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On the q-exponential of matrix q-Lie algebras

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Abstract: In this paper, we define several new concepts in the borderline between linear algebra, Lie groups and q-calculus. We first introduce the ring epimorphism τ, the set of all inversions of the basis q, and then the important q-determinant and corresponding q-scalar products from an earlier paper. Then we discuss matrix q-Lie algebras with a modified q-addition, and compute the matrix q-exponential to form the corresponding $n \times n$ matrix, a so-called q-Lie group, or manifold, usually with q-determinant 1. The corresponding matrix multiplication is twisted under τ, which makes it possible to draw diagrams similar to Lie group theory for the q-exponential, or the so-called q-morphism. There is no definition of letter multiplication in a general alphabet, but in this article we introduce new q-number systems, the biring of q-integers, and the extended q-rational numbers. Furthermore, we provide examples of matrices in $su_q(4)$, and its corresponding q-Lie group. We conclude with an example of system of equations with Ward number coefficients.

Keywords: Ring morphism, q-determinant, Nova q-addition, q-exponential function, q-Lie algebra, q-trace, biring

MSC: Primary 17B99; Secondary 17B37, 33D15

1 Introduction

The purpose of this article is to introduce the new concept of a q-matrix Lie algebra, and the corresponding q-Lie group, defined as the set of q-exponentials of the q-Lie algebra. We find that our q-additions fit naturally in the new context, since they virtually replace all additions for the ordinary case, especially for the so-called q-morphisms, or the q-exponentials. In the first article [2], we introduced the inversion operator τ, together with a general $n \times n$ q-determinant, with the purpose that our q-Lie groups $SO_q(2)$ and $SU_q(2)$ should have q-determinant 1. In practise, however, many definitions of q-determinants and many definitions of the basis inversion are necessary, since the q-orthogonalities look differently from case to case. We postpone the definition of the set $\mathrm{GL}_q(n, \mathbb{R})$, which refers to all q-Lie groups, until after the definitions of q-determinants. The two definitions of matrix multiplications, which precede $\mathrm{GL}_q(n, \mathbb{R})$, actually only refer to the latter set. For the q-Lie algebras, there are no matrix multiplications, but a slightly different q-addition. As in the ordinary case, for obvious reasons, we decided to define the q-Lie algebras before the q-Lie groups.

It goes without saying that all computations in this paper apply to the so-called q-real numbers from [3], which are also defined in the paper. The corresponding umbral calculus, with definitions of the alphabet, which would take too long to reproduce here, can also be found in [3]. The q-natural numbers $\mathbb{N}_{\oplus_q}$, which appear in $su_q(n)$, are also defined in [3]. This paper is organized as follows: In section 1 we give a general introduction and the first definitions. Furthermore, we define two important q-analogues of $\mathbb{Z}$ and $\mathbb{Q}$, which extend previous q-numbers [3] and [5]. We show that the first object is an extension of a graded commutative biring, which will be used for the solution of a general q-analogue of a linear system of equations in section 6.

In section 2 we prove that τ is a ring epimorphism and give corresponding definitions of q-scalar and q-vector products and of q-determinants.

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In section 3 we come to the q-Lie algebras, which are very similar to matrix Lie algebras. The only difference is that the q-additions occur both as matrix elements in the q-Lie algebras and as q-analogues of direct sums of Lie algebras (or ideals). However, we will not always use the same symbol for q-addition between letters as for q-addition between matrix Lie algebras, see formula (74). The reason is that matrices in general have noncommutative multiplication. Although it is a kind of q-addition, we will strive to keep the same nomenclature for Lie algebras as in the ordinary case. In section 4 we come to the important q-Lie groups, which have very similar properties as groups, but with the important difference that there are two multiplications. All q-Lie groups are manifolds.

In section 5 we present the first q-analogue of $SU(4)$ and the corresponding maximal torus; we note that there are many definitions of these objects in the literature. We also give some other examples of q-Lie groups.

In section 6 we show that a q-analogue of linear systems of equations has a set of solutions, similar to the ordinary case, which will have later applications for so-called q-symmetric spaces.

We now start with the definitions, compare with the book [3].

Definition 1. The q-analogue and q-factorial are given by

$$\{a\}_q \equiv \frac{1-q^a}{1-q};\ \{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}\backslash\{1\}, \tag{1}$$

The Gauss q-binomial coefficient is given by

$$\binom{n}{k}_q \equiv \frac{\{n\}_q!}{\{n-k\}_q!\{k\}_q!}, k = 0, 1, \dots, n. \tag{2}$$

The q-derivative is defined by

$$\mathrm{D}_q\varphi(x) \equiv \frac{\varphi(x)-\varphi(qx)}{(1-q)x}. \tag{3}$$

The q-exponential functions are defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}z^k;\ \mathrm{E}_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{\{k\}_q!}z^k. \tag{4}$$

The q-trigonometric functions are defined by

$$\mathrm{Cos}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(ix) + \mathrm{E}_q(-ix)). \tag{5}$$

$$\mathrm{Sin}_q(x) \equiv \frac{1}{2i}(\mathrm{E}_q(ix) - \mathrm{E}_q(-ix)). \tag{6}$$

Definition 2. Let a and b belong to a commutative monoid. The Nalli–Ward–AlSalam q-addition (NWA) is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n}\binom{n}{k}_q a^k b^{n-k},\ (a \ominus_q b)^n \equiv \sum_{k=0}^{n}\binom{n}{k}_q a^k(-b)^{n-k} \tag{7}$$

The Jackson–Hahn–Cigler q-addition (JHC) is given by

$$(a \boxplus_q b)^n \equiv \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}\, a^{n-k}b^k,\ (a \boxminus_q b)^n \equiv \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}\, a^{n-k}(-b)^k. \tag{8}$$

Definition 3. We use the alphabet [3, p. 98], with zero denoted by θ. Let M be a subset of this alphabet. Then $\langle M\rangle$ denotes the set generated by M together with the four operations $\oplus_q, \boxplus_q, \ominus_q, \boxminus_q$.

