

Some applications of representation theory in
homogeneous dynamics and automorphic functions

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Abstract

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This thesis consists of an introduction and five papers in the general area of dynamics and functions on homogeneous spaces. A common feature is that representation theory plays a key role in all articles.

Papers I-IV are concerned with the effective equidistribution of translates of pieces of subgroup orbits in quotient spaces of semisimple Lie groups by discrete subgroups. In Paper I we focus on finite-volume quotients of $SL(2, \mathbb{C})$ and study the speed of equidistribution for expanding translates orbits of horospherical subgroups. Paper II also studies the effective equidistribution of translates of horospherical orbits, though now in the setting of a quotient of a general semisimple Lie group by a lattice subgroup. Like Paper II, Paper III considers effective equidistribution in quotients of general semisimple Lie groups, but now studies translates of orbits of symmetric subgroups. In all these papers we show that the translates equidistribute with the same exponential rate as for the decay of the corresponding matrix coefficients of the translating subgroup. In Paper IV we consider the effective equidistribution of translates of pieces of horospheres in infinite-volume quotients of groups $SO(n, 1)$ by geometrically finite subgroups, and improve the dependency on the spectral gap for certain known effective equidistribution results.

In Paper V we study the Fourier coefficients of Eisenstein series for generic non-cocompact cofinite Fuchsian groups. We use Zagier's renormalization of certain divergent integrals to enable use of the so-called triple product method, and then combine this with the analytic continuation of irreducible representations of $SL(2, \mathbb{R})$ due to Bernstein and Reznikov.

Keywords: automorphic functions, effective equidistribution, Eisenstein series, homogeneous dynamics, horospheres, lattices, Lie groups, representation theory

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To Gunta

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I S. C. EDWARDS, On the rate of equidistribution of expanding horospheres in finite-volume quotients of $SL(2, \mathbb{C})$.
Journal of Modern Dynamics, Volume 11, 2017, pp. 155-188.
- II S. C. EDWARDS, On the rate of equidistribution of translates of horospheres in $\Gamma \backslash G$.
Manuscript, 2017.
- III S. C. EDWARDS, On the equidistribution of translates of orbits of symmetric subgroups in $\Gamma \backslash G$.
Manuscript, 2018.
- IV S. C. EDWARDS, Effective equidistribution of horospheres in infinite-volume quotients of $SO(n, 1)$ by geometrically finite groups.
Manuscript, 2018.
- V S. C. EDWARDS, Renormalization of integrals of products of Eisenstein series and analytic continuation of representations.
Manuscript, 2018.

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1. Introduction

The central parts of this thesis are the following introduction, in which we survey some of the key concepts and objects related to the thesis, a short summary of the articles, and finally the five articles which make up the majority of the thesis.

1.1 Homogeneous spaces

The central objects of study in this thesis are *homogeneous spaces* of Lie groups. A homogeneous space of a Lie group G is a non-empty set X equipped with a transitive right G -action, which we denote by “ \cdot ”. This means that for a given element $x \in X$, $x \cdot g$ is also an element of X for every $g \in G$, every other element $y \in X$ may be written as $y = x \cdot h$ for some $h \in G$, and

$$(x \cdot g) \cdot h = x \cdot (gh) \quad \forall g, h \in G, x \in X.$$

For a point $x \in X$, we let $\text{Stab}_G(x)$ denote the *stabiliser subgroup* of x in G , i.e. $\text{Stab}_G(x) = \{g \in G : x \cdot g = x\}$. By the *orbit-stabiliser theorem* the map $\iota : \text{Stab}_G(x) \backslash G \rightarrow X$ given by $\iota(\text{Stab}_G(x)g) = x \cdot g$ is a bijection (recall that for a subgroup $H < G$, the quotient space $H \backslash G$ is the set of right H -cosets in G : $H \backslash G = \{Hg : g \in G\}$). Moreover, $\text{Stab}_G(x) \backslash G$ is itself a homogeneous space, with a natural transitive right G -action given by right multiplication: $\text{Stab}_G(x)h \cdot g = \text{Stab}_G(x)hg$, and $\iota(\text{Stab}_G(x)h \cdot g) = \iota(\text{Stab}_G(x)h) \cdot g$. Thus, in order to study many properties of the G -action on X , one can instead consider the corresponding action on the quotient space $\text{Stab}_G(x) \backslash G$.

