in $D = d+1$ space-time dimensions a totally antisymmetric field with d indices coupled to its natural source, the $d-1$ spatial dimensional membrane. Just as in the case that an electric field can create particle pairs, the antisymmetric tensor field may create membranes. Now, couple this system to gravity and include a cosmological constant λ representing a collection of physical processes, such as the mechanism of spontaneous symmetry breaking. The resulting cosmological constant Λ, is now a combination of λ and a term proportional to the energy density associated with the antisymmetric field. We will show that the spontaneous creation of closed membranes, tends to reduce the cosmological constant Λ to a lower value. The mechanism of membrane creation may evolve continuously in this manner, but as soon as $\Lambda \leq 0$ the process goes to a halt. This implies that the cosmological constant is neutralized by a natural mechanism.

1.1 Creation Mechanism

We shall begin by presenting a qualitative description of the membrane creation mechanism, and how the presence of it, changes the cosmological constant. Most of the technical analysis is reserved for later sections, but it may be helpful to establish some mathematical notation and derive the equations of motion for the antisymmetric field. Thus let $x^\mu = z^\mu(\xi)$ be the d-dimensional history of the membrane in $D = d+1$ space-time dimensions as a function of its coordinates $\xi^a = 0, \cdots, d-1$, and let ${}^d g_{ab} = g_{\mu\nu} z^\mu{}_{,a} z^\nu{}_{,b}$ be the induced metric. Also let $A_{\mu_1\cdots\mu_d}$ denote the components of the antisymmetric tensor field, with the associated field strength $F = dA$. The action we will use is

$$\begin{aligned} S^E &= m \int d^d\xi \sqrt{{}^d g} - \frac{1}{2D!} \int d^D x \sqrt{g} F_{\mu_1\cdots\mu_D} F^{\mu_1\cdots\mu_D} \\ &+ \frac{e}{d!} \int d^d\xi A_{\mu_1\cdots\mu_d} z^{\mu_1}{}_{,a_1} \cdots z^{\mu_d}{}_{,a_d} \epsilon^{a_1\cdots a_d} + \frac{1}{d!} \int d^D x \sqrt{g} \partial_{\mu_1} \left(F^{\mu_1\cdots\mu_D} A_{\mu_2\cdots\mu_D} \right) \\ &+ S^{E,grav}(\lambda) \end{aligned} \tag{1.1.1}$$

where m is the mass per $(d-1)$ unit volume, and e is the coupling constant between the membrane and the antisymmetric tensor field. Here, $S^{E,grav}(\lambda)$ is the gravitational action, including an explicit contribution upon λ which is later assumed to be negative. The total derivative in (1.1.1) is included to ensure that the action has well defined variations with respect to $A_{\mu_1\cdots\mu_d}$.
The field strength tensor can in general be written as $F^{\mu_1\cdots\mu_D} = g^{-1/2} E \epsilon^{\mu_1\cdots\mu_D}$ for some scalar field E. With this notation, the equations of motion from (1.1.1) for the antisymmetric tensor field are

$$(\partial_{\mu_1} E) \varepsilon^{\mu_1\cdots\mu_D} = -e \int d^d\xi \, \delta^D(x - z(\xi)) z^{\mu_2}_{,a_1} \cdots z^{\mu_D}_{,a_d} \varepsilon^{a_1\cdots a_d}. \tag{1.1.2}$$

This equation shows that away from the source, the scalar field E on either side of the membrane is a constant, and the two values of E differ in magnitude of $|e|$. Substitution of (1.1.2) into (1.1.1) shows that away from the membrane, the antisymmetric tensor field contributes a positive cosmological constant term, proportional to the energy density E^2. So the membrane divides space-time into two regions of different energy densities, and each will be a portion of de Sitter or anti-de Sitter space characterized by some value of the total cosmological constant Λ.

If the initial space-time is in a metastable state, a first order phase transition to a true state will occur, where the true state quickly expands as a "bubble" within the metastable state. The energy density of the initial metastable state must be less than the true state, as demanded by the conservation of energy, since some of the energy is used for the creation of the membrane. The probability per space-time volume, that a bubble will form, is given by

$$P \sim e^{-B/\hbar}, \qquad B = S_E[\text{instanton}] - S_E[\text{background}], \tag{1.1.3}$$

where $S_E[\text{instanton}]$ is the euclidean action evaluated at the instanton solution to the euclidean equations of motion, in which the solution is the interpolation between real classical motions of the system, and so provides a semi-classical path by which the system tunnels from one region to the other. Also $S_E[\text{background}]$ is the background which corresponds to the metastable state, together with the configuration of the classical single bubble. Once the membrane is created, it expands rapidly and coalesces with other such membrane. This is the essential mechanism in how the cosmological constant is altered.

1.2 Neutralization of The Cosmological Constant

Accordingly to the conservation of energy, the energy density of the metastable state must be strictly larger than the stable state. Suppose we label the metastable and stable state as outside and inside, respectively. It can then be shown that for the realization of the membrane creation process, the magnitude of the outside scalar field must obey the inequality

$$|E_o| > \frac{1}{2}|e|, \tag{1.2.1}$$

and if E_o is large enough so that (1.2.1) is satisfied, then membrane creation will on the average reduce the magnitude of E, and by consequence reduce the cosmological constant, $\Lambda_o > \Lambda_i$. On the other hand, if instead $\Lambda_i \leq 0$, then for some values of the parameters and initial field values E_o, Λ_o, the membrane cannot be produced at all. This may happen, even if E_o is relatively large, satisfying (1.2.1).

The properties of membrane production provide a natural mechanism by which the cosmological constant may be reduced to a value near zero. Suppose the initial cosmological constant Λ_o is positive, with λ having a arbitrary negative value, and suppose E_o is large enough satisfying (1.2.1). Then membranes will be produced which generally lowers Λ, and this process will continue as long as $\Lambda > 0$. But as soon as the value of Λ falls at or below zero, for some parameters and fields E, Λ, the neutralization process cannot continue and the membrane creation process turns to a halt. We will show that the membrane creation is halted if the parameters and the fields Λ_o, E_o satisfy a general inequality. Satisfaction of this inequality requires, however, the mass density m of the membrane to be large, and the magnitude of the coupling constant $|e|$ to be small. It turns out that the rate for membrane creation is decreased as either m is large or $|e|$ is small, at least for those processes which reduces the cosmological constant. Then as the membrane creation stops, when Λ is small, also forces the rate of membrane creation to be small.

2 Dynamics

In arriving at the description of the dynamics involving membrane creation, and so how the cosmological constant is changed, one may follow the indispensable treatment of such dynamics through the theory of action. All treatment occur in euclidean space where timelike variables are anti-clockwise Wick rotated, and by the reality of the energy density associated with the antisymmetric tensor field, the anti-symmetric field components must be rotated in the same amount, but clockwise. Let (U, z) be a chart on a manifold M, where $\dim(M) = d+1$. Let $\xi \in M$, then $z : \xi \to x$ is a 1-1 map of U onto some subset of $\mathbb{R}^d$. Denote this mapping by $z^\mu = x^\mu(\xi)$ where z^μ, $\mu = 0, 1, \cdots, d$, specifies the d-dimensional history of a membrane in $D = d+1$ space-time dimensions as a function of its coordinates $\xi = 0, 1, \cdots, d-1$, and let ${}^d g_{ab} = g_{\mu\nu} z^\mu{}_{,a} z^\nu{}_{,b}$ be the induced metric. Also let the d-form A denote the antisymmetric field tensor with field strength $F = dA$.

2.1 Action Functional

The action used throughout is

$$S^E = +m \int_M \eta + e \int_{\partial M} \phi^* A - \frac{1}{2} \int_M F \wedge \star F + \int_M d\,(A \wedge \star F) + S^{\mathrm{E,grav}}(\lambda), \tag{2.1.1}$$

where the mass of the membrane is m, e is the coupling between the antisymmetric field and gravity, and κ is constant. The term in(2.1.1) which is a total exterior derivative, is included to counteract terms in the action functional and so ensure well defined variations of (2.1.1) with respect to A. Here $S^{\mathrm{E,grav}}(\lambda)$ is the euclidean gravitational action

$$S^{\mathrm{E,grav}}(\lambda) = -\frac{1}{2\kappa} \int_M (\mathscr{R} - 2\omega\lambda) + \frac{1}{\kappa} \int_{\partial M} \mathscr{K}, \tag{2.1.2}$$

where λ is the cosmological constant, $\mathscr{R} = \mathscr{R}^{ij} \wedge e_i \wedge e_j$ is the Ricci scalar, and ∂M denotes the boundary of M. All integration is taken with respect to the orientation of the manifold M. The boundary term in (2.1.2) is the Gibbons-Hawking-York term, included to counteract the Palatini identity obtained from varying the Einstein-Hilbert action. This matching gives also a definition of the extrinsic curvature K, where $\mathscr{K} := \sigma K$. The volume-forms as viewed in (2.1.1), are defined as $\eta := d^d\xi \sqrt{{}^d g}$, $\omega := d^D x \sqrt{g}$, and $\sigma := d^d x \sqrt{h}$, where we have used a shorthand, e.g. $d^d x := dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{d-1}$.

The standard action functional analysis involves the infinitesimal change in the fields, and the equations of motion are obtained from the principle of extremal action, that is the action is constant with respect to first order changes in the fields involved. In our case the fields involved are the antisymmetric field, gravity, and the membrane itself.

Without further ado, the first order change in the action (2.1.1) with respect to the antisymmetric field is

$$\delta_A S^E = e \int_{\partial M} \delta_A \phi^* A + (-1)^d \int_M \delta_A (A \wedge d \star F), \tag{2.1.3}$$

where we have used the identity $d(a \wedge b) = da \wedge b + (-1)^m a \wedge db$, for a m-form a, to shift the exterior derivative over to the dual field $\star F$. The pullback $\phi^* A$ can be expanded as,

$$\phi^* A = \frac{1}{d!} A_{\mu_1 \cdots \mu_d} z^{\mu_1}_{,a_1} \cdots z^{\mu_d}_{,a_d} \varepsilon^{a_1 \cdots a_d} d^d \xi, \tag{2.1.4}$$

where we have defined,

$$(A_\phi)_{\mu_1 \cdots \mu_d} := A_{\mu_1 \cdots \mu_d}, \qquad \frac{\partial z^\mu}{\partial \xi^a} := z^\mu_{,a}. \tag{2.1.5}$$

Suppose for now that $d\star F$ is an $(n-d)$-form, then the exterior product of $\delta_A A$ with $d\star F$ is

$$\begin{aligned}\delta_A A\wedge d\star F &= \frac{1}{d!(n-d)!}\delta_A A_{\mu_1\cdots\mu_d}(d\star F)_{\mu_{d+1}\cdots\mu_{n-d}}\varepsilon^{\mu_1\cdots\mu_d\mu_{d+1}\cdots\mu_{n-d}}d^n x, \\ &= \frac{1}{d!}\delta_A A_{\mu_1\cdots\mu_d}(\star d\star F)^{\mu_1\cdots\mu_d}d^n x,\end{aligned} \tag{2.1.6}$$

where (A.4.1) has been used. The local chart dimensions have been kept at n in order to make sense of the calculation. At the end of the day, one may limit to intended dimensions, in our case $n\to D$. Assuming that the fields A, F, vanishes at ∂M, using Stokes' theorem

$$0=\int_M d(A\wedge\star F)=\int_M(F\wedge\star F-A\wedge\star\star d\star F)\,. \tag{2.1.7}$$

where we have used the property $\star\star d\star F=(-1)^{(D-1)(n-D+1)}(d\star F)$. By (A.4.4), (2.1.7) is the same as $(F,F)-(A,\star d\star F)=0$, thus by (2.1.6),

$$\begin{aligned}\delta_A A\wedge d\star F &= \frac{1}{d!}(d\delta_A A)_{\mu_1\cdots\mu_D}F^{\mu_1\cdots\mu_D}d^D x \\ &= -\frac{1}{d!}\partial_{\nu_1}E\,\varepsilon^{\nu_1\cdots\nu_D}\delta_A A_{\nu_2\cdots\nu_D}d^D x+\cdots\,,\end{aligned} \tag{2.1.8}$$

where the relation $F^{\mu_1\cdots\mu_D}=E|g|^{-1/2}\varepsilon^{\mu_1\cdots\mu_D}$ has been used. Combining (2.1.8) and the variation of (2.1.4), (2.1.3) becomes

$$\begin{aligned}\delta_A S^E = \int d^D x\frac{1}{d!}\delta_A A_{\mu_2\cdots\mu_D}\left(e\int d^d\xi\delta^D(x-z(\xi))z^{\mu_2}_{,a_1}\cdots z^{\mu_D}_{,a_d}\varepsilon^{a_1\cdots a_d}+\partial_{\mu_1}E\varepsilon^{\mu_1\cdots\mu_D}\right) \\ +\text{ surface terms.}\end{aligned} \tag{2.1.9}$$

By the principle of extremal action, the equations of motion are simply

$$(\partial_{\mu_1}E)\,\varepsilon^{\mu_1\cdots\mu_D}=-e\int d^d\xi\,\delta^D(x-z(\xi))z^{\mu_2}_{,a_1}\cdots z^{\mu_D}_{,a_d}\varepsilon^{a_1\cdots a_d}. \tag{2.1.10}$$

2.2 Gravitational Equations of Motion

In (2.1.1) the field strength F and the induced metric determinant ${}^d g$, have intrinsic dependence upon the metric tensor. Hence, the variation of (2.1.1) with respect to the metric tensor is

$$\begin{aligned}\delta_g S^E &= m\,\delta_g\int_M\eta-\frac{1}{2}\delta_g\int_M F\wedge\star F+\delta_g S^{\mathrm{E,grav}}(\lambda), \\ &= m\int d^D x\,d^d\xi\,\delta^D(x-z(\xi))\delta_g\sqrt{{}^d g}-\frac{1}{2}\int d^D x\,E^2\delta_g\sqrt{g}+\delta_g S^{\mathrm{E,grav}}(\lambda),\end{aligned} \tag{2.2.1}$$

where

$$\delta_g S^{\mathrm{E,grav}}=-\frac{1}{2\kappa}\int_M\delta\left(\mathscr{R}-2\Lambda\right)+\frac{1}{\kappa}\int_{\partial M}\delta\mathscr{K}\,. \tag{2.2.2}$$

The variation of the Ricci scalar $\mathscr{R}$ can be resolved as

$$\begin{aligned}\delta\mathscr{R} &\equiv \delta g^{ik}\mathscr{R}^j{}_k\wedge\star(\partial_i\wedge\partial_j)+g^{ik}\delta\left[\mathscr{R}^j{}_k\wedge\star(\partial_i\wedge\partial_j)\right] \\ &= R^{jk}\delta g_{jk}\omega+g^{ik}\delta\left[\mathscr{R}^j{}_k\wedge\star(\partial_i\wedge\partial_j)\right],\end{aligned} \tag{2.2.3}$$

where we have used (A.4.4),(A.7.16), and defined $\omega := d^D x\sqrt{g}$. Using the explicit expression (A.7.12), one can deduce that

$$\begin{aligned} g^{ik}\delta\left[\mathscr{R}^j{}_k \wedge \star(\partial_i \wedge \partial_j)\right] &= g^{ik}\delta\left[\left(d\Gamma^j{}_k + \Gamma^j{}_l \wedge \Gamma^l{}_k\right) \wedge \star(\partial_i \wedge \partial_j)\right] \\ &= g^{ik}(\nabla_i \delta\Gamma^j{}_{kj} - \nabla_j \delta\Gamma^j{}_{ki})\omega - \frac{1}{2} R g^{ik}\delta g_{ik}\omega, \end{aligned} \tag{2.2.4}$$

where $\omega := d^D x\sqrt{g}$. Hence, by substitution of (2.2.4) in (2.2.2), we find

$$\begin{aligned} \delta_g S^{\mathrm{E,grav}} &= -\frac{1}{2\kappa}\int_M \omega\left[\delta g_{jk}\left(R^{jk} - \frac{1}{2}R\, g^{jk} + \lambda g^{jk}\right)\right] \\ &\quad + g^{mk}\left(\nabla_m \delta\Gamma^i{}_{ik} - \nabla_i \delta\Gamma^i{}_{mk}\right) + \frac{1}{\kappa}\int_{\partial M}\delta\mathscr{K} \\ &= -\frac{1}{2\kappa}\int_M \omega\,\delta g_{jk}\left(R^{jk} - \frac{1}{2}R\, g^{jk} + \lambda g^{jk}\right) - \frac{1}{\kappa}\int_{\partial M}\sigma\left(\frac{1}{2}n_k V^k - \delta K\right), \end{aligned} \tag{2.2.5}$$

where we have used Stokes theorem to change the integration over to the boundary. Here $\mathscr{K} := \sigma\delta K := d^d x\sqrt{h}\delta K$ where K is the trace of the extrinsic curvature, and V^k is the defined by,

$$V^k \equiv g^{jk}\delta\Gamma^i{}_{ij} - g^{ji}\delta\Gamma^k{}_{ij}, \tag{2.2.6}$$

and n_k are the components of a normal vector perpendicular to the surface of M.

