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# Topics in Five Dimensional Supersymmetry and Topological Quantum Field Theories

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### **Abstract**

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We explore five-dimensional supersymmetric quantum field theories and topological quantum field theories. In five dimensions, we describe a framework to consistently define a supersymmetric theory on a curved spacetime manifold. This is the starting point for the so called supersymmetric localization procedure which we exploit to compute the partition function on a large class of five-dimensional backgrounds. We discuss the geometric structure behind our result and analyze various extension of this program. Next, we present a seven-dimensional bulk topological field theory of abelian three-form potentials with a single derivative Chern-Simons-like action. We show how the edge modes can be understood from the perspective of a six-dimensional generalization of the fractional quantum Hall effect and compute an analog of the Laughlin wavefunction. We also explain why this has interesting implications for the study of six-dimensional superconformal field theories. Finally, we review a new approach to integrable lattice models in 1+1 dimensions based on a four-dimensional Chern-Simons-like theory and propose a strategy for its string theory embedding.

*Keywords:* Supersymmetric Quantum Field Theories, Topological Quantum Field Theories, String Theory.

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*Dedicated to my parents*

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# List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I J. Qiu, L. Tizzano, J. Winding and M. Zabzine, “Gluing Nekrasov partition functions,” *Commun. Math. Phys.* **337** (2015) no. 2, 785, arXiv:1403.2945 [hep-th].
- II J. Qiu, L. Tizzano, J. Winding and M. Zabzine, “Modular properties of full 5D SYM partition function,” *JHEP* **1603** (2016) 193, arXiv:1511.06304 [hep-th].
- III J. J. Heckman and L. Tizzano, “6D Fractional Quantum Hall Effect,” arXiv:1708.02250 [hep-th].

Papers not included in the thesis.

- IV L. Tizzano and J. Winding, “Multiple sine, multiple elliptic gamma functions and rational cones,” arXiv:1502.05996 [math.CA].
- V F. Apruzzi, G. Dibitetto and L. Tizzano, “A new 6d fixed point from holography,” *JHEP* **1611** (2016) 126 arXiv:1603.06576 [hep-th].
- VI F. Apruzzi, J. J. Heckman, D. R. Morrison and L. Tizzano, “4D Gauge Theories with Conformal Matter,” arXiv:1803.00582 [hep-th].

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# 1. Introduction

The overarching themes of this thesis are the characterization of non-perturbative aspects of supersymmetric quantum field theories and the study of topological quantum field theories. In particular, we will show how these two subjects are not separate entities but instead have many interesting connections which influenced various recent developments in theoretical physics.

One of the most intriguing features in the study of supersymmetric field theories is that certain physical quantities can be computed *exactly*. Indeed, being able to identify a class of observables which are invariant under a subset of supersymmetries in the problem yields powerful tools to constrain their non-perturbative dynamics. This strategy can be used to compute a large class of observables such as superpotentials, partition functions, BPS indices and chiral rings and to use them for sharpening our understanding of quantum field theories beyond perturbation theory.

Guided by this principle, the initial part of this work is devoted to the analysis of *supersymmetric localization* for 5D gauge theories. The discovery of consistent quantum field theories in five and six dimensions is one of the most important predictions of string theory [1, 2, 3]. These objects are isolated high energy fixed points of the RG flow and as such do not admit a weakly coupled description. As a consequence, developing exact results in five dimensions goes beyond the realm of traditional quantum field theory and gives us a useful way to actually test predictions from string theory.

Supersymmetric localization techniques were first introduced in the context of topological quantum field theories with the goal of obtaining a physical description of four-manifolds invariants [4]. A crucial element in this construction is the so called topological twisting which takes a supersymmetric quantum field theory and mixes part of its internal symmetries. The twisted theory can be described by a set of redefined fields which underlies the geometrical structure of the twist. Therefore, twisted topological quantum field theories have a valid description for any curved Riemannian manifold  $\mathcal{M}$  and their infinite-dimensional path integral reduces to a finite sum over localized fields configurations.

In chapter 2 we explain why topological twisting is *not* the only possible way to consistently define a supersymmetric theory on curved spacetime.

As a result, localization can be now applied to *physical* supersymmetric quantum field theories. A famous result in modern supersymmetric localization is the exact evaluation of the partition function for a 4D  $\mathcal{N} = 2$  theory on a four-sphere  $S^4$  [5]. Since then, a large body of work extended this calculation to different supersymmetric theories with various amounts of supersymmetries and different dimensions.

Both papers I and II are focussed on the study of 5D  $\mathcal{N} = 1$  supersymmetric gauge theories defined on a large class of five-manifolds called toric Sasaki-Einstein manifolds. These manifolds have been known for quite some time in the string theory literature as their metric cone is Calabi-Yau. In order to compute the partition function on such backgrounds we made extensive use of the toric action. The final result exhibits a peculiar factorization property which was thoroughly analyzed. In particular, we showed how the partition function receives contributions uniquely from elementary degrees of freedom concentrated along the fixed points of the toric action and how these can be organized in holomorphic building blocks. The results of II generalize this setup to allow for five-dimensional Abelian instantons contributions. This hints to the possibility of factorization being a robust property which also holds at a full non-perturbative level.

Let us now highlight a few interesting examples of potential applications of these results. Reducing the five-dimensional theory along a circle fiber of the Sasaki-Einstein manifold opens up the possibility of calculating new supersymmetric partition functions for 4D toric surfaces. A more ambitious goal is to probe the UV behavior of these theories and their relationship with 6D superconformal field theories. A characteristic feature of these theories is the presence of tensionless strings in their low-energy spectrum. Quite interestingly, we can consider 6D  $(1, 0)$  superconformal field theories compactified on a circle and relate the elliptic genus of their instantonic strings with the holomorphic blocks obtained in the 5D localization setup.

The study of six-dimensional superconformal field theories is also very influential for topological quantum field theories. Six-dimensional tensionless strings carry a lattice of string charges and an associated pairing which can be classified and have a precise geometrical meaning. The goal of paper III was to describe such topological data from the perspective of a 7D topological quantum field theory of abelian three-forms potentials. An interesting byproduct of our analysis is that the appearance of continued fractions in the spectrum of string excitations is interpreted much like filling fractions above the ground state in the fractional quantum Hall effect. This approach can be further extended by considering a top-down approach which recovers the 7D TQFT directly from M-theory. This provides a unified description for the low energy dynamics of a

multitude of topological systems in various dimensions, characterized by similar discrete data encoded in the compactification geometry.

The final chapter of this thesis is mostly a review of recent developments in the study of Chern-Simons theory [6]. One of the original motivation behind the study of such theory was to explain the fascinating connections between 1+1D integrable lattice models, quantum groups and knot theory. Integrable scattering processes are labeled by a complex “spectral parameter” which appears explicitly in the  $R$ -matrix. Unfortunately, it does not seem possible to derive the spectral parameter using 3D Chern-Simons theory. This problem has been solved thanks to a new 4D version of Chern-Simons theory giving geometrical origin to the spectral parameter. The novel topological quantum field theories has many surprising aspects which can only be fully explained by a non-perturbative formulation. Finally, we describe why string theory is the perfect tools for addressing all these aspects.

## 2. Supersymmetry on a Curved Background

In this chapter we give a general overview of the systematic framework for constructing and analyzing supersymmetric field theories on curved spacetime manifolds. We follow closely [7].

### 2.1 Background Fields and Conserved Currents

In the study of quantum field theory (QFT) it is often very useful to introduce non-dynamical sources or background fields. In this way, the effect of symmetries can be analyzed by assigning new transformation rules to these fields. For supersymmetric quantum field theories this strategy has interesting consequences as the new fields are assigned to a supersymmetric multiplet. As a result, it is possible to keep track of their effect on protected or BPS quantities. The most famous example of this kind was obtained in a groundbreaking paper [8]. There, a  $\mathcal{N} = 1$  nonrenormalization theorem was derived as a consequence of the effective superpotential being a holomorphic function of the coupling constants contained in a background chiral superfield. Using this theorem it is possible to derive *exactly* the superpotential for a vast class of theories as reviewed in a classic set of lectures [9].

Background fields couple to existing degrees of freedom through conserved currents. For example, a background gauge field  $V_\mu$  couples to a conserved U(1) current  $J_\mu$  as follows:

$$\Delta\mathcal{L} = V^\mu J_\mu + \mathcal{O}(V^2). \quad (2.1)$$

The  $\mathcal{O}(V^2)$  are needed to enforce current conservation. A different option is to study how the stress-energy momentum tensor  $T_{\mu\nu}$  couples to a background spacetime metric  $g_{\mu\nu}$ . For a flat Euclidean space we have  $g_{\mu\nu} = \delta_{\mu\nu}$  and under a small deformation of the metric  $g_{\mu\nu} \rightarrow g_{\mu\nu} + \Delta g_{\mu\nu}$  we have:

$$\Delta\mathcal{L} = \Delta g^{\mu\nu} T_{\mu\nu} + \mathcal{O}(\Delta g^2). \quad (2.2)$$

Again, after tuning the  $\mathcal{O}(\Delta g^2)$  terms, the Lagrangian is invariant under diffeomorphisms acting on the background metric and as such it can be studied on an arbitrary Riemannian manifold  $\mathcal{M}$ . For a given QFT on  $\mathcal{M}$  it is very interesting to compute its partition function:

$$Z_{\mathcal{M}}[V_\mu, g_{\mu\nu}, \dots] = \int [D\Phi] e^{-S_{\mathcal{M}}[\Phi, V_\mu, g_{\mu\nu}, \dots]}, \quad (2.3)$$

where the ellipses represent extra background fields to be considered, indeed these extra fields play a crucial role for supersymmetric theories. There are a number of difficulties in computing (2.3). First, there is an IR divergence which can be cured by choosing  $\mathcal{M}$  to be a compact manifold<sup>1</sup>. At the same time, there is also a UV divergence for which a short-distance cutoff is needed. For UV complete theories we only need to introduce a finite number of new terms. The scheme independent part of  $Z_{\mathcal{M}}$  is very important as it can be used to calculate various correlation functions of local operators on  $\mathcal{M}$ . For topological quantum field theories (TQFTs) the partition function can also be used to compute correlation functions of non-local observables. A typical example is 3D Chern-Simons where there are no local observables but the partition function on  $\mathcal{M}$  still captures useful information [6].

## 2.2 Supersymmetric Partition Function

For interacting QFTs the partition function (2.3) cannot be computed exactly. The best option would be to consider a supersymmetric extension of this calculation where it is known that BPS observables can be understood non-perturbatively. Unfortunately coupling a supersymmetric theory to a background metric  $g_{\mu\nu}$  leads to complete supersymmetry breaking. A simple argument shows that a coupling like (2.2) would be prohibited in a supersymmetric theory as the stress-energy tensor is not a BPS observable for every supercharge  $Q$ , i.e.  $[Q, T_{\mu\nu}] \neq 0$ . In order to preserve rigid supersymmetry on a curved background  $\mathcal{M}$  we need to solve the following differential equation for every curved space supercharge:

$$\nabla_{\mu}\zeta = 0. \tag{2.4}$$

Solutions to (2.4) are called covariantly constant spinors  $\zeta$  on  $\mathcal{M}$ . There are very few examples of interesting compact manifolds satisfying this condition. In four dimensions, these are  $T^4$  and  $K3$  surfaces. Luckily, there is a clever way to find a larger set of supersymmetric backgrounds that was discovered in [10]. This new approach is clearly inspired by [8] and the role of extra background fields will be crucial here. As a result, it is still possible to think about a consistent supersymmetric theory on a curved space  $\mathcal{M}$  with a supersymmetric Lagrangian  $\mathcal{L}_{\mathcal{M}}$ .