Definition 4. The basic construction which replaces the real numbers as function arguments in q-trigonometric functions etc. are the q-real numbers $\mathbb{R}_q$, which are defined as follows:

$$\mathbb{R}_q \equiv \langle\mathbb{R}\rangle. \tag{9}$$

In [3, p. 167] we introduced the following concept:

Definition 5. There is a Ward number $\overline{n}_q$

$$\overline{n}_q \sim 1 \oplus_q 1 \oplus_q \ldots \oplus_q 1, \tag{10}$$

where the number of 1 in the RHS is n. Let $(\mathbb{N}_{\oplus_q}, \oplus_q, \odot_q)$ denote the semiring of Ward numbers $\overline{k}_q,\ k \geq 0$ together with two binary operations: $\oplus_q$ is the usual Ward q-addition. The multiplication $\odot_q$ is defined as follows:

$$\overline{n}_q \odot_q \overline{m}_q \sim \overline{nm}_q, \tag{11}$$

where $\sim$ denotes the equivalence in the alphabet.

We will now extend this semiring to a biring. Therefore we first define our general biring. The following definition prepares for the biring in theorem 1.1.

Definition 6. Assume that $R \equiv R_1 \cup -(R_1)$, a gradation. A graded commutative biring is a set $(R, \oplus, \boxplus, \odot, 0)$, with two binary operations $\oplus$ and $\odot$ on R, a dual addition $\boxplus$, a zero 0, (and a unit 1), which satisfy the following axioms. For each elements $a, b, c \in R$:

1. Additive associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
2. Additive commutativity: $a \oplus b = b \oplus a$.
3. Additive identity: There exists an element $0 \in R$ such that $0 \oplus a = a \oplus 0 = a$.
4. Additive inverse: There exists an element -a$\in R$ such that $a \oplus (-a) = (-a) \oplus a = 0$.
5. Multiplicative identity: There exists an element $1 \in R$ such that $a \odot 1 = 1 \odot a = a$.
6. Multiplicative associativity: $(a \odot b) \odot c = a \odot (b \odot c)$.
7. distributivity: $a \odot (b \oplus c) = a \odot b \oplus a \odot c$.
8. Multiplicative commutativity: $a \odot b = b \odot a$.

We assume that for b or c equal to $-d$, $d \in R_1$, we may replace $\oplus$ by $\boxplus$.

We can now extend the q-addition with JHC to obtain a graded commutative biring.

Definition 7. Let $(\mathbb{Z}_q, \oplus_q, \boxplus_q, \odot_q, \overline{0}_q)$ denote ± the Ward numbers, i.e. $\mathbb{Z}_q \equiv \mathbb{N}_{\oplus_q} \cup -\mathbb{N}_{\oplus_q}$, where there are two inverse q-additions $\oplus_q$ and $\boxplus_q$. $\overline{0}_q$ denotes the zero θ, and $\overline{1}_q$ denotes the multiplicative identity. The dual addition is defined by

$$\overline{n}_q \boxplus_q -\overline{m}_q \sim \overline{n-m}_q, n \geq m. \tag{12}$$

Furthermore, the multiplication $\odot_q$ is defined by (11) and

$$\overline{n}_q \odot_q -\overline{m}_q \sim -\overline{nm}_q. \tag{13}$$

Finally, we define

$$\overline{-m}_q \equiv -\overline{m}_q. \tag{14}$$

Theorem 1.1. *An extension of [3, p. 167]. Assume that $\mathbb{Z}_q$ is defined by the previous definition. Then $(\mathbb{Z}_q, \oplus_q, \boxplus_q, \odot_q, \overline{0}_q)$ is a graded commutative biring.*

Proof. The proof is achieved for three elements $\overline{n}_q, \overline{m}_q, \overline{k}_q \in \mathbb{N}_{\oplus_q}$ by [3, p. 167]. Instead, choose three elements $\overline{n}_q, -\overline{m}_q, \overline{k}_q \in \mathbb{Z}_{\oplus_q}$. The associativity for $(\mathbb{Z}_{\oplus_q}, \oplus_q, \boxplus_q)$ is shown as follows:

$$(\overline{n}_q \boxplus_q -\overline{m}_q) \oplus_q \overline{k}_q \overset{\text{by}(12)}{\sim} \overline{n-m}_q \oplus_q \overline{k}_q \sim \overline{(n-m)+k}_q, n \geq m. \tag{15}$$

$$\overline{n}_q \oplus_q (\overline{k}_q \boxplus_q -\overline{m}_q) \overset{\text{by}(12)}{\sim} \overline{n}_q \oplus_q \overline{k-m}_q \sim \overline{n+(k-m)}_q, k \geq m. \tag{16}$$

The associativity for $(\mathbb{Z}_{\oplus_q}, \odot_q)$ is shown as follows:

$$(\overline{n}_q \odot_q -\overline{m}_q) \odot_q \overline{k}_q \overset{\text{by}(13)}{\sim} -\overline{nm}_q \odot_q \overline{k}_q \overset{\text{by}(13)}{\sim} -\overline{nmk}_q. \tag{17}$$

$$\overline{n}_q \odot_q (-\overline{m}_q \odot_q \overline{k}_q) \overset{\text{by}(13)}{\sim} \overline{n}_q \odot_q -\overline{mk}_q \overset{\text{by}(13)}{\sim} -\overline{nmk}_q. \tag{18}$$

The identity is $\overline{1}_q$. The commutativity for $(\mathbb{Z}_{\oplus_q}, \odot_q)$ follows from (13).
The distributive law is proved as follows:
Assume that $k \geq m$.

$$\overline{n}_q \odot_q (\overline{k}_q \boxplus_q -\overline{m}_q)) \overset{\text{by}(12)}{\sim} \overline{n}_q \odot_q \overline{k-m}_q \sim \overline{n(k-m)}_q. \tag{19}$$

$$(\overline{n}_q \odot_q \overline{k}_q) \boxplus_q (\overline{n}_q \odot_q -\overline{m}_q) \overset{\text{by}(13)}{\sim} \overline{-nm}_q \oplus_q \overline{nk}_q \overset{\text{by}(12)}{\sim} \overline{(nk-nm)}_q. \tag{20}$$

□

Definition 8. [5] Let $\mathbb{Q}_{\oplus_q}$ denote the set of objects of the following type:

$$\frac{\overline{m}_q}{\overline{n}_q},\ \overline{n}_q \nsim \overline{0}_q, \tag{21}$$

together with a linear functional

$$v,\ \mathbb{R}[x] \times \mathbb{Q}_{\oplus_q} \to \mathbb{R}, \tag{22}$$

called the evaluation. If $v(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$v\left(\frac{\overline{m}_q}{\overline{n}_q}\right) \equiv \sum_{k=0}^{\infty} a_k \frac{(\overline{m}_q)^k}{(\overline{n}_q)^k}. \tag{23}$$

Definition 9. Let $\mathbb{Q}_{\oplus_q}[\mathbb{N}]$ denote q-rational numbers, where we can replace Ward numbers in the numerator by products of Ward numbers. These products are denoted by $\cdot$. Let $\mathbb{Q}^{\star}_{\oplus_q}$ denote the set of objects generated by $\mathbb{Q}_{\oplus_q}[\mathbb{N}]$, together with the two operators $\oplus_q, \boxplus_q$,

$$\mathbb{Q}^{\star}_{\oplus_q} \equiv \langle \mathbb{Q}_{\oplus_q}[\mathbb{N}] \rangle. \tag{24}$$

The following equation can be used for solutions of q-linear systems of equations in section 6.