The homogeneous spaces that are studied in this thesis are, for the most part, quotient spaces $X = \Gamma \backslash G$, where Γ is a discrete subgroup of G and the G -action is the natural right multiplication. We adopt the common practice of suppressing the “ \cdot ” when writing the result of the action of an element of G on an element of X , and instead simply write xg for $x \cdot g$. Since, for any $x = \Gamma g_0 \in \Gamma \backslash G$, $\text{Stab}_G(x) = g_0^{-1} \Gamma g_0$ is a closed subgroup of G , by [19, Theorem 21.20] there exists a *unique* smooth manifold structure on X under which the group action is smooth. Furthermore, we will see that the group action preserves a natural measure on X . As such, the homogeneous spaces considered here are special cases of a more general construction. They are, however, ubiquitous throughout many fields of mathematics; for example, the n -torus \mathbb{T}^n may be realised as $\mathbb{T}^n = \mathbb{Z}^n \backslash \mathbb{R}^n$. Other important examples of

homogeneous spaces of this type include the space of unimodular lattices in \mathbb{R}^n , which may be realised as the quotient $\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$, and quotients $\Gamma \backslash \mathrm{SO}(n, 1)$, which may be viewed as the orthonormal frame bundle $\mathcal{F}_0(\mathcal{M}_\Gamma)$ of a hyperbolic n -orbifold \mathcal{M}_Γ .

As noted above, homogeneous spaces X of the form $X = \Gamma \backslash G$ possess several different structures under which the G -action is well-behaved. The study of how these structures interact with the group action thus leads, in a natural way, to an interplay of ideas in algebra, analysis, geometry, and number theory.

1.2 Homogeneous dynamics

Homogeneous dynamics refers to the study of *dynamical systems* on homogeneous spaces. Over the last 30 years, homogeneous dynamics has developed into a vibrant and highly active field of mathematical research. This is in part due to the fact that several important problems in number theory have been successfully resolved (or had significant progress made towards) by reinterpreting them in terms of homogeneous dynamics. Perhaps the most famous example of this is Margulis' proof of the Oppenheim conjecture [21, 22]. We refer to [15] for a more comprehensive survey of the field of homogeneous dynamics and its applications to number theory.

Let G be a semisimple Lie group and Γ a discrete subgroup of G . Given a one-parameter subgroup $\{g_t\}_{t \in \mathbb{R}}$ of G , one defines a function $\Phi : \mathbb{R} \times \Gamma \backslash G \rightarrow \Gamma \backslash G$ by

$$\Phi(t, x) := xg_t \quad \forall t \in \mathbb{R}, x \in \Gamma \backslash G.$$

The triple $(\Gamma \backslash G, \mathbb{R}, \Phi)$ is then a dynamical system (cf. [12]); indeed, we have $\Phi(0, x) = x$ and $\Phi(s+t, x) = \Phi(s, \Phi(t, x))$ for all $s, t \in \mathbb{R}$ and $x \in \Gamma \backslash G$. For the sake of notational simplicity, we write $g_\mathbb{R}$ in place of the triple $(\Gamma \backslash G, \mathbb{R}, \Phi)$, and xg_t for $\Phi(t, x)$. The long-term behaviour of orbits of dynamical systems of this type is one of the central questions in homogeneous dynamics; that is to say, the goal is to understand the topological and “statistical” (see below) properties of $\{\Phi(t, x)\}_{t \in [0, T]} \subset \Gamma \backslash G$ as $T \rightarrow \infty$ for various choices of $x \in \Gamma \backslash G$.

Since G is semisimple, it is *unimodular*, see [17, Chapter 8]; there exists a unique one-dimensional vector space of *left Haar measures* μ_l such that $\mu_l(gA) = \mu_l(A)$ for all Borel sets $A \subset G$ and all $g \in G$, and this vector space coincides with the space of *right Haar measures* μ_r with the property that $\mu_r(Ag) = \mu_r(A)$ for all A, g as before (here $gAh = \{gah : a \in A\}$ for all $g, h \in G, A \subset G$). Thus every left Haar measure μ is also a right Haar measure, allowing us to (unambiguously) simply refer to μ as a Haar measure on G . The measure μ induces a measure $\mu_{\Gamma \backslash G}$ on $\Gamma \backslash G$ as follows: firstly, let $F \subset G$ be a Borel fundamental domain for Γ in G (i.e. $\#(\Gamma g \cap F) = 1$ for all $g \in G$), and $\pi : G \rightarrow \Gamma \backslash G$ is the natural projection map given by $\pi(g) := \Gamma g$ for all

$g \in G$. Then for a Borel subset $B \subset \Gamma \backslash G$, define $\mu_{\Gamma \backslash G}(B) := \mu(F \cap \pi^{-1}(B))$. Note that $\mu_{\Gamma \backslash G}$ is independent of the choice of F . Moreover, since μ is a right Haar measure, $\mu_{\Gamma \backslash G}$ is G -invariant: $\mu_{\Gamma \backslash G}(Bg) = \mu_{\Gamma \backslash G}(B)$ for all $g \in G$ and Borel subsets $B \subset \Gamma \backslash G$.