A properly defined supersymmetric theory on  $\mathcal{M}$  opens up new possibilities for computing  $Z_{\mathcal{M}}$ . By now there is a huge body of literature devoted to the so called supersymmetric localization procedure which

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<sup>1</sup>Generally this is not enough for  $Z_{\text{IR}}$  to be finite as the theory can still have bosonic zero modes.

has the evaluation of  $Z_{\mathcal{M}}$  as one of its fundamental goals. A very detailed review on these aspects can be found at [5]. Let us briefly review the localization argument. Consider a supersymmetric theory  $\mathcal{L}_{\mathcal{M}}$  invariant under a nilpotent<sup>2</sup> supercharge  $Q$ , i.e.  $Q^2 = 0$ . The path integral expression for  $Z_{\mathcal{M}}$  can be deformed in a supersymmetric way by:

$$Z_{\mathcal{M}}(t) = \int [\mathcal{D}\Phi] e^{-S_{\mathcal{M}} + t \delta S_{\mathcal{M}}}, \quad (2.5)$$

where  $\delta S_{\mathcal{M}} = \{Q, \mathcal{O}\}$  for some fermionic operator  $\mathcal{O}$ . It can be shown that  $Z_{\mathcal{M}}(t)$  does not depend on  $t$  as the deformation is Q-exact<sup>3</sup>:

$$\frac{d}{dt} Z_{\mathcal{M}}(t) = \langle \{Q, \mathcal{O}\} \rangle = 0. \quad (2.6)$$

Notice that  $Z_{\mathcal{M}} = Z_{\mathcal{M}}(0)$  can be computed for any value of  $t$ . In particular, the limit  $t \rightarrow \infty$  is particularly interesting as it allows for a saddle point approximation. After choosing an appropriate  $\mathcal{O}$ , the infinite-dimensional path integral (2.5) is then reduced to finite sum over semiclassical saddle configurations in the deformed theory.

Historically, the first realization of curved supersymmetry was established through topological twisting [4]. This will be discussed in chapter 4. In a supersymmetric theory with a continuous R-symmetry  $G_R$  we can define a supercharge  $Q$  which is a singlet under the diagonal embedding  $(G_R \times G_{\text{hol}})|_{\text{diag}}$ , where  $G_{\text{hol}}$  denotes the Riemannian holonomy group of  $\mathcal{M}$ . Two paradigmatic examples are the topological twist of 4D  $\mathcal{N} = 2$  theories [4] and the A and B twist of 2D  $\mathcal{N} = (2, 2)$  theories [12, 13] which led to revolutionary results in mathematical physics. Although not manifest, it can be shown that the partition function of twisted topological field theories is independent of the metric. Since BPS observables are elements of Q-cohomology, these theories are also referred to as cohomological field theories to distinguish them from the more traditional topological theories of Chern-Simons type.

More recently there has been a lot of interest in two backgrounds which lie outside the original description of topological twisting. First, the  $\Omega$ -background introduced in [14, 15] which is an equivariant deformation of  $\mathbb{R}_{\epsilon_1, \epsilon_2}^4$  rotating the two orthogonal  $\mathbb{R}^2$  planes. Finally, the  $S^4$  background of [16], exhibiting a  $\text{OSp}(2|4)$  algebra, which was the first comprehensive example of supersymmetric localization in the modern sense. Both these supersymmetric backgrounds are best understood using the procedure outlined in the following section.

<sup>2</sup>More generally [5], the nilpotent condition can be relaxed to a  $Q$  which squares to a Killing vector on  $\mathcal{M}$ .

<sup>3</sup>There are some unfortunate situations where the path integral does not converge fast enough and the equality is no longer valid [11].

## 2.3 General Overview

A general prescription for analyzing curved supersymmetric backgrounds in a systematic way has been pioneered in [10]. The stress-energy tensor is what causes problems when it is coupled to a background metric. Recall that, in a supersymmetric theory  $T_{\mu\nu}$  sits in a supersymmetric multiplet together with bosonic and fermionic partners which we denote respectively by  $\mathcal{J}_B^i$  and  $\mathcal{J}_F^i$ . At this stage, some background bosonic fields  $\mathcal{B}^i$  could be introduced through the following coupling:

$$\Delta\mathcal{L} = \Delta g^{\mu\nu} T_{\mu\nu} + \sum_i \mathcal{B}^i \mathcal{J}_B^i + \dots, \quad (2.7)$$

the ellipses here being higher order background field terms. A crucial point is that for specific choices of  $\Delta g^{\mu\nu}$  and  $\mathcal{B}^i$  there can still be some fraction of supersymmetry preserved by  $\Delta\mathcal{L}$ . This is due to cancellations between supersymmetry variations of  $T_{\mu\nu}$  and  $\mathcal{J}_B^i$ .

As explained in [10], the best way to understand this problem is obtained by constructing a new supermultiplet containing both  $g_{\mu\nu}$  and  $\mathcal{B}^i$ . The background fermionic partners  $\mathcal{B}_F^i$  are set to zero in (2.7). An important remark is that all the sources are considered as part of an *off-shell* supergravity multiplet. In particular the values of auxiliary fields are kept arbitrary and their equations of motions are not solved. Their coupling to various supermultiplets is performed using a formulation of supergravity known as linearized supergravity, i.e. an expansion at leading order in  $\frac{1}{M_P}$  in Planck mass. See [17] for an introduction to this approach. Since the background auxiliary fields are kept non-dynamical, the approach of [10] has been also referred to as a rigid limit of off-shell supergravity, where the gravitational degrees of freedom are frozen in the limit where  $M_P \rightarrow \infty$ .

In order to preserve supersymmetry, the variations of fermionic background sources  $\mathcal{F}^i$  related to  $\mathcal{J}_F^i$  should satisfy:

$$\delta_Q \mathcal{F}^i = 0. \quad (2.8)$$

It is quite interesting to analyze various consequences of the above equations. When the supersymmetric theory is coupled to off-shell supergravity, the fermionic sources  $\mathcal{J}_F^i$  always include at least one gravitino  $\Psi_{\mu\alpha}$ . Imposing (2.8) corresponds to solving a new differential equation for the spinor  $\zeta$ :

$$\nabla_\mu \zeta + \dots = 0, \quad (2.9)$$

as follows from the usual form of supersymmetric variations that a gravitino enjoys. Equation (2.9) is an extension of (2.4) known as generalized Killing spinor equation. Notice that a given curved supersymmetric configuration might solve (2.9) for just a subset of the original supercharges  $Q$ .

A supersymmetric field theory might admit multiple stress-tensor supermultiplet descriptions that could couple to different off-shell supergravities leading to multiple choices of curved supersymmetric backgrounds. Interestingly, the background metric does not uniquely specify a curved supersymmetric background as there can be inequivalent choices of background bosonic fields for the same geometry. In some sense, background bosonic fields retain all the most important physical information regarding a background as they influence the evaluation of the partition function. Moreover, since the auxiliary fields are non-dynamical there is more freedom in setting their values, in particular when computing partition functions it might be useful to give them complex values.

Finally, note that from the point of view of flat space supersymmetry the curved space Lagrangian is always ambiguous. Indeed, as remarked in [10], there can be terms coupled to powers of the curvature or to scale factors in the metric. The key point is that it is only important to specify all the terms that *must* be included to a flat space Lagrangian to preserve supersymmetry. In the next chapter we will study how to do so for a five-dimensional supersymmetric theory.



### 3. Five-Dimensional Theories

In this chapter we first review general aspects of 5D supersymmetric theories. Then, we describe how to obtain rigid supersymmetric backgrounds with a special focus on Sasaki-Einstein manifolds. Finally we discuss how to compute the supersymmetric partition function on such backgrounds.

#### 3.1 Supersymmetry in Five Dimensions

One of the most surprising results in string theory is the discovery of qualitatively new quantum field theories in five dimensions. Indeed, a traditional approach would suggest to rule out the study of 5D gauge theories as these are non-renormalizable systems flowing to free fixed points at low energies. However, supersymmetric gauge theories in five dimensions can have non trivial UV superconformal fixed points from which they flow out. As demonstrated in a classic set of works [18, 19, 20, 21] a combination of field and string theory arguments leads to the discovery of a large class of such fixed points. From the point of view of string theory it is quite obvious that these models enjoy UV global symmetry enhancement, while it is much more subtle to understand why this is the case using QFT arguments<sup>1</sup>.

An instructive example is the  $\mathcal{N} = 1$  SU(2) theory with  $N_f \leq 7$  flavors exhibiting a global symmetry enhancement to  $E_{N_f+1}$ . This can be engineered using M-theory on a noncompact Calabi-Yau threefold  $X$  with a divisor  $D$  collapsing to a point. In this example, the local geometry of  $D$  is that of a del Pezzo surface  $dP_k$  i.e.  $\mathbb{P}^2$  blown-up at  $0 \leq k \leq 8$  points. For  $k \geq 1$  the number of blow-ups is related to the number of flavors  $N_f = k - 1$  in the SU(2) theory and there is a perfect matching of the symmetry enhancement pattern. When  $N_f = 8$  the situation changes drastically as the UV completion is no longer a truly 5D SCFT but a 6D (1,0) E-string theory on a circle [23]. Using string dualities, it is also

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<sup>1</sup> One of the hallmark of 5D supersymmetric theories is the presence of an unusual conserved current  $J_\mu = \star(F \wedge F)$  giving rise to a topological  $U(1)_I$  symmetry. Under this symmetry there are charged non-perturbative states called “instantons”. Quite often, flavor symmetries combine with the  $U(1)_I$  symmetry to generate an even larger UV global symmetry [22].

possible to study maximally supersymmetric 5D  $\mathcal{N} = 2$  theories with any gauge group  $G$ . In this case it has been shown that the only possible completion is a 6D  $(2, 0)$  theory compactified on a circle [24, 25]. The set of 5D fixed points can be further extended by allowing  $(p, q)$ -fivebrane webs and related quiver constructions [26].

Five-dimensional superconformal field theories admit supersymmetric relevant deformations that allow us to study them with weakly coupled supersymmetric Lagrangians of interest in this thesis. As remarked in section 2.1, BPS observables such as indices and supersymmetric partition functions are protected along the RG flow and for this reason there is a great deal of interest in calculating them exactly [27, 28, 29, 30, 31, 32].

The low energy massless degrees of freedom for 5D  $\mathcal{N} = 1$  supersymmetry consists of vector multiplets with a gauge algebra  $G$  and hypermultiplets in a representation  $\mathbf{R}$  of  $G$ . Interestingly, vector multiplets have *real* scalars  $\phi^i$  that upon reduction to four dimensions combines with the fifth component of the gauge fields  $A_5^i$  to make  $\phi^i$  complex as in 4D  $\mathcal{N} = 2$  vector multiplets. All these theories have interesting Coulomb branches parametrized by expectation values of  $\phi^i$  in the Cartan subalgebra of  $G$ . For this reason the Coulomb branch has dimension equals to  $r = \text{rank}(G)$ . There can also be Higgs branches where the hypermultiplets scalars can vary, these are hyper-Kähler manifolds. The most general Lagrangian on the Coulomb branch is derived from a prepotential  $\mathcal{F}(\mathcal{A}_i)$  which is (locally) a function of vector superfields  $\mathcal{A}_i$ :

$$\mathcal{F} = h_{ij} \mathcal{A}^i \mathcal{A}^j + d_{ijk} \mathcal{A}^i \mathcal{A}^j \mathcal{A}^k, \quad (3.1)$$

with reality constraints that fix  $h_{ij}$  and  $d_{ijk}$  to be real constants. Moreover, because of gauge invariance  $d_{ijk}$  is further restricted to be integrally quantized [21]. Now, at a generic point on the Coulomb branch the gauge group is broken to its maximal torus  $U(1)^r$  and the low energy dynamics is governed by an Abelian theory. The full 1-loop exact quantum prepotential is a cubic polynomial of  $\phi^i$  given by [21]:

$$\mathcal{F} = \frac{1}{2g^2} h_{ij} \phi^i \phi^j + \frac{k}{6} d_{ijk} \phi^i \phi^j \phi^k + \frac{1}{12} \left( \sum_{e \in \text{Root}} |e \cdot \phi|^2 + \sum_j \sum_{w \in \mathbf{R}_j} |w \cdot \phi + m_j|^3 \right), \quad (3.2)$$

where  $\mathbf{R}_j$  denotes the set of weights for a  $j$ -th hypermultiplet in representation  $\mathbf{R}$  of  $G$ ,  $h_{ij} = \text{Tr}(T_i T_j)$  and  $d_{ijk} = \text{Tr}_{\mathbf{F}}(T_i \{T_j, T_k\})$  where  $\mathbf{F}$  denotes the fundamental representation. Notice that for a single hypermultiplet in adjoint representation there are no cubic terms in the prepotential. The first two terms in (3.2) follow from (3.1) while the

last two terms are generated at the quantum level by integrating out charged fermions on the Coulomb branch [33]. The 1-loop contributions give rise to a renormalized effective gauge coupling obtained from the prepotential:

$$\frac{1}{(g_{\text{eff}}^2)_{ij}} = \partial_i \partial_j \mathcal{F}. \quad (3.3)$$

From here it is also possible to obtain the exact metric on the Coulomb branch:

$$ds^2 = (g_{\text{eff}}^{-2})_{ij} d\phi^i d\phi^j. \quad (3.4)$$

By analyzing the structure of the prepotential (3.2) it is possible to attempt a classification of low-energy 5D gauge theories which might admit a 5D SCFT fixed point [21]. The main logic behind this classification is that for a 5D gauge theory with a well defined UV completion it should be possible to reach the most singular point of the Coulomb branch while maintaining the metric positive semidefinite. A confusing aspect in this strategy is that the list of possible candidates meeting this requirement seems to be very short as the only possibilities are theories with a single gauge group factor and a strict upper bound on flavor number. For example this classification rules out quiver theories with  $G = \prod_i G_i$  which are very common examples of SCFTs engineered in string theory [26]. A resolution for this apparent paradox has been put forward only very recently [34, 35]. The key loophole is that there can be regions of the moduli space where  $g_{\text{eff}}^{-2}$  becomes formally negative, and before reaching such regions effective field theory arguments break down and are no longer valid. There is a clear way to understand this phenomenon from a geometrical point of view related to the Kähler cone of Calabi-Yau 3-folds used to engineer 5D theories [19]. Clearly, having a very large set of 5D SCFTs is a very motivating aspect for developing new quantum field theory tools in 5D as they might become essential to further elucidate their microscopic aspects.