Lemma 1.2.

$$\boxplus_q \boxplus_q = \boxminus_q \boxminus_q = \oplus_q. \tag{25}$$

Proof. Use the fact that $\boxplus_q$ is the inverse operator to $\oplus_q$. □

We can combine (25) with combinations of minus signs in an obvious way. We assume that the following equations from [3, p. 99-100] are known: If

$$\alpha \oplus_q \beta \sim \gamma \tag{26}$$

or

$$\alpha \boxplus_q \beta \sim \gamma, \tag{27}$$

we can compute α explicitly. By (25) the solutions are

$$\alpha \sim \gamma \boxminus_q \beta \text{ or } \alpha \sim \gamma \ominus_q \beta. \tag{28}$$

We can calculate β explicitly from (27) as

$$-\beta \sim \alpha \boxminus_q \gamma. \tag{29}$$

2 The ring morphism τ, scalar products and q-determinants

We will now generalize the theory from the two previous articles [2] and [4] and define τ as a ring morphism.

Definition 10. Let $+$ and $\oplus$ denote addition of functions in two different sets F and $F_{\frac{1}{q}}$ with real argument. Let $\cdot$ and $\star$ denote multiplication of functions in two different sets F and $F_{\frac{1}{q}}$ with real argument. We define two sets of functions:

$$(F,+,\cdot,1) \equiv \mathbb{R}[\mathrm{Sin}_q,\mathrm{Cos}_q,\mathrm{Sinh}_q,\mathrm{Cosh}_q,\mathrm{E}_q], \tag{30}$$

$$(F_{\frac{1}{q}},\oplus,\star,E) \equiv \mathbb{R}[\mathrm{Sin}_q,\mathrm{Cos}_q,\mathrm{Sinh}_q,\mathrm{Cosh}_q,\mathrm{E}_q,\mathrm{Sin}_{\frac{1}{q}},\mathrm{Cos}_{\frac{1}{q}},\mathrm{Sinh}_{\frac{1}{q}},\mathrm{Cosh}_{\frac{1}{q}},\mathrm{E}_{\frac{1}{q}}]. \tag{31}$$

The letters in both F and $F_{\frac{1}{q}}$ are in $\mathbb{R}$.

Theorem 2.1. $(F,+,\cdot,1)$ *is a commutative ring.*

Proof. This follows since the elements are linear combinations of real numbers. □

Theorem 2.2. $(F_{\frac{1}{q}},\oplus,\star,E)$ *is a commutative ring.*

Proof. This is proved similarly. □

Definition 11. Let $\Phi \subset \mathbb{R}$, $|\Phi| < \infty$ denote a finite set of arbitrary letters. Let $f \equiv \prod_i f_i \in F$. The function τ_Φ is the function $F \mapsto F_{\frac{1}{q}}$, which operates on the functions f_i in the following way:

$$\begin{cases} q \to \frac{1}{q}, & \text{if } f_i \text{ depends on } \Phi; \\ \mathrm{I}, & \text{if } f_i \text{ is independent of } \Phi, \end{cases} \tag{32}$$

where I denotes the identity operator.

Then we have

Theorem 2.3. *The function* τ_Φ *is a ring morphism.*

$$\tau_\Phi(f_1+f_2) = \tau_\Phi(f_1) \oplus \tau_\Phi(f_2), \tag{33}$$

$$\tau_\Phi(f_1 \cdot f_2) = \tau_\Phi(f_1) \star \tau_\Phi(f_2), \tag{34}$$

$$\tau_\Phi(1) = E. \tag{35}$$

Proof. Formulas (33) and (34) follow at once, since we first add or multiply elements on F and then invert the basis. This is the same as first inverting the basis and then adding or multiplying. The unit 1 is independent of q, and formula (35) follows. □

Theorem 2.4. *The function* τ_Φ *is a ring epimorphism.*

Proof. Assume that $\Gamma \in F_{\frac{1}{q}}$, and $\Psi \supset \Phi$ contains all letters in the alphabet. Then we can always find an element $\Delta \in F$ which maps to Γ by τ_Φ because of the completeness of the ring F. □

We now come to the first matrix definitions. As before, matrix elements are denoted (i,j) and range from 0 to $n-1$.

Definition 12. For $q \neq 1$, we have two matrix multiplications $\cdot$ and $\cdot_q$. $\cdot$ is the ordinary matrix multiplication and $\cdot_q$ is defined as follows: for two $n \times n$ matrices $A(q)$ and $B(q)$, with entries a_{ij} and b_{ij}, we define

$$(a \cdot_q b)_{ij} \equiv \sum_{m=0}^{n-1} a_{im}\tau(b_{mj}). \tag{36}$$

We can modify this product in the following way:

$$(a \cdot_{\Phi;q} b)_{ij} \equiv \sum_{m=0}^{n-1} a_{im}\tau_{\Phi}(b_{mj}), \tag{37}$$

where τ_{Φ} is defined by formula (32).

Definition 13. Let

$$\xi(x) \equiv \begin{cases} \tau(x) \text{ if } n \text{ is even,} \\ \text{I, the identity if } n \text{ is odd.} \end{cases} \tag{38}$$

The q-determinant of an $n \times n$ matrix $\alpha = [a_{ij}]_{i,j=0}^{n-1}$ (the first index denotes the row) is defined by the formula

$$\det_q \alpha \equiv \sum_{\pi \in S_n} \text{sign}(\pi) a_{0\pi(0)} \tau(a_{1\pi(1)}) \ldots \xi(a_{n-1\pi(n-1)}). \tag{39}$$

In other words, τ is applied to the matrix elements with first index odd.