2.3 The Gibbons-Hawking-York Term

The surface integral in the gravitational part of the euclidean action (2.1.1), is included as a counter-term to ensure that the equations of motion have a nicer form. In such a way the system have a description of extrinsic curvature, defined through first order variations of the connection 1-forms. The simplification of the scalar $n_k V^k$, will therefore be the object of interest, and resolved to match the counter-term. One could take a prior definition of the extrinsic curvature, and see if the counterterm is matched exactly to it, but assume that no such definition is performed. This is the way we follow now. Take the basis to be $\{\partial_i\}_{i=0}^d$ where the local coordinate system is $(x^0, \cdots, x^D)$. Hence by definition, the scalar $n_k V^k$ is

$$n_k V^k = n_k g^{jk}\delta\Gamma^i{}_{ij} - n_k g^{ji}\delta\Gamma^k{}_{ij}. \tag{2.3.1}$$

Now define the transverse metric components, h_{ij}, given by

$$\begin{aligned} g_{ij} &\equiv h_{ij} + n_i n_j, \\ g^{ij} &\equiv h^{ij} + n^i n^j, \end{aligned} \tag{2.3.2}$$

where $n_i n^i = 1$. The metric components h_{ij} has the following properties,

$$\begin{aligned} h_{ij}n^j &= 0, \\ h_{ij}h^{ik} &= \delta^k_j, \\ h_{ij}h^{ij} &= d, \end{aligned} \tag{2.3.3}$$

thus the transverse metric has dimensions equal to d, and it annihilates the normal vector components. The transverse metric thus is intrinsic to the boundary ∂M. In order to obtain the desired explicit result of $n_k V^k$, one notices that

$$0 = \nabla_k\left(n^i n_i\right) = g^{ij}\left(n_j \nabla_k n_i + n_i \nabla_k n_j\right), \tag{2.3.4}$$

that is $n^i \nabla_k n_i = 0$. The variation of the connection 1-form components are

$$
\begin{aligned}
g^{jk}\delta\Gamma^i{}_{ij} - g^{ij}\delta\Gamma^k{}_{ij} &= \frac{1}{2} g^{jk} g^{im} \delta\left(g_{mi,j} + g_{jm,i} - g_{ij,m}\right) \\
&- \frac{1}{2} g^{mk} g^{ij} \delta\left(g_{mi,j} + g_{jm,i} - g_{ij,m}\right), \\
&= g^{jk} g^{im}\left(\delta g_{mi,j} - \delta g_{ji,m}\right),
\end{aligned}
\tag{2.3.5}
$$

where the Christoffel symbols are defined by (A.5.8). Thus (2.3.1) is simplified to,

$$
\begin{aligned}
n_k V^k &= n^j g^{im}\left(\delta g_{mi,j} - \delta g_{ji,m}\right) \\
&= n^j h^{im}\left(\delta g_{mi,j} - \delta g_{ji,m}\right).
\end{aligned}
\tag{2.3.6}
$$

The variation of the metric tensor components, δg_{ij}, vanishes by assumption everywhere on the boundary ∂M. Therefore it is also true that the tangential derivatives vanishes there, $\delta g_{ij,k} = 0$. This implies that the transverse metric is transverse to the tangential derivatives, $h^{ik}\delta g_{ij,k} = 0$, and so (2.3.6) becomes

$$
\begin{aligned}
n_k V^k &= n^j h^{im}\left(\delta g_{mi,j} - \delta g_{jm,i} - \delta g_{ji,m}\right) \\
&= -2h^{im} n_k \delta\Gamma^k{}_{im}.
\end{aligned}
\tag{2.3.7}
$$

Using that $\delta\left(h^{ik} n_{m,i}\right) = 0$ on ∂M, one finds

$$
\begin{aligned}
n_k V^k &= 2h^{im}\delta\left(n_{i,m} - n_k \Gamma^k{}_{im}\right) + 2n_{m,i}\delta h^{im} + 2h^{im}\Gamma^k{}_{im}\delta n_k \\
&= 2\left(h^{im} - n^i n^m\right)\delta\left(\nabla_m n_i\right) \\
&= 2\delta\left(\nabla_m n^m\right).
\end{aligned}
\tag{2.3.8}
$$

So the boundary term in the variation (2.2.5) together with (2.3.8), gives

$$
-\frac{1}{\kappa}\int_{\partial M} \sqrt{h}\,\left(\delta\left(\nabla_m n^m\right) - \delta K\right) = -\frac{1}{\kappa}\int_{\partial M} \sqrt{h}\,\delta\left(\nabla_m n^m - K\right).
\tag{2.3.9}
$$

The vanishing of the boundary term in (2.2.5), implies that $K := \nabla_i n^i$, and (2.2.1) becomes

$$
\begin{aligned}
\delta_g S^E &= +m\int d^D x\, d^d\xi\, \delta g_{jk}\, \delta^D\left(x - z(\xi)\right) \frac{g}{2\sqrt{{}^d g}} \frac{\delta^d g}{\delta g} g^{jk} \\
&- \frac{1}{2}\int d^D x\, \delta g_{jk} \frac{\sqrt{g}}{2} g^{jk}\, E^2 - \frac{1}{2\kappa}\int d^D x \sqrt{g}\, \delta g_{jk}\left(G^{jk} + \lambda\, g^{jk}\right) \\
&= -\int d^D x\, \sqrt{g}\, \delta g_{jk} \left\{ \int d^d\xi\, \frac{m\sqrt{{}^d g}}{2\sqrt{g}} \delta^D(x - z(\xi)) g^{jk} + \frac{1}{2\kappa}\left[G_{jk} + \left(\lambda + \frac{1}{2}\kappa E^2\right) g^{jk}\right]\right\}.
\end{aligned}
\tag{2.3.10}
$$

Keeping in mind that indices are all euclidean, the equations of motion with respect to variations of the metric tensor g, are

$$
G_{\mu\nu} = -\Lambda\, g_{\mu\nu} - \kappa m \int d^d\xi\, \frac{\sqrt{{}^d g}}{\sqrt{g}} \delta^D(x - z(\xi)) g_{\mu\nu},
\tag{2.3.11}
$$

where $\mu = 0, 1, \cdots, D$, and we have defined the Einstein tensor components and an effective cosmological constant as, $G_{\mu\nu} \equiv R_{\mu\nu} - R g_{\mu\nu}/2$ and $\Lambda \equiv \lambda + \kappa E^2/2$, respectively.

2.4 Membrane Equations of Motion

To complete the equations of motion, derived from (2.1.1), we need to vary the euclidean action with respect to the membrane history, that is the z^μ's, keeping excluding terms at a constant. The terms which carries explicit factors of ${}^d g$ and z^μ, are thus considered but also an implicit variation of the components of $\phi^* A$ are included. At a glance the variation of (2.1.1) with respect to the membrane history is

$$\begin{aligned} \delta_z S^E = & -\frac{m}{2}\int d^d\xi\, \sqrt{{}^d g}{}^d g^{ab}\delta\, {}^d g_{ab} \\ & + \frac{e}{d!}\int d^d\xi \left[A_{\mu_1\cdots\mu_d}\delta\left(z,^{\mu_1}_{a_1}\cdots z,^{\mu_d}_{a_d}\right)\varepsilon^{a_1\cdots a_d} + (\delta A_{\mu_1\cdots\mu_d})\left(z,^{\mu_1}_{a_1}\cdots z,^{\mu_d}_{a_d}\right)\varepsilon^{a_1\cdots a_d}\right], \end{aligned} \tag{2.4.1}$$

The term $\sqrt{{}^d g}{}^d g^{ab}\delta^d g_{ab}$ can be expanded as

$$\sqrt{{}^d g}{}^d g^{ab}\delta\, {}^d g_{ab} = \sqrt{{}^d g}{}^d g^{ab}\delta\left(g_{\mu\nu}z^\mu_{,a}z^\nu_{,b}\right) = \sqrt{{}^d g}{}^d g^{ab}g_{\mu\nu,\alpha}\delta z^\alpha z^\mu_{,a}z^\nu_{,b} + 2\,{}^d g^{ab}g_{\mu\nu}z^\mu_{,a}\delta z^\nu_{,b}. \tag{2.4.2}$$

Using integration by parts, one can resolve (2.4.2) to be

$$\begin{aligned} \sqrt{{}^d g}{}^d g^{ab}\delta\, {}^d g_{ab} = & \sqrt{{}^d g}{}^d g^{ab}g_{\mu\nu,\alpha}\delta z^\alpha z^\mu_{,a}z^\nu_{,b} + 2\big(\sqrt{{}^d g}{}^d g^{ab}g_{\mu\nu}z^\mu_{,a}\delta z^\nu\big)_{,b} \\ & - 2g_{\mu\nu}\big(\sqrt{{}^d g}{}^d g^{ab}z^\mu_{,a}\big)_{,b}\delta z^\nu - 2\sqrt{{}^d g}{}^d g^{ab}g_{\mu\alpha,\nu}z^\mu_{,a}z^\nu_{,b}\delta z^\alpha. \end{aligned} \tag{2.4.3}$$

The last term in (2.4.3) have the symmetry that

$${}^d g^{ab}g_{\mu\alpha,\nu}z^\mu_{,a}z^\nu_{,b} = \frac{1}{2}\big(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu}\big){}^d g^{ab}z^\mu_{,a}z^\nu_{,b}, \tag{2.4.4}$$

thus (2.4.3) becomes

$$\begin{aligned} \sqrt{{}^d g}{}^d g^{ab}\delta\, {}^d g_{ab} = & -\sqrt{{}^d g}{}^d g^{ab}\big(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\big)\delta z^\alpha z^\mu_{,a}z^\nu_{,b} - 2g_{\beta\alpha}\big(\sqrt{{}^d g}{}^d g^{ab}z^\beta_{,a}\big)_{,b}\delta z^\alpha \\ & + \text{boundary terms.} \end{aligned} \tag{2.4.5}$$

In (2.4.5), the first term can be written as the components of the connection 1-forms, as defined in (A.5.8). Hence (2.4.5) becomes

$$\begin{aligned} & = -2g_{\alpha\beta}\sqrt{{}^d g}\,\Gamma^\beta{}_{\mu\nu}z^\mu_{,a}z^\nu_{,b}\delta z^\alpha - 2g_{\beta\alpha}\big(\sqrt{{}^d g}{}^d g^{ab}z^\beta_{,a}\big)_{,b}\delta z^\alpha + \text{boundary terms}, \\ & = -2g_{\alpha\beta}z^\mu_{,a}\nabla_\mu\left(\sqrt{{}^d g}{}^d g^{ab}z^\beta_{,b}\right)\delta z^\alpha + \text{boundary terms}, \end{aligned} \tag{2.4.6}$$

The variation of the second term of (2.4.1) can be resolved through manipulations, as

$$\begin{aligned} & \int d^d\xi\left[d\times A_{\mu_1\cdots\mu_d}\delta\left(z,^{\mu_1}_{a_1}\right)z,^{\mu_2}_{a_2}\cdots z,^{\mu_d}_{a_d} + \delta z^{\nu_1}\partial_{\nu_1}A_{\mu_1\cdots\mu_d}z,^{\mu_1}_{a_1}\cdots z,^{\mu_d}_{a_d}\right]\varepsilon^{a_1\cdots a_d} \\ & = \int d^d\xi\,\delta z^{\nu_1}\left[\partial_{\nu_1}A_{\mu_1\cdots\mu_d} - d\partial_{\mu_1}A_{\nu_1\cdots\mu_d}\right]z,^{\mu_1}_{a_1}\cdots z,^{\mu_d}_{a_d}\varepsilon^{a_1\cdots a_d} + \text{boundary terms}, \\ & = \frac{1}{d!}\int d^d\xi\,\delta z^{\nu_1}\partial_{[\nu_1}A_{\mu_1\cdots\mu_d]}z,^{\mu_1}_{a_1}\cdots z,^{\mu_d}_{a_d}\varepsilon^{a_1\cdots a_d} + \text{boundary terms}, \\ & := \int d^d\xi\,\sqrt{g}\,\delta z^{\nu_1}E\varepsilon_{\nu_1\mu_1\cdots\mu_d}z,^{\mu_1}_{a_1}\cdots z,^{\mu_d}_{a_d}\varepsilon^{a_1\cdots a_d} + \text{boundary terms} \end{aligned} \tag{2.4.7}$$

where we have used the definition of the anti-symmetrization operation, the identification $F_{\nu_1\mu_1\cdots\mu_d} = E\,\sqrt{g}\,\varepsilon_{\nu_1\mu_1\cdots\mu_d}$. Bringing everything together we can solve the equations of motion for (2.4.1), as

$$z^{\mu}_{,a}\nabla_{\mu}\left(\sqrt{{}^{d}g}^{d}g^{ab}z^{\alpha}_{,b}\right) = -\frac{eE}{md!}\sqrt{g}\,g^{\alpha\mu_1}z_{,a_1}^{\mu_2}\cdots z_{,a_d}^{\mu_D}\,\varepsilon_{\mu_1\mu_2\cdots\mu_D}\varepsilon^{a_1\cdots a_d}. \tag{2.4.8}$$

Bringing together (2.1.10), (2.3.11), (2.4.8), we arrive at the complete set of equations,

$$(\partial_{\mu_1}E)\,\varepsilon^{\mu_1\cdots\mu_D} = -e\int d^d\xi\,\delta^D(x-z(\xi))z^{\mu_2}_{,a_1}\cdots z^{\mu_D}_{,a_d}\varepsilon^{a_1\cdots a_d}, \tag{2.4.9a}$$

$$G_{\mu\nu} = -\Lambda\,g_{\mu\nu} - \kappa m\int d^d\xi\,\frac{\sqrt{{}^{d}g}}{\sqrt{g}}\delta^D(x-z(\xi))z_{\mu,a}{}^{d}g^{ab}z_{\nu,b}, \tag{2.4.9b}$$

$$z^{\mu}_{,a}\nabla_{\mu}\left(\sqrt{{}^{d}g}^{d}g^{ab}z^{\alpha}_{,b}\right) = -\frac{eE}{md!}\sqrt{g}\,g^{\alpha\mu_1}z^{\mu_2}_{,a_1}\cdots z^{\mu_D}_{,a_d}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\varepsilon^{a_1\cdots a_d}. \tag{2.4.9c}$$

The first equation in (2.4.9) relates the rate of change of the electric field E, and if the system does not satisfy the classical membrane history, E remain constant. Any crossing, between the two regions as separated, results in a discontinuous jump from one space-time to another thus any contribution to the change in E stems via the membrane. The second describes the space-times on either side of the membrane, where either side follow the equations of general relativity, with the exception that the cosmological constant is now dependent on E. Also in this situation, a discontinuity occurs from any crossing between the two regions. The third equation describes in some sense, the acceleration of the the membrane history z^{μ} where the rate of change of the $z^{\mu}_{,a}$'s is proportional to the charge coupling e, the electric field E, the membrane mass density m, the metric tensor, and the $z^{\mu}_{,a}$'s themselves.

3 Solving the Equations

Now that the complete set of equations of motion (2.4.9) are in hand, the solutions of (2.4.9) are of our interest. As previously mentioned, two equations have singular terms which needs to be considered, and we leave this to the end of the section. When the membrane history is not the classical path the singular terms vanishes, and by taking the trace of the gravitational equations of motion in (2.4.9), leads to

$$R = \frac{2D}{D-2}\Lambda = \frac{2D}{D-2}\left(\lambda + \frac{1}{2}\kappa E^2\right). \tag{3.0.1}$$

This equation remains true for two values of the electric field E, and we can make sense of this because of the separated regions of corresponding space-times, by the membrane as represented by the singularities in (2.4.9). Suppose we label these two regions by "inside" and "outside", so that E_o, E_i, is the electric field on the outside and inside respectively. By this, the scalar curvature R is defined by values on the separate regions. Denote these curvatures by $R_{i,o}$, then the "cosmological constants" are

$$\begin{aligned}\Lambda_o &= \lambda + \frac{1}{2}\kappa E_o^2, \\ \Lambda_i &= \Lambda_o - \frac{1}{2}\kappa\left(E_o^2 - E_i^2\right),\end{aligned} \tag{3.0.2}$$

where we have eliminated R_i from Λ_i, in terms of Λ_o. In analogy with how the step function is defined at a discontinuity, the value of the electric field at the membrane history, E_{on}, may be defined analogous as the average of the inside and outside electric fields,

$$E_{on} := \frac{1}{2}(E_o + E_i). \tag{3.0.3}$$

To find E_i in terms of E_o we need to resolve the difference $E_i - E_o$, and this is related to the analysis on what occurs when crossing the inside and outside regions. The first equation of (2.4.9), can be contracted by the permutation symbol to yield

$$\partial_\beta E = -\frac{e}{d!}\int d^d\xi\, \delta^D\left(x - z(\xi)\right) z^{\mu_2}_{,a_1} \cdots z^{\mu_D}_{,a_d} \varepsilon_{\beta\mu_2\cdots\mu_D}\varepsilon^{a_1\cdots a_d}, \tag{3.0.4}$$

where the property (A.8.4) has been used. Let us define the unit normal vector components as

$$-\epsilon n_{\mu_1} \equiv \frac{1}{d!} z^{\mu_2}_{,a_1} \cdots z^{\mu_D}_{,a_d} \epsilon_{\mu_1\mu_2\cdots\mu_D}\epsilon^{a_1\cdots a_d}, \tag{3.0.5}$$

where $\epsilon_{\alpha\mu_2\cdots\mu_D} := g^{1/2}\varepsilon_{\alpha\mu_2\cdots\mu_D}$, $\epsilon^{a_1\cdots a_d} := {}^d g^{-1/2}\,\varepsilon^{a_1\cdots a_d}$ are the covariant permutation tensor components, $\epsilon = \pm 1$. Then by using (3.0.5) in (3.0.4), one finds the equation

$$\partial_\beta E = +e\epsilon \int d^d\xi\, \frac{\sqrt{{}^d g}}{\sqrt{g}} \delta^D(x - z(\xi))\, n_\beta, \tag{3.0.6}$$

and on the world volume it may be written as

$$n^\beta \partial_\beta E = e\epsilon \int d^d\xi \frac{\sqrt{{}^d g}}{\sqrt{g}} \delta^D(x - z(\xi)). \tag{3.0.7}$$

Integrating (3.0.7), reveals that the difference in the outside and inside electric fields, are related by

$$E_o - E_i = e\epsilon. \tag{3.0.8}$$

The constant ϵ has the ambiguity of being within a sign, just for the matter of what we would call "inside" or "outside" is of choice. As one will encounter later on, the initial configuration of the tunneling process, dictates a choice of what will be the inside and the outside. As an example; suppose the initial electric field is E_o, then since the energy density is proportional to the square of the electric field, energy conservation states that,

$$E_o^2 - E_i^2 = (E_o + E_i)(E_o - E_i) = \frac{1}{2} E_{on} e\epsilon > 0, \tag{3.0.9}$$

where we have used (3.0.3), (3.0.8). By the assumption that the charge coupling is strictly positive, $e > 0$, one must have $\epsilon = +1$. If the inside and outside are exchanged, so does the sign of (3.0.9) and consequently $\epsilon = -1$. With the relation (3.0.8), the electric field on the worldline (3.0.3) takes the form

$$E_{on} = E_o - \frac{1}{2} e\epsilon, \tag{3.0.10}$$

where we have eliminated the inside electric field E_i, in (3.0.3). The inside electric field can also be eliminated in (3.0.2), to yield Λ_i in terms of the outside electric field, as

$$\Lambda_i = \Lambda_o - \kappa \left(\epsilon e E_o - \frac{e^2}{2} \right). \tag{3.0.11}$$

This concludes the configuration parameters $\Lambda_{i,o}$, and the electric fields $E_{i,o}$ and E_{on}.