The dynamical system $g_{\mathbb{R}}$ is thus in fact a *measure-preserving* system; $\mu_{\Gamma \backslash G}(Bg_t) = \mu_{\Gamma \backslash G}(B)$ for all Borel subsets $B \subset \Gamma \backslash G$. If $\mu(F) < \infty$, we call Γ a *lattice*, and assume that μ has been chosen so that $\mu_{\Gamma \backslash G}$ is a probability measure. This allows one to use the methods of *ergodic theory* (cf. [12]) when studying problems related to the behaviour of the dynamical system. For lattices $\Gamma < G$, recall that the orbit of a point $x \in \Gamma \backslash G$ is said to become *equidistributed* with respect to the probability measure $\mu_{\Gamma \backslash G}$ if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(xg_t) dt = \int_{\Gamma \backslash G} f(y) d\mu_{\Gamma \backslash G}(y) \quad \forall f \in C_c(\Gamma \backslash G).$$

Note that if the orbit $\{xg_t\}_{t \in \mathbb{R}_{\geq 0}}$ becomes equidistributed with respect to $\mu_{\Gamma \backslash G}$, then

$$\overline{\{xg_t\}_{t \in \mathbb{R}_{\geq 0}}} = \Gamma \backslash G. \quad (1.1)$$

In a similar fashion, given any probability measure ν on $\Gamma \backslash G$ (again in the case of general discrete subgroups $\Gamma < G$) we say that $xg_{\mathbb{R}_{\geq 0}}$ becomes equidistributed with respect to ν if $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(xg_t) dt = \int_{\Gamma \backslash G} f d\nu$ for all $f \in C_c(\Gamma \backslash G)$. As in (1.1) above, if $xg_{\mathbb{R}_{\geq 0}}$ becomes equidistributed with respect to a probability measure ν , then $\overline{xg_{\mathbb{R}_{\geq 0}}} = \text{supp } \nu$. Establishing results regarding the equidistribution of $xg_{\mathbb{R}_{\geq 0}}$ for various choices of x , subgroups $g_{\mathbb{R}}$, and measures ν is a major direction of study in homogeneous dynamics, cf. [15]. In the case that Γ is a lattice and $g_{\mathbb{R}}$ is *unipotent*, important results of Ratner [27, 28] (proving conjectures due to Raghunathan; a special case having been proved by Margulis in his proof of the Oppenheim conjecture) from the early 1990s give a complete classification of the possible measures that orbits $xg_{\mathbb{R}_{\geq 0}}$ can become equidistributed with respect to. Ratner's theorems have since become a centrepiece of homogeneous dynamics, with much research being focussed on applying, generalising, and strengthening them in various situations.

1.3 Automorphic functions on the hyperbolic plane

The Poincaré half-plane model of two-dimensional hyperbolic geometry consists of the following subset of the complex plane:

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\},$$

equipped with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$, i.e. the length $|\gamma|$ of a C^1 path $\gamma: [0, 1] \rightarrow \mathbb{H}$ is given by

$$|\gamma| := \int_0^1 \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} dt.$$

We denote the corresponding distance on \mathbb{H} by dist :

$$\text{dist}(z, w) = \text{arcosh} \left(1 + \frac{|z - w|^2}{2\text{Im}(z)\text{Im}(w)} \right) \quad \forall z, w \in \mathbb{H}.$$

The group $\text{PSL}(2, \mathbb{R}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$ acts on \mathbb{H} by Möbius transformations:

$$g \cdot z = \frac{az + b}{cz + d} \quad \forall g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R}), z \in \mathbb{H},$$

and this action is isometric;

$$\text{dist}(g \cdot z, g \cdot w) = \text{dist}(z, w) \quad \forall g \in \text{PSL}(2, \mathbb{R}), z, w \in \mathbb{H}.$$

Given a discrete subgroup $\Gamma < \text{PSL}(2, \mathbb{R})$ and $k \in \mathbb{Z}$, a function $f : \mathbb{H} \rightarrow \mathbb{C}$ is said to be *automorphic of weight k* (with respect to Γ) if

$$f(\gamma \cdot z) = \left(\frac{cz + d}{|cz + d|} \right)^{2k} f(z) \quad \forall \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \mathbb{H}.$$

Number theory is a particularly rich (and the primary) source of automorphic functions, cf., e.g., [1, 33]. However, they also occur, and have applications, in other areas of mathematics, for example in relation to *Ramanujan graphs* in network theory cf., e.g., [30]. In the most classical case, one considers the lattice $\Gamma = \text{PSL}(2, \mathbb{Z})$ and functions f that are automorphic with respect to $\text{PSL}(2, \mathbb{Z})$ (for some $k \in \mathbb{Z}$), holomorphic, and have a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \tag{1.2}$$

(observe that since $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$, $f(z + 1) = f(z)$ for all functions f that are automorphic with respect to $\text{PSL}(2, \mathbb{Z})$). Examples of important automorphic functions for $\text{PSL}(2, \mathbb{Z})$ from number theory include powers of *Jacobi theta functions*, and *Ramanujan's tau function*. Another class of important automorphic functions for $\text{PSL}(2, \mathbb{Z})$ are *Maass wave forms*; these are eigenfunctions of the hyperbolic Laplacian $\Delta = y^{-2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ that are also automorphic functions f of weight zero with respect to $\text{PSL}(2, \mathbb{Z})$, and satisfy a growth condition $|f(x + iy)| \ll y^A$ for all $x + iy \in \mathbb{H}$ for some $A > 0$. Assuming f is a Maass wave form such that $-\Delta f = s(1 - s)f$ for some $s \in \mathbb{C}$, f has a Fourier decomposition similar to (1.2):

$$f(x + iy) = c_0 y^s + d_0 y^{1-s} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi |n| y) \tag{1.3}$$

(here $K_\mu(z)$ denotes the *K-Bessel function*, cf. [11, Chapter 8.4]).