The following variant of 3×3 q-determinant will also be used:

$$|\alpha|_{s,t;q} \equiv \sum_{\pi \in S_3} \text{sign}(\pi) a_{0\pi(0)} \tau_s(a_{1\pi(1)}) \tau_t(a_{2\pi(2)}). \tag{40}$$

Remark 1. For complex matrices we change τ to its complex conjugate as in the definition of SU(N).

In particular this definition applies to vector products.

Example 1. Put

$$\mathbf{x}_s \equiv \mathrm{D}_{q,s}\mathbf{x} \text{ and } \mathbf{x}_t \equiv \mathrm{D}_{q,t}\mathbf{x}. \tag{41}$$

The q-deformed vector product of

$$\mathbf{x}_t = (\text{Sinh}_q(s)\text{Cos}_q(t), \text{Sinh}_q(s)\text{Sin}_q(t), 1), \tag{42}$$

and

$$\mathbf{x}_s = (-\text{Cosh}_q(s)\text{Sin}_q(t), \text{Cosh}_q(s)\text{Cos}_q(t), 0) \tag{43}$$

is given by

$$\mathbf{x}_t \times_q \mathbf{x}_s \equiv ||_{t,-;q} \equiv \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \text{Sinh}_q(s)\text{Cos}_{\frac{1}{q}}(t) & \text{Sinh}_q(s)\text{Sin}_{\frac{1}{q}}(t) & 1 \\ -\text{Cosh}_q(s)\text{Sin}_q(t) & \text{Cosh}_q(s)\text{Cos}_q(t) & 0 \end{vmatrix} = \text{Cosh}_q(s)(-\text{Cos}_q(t), -\text{Sin}_q(t), \text{Sinh}_q(s)). \tag{44}$$

We can now finally, mainly for symbolic purposes, define a q-analogue of the general linear group, which will be used in section 4.

Definition 14. The set $\text{GL}_q(n, \mathbb{R})$ is defined by

$$\text{GL}_q(n, \mathbb{R}) \equiv \{A \in (F, +, \cdot, 1)^{(n,n)} | det_1 A \neq 0\}. \tag{45}$$

3 Basic definitions for matrix q-Lie algebras

We start with some definitions. We only consider matrix q-Lie algebras, which we call q-Lie algebras. These are q-analogues of matrix Lie algebras denoted by $\mathfrak{g}_q$.

Definition 15. Let $\mathcal{M}_n(\mathbb{R}_{\cdot q'}[\mathbb{Z}_q]^{(n,n)})$ denote (a q-analogue of) $\{g|g \subset \mathfrak{gl}(n)\}$, i.e. the set of $n \times n$ matrices with entries in a Lie algebra.

Then $\mathcal{M}_n(\mathbb{R}_{\cdot q'}[\mathbb{Z}_q]^{(n,n)})$ is a bi $\mathbb{R}$-vector space with the operations of $\oplus_{\cdot q'}$, matrix addition and $\mathbb{R}$-scalar multiplication. The zero vector is the $n \times n$ zero matrix $\mathrm{O}_{n,n;q}$, which will often be abbreviated as O, when we know the size of the matrix.

Definition 16. A *q-Lie algebra* is a set of matrices

$$\mathfrak{gl}_{n,\mathbb{C};\mathbb{Z}_q} \equiv \{A \in \mathcal{M}_n(\mathbb{R}_{\cdot q'}[\mathbb{Z}_q]^{(n,n)}), \tag{46}$$

with properties similar to Lie algebras, which is a vector space in $\mathbb{N}^2$ with an antisymmetric bilinear bracket operation $[\cdot,\cdot] : \mathfrak{g}_q \times \mathfrak{g}_q \mapsto \mathfrak{g}_q$ defined by

$$[A, B] \equiv AB - BA. \tag{47}$$

Something about the notation: we denote direct sums of matrices by $\oplus_{\cdot q'}$ or $\oplus_q$. The notation $\oplus_{\cdot q'}$ denotes direct sum of two matrices in the context of Lie algebra, and the notation $\oplus_q$ denotes sums of commuting matrices. The fact that the matrices of the q-Lie algebras do not commute is unimportant since we always multiply the q-Lie groups in a certain order. In the following, we write $\oplus_q = \{\oplus_q, \oplus_{\cdot q'}\}$. We can solve for the compact part:

$$\mathfrak{k} = \mathfrak{g}_q \boxminus_{\cdot q'} \mathfrak{p}. \tag{48}$$

Definition 17. A subspace $\mathfrak{h}_q$ of a q-Lie algebra $\mathfrak{g}_q$ is said to be a *q-Lie subalgebra* if it is closed under the Lie bracket.

Definition 18. Assume that a basis for a q-Lie algebra is fixed. Let A_i, A_j, A_k be some basis vectors in a q-Lie algebra. The structure constants c_{ij}^k are defined by

$$[A_i, A_j] = \sum_{k=1}^{r} c_{ij}^k A_k. \tag{49}$$

We will see one example of these structure constants in section 5.

Definition 19. The *commutator series* of $\mathfrak{g}_q$ is defined as follows: let $\mathfrak{g}_q^0 \equiv \mathfrak{g}_q, \mathfrak{g}_q^1 \equiv [\mathfrak{g}_q, \mathfrak{g}_q], \ldots, \mathfrak{g}_q^{n+1} \equiv [\mathfrak{g}_q^n, \mathfrak{g}_q^n]$. We call $\mathfrak{g}_q$ *solvable* if $\mathfrak{g}_q^n = 0$ for some n.

Definition 20. The *lower central series* of $\mathfrak{g}_q$ is defined as follows: let $\mathfrak{g}_{q;0} \equiv \mathfrak{g}_q, \mathfrak{g}_{q;1} \equiv [\mathfrak{g}_q, \mathfrak{g}_q], \ldots, \mathfrak{g}_{q;n+1} \equiv [\mathfrak{g}_q, \mathfrak{g}_{q;n}]$. We call $\mathfrak{g}_q$ *nilpotent* if $\mathfrak{g}_{q;n} = 0$ for some n.

Example 2. The q-Lie algebra of upper-triangular matrices in $\mathfrak{g}_q$ is solvable.

Definition 21. The derived q-Lie algebra is the q-subalgebra of $\mathfrak{g}_q$, denoted $\mathfrak{g}_q'$, which consists of all Lie brackets of pairs of elements of $\mathfrak{g}_q$.