4 Instanton Solutions

Some of the parameters to describe the system, has now been found, including the dynamical expressions of Λ, the electric fields in terms of ϵ and the charge coupling e. There are still a few parameters missing to the complete picture, as we will now resolve. Some of the possible freedoms have been left out in the calculations in the previous section, such as the inclusion of the extrinsic curvature. This is of no surprise, since we have left out the third equation of (2.4.9). Now, this equation depends upon a product of velocities, and one should extract the normal vector components from these terms. Without further ado, define the pullback of the extrinsic curvature as

$$K_{ab} = z^{\mu}_{,a} z^{\nu}_{,b} \nabla_{\mu} n_{\nu}, \tag{4.0.1}$$

where its trace is defined as $K := {}^{d}g^{ab} K_{ab}$. The velocities of the trajectories z^{μ} are orthonormal to the unit normal components, that is $z^{\mu}_{,a} n_{\mu} = 0$. Hence, one can compose the expression

$$\sqrt{{}^{d}g}\,{}^{d}g^{ab} z^{\mu}_{,a} n_{\mu} = 0, \tag{4.0.2}$$

By simply covariantly differentiating (4.0.2) by the operator $z^{\alpha}_{,b}\nabla_{\alpha}$, one resolves (4.0.2) to be the same as

$$z^{\alpha}_{,b}\nabla_{\alpha}\left(\sqrt{{}^{d}g}\,{}^{d}g^{ab} z^{\mu}_{,a}\right) n_{\mu} = -\sqrt{{}^{d}g}\,{}^{d}g^{ab} z^{\alpha}_{,b} z^{\mu}_{,a} \nabla_{\alpha} n_{\mu} = -\sqrt{{}^{d}g} K_{on}, \tag{4.0.3}$$

where we have defined K_{on} as the trace of the extrinsic curvature, defined on the worldline. Now, comparing (4.0.3) with the third equation of (2.4.9), this relation follows:

$$K_{on} = -\frac{e\epsilon E_{on}}{m}. \tag{4.0.4}$$

so the trace of the extrinsic curvature, on the worldline, is proportional to the charge coupling e, the membrane mass m, the electric field. In the next section we shall follow an analogous analysis for the electric fields, to find the relations between the extrinsic curvatures on the outside and inside, denoted by K_o and K_i respectively. We will begin with a decomposition of space-time and further on define a Hamiltonian in deriving the equations of motion for the extrinsic curvatures.

4.1 Decomposition of Space-time

The dynamics of the metric tensor components, can be understood through the gravitational equations of motion in (2.4.9), and in general it is true that the theory will remain invariant under general coordinate transformations. As we will now follow, the space-time will be separated in a particular way such that time is be singled out, separated from d spatial dimensions. This procedure may be a necessary step in recasting the gravitational equations of motion to canonical form. The treatment in this section will begin with a mathematical exercise, defining the separation in time and foliations, or sheets. Then we will resolve what quantities are intrinsic to the foliations at constant time, to follow up by defining an action principle in canonical variables. Given a manifold M equipped with an euclidean metric. The *foliation* of M into $\Sigma \times \mathbb{R}$, can be defined as,

$$\begin{aligned} \forall t \in \mathbb{R}, \qquad & \Sigma_t \equiv \{p \in M, t(p) = t\}, \\ & \Sigma_t \cap \Sigma_{t'} = \emptyset, \end{aligned} \tag{4.1.1}$$

where t is smooth such that the gradient of t, between the sheets defined by (4.1.1), is non-zero. Take as the basis $\{\partial_\mu\}_{\mu=0}^{d}$ of T_pM associated with the coordinates z^ν, that is we have

$$\begin{aligned}\partial_t &\equiv \frac{\partial}{\partial t} = \frac{\partial}{\partial z^0}, \\ \partial_i &\equiv \frac{\partial}{\partial z^i}, \qquad i \in [1, \cdots, D].\end{aligned} \tag{4.1.2}$$

For any $i = 1, \cdots, d$ the vector ∂_i is tangent to $t = c^0$, $z^i = c^i$, where c^0, c^i, are constants such that $c^i \neq c^j$, $i \neq j$, holds, that is

$$\partial_i \in T_pM(\Sigma_t), \qquad i = 1, \cdots, D. \tag{4.1.3}$$

The dual basis $\{dx^\mu\}_{\mu=0}^{d}$ in T_p^*M, satisfies the completion relation $(dx^\mu, \partial_\nu) = \delta_\nu^\mu$, where $(\cdot,\cdot)$ is the inner product operation, in particular $(dt, \partial_t) = 1$. Generally, the vector ∂_t can be written as a linear expression, involving the terms of a normal 1-form $n \equiv \alpha dt$, $dt \in T_p^*M(\Sigma_t)$, and a vector $\beta \in T_pM(\Sigma_t)$. That is we can write

$$\partial_t \equiv \alpha n + \beta, \tag{4.1.4}$$

where β is called the *shift*, and α is called the *lapse*. Since $(n, n) = 1$ and $(dt, \partial_t) = 1$, the inner product of the 1-form dt and the vector β, is

$$(dt, \beta) = (dt, \partial_t) - \alpha(dt, n) = 1 - (n, n) = 0. \tag{4.1.5}$$

Thus $(n, \beta) = 0$. Now, $\beta = \beta^i \partial_i$ so from (4.1.4)

$$1 \cdot \partial_t - \alpha n^t \partial_t - \alpha n^i \partial_i - \beta^i \partial_i = 0, \tag{4.1.6}$$

where $1 \in \mathbb{R}$. Equating the vector components in (4.1.6), gives

$$\begin{aligned}n^t &= \alpha^{-1}, \\ n^i &= \beta^i \alpha^{-1}.\end{aligned} \tag{4.1.7}$$

Since the normal 1-form can be expanded as $n = \alpha dt$, the components are simply $n_t = \alpha$, $n_i = 0$, and so the contravariant, and covariant normal vector components are given, respectively, by

$$\begin{aligned}n^\mu &= \left(\frac{1}{\alpha}, \frac{\beta^1}{\alpha}, \cdots, \frac{\beta^D}{\alpha}\right), \\ n_\mu &= (\alpha, 0, \cdots, 0)\,.\end{aligned} \tag{4.1.8}$$

The metric tensor components can now be expressed in terms of lapse and shift, as

$$\begin{aligned}g_{tt} &= g(\partial_t, \partial_t) = g\left(\alpha n + \beta, \alpha n + \beta\right) = \alpha^2 g_{\mu\nu} n^\mu n^\nu + g_{ij}\beta^i\beta^j = \alpha^2 + \beta^i\beta_i, \\ g_{ti} &= g(\alpha n + \beta, \partial_i) = g_{ij}\beta^j = \beta_i, \\ g_{ij} &\equiv h_{ij},\end{aligned} \tag{4.1.9}$$

and so the $d + 1$-decomposed metric is

$$\begin{aligned}ds^2 = g_{\mu\nu} dz^\mu dz^\nu &= (\alpha^2 + h_{ij}\beta^i\beta^j)dt^2 + 2h_{ij}\beta^j dt dz^i + h_{ij} dz^i dz^j, \\ &= \alpha^2 dt^2 + h_{ij}\left(dz^i + \beta^i dt\right)\left(dz^j + \beta^j dt\right).\end{aligned} \tag{4.1.10}$$

This is the metric we shall now focus onward. By defining the pullback of dz^i, at constant t, as

$$dz^i = \left(\frac{\partial z^i}{\partial \xi^a}\right)_t d\xi^a, \qquad a = 1, \cdots, d, \tag{4.1.11}$$

the metric can be written in terms of the history parameters, ξ's, as

$$ds^2 = \alpha^2 dt^2 + h_{ab}\left(d\xi^a + \beta^a dt\right)\left(d\xi^b + \beta^b dt\right), \tag{4.1.12}$$

where

$$h_{ab} = g_{ij}\left(\frac{\partial z^i}{\partial \xi^a}\right)_t \left(\frac{\partial z^j}{\partial \xi^b}\right)_t := g_{ij} E^i_a E^j_b, \tag{4.1.13}$$

is the pullback of the metric tensor components. The inverse metric tensor components $g^{\mu\nu}$, can be found by noticing that

$$\begin{aligned} \frac{1}{\alpha} &= n^t = g^{tt} n_t = g^{tt}\alpha, \\ \frac{\beta^b}{\alpha} &= n^b = g^{bt} n_t, \\ \delta^a_c &= g_{cb} g^{ba} + g_{ct} g^{ta} = h_{cb} g^{ba} + \beta_c \beta^a \alpha^{-2}, \end{aligned} \tag{4.1.14}$$

which in turn reveals,

$$\begin{aligned} g^{tt} &= \alpha^{-2}, \\ g^{at} &= \alpha^{-2}\beta^a, \\ g^{ab} &= h^{ab} + \alpha^{-2}\beta^a\beta^b, \end{aligned} \tag{4.1.15}$$

where we have used (2.3.2). Using Cramer's rule, one can relate the $d+1$ determinant $g = \det(g_{\alpha\beta})$, to the pullback determinant, $h = \det(h_{ab})$. The result is

$$\sqrt{g} = \alpha\sqrt{h}, \tag{4.1.16}$$

that is the g determinant is proportional to the lapse and the determinant h.
The intrinsic features of the foliations Σ_t are of particular interest, since the resulting relations between the curvature scalar R and the extrinsic curvature scalar K, yields a nice form. By (2.3.2), the attempt of lowering the unit normal with $h_{\mu\nu}$, only nullifies it, but the operation of $h_{\mu\nu}$ on any 1-form $\omega \in \Sigma_t$, is a mapping back to Σ_t. We thus have,

$$\begin{aligned} h^\mu{}_\nu n^\nu &= 0, \\ h^\mu{}_\nu \omega_\mu &= \omega_\nu, \end{aligned} \tag{4.1.17}$$

for $\omega \in \Sigma_t$ and $n \perp \Sigma_t$. We may therefore define the projection of an arbitrary (m,n)-tensor, to be

$$\mathcal{P}T^{i_1 i_2 \cdots i_m}{}_{j_1 j_2 \cdots j_n} := h^{i_1}{}_{i'_1} h^{i_2}{}_{i'_2} \cdots h^{i_m}{}_{i'_m} h^{j'_1}{}_{j_1} h^{j'_2}{}_{j_2} \cdots h^{j'_n}{}_{j_n} T^{i'_1 i'_2 \cdots i'_m}{}_{j'_1 j'_2 \cdots j'_n}, \tag{4.1.18}$$

where $\mathcal{P}$ is defined as the operation of projection. The definition of (4.1.18) enables the projection of arbitrary tensor components. The interest in our hands, is how we may relate the curvature scalar R, to the extrinsic curvature scalar K. Naturally, since R is contractions of the Riemann tensor components, the projection of $R^{\tau}{}_{\alpha\mu\nu}$ on Σ_t, has the formal expression,

$$\bar{R}^{\tau}_{\alpha\mu\nu}v^{\alpha} = \mathcal{P}\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu v^{\tau}\right), \tag{4.1.19}$$

where the bar denotes dependence on the transverse metric, and $v \in \Sigma_t$, $v^{\tau}n_{\tau} = 0$. The expansion of the terms on the right hand side of (4.1.19), are by definition

$$\begin{aligned}\mathcal{P}\left(\nabla_\mu\nabla_\nu v^{\tau}\right) &= h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}h^{\tau}{}_{\tau'}\nabla_{\mu'}(h^{\alpha}{}_{\nu'}h^{\tau'}{}_{\beta}\nabla_\alpha v^{\beta}) \\ &= h^{\mu'}{}_{\mu}h^{\tau}{}_{\tau'}(\nabla_{\mu'}h^{\tau'}{}_{\beta})\nabla_\nu v^{\beta} + h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}(\nabla_{\mu'}h^{\alpha}{}_{\nu'})\nabla_\alpha v^{\tau} \\ &+ h^{\mu'}{}_{\mu}h^{\alpha}{}_{\nu}h^{\tau}{}_{\beta}\nabla_{\mu'}\nabla_\alpha v^{\beta}.\end{aligned} \tag{4.1.20}$$

Using the covariant derivative of the projection operators, $\nabla_\mu(h^{\alpha}{}_{\beta}) = -\nabla_\mu(n^{\alpha}n_{\beta})$, one can find the following relations

$$\begin{aligned}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}h^{\tau}{}_{\beta}(\nabla_{\mu'}h^{\alpha}{}_{\nu'})\nabla_\alpha v^{\beta} &= KK_{\mu\nu}v^{\tau}, \\ h^{\mu'}{}_{\mu}h^{\alpha}{}_{\nu}h^{\tau}{}_{\tau'}(\nabla_{\mu'}h^{\tau'}{}_{\beta})\nabla_\alpha v^{\beta} &= K_{\nu\lambda}K_{\mu}{}^{\tau}v^{\lambda},\end{aligned} \tag{4.1.21}$$

where we have used that $h^{\nu'}{}_{\nu}h^{\beta}{}_{\nu'} = h^{\beta}{}_{\nu}$. Hence, by combining the analysis of (4.1.21), with (4.1.19), yields the projection of the Riemann tensor components as

$$\begin{aligned}\bar{R}^{\tau}_{\alpha\mu\nu}v^{\alpha} &= h^{\mu'}{}_{\mu}h^{\alpha}{}_{\nu}h^{\tau}{}_{\beta}(\nabla_{\mu'}\nabla_\alpha - \nabla_\alpha\nabla_{\mu'})v^{\beta} + (K_{\nu\beta}K_{\mu}{}^{\tau} - K_{\mu\beta}K_{\nu}{}^{\tau})v^{\beta} \\ &= h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}h^{\tau}{}_{\beta}R^{\beta}{}_{\alpha'\mu'\nu'}h^{\alpha'}{}_{\alpha}v^{\alpha} + (K_{\nu\alpha}K_{\mu}{}^{\tau} - K_{\mu\alpha}K_{\nu}{}^{\tau})v^{\alpha}.\end{aligned} \tag{4.1.22}$$

Thus we conclude that the projection of the Riemann tensor components are

$$\bar{R}^{\tau}_{\alpha\mu\nu} = h^{\tau}{}_{\tau'}h^{\alpha'}{}_{\alpha}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}R^{\tau'}{}_{\alpha'\mu'\nu'} + K_{\mu}{}^{\tau}K_{\nu\alpha} - K_{\nu}{}^{\tau}K_{\mu\alpha}. \tag{4.1.23}$$

In (4.1.19) we see a direct relation between the Riemann tensor components, and the extrinsic curvature, that is the Riemann tensor components are separated in two parts, intrinsic to the foliations Σ_t. Taking the trace, $\tau = \mu$ of (4.1.23), reveals the relation

$$\bar{R}_{\alpha\nu} = h^{\mu'\tau'}h^{\alpha'}{}_{\alpha}h^{\nu'}{}_{\nu}R_{\tau'\alpha'\mu'\nu'} + K_{\mu}{}^{\mu}K_{\nu\alpha} - K_{\nu}{}^{\mu}K_{\mu\alpha}. \tag{4.1.24}$$