In a similar manner, one may also define holomorphic automorphic functions and Maass wave forms for other lattices $\Gamma < \mathrm{PSL}(2, \mathbb{R})$. If Γ contains elements that are conjugate (in $\mathrm{PSL}(2, \mathbb{R})$) to $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then such functions will have Fourier decompositions of a similar nature to (1.2) and (1.3).

The study of properties of automorphic functions, their Fourier expansions, and the automorphic spectrum of Δ (as well as their natural generalisations; see below) is a major field of research. We note that due to their connections to number theory, automorphic functions for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, as well as for other *arithmetic* lattices, are particularly well-studied. For *non-arithmetic* Γ , the situation is more complicated, and many results that have been established for arithmetic lattices remain wide open for generic lattices. A famous example of this is the *Ramanujan-Petersson conjecture* (proved by Deligne [4, 5] for $\mathrm{PSL}(2, \mathbb{Z})$, and subsequently for *congruence subgroups* of $\mathrm{PSL}(2, \mathbb{Z})$ by Deligne and Serre [6]) regarding the growth of the Fourier coefficients a_n in (1.2).

Using the *Iwasawa decomposition* (see [17, Chapter 6]) of $\mathrm{SL}(2, \mathbb{R})$, letting

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad A = \left\{ a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} : y \in \mathbb{R}_{>0} \right\},$$

$$K = \left\{ k_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} = \mathrm{SO}(2),$$

we have $\mathrm{SL}(2, \mathbb{R}) = NAK$ and $N \cap A = N \cap K = A \cap K = \{e\}$; hence every element g may be written as $g = n_x a_y k$ for some uniquely determined $(x, y, \theta) \in \mathbb{R} \times \mathbb{R}_{>0} \times (\mathbb{R}/2\pi\mathbb{Z})$. Letting $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ be a lattice and f an automorphic function of weight $j \in \mathbb{Z}$ on \mathbb{H} for Γ , we define a function f_G on G by

$$f_G(n_x a_y k_\theta) := f(x + iy) e^{-2ij\theta}.$$

Letting $\tilde{\Gamma} < \mathrm{SL}(2, \mathbb{R})$ denote the inverse image of Γ under the map from $\mathrm{SL}(2, \mathbb{R})$ to $\mathrm{PSL}(2, \mathbb{R})$ given by $g \mapsto \pm g$, we have

$$\left(\pm (n_x a_y k_\theta) \right) \cdot i = x + iy \quad \forall x + iy \in \mathbb{H}, k \in K,$$

and

$$\gamma n_x a_y = n_{\mathrm{Re}((\pm\gamma) \cdot z)} a_{\mathrm{Im}((\pm\gamma) \cdot z)} \begin{pmatrix} \frac{cx+d}{|cz+d|} & -\frac{cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix}$$

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}, z = x + iy \in \mathbb{H}.$$

This gives

$$f_G(\gamma n_x a_y k_\theta) = f((\pm\gamma) \cdot z) \left(\frac{cz+d}{|cz+d|} \right)^{-2j} e^{-2ij\theta} = f(z) e^{-2ij\theta} = f_G(n_x a_y k_\theta)$$

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}, z = x + iy \in \mathbb{H}, \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

i.e. $f_G(\gamma g) = f_G(g)$ for all $g \in \mathrm{SL}(2, \mathbb{R})$, $\gamma \in \tilde{\Gamma}$. We may then define a function \tilde{f} on $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R})$ by $\tilde{f}(\Gamma g) := f_G(g)$. Thus: the map $f \mapsto \tilde{f}$ allows us to identify automorphic functions for Γ on \mathbb{H} with functions on the homogeneous space $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R})$. One can therefore generalise (in a natural way) the notion of an automorphic function to mean any function on a homogeneous space $\Gamma \backslash G$ for some Lie group G and a discrete subgroup $\Gamma < G$.

1.4 Group representations

A *representation* of a group G is a pair (π, V) , where V is a vector space (over a field \mathcal{K}) and π is a group homomorphism from G to $\mathrm{GL}(V)$, hence

$$\begin{aligned} \pi(gh)\mathbf{v} &= \pi(g)(\pi(h)\mathbf{v}), & \pi(g)(a\mathbf{u} + b\mathbf{v}) &= a\pi(g)\mathbf{u} + b\pi(g)\mathbf{v} \\ & & \forall g, h \in G, \mathbf{u}, \mathbf{v} \in V, a, b \in \mathcal{K}. \end{aligned}$$

The study of group representations has grown to be one of the major areas of mathematical research, both because it allows one to study the group structure of an abstract group by representing as a group of linear operators, and also because many objects occurring in a wide range of topics in mathematics can be interpreted as group representations.