## 3.2 Sasaki-Einstein Background

Motivated by chapter 2, it is natural to study how to preserve curved supersymmetry for five-dimensional theories. This was developed in a sequence of papers [36, 37, 38] which we follow closely. The first step consists in identifying a five-dimensional stress-tensor supermultiplet. Notice that in 5D there is no complete classification of supersymmetric stress-tensor multiplets. The simplest known option is a five-dimensional version of the Sonnius multiplet, usually discussed in the context of 4D

$\mathcal{N} = 2$  supersymmetry<sup>2</sup>:

$$(C, \psi_\alpha^i, X^{ij}, W_{\mu\nu}, R_\mu^{ij}, S_{\mu\alpha}^i, J_\mu, T_{\mu\nu}). \quad (3.5)$$

In this notation  $i, j$  are  $SU(2)_R$  indices,  $\mu, \nu$  spacetime indices and  $\alpha$  spinorial indices. The corresponding background fields are part of an off-shell supergravity multiplet first considered in [39, 40]:

$$(K, \eta_\alpha^i, t^{ij}, \mathcal{V}_{\mu\nu}, (V_\mu)^{ij}, \Psi_{\mu\alpha}, \mathcal{A}_\mu, g_{\mu\nu}). \quad (3.6)$$

The supersymmetric values of background fields needed for preserving supersymmetry are found by setting to zero the supersymmetric variations of fermionic fields in the supergravity multiplet. Following the notation of chapter 2 we denote the supersymmetry parameter by  $\zeta$ , in this way the supersymmetric variations read:

$$\delta_\zeta \Psi = 0, \quad \delta_\zeta \eta = 0. \quad (3.7)$$

The first of these equations is called *gravitino* equation while the second is known as *dilatino* equation. They imply two differential equations for the bosonic background fields in (3.6). Crucial information on the background is contained in the gravitino equation as it implies the Killing spinor equation discussed previously<sup>3</sup>:

$$\nabla_\mu \zeta_i = t_i^j \Gamma_\mu \zeta_j + \mathcal{F}_{\mu\nu} \Gamma^\nu \zeta_i + \frac{1}{2} \mathcal{V}^{\nu\rho} \Gamma_{\mu\nu\rho} \zeta_i, \quad (3.8)$$

where  $\mathcal{F}$  is the field strength for  $\mathcal{A}$  in (3.6) and  $\nabla_\mu$  also contains the  $SU(2)_R$  gauge field  $V_\mu$ . At this point it is possible to analyze systematically this equation and to attempt a classification of all the possible backgrounds admitting 5D supersymmetry. This was the ultimate goal of [36], but unfortunately a complete classification of all five-dimensional backgrounds is not yet available. For the purposes of this thesis this is not worrisome as we are interested in one of the simplest known solutions to this equation. Indeed the focus of I and II is on Sasaki-Einstein<sup>4</sup> backgrounds  $\mathcal{M}$  which can be obtained by the following choice of background fields:

$$V = \mathcal{F} = \mathcal{V} = 0, \quad t_i^j = \frac{i}{2r} (\sigma_3)_i^j, \quad (3.9)$$

<sup>2</sup>Actually, it is not known whether there are other interesting examples. I would like to thank Guido Festuccia for discussions on this point.

<sup>3</sup>The dilatino equation is not discussed here as it is only needed for demanding the closure of rigid supersymmetric algebra. Interested readers might consult [36] for additional details.

<sup>4</sup>Consider a metric cone over a five-manifold  $\mathcal{M}$  defined as  $C(\mathcal{M}) = \mathcal{M} \times \mathbb{R}_+$  with  $g_{C(\mathcal{M})} = dt^2 + t^2 g_{\mathcal{M}}$  and  $t$  a local coordinate along  $\mathbb{R}_+$ . If  $C(\mathcal{M})$  is Kähler then  $\mathcal{M}$  is said to be Sasaki. If  $C(\mathcal{M})$  is, in addition, Calabi-Yau then  $\mathcal{M}$  is said to be Sasaki-Einstein. Detailed accounts on Sasakian geometry can be found at [41, 42].

where  $r$  is the overall scale factor of the background and  $\sigma_3 = \text{diag}[1, -1]$ . With this choice the Killing spinor equation is significantly simplified:

$$\nabla_\mu \zeta_i = \frac{i}{2r} (\sigma_3)_i^j \Gamma_\mu \zeta_j. \quad (3.10)$$

As described in [43], out of the solutions to (3.10) it is possible to extract an important geometric quantity known as the *Reeb* vector field:

$$R^\mu = \zeta_i \Gamma^\mu \zeta^i. \quad (3.11)$$

Notice that  $R$  squares to 1 and it is a Killing vector field. The metric dual of this object is known as *contact* one-form:

$$\kappa_\mu = g_{\mu\nu} R^\nu. \quad (3.12)$$

At this stage, curved supersymmetric variations for a 5D vector multiplet can be introduced in the following way:

$$\begin{aligned} \delta A_\mu &= i \zeta_i \Gamma_\mu \lambda^i, \\ \delta \sigma &= i \zeta_i \lambda^i, \\ \delta \lambda_i &= -\frac{1}{2} (\Gamma^{\mu\nu} \zeta_i) F_{\mu\nu} + (\Gamma^\mu \zeta_i) D_\mu \sigma - \zeta^j D_{ji} + 2t_i^j \zeta_j \sigma, \\ \delta D_{ij} &= -i \zeta_i \Gamma^\mu D_\mu \lambda_j + [\sigma, \zeta_i \lambda_j] + i t_i^k \zeta_k \lambda_j + (i \leftrightarrow j). \end{aligned} \quad (3.13)$$

The corresponding rigid supersymmetric invariant action on  $\mathcal{M}$  is given by:

$$\begin{aligned} S_{(5D, \text{Vec})} &= \frac{1}{g_{\text{YM}}^2} \int_{\mathcal{M}} \text{Tr} \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu \sigma D^\mu \sigma - \frac{1}{2} D_{ij} D^{ij} + 2\sigma t^{ij} D_{ij} \right. \\ &\quad \left. - 10t^{ij} t_{ij} \sigma^2 + i \lambda_i \Gamma^\mu D_\mu \lambda^i - \lambda_i [\sigma, \lambda^i] - i t^{ij} \lambda_i \lambda_j \right]. \end{aligned} \quad (3.14)$$

A very similar analysis can be carried out for a hypermultiplet, this was indeed performed in I. See also [43]. Without further ado, we move to illustrating the main logic behind localization computations on Sasaki-Einstein backgrounds.

### 3.3 Localization Results

Since the first paper on supersymmetric localization [16] it has been quite useful to rewrite the supersymmetric variations (3.13) into a cohomological form. This procedure makes the analogy with twisted topological theories more manifest and streamline the notation. The basic step is to introduce the following combinations:

$$\begin{aligned} \Psi_\mu &= \zeta_i \Gamma_\mu \lambda^i, \\ \chi_{\mu\nu} &= \zeta_i \Gamma_{\mu\nu} \lambda^i + R_{[\mu} \zeta_i \Gamma_{\nu]} \lambda^i. \end{aligned} \quad (3.15)$$

In this procedure  $\Psi_\mu$  becomes a differential 1-form and  $\chi_{\mu\nu}$  is a 2-form satisfying further conditions:

$$\begin{aligned}\iota_R\chi &= 0, \\ \iota_R(\star\chi) &= \chi.\end{aligned}\tag{3.16}$$

Using these new variables, the supersymmetric variations (3.13) can be expressed as:

$$\begin{aligned}\delta A &= i\Psi, \\ \delta\Psi &= -\iota_R F + D\sigma, \\ \delta\chi &= H, \\ \delta H &= -i\mathcal{L}_R^A\chi - [\sigma, \chi], \\ \delta\sigma &= -i\iota_R\Psi,\end{aligned}\tag{3.17}$$

where  $\mathcal{L}_R^A$  is the covariant Lie derivative along the Reeb vector field  $d_A\iota_R + \iota_R d_A$ . From (3.17) it is easy to verify that:

$$\delta^2 = -i\mathcal{L}_R^A + i\mathcal{G}_\sigma,\tag{3.18}$$

where  $\mathcal{G}_\sigma$  is an infinitesimal gauge transformation with parameter  $\sigma$ . The supersymmetric action can also be rewritten in a more transparent way:

$$S_{(5D, \text{Vec})} = S_{(5D, \text{Closed})} + \delta V,\tag{3.19}$$

where  $S_{(5D, \text{Closed})}$  is a  $\delta$ -closed topological term obtained by supersymmetrizing a 5D lift of Chern-Simons term known in the literature as  $CS_{3,2}(A)$  [27, 28]. The exact part of  $S_{(5D, \text{Vec})}$  is given by:

$$V = \int \text{Tr} \left[ \Psi \wedge \star(-\iota_R F - D\sigma) - \frac{1}{2}\chi \wedge \star H + 2\chi \wedge \star F + \sigma \kappa \wedge d\kappa \wedge \chi \right].\tag{3.20}$$

Localization requires a deformation term for the action that allows to express the partition function as in (2.5). The deformation term, denoted by  $\delta\mathcal{S}_{\mathcal{M}}$ , is usually called a localization term. In this particular example its bosonic part  $\delta\mathcal{S}_{\mathcal{M}}|_{\text{bos}}$  is written as:

$$\delta\mathcal{S}_{\mathcal{M}}|_{\text{bos}} = \text{Tr} \int \iota_R F \wedge \star(\iota_R F) - (D\sigma) \wedge \star(D\sigma) - \frac{1}{2}H \wedge \star H + 2F_H^+ \wedge \star H,\tag{3.21}$$

where  $F_H^+ = (1 + \iota_R\star)F$  is the horizontal self-dual part of  $F$  and identities (3.16) are used in the derivation. It is easy to see that  $\delta\mathcal{S}_{\mathcal{M}}|_{\text{bos}}$  comes from the bosonic part of  $\delta V|_{\text{bos}}$ , obtained from (3.20), after dropping the final term. The localization term can be further simplified as the auxiliary field  $H$  is not dynamical and can be integrated out. Moreover,

it is also necessary to Wick rotate the scalar field  $\sigma \rightarrow i\sigma$  to get a sum of positive square terms:

$$\delta\mathcal{S}_{\mathcal{M}}|_{\text{bos}} = \text{Tr} \int \iota_R F \wedge \star(\iota_R F) + (D\sigma) \wedge \star(D\sigma) + 2F_H^+ \wedge \star F_H^+. \quad (3.22)$$