The intrinsic Ricci curvature scalar, is the contraction of (4.1.24), hence

$$\begin{aligned}\bar{R} = h^{\alpha\nu}\bar{R}_{\alpha\nu} &= h^{\mu'\tau'}h^{\alpha'\nu'}R_{\tau'\alpha'\mu'\nu'} + K^2 - K^{\alpha\beta}K_{\alpha\beta} \\ &= R - 2n^{\alpha}n^{\beta}R_{\alpha\beta} + K^2 - K^{\alpha\beta}K_{\alpha\beta}.\end{aligned} \tag{4.1.25}$$

We may find other expressions by considering different contractions with the Riemann tensor components. Suppose the following projection is realized:

$$h^{\alpha}{}_{\alpha'}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}R^{\alpha'}{}_{\beta\mu'\nu'}n^{\beta} = h^{\alpha}{}_{\alpha'}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}[\nabla_{\mu'}, \nabla_{\nu'}]n^{\alpha'}. \tag{4.1.26}$$

By similar analysis, one can find

$$h^{\alpha}{}_{\alpha'}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}\nabla_{\mu'}\nabla_{\nu'}n^{\alpha'} = h^{\alpha}{}_{\alpha'}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}\nabla_{\mu'}K_{\mu'}{}^{\alpha'} = \bar{\nabla}_\mu K_{\nu}{}^{\alpha}. \tag{4.1.27}$$

Hence (4.1.26) is

$$n^{\beta}h^{\alpha}{}_{\alpha'}h^{\mu'}{}_{\mu}h^{\nu'}{}_{\nu}R^{\alpha'}{}_{\beta\mu'\nu'} = \bar{\nabla}_\mu K_{\nu}{}^{\alpha} - \bar{\nabla}_\nu K_{\mu}{}^{\alpha}. \tag{4.1.28}$$

A contraction, $\nu = \alpha$, in (4.1.28) reveals

$$n^\beta h^{\nu'}{}_{\alpha'} h^{\mu'}{}_\mu R^{\alpha'}{}_{\beta\mu'\nu'} = \bar{\nabla}_\mu K - \bar{\nabla}_\alpha K^\alpha{}_\mu, \tag{4.1.29}$$

where we have used the defining properties of the Riemann tensor components. The simplification of (4.1.25), may be found by noticing that

$$\begin{aligned} K_{\alpha\beta}K^{\alpha\beta} &= (\nabla_\alpha n^\beta)(\nabla_\beta n^\alpha) \\ &= \nabla_\alpha(n^\beta \nabla_\beta n^\alpha) - n^\beta[\nabla_\alpha, \nabla_\beta]n^\alpha - n^\beta \nabla_\beta \nabla_\alpha n^\alpha \\ &= \nabla_\alpha(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta) - n^\alpha n^\beta R_{\alpha\beta} + K^2, \end{aligned} \tag{4.1.30}$$

and by the result of (4.1.30), we can eliminate the contraction $n^\alpha n^\beta R_{\alpha\beta}$ in (4.1.25), to find

$$R - 2\Lambda = \bar{R} - 2\Lambda + 2\nabla_\alpha(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta) + K^2 - K_{\alpha\beta}K^{\alpha\beta}, \tag{4.1.31}$$

where we have subtracted terms proportional to the cosmological constant, on both sides of (4.1.25) and rearranged terms. (4.1.31) is the relation we have sought for, which shows a relation of the Ricci curvature scalar R to the extrinsic curvature scalar K. The left hand side of (4.1.31) is just the integrand of the gravitational action, ignoring the Gibbons-Hawking-York term. Thus if we integrate both sides of (4.1.31) over space-time, we find

$$\begin{aligned} S_{\Lambda EH} = &-\frac{1}{2\kappa}\int d^D x\sqrt{g}(\bar{R} - 2\Lambda + K^2 - K_{ab}K^{ab}) \\ &-\frac{1}{\kappa}\int d^d x\sqrt{h} n_\alpha(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta) \\ &:= S_{GC} - S_{GHY}, \end{aligned} \tag{4.1.32}$$

where we have defined the Gauss-Codazzi action, S_{GC}, as,

$$S_{GC} \equiv -\frac{1}{2\kappa}\int d^D x\sqrt{g}(\bar{R} - 2\Lambda + K^2 - K_{ab}K^{ab}), \tag{4.1.33}$$

$S_{\Lambda EH}$ is the Einstein-Hilbert action with cosmological constant, and S_{GHY} is just the Gibbons-Hawking-York term. If we examine (4.1.32) we see that $S_{\Lambda EH} + S_{GHY} = S_{GC}$. This implies that the equation of motion for $S_{\Lambda EH}$ is the same as for S_{GC}. Using (4.1.16) in (4.1.33), gives the action

$$S_{ADM}[h_{ab}, \alpha, \beta^a] = -\frac{1}{2\kappa}\int d^D x\sqrt{h}\,\alpha\left(\bar{R} - 2\Lambda + K^2 - K_{ab}K^{ab}\right), \tag{4.1.34}$$

where we have explicitly denoted this form of the action, as the ADM-action, and denoted the extrinsic curvature in spatial indices. It is a functional of the induced metric tensor components, the shift components, and the lapse. We may now change to an appropriate coordinate system. Let us pick the spatial coordinates as the foliations Σ_t themselves, which we will call *ADM-coordinates*. This gives the simplifications $E^t_a = 0$, $E^b_a = \delta^b_a$. The extrinsic curvature (4.0.1), in this coordinate system, becomes

$$\begin{aligned} K_{ab} = -\Gamma^t{}_{ab} n_t &= \frac{1}{2\alpha}(\bar{\nabla}_a \beta_b + \bar{\nabla}_b \beta_a - \dot{h}_{ab}) \\ &= -\frac{1}{2}\pounds_n h_{ab}, \end{aligned} \tag{4.1.35}$$

where we have defined $\dot{h}_{ab} := \partial_t h_{ab}$, $n = n^\mu \partial_\mu$, and recognized the Lie-derivative of the transverse metric with respect to the unit normal. (4.1.34). One can notice that,

$$K_{ab}K^{ab} - K^2 = \frac{1}{2} K_{ab}K_{cd} \left(h^{ac}h^{bd} + h^{bc}h^{ad} - 2h^{ab}h^{cd} \right), \tag{4.1.36}$$

where we have invoked the symmetry of the extrinsic curvature, to symmetrize the right hand side of (4.1.36). The set of factors of the transverse metric components in (4.1.36), can be defined as

$$G^{abcd} := \frac{1}{2} \left(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd} \right), \tag{4.1.37}$$

which is known as the deWitt metric tensor components. If S_{ab} are fully symmetric tensor components, then

$$G^{abcd}S_{cd} = S^{ab} - Sh^{ab}, \tag{4.1.38}$$

where the inverse operation where $S := h^{cd}S_{cd}$. Using (4.1.36) in (4.1.34), yields

$$S_{ADM}[h_{ab}, \alpha, \beta^a] = \frac{1}{\kappa} \int d^D x \, \sqrt{h} \, \alpha \left(\bar{R} - 2\Lambda - G^{abcd}K_{ab}K_{cd} \right). \tag{4.1.39}$$

To reach the structure of the Hamiltonian, we need to define a Lagrangian density of the theory (4.1.39), hence

$$\mathcal{L}_{ADM}(h_{ab}, \dot{h}_{ab}, \alpha, \beta^a) := \alpha\sqrt{h} \left(\bar{R} - 2\Lambda - G^{abcd}K_{ab}K_{cd} \right), \tag{4.1.40}$$

where we regard the Lagrangian density as a function of the lapse, shift, the transverse metric and its time derivative. The shift and the lapse are cyclic coordinates, and therefore the corresponding momenta vanishes. Thus, the only non-zero momenta is the one for the transverse metric. We define the momenta conjugate to h_{ab}, as

$$\Pi^{ab} := \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{h}_{ab}}, \tag{4.1.41}$$

so by definition the given momenta from (4.1.40), is

$$\Pi^{ab} = -\sqrt{h} G^{abcd}K_{cd} = -\sqrt{h} \left(K^{ab} - h^{ab}K \right) \tag{4.1.42}$$

By inverting Π^{ab} in terms of K^{ab}, defining $h_{ab}\Pi^{ab} \equiv \Pi$, we find the extrinsic curvature and the time rate of change of h_{ab}, to be

$$\begin{aligned} K^{ab} &= \frac{1}{\sqrt{h}} \left(\Pi^{ab} - \frac{\Pi}{d-1} h^{ab} \right), \\ \dot{h}_{ab} &= \bar{\nabla}_a \beta_b + \bar{\nabla}_b \beta_a - \frac{2\alpha}{\sqrt{h}} \left(\Pi_{ab} - \frac{\Pi}{d-1} h_{ab} \right). \end{aligned} \tag{4.1.43}$$

Now, If we define the ADM-Hamiltonian density as the following:

$$\mathcal{H}_{ADM}(\alpha, \beta^a, h_{ab}, \Pi^{ab}) = \Pi^{ab}\dot{h}_{ab} - \mathcal{L}_{ADM}(\alpha, \beta^a, \Pi^{ab}, h_{ab}), \tag{4.1.44}$$

where the expression in (4.1.44) is analogous to the Legendre transformation, where in our case the coordinates are the transverse metric components h_{ab}, and the momenta is Π^{ab}. The Hamiltonian is the spatial integration over the Hamiltonian density,

$$H_{ADM} := \int d^d x \mathcal{H}_{ADM}, \tag{4.1.45}$$

where if we substitute for $\dot{h}_{ab}$, the momenta, we find the *ADM-Hamiltonian* density to be

$$\mathcal{H}_{ADM} = 2\Pi^{ab}\bar{\nabla}_a\beta_b - \alpha\sqrt{h}\left[\bar{R} - 2\Lambda + \frac{1}{h}\left(\Pi_{ab}\Pi^{ab} - \frac{\Pi^2}{d-1}\right)\right]. \tag{4.1.46}$$

In (4.1.46), the independence of $\dot{\alpha}$ and $\dot{\beta}^a$, reveals the Hamiltonian constraints:

$$\begin{aligned} \mathcal{H}^b &= 2\bar{\nabla}_a\Pi^{ab} = 0, \\ \mathcal{H}^0 &= \sqrt{h}\left[\bar{R} - 2\Lambda + \frac{1}{h}\left(\Pi_{ab}\Pi^{ab} - \frac{\Pi^2}{d-1}\right)\right] = 0. \end{aligned} \tag{4.1.47}$$

The ADM-Hamiltonian equations of motion are found by vary (4.1.46), with respect to the fields except the lapse and the shift since the coefficients returns the constraints. Thus,

$$\begin{aligned} \delta\mathcal{H}_{ADM} &= 2\delta\Pi^{ab}\bar{\nabla}_a\beta_b + 2\Pi^{ab}\delta\left(\bar{\nabla}_a\beta_b\right) - \alpha\delta\mathcal{H}^0 \\ &= 2\delta\Pi^{ab}\bar{\nabla}_a\beta_b + 2\Pi^{ab}\bar{\nabla}_a\delta\beta_b - 2\Pi^{ab}\beta_c\delta\bar{\Gamma}^c_{ab} - \alpha\delta\mathcal{H}^0, \end{aligned} \tag{4.1.48}$$

where the variations with respect to the shift and the lapse, are simply ignored, since they just generates the solutions (4.1.47). Further analysis of (4.1.48) shows that

$$\begin{aligned} \Pi^{ab}\beta_c\delta\bar{\Gamma}^c_{ab} :&= \frac{1}{2}\Pi^{ab}\beta^d\left(\bar{\nabla}_b\delta h_{da} + \bar{\nabla}_d\delta h_{ab} - \bar{\nabla}_a\delta h_{bd}\right) \\ &= \frac{1}{2}\bar{\nabla}_d\left(\Pi^{ab}\beta^d\delta h_{ab}\right) - \frac{1}{2}\bar{\nabla}_d\left(\Pi^{ab}\beta^d\right)\delta h_{ab}, \end{aligned} \tag{4.1.49}$$

where the definition of the variation of the Christoffel symbols, have been used. What comes to the variation of the Hamiltonian density $\mathcal{H}^0$, it is dependent upon the transverse fields and the corresponding momenta. Varying $\mathcal{H}^0$, one finds

$$\begin{aligned} \delta\mathcal{H}^0 = &\frac{1}{2}\sqrt{h}h^{cd}\delta h_{cd}\left[\bar{R} - 2\Lambda - \frac{1}{h}\left(\Pi_{ab}\Pi^{ab} - \frac{\Pi^2}{d-1}\right)\right] + \sqrt{h}\delta\bar{R} \\ &+ \frac{1}{\sqrt{h}}\delta\left(\Pi_{ab}\Pi^{ab} - \frac{\Pi^2}{d-1}\right). \end{aligned} \tag{4.1.50}$$

We can further expand:

$$\begin{aligned} \delta\left(\Pi_{ab}\Pi^{ab} - \frac{\Pi^2}{d-1}\right) &= \delta\left[h_{ac}h_{bd}\left(\Pi^{cd}\Pi^{ab} - \frac{\Pi^{ac}\Pi^{bd}}{d-1}\right)\right] \\ &= 2\delta h_{ab}\left(\Pi^{ac}\Pi_c{}^b - \frac{\Pi\Pi^{ab}}{d-1}\right) + 2\delta\Pi^{ab}\left(\Pi_{ab} - h_{ab}\frac{\Pi}{d-1}\right), \end{aligned} \tag{4.1.51}$$

where we have defined $\Pi_c{}^b := h_{ca}\Pi^{ab}$. The variation of the Ricci curvature $\bar{R}$ is,

$$\begin{aligned} \delta\bar{R} &= \bar{R}_{ab}\delta h^{ab} + h^{ab}\delta\bar{R}_{ab} = \bar{R}_{ab}\delta h^{ab} + h^{ab}\left[\bar{\nabla}_c\left(\delta\bar{\Gamma}^c_{ab}\right) - \bar{\nabla}_b\left(\delta\bar{\Gamma}^c_{ac}\right)\right] \\ &= \bar{R}_{ab}\delta h^{ab} + \left(h^{ab}\bar{\nabla}^2 - \bar{\nabla}^a\bar{\nabla}^b\right)\delta h_{ab}, \end{aligned} \tag{4.1.52}$$

where $\bar{\nabla}^2 := h^{ab}\bar{\nabla}_a\bar{\nabla}_b$. Bringing together (4.1.49), (4.1.51), (4.1.52), in (4.1.50), gives

$$\begin{aligned}
\delta\mathcal{H}_{ADM} = \delta\pi^{ab}\left[\bar{\nabla}_a\beta_b + \bar{\nabla}_b\beta_a - \frac{2\alpha}{\sqrt{h}}\left(\Pi_{ab} - \frac{\Pi}{d-1}h_{ab}\right)\right] \\
+ \delta h_{ab}\Bigg\{2\Pi^{bc}\bar{\nabla}_c\beta^a - \bar{\nabla}_d\left(\Pi^{ab}\beta^d\right) + \sqrt{h}\alpha\left(\bar{R}^{ab} - \frac{1}{2}h^{ab}\left(\bar{R} - 2\Lambda\right)\right) \\
- \frac{2\alpha}{\sqrt{h}}\left[\Pi^{ac}\Pi^b{}_c - \frac{\Pi\Pi^{ab}}{d-1} - \frac{h^{ab}}{4}\left(\Pi_{cd}\Pi^{cd} - \frac{\Pi^2}{d-1}\right)\right] + \sqrt{h}\left(h^{ab}\bar{\nabla}^2 - \bar{\nabla}^a\bar{\nabla}^b\right)\alpha\Bigg\} \\
+ 2\bar{\nabla}_d\left(\Pi^{bd}\beta^a\delta h_{ab} - \frac{1}{2}\Pi^{ab}\beta^d\delta h_{ab} + \Pi^d{}_a\delta\beta^a\right) - \sqrt{h}\left(h^{ab}\bar{\nabla}^2 - \bar{\nabla}^a\bar{\nabla}^b\right)(\alpha\delta h_{ab}).
\end{aligned} \tag{4.1.53}$$

The coefficient of $\delta\Pi^{ab}$ in (4.1.53), is just the definition of $\dot{h}_{ab}$ in (4.1.43), which are the equations of motion for the transverse metric. The principle of extremal action, reveals that the equations of motion from varying S_{ADM}, are

$$\begin{aligned}
\dot{h}_{ab} :=& \bar{\nabla}_a\beta_b + \bar{\nabla}_b\beta_a - \frac{2\alpha}{\sqrt{h}}\left(\Pi_{ab} - \frac{\Pi}{d-1}h_{ab}\right), \\
\dot{\Pi}^{ab} :=& \bar{\nabla}_c\left(\Pi^{ab}\beta^c\right) - \Pi^{bc}\bar{\nabla}_c\beta^a - \Pi^{ac}\bar{\nabla}_c\beta^b - \sqrt{h}\alpha\left(\bar{R}^{ab} - \frac{1}{2}h^{ab}\left(\bar{R} - 2\Lambda\right)\right) \\
&+ \frac{2\alpha}{\sqrt{h}}\left[\Pi^{ac}\Pi^b{}_c - \frac{\Pi\Pi^{ab}}{d-1} - \frac{h^{ab}}{4}\left(\Pi_{cd}\Pi^{cd} - \frac{\Pi^2}{d-1}\right)\right] - \sqrt{h}\left(h^{ab}\bar{\nabla}^2 - \bar{\nabla}^a\bar{\nabla}^b\right)\alpha, \\
&\mathcal{H}^\mu(h_{ab}, \Pi^{ab}) = 0,
\end{aligned} \tag{4.1.54}$$

where $\mathcal{H}^\mu := (\mathcal{H}^0, \mathcal{H}^b)$ are the Hamiltonian constriants. The set of equations (4.1.54) are equivalent to the outside/inside gravitational set of equations (2.4.9), that is the second set of equations of motion in (2.4.9), have been recasted into terms of canonical theory.