The vector spaces for the group representations encountered in this thesis all have either \mathbb{C} or \mathbb{R} as the underlying field of scalars. Furthermore, the representations we consider carry the additional structure of being representations of *topological groups* on *topological vector spaces*: for all sequences $\mathbf{v}_n \rightarrow \mathbf{v}$ (in V) and $g_n \rightarrow g$ (in G), we require that $\pi(g_n)\mathbf{v}_n \rightarrow \pi(g)\mathbf{v}$ (in V). Most of these representation will carry *even more* structure: they are *unitary representations* of semisimple Lie groups G , i.e. V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and

$$\langle \pi(g)\mathbf{u}, \pi(g)\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \quad \forall g \in G, \mathbf{u}, \mathbf{v} \in V.$$

Unitary representations are particularly well-behaved: given an arbitrary unitary representation, one can “decompose” it into so-called *irreducible* representations, which are (reasonably) well-understood (see, for example, [16, Chapter 1] for an introduction to the representation theory of semisimple Lie groups).

Given a semisimple Lie group G and a lattice $\Gamma < G$, a particular unitary representation that occurs naturally in connection with Sections 1.2 and 1.3 is $(\rho, L^2(\Gamma \backslash G))$, where $L^2(\Gamma \backslash G) = L^2(\Gamma \backslash G, \mu_{\Gamma \backslash G})$ and

$$[\rho(g)f](x) := f(xg) \quad \forall g \in G, f \in L^2(\Gamma \backslash G), \mu_{\Gamma \backslash G}\text{-almost every } x \in \Gamma \backslash G.$$

This allows many problems in homogeneous dynamics and automorphic functions to be reinterpreted in terms of (and then attacked using methods from) representation theory.

2. Summary of papers

2.1 Paper I

In Paper I, we study the *effective equidistribution* of expanding translates of pieces of *horospherical* orbits in finite-volume quotients of $\mathrm{SL}(2, \mathbb{C})$. The setup is as follows: we denote $G = \mathrm{SL}(2, \mathbb{C})$ and let Γ be a lattice in G . For $t \in \mathbb{R}$, we define

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in G,$$

and let $A = \{g_t\}_{t \in \mathbb{R}}$; note that this is a one-parameter subgroup of G . By the *Howe-Moore vanishing theorem* [14], for all $f_1, f_2 \in L^2(\Gamma \backslash G)$, we have

$$\lim_{t \rightarrow \pm\infty} \int_{\Gamma \backslash G} [\rho(g_t) f_1](x) f_2(x) d\mu_{\Gamma \backslash G}(x) = \int_{\Gamma \backslash G} f_1 d\mu_{\Gamma \backslash G} \int_{\Gamma \backslash G} f_2 d\mu_{\Gamma \backslash G} \quad (2.1)$$

Furthermore, by placing certain restrictions on f_1, f_2 one can obtain a quantitative version of (2.1); for these functions, the integral in the left-hand side of (2.1) equals the product of the integrals in the right-hand side plus an error term that decays exponentially in t .

By (2.1), the action of A on the probability space $(\Gamma \backslash G, \mu_{\Gamma \backslash G})$ is *mixing*, and hence *ergodic* (cf., e.g., [8, Chapter 2]). By *Birkhoff's ergodic theorem*, we then have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(xg_{\pm t}) dt = \int_{\Gamma \backslash G} f d\mu_{\Gamma \backslash G}$$

for all $f \in L^1(\Gamma \backslash G)$ and $\mu_{\Gamma \backslash G}$ -almost every $x \in \Gamma \backslash G$. While this shows that the subset $\{xg_{-t}\}_{t \in [0, T]}$ becomes equidistributed in $\Gamma \backslash G$ for almost all x , we consider another type of subset related to g_t that becomes equidistributed in $\Gamma \backslash G$ for *all* x .

Define $N < G$ by

$$N := \{h \in G : \lim_{t \rightarrow -\infty} g_t h g_{-t} = e\}. \quad (2.2)$$

N is called the *expanding horospherical subgroup* with respect to g_t . Note that

$$N = \{n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\}.$$

For a set $B \subset \mathbb{C}$, $x \in \Gamma \backslash G$, and $t \in \mathbb{R}_{\leq 0}$ we consider the subset $\{x n_z g_t\}_{z \in B}$ of $\Gamma \backslash G$. This set is thus a piece of the orbit of x under the horospherical subgroup

N , which has been translated by g_t . Again using the Howe-Moore theorem, combined with *Margulis' thickening technique* [23], one can show that for any relatively compact subset of positive Lebesgue measure $B \subset \mathbb{C}$,

$$\lim_{t \rightarrow -\infty} \frac{1}{m(B)} \int_B f(xn_z g_t) dm(z) = \int_{\Gamma \backslash G} f d\mu_{\Gamma \backslash G} \quad (2.3)$$

$$\forall x \in \Gamma \backslash G, f \in L^2(\Gamma \backslash G) \cap C(\Gamma \backslash G),$$

where m denotes the Lebesgue measure on \mathbb{C} .