The simplest matrix q-Lie algebras are the same as in [7, p. 45-49]. We list some of them here.

In the first two cases the derived algebra is one-dimensional:

$$I \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, J \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K \equiv \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

$$I \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, J \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{51}$$

In the third case the derived algebra is two-dimensional:

$$M \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, K \equiv \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{52}$$

There is also the following example of a solvable but not nilpotent affine q-Lie algebra from [1, p. 197]:

$$I \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, J \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{53}$$

We will use almost the same q-Lie algebras as for Lie algebras. Then we use the q-exponential to compute an element of the corresponding q-Lie group.

We are now going to describe the phenomenon direct sum of two matrices in the context of q-Lie algebras.

Example 3. When A and B are q-Lie algebras, we have a formula for a q-analogue of certain continuous-time Markov processes:

$$\mathrm{E}_q(A \oplus_q B) = \mathrm{E}_q(A)\mathrm{E}_q(B). \tag{54}$$

Definition 22. A subset B of a q-Lie algebra L is said to be a *q-ideal* if it is a vector subspace of L under q-addition, and $[X, Y] \in B$ for any $X \in B$ and $Y \in L$.

The following definition is very important as in the ordinary case.

Definition 23. A q-Lie algebra $\mathfrak{g}_q$ is called *semisimple* if its only solvable q-ideal is 0.

Definition 24. Consider

$$L = L_1 \oplus_q L_2 \oplus_q \cdots L_k, \tag{55}$$

where all L_i are simple q-ideals in L. The q-direct sum of the q-Lie algebras L_i is the vector space $\sum \oplus_q L_i$ with component-wise addition, scalar multiplication, and product $(\sum \oplus_q a_i)(\sum \oplus_q b_i) = \sum a_i b_i$. The q-tensor product of the q-Lie algebras L_i is the q-tensor product

$$L_1 \otimes_q L_2 \otimes_q \cdots L_k \tag{56}$$

of the vector spaces $L_1, \ldots, L_k$, together with the bilinear product defined by

$$(a_1 \otimes_q a_2 \otimes_q \cdots a_k)(b_1 \otimes_q b_2 \otimes_q \ldots b_k) \equiv a_1 b_1 \otimes_q a_2 b_2 \otimes_q \cdots a_k b_k. \tag{57}$$

Definition 25. The Lie bracket for the q-direct sum $\mathfrak{g}_{q;1} \oplus_q \mathfrak{g}_{q;2}$ is

$$[(X_1, X_2), (Y_1, Y_2)]_{\mathfrak{g}_{q;1} \oplus_q \mathfrak{g}_{q;2}} \equiv ([X_1, Y_1]_{\mathfrak{g}_{q;1}}, [X_2, Y_2]_{\mathfrak{g}_{q;2}}). \tag{58}$$

Definition 26. Let $\mathfrak{h}_q \subset \mathfrak{g}_q$ be a q-ideal and $\pi : \mathfrak{g}_q \mapsto \mathfrak{g}_q/\mathfrak{h}_q$ denote projection onto the vector space quotient. Then the bracket $[\pi(x), \pi(y)] \equiv \pi([x, y])$ is well-defined and defines the *quotient algebra* $\mathfrak{g}_q/\mathfrak{h}_q$.

4 q-Lie groups

Definition 27. Denote the functions which are infinitely q-differentiable in all letters (variables) by C_q^∞.

Example 4. We explain what it means to be q-differentiable in all letters. Put

$$x \sim \alpha \oplus_q \beta \boxminus_q \gamma,\ f(x) \equiv \mathrm{E}_q(x). \tag{59}$$

Then we have

$$\mathrm{D}_{q,x} f(x) = \mathrm{D}_{q,\alpha} f(x) = \mathrm{D}_{q,\beta} f(x) = f(x). \tag{60}$$

The function $f(x)$ is q-differentiable in all letters.

Definition 28. A q-Lie group $(G_{n,q,\cdot,\cdot_q}, \mathrm{I}_g) \supset \mathrm{E}_q(\mathfrak{g}_q)$, is a possibly infinite set of matrices $\in \mathrm{GL}_q(n, \mathbb{R})$, with two associative multiplications: $\cdot$, and the twisted $\cdot_q$. Each q-Lie group has a unit, denoted by I_g. Each element $\Phi \in G$ has an inverse Φ^{-1} with the property $\Phi \cdot_q \Phi^{-1} = \mathrm{I}_g$.

Theorem 4.1. *A q-Lie group is also a manifold (with boundary). The boundary comes from the limits of some variables, usually angles for q-trigonometric functions.*

Proof. It is an open subset of $\mathbb{R}^{n^2}$. The functions in question in $(F, +, \cdot, 1)$ all belong to C^∞ (and C_q^∞). □

Many of the following theorems have their origin in the corresponding q-Lie algebras, and should sometimes be interpreted in a formal sense.

Definition 29. If $(G_1, \cdot_1, \cdot_{1:q})$ and $(G_2, \cdot_2, \cdot_{2:q})$ are two q-Lie groups, then $(G_1 \times G_2, \cdot, \cdot_q)$ is a q-Lie group called the product q-Lie group. This has group operations defined by

$$(g_{11}, g_{21}) \cdot (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}), \tag{61}$$

and

$$(g_{11}, g_{21}) \cdot_q (g_{12}, g_{22}) = (g_{11} \cdot_{1:q} g_{12}, g_{21} \cdot_{2:q} g_{22}). \tag{62}$$

Definition 30. If $(G_{n,q}, \cdot, \cdot_q)$ is a q-Lie group and $H_{n,q}$ is a nonempty subset of $G_{n,q}$, then $(H_{n,q}, \cdot, \cdot_q)$ is called a q-Lie subgroup of $(G_{n,q}, \cdot, \cdot_q)$ if

1.
$$\Phi \cdot \Psi \in H_{n,q} \text{ and } \Phi \cdot_q \Psi \in H_{n,q} \text{ for all } \Phi, \Psi \in H_{n,q}. \tag{63}$$

2.
$$\Phi^{-1} \in H_{n,q} \text{ for all } \Phi \in H_{n,q}. \tag{64}$$

3. $H_{n,q}$ is a submanifold of $G_{n,q}$.