4.2 Time Gauge

In the previous subsection, we constructed a systematic analysis and separated space-time in spatial foliations, separated by a temporal vector. What one can now do, is to pick the explicit forms of the lapse and the shift. As this is just a re-definition of the distances between the sheets of $t =$ constant, the space-time remains complete in sense of freedom of choice for α and β^a. One of these choices are called the *time gauge*, where if $x^a(\xi)$ are vectors in Σ_t, the vector ∂_t is defined to be perpendicular to the velocities $x^a{}_{,b} \in TM(\sigma_t)$. This enforces $\alpha = 1$ and $\beta^a = 0$, that is $\partial_t = dt$. Also, using the definition of the trace of the momenta, $\Pi^{ab} - \Pi h^{ab}/d = 0$, the equations of motion attains the simpler form,

$$\begin{aligned}
\dot{h}_{ab} :=& \frac{2}{\sqrt{h}}\frac{\Pi}{d(d-1)}h_{ab}, \\
\dot{\Pi}^{ab} :=& -\sqrt{h}\left[\bar{R}^{ab} - \frac{1}{2}h^{ab}\left(\bar{R} - 2\Lambda\right) + \frac{h^{ab}}{h}\frac{d-4}{d-1}\left(\frac{\Pi}{2d}\right)^2\right] \\
&\mathcal{H}^\mu(h_{ab}, \Pi^{ab}) = 0,
\end{aligned} \tag{4.2.1}$$

and the variation of the Hamiltonian density (4.1.53) becomes

$$\delta\mathcal{H}_{ADM} = \sqrt{h}\bar{\nabla}_c\left(h^{ab}\delta\bar{\Gamma}^c_{ab} - h^{ac}\delta\bar{\Gamma}^b_{ab}\right). \tag{4.2.2}$$

In the time gauge, the variation of the ADM-action, becomes

$$\begin{aligned}\delta S_{ADM} &= -\frac{1}{2\kappa}\int d^D x \left[\partial_t(\Pi^{ab}\delta h_{ab}) + \sqrt{h}\bar{\nabla}_c\left(h^{ab}\delta\bar{\Gamma}^c_{ab} - h^{ac}\delta\bar{\Gamma}^b_{ab}\right)\right] \\ &= G(t_2) - G(t_1),\end{aligned} \tag{4.2.3}$$

where

$$G(t) := -\frac{1}{2\kappa}\int d^d x \Pi^{ab}\delta h_{ab}, \tag{4.2.4}$$

are the generators of spatial translations. In terms of the trace of the momentum, we find the Hamiltonian constraint $\mathcal{H}^0$ to be the same as

$$\Pi^2 = hd(d-1)\left(\bar{R} - 2\Lambda\right). \tag{4.2.5}$$

The explicit form of the trace of the momentum, is what we need to complete the instanton solutions the metric (4.1.12). At constant scalar curvature $t^2 + \bar{\rho}^2 = k^{-1}$, ADM coordinates and time gauge, the metric becomes

$$ds^2 = dt^2 + h_{ab}dx^a dx^b = dt^2 + d\bar{\rho}^2 + \bar{\rho}^2 d\Omega_{d-1}, \tag{4.2.6}$$

where the assumption of the space to be of maximal symmetry, has been made. The $d\Omega_{d-1}$ is the $d-1$ spherical angular terms, where the components are defined through

$$\begin{aligned} h_{\bar{\rho}\bar{\rho}} &= 1, \\ h_{\phi_1\phi_1} &= \bar{\rho}^2, \\ h_{\phi_i\phi_i} &= \bar{\rho}^2 \sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{i-1}, \end{aligned} \tag{4.2.7}$$

$2 \leq i \leq d-1$. The Ricci scalar R in D dimensions, determine the form of k, and has the simple form,

$$R = kD(D-1), \tag{4.2.8}$$

and Einstein's equations in $D = d+1$ space-time are $G + D\Lambda = 0$. Therefore, we can identify k by comparing Einstein's equations with (4.2.8). For $\Lambda > 0$, a coordinate change $\sqrt{\Lambda'}\bar{\rho} = \sin\psi$, transforms (4.2.6) into,

$$\Lambda' ds^2 := d\psi^2 + \sin^2\psi d\Omega_{d-1}, \tag{4.2.9}$$

where we have defined $\Lambda' := 2\Lambda/d(d-1)$. The transverse metric is now

$$\begin{aligned} h_{\psi\psi} &= (\Lambda')^{-1}, \\ h_{\phi_1\phi_1} &= (\Lambda')^{-1}\sin^2\psi, \\ h_{\phi_i\phi_i} &= (\Lambda')^{-1}\sin^2\psi \sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{i-1}, \end{aligned} \tag{4.2.10}$$

thus the metric, is subject to an overall scale transformation. If $\Lambda < 0$, exchange the trigonometric functions by hyperbolic ones. Next is to find the Ricci scalar $\bar{R}$, given the metric (4.2.9), and in the ADM-system with Gaussian coordinates; the first partial derivatives of the transverse metric is zero. Hence, one can ignore the product of Christoffel symbols as appearing in the Riemann tensor components. What is left for the Ricci scalar is the simple expression

$$\begin{aligned}\bar{R} &= h^{ab}\bar{R}_{ab} = \partial_c\bar{\Gamma}^c_{ab} - \partial_b\bar{\Gamma}^c_{ac} \\ &= \frac{1}{2}h^{ab}\partial_c\left(h^{cd}h_{ab,d}\right) - h^{ab}\partial_a\left[\frac{1}{\sqrt{h}}\partial_b\sqrt{h}\right].\end{aligned} \tag{4.2.11}$$

Due to the spherical symmetry of our system, there are simplifications to be made. Suppose we perform a rotation $(\psi, \phi_1, \cdots, \phi_{d-1}) \to (\psi', \phi_{1'}, \cdots, \phi_{(d-1)'})$. Then the component $\bar{R}_{\psi\psi}$ transforms as

$$\bar{R}_{\psi'\psi'} = \left(\frac{\partial\psi}{\partial\psi'}\right)^2 \bar{R}_{\psi\psi} + \sum_{j=1}^{d-1} \left(\frac{\partial\phi_j}{\partial\psi'}\right)^2 \bar{R}_{\phi_j\phi_j} = \bar{R}_{\psi\psi}, \tag{4.2.12}$$

as it should be invariant under such coordinate transformations. The form dl'^2 in (4.2.9), transforms as

$$\begin{aligned} dl'^2 &= d\psi^2 + \sin^2\psi d\Omega_{d-1} \\ &+ \left(\frac{\partial\psi}{\partial\psi'}\right)^2 d\psi'^2 + \sin^2\psi\, d\psi'^2 \left[\left(\frac{\partial\phi_1}{\partial\psi'}\right)^2 + \sum_{i=2}^{d-1}\prod_{j=1}^{i-1}\left(\frac{\partial\phi_i}{\partial\psi'}\right)^2 \sin^2\phi_j\right] + \cdots \end{aligned} \tag{4.2.13}$$

and as dl'^2 is an invariant with respect to the transformations in (4.2.13), the constraint on the rotations are

$$\left(\frac{\partial\psi}{\partial\psi'}\right)^2 + \sin^2\psi \left[\left(\frac{\partial\phi_1}{\partial\psi'}\right)^2 + \sum_{i=2}^{d-1}\prod_{j=1}^{i-1}\left(\frac{\partial\phi_i}{\partial\psi'}\right)^2 \sin^2\phi_j\right] = 1. \tag{4.2.14}$$

Substitution of (4.2.14) in (4.2.12), and rearranging terms, gives,

$$\begin{aligned} 0 &= \left(\frac{\partial\phi_1}{\partial\psi'}\right)^2 \left(\bar{R}_{\phi_1\phi_1} - \sin^2\psi\bar{R}_{\psi\psi}\right) \\ &+ \sum_{j=1}^{d-1}\left[\left(\frac{\partial\phi_i}{\partial\psi'}\right)^2 \left(\bar{R}_{\phi_i\phi_i} - \bar{R}_{\psi\psi}\sin^2\psi\prod_{j=1}^{i-1}\sin^2\phi_j\right)\right]. \end{aligned} \tag{4.2.15}$$

As the general transformations are non-zero, we conclude that the coefficients must be zero, and hence

$$\begin{aligned} &\bar{R}_{\phi_1\phi_1} - \bar{R}_{\psi\psi}\sin^2\psi = 0, \\ &R_{\phi_2\phi_2} - \bar{R}_{\psi\psi}\sin^2\psi\sin^2\phi_1 = 0, \\ &\bar{R}_{\phi_3\phi_3} - \bar{R}_{\psi\psi}\sin^2\psi\sin^2\phi_1\sin^2\phi_2 = 0, \\ &\quad\vdots \end{aligned} \tag{4.2.16}$$

Contraction of (4.2.16) with the inverse of (4.2.10), excluding Λ', reveals that $\bar{R}^{\phi_i}_{\phi_i} - \bar{R}_{\psi\psi} = 0$, for $i = 1, 2, \cdots, d-1$. This implies the simple relation $\bar{R} - d\bar{R}_{\psi\psi} = 0$, that is to calculate the Ricci scalar $\bar{R}$ we need only to find the explicit expression of the component $\bar{R}_{\psi\psi}$. From (4.2.11) we find

$$\bar{R} = d\bar{R}_{\psi\psi} = -d\Lambda'\partial_\psi\left[\frac{1}{\sqrt{h}}\partial_\psi\sqrt{h}\right] = \frac{d(d-1)}{\rho^2}. \tag{4.2.17}$$

Substitution of (4.2.17) in (4.2.5), and passing the limit of $\rho \to \bar{\rho}$, gives the final expression for the trace of the momenta as

$$\Pi_{i,o} = -\sigma_{i,o}\, d(d-1)\sqrt{h}\left(\bar{\rho}^{-2} - \frac{2\Lambda_{i,o}}{d(d-1)}\right)^{1/2}, \tag{4.2.18}$$

where $\sigma_{i,o} := \pm 1$ are defined such that, for example, $\sigma_i = +1 (\sigma_i = -1)$ if the unit normal points in increasing (decreasing) area of the d-spheres concentric with the membrane in the inside region. $\bar{\rho}$ is the proper radius of the membrane, as measured in the inside and outside regions. The integration of the gravitational equations of motion in (2.4.9), over a small proper distance just encompassing the history, picks up the singularity at the history. The difference of the trace of the momenta, as defined in the outside and inside regions, are defined as

$$\Pi_o - \Pi_i = \sqrt{h}\kappa m d. \tag{4.2.19}$$

The average of Π, Π_{on}, is found from (3.0.10) and (4.1.43), as

$$\Pi_o + \Pi_i = -\frac{2e\sqrt{h}\epsilon E_{on}}{m(d-1)}, \tag{4.2.20}$$

thus by combining (4.2.20) with (4.2.19), the Π's in terms of the electric field, the mass of the membrane, the charge coupling, is

$$\begin{aligned} h^{-1/2}\Pi_o &= \frac{1}{2}\kappa m d - \frac{e\epsilon E_{on}}{m}(d-1), \\ h^{-1/2}\Pi_i &= \frac{1}{2}\kappa m d + \frac{e\epsilon E_{on}}{m}(d-1). \end{aligned} \tag{4.2.21}$$

So by (4.2.21) and (4.2.18), the signs of $\sigma_{i,o}$ is found to be

$$\begin{aligned} \sigma_o &= \text{sign}\left[\frac{1}{d}\left(e\epsilon E_o - \frac{1}{2}e^2\right) - \frac{\kappa m^2}{2(d-1)}\right], \\ \sigma_i &= \text{sign}\left[\frac{1}{d}\left(e\epsilon E_o - \frac{1}{2}e^2\right) + \frac{\kappa m^2}{2(d-1)}\right]. \end{aligned} \tag{4.2.22}$$

Now, to find the proper radius of the membrane, one can pick either the outside or the inside Π as defined in (4.2.21), and compare it with (4.2.18). Then by simple calculations one reveals that

$$\bar{\rho}^{-2} = \frac{2\Lambda_o}{d(d-1)} + \frac{1}{m^2}\left[\left(e\epsilon E_o - \frac{1}{2}e^2\right) - \frac{\kappa m^2}{2(d-1)}\right]^2. \tag{4.2.23}$$

This concludes the instanton solutions for the theory (2.1.1).

4.3 Complete Set of Configuration Parameters

The instanton solutions from the euclidean action (2.1.1), is defined by the group of equations,

$$\Lambda_o = \lambda + \frac{1}{2}\kappa {E_o}^2, \tag{4.3.1a}$$

$$\Lambda_i = \Lambda_o - \kappa\left(\epsilon e E_o - \frac{1}{2}e^2\right), \tag{4.3.1b}$$

$$E_i = E_o - \epsilon e, \tag{4.3.1c}$$

$$E_{on} = E_o - \frac{1}{2}\epsilon e, \tag{4.3.1d}$$

$$\sigma_o = \text{sign}\left[\frac{1}{d}\left(\epsilon e E_o - \frac{1}{2}e^2\right) - \frac{\kappa m^2}{2(d-1)}\right], \tag{4.3.1e}$$

$$\sigma_i = \text{sign}\left[\frac{1}{d}\left(\epsilon e E_o - \frac{1}{2}e^2\right) + \frac{\kappa m^2}{2(d-1)}\right], \tag{4.3.1f}$$

$$\bar{\rho} = \left\{\frac{2\Lambda_o}{d(d-1)} + \frac{1}{m^2}\left[\frac{1}{d}\left(\epsilon e E_o - \frac{1}{2}e^2\right) - \frac{\kappa m^2}{2(d-1)}\right]^2\right\}^{-1/2}, \tag{4.3.1g}$$

which relates the different parameters that defines the configurations of the systems. A nice feature of the collection (4.3.1) is that the inside can be recast in terms of the outside, so the set of independent equations in (4.3.1) is actually 5, rather than 7. As the proper radius is a non-negative quantity, we find a first restriction

$$\left[\frac{1}{d}\left(\epsilon e E_o - \frac{1}{2}e^2\right) - \frac{\kappa m^2}{2(d-1)}\right]^2 \geq -\frac{2m^2\Lambda_o}{d(d-1)}, \tag{4.3.2}$$

which is automatically satisfied if $\Lambda_o \geq 0$. How many types of configurations exists for the instanton? For a given region, $\sigma = \pm 1$, $\Lambda > 0$, $\Lambda \leq 0$. This returns 16 configurations. But these configurations are not independent of each other. Suppose $\sigma_o = +1$, then the relation between the inside and outside is $\Lambda_o < \Lambda_i$. That means that for this particular choice of σ_o, it is not possible for having $\Lambda_i > 0$ and $\Lambda_o < 0$. Now, let $\sigma_i = -1$, then it is not possible to have $\Lambda_i < 0$ and $\Lambda_o > 0$. Due to the fact that $\sigma_i \geq \sigma_o$, realizes that $\sigma_i = -1$ and $\sigma_o = -1$, in the same configuration, is also not possible. This exhaust the redundant configurations, reducing the 16 to 10.

The possible solutions may be arranged in groups, labeling the instantons accordingly to types:

- Type 1: All instantons, excluding the configurations $\Lambda_i \leq 0$, $\sigma_i = -1$, or $\Lambda_o \leq 0$, $\sigma_o = -1$,
- Type 2: $\Lambda_{i,o} \leq 0$, $\sigma_o = -1$, but excluding Type 3,
- Type 3: $\Lambda_o \leq 0$, $\sigma_o = -1$, with $\Lambda_i \leq 0$. $\sigma_i = -1$

Each of these types will be investigated, clarifying what configurations are possible for membrane creation to take place.