In Paper I, we prove an *effective* version of (2.3); we show that for all $x, \in \Gamma \backslash G$, $t \leq 0$, f in an appropriate *Sobolev space*, and relatively compact subsets $B \subset \mathbb{C}$ whose boundary satisfies a certain Lipschitz condition, we have that the integral $\frac{1}{m(B)} \int_B f(xn_z g_t) dm(z)$ equals the right-hand side of (2.3) plus an error term that decays exponentially with respect to t . Moreover, the rate of exponential decay we obtain matches that given in the corresponding quantitative version of (2.1).

Similar results have previously been established for translates of closed horocycles on finite-volume hyperbolic surfaces; these have been studied using spectral theory by (amongst others) Hejhal [13], Sarnak [29], Selberg (unpublished), Strömbergsson [35], and Zagier [37]. In [36], Södergren generalised these results to translates of closed horospheres in higher-dimensional hyperbolic manifolds.

In order to prove our results, we use the *method of Burger*. This is a representation-theoretic method, first used in [3] to study the equidistribution of horocycles in quotients $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$, where Γ is a *geometrically finite convex cocompact* subgroup of $\mathrm{SL}(2, \mathbb{R})$. Strömbergsson later used the same method to study the equidistribution of horocycles in $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$, for all lattices $\Gamma < \mathrm{SL}(2, \mathbb{R})$; cf. [34]. The idea behind the method is as follows: for $f \in L^2(\Gamma \backslash G) \cap C(\Gamma \backslash G)$, let $f_t = \frac{1}{m(B)} \int_B \rho(n_z g_t) f dm(z) \in L^2(\Gamma \backslash G)$. The quantity from (2.3) that we are interested in is then given by $f_t(x)$ ($x \in \Gamma \backslash G$). The decomposition of the unitary representation $(\rho, L^2(\Gamma \backslash G))$ into irreducible representations and the fact that (by Schur's lemma) the centre of the universal enveloping algebra of the Lie algebra of G acts by scalars on *smooth vectors* of irreducible representations are then used to express an identity that captures the asymptotic behaviour of f_t (as a vector in $L^2(\Gamma \backslash G)$). In order to study the pointwise behaviour of f_t , we then use an automorphic Sobolev inequality due to Bernstein and Reznikov [2].

2.2 Paper II

In Paper II, we generalise Burger's method to study the equidistribution of translates of pieces of horospherical orbits in homogeneous spaces $\Gamma \backslash G$ for arbitrary semisimple Lie groups G and lattices $\Gamma < G$ that satisfy a certain

technical condition. We let G be such a group, and choose a one-parameter subgroup $\{g_t\}_{t \in \mathbb{R}} < G$ such that each operator Ad_{g_t} (acting on the Lie algebra of G) is diagonalizable over \mathbb{R} . This ensures that N , the expanding horospherical subgroup with respect to g_t (defined as in (2.2)), is the unipotent radical of some parabolic subgroup of G . Consequently, this enables us to use the Harish-Chandra isomorphism to construct differential equations related to g_t from the centre of the universal enveloping algebra of the Lie algebra of G ; this is key to the method.

The main difference from Paper I is that (for Lie algebraic reasons) in order to get the relevant translates to equidistribute with the same rate of exponential decay as the *matrix coefficients* of g_t , instead of integrating over subsets of N , we must consider averages against smooth, compactly supported test functions on N , i.e. we study integrals of the form

$$\int_N f(xng_t)\chi(n) d\mu_N(n),$$

where $x \in \Gamma \backslash G$, $f \in C(\Gamma \backslash G) \cap L^2(\Gamma \backslash G)$, $\chi \in C_c^\infty(N)$ and μ_N denotes a Haar measure on N (note that N is unimodular). Furthermore, we must also combine the underlying method of [3] (and Paper I) with an iterative procedure so as to obtain the desired rate.

2.3 Paper III

Like Papers I and II, Paper III is concerned with the effective equidistribution in homogeneous spaces $\Gamma \backslash G$ (as before, G is a semisimple Lie group and Γ is a lattice in G) of translates of pieces of orbits of subgroups of G . However, Paper III studies the translates of orbits of *symmetric* subgroups of G , instead of horospherical subgroups. A symmetric subgroup $S < G$ is defined to be the identity component of the subgroup of fixed points for some Lie group involution σ ; i.e. S is the identity component of G_σ , where

$$G_\sigma := \{g \in G : \sigma(g) = g\}$$

(since σ is an involution, $\sigma(gh) = \sigma(g)\sigma(h)$ and $\sigma^2(g) = g$ for all $g, h \in G$). We once again use the method of Burger to show that translates of orbits of points $x \in \Gamma \backslash G$ under S by certain one-parameter subgroups $\{g_t\}_{t \in \mathbb{R}}$ equidistribute with the same exponential rate as the matrix coefficients decay. The main arguments are quite similar to those of Paper II; the structure theory of symmetric subgroups permits us to associate a parabolic subgroup of G (and hence also a horospherical subgroup) to the translate being considered. This permits relatively straightforward modifications to be made to the proofs of the central results of Paper II that allow them to be carried over to the case of symmetric subgroups.