The semiclassical saddles which contribute to the path integral are given by solutions to:

$$\iota_R F = 0, \quad F_H^+ = 0, \quad D\sigma = 0. \quad (3.23)$$

The most interesting equations in (3.23) are the first two which can be described by a unique equation:

$$\star F = -\kappa \wedge F. \quad (3.24)$$

In simple terms this is just a 5D lift of the famous 4D self-dual instanton equation and for this reason these solutions were dubbed *contact instantons* [27]. The moduli problem for this set of equations has not been studied in detail and it is an interesting open problem. For more details see [43]. As opposed to the famous example of  $S^4$  localization [16], in this setup it is not yet possible to establish rigorously that the only singular profile solutions of (3.23) are localized on *closed Reeb orbits*<sup>5</sup> of the Sasaki-Einstein background. Nevertheless here it is assumed that there are no other singular solutions. Motivating arguments come from 3D contact manifolds which have localized singular solutions as shown by [44] and from toric surfaces whose instanton partition function has been calculated in [45]. The smooth locus solutions are given by:

$$A = 0, \quad \sigma = ia, \quad (3.25)$$

where  $a$  is a constant. The classical part of the action evaluated around these configurations gives a Gaussian factor:

$$S_{(5D, \text{Vec})}(a) = -\frac{8 \text{Vol}(\mathcal{M})}{g_{\text{YM}}^2} \text{Tr}[a^2]. \quad (3.26)$$

The next step consists in linearizing the differential operator (3.18) around (3.25) in order to consider 1-loop contributions to the path integral. Physically, we would like to compute the functional determinant of  $\delta^2$  and take care of gauge fixing. This involves a number of technicalities that we will not explain here, we refer the reader to the original paper [16] or to [43] for additional details. For a 5D vector multiplet the perturbative partition function is given by:

$$Z_{(5D, \text{Vec})}^{\text{pert}} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3}{g_{\text{YM}}^2} \text{Tr}[a^2]} \det'_{\text{adj}} \text{sdet}_{\Omega_H^{(0, \bullet)}}(-i\mathcal{L}_R - \mathcal{G}_a), \quad (3.27)$$

<sup>5</sup>Let  $Y$  be a closed contact manifold, a closed Reeb orbit is defined as a  $T$ -periodic map  $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow Y$ , for some  $T > 0$ , for which  $\dot{\gamma}(t) = R(\gamma(t))$ .

where the integration is intended over the Cartan  $\mathfrak{t}$  of the gauge group. The superdeterminant of equation (3.27) should be computed over a space of differential forms with values in  $\mathfrak{g}$ . As explained in I, for  $\mathcal{M}$  a Sasaki-Einstein five-manifold, this reduces to the Kohn-Rossi differential complex  $\Omega_H^{(0,\bullet)}$ . Even though the space of differential forms  $\Omega_H^{(0,\bullet)}$  is infinite-dimensional there is still hope to compute (3.27) because of supersymmetry. As for other famous examples of localization, the cancellations between bosonic and fermionic degrees of freedom in (3.27) leads to a striking consequence: the only physical modes contributing to the 1-loop determinant are contained in the Kohn-Rossi *cohomology*<sup>6</sup>  $H_{\bar{\partial}_H}^{(0,p)}$ .

Let us now specialize to a *toric Sasaki-Einstein*  $\mathcal{M}$  having a  $G = \text{U}(1)^3$  isometry, in this setup the Reeb vector field is a linear combination of these  $\text{U}(1)$ 's. Moreover, it can be shown that the  $\bar{\partial}_H$ -complex is invariant under  $G$ . As a consequence,  $\bar{\partial}_H$ -cohomology is organized under representations of  $G$ :

$$H_{\bar{\partial}_H}^{(0,p)} = \bigoplus_i m_i^p R_i, \quad m_i^p \in \mathbb{Z}_{\geq 0}, \quad (3.28)$$

where  $m$ 's are multiplicities of the representations  $R_i$ . For  $G = \text{U}(1)^3$ , the representations are labelled by three charges organized into an integer valued 3-vector. A noticeable observation by [48], further corroborated by I, relates the Kohn-Rossi cohomology with a more familiar Dolbeault cohomology on the *Calabi-Yau cone* over  $\mathcal{M}$ , i.e. there is a new map such that:

$$H_{\bar{\partial}_H}^{(0,\bullet)}(\mathcal{M}) \rightarrow H_{\bar{\partial}}^{(0,\bullet)}(C(\mathcal{M})). \quad (3.29)$$

On a Calabi-Yau cone  $C(\mathcal{M})$  there are no holomorphic one-form to be counted i.e.  $H_{\bar{\partial}_H}^{(0,1)}(\mathcal{M}) = 0$  after restriction. Conversely, there are interesting contributions from *holomorphic functions* and in order to characterize them a few elements of toric geometry are needed<sup>7</sup>. Let  $\mu$  be the moment map for the three torus actions, then due to the cone structure on  $C(\mathcal{M})$ , the image of  $\mu$  will also be a cone in  $\mathbb{R}^3$  denoted by  $C_\mu(\mathcal{M})$ :

$$C_\mu(\mathcal{M}) = \{\vec{r} \in \mathbb{R}^3 | \vec{r} \cdot \vec{v}_i \geq 0, i = 1, \dots, k\}, \quad (3.30)$$

where  $v_i$ 's are inward pointing vectors of the  $k$  faces of  $C_\mu(\mathcal{M})$ . See figure 3.1.

Internal points of (3.30) are in one to one correspondence with holomorphic functions. More precisely, a point in  $C_\mu(\mathcal{M})$  of coordinates

<sup>6</sup>The Kohn-Rossi cohomology of  $\bar{\partial}_H$  is a restriction of the Dolbeault operator on the metric cone  $C(\mathcal{M})$  to its boundary  $\mathcal{M}$ . See [46, 47].

<sup>7</sup>See [49].



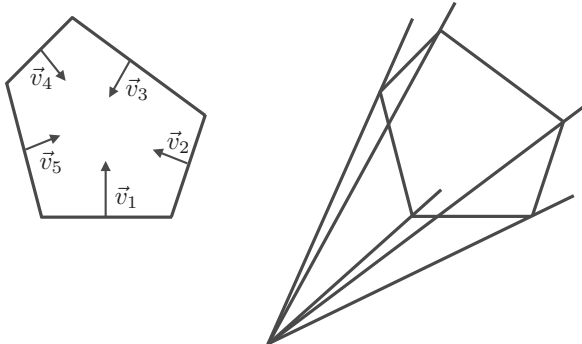


Figure 3.1. On the left a pentagonal base with inwards pointing vectors  $\vec{v}_i, i = 1, \dots, 5$ . On the right the corresponding moment cone.

$(n_1, n_2, n_3)$  is mapped to a monomial  $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ . As remarked above, a Reeb vector field acts on  $(z_1, z_2, z_3)$  with eigenvalues  $(R_1, R_2, R_3)$ , in particular each monomial has an eigenvalue given by  $\vec{n} \cdot R = n_1 R_1 + n_2 R_2 + n_3 R_3$ . For holomorphic two-form contributions, it is useful to exploit the Sasaki-Einstein condition which requires a vector  $\vec{\xi}$  such that:

$$\vec{\xi} \cdot \vec{v}_i = 1, \quad \forall i. \quad (3.31)$$

On  $C(\mathcal{M})$  there is a top holomorphic 3-form  $\Omega$  whose charges are given by  $\vec{\xi}$ . This implies that the eigenvalues of holomorphic  $(0, 2)$ -forms can be written as  $-(\vec{m} + \vec{\xi}) \cdot R$  for  $\vec{m} \in C_\mu(\mathcal{M})$ . Finally, it is now possible to combine all contributions to the superdeterminant as:

$$\text{sdet}_{\Omega_H^{(0, \bullet)}}(-i\mathcal{L}_R + x) = \prod_{\vec{n} \in C_\mu(\mathcal{M}) \cap \mathbb{Z}^3} (\vec{n} \cdot \vec{R} + x)(\vec{n} \cdot \vec{R} - x + \vec{\xi} \cdot \vec{R}) \equiv S_3^C(x|R), \quad (3.32)$$

where we defined this contribution as a generalized triple sine function  $S_3^C$  associated to  $C_\mu(\mathcal{M})$ . These functions have been discussed at length in I and V. Probably, the most known example is for  $\mathcal{M} = S^5$  which implies  $C_\mu(\mathcal{M}) = \mathbb{R}_{\geq 0}^3$  leading to an ordinary triple sine function  $S_3$ . Indeed, it is well known that the 1-loop determinant of a vector multiplet on  $S^5$  can be written as a triple sine function [32]. A final comment regards the notation adopted in (3.27), the determinant in adjoint representation of  $G$  can be thought of as:

$$\det_{\text{Adj}} S_3^C(x|R) = \prod_{\beta} S_3^C(i\langle x, \beta \rangle | R), \quad (3.33)$$

where  $\beta$  runs over all the roots of  $\mathfrak{g} = \text{Lie}(G)$ . In addition, the superscript  $'$  denotes scalar zero-modes being excluded from the final answer:

$$Z_{(5D, \text{Vec})}^{\text{pert}} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3}{g_{\text{YM}}^2} \text{Tr}[a^2]} \det'_{\text{adj}} S_3^C(ia|R). \quad (3.34)$$

From here it is of course possible to repeat the calculation with hypermultiplets in representation  $R$ , this was discussed in I and [28, 43]. The complete final result for a 5D gauge theory at perturbative level is given by:

$$Z_{(5D)}^{\text{pert}} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3}{g_{\text{YM}}^2} \text{Tr}[a^2]} \frac{\det'_{\text{adj}} S_3^C(ia|R)}{\det_R S_3^C(i(a+m) + \xi \cdot R/2|R)}. \quad (3.35)$$

There are powerful matrix model techniques that can be used to analyze this object. For recent developments on the subject see [50] and references therein.

Expressing the partition function (3.35) in terms of triple sine functions turns out to be very helpful in clarifying its physical meaning. Starting from an observation in [32], in I it was shown how it is possible to express the newly introduced triple sine functions as:

$$S_3^C(z|R) \sim \prod_{\nu \in \text{vertices}} (e^{2\pi i \beta_\nu z} | e^{2\pi i \beta_\nu \epsilon'_1}, e^{2\pi i \beta_\nu \epsilon'_2}), \quad (3.36)$$

where the above product is taken over all vertices in  $C_\mu(\mathcal{M})$  and we allowed  $R$  to get a complex value<sup>8</sup>. The right hand side of (3.36) is expressed in terms of a special function called multiple q-factorial<sup>9</sup>. Intuitively, the global geometry of  $\mathcal{M}$  is treated as a product of local patches with geometry  $\mathbb{C}^2 \times_\epsilon S^1$ , in which  $\beta$  denotes the radius of  $S^1$  and  $(z_1, z_2)$  are local coordinates on  $\mathbb{C}^2$ . Moreover,  $\times_\epsilon$  is a short-hand notation for

<sup>8</sup>More precisely,  $R$  can have generic imaginary part but its real part must lie within the dual cone condition i.e.  $R = \sum_i \lambda_i \vec{v}_i$ ,  $\lambda_i > 0, \forall i$ .