4.4 Extremizing of The Action

If the equations of motion for the antisymmetric field, are satisfied, (2.1.1) reduces to

$$S^E = m\int d^dx\sqrt{h} - \frac{1}{2\kappa}\int d^Dx\sqrt{h}(R - 2\Lambda) + \frac{1}{\kappa}\int d^dx\sqrt{h}\frac{\Pi}{d-1}, \tag{4.4.1}$$

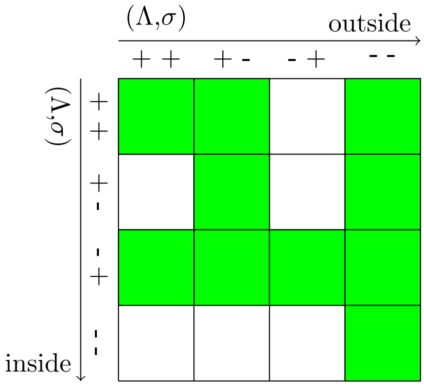

Figure 1: Indicated by green color; the possible configuration values for Λ and σ. For example the set $+-$ means $\Lambda \geq 0, \sigma = -1$.

where time gauge, and ADM coordinates, are implied. Suppose we are residing in either the inside region or the outside region. Then, from the gravitational equations of motion in (2.4.9), $R = 2D\Lambda/(d-2)$. Substitution of this into (4.4.1) gives the classical euclidean action as

$$\begin{aligned} S^E &= m \int d^d x \sqrt{h} - \frac{2}{\kappa(d-1)} \int d^D x \sqrt{h} \Lambda(x) + \frac{1}{\kappa} \int d^d x \sqrt{h} \frac{\Pi}{d-1} \\ &:= m A_d(\bar{\rho}) - \frac{2\Lambda}{\kappa(d-1)} V_D(\bar{\rho}, \sigma, \Lambda) + \frac{1}{\kappa(d-1)} A_d(\bar{\rho}) \Pi(\bar{\rho}, \sigma, \Lambda), \end{aligned} \tag{4.4.2}$$

where we have defined

$$\begin{aligned} A_d(\bar{\rho}) &:= \int d^d x \sqrt{h} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \bar{\rho}^d, \\ V_d(\bar{\rho}) &:= \int d^D x \sqrt{g}. \end{aligned} \tag{4.4.3}$$

Suppose Λ_o, σ_o, is the false vacua, that is let it define the background from which the tunneling occur. Then as the net phase contribution to the probability is $B = S^E[\text{Instanton}] - S^E[\text{Background}]$, we find B as

$$B = m A_d(\bar{\rho}) - \frac{2\Lambda}{\kappa(d-1)} V_D(\bar{\rho}, \sigma_i, \Lambda_i) - \frac{d\sigma_i}{\kappa} A_d(\bar{\rho}) \left[\bar{\rho}^{-2} - \frac{2\Lambda_i}{d(d-1)} \right]^{1/2} - (i \leftrightarrow o), \tag{4.4.4}$$

where we have used the definition for the trace of the momenta. The task now is to maximize B with respect to the proper radius $\bar{\rho}$. Suppose that the background and instanton space-time, is de-Sitter: $\Lambda_{i,o} > 0$, and $\sigma_i = +1$. Then, by differentiating with respect to the proper radius, one can resolve,

$$-\frac{\bar{\rho}\kappa}{A_d(\bar{\rho})} \frac{\partial B}{\partial \bar{\rho}} = \kappa m d + \frac{d\Pi_i}{(d-1)\sqrt{h}} - \frac{d^2(d-1)\sqrt{h}}{\bar{\rho}^2 \Pi_i} - \frac{d\bar{\rho}\Lambda'}{A_d(\bar{\rho})} \frac{\partial V_D}{\partial \bar{\rho}} - (i \leftrightarrow o). \tag{4.4.5}$$

Using the metric (4.2.9), valid for de-Sitter space-time, together with $\sigma_i = +1$, we can solve for the inside volume and find

$$V_D(\bar{\rho}, \sigma_i, \Lambda_i) = \int_{\text{inside}} d^D x \sqrt{g} = |\Lambda_i'|^{-D/2} \frac{A_d(\bar{\rho})}{\bar{\rho}^d} \left| \int_1^{b(\bar{\rho})} d(\cos\psi) \sin^{d-1}\psi \right|, \tag{4.4.6}$$

where $b(\bar{\rho}) := \sigma_i(1 - \Lambda_i'\bar{\rho}^2)^{1/2}$. Differentiation of (4.4.6) with respect to $\bar{\rho}$, gives

$$\begin{aligned} \frac{\partial V_{D,i}}{\partial \bar{\rho}} &= |\Lambda_i'|^{-D/2} \frac{A_d(\bar{\rho})}{\bar{\rho}^d} \mathrm{sign}\left(\int_1^{b(\bar{\rho})} d(\cos\psi) \sin^{d-1}\psi \right) \frac{\partial}{\partial \bar{\rho}} \int_1^{b(\bar{\rho})} d(\cos\psi) \sin^{d-1}\psi \\ &= \frac{d(d-1)\sqrt{\hbar}}{\Pi_i} \frac{A_d(\bar{\rho})}{\bar{\rho}}. \end{aligned} \tag{4.4.7}$$

Using the results from (4.4.7) in (4.4.5), the extremal of B with respect to $\bar{\rho}$, reveals,

$$0 = -\frac{\bar{\rho}\kappa}{A_d(\bar{\rho})} \frac{\partial B}{\partial \bar{\rho}} = \kappa m d + \frac{\Pi_i}{\sqrt{\hbar}} - \frac{\Pi_o}{\sqrt{\hbar}}. \tag{4.4.8}$$

But (4.4.8) is just the same as (4.2.19), confirming the equality of the results.

5 Neutralization of The Cosmological Constant

By the result of (4.4.6), the volume function V_D is proportional to the magnitude of the integral over ψ. The magnitude itself is a function of $b(\bar{\rho}, \Lambda, \sigma)$, that is the parameter values governs the character of the volume function. Looking at (4.4.7), the change of volume with respect to proper radius is

$$\frac{\partial}{\partial \bar{\rho}} V_D(\bar{\rho}, \sigma, \Lambda) \sim -\sigma \left(1 - \Lambda' \bar{\rho}^2\right)^{-1/2} \bar{\rho}^{d-1}. \tag{5.0.1}$$

For Type 2 instantons, the instanton volume remains finite up to the proper radius. The background volume attains the values $\Lambda_o \leq 0$, $\sigma_o = -1$. Thus for large $\bar{\rho}$, integrating (5.0.1) with respect to $\bar{\rho}$ results in

$$V_D(\bar{\rho}, -1, \Lambda_o \leq 0) \sim +|\Lambda_o'|^{-1/2} \frac{A_d(\bar{\rho})}{d} + V_c, \tag{5.0.2}$$

where $\partial_{\bar{\rho}} V_c = 0$. So the volume diverges at large $\bar{\rho}$. This means that, as the instanton remains bounded as $\bar{\rho} \to \infty$ the background blows up, $B \to \infty$, and so $P \sim e^{-B/\hbar} \to 0$. For Type 3 instantons, both the instanton and background action, diverges at large proper radius, thus one must look at the difference in the actions. One may find that

$$B \sim \frac{d-1}{\kappa} \Lambda_d(\bar{\rho}) \left(|\Lambda_o'|^{1/2} - |\Lambda_i'|^{1/2} \right). \tag{5.0.3}$$

As $\sigma_i = -1$, (4.3.1e) and (4.3.1f) shows that $|\Lambda_o| > |\Lambda_i|$, hence (5.0.3) has the property $B \to \infty$, and therefore $P \to 0$. In conclusion, the only configurations possible for membrane creation to take place are the Type 1 instantons, that is those green squares of coordinates in (4.3), excluding configurations of inside or outside values of $(--)$.

5.1 Mass and Charge Dependence

So far, the treatment of the theory has governed the viability of the instanton parameters to imply a finite phase contribution to the probability of a membrane to be created. No major assumption on the mass m of the membrane itself nor the charge coupling e, has been made. We concluded that the phase B remains bounded for Type 1 instantons, so the treatment of $B_1 := B(\text{Type 1})$ is of interest. Without further ado, by differentiating equation (4.4.4) with respect to the membrane mass m, produces

$$\frac{\partial B_1}{\partial m} = A_d(\bar{\rho}), \tag{5.1.1}$$

that is rate of change of B_1 with respect to the proper radius, is proportional to the surface area A_d of the membrane. This implies that B is monotonically growing with increasing mass and so the probability of a membrane to be created, decreases with increasing m. By differentiating the euclidean action (2.1.1), with respect to the magnitude of the charge coupling e, one finds

$$\frac{\partial S_E}{\partial e} = \frac{\partial S_E}{\partial |e|}\mathrm{sign}(e) = \int_{\partial M} A = \int_M dA = \int_M F, \tag{5.1.2}$$

where $F = dA$ is the field strength. The integral over ∂M covers the instanton on the inside, and the membrane itself. The inclusion of the surface term in (2.1.1) patches the inside and outside, as the inside can not be covered with A on a single patch. By assumption, this surface term is away from the instanton, residing in the outside region at some fixed $E = E_o$ as a boundary condition. Therefore, patch 1 covers the inside, the membrane and a finite portion of the outside. Patch 2 covers the outside region of the instanton. By consequence, the patches must cover the inside, but not necessarily the outside.

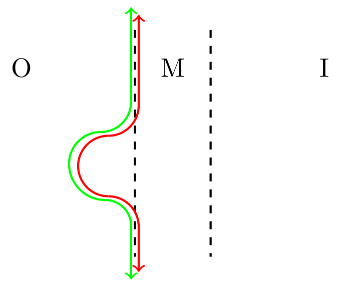

Figure 2: Integration over the potentials: O := outside, I := inside, M := membrane. All regions to the right of the red line is patch 1, and the region to the left of the green line, is patch 2. Integrations are with respect to the distance between the solid lines, where the distance tends to zero. The arrows extend such that the solid lines covers the respective integration regions.

From (3.0.5) the difference in the given surface orientation to the induced orientation, is $-\epsilon$, thus the integration in (5.1.2) is

$$\begin{aligned}\frac{\partial B_1}{\partial |e|} &= -\text{sign}(e)\epsilon \int_{inside} d^D x\sqrt{h}E = -\text{sign}(e)(\epsilon E_o - e)V_D(\bar{\rho}, \sigma_i, \Lambda_i), \\ &= -[\epsilon\, \text{sign}(eE_o)|E_o| - |e|]V_D(\bar{\rho}, \sigma_i, \Lambda_i).\end{aligned} \tag{5.1.3}$$

where we have used (4.3.1). Now, suppose the magnitude of the outside electric field E_o, is small compared to the magnitude of the charge coupling e, that is $|E_o| < |e|$. From the set of equations (4.3.1) with $\sigma_o = +1$, one finds that $\epsilon E_o - \frac{1}{2}e > 0$ hence,

$$\epsilon\, |E_o|\text{sign}(eE_o) - |e| = e\frac{\epsilon E_o}{|e|} - |e| > -\frac{|e|}{2}, \tag{5.1.4}$$

showing that $\partial B_1/\partial |e| > 0$. Thus B_1 increases with increasing $|e|$, that is, the probability for membrane creation, decreases with increasing $|e|$, where $|E_o| < |e|$. If on the other hand, $|E_o| > e$, then for $\epsilon = +\text{sign}(eE_o)$, $\sigma_i = +1$, we have $\partial B_1/\partial |e| > 0$ and hence the probability increases with increasing $|e|$. For $\epsilon = -\text{sign}(eE_o)$ the probability will decrease with increasing $|e|$. For the reduction process of Λ, we will direct a particular interest in the case $\epsilon = +\text{sign}(eE_o)$, $|E_o| \ll |e|$, where the probability increases with increasing magnitude of the charge coupling.

5.2 Reduction of The Cosmological Constant

Given a particular set of variables, ϵ, e, m, and E_o; suppose that $\Lambda_i > 0$. As previously found, $\sigma_o = +1$ forces the inequality $\Lambda_i < \Lambda_o$. Similarly, for $\sigma_o = -1$, $\Lambda_i > \Lambda_o$. Now, assume that $|E_o|$ is small, that is, $2|E_o| < |e|$. From (4.3.1) one can resolve,

$$\text{sign}(\Lambda_o - \Lambda_i) \sim \text{sign}\left(\frac{1}{2}\kappa e^2\right) > 0, \tag{5.2.1}$$

thus whenever $2|E_o| < |e|$, the membrane production always increases the cosmological constant. But if $2|E_o| > |e|$, then

$$\text{sign}(\Lambda_o - \Lambda_i) = \text{sign}(\epsilon e E_o), \tag{5.2.2}$$

so the cosmological constant increases when $\text{sign}(\epsilon e E_o) = -1$, and decreases when $\text{sign}(\epsilon e E_o) = +1$. Which of these values of $\text{sign}(\epsilon e E_o)$, characterizing the membrane creation processes, is more likely to occur? Define the difference in B evaluated at $\epsilon = +1$ and $\epsilon = -1$, respectively, as

$$D(m) := B(\epsilon = +\text{sign}(eE_o)) - B(\epsilon = -\text{sign}(eE_o)). \tag{5.2.3}$$

We will show that the function $D(m)$ is always negative, that is the phase contribution to the probability, is larger for $\epsilon = -1$, and consequently increases the probability itself. Differentiating (5.2.3) with respect to the membrane mass m, gives the expression

$$\begin{aligned}\frac{\partial D(m)}{\partial m} &= \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}\left\{\left(\frac{2\Lambda_o}{d(d-1)} + \frac{1}{m^2}\left[\frac{1}{d}\left(|eE_o| - \frac{e^2}{2}\right) - \frac{\kappa m^2}{2(d-1)}\right]^2\right)\right\}^{-d/2} \\ &- \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}\left\{\left(\frac{2\Lambda_o}{d(d-1)} + \frac{1}{m^2}\left[\frac{1}{d}\left(|eE_o| + \frac{e^2}{2}\right) + \frac{\kappa m^2}{2(d-1)}\right]^2\right)\right\}^{-d/2},\end{aligned} \tag{5.2.4}$$

where we have used (5.1.1), (4.4.3), and (4.3.1g). This expression can not be zero, for any m, thus $D(m)$ is a monotonic function of the mass m. In the limit of large membrane mass, $m \to \infty$, we find from (4.3.1), with $\sigma_o = -1$, $\sigma_i = +1$, and $\bar{\rho} \to 0$, the asymptotic of (5.2.3) to be,

$$D(m) \sim m^{-1} \to 0, \tag{5.2.5}$$

as $m \to \infty$. In the massless limit, $m \to 0$, the sign of σ_i and σ_o, can be found from (4.3.1) to be equivalent, and equal to,

$$\sigma_i = \sigma_o = \frac{1}{d}\mathrm{sign}\left(\epsilon e E_o - \frac{e^2}{2}\right). \tag{5.2.6}$$

The sign of σ_i and σ_o is determined purely from the sign of the factor $\epsilon e E_o$. Thus $\sigma_i = \sigma_o = \mathrm{sign}(\epsilon e E_o)$. As $\bar{\rho}$ tends to zero, $m \to 0$, we find that

$$B(\epsilon) \sim -\frac{2\Lambda_i}{\kappa(d-1)} V_D(0, \mathrm{sign}(\epsilon e E_o), \Lambda_i) + \frac{2\Lambda_i}{\kappa(d-1)} V_D(0, \mathrm{sign}(\epsilon e E_o), \Lambda_o). \tag{5.2.7}$$

For $\sigma_i = +1$, this expression vanishes, but for $\sigma_i = -1$, the volume functions in (5.2.7) are the volume of de-Sitter space, which is proportional to $(\Lambda)^{-D/2}$. Then, since (5.2.2) holds,

$$B(\epsilon = -\mathrm{sign}(\epsilon e E_o)) \sim -\left[(\Lambda_i)^{-d/2} - (\Lambda_o)^{-d/2}\right] > 0. \tag{5.2.8}$$

Hence, since B vanishes for $\mathrm{sign}(\epsilon e E_o) = +1$, one can resolve,

$$D(m) < 0, \qquad m \to \infty, \tag{5.2.9}$$

and consequently the probability of a membrane to be created, is more likely to occur when $-\mathrm{sign}(\epsilon e E_o) = +1$, that is $-\mathrm{sign}(\Lambda_o - \Lambda_i) = +1$. But this is the same as $\Lambda_i < \Lambda_o$, thus the cosmological constant is reduced in the membrane creation. This concludes the proposition that $D(m) < 0$.