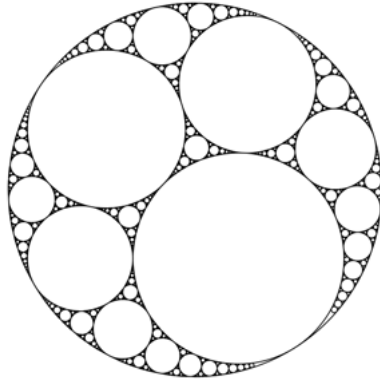


Figure 2.1. A bounded Apollonian circle packing.

The equidistribution of translates of symmetric subgroups is a much-studied topic in homogeneous dynamics. One reason for this is due to applications to counting problems on *affine symmetric spaces*. An affine symmetric space is a homogeneous space G/S , where S is a symmetric subgroup of a semisimple Lie group G . Duke, Rudnick, and Sarnak [7], and Eskin and McMullen [9], showed how to relate counting problems for Γ -orbits on G/S (for the natural *left* G -action on G/S) to the equidistribution of translates of S -orbits in $\Gamma \backslash G$. While many results regarding the equidistribution of symmetric translates are proved with the goal of obtaining similar counting results, our motivation has instead been on understanding the precise asymptotic behaviour of the translates.

2.4 Paper IV

As in Papers I through III, in Paper IV, we use the method of Burger to study the effective equidistribution of translates of pieces of orbits in homogeneous spaces $\Gamma \backslash G$. We once again consider the translates of horospherical orbits, though now consider $G = \mathrm{SO}_0(n, 1)$ (i.e. G is the identity component of $\mathrm{SO}(n, 1)$), and discrete subgroups $\Gamma < G$ which need no longer be lattices, but instead are *geometrically finite* (hence the measures $\mu_{\Gamma \backslash G}$ are not necessarily finite). The study of dynamics on infinite-volume quotients $\Gamma \backslash G$ has grown in recent years; this is due to recent interest in counting problems for *thin groups*, to which such dynamics are connected, cf. [25].

One particularly beautiful example of this in *Apollonian circle packings* (see Figure 2.1). In [18], Kontorovich and Oh proved that translates of closed horospheres equidistribute in $\Gamma \backslash \mathrm{SO}_0(3, 1)$ with respect to what is now called the *Burger-Roblin* measure, and used this to obtain leading-term asymptotics

for various counting problems related to Apollonian packings, for example counting the number of circles in a given packing with curvature less than or equal to $T > 0$ (as $T \rightarrow \infty$). Lee and Oh [20] subsequently proved effective versions of the equidistribution results of [18]; this enabled them to obtain an error term with a power savings for counting problems related to Apollonian circle packings. Using slightly different (but still related) methods, Mohammadi and Oh [24] proved effective equidistribution results for translates of orbits of horospherical and symmetric subgroups in $\Gamma \backslash \mathrm{SO}_0(n, 1)$ for all $n \geq 2$ and geometrically finite Γ that satisfy certain *spectral gap* conditions.

The main goal of Paper IV is to study the dependency of the error term in the equidistribution results of [24] on the spectral gap. Using the method of Burger instead of the thickening techniques of [24], we are able to improve the dependency on said spectral gap to match the corresponding rate of decay of matrix coefficients. However, unlike in the case that $\Gamma < \mathrm{SO}_0(n, 1)$ is a lattice, the subrepresentation of $(\rho, L^2(\Gamma \backslash \mathrm{SO}_0(n, 1)))$ whose matrix coefficients have the slowest rate of decay also contributes to the error term. For this reason, we are not able to improve the effective equidistribution results of [24] for all functions in $C_c^\infty(\Gamma \backslash \mathrm{SO}_0(n, 1))$ (since the quotients $\Gamma \backslash \mathrm{SO}_0(n, 1)$ can have infinite volume, it is convenient to consider only functions with compact support); our results yield genuinely new equidistribution statements only for functions in the orthogonal complement to the subrepresentations mentioned above.

In the special case $G = \mathrm{SO}_0(3, 1)$, and considering only translates of closed horospheres, we combine the method of Burger with results of Lee and Oh [20] regarding functions in the subrepresentation whose matrix coefficients have the slowest rate of exponential decay, and do in fact obtain an improvement in the effective equidistribution results of [20] for all $\mathrm{SO}(2)$ -invariant functions in $C_c^\infty(\Gamma \backslash G)$. This enables us to improve the error terms in some counting problems of a similar nature to (and including) those for Apollonian circle packings mentioned above.