Definition 31. The q-Lie group $(G_{n,q}, \cdot, \cdot_q)$ acts on the set X if there are two functions

$$\Psi_q : G_{n,q} \times X \mapsto X$$

and

$$\Phi_q : G_{n,q} \times X \mapsto X$$

such that, when we write $g(x)$ instead of $\Psi_q(g, x)$, and $g_q(x)$ instead of $\Phi_q(g, x)$ we have
1. $(g_1 \cdot g_2)(x) = g_1((g_2)(x))$ for all $g_1, g_2 \in G_{n,q},\ x \in X$,
2. $(g_1 \cdot_q g_2)(x) = (g_1)_q((g_2)(x))$ for all $g_1, g_2 \in G_{n,q},\ x \in X$,
3. $\mathrm{I}_g(x) = x,\ x \in X$.

The q-Lie group $G_{n,q}$ acts faithfully on X if the only element of $G_{n,q}$, which fixes every element of X under the two operations $\Psi_q(g, x)$ and $\Phi_q(g, x)$, is the identity.

Theorem 4.2. *If $G_{n,q}$ acts on a set X and $x \in X$, then*
Stab $x = \{g \in G_{n,q} | (g(x) = x) \wedge (g_q(x) = x)\}$ *is a q-Lie subgroup of $G_{n,q}$, called the stabilizer of x.*

Definition 32. A q-Lie subgroup $H_{n,q}$ of a q-Lie group $G_{n,q}$ is called a normal q-Lie subgroup of $G_{n,q}$ if

$$\phi^{-1} \cdot_q \Psi \cdot_q \phi \in H_{n,q} \text{ for all } \phi \in G_{n,q} \text{ and } \Psi \in H_{n,q}. \tag{65}$$

Definition 33. An invertible mapping $f : (G_{n,q}, \cdot, \cdot_q) \to (H_{n,q}, \cdot, \cdot_q)$ is called a q-Lie group morphism between $(G_{n,q}, \cdot, \cdot_q)$ and $(H_{n,q}, \cdot, \cdot_q)$ if

$$f(\phi \cdot \psi) = f(\phi) \cdot f(\psi),\ \phi, \psi \in \mathbb{R}_q,\ \text{ and } f(\phi \cdot_q \psi) = f(\phi) \cdot_q f(\psi),\ \phi, \psi \in \mathbb{R}_q. \tag{66}$$

A q-Lie group morphism is called q-Lie group isomorphism if it is one-to-one.

Theorem 4.3. *If $f : (G_{n,q}, \cdot, \cdot_q) \to (H_{n,q}, \cdot, \cdot_q)$ is a q-Lie group morphism, then*

$$f(\mathrm{I}_g) = \mathrm{I}_h \text{ and } f(\phi^{-1}) = f(\phi)^{-1} \text{ for all } \phi \in G_{n,q}. \tag{67}$$

Proof. 1. $f(\mathrm{I}_g) = f(\mathrm{I}_g \cdot \mathrm{I}_g) = f(\mathrm{I}_g) \cdot f(\mathrm{I}_g) = \mathrm{I}_h$.
2. $f(\phi) \cdot_q f(\phi^{-1}) = f(\phi \cdot_q \phi^{-1}) = f(\mathrm{I}_g) = \mathrm{I}_h$. We find the desired result. □

Definition 34. If $f : (G_{n,q}, \cdot, \cdot_q) \to (H_{n,q}, \cdot, \cdot_q)$ is a q-Lie group morphism, the kernel of f, which we denote by Ker$f$, is the set of elements of $G_{n,q}$ that are mapped by f to the identity of $H_{n,q}$.

Theorem 4.4. *Let $f : (G_{n,q}, \cdot, \cdot_q) \to (H_{n,q}, \cdot, \cdot_q)$ be a q-Lie group morphism. Then*
1. *Kerf is a normal q-Lie group of $G_{n,q}$,*
2. *f is injective if and only if Kerf $= I_G$.*

Proof. We begin by showing that Ker$f$ is a q-Lie subgroup.
Assume that α and $\beta \in$ Ker$f$ so that $f(\alpha) = f(\beta) = \mathrm{I}_h$.
Then

$$f(\alpha \cdot \beta) = f(\alpha) \cdot f(\beta) = I_H \cdot \mathrm{I}_h = \mathrm{I}_h, \text{ and } \alpha \cdot \beta \in \mathrm{Ker}f = f(\alpha \cdot_q \beta)$$

Furthermore,

$$f(\alpha^{-1}) = f(\alpha)^{-1} = \mathrm{I}_h^{-1} = \mathrm{I}_h \text{ and } \alpha^{-1} \in \mathrm{Ker}f.$$

Assume that $\alpha \in$ Ker$f$ and $g \in G_q$, then

$$f(g^{-1} \cdot_q \alpha \cdot_q g) = f(g^{-1}) \cdot_q f(\alpha) \cdot_q f(g) = f(g^{-1}) \cdot_q \mathrm{I}_h \cdot_q f(g) = f(g^{-1}) \cdot_q f(g) = \mathrm{I}_h.$$

This implies that $g^{-1} \cdot_q \alpha \cdot_q g \in$ Ker$f$ and Ker$f$ is a normal q-Lie subgroup of $G_{n,q}$. The second statement is proved in the same way as for groups. □

Theorem 4.5. *For any q-Lie group morphism $f : (G_{n,q}, \cdot, \cdot_q) \to (H_{n,q}, \cdot, \cdot_q)$, the image of f,* Imf $=$ *$\{f(\phi) | \phi \in G_{n,q}\}$ is a q-Lie subgroup of $H_{n,q}$.*

Proof. This is again similar to the proof for groups. Assume that $f(g_1), f(g_2) \in$ Imf. It follows that $f(g_1) \cdot f(g_2) = f(g_1 \cdot g_2) \in$ Imf, and $f(g_1) \cdot_q f(g_2) = f(g_1 \cdot_q g_2) \in$ Imf. Also we have $f(g_1)^{-1} = f(g_1^{-1}) \in$ Imf. □

The morphism

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}^* \tag{68}$$

corresponds to the q-Lie group morphisms

$$\mathrm{E}_q : (\mathbb{R}_q, \oplus_q) \longrightarrow \mathbb{R}^*, \tag{69}$$

$$\mathrm{E}_q(x \oplus_q y) = \mathrm{E}_q(x)\mathrm{E}_q(y) \tag{70}$$

and

$$\mathrm{E}_q : (\mathbb{R}_q; \oplus_q, \boxplus_q) \longrightarrow \mathbb{R}^\star, \tag{71}$$

$$\mathrm{E}_q(x \oplus_q y) = \mathrm{E}_q(x)\mathrm{E}_q(y), \tag{72}$$

$$\mathrm{E}_q(x \boxplus_q y) = \mathrm{E}_q(x)\mathrm{E}_{\frac{1}{q}}(y). \tag{73}$$

When x and y are so-called q-Lie algebras, we call these mappings q-one-parameter subgroups, for which see next subchapter.