As we know the specific configuration of which membrane creation process is more probable, the care of asking for a simple algorithm to reduce the cosmological constant, is concerned. Thus we want to find the range of certain configuration values, especially Λ and E. The following shows how these values changes with successive created membranes, and we suppose $\sigma_o = +1$. Suppose that $\Lambda_o \leq 0$. By the instanton solutions (4.3.1), the proper radius can only be non-negative. We find

$$|eE_o| \geq \frac{e^2}{2} + \frac{kdm^2}{2(d-1)} + m\sqrt{\frac{-2\Lambda_o}{d(d-1)}}, \tag{5.2.10}$$

where we have solved the equation for $\bar{\rho}$ in (4.3.1), in terms of the magnitude of the initial electric field, $|E_o|$. The sign of the second equation in (4.3.1), indicates as previously that we must have $\Lambda_i < \Lambda_o$, that is membrane creation always reduces the cosmological constant. Suppose now that the initial electric field E_o, is large enough, so that $\Lambda_o = \lambda + \frac{1}{2}\kappa E_o^2 > 0$, also let $\lambda < 0$, and assume that $|e|$ is very small, compared to $|E_o|$. From (4.3.1), the consequence of the creation of a membrane, the outside cosmological constant and magnitude of the electric field, changes as,

$$\begin{aligned} \Lambda_o &\to \Lambda_i \approx \Lambda_o - \kappa|eE_o|, \\ |E_o| &\to |E_i| \approx |E_o| - |e|, \end{aligned} \tag{5.2.11}$$

thus the magnitude of the outside cosmological constant, gets reduced by a factor proportional to the magnitude of eE_o, and the magnitude of the outside electric field, is reduced by the magnitude of the charge coupling. This process can be repeated in recursion, and we can build up the reduction process by successive creation of membranes. That is, we have a continuous process, governed by the general rules:

$$\begin{aligned} \Lambda &\to \Lambda - \kappa|eE|, \\ |E| &\to |E| - |e|, \end{aligned} \tag{5.2.12}$$

subject to the restriction that the outside electric field is large enough, so there is energy to dispense in the membrane creation process. The reduction continuous freely, even to a point where $\Lambda_o \leq 0$, and at that point (5.2.10) is satisfied. The inequality (5.2.10) will certainly not hold when the inequality $\leq$ is changed to $>$, that is the evolution of E and Λ, comes to a halt when (5.2.10) no longer holds. Suppose that $E = E_f$, and $\Lambda = \Lambda_f$, are the values of the electric field and the cosmological constant, respectively, when such a halt occur. Then by reversing the inequality in (5.2.10), and taking the magnitude $|E|$ to be large compared to $|e|$, we obtain at the halt:

$$|eE_f| < \frac{kdm^2}{2(d-1)}. \tag{5.2.13}$$

From the general rule (5.2.12), we find that Λ_f can take values in the range,

$$-k|eE_f| < \Lambda_f \leq 0. \tag{5.2.14}$$

6 Conclusion

The mechanism of neutralizing the cosmological constant through membrane creation, indeed seems to be a possible candidate for a natural process, but the conditions on parameters and fields, are of very specific nature. What comes from experimental data, indicates the current space-time to be de-Sitter, and this sheds hope given that the most probable process of neutralization starts from de-Sitter space-time. In the main paper given by Brown and Teitelboim[1], the analysis is thorough but lack some clarification and motivation in the derivations. Indeed, the inclusion on the definitions of distributions on the manifold, is a subject left out in this paper, but the main objective with this thesis was to understand how the mechanics can be realized, using intuitive standard methods as in Brown and Teitelboim[1]. The book by Castillo[4] contains many of the theoretical tools which lies in my interest, but has been left out due to time investment.

7 Appendix

A Mathematical Definitions

Let M be a orientable, differentiable manifold, and denote the set of p-forms on M by $\Lambda^k(M)$. Suppose $a \in \Lambda^p(M)$, and $b \in \Lambda^q(M)$. We define the exterior product of the p-form a with the q-form b, as

$$a \wedge b \equiv \frac{(p+q)!}{p!q!} A(a \otimes b), \tag{A.0.1}$$

where A is defined as the antisymmetrization operator, given by,

$$AT(\mathbf{X}_1, \cdots \mathbf{X}_k) \equiv \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sign}\, \sigma) T(\mathbf{X}_{\sigma(1)}, \cdots \mathbf{X}_{\sigma(k)}). \tag{A.0.2}$$

Here, T is an arbitrary tensor field, $\mathbf{X_i}$ are vector fields on M, S_k is the group of all permutations of the numbers $(1, 2, \cdots, k)$, and the sign is -1 if this permutation is odd, and $+1$ if the permutation i even. Now, a covariant tensor t_x of rank $d+1$, at the point x, is defined through the mapping $t_x : T_x(M) \times \cdots \times T_x(M) \to \mathbb{R}$, where $T_x(M)$ is the tangent space at $p \in M$. Thus given a coordinate chart on M, $(x^0, \cdots, x^D)$,

$$\begin{aligned} t_x(\mathbf{X}_1, \mathbf{X}_2 \cdots, \mathbf{X}_{d+1}) &= t_x \left(dx^{i_1}(\mathbf{X}_1) \frac{\partial}{\partial x^{i_1}}, dx^{i_2}(\mathbf{X}_2) \frac{\partial}{\partial x^{i_2}} \cdots, dx^{i_{d+1}}(\mathbf{X}_{d+1}) \frac{\partial}{\partial x^{i_{d+1}}} \right) \\ &= t_x \left(\partial_{i_1}, \partial_{i_2}, \cdots, \partial_{i_{d+1}} \right) dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_{d+1}}(\mathbf{X}_1, \mathbf{X}_2 \cdots, \mathbf{X}_{d+1}), \end{aligned} \tag{A.0.3}$$

which shows that

$$t_x = (t_x)_{i_1, i_2, \cdots, i_{d+1}} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_{d+1}}. \tag{A.0.4}$$

The anti-symmetrization operator A, as defined in (A.0.2), when operated from the left on a tensor product of forms, gives the exterior product as defined in (A.0.1). Since (A.0.4) is a tensor product of 1-forms, and the components $(t_x)_{i_1, i_2, \cdots, i_{d+1}}$ are totally skew-symmetric in all their indices and $t_x = A(t_x)$. We have

$$\begin{aligned} t_x &= A(t_x) \\ &= (t_x)_{i_1, i_2, \cdots, i_{d+1}} A \left(dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_{d+1}} \right) \\ &= \frac{1}{(d+1)!} (t_x)_{i_1, i_2, \cdots, i_{d+1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{d+1}}. \end{aligned} \tag{A.0.5}$$

A.1 Exterior Derivative

Let $g \in C^\infty(M)$; the differential of g at the point $y \in M$, dg_y, is defined by

$$dg_y(\mathbf{X}_y) \equiv \mathbf{X}_y[g], \qquad \mathbf{X}_y \in T_y(M). \tag{A.1.1}$$

Let $g_x \in T_x(M) \times \cdots \times T_x(M)$, be a contravariant tensor of rank $(k, 0)$, and take the chart $(x^0, \cdots, x^D)$. The differential of g_x becomes,

$$\begin{aligned} dg_x &= \mathbf{X}_x [g] \\ &= dx^i \frac{\partial}{\partial x^i} g_{j_1 j_2 \cdots j_k} \left[dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_k} \right] \\ &= \frac{\partial}{\partial x^i} g_{j_1 j_2 \cdots j_k} \left[dx^i \otimes dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_k} \right]. \end{aligned} \tag{A.1.2}$$

Anti-symmetrization of dg_x, Adg_x, returns $Adg_x = dg_x$, thus,

$$\begin{aligned}
dg_x &= A(dg_x) \\
&= A\left(\partial_i \, g_{j_1 j_2 \cdots j_k} \left[dx^i \otimes dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_k}\right]\right) \\
&= \frac{1}{(k+1)!} \sum_{\sigma \in S_k} (\operatorname{sign} \sigma) \partial_{\sigma(i)} g_{\sigma(j_1),\sigma(j_2),\cdots,\sigma(j_k)} \left[dx^{\sigma(i)} \otimes dx^{\sigma(j_1)} \otimes dx^{\sigma(j_2)} \otimes \cdots \otimes dx^{\sigma(j_k)}\right] \\
&= \partial_{[j_0} g_{j_1 \cdots j_k]} dx^{j_0} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_k},
\end{aligned} \tag{A.1.3}$$

where we have defined,

$$\partial_{[i_0} g_{i_1 \cdots i_k]} \equiv \frac{1}{(k+1)!} \varepsilon_{i_0 i_1 \cdots i_k}{}^{j_0 j_1 \cdots j_k} \partial_{j_0} g_{j_1 \cdots j_k}, \tag{A.1.4}$$

where $\varepsilon^{\cdots}{}_{\cdots}$ is the Levi-Civita symbol.

A.2 Lie Derivative

Let ϕ be a one-parameter group of transformations on M with X being the infinitesimal generator with respect to ϕ, and let t be a tensor field of type (0,k) on M. If the limit $\lim_{h\to 0} h^{-1}(\phi_h^*(t) - t)$ exists, it is called the lie derivative with respect to X and is denoted by $\pounds_X t$. Let $f \in C^\infty(M)$, $\psi* : M \to N$ be a differentiable map, and define the pullback of *df* as $\psi^* df = d(\psi^* f)$. Then the Lie derivative on *df* with respect to the infinitesimal generator X is

$$\pounds_X df \equiv \lim_{h\to 0} \frac{\phi_h^*(df) - df}{h} = \lim_{h\to 0} \frac{d\phi_h^* f - df}{h} = d(\pounds_X f). \tag{A.2.1}$$

The lie derivative on f with respect to X is,

$$\begin{aligned}
\pounds_X f(x) &= \lim_{h\to 0} \frac{\phi_h^* f(x) - f(x)}{h} = \lim_{h\to 0} \frac{f(\phi(h,x)) - f(x)}{h} \equiv \frac{d\phi}{dh}(h,x)\big|_{h=0}[f] \\
&= X_x[f] = (Xf)(x),
\end{aligned} \tag{A.2.2}$$

thus $\pounds_X f = Xf$. For a $(0,k)$-tensor t and a function f, we define $\pounds_X(ft) \equiv f\pounds_X t + (\pounds_X f)t$. Now, the Lie derivative on the $(0,k)$-tensor t with respect to X, in some local chart, is

$$\begin{aligned}
\pounds_X t &= \pounds_X \left(t_{i_1 i_2 \cdots i_k} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k}\right) \\
&= (X t_{i_1 i_2 \cdots i_k}) dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} \\
&\quad + t_{i_1 i_2 \cdots i_k} [d(\pounds x^{i_1}) \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} + \cdots + dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes d(\pounds x^{i_k})]
\end{aligned} \tag{A.2.3}$$

A.3 Pull-back

Let $\psi : M \to N$ be a differentiable map. This induces a differentiable mapping $\bar{\psi} : T_p(M) \to T_q(N)$, defined by $\bar{\psi}(X) \equiv \psi_*(X)$, for $X \in T_p(M)$. Then, let t be a covariant tensor of rank $(0,k)$, on N. The pullback on the tensor t, at $p \in M$, is defined as

$$(\psi^* t)\left(X^1, \cdots, X^k\right) \equiv t_\psi \left(\psi_* X^1, \cdots, \psi_* X^k\right), \tag{A.3.1}$$

where $X^i \in T_p(M)$, $p \in M$. We see that $\psi_* X^i \in T_q(N)$, thus if $(x^0, \cdots, x^k)$ is a chart in M, $(y^0, \cdots, y^l)$ is a chart in N, then we can expand the vector $\psi_* X^1$ as

$$\psi_* X^1 = dy^i(X^1) \frac{\partial}{\partial y^i} \; \equiv \; X^1[y^i] \frac{\partial}{\partial y^i} = dx^{j_1}(X^1) \frac{\partial}{\partial x^{j_1}} [y^i] \frac{\partial}{\partial y^i}, \tag{A.3.2}$$

thus the pullback of t, becomes

$$\begin{aligned}(\psi^* t)\left(X^1, \cdots, X^k\right) &= t_\psi\left(\psi_* X^1, \cdots, \psi_* X^k\right) \\ &= t_\psi\left(\frac{\partial y^{i_1}}{\partial x^{j_1}} dx^{j_1}(X^1)\frac{\partial}{\partial y^{i_1}}, \cdots, \frac{\partial y^{i_k}}{\partial x^{j_k}} dx^{j_k}(X^k)\frac{\partial}{\partial y^{i_k}}\right), \\ &= t_\psi\left(\frac{\partial}{\partial y^{i_1}}, \cdots, \frac{\partial}{\partial y^{i_k}}\right)\frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_k}}{\partial x^{j_k}} dx^{j_1} \otimes \cdots \otimes dx^{j_k}(X^1, \cdots, X^k),\end{aligned} \tag{A.3.3}$$

and we conclude that,

$$\psi^* t = t_\psi\left(\frac{\partial}{\partial y^{i_1}}, \cdots, \frac{\partial}{\partial y^{i_k}}\right)\frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_k}}{\partial x^{j_k}} dx^{j_1} \otimes \cdots \otimes dx^{j_k}. \tag{A.3.4}$$

A.4 Hodge Dual

The Hodge dual is defined as the mapping $\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$, where if ω is a p-form on M, $\omega \in \Lambda^p(M)$. Let $\alpha \in \Lambda^k(M)$, and $\beta \in \Lambda^{n-k}(M)$, then since $\alpha \wedge \beta$ is defined, there exists a function $g \in C^\infty(M)$, such that $\alpha \wedge \beta = f\eta$. By definition:

$$\begin{aligned}\alpha \wedge \beta &= \alpha_{i_1,\cdots,i_k}\beta_{i_{k+1},\cdots,i_n} dx^{i_1} \wedge \cdots \wedge dx^{j_k} \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{j_n}, \\ &= \alpha_{i_1,\cdots,i_k}\beta_{i_{k+1},\cdots,i_n}\varepsilon^{i_1\cdots i_k}\varepsilon^{j_{k+1}\cdots j_n} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,\end{aligned} \tag{A.4.1}$$

We define g to be the fully symmetric covariant tensor of rank $(0,2)$, on M, that is
$g(X,Y) = g(Y,X)$, and $g(X,X) \geq 0$, $X, Y \in T_p(M)$. Take the coordinate chart $(x^0, \cdots, x^D)$, defined at $p \in M$. Then the basis of $T_p(M)$ are $\{\partial/\partial x^i\}_{i=0}^D$. We define the contraction $\rfloor$ of the vector field X, $X \in T_p(M)$, with the metric, $X\rfloor g$, $X \in T_p(M)$, as the operation in which $X\rfloor g$ is a covariant tensor of rank $(0,1)$. If $X = X^i\partial/\partial x^i$, locally, we find,

$$\begin{aligned}X\rfloor g &= X^i\frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) dx^k \otimes dx^l \\ &= 2X^i g_{kl}\,\delta^k_i dx^l = 2X^i g_{il} dx^l.\end{aligned} \tag{A.4.2}$$

The determinant of g_{ij} is defined here, to never be zero. If ω is a co-vector field, locally given by $\omega = \omega_i dx^i$, then there exist only one vector field, X, such that $\alpha = (1/2)X\rfloor g$, that is, $\alpha_l = X^i g_{il}$. g_{ij} has an inverse, defined by $g_{ij}g^{jl} = \delta^l_i$. Thus the metric tensor component g_{ij} is thought of a operator that changes co-vectors into vectors, and vice versa. Thus by definition,

$$\varepsilon_{i_1\cdots i_k} g^{i_1 j_{k+1}} \cdots g^{i_k j_n} = det(g^{ij})\varepsilon^{j_{k+1}\cdots j_n} = \left(det(g_{ij})\right)^{-1}\varepsilon^{j_{k+1}\cdots j_n}. \tag{A.4.3}$$

This gives for (A.4.1):

$$\begin{aligned}\alpha \wedge \beta &= \alpha_{i_1\cdots i_k}\beta_{i_{k+1}\cdots i_n}\sqrt{|\det(g_{ij})|}\varepsilon^{i_1\cdots i_k}\varepsilon_{l_1\cdots l_k} g^{l_1 j_{k+1}} \cdots g^{l_k j_n}\eta \\ &\equiv f\eta.\end{aligned} \tag{A.4.4}$$

The Hodge dual, $\star$, is also defined as the inner product,

$$f = (\star\alpha, \beta) \equiv k!(\star\alpha)_{j_1\cdots j_k}\beta_{i_{k+1}\cdots i_n} g^{j_1 i_{k+1}} \cdots g^{j_k i_n}, \tag{A.4.5}$$

thus comparing (A.4.4) and (A.4.5), we find

$$(\star\alpha)_{j_{k+1}\cdots j_n} = \frac{1}{k!}\alpha^{i_1\cdots i_k}\sqrt{|\det(g_{ij})|}\varepsilon_{i_1\cdots i_k j_{k+1}\cdots j_n}. \tag{A.4.6}$$

A.5 Covariant Derivative

Let M be a differentiable manifold. A connection on M assign to each vector field X, $X \in \mathfrak{X}(M)$ an operator ∇_X from $\mathfrak{X}(M)$ into itself, such that for all $X, Y, Z \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$, $f \in C^\infty(M)$,

$$\begin{aligned} \nabla_X(aY + bZ) &= a\nabla_X Y + b\nabla_X Z, \\ \nabla_X(fY) &= f\nabla_X Y(Xf)Y, \\ \nabla_{aX+bY} Z &= a\nabla_X Z + b\nabla_Y Z, \\ \nabla_{fX} Y &= f\nabla_X Y. \end{aligned} \tag{A.5.1}$$

Take a chart $(x^1, \cdots, x^n)$ of M, and let $Y = Y^i\partial_i$, $X = X^j\partial_j$ then

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i\partial_i} Y^j\partial_j \equiv X^i\nabla_{\partial_i}(Y^j\partial_j) \\ &= X^i\left(\frac{\partial Y^j}{\partial x^i}\partial_j + Y^j\nabla_{\partial_i}\partial_j\right) = X^i\left(\frac{\partial Y^k}{\partial x^i} + Y^j(\nabla_{\partial_i})^k_j\right)\partial_k. \end{aligned} \tag{A.5.2}$$

The n^3 number of differentiable functions, $(\nabla_{\partial_i})^k_j \equiv \Gamma^k{}_{ij}$ characterizes the connection in this coordinate system. Let M be a Riemannian Manifold, there exists an unique connection, with vanishing torsion, $0 = T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y]$, and such that $\nabla_X g = 0$. Let $X = X^i\partial_i$, $g = g_{ij}dx^i \otimes dx^j$, then by definition,

$$\begin{aligned} 0 = \nabla_X g &= X^i\nabla_{\partial_i}\left(g_{jk}dx^j \otimes dx^k\right) \\ &= X^i\left(\nabla_{\partial_i} g_{jk}dx^j \otimes dx^k + g_{jk}\nabla_{\partial_i}dx^j \otimes dx^k + g_{jk}dx^j \otimes \nabla_{\partial_i}dx^k\right). \end{aligned} \tag{A.5.3}$$