<sup>9</sup>For  $|q_1|, \dots, |q_k| < 1$  the multiple q-factorial is an analytic function of  $x$  and  $q$  with the following infinite product representation:

$$(x, q_1, \dots, q_k)_\infty = \prod_{n_1, \dots, n_k=0}^{\infty} (1 - x q_1^{n_1} \dots q_k^{n_k}).$$

This can be extended to  $|q_1|, |q_2|, \dots, |q_j| > 1$  and  $|q_{j+1}|, |q_{j+2}|, \dots, |q_k| < 1$  by

$$(x, q_1, \dots, q_k)_\infty = \prod_{n_1, \dots, n_k=0}^{\infty} (1 - x q_1^{-(n_1+1)} \dots q_j^{-(n_j+1)} q_{j+1}^{n_{j+1}} \dots q_k^{n_k})^{(-1)^j}.$$

When any  $|q_i| = 1$  the function is not well defined.

what is known as  $\Omega$ -background [14]. Whenever a circle is periodically identified in this background, there is a corresponding twist by a factor of  $\epsilon_i$  on the two orthogonal planes in  $\mathbb{C}_i \subset \mathbb{C}^2$ , using local coordinates:

$$[\theta, z_1, z_2] \simeq [\theta + 2\pi, z_1 e^{2\pi i \beta \epsilon_1}, z_2 e^{2\pi i \beta \epsilon_2}]. \quad (3.37)$$

In I it is shown how to determine  $(\beta_\nu, \epsilon'_{1,2})$  using toric geometry on  $\mathcal{M}$ , in this way it is possible to generate many interesting examples. Paper II further elaborates these ideas for a different five-dimensional example, a 5D  $\mathcal{N} = 2^*$  Abelian theory where maximal supersymmetry is deformed by the addition of an adjoint hypermultiplet. Quite miraculously, the complete non-perturbative partition function can be computed exactly in this model as shown in [15]. As described in II, on a Sasaki-Einstein background the partition functions of Abelian instantons can be formulated in terms of generalized Double Elliptic Gamma functions<sup>10</sup> satisfying factorization properties similar to (3.36).

Coming to a physical interpretation, it should be noted that the right hand side of (3.36) is a 1-loop partition function for a vector multiplet in a  $\Omega$ -background (3.37) [14, 15]. The main message behind (3.36) is that a complicated partition function on a Sasaki-Einstein background  $\mathcal{M}$  is reduced to a product of elementary building blocks “glued” together through a geometric prescription. These building blocks appears in different forms in every known localization computation and have been dubbed *holomorphic blocks*. By now, there is a huge literature on the subject, see for example [52, 53, 54]. An interesting future direction in the field would be to derive the 5D holomorphic blocks for general curved backgrounds  $\mathcal{M}$  using the duality domain-walls approach of [55]. Recently this analysis was performed successfully for a 3D  $\mathcal{N} = 2$  theory in [56].

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<sup>10</sup>See V and [51] for mathematical details.

## 4. Topological Quantum Field Theories

In this chapter we shift focus and introduce topological quantum field theories (TQFTs) as they will play an important role in the remaining part of this thesis. The history of topological quantum field theory started with the foundational paper [6] on Chern-Simons theory. Since then, TQFTs have become a very useful theoretical laboratory where to take a privileged look at a large class of physics problem.

By now, there is a beautiful interplay between properties of symmetries  $G_i$ 's in quantum field theory and topological quantum field theory. The most illustrative example comes from the study of quantum anomalies which are captured by topological actions involving certain characteristic classes of  $G_i$ -bundles<sup>1</sup>. On the contrary, it is not widely appreciated how topological quantum field theories are important in determining subtle issues on the global aspects of the gauge group in quantum field theory. The prime example to keep in mind is the difference between  $U(1)$  and  $\mathbb{R}$  [59]. It is often stated that the gauge group is  $U(1)$  or  $\mathbb{R}$  depending on what particular value is assigned to the charges in the system. Consider a quantum field theory with  $G = U(1)$  with only integer charged electrons. In this example the global group would be called  $U(1)$ . Conversely, a system with particles of charge  $\sqrt{2}$  would then have a global group  $\mathbb{R}$ . However, the value of charges in the system is a secondary issue and as such comes after specifying the global aspect of the group. In fact, the main attention should rather be on what *observables* the gauge theory has. In the  $\mathbb{R}$  gauge theory the observables are *Wilson lines*  $e^{ir \oint A}$  with charge any real number  $r \in \mathbb{R}$ . They can be thought of as the worldline of a very massive particle with charge  $r$ . With this gauge group there are no *'t Hooft operators*.<sup>2</sup> On the other hand, for a  $U(1)$  gauge theory there are Wilson lines  $e^{in \oint A}$  with only integer charges  $n \in \mathbb{Z}$  and there are also *'t Hooft operators*.

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<sup>1</sup> Interestingly, QFTs might also enjoy “generalized global symmetries” associated to extended objects for which there are corresponding anomalies. See [57, 58].

<sup>2</sup> A *'t Hooft operator* is the worldline of a magnetic monopole. One way to define it is to consider a line  $l$  and to surround it with a 2-sphere  $S^2$  supporting a non-trivial flux  $F$  such that  $\int_{S^2} F = 2\pi N$  where  $N \in \mathbb{Z}$ .

A proper definition of gauge theory requires to specify what kind of fluxes are allowed and correspondingly what kind of *operators* are allowed<sup>3</sup>. A final important example in 4D comes from considering  $SU(N)$  versus  $SU(N)/\mathbb{Z}_N$ , these are really different theories with a very different set of observables as studied carefully in [60]. The real payoff of topological quantum field theories is that all these questions can be formulated using a topological action with discrete symmetries. A famous example is the  $\mathbb{Z}_p$ -gauge theory which was studied in [61, 62, 59, 63]. The analysis of global aspects of a gauge group is then reformulated in terms of properties of non-local observables in a TQFT.

The study of modern condensed matter theory has been heavily influenced by the use of topological methods. By now, it is widely believed that all gapped phases of matter are described by a long-range topological theory. A famous example in this sense is the fractional quantum Hall effects (FQHE) which can be modeled in terms of 3D Chern-Simons theory and that has been experimentally observed since the 80's. See chapter 5 for more details. Also, recent developments in the study of discrete anomalies in condensed matter physics have brought to discover new phases of matter called symmetry protected topological phases ( $G$ -SPT). In particular many of these result have been highly influential on some recent works on QCD phases [64].

## 4.1 Twisted Supersymmetry

The previous section have been primarily focussed on topological theories whose action is manifestly topological and does not depend on explicit details of the spacetime metric. However, this turns out to be only a partial description as there is another well known way to define a topological theory that heavily makes use of supersymmetry. As mentioned at the end of section 2.2, this strategy is called topological twisting [4]. The most famous example is probably the topological twisting of 4D  $\mathcal{N} = 2$  super Yang-Mills theory with the following symmetries:

$$SU(2)_+ \times SU(2)_- \times SU(2)_R, \quad (4.1)$$

where  $SU(2)_+ \times SU(2)_-$  is the 4D Lorentz group while  $SU(2)_R$  is the R-symmetry group. A topologically twisted theory is specified by the choice of a new Lorentz group  $SU(2)_+ \times SU(2)_\Delta$  such that:

$$SU(2)_+ \times SU(2)_\Delta \subset SU(2)_+ \times SU(2)_- \times SU(2)_R, \quad (4.2)$$

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<sup>3</sup> A similar important example is related to 2D orbifold theories. These theories can be thought of as discrete gauge theories, notice that the difference between  $U(1)$  and  $\mathbb{R}$  comes from the center  $\mathbb{Z}$ . Once we mod out by  $\mathbb{Z}$ , only  $\mathbb{Z}$ -invariant operators should be considered and twisted sectors must be included. In this analogy, 't Hooft lines play the role of twisted sectors.

where  $SU(2)_\Delta = SU(2)_- \times SU(2)_R|_{\text{diag}}$  is a diagonal subgroup. This means that the Lorentz spins are modified in the following way:

$$(\mathbf{n}_+, \mathbf{n}_-) \Rightarrow (\mathbf{n}_+, \mathbf{n}_- \otimes \mathbf{n}_R). \quad (4.3)$$

Correspondingly the supersymmetry charges  $Q_\alpha^i, \bar{Q}_{\dot{\alpha}}^j$  transform as:

$$Q_\alpha^i : (\mathbf{2}, \mathbf{1}, \mathbf{2}) \rightarrow (\mathbf{2}, \mathbf{2}) \Rightarrow G_\mu, \quad (4.4)$$

$$\bar{Q}_{\dot{\alpha}}^j : (\mathbf{1}, \mathbf{2}, \mathbf{2}) \rightarrow (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \Rightarrow \bar{Q}, \bar{Q}_{\mu\nu}^+, \quad (4.5)$$

which means that there is a twisted supersymmetry algebra given by:

$$\{\bar{Q}, G_\mu\} = \partial_\mu, \quad (4.6)$$

with  $\bar{Q}^2 = 0$ . This is precisely the necessary conditions for localization discussed in section 2.2. The work [4] shows how 4D  $\mathcal{N} = 2$  super Yang-Mills theory can be formulated on any 4-manifold  $\mathcal{M}_4$ , while preserving  $\bar{Q}$ . In this way, observables annihilated by  $\bar{Q}$  become distinguished. In particular, the supersymmetric partition function localizes on the moduli space of anti-self dual instantons (ASD) on  $\mathcal{M}_4$ :

$$\mathcal{M}_{\text{ASD}} = \{A \in \mathcal{A} | F^+ = 0\} / \mathcal{G}, \quad (4.7)$$

where  $\mathcal{A}$  is the space of irreducible connections on  $\mathcal{M}_4$  and  $\mathcal{G}$  is the infinite-dimensional gauge symmetry. By (4.6), correlation functions of protected operators are independent of the metric on  $\mathcal{M}_4$  and capture smooth structure invariants of  $\mathcal{M}_4$ . Consider  $G = SO(3)$  and a tower of observables constructed from  $\mathcal{O} = \text{Tr}(\Phi^2)$ . Then the correlation function of these observables generates the celebrated Donaldson polynomial invariants of  $\mathcal{M}_4$ . See [65, 66, 67, 68] for a selected list of pedagogical references on the subject. The discovery of twisted supersymmetry led to profound revolution in the world of mathematics and tied together quantum field theory, string theory and geometry. There are many other versions of topological twist leading to different counting problems and to different insights on mathematics. See [69, 70, 71, 72, 73, 74] for a sample of selected results.

## 5. 6D FQHE

In this chapter we present a 6D generalization of the fractional quantum Hall effect involving membranes coupled to a three-form potential in the presence of a large background four-form flux. The low energy physics is governed by a bulk 7D topological field theory of abelian three-form potentials with a Chern-Simons action coupled to a 6D anti-chiral theory of Euclidean effective strings.

### 5.1 A Bulk-Boundary Correspondence

A striking result in modern condensed matter theory relates the 1 + 1D fractional quantum Hall effect and 2 + 1D Chern-Simons theory using a bulk-boundary correspondence [75]. Most of the material presented in this section can be found in a beautiful set of lectures [76].

The starting point is a three-dimensional topological action of abelian one-forms  $a^I$  with Chern-Simons interaction and an integral symmetric matrix of couplings  $K_{IJ}$ :

$$S_{(3D)}[a^I] = \frac{K_{IJ}}{4\pi i} \int_{\mathcal{M}_3} a^I \wedge da^J, \quad (5.1)$$

where  $\mathcal{M}_3$  is a Lorentzian 3-manifold. For this particular example we will assume that  $\mathcal{M}_3$  is a long stretched neck with boundary  $\mathbb{R}^2$ , i.e.  $\mathcal{M}_3 = \mathbb{R}_{\leq 0} \times \mathbb{R}^2$ . See figure 5.1.