Definition 35. The set $\mathrm{GL}'_{n;q} \subset \mathrm{GL}_q(n, \mathbb{R})$ is defined as the image $\mathrm{E}_q(X)$ for $X \in \mathcal{M}_n(\mathbb{R}_{'q'})$. We assume that there exist two matrix multiplications in $\mathrm{GL}'_{n;q}$: $\cdot$ and $\cdot_q$, such that $\mathrm{E}_q(X) \cdot_q \mathrm{E}_q(-X) = \mathrm{I}_n$, the unit matrix. The second multiplication $\cdot_q$ is given by formula (36).

We have the following commutative diagram:

$$\begin{array}{ccc} U \times V & \xrightarrow{\mathrm{E}_q} & \mathrm{E}_q(U) \times \mathrm{E}_q(V) \\ \downarrow \oplus_{'q'} & & \downarrow \\ U \oplus_{'q'} V & \xrightarrow[\mathrm{E}_q]{} & \mathrm{E}_q(U \oplus_{'q'} V) \end{array} . \tag{74}$$

With the following notation

$$T \equiv \mathrm{E}_q(\mathfrak{t}),\ G \equiv \mathrm{E}_q(\mathfrak{g}_q),\ P \equiv \mathrm{E}_q(\mathfrak{p}), \tag{75}$$

we have the following expression for a so-called q-symmetric space:

$$T = G/P, \tag{76}$$

where $\mathfrak{g}$ is a real form of a simple complex q-Lie algebra, $\mathfrak{g} = \mathfrak{t} \oplus_{'q'} \mathfrak{p}$, with $[\mathfrak{p}, \mathfrak{p}] \leq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{t}] \leq \mathfrak{t}$, and $[\mathfrak{t}, \mathfrak{t}] \leq \mathfrak{p}$.

5 Examples: $su_q(n)$, $SU_q(4)$, $SU_q(2)$ and $SO_q(4)$

5.1 The q-Lie algebra $su_q(n)$

We now turn to the q-Lie algebra $su_q(n)$. We note that our definition is different from the physicist notation [6]. First we define the q-trace.

Definition 36. The q-trace of an $n \times n$ matrix A is defined by

$$tr_q A \equiv A_{00} \oplus_q A_{11} \oplus_q \ldots \oplus_q A_{n-1,n-1},\ \text{where } \overline{n}_q \oplus_q -\overline{n}_q \sim \theta, \tag{77}$$

the zero of the alphabet.

Definition 37. Let $A^\star$ denote conjugated transpose. Then the q-Lie algebra $su_q(n)$ is defined by

$$su_q(n) \equiv \{A \in \mathbb{C}[\mathbb{Z}_q]^{(n,n)} | tr_q(A) \sim \theta,\ A^\star = -A^T\}. \tag{78}$$

Definition 38. The basis for $su_q(4)$ consists of the following 15 matrices:

$$\lambda_1 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_2 \equiv \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_3 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_4 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 \equiv \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_6 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_8 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\overline{2}_q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_9 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \lambda_{10} \equiv \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_{11} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \lambda_{12} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$
$$\lambda_{13} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \lambda_{14} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},$$
$$\lambda_{15} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\overline{3}_q. \end{pmatrix} \tag{79}$$

5.2 $SU_q(4)$

We now compute some of the matrices in $SU_q(4)$, all with q-determinant 1. The general formula is

$$U_{t;i;q} \equiv \mathrm{E}_q(\lambda_i t). \tag{80}$$

$$
\begin{aligned}
&U_{t;1;q} = \begin{pmatrix} \mathrm{Cos}_q t & \mathrm{Sin}_q t & 0 & 0 \\ -\mathrm{Sin}_q t & \mathrm{Cos}_q t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_{t;2;q} = \begin{pmatrix} \mathrm{Cos}_q t & i\mathrm{Sin}_q t & 0 & 0 \\ i\mathrm{Sin}_q t & \mathrm{Cos}_q t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&U_{t;3;q} = \begin{pmatrix} \mathrm{E}_q(t) & 0 & 0 & 0 \\ 0 & \mathrm{E}_q(-t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_{t;4;q} = \begin{pmatrix} \mathrm{Cos}_q t & 0 & \mathrm{Sin}_q t & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{Sin}_q t & 0 & \mathrm{Cos}_q t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&U_{t;5;q} = \begin{pmatrix} \mathrm{Cos}_q t & 0 & i\mathrm{Sin}_q t & 0 \\ 0 & 0 & 0 & 0 \\ i\mathrm{Sin}_q t & 0 & \mathrm{Cos}_q t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_{t;9;q} = \begin{pmatrix} \mathrm{Cos}_q t & 0 & 0 & \mathrm{Sin}_q t \\ 0 & \mathrm{Cos}_q t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{Sin}_q t & 0 & 0 & 0 \end{pmatrix}, \\
&U_{t;15;q} = \begin{pmatrix} \mathrm{E}_q(t) & 0 & 0 & 0 \\ 0 & \mathrm{E}_q(t) & 0 & 0 \\ 0 & 0 & \mathrm{E}_q(t) & 0 \\ 0 & 0 & 0 & \mathrm{E}_q(-t\overline{3}_q) \end{pmatrix}.
\end{aligned}
\tag{81}
$$

Note that there is a juxtaposition (multiplication) in $-t\overline{3}_q$.