One can see that, $Y = Y^k\partial_k$,

$$\begin{aligned} \left(\nabla_{\partial_i}dx^j\right)(Y) &= \nabla_{\partial_i}(dx^jY\partial_k) - dx^j\nabla_{\partial_i}Y \\ &= \partial_i Y^j - dx^j\left(\partial_i Y^k + Y^m\Gamma^k{}_{mi}\right)\partial_k \\ &= -Y^m\Gamma^j{}_{mi} = -\Gamma^j{}_{mi}dx^m(Y), \end{aligned} \tag{A.5.4}$$

thus, $\nabla_{\partial_i}dx^j = -\Gamma^j{}_{mi}dx^m$, and we can resolve (A.5.3) as,

$$\begin{aligned} 0 = \nabla_X g &= X^i\left(\nabla_{\partial_i} g_{jk}dx^j \otimes dx^k - g_{jk}\Gamma^j{}_{mi}dx^m \otimes dx^k - g_{jk}\Gamma^k{}_{mi}dx^j \otimes dx^m\right) \\ &= X^i\left(\nabla_{\partial_i} g_{mk} - g_{jk}\Gamma^j{}_{mi} - g_{mj}\Gamma^j{}_{ki}\right)dx^m \otimes dx^k, \end{aligned} \tag{A.5.5}$$

hence we find the relation,

$$\nabla_{\partial_i} g_{mk} - g_{jk}\Gamma^j{}_{mi} - g_{mj}\Gamma^j{}_{ki} = 0. \tag{A.5.6}$$

To resolve the connection coefficients, to the metric coefficients, we write,

$$\begin{aligned} \partial_i g_{mk} + \partial_k g_{im} - \partial_m g_{ki} &= \nabla_{\partial_i} g_{mk} + \nabla_{\partial_k} g_{im} - \nabla_{\partial_k m} g_{ki} \\ &= g_{jk}\Gamma^j{}_{mi} + g_{mj}\Gamma^j{}_{ki} + g_{ji}\Gamma^j{}_{km} + g_{ij}\Gamma^j{}_{mk} - g_{jm}\Gamma^j{}_{ik} - g_{kj}\Gamma^j{}_{im} \\ &= 2g_{ij}\Gamma^j{}_{km}. \end{aligned} \tag{A.5.7}$$

So we conclude that the Riemann connection, with no torsion, under the condition $\nabla_X g = 0$, is

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{im}\left(g_{mj,k} + g_{km,j} - g_{jk,m}\right), \tag{A.5.8}$$

where we have used the notation $\partial_i g_{jk} \equiv g_{jk,i}$. The transformation property of the Christoffel symbols, in a local coordinate system x^ν, are given by,

$$\Gamma^{i'}{}_{j'\sigma'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{\sigma'}}\Gamma^i{}_{jk} + \frac{\partial x^{i'}}{\partial x^k}\frac{\partial x^k}{\partial x^{j'}\partial x^{k'}}, \tag{A.5.9}$$

thus the Christoffel symbols are not tensors. By considering another Christoffel symbol, $\underline{\Gamma}$, with respect to the same local coordinate system as for Γ, defining a different connection, the difference of $\underline{\Gamma}$ and Γ,

$$\begin{aligned}\delta\Gamma^{i'}{}_{j'k'} &\equiv \underline{\Gamma}^{i'}{}_{j'k'} - \Gamma^{i'}{}_{j'k'} \qquad\qquad \text{(A.5.10)}\\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\left(\underline{\Gamma}^i_{jk} - \Gamma^i{}_{jk}\right) + \frac{\partial x^{i'}}{\partial x^k}\frac{\partial x^k}{\partial x^{j'}\partial x^{k'}} - \frac{\partial x^{i'}}{\partial x^k}\frac{\partial x^k}{\partial x^{j'}\partial x^{k'}}\\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\delta\Gamma^i{}_{jk},\end{aligned}$$

we see that the difference transform as a tensor.

A.6 Curvature Tensor

The curvature tensor R, of the connection ∇, is the mapping $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by

$$R(X,Y) \equiv \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \tag{A.6.1}$$

where $[X,Y] \equiv XY - YX$ is the Lie bracket. R is a rank $(1,3)$ tensor, it is not a tensor in the sense of that it does not map to a real number, but it is equivalent to the operation $R(X,Y)Z$, $X,Y,Z \in \mathfrak{X}(M)$. Take function $f \in C^\infty(M)$, then the equation,

$$R(X,Y)(fZ) = fR(X,Y)Z, \tag{A.6.2}$$

holds. Let With $(x^1,\cdots,x^D)$ be a chart. We find for $X = X^i\partial_i$, $Y = Y^j\partial_j$, and $Z = Z^k\partial_k$:

$$\begin{aligned}R(X,Y)Z &= \nabla_X(\nabla_Y Z) - (X \leftrightarrow Y) \qquad\qquad \text{(A.6.3)}\\ &= \nabla_X\left[Y^j(\partial_j Z^k)\partial_k + Y^j Z^k\nabla_{\partial_j}\partial_k)\right] - (X \leftrightarrow Y)\\ &= X^i\big[(\partial_i Y^j)(\partial_j Z^k)\partial_k + Y^j(\partial_i\partial_j Z^k)\partial_k + Y^j(\partial_j Z^k)\nabla_{\partial_i}\partial_k\\ &+ (\partial_i Y^j)Z^k\nabla_{\partial_j}\partial_k + Y^j(\partial_i Z^k)\nabla_{\partial_j}\partial_k + Y^j Z^k\nabla_{\partial_i}\nabla_{\partial_j}\partial_k\big] - (X \leftrightarrow Y)\\ &= (X(Y^j) - Y(X^j)](\partial_j Z^k)\partial_k\\ &+ (X^i Y - Y^i X)(Z^k)\Gamma^l{}_{ik}\partial_l\\ &+ (X(Y^j) - Y(X^j))Z^k\Gamma^l{}_{jk}\partial_l\\ &+ (Y^j X - X^j Y)(Z^k)\Gamma^l{}_{jk}\partial_l\\ &+ X^i Y^j Z^k\left(\partial_i\Gamma^l{}_{jk} - \partial_j\Gamma^l{}_{ik} + \Gamma^m{}_{jk}\Gamma^l{}_{im} - \Gamma^m{}_{ik}\Gamma^l{}_{jm}\right)\partial_l.\end{aligned}$$

We see that by a change of dummy index, two lines cancel, and by regrouping, we find

$$\begin{aligned}R(X,Y)Z &= (X(Y^j) - Y(X^j))(\partial_j Z^l + \Gamma^l{}_{jk}Z^k)\partial_l + X^i Y^j Z^k R^l_{kji}\partial_l\\ &= \partial_l(X(Y^j\nabla_j) - Y(X^j\nabla_j))Z^l + X^i Y^j Z^k R^l_{kij}\partial_l,\end{aligned}$$

where we have defined a covariant derivative for vector components,

$$\nabla_j Z^l \equiv \partial_j Z^l + \Gamma^l{}_{jk}Z^k, \tag{A.6.4}$$

and the Riemann tensor components,

$$R^l_{ijk} \equiv \Gamma^l{}_{ik,j} - \Gamma^l{}_{ij,k} + \Gamma^m{}_{ki}\Gamma^l{}_{jm} - \Gamma^m{}_{ji}\Gamma^l{}_{km}. \tag{A.6.5}$$

Contraction of l and j, in (A.6.5), defines the Ricci tensor components as,

$$R_{ik} \equiv g^{jm}g_{lm}R^l{}_{ijk} = \delta^j_l R^l{}_{ijk} = R^j{}_{ijk}, \tag{A.6.6}$$

and the contraction of the Ricci tensor, yields the Ricci scalar, defined by $R \equiv g^{ik}R_{ik} = R^i{}_i$

A.7 Forms

Let e_i be a local basis, for the vector fields. If ∇ is a connection on M, the connection forms Γ^i_j, with respect to the basis e_i, are the n^2 1-forms defined by

$$\Gamma^i{}_j(X) \equiv \theta^i(\nabla_X e_j), \tag{A.7.1}$$

for $X \in \mathfrak{X}(M)$, and $\theta^i e_j = \delta^i_j$. The n^3 Christoffel symbols, $\Gamma^i{}_{jk}$, are thus defined by $\Gamma^i{}_{jk} \equiv \Gamma^i{}_j(e_k)$. We define the torsion 2-forms as,

$$T^i(X,Y) \equiv \frac{1}{2}\theta^i(T(X,Y)). \tag{A.7.2}$$

Let $X = X^j e_j$, $Y = X^k e_k$, then by definition,

$$\begin{aligned} T^i(X,Y) &= \frac{1}{2}\theta^i\left(\nabla_X Y - \nabla_Y X - [X,Y]\right) \\ &= \frac{1}{2}\theta^i\left(\nabla_X(Y^k e_k) - \nabla_Y(X^j e_j) - X^j Y^k[e_j,e_k]\right) \\ &= \frac{1}{2}\theta^i((\nabla_X Y^k)e_k + Y^k\nabla_X e_k - (\nabla_Y X^j)e_j - X^j\nabla_Y e_j) \\ &= \frac{1}{2}\theta^i((\nabla_X Y^k)e_k - (\nabla_Y X^j)e_j) - \frac{1}{2}(Y^k(\nabla_X\theta^i)e_k - X^j(\nabla_Y\theta^i)e_j). \end{aligned} \tag{A.7.3}$$

Since,

$$\begin{aligned} \Gamma^i{}_j(X) = \theta^i(\nabla_X e_j) &= \nabla_X(\theta^i e_j) - (\nabla_X\theta^i)e_j \\ &= \nabla_X\left(\delta^i_j\right) - (\nabla_X\theta^i)e_j = -(\nabla_X\theta^i)e_j \\ \implies -\Gamma^i{}_j(X)\theta^k &= (\nabla_X\theta^i)e_j\theta^k = (\nabla_X\theta^i)\delta^k_j, \end{aligned} \tag{A.7.4}$$

thus we find $\nabla_X\theta^i = -\Gamma^i{}_j(X)\theta^j$, and so,

$$T^i(X,Y) = \frac{1}{2}\theta^i((\nabla_X Y^k)e_k - (\nabla_Y X^j)e_j) + \frac{1}{2}(\Gamma^i{}_j(X)\theta^j(Y) - \Gamma^i{}_j(Y)\theta^j(X)) \tag{A.7.5}$$

The exterior derivative of a k-form α on M, is defined as,

$$\begin{aligned} (k+1)d\alpha(X_1,\cdots,X_{k+1}) \equiv& \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\alpha(X_1,\cdots,\hat{X}_i,\cdots X_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}), \end{aligned} \tag{A.7.6}$$

where the vector fields denoted by $\hat{}$, should be omitted. Hence, form a 1-form θ^i, X, Y in our basis,

$$2d\theta^i(X,Y) = X\theta^i(Y) - Y\theta^i(X) - \theta^i[X,Y] = X\theta^i(Y) - Y\theta^i(X), \tag{A.7.7}$$

and the torsion 2-form yields,

$$\begin{aligned} T^i(X,Y) &= \frac{1}{2}(X\theta^i(Y) - Y\theta^i(X)) + \frac{1}{2}(\Gamma^i{}_j \otimes \theta^j - \theta^j \otimes \Gamma^i{}_j)(X,Y) \\ &= (d\theta^i + \Gamma^i{}_j \wedge \theta^j)(X,Y), \end{aligned} \tag{A.7.8}$$

that is,

$$T^i = d\theta^i + \Gamma^i{}_j \wedge \theta^j. \tag{A.7.9}$$

Similarly, define the curvature 2-forms

$$\mathscr{R}^i{}_j(X,Y) \equiv \frac{1}{2}\theta^i(R(X,Y)e_j), \tag{A.7.10}$$

where $R(X,Y)$ is defined as (A.6.1), hence,

$$\begin{aligned} 2\mathscr{R}^i{}_j(X,Y) &= \theta^i(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})e_j, \\ &= \theta^i\left[\nabla_X(\Gamma^k{}_j(Y)e_k) - \nabla_Y(\Gamma^k{}_j(X)e_k)\right] - \Gamma^k{}_j([X,Y]), \\ &= \nabla_X\Gamma^i{}_j(Y) - (\nabla_X\theta^i)\Gamma^k{}_j(Y)e_k - \nabla_Y\Gamma^i{}_j(X) + (\nabla_Y\theta^i)\Gamma^i{}_j(X)e_k - \Gamma^i{}_j([X,Y]) \\ &= \nabla_X\Gamma^i{}_j(Y) - \nabla_Y\Gamma^i{}_j(X) - \Gamma^i{}_j([X,Y]) + \left(\Gamma^i{}_l(X)\theta^l\Gamma^k{}_j(Y) - \Gamma^i{}_l(Y)\theta^l\Gamma^k{}_j(X)\right)e_k \\ &= 2\left(d\Gamma^i{}_j + \Gamma^i{}_l \wedge \Gamma^l{}_j\right)(X,Y). \end{aligned} \tag{A.7.11}$$

Thus we find that,

$$\mathscr{R}^i{}_j = d\Gamma^i{}_j + \Gamma^i{}_l \wedge \Gamma^l{}_j. \tag{A.7.12}$$

By taking the differential of (A.7.8), and using (A.7.12), we find,

$$\begin{aligned} dT^i &= d(\Gamma^i{}_j \wedge \theta^j) = d\Gamma^i{}_j \wedge \theta^j - \Gamma^i{}_j \wedge d\theta^j \\ &= \mathscr{R}^i{}_j \wedge \theta^j - \Gamma^i{}_l \wedge \left(T^l - d\theta^l\right) - \Gamma^i{}_j \wedge d\theta^j. \end{aligned} \tag{A.7.13}$$

If the torsion 2-form vanishes, one can resolve that

$$0 = \mathscr{R}^i{}_j \wedge \theta^j, \tag{A.7.14}$$

and from (A.7.9), we find

$$0 = \Gamma^i{}_j \wedge \theta^j = \Gamma^i{}_{jk}\theta^k \wedge \theta^j, \tag{A.7.15}$$

hence the connection is symmetric in its lower indices, $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$. We can also write the curvature tensor, as the exterior product,

$$\mathscr{R}^i{}_j = \frac{1}{2}R^i{}_{jlk}\theta^l \wedge \theta^k. \tag{A.7.16}$$

Taking the differential of the curvature 2-form, using (A.7.12), gives

$$\begin{aligned} d\mathscr{R}^i{}_j &= 0 + d\Gamma^i{}_l \wedge \Gamma^l{}_j - \Gamma^i{}_l \wedge d\Gamma^l{}_j \\ &= \left(\mathscr{R}^i{}_l - \Gamma^i{}_k \wedge \Gamma^k{}_l\right) \wedge \Gamma^l{}_j - \Gamma^i{}_l \wedge \left(\mathscr{R}^l{}_j - \Gamma^l{}_k \wedge \Gamma^k{}_j\right) \\ &= \mathscr{R}^i{}_l \wedge \Gamma^l{}_j - \Gamma^i{}_l \wedge \mathscr{R}^l{}_j. \end{aligned} \tag{A.7.17}$$

Substituting (A.7.16) in (A.7.14), we find

$$0 = R^i{}_{kjl}\theta^k \wedge \theta^l \wedge \theta^j, \tag{A.7.18}$$

which is the same as the first Bianchi identities, $R^i{}_{[kjl]} = 0$. We can write the equal constraint,

$$0 = R^i{}_{kjl} + R^i{}_{jlk} + R^i{}_{lkj}. \tag{A.7.19}$$

A.8 Generalized Kronecker Delta

By defining the generalized Kronecker delta,

$$\delta^{\mu_1\cdots\mu_D}_{\nu_1\cdots\nu_D} \equiv \sum_{\sigma\in\mathcal{S}_D} \text{sign}(\sigma)\delta^{\mu_1}_{\nu_{\sigma(1)}}\cdots\delta^{\mu_D}_{\nu_{\sigma(D)}}, \tag{A.8.1}$$

where $\mathcal{S}_\mathcal{D}$ is the group of all the permutations of the numbers $(0, 1, \cdots, D)$, one can determine a relation between (A.8.1) and the Levi-Civita symbol, as

$$\varepsilon^{\mu_1\cdots\mu_D}\varepsilon_{\nu_1\cdots\nu_D} = \delta^{\mu_1\cdots\mu_D}_{\nu_1\cdots\nu_D}, \tag{A.8.2}$$

so by contraction,

$$\varepsilon^{\mu_1\mu_2\cdots\mu_D}\varepsilon_{\beta\mu_2\cdots\mu_D} = \delta^{\nu_1}_{\beta}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_D}_{\mu_D}\sum_{\sigma\in\mathcal{S}_D} \text{sign}(\sigma)\delta^{\mu_1}_{\nu_{\sigma(1)}}\cdots\delta^{\mu_D}_{\nu_{\sigma(D)}}. \tag{A.8.3}$$

One can express the right hand side of (A.8.3) to find that

$$\varepsilon^{\mu_1\mu_2\cdots\mu_D}\varepsilon_{\beta\mu_2\cdots\mu_D} = d!\delta^{\mu_1}_{\beta}. \tag{A.8.4}$$

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