Since  $\mathcal{M}_3$  has boundary it is now important to determine which boundary conditions should be imposed on  $a^I$ 's. It can be shown that after imposing  $a^I = -i \star_{2D} a^I$  the action (5.1) reduces to a 1 + 1D chiral boson. Contrarily to other dimensions, a *useful* action for a chiral 2D boson can be written explicitly in a mild non-local way. This is usually called Floreanini-Jackiw action [77]. Notice that, just by employing a simple topological field theory we can derive this action in a much more elegant way [78]. Moreover, the 3D abelian CS theory has other quite convenient “practical uses”. For example, switching on a background flux  $F = dA$  through  $\partial\mathcal{M}_3$  leads to:

$$S_{(3D)}[A, a^I] = \frac{K_{IJ}}{4\pi i} \int_{\mathcal{M}_3} a^I \wedge da^J + \frac{1}{2\pi i} \int_{\mathcal{M}_3} A \wedge \nu_I da^I, \quad (5.2)$$

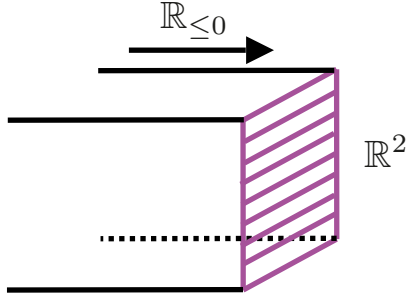


Figure 5.1. A 3-manifold  $\mathcal{M}_3$  given by the product of a half-line  $\mathbb{R}_{\leq 0}$ , parametrizing the “time” variable, and a spatial  $\mathbb{R}^2$ .

from which it is easy to derive the so called *fractional conductivity*<sup>1</sup>:

$$\sigma_{xy} = \nu_I (K^{-1})^{IJ} \nu_J. \quad (5.4)$$

Now, for a  $\nu = [1, 0, \dots, 0]$  and

$$K = \begin{bmatrix} x_1 & -1 & & & \\ -1 & x_2 & -1 & & \\ & -1 & \dots & -1 & \\ & & -1 & x_{k-1} & -1 \\ & & & -1 & x_k \end{bmatrix}, \quad (5.5)$$

there is a continued fraction spectrum:

$$\frac{p}{q} = x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \dots}}. \quad (5.6)$$

More precisely, in condensed matter applications there is an entire hierarchy given by:

$$\frac{p}{q} = x_1 \mp \frac{1}{x_2 \mp \frac{1}{x_3 \mp \dots}}, \quad (5.7)$$

<sup>1</sup>Consider a particle of charge  $e$  in a constant magnetic field pointing in the  $z$ -direction. Next, apply a small electric field in the  $x$ -direction and measure its current through the  $y$ -direction. The *Hall conductivity* is defined as the ratio:

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \frac{p}{q}, \quad (5.3)$$

where  $p, q \in \mathbb{Z}$ . The integer quantum Hall effect happens for  $q = 1$ . When  $q > 1$  the electrons fractionalizes and give rise to the fractional quantum Hall effect.



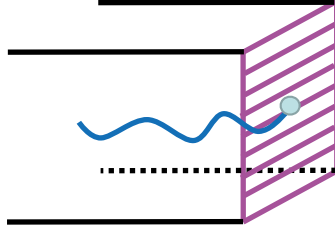


Figure 5.2. State insertion by a 3D Wilson line ending on the boundary  $\partial\mathcal{M}_3$ .

and there is an interesting interpretation for them: (+) signs correspond to “Holes” while (-) signs are interpreted as “Particles”. Strikingly, from 3D Chern-Simons it is also possible to derive the many-body wavefunction for fractional Quantum Hall Effect [75]. A free chiral boson in 2D has the following two point function:

$$\langle\phi(z)\phi(w)\rangle = -\frac{1}{m}\log(z-w). \quad (5.8)$$

All states in this 2D CFT can be described from a three-dimensional perspective. Indeed, it is convenient to introduce an operator  $\Phi(z)$  defined as:

$$\Phi(z) = : e^{im\phi(z)} :, \quad (5.9)$$

which can be thought of as a 3D Wilson line supported on a path which ends at  $\partial\mathcal{M}_3$ . See figure 5.2.

As shown by [75] the many-body wavefunction is captured by a CFT correlation function:

$$\Psi(z_1, \dots, z_N) = \langle\Phi(z_1) \dots \Phi(z_N) \exp(-i \int_{\mathbb{D}_2} F \wedge \phi)\rangle, \quad (5.10)$$

where  $\mathbb{D}_2$  is a solid disk of radius  $R$ . Everything reduces to the calculation of two-point functions using (5.8):

$$\Psi(z_1, \dots, z_N) = \Psi_{\text{Laughlin}} \Psi_{\text{Landau}} = \prod_{i < j} (z_i - z_j)^m e^{-\sum_i |z_i|^2 / 4l_B^2}. \quad (5.11)$$

The final result is factorized in two pieces, the first is known as Laughlin wavefunction which was first discovered in [79] and it was very important

for subsequent theoretical developments in quantum Hall physics [80, 81, 82]. The second contribution is that of an electron moving in a magnetic field  $B$  with a background charge density set by  $l_B \sim 1/\sqrt{F}$ . There are two amazing consequences of this result, the first is that by using some arguments in high energy theory, in particular CFT techniques and topological quantum field theories, it is possible to give a first principle derivation of a phenomenon which is observed in real labs. The second one, probably more appealing for a high energy theorist, is that physics of a chiral 2D CFT is characterized in terms of a many-body wavefunction. In the next section we explain how this analogy can be quite fruitful in the study of other known examples of chiral CFTs.

## 5.2 7D Chern-Simons Theory

Among several AdS/CFT early practitioners [83, 84, 85, 62, 86] it has been known for a long time that a seven-dimensional Chern-Simons action:

$$S_{(7D)}[c^I] = \frac{\Omega_{IJ}}{4\pi i} \int_{\mathcal{M}_7} c^I \wedge dc^J, \quad (5.12)$$

could be a good starting point for describing 6D systems with chiral 2-forms. Here  $c^I$ 's are a collection of 3-form potentials subject to gauge redundancy  $c^I \rightarrow c^I + db^I$  and  $\mathcal{M}_7$  is again a Lorentzian manifold with cylindrical geometry:  $\mathcal{M}_7 = \mathbb{R}_{\text{time}} \times \mathcal{M}_6$ . For any three-cycle  $S \in H_3(\mathcal{M}_6, \mathbb{Z})$  there is a corresponding operator defined by:

$$\Phi_m(S) = \exp\left(im_I \int_S c^I\right), \quad (5.13)$$

where  $m_I$  is a vector of charges. The pairing  $\Omega_{IJ}$  defines an integral lattice  $\Lambda$  and the  $m_I$  take values in its dual  $\Lambda^*$ . Given two operators as  $\Phi_m(S)$  and  $\Phi_n(T)$  as in (5.13) they satisfy the so-called *braid relations* [84]:

$$\Phi_m(S) \Phi_n(T) = \Phi_n(T) \Phi_m(S) \times \exp\left(2\pi i \left(m_I (\Omega^{-1})^{IJ} n_J\right) (S \cdot T)\right), \quad (5.14)$$

where  $S \cdot T$  is the intersection pairing for 3-cycles in  $\mathcal{M}_6$ . Because of (5.14) the ground state degeneracy of the system depends on the topology of  $\mathcal{M}_6$ . This is a sign of *topological order*, a feature of certain gapped systems studied extensively in condensed matter theory. The interested reader might find it useful to consult [87, 88, 89]. In this setup it is interesting to understand what happens when chiral boundary conditions are enforced at the boundary of  $\mathcal{M}_7$ :

$$\star_{6D} c^I|_{\partial\mathcal{M}_7} = ic^I|_{\partial\mathcal{M}_7} \quad (5.15)$$

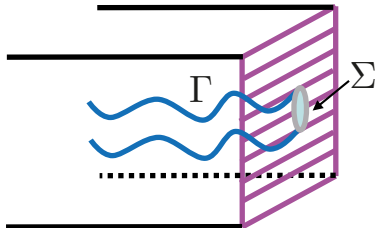


Figure 5.3. Insertion of a membrane wrapping a Riemann surface  $\Sigma$  on the boundary  $\partial\mathcal{M}_7$  via a three-form supported along a 3-chain  $\Gamma$ .

i.e. the  $c^I$ 's are *anti-self dual* at the boundary. Moreover, the  $c^I$ 's are considered here as three-form field strengths for two-forms  $b^I$ 's:

$$c^I = db^I. \quad (5.16)$$

As for 3D Chern-Simons there are physical degrees of freedom associated with the  $c^I$ 's, these are 2+1D membranes and can be thought as an analog of “electrons” for the 7D system.

In this sense, for a given three-chain  $\Gamma$  there is an associated operator:

$$\Phi_m(\Gamma) = \exp\left(im_I \int_{\Gamma} c^I\right), \quad (5.17)$$

which represents a bulk insertion of a membrane of charge  $m_I$ . When  $\Gamma$  has a topology like  $\Gamma = \mathbb{R}_{\leq 0} \times \Sigma$ , with  $\Sigma$  a Riemann surface lying at  $\partial\mathcal{M}_7$  as in figure 5.3, there is a corresponding boundary operator denoted by:

$$\Phi_m(\Gamma) = \exp\left(im_I \int_{\Gamma} c^I\right). \quad (5.18)$$

The bulk insertion of a membrane operator supported on a 3-chain  $\Gamma$  corresponds to a Euclidean string coupled to  $b^I$  and wrapping  $\Sigma$  on the boundary. As shown in III there are useful “practical uses” also for this theory. Consider a large 4-form background flux  $G = dC$ , this couple to the original action in the following way:

$$S_{(7D)}[c^I, C] = \frac{\Omega_{IJ}}{4\pi i} \int_{\mathcal{M}_7} c^I \wedge dc^J + \frac{\nu_I}{2\pi i} \int_{\mathcal{M}_7} C \wedge dc^I. \quad (5.19)$$

From (5.19) it is possible to derive the fractional conductivity following similar steps as in section 5.1, this leads to a hierarchy given again by a continued fraction:

$$\frac{p}{q} = x_1 \mp \frac{1}{x_2 \mp \frac{1}{x_3 \mp \dots}}. \quad (5.20)$$

Although this is exactly the same formula as before it carries a quite different physical interpretation. In the setup of III the (-) signs are interpreted as “brane” while (+) signs were dubbed “dranes” i.e. the negation of a brane. This picture is quite interesting as it is known that 6D SCFTs possess a spectrum of tensionless strings described by D3 branes wrapped on 2-cycles with  $\mathbb{P}^1$  topology and negative self-intersection [1]. The 7D Chern-Simons setup gives an effective description of this phenomenon in terms of filling fractions for excitations above the ground state of a quantum many body system localized at the boundary of  $\mathcal{M}_7$ . A natural question is how to characterize the many body wavefunction for this system. To do that it is necessary to introduce a background flux operator  $\Phi_{\text{bkgd}}$ :

$$\Phi_{\text{bkgd}} = \exp\left(\frac{\nu_I}{2\pi i} \int_{\mathbb{D}_7} c^I \wedge G\right) = \exp\left(\frac{\nu_I}{2\pi i} \int_{\mathbb{D}_6} b^I \wedge G\right), \quad (5.21)$$

where  $\mathbb{D}_7 = \mathbb{R}_{\leq 0} \times \mathbb{D}_6$ . Now, the ultimate goal is to determine the following correlation function:

$$\Psi(\Sigma_1, \dots, \Sigma_N) = \langle \Phi^{(1)} \dots \Phi^{(N)} \Phi_{\text{bkgd}} \rangle. \quad (5.22)$$

Since the theory on  $\partial\mathcal{M}_7$  is free this boils down to evaluating:

$$\Psi_{\text{Laughlin}} = \prod_{1 \leq i < j \leq N} \langle \Phi^{(i)} \Phi^{(j)} \rangle_{6D} \times \prod_{1 \leq i \leq N} \Psi_{\text{Landau}}^{(i)}, \quad (5.23)$$

where the unnormalized Landau wavefunction for a single membrane moving in a background four-form flux can be defined as:

$$\Psi_{\text{Landau}}^{(i)} = \langle \Phi_{\text{bkgnd}} \Phi^{(i)} \rangle_{6D}. \quad (5.24)$$

Although (5.23) looks very similar to (5.10) its calculation is not as straightforward since it involves many non-trivial data about bulk geometry. The goal of III was to study various interesting limits of this expression. An obvious difference between electrons and membranes is that the latter objects carry a natural tension denoted by  $T_{M2}$ :

$$T_{M2} = \frac{1}{(2\pi l_*)^3}, \quad (5.25)$$

which depends on  $G$ . The strength of  $G$  governs the behavior of the effective degrees of freedom, indeed for a big value of the background

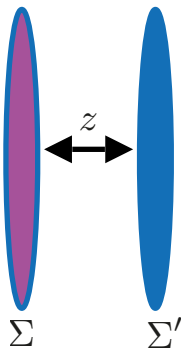


Figure 5.4. Two large flat surfaces  $\Sigma$  and  $\Sigma'$  separated along  $z$ .

flux the membrane will puff up to a large rigid object while for a small value of the flux the correct behavior would be that of a point particle. As a consequence of working with a 4-form flux, there is a tensor of magnetic lengths which intuitively can be described as:

$$l_{\perp} \sim \frac{1}{\sqrt{G}}, \quad (5.26)$$

where  $\perp$  denotes the directions transverse to  $\Sigma$ . An instructive example is that of two flat surfaces  $\Sigma, \Sigma'$  close to each other in the large  $G$  limit, in this case the global six-dimensional geometry can be thought of as  $\mathcal{M}_6 = \mathbb{R}^2 \times \Sigma \times \Sigma'$ . The separation of  $\Sigma$  and  $\Sigma'$  on  $\mathbb{R}^2$  is parametrized by complex variables  $z, \bar{z}$ . See figure 5.4.