Theorem 5.1. *For the q-Lie algebra $su_q(n)$, the matrix elements $-\overline{N-1}_q$ in the basis do not contribute to the commutator, because they will always be multiplied with zeros. Because of the bilinearity, this applies to all elements in $su_q(n)$.*

5.3 The q-Lie group $SU_q(2)$

In an earlier paper [2] we introduced a q-Lie group $SU_q(2)$, together with its maximal torus. We showed that $(SU_q(2), \cdot, \cdot_q)$ in the form of a maximal torus is closed under the two operations. Here we consider a slightly different form:

Definition 39. Let $\xi(q)$ denote the first zero > 0 of the function $\mathrm{Sin}_q(\psi)$, and let $\xi(q,2)$ denote the second zero > 0 of the function $\mathrm{Sin}_q(\psi)$. The general form for the q-Lie group $SU_q(2)$ is

$$
U_{\psi,\alpha} \equiv \begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi)\mathrm{E}_q(i\alpha) \\ \mathrm{Sin}_q(\psi)\mathrm{E}_q(-i\alpha) & \mathrm{Cos}_q(\psi)) \end{pmatrix}
\tag{82}
$$

for $0 \le \psi < \xi(q)$, $0 \le \alpha < \xi(q,2)$, with q-determinant 1.

Theorem 5.2. *Let the matrices A and B have matrix elements a_{ij} and b_{ij} respectively. Each element $U_{\psi,\alpha}$ has an inverse $U_{-\psi,\alpha}$ under the following matrix multiplication:*

$$
A \cdot_q B \equiv \begin{pmatrix} a_{00}\tau b_{00} + a_{01}\tau b_{10} & a_{00}b_{01} + a_{01}b_{11} \\ a_{10}b_{00} + a_{11}b_{10} & a_{10}\tau b_{01} + a_{11}\tau b_{11} \end{pmatrix},
\tag{83}
$$

which is a mixture of ordinary matrix multiplication and formula (36).

5.4 The q-Lie group $SO_q(4)$

We can now construct arbitrary q-Lie groups by the tensor product (61), we give one example:

Definition 40. The q-Lie group $SO_q(4)$ is defined by

$$SO_q(4) \equiv SU_q(2) \times SU_q(2). \tag{84}$$

6 q-linear system of equations

We will now describe a q-linear system of equations with Ward numbers by an unorthodox multiplication of Ward numbers by q-additions and unknown q-rational numbers, where additions and subtractions are replaced by the four q-additions.

Definition 41. Matrix elements will always be denoted (i, j). Here i denotes the row and j denotes the column. The matrix elements range from 1 to n.

Assume that the letters $\{\alpha_{i,j}\}_{i,j=1}^{n} \in (\mathbb{N}_{\oplus_q}, \oplus_q, \odot_q)$, and $x_1, \ldots, x_n \in \mathbb{Q}^{\star}_{\oplus_q}$. Furthermore, let the matrix of q-additions be given by $Y = \{y_{ij}\}_{i,j=1}^{n}$, where I denotes the unit operator. Let A be the $n\times 2n$-matrix with matrix elements $\alpha_{ij} \times y_{ij}$ and let X be the n-vector with matrix elements x_j. Then we define the matrix multiplication

$$A \cdot_q X_i \equiv \sum_{m=1}^{n} (\alpha_{i,m} \times y_{i,m}) X_m. \tag{85}$$

We assume that $y_{i,1} = \mathrm{I}$, $y_{i,j} \neq \mathrm{I}$, $j \neq 1$. The matrix element $(Y \cdot_q X)_i$ then denotes the vector

$$X_1, y_{i,2}X_2, \ldots, y_{i,n}X_n, \tag{86}$$

where $y_{i,m}$ denotes the respective q-addition, which precedes X_m.

Note that the sign of letters is included in $y_{i,m}$, so that we do not need $\mathbb{Z}_{\oplus_q}$.

Theorem 6.1. *Assume that the letters $\beta_1, \ldots, \beta_n \in (\mathbb{N}_{\oplus_q}, \oplus_q, \odot_q)$, and let B be the n-vector with components β_j. Furthermore, assume that the coefficient matrix for $q = 1$ is nonsingular, $det\{A_{i,j}\}_{i,j=1}^{n}|q = 1 \neq 0$. Then the q-linear system of equations*

$$A \cdot_q X \sim B \tag{87}$$

has $\prod_{k=2}^{n} k^2$ n-vector solutions in $\mathbb{Q}^{\star}_{\oplus_q}$, counted with multiplicity.

Proof. These solutions correspond to the rational solution of the corresponding linear system of equations. To find one solution, use Gaussian elimination by formulas (28) and (29), using the four q-additions between the respective equations, and by (25) two JHC become an NWA. The number of possible solutions is then found by the multiplication principle. □

To explain this we give an example.

Example 5. Give one solution to the q-linear system of equations

$$\begin{cases} x \oplus_q z \boxminus_q y \sim \overline{4}_q, \\ \overline{2}_q x \oplus_q z \boxminus_q \overline{3}_q y \sim \overline{1}_q, \\ \overline{3}_q x \oplus_q \overline{2}_q y \sim \overline{7}_q. \end{cases} \tag{88}$$

The corresponding linear system of equations has solution $x = 1, y = 2, z = 5$. We solve for y by the third equation using (29) to give

$$y \sim \frac{\overline{7}_q \boxminus_q \overline{3}_q x}{\overline{2}_q}. \tag{89}$$

By (89), again using (29), we obtain

$$\begin{cases} x \oplus_q z \boxminus_q \frac{\overline{7}_q \boxminus_q \overline{3}_q \cdot x}{\overline{2}_q} \sim \overline{4}_q \\ \overline{2}_q x \oplus_q z \boxminus_q \overline{3}_q \cdot \frac{\overline{7}_q \boxminus_q \overline{3}_q \cdot x}{\overline{2}_q} \sim \overline{1}_q. \end{cases} \tag{90}$$

Now compute the first equation $\boxminus_q$ the second using (28) and (25).

$$\boxminus_q x \oplus_q \overline{7}_q \boxminus_q \overline{3}_q \cdot x \sim \overline{3}_q \Leftrightarrow x \sim \overline{1}_q. \tag{91}$$

Formula (89) then gives

$$y \sim \frac{\overline{4}_q}{\overline{2}_q}. \tag{92}$$

Finally, the first formula (90) gives

$$z \sim \overline{3}_q \oplus_q \frac{\overline{4}_q}{\overline{2}_q}. \tag{93}$$

7 Discussion

In this article we have combined two mathematical subjects, which are ubiquitous in theoretical physics, compare [6], [3]. We note furthermore that these considerations can be extended to complex reflection groups and hyperplanes, where similar matrices occur. These interesting subjects, as well as q-differential geometry and q-symmetric spaces will be covered in forthcoming articles.

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