Integrating over the Riemann surfaces leads to a Laughlin wavefunction given by:

$$\Psi_{\text{Laughlin}} \subset \left\langle e^{im \int_{\Sigma} b} e^{im' \int_{\Sigma'} b} \right\rangle_{6D} \sim z^{\rho} \bar{z}^{\tilde{\rho}}, \quad (5.27)$$

where  $\rho$  and  $\tilde{\rho}$  depends on the decomposition of  $b$  in terms of harmonic self-dual and anti-self dual 2-forms. In particular, this implies that there is an explicit dependence on the metric. Nevertheless, there still is a meaningful quantity protected by topology [90]:

$$\rho - \tilde{\rho} = \frac{mm'}{\Omega} \times (\Sigma \cdot \Sigma')|_{z=0}, \quad (5.28)$$

where  $(\Sigma \cdot \Sigma')|_{z=0}$  is the intersection pairing of two-cycles on a Kähler surface. Returning to the evaluation of (5.27), it is worth noticing that there is a rather close similarity to the case of a 2D chiral boson. The

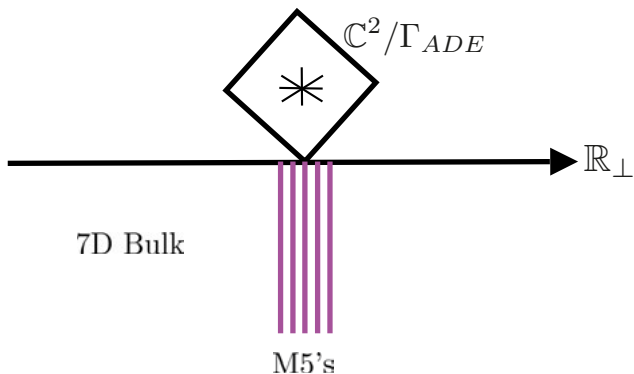


Figure 5.5. A stack of M5 branes probing a  $\mathbb{C}^2/\Gamma_{ADE}$  singularity seen as a 6D domain-wall along the  $\mathbb{R}_\perp$  coordinate.

main difference is that the precise values of the exponents  $\rho$  and  $\tilde{\rho}$  also require information about the explicit choice of metric as well as the dynamics of the membranes moving in a background charge density. Indeed, it is well known that for a six manifold  $\mathcal{M}_6 = \mathbb{R}^2 \times \mathbb{P}^2$  this is actually *just* a 2D chiral boson.

### 5.3 Further Directions

The main motivation for studying a 7D Chern-Simons theory in III was to get a physical explanation for certain characteristic features of 6D SCFTs that have been observed in recent years [91, 92, 93, 94, 95]. Indeed, it is known that M-theory has an 11D Chern-Simons term whose reduction to 7D dimensions leads to a bulk action containing terms like:

$$S_{11D,\text{bulk}} \supset \frac{\Omega_{IJ}}{4\pi i} \int_{\mathcal{M}_7} c^I \wedge dc^J + \frac{\mu_I}{4\pi i} \int_{\mathcal{M}_7} c^I \wedge \text{Tr}(F \wedge F) . \quad (5.29)$$

The first term in (5.29) has been discussed at length in section 5.2 while the second term is a 7D lift of a Green-Schwarz coupling [96, 97, 98] which has also been recently studied in the context of 6D (1,0) SCFTs [99]. In III it is conjectured that a particular “gapped” limit of a 7D  $\mathcal{N} = 1$  gravitino multiplet would be a good starting point for thinking about the 7D Chern-Simons theory in a supersymmetric way. See figure 5.5. An alternative proposal relating M5 branes to the fractional quantum Hall effect has been proposed in [100].

Finally, the higher-dimensional point of view could be seen as a laboratory to understand and classify a large class of lower-dimensional

topological systems whose matrix of couplings are determined in terms of geometrical data of the internal directions.

## 6. Spin Chains and TQFTs

We review how solutions of the Yang-Baxter equation associated to Lie groups can be deduced in a systematic way using a four-dimensional topological gauge theory. We conclude the chapter with some comments on the string theory origin of such result.

### 6.1 Gauge Theory and Integrable Lattice Models

The study of integrable lattice models has been a very active line of research since the early days of quantum mechanics. Most of the classic results in this topic can be found in [101]. Elastic scattering process in 1+1D can be drawn as the left diagram in figure 6.1. Particles have quantum numbers, labeled by elements of a vector space  $V$ , denoted by  $e_i$  with  $i = 1, \dots, \dim(V)$ . Moreover, each incoming state is also characterized by a (complex) spectral parameter  $z$ . All the interactions among internal states are governed by a matrix which depends on  $z$ , for this reason it is crucially important to determine the exact form of:

$$R(z_1, z_2) : V \otimes V \rightarrow V \otimes V, \quad (6.1)$$

which is also known of as *R-matrix*. A basic working assumption<sup>1</sup> is that the integrable systems considered here will have an *R-matrix* which only depends on  $(z_1, z_2)$  through their difference  $z_1 - z_2$ .

The power of integrability emerges quite naturally when more incoming states interacts with each other. For example, consider an elastic scattering with three incoming states as in the left hand side of figure 6.2. Because of integrability, the S-matrix structure is constrained in such a way that exchanging the incoming channels as in the right hand side of figure 6.2 leads to the Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad (6.2)$$

where  $R_{ij}$  with  $i, j = 1, 2, 3$  is an abbreviation for  $R_{ij}(z_i - z_j)$ . A complete classification of solutions for general Yang-Baxter equation (YBE) is not known. However, it is possible to introduce a *quasi-classical R-matrix* defined as:

$$R_{\hbar} = I + \hbar r(z) + \mathcal{O}(\hbar^2), \quad (6.3)$$

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<sup>1</sup>Indeed, there are interesting systems which do not have this property, see [102, 103].



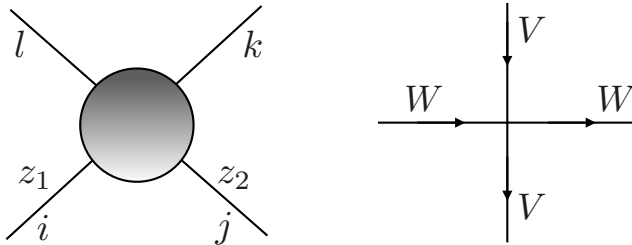


Figure 6.1. On the left an usual scattering diagrams. On the right a XXX spin-chain.

where  $r(z)$  is known as classical  $R$ -matrix and it satisfies a classical Yang-Baxter equation (cYBE) that can be found by substituting (6.3) in (6.1). In this way, the situation improves drastically since a complete classification of solutions to cYBE has been worked out in [104]. Given a classical  $R$ -matrix expressed as:

$$r(z) = \sum_{a,b} r_{a,b}(z)(t^a \otimes t^b), \quad (6.4)$$

where  $t^a$  is a basis for a Lie algebra  $\mathfrak{g}$  and  $r_{a,b}(z)$  satisfies a non-degeneracy condition i.e.  $\det_{a,b} r_{a,b}(z) \neq 0$ , all possible solutions to cYBE can be classified by studying the poles of  $r(z)$ . Indeed, in [104] it was shown that poles of  $r(z)$  span a lattice whose dimension can be either 0, 1 or 2. The corresponding solutions to classical Yang-Baxter equation are given by rational, trigonometric or elliptic functions.

A very simple but important example is the XXX spin chain as drawn on the right hand side of 6.1. Incoming particles have quantum numbers labeled by two vector spaces  $V = W = \mathbb{C}^2$ . The  $R$ -matrix acts as an intertwiner of  $V$  and  $W$  i.e.  $R : V \otimes W \rightarrow W \otimes V$ . Because of integrability the system can be solved exactly and the  $R$ -matrix is found to be:

$$R_{\hbar}(z) = w \otimes v + \frac{\hbar}{z} c(w \otimes v), \quad (6.5)$$

where  $v \in V$ ,  $w \in W$  and  $c \in \mathfrak{sl}_2 \times \mathfrak{sl}_2$  is the quadratic Casimir.

A closer inspection to figure 6.2 reveals a surprising connection. In knot theory it is customary to classify and distinguish different knots using

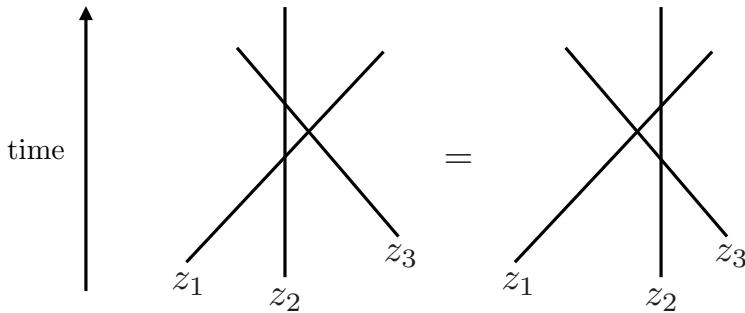


Figure 6.2. A famous illustration of Yang-Baxter equation.

the so called Reidemeister moves. In particular two knot diagrams belonging to the same knot can be shown to be equivalent by a sequence of three different moves. Strikingly, one of these moves is portrayed exactly in the same way as the diagrammatic Yang-Baxter equation. Following this intuition, it is natural to wonder whether 3D Chern-Simons theory could have anything to do with integrable lattice models [105, 6]. Unfortunately, Chern-Simons theory by itself does not seem able to capture the spectral parameter or to be more precise it does only capture it in a very degenerate limit  $z \rightarrow i\infty$ . For nearly 30 years, a topological quantum field theory interpretation of the spectral parameter has been missing.

Recently, the problem of finding a correct topological quantum field theory interpretation for the spectral parameter has been solved in a series of mathematical papers [106, 107]. See [108, 109] for a physics adaptation. The central idea behind this proposal is very simple, consider a four-dimensional action given by:

$$S_{(4D)}[A] = \frac{1}{2\pi} \int_{\mathcal{M}_4} dz \wedge \text{CS}(A), \quad (6.6)$$

where  $\mathcal{M}_4 = \mathbb{R}^2 \times \mathbb{C}^2$  has local coordinates  $(x, y, z, \bar{z})$  and the connection 1-form  $A$  has no explicit dependence on  $A_z$ :

$$A = A_x dx + A_y dy + A_{\bar{z}} d\bar{z}. \quad (6.7)$$

Intuitively, by lifting the problem to four dimensions it is possible to give a clear geometrical description of the spectral parameter. At this point, it is quite useful to describe a number of surprising consequences

of (6.6). First, the action does not have four-dimensional nor three-dimensional diffeomorphisms invariance but it is only invariant under two-dimensional diffeomorphisms along  $\mathbb{R}^2$ . In addition, there is no notion of reality as the action functional is a complex function of  $(A_x, A_y, A_{\bar{z}})$ . The action is non-renormalizable by power counting and its coupling constant, denoted by  $\hbar$ , has dimension:  $[\hbar] = [L] = [M]^{-1}$ . Contrarily to 3D Chern-Simons,  $\hbar$  does not have a quantization condition. Finally, upon integration by parts, (6.6) can be rewritten as a position dependent  $\theta$ -term:

$$S_{(4D)}[A] = -\frac{1}{2\pi} \int_{\mathcal{M}_4} z \text{Tr}(F \wedge F). \quad (6.8)$$

A simple exercise shows that the equations of motions from (6.6) are given by:

$$F_{xy} = 0, \quad F_{x\bar{z}} = F_{y\bar{z}} = 0. \quad (6.9)$$

Therefore, the theory is topological on  $\mathbb{R}^2$  and holomorphic along  $\mathbb{C}$ . More generally, a topological theory with similar features<sup>2</sup> can be written for any smooth oriented 2-fold  $\Sigma$  and any complex 1-fold  $C$  with a meromorphic 1-form  $\omega$ :

$$S_{(4D)}[A] = \frac{1}{2\pi} \int_{\Sigma \times C} \omega \wedge CS(A). \quad (6.10)$$

The limit  $\hbar \rightarrow \infty$  is not well defined in perturbation theory, for this reason  $\omega$  is not allowed to have zeroes. Then, by Riemann-Roch theorem, there is a limited number of possibilities for a genus  $g$  complex 1-fold  $C$ . In particular, for  $g = 0$  there can be a double pole which gives  $C = \mathbb{C}$  and  $\omega = dz$  or two simple poles leading to  $C = \mathbb{C}^*$  and  $\omega = \frac{dz}{z}$ . Conversely, for  $g = 1$  there are no poles and  $C = E$  where  $E$  denotes a complex elliptic curve with  $\omega = dz$ . Notice that, this exhausts all the possible choices of  $C$ .

Consider a Wilson loop observable defined by:

$$W_\rho(K) = \text{Tr}_\rho \text{Pexp} \left( \oint_K A \right), \quad (6.11)$$

where  $K$  is a non-arbitrary loop in  $\Sigma \times C$ . Non-arbitrary means that  $K$  needs to be a loop in  $\Sigma$  at a fixed value  $z = z_*$  in  $C$ . Quite interestingly, perturbative calculations involving these observables are quite simple as (6.6) is IR free. Here we follow [108] and derive a crucial example involving two intersecting Wilson loop observables exchanging a gluon. The first step consists of a metric rescaling on  $\Sigma$  by a large factor, in

<sup>2</sup>Similar theories have appeared also in the context of topological twisting. See [110, 111].

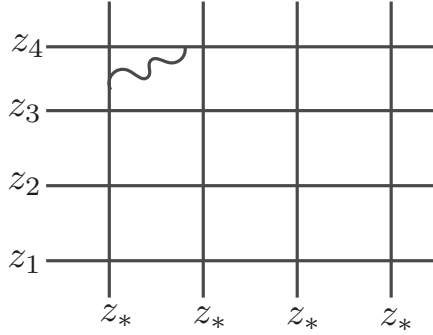


Figure 6.3. Typical configuration of lines in  $\Sigma \times \mathbb{C}$  for the 4D TQFT.

this way two intersecting Wilson loops become infinitely stretched lines on  $\mathbb{R}^2$ . By the same token, “long-distance effects” can be safely ignored and the only contribution to be considered is localized at the intersection between two lines. When  $C = \mathbb{C}$ , a typical configuration of multiple non-arbitrary lines in  $\mathbb{R}^2 \times \mathbb{C}$  together with their gluon exchange is shown in figure 6.3.

In what follows it is useful to use a metric on  $\mathbb{R}^2 \times \mathbb{C}$  such that:

$$ds^2 = dx^2 + dy^2 + dzd\bar{z} = g_{\mu\nu}dx^\mu dx^\nu, \quad (6.12)$$

and a metric tensor  $g_{\mu\nu}$  given by:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}. \quad (6.13)$$

The most convenient choice of gauge-fixing condition in these coordinates is an analog of Lorentz gauge written as:

$$\partial^x A_x + \partial^y A_y + 4\partial^z A_{\bar{z}} = 0. \quad (6.14)$$

In this gauge, the four-dimensional propagator for two gauge fields  $A$  and  $A'$  is given by:

$$\langle AA' \rangle_{xy} = \frac{1}{2\pi} \frac{2(\bar{z} - \bar{z}')}{((x - x')^2 + (y - y')^2 + |z - z'|^2)^2}, \quad (6.15)$$

$$\langle AA' \rangle_{x\bar{z}} = \frac{1}{2\pi} \frac{y - y'}{((x - x')^2 + (y - y')^2 + |z - z'|^2)^2}, \quad (6.16)$$

$$\langle AA' \rangle_{y\bar{z}} = \frac{1}{2\pi} \frac{x - x'}{((x - x')^2 + (y - y')^2 + |z - z'|^2)^2}. \quad (6.17)$$

Moreover, the propagator can be defined as a two-form on  $\mathbb{R}^2 \times \mathbb{C}$ :

$$P(x, y, z, \bar{z}) \equiv \frac{1}{2\pi} (xdy \wedge d\bar{z} + yd\bar{z} \wedge dx + 2\bar{z}dx \wedge dy) \times \frac{1}{(x^2 + y^2 + z\bar{z})^2}. \quad (6.18)$$

So far, Lie algebra indices have been omitted everywhere from the presentation. A four-dimensional bulk gauge field  $A_\mu^a$  couples to a Wilson line in a representation  $\rho$  through a Lie algebra matrix  $t_{a,\rho}$ . As such, the correlator of a Wilson line observable, supported on a x-axis line  $K_1$  with  $z = z_1$ , intersecting a second observable localized on a y-axis line  $K_2$  with  $z = z_2$  is given by:

$$\langle W_\rho(K_1)W_{\rho'}(K_2) \rangle_{4D} = \hbar c_{\rho,\rho'} \int dx dy' P(x - x', y - y', z_1 - z_2, \bar{z}_1 - \bar{z}_2), \quad (6.19)$$

where  $c_{\rho,\rho'}$  is expressed as:

$$c_{\rho,\rho'} = \sum_a t_{a,\rho} \otimes t_{a,\rho'}. \quad (6.20)$$

Here  $c_{\rho,\rho'}$  can be viewed as the image of an element  $c = \sum_a t_a \otimes t_a$  of of  $\mathfrak{g} \otimes \mathfrak{g}$  in the representation  $\rho \otimes \rho'$ . The integral in (6.19) can be easily evaluated:

$$\begin{aligned} \langle W_\rho(K_1)W_{\rho'}(K_2) \rangle_{4D} &= \hbar c_{\rho,\rho'} \frac{1}{2\pi} \int dx dy \frac{2(\bar{z}_1 - \bar{z}_2)}{(x^2 + y^2 + |z_1 - z_2|^2)^2} \\ &= \frac{\hbar c_{\rho,\rho'}}{z_1 - z_2}. \end{aligned} \quad (6.21)$$

Amazingly, a gauge theory calculation reproduces the XXX spin-chain  $R$ -matrix semiclassical expansion (6.5):

$$R_{\hbar}(z_1, z_2) = I + \hbar r(z_1, z_2) + \mathcal{O}(\hbar^2) = I + \frac{\hbar c_{\rho,\rho'}}{z_1 - z_2} + \mathcal{O}(\hbar^2). \quad (6.22)$$

Furthermore, by general theorems [104] it is known that this is enough to determine the full rational  $R$ -matrix. Finally, the missing link between integrable lattice models and gauge theories has been found.

As remarked above, the full four-dimensional formulation was crucial. More precisely, the results of [106, 107] have shown rigorously how to make sense of perturbative calculations using formal arguments in BV formalism. Moreover, the works [108, 109] have put these works on a solid physical perspective. By now, all these insights seem to point out that even though the original 4D theory is non-renormalizable its perturbation theory seems to converge. In the next section we describe how twisted supersymmetry and brane configurations in string theory are a promising tool to get a first-principle understanding of such phenomenon.

## 6.2 String Theory Formulation

A program for understanding the non-perturbative formulation of complex 3D Chern-Simons theory and its relationship to string theory has been put forward starting with [112, 113] and culminating in [114]. In all these papers there is an important interplay between supersymmetric brane configurations and twisted topological field theories. In particular, a new type of topological coupling for 4D  $\mathcal{N} = 4$  super Yang-Mills theory was first found in [115, 116] by studying configurations of D3 branes ending on NS5 branes. This coupling is a generalization of the  $\theta$ -angle term and can be understood in terms of a topologically twisted version of  $\mathcal{N} = 4$  SYM [74]. Moreover, the supersymmetric partition function for a twisted  $\mathcal{N} = 4$  theory on a four-manifold with boundary localizes to 3D complex Chern-Simons at the boundary. As shown in [114], all these topics can be elegantly unified using various twisted compactification of a six-dimensional  $(2, 0)$  SCFT.

This strategy needs to be modified in order to describe the 4D version of Chern-Simons theory introduced in section 6.1. The appropriate brane system is now given by a configuration of D4-NS5 branes with a Ramond-Ramond flux turned on. These backgrounds have a long history initiated in [117]. More recently, they have been used in the context of supersymmetric field theory in [118, 119, 120, 121] where they were called “flux-trap” backgrounds. A fluxtrap gives a string theory description for the equivariant parameters used for  $\Omega$ -deformed supersymmetric gauge theories. The twisted complex planes are now part of the ten-dimensional geometry and all the branes are trapped at fixed point of the rotation isometry.

There are some promising hints that the above brane configuration can be further described using little string theory (LST) [122]. In a convenient

string duality frame this has a low-energy description in terms of 6D (1,1) super Yang-Mills theory. Using supersymmetric localization, the partition function of this system can be shown to localize to the 4D TQFT thus giving rise to a well defined non-perturbative definition [123].

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I always thought about my PhD work as a long journey. Along the road I experienced all sorts of feelings, passing from moments of complete pessimism and discouragement to great enthusiasm and excitement. I am deeply grateful to everyone who crossed my path and gave me precious advices and instructions, it is thank to you that I am here today.

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## 8. Svensk Sammanfattning

Att fysikaliska modeller har exakta lösningar spelar en avgörande roll för vår förståelse av naturen. Faktum är att bara genom att studera ett enkelt system kan vi ofta lära oss viktiga egenskaper hos mer komplicerade modeller. Ett exempel på detta är vätagasatomen som har lärt oss grundläggande aspekter av kvantmekanik. Min forskning handlar om att försöka skapa nya exakta lösningar inom kvantfältteorier, det universella språket inom modern teoretisk fysik.

Att studera exakta lösningar är grundläggande för att övervinna kvantfältteoriens begränsningar och sedan kunna använda det som ett verktyg när vi vill beskriva naturen. Detta blev tydligt sedan begynnelsen av kvantkromodynamik eller QCD, den fysikaliska teori som förklarar atomkärnans struktur. Vid låga energier sker en stark växelverkan och det finns en kraft med oändlig räckvidd som "fjättrar" kvarkarna. Denna region kan inte observeras genom en approximation i störningsteori och "fjättringen" (confinement) av kvarkar är fortfarande ett av de mest svårlösta problemen i teoretisk fysik. Ett annat exempel på detta är fasövergångar hos supraledare som vid höga temperaturer uppvisar ett liknande beteende. Slutligen ger strängteori upphov till ett stort utbud av kvantfältteorier vars beteende saknar en sektion med svag växelverkan.

Under min tid som doktorand har jag varit intresserad av de tänkbara sätt man kan använda för att karakterisera den icke-perturbativa dynamiken hos olika kvantfältteorier med hjälp av verktyg såsom supersymmetri och geometri. Supersymmetriska kvantfältteorier är användbara teoretiska laboratorier där vi kan fördjupa vår förståelse av viktiga fenomen som också existerar i icke-supersymmetriska teorier. Samtidigt är många topologiska egenskaper hos fysiska system universella och överlever trots att det inte finns en region med svag växelverkan.

Ett av målen med denna avhandling var att studera dynamiken hos supersymmetriska teorier i rumtid med ett stort antal dimensioner. Detta är en av utmaningarna i teoretisk fysik idag, eftersom störningsteori inte gäller, och vi istället måste använda konstruktioner från strängteori för att ens påvisa existensen av sådana teorier. Tack vare att partitionsfunktionen är en supersymmetrisk skyddad kvantitet kan vi använda den för att undersöka teorier vid höga energier, något som annars är omöjligt.

Detta är ett viktigt steg i processen att identifiera motsvarigheterna mellan variabler i lokala kvantfältteorier och strängdynamik.

Något som kännetecknar supersymmetriska fältteorier i högre dimensioner är existensen av strängliknande frihetsgrader vid låga energier. Ytterligare ett mål för denna avhandling var att beskriva sådana strängar genom att upprätta en korrespondens mellan en topologisk teori i den högre dimensionella rymden "bulken" och rand-frihetsgrader. I denna analys utläses spektrumet för rand-excitationer på ett sätt som i stor utsträckning liknar låg-energi-excitationer som är ovanför grundtillståndet hos ett system inom den kondenserade materiens fysik.

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