2.5 Paper V

Paper V is concerned with the growth properties of Fourier coefficients of *Eisenstein series* for the hyperbolic plane. We recall that for a non-cocompact lattice $\Gamma \subset G = \mathrm{PSL}(2, \mathbb{R})$, to each *parabolic fixed point* $\eta \in \partial_\infty \mathbb{H} = \mathbb{R} \cup \{\infty\}$, we define the Eisenstein series $E_\eta(z, s)$, where $(z, s) \in \mathbb{H} \times \{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) > 1\}$, by

$$E_\eta(z, s) := \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \mathrm{Im}(h_\eta \gamma \cdot z)^s,$$

where $\Gamma_\eta = \{\gamma \in \Gamma : \gamma \cdot \eta = \eta\}$, and $h_\eta \in G$ is such that $h_\eta \cdot \eta = \infty$ and $h_\eta \Gamma_\eta h_\eta^{-1} = \{n_x : x \in \mathbb{Z}\}$ (that η is a parabolic fixed point for Γ ensures that such an h_η exists). For each $z \in \mathbb{H}$, $s \mapsto E_\eta(z, s)$ has a meromorphic continuation in s to the entire complex plane, and the location of the poles of

$s \mapsto E_\eta(z, s)$ are independent of z . On the other hand, for each fixed $s \in \mathbb{C}$ that is not a pole, $z \mapsto E_\eta(z, s)$ is an automorphic function of weight zero with respect to Γ . Furthermore, each such $E_\eta(z, s)$ is a Maass wave form: $-\Delta E_\eta(x + iy, s) = s(1 - s)E_\eta(x + iy, s)$. As such, each $E_\eta(z, s)$ has a Fourier decomposition similar to (1.3) (in fact, $E_\eta(z, s)$ has one such decomposition for each cusp of $\Gamma \backslash \mathbb{H}$). In Paper V, we prove formulas for sums of the form $\sum_{0 < |n| \leq N} |a_n|^2$ (the numbers a_n being the Fourier coefficients for the Eisenstein series, cf. (1.3)) as $N \rightarrow \infty$ for Eisenstein series $E_\eta(z, 1/2 + it)$ (where $t \in \mathbb{R}$) for generic (in particular, non-arithmetic) lattices $\Gamma < G$. These formulas lead to the bound $|a_n| \ll_\varepsilon n^{\frac{5}{12} + \varepsilon}$ for the individual Fourier coefficients.

The Eisenstein series with $\operatorname{Re}(s) = \frac{1}{2}$ span the *continuous spectrum* of Δ in $L^2(\Gamma \backslash \mathbb{H})$ ($L^2(\Gamma \backslash \mathbb{H})$ is identified with $L^2(\mathcal{F}, \mu)$, where $\mathcal{F} \subset \mathbb{H}$ is a reasonable fundamental domain for the action of Γ on \mathbb{H} , and $d\mu(x + iy) = \frac{dx dy}{y^2}$); each individual Eisenstein series $E_\eta(\cdot, \frac{1}{2} + it)$ is *not* in $L^2(\Gamma \backslash \mathbb{H})$, but appropriate continuous superpositions of them are. For the most part, previous results regarding the Fourier coefficients of Maass wave forms for generic Γ have focussed on *Maass cusp forms*; these are Maass wave forms that are of *rapid decay* in all of the cusps of $\Gamma \backslash \mathbb{H}$. The “standard”, or *Hecke*, bound on the Fourier coefficients for Maass cusp forms is $|a_n| \ll |n|^{1/2}$.

The first improvement over the standard bound for Maass cusp forms for generic Γ is due to Sarnak; in [31], he combined the *Rankin-Selberg method* [26, 32] with a clever use of analytic continuation to obtain the bound $|a_n| \ll_\varepsilon |n|^{5/12 + \varepsilon}$. The analytic continuation in [31] was reinterpreted in terms of representation theory by Bernstein and Reznikov in [2], enabling them to develop representation-theoretic methods (in particular, *G*-invariant Sobolev norms) to further reduce the bound on the Fourier coefficients for Maass cusp forms to $|a_n| \ll_\varepsilon |n|^{1/3 + \varepsilon}$. This bound matches the best available bound for the Fourier coefficients of *holomorphic cusp forms* for generic lattices Γ , due to Good [10].

The aim of Paper V is to develop the Rankin-Selberg method and a theory of analytic continuation of representations as in [2] to go beyond the Hecke bound for the Fourier coefficients for Eisenstein series for generic lattice $\Gamma < G$. Since the Rankin-Selberg method involves studying *triple products* of Maass wave forms, and Eisenstein series are not in $L^2(\Gamma \backslash \mathbb{H})$, the integrals that a naïve attempt to use the Rankin-Selberg method suggests one should consider are not finite. For this reason, we follow Zagier [38], and instead use *renormalized integrals*; these allow one to give meaning to certain integrals that ordinarily would not have a finite value. Viewing Eisenstein series as the images in $C^\infty(\Gamma \backslash G)$ of certain vectors in irreducible unitary representations of G allows us to then make use of the results of [2] regarding the analytic continuation of representations of $\operatorname{SL}(2, \mathbb{R})$. We are, however, unable to make use of *G*-invariant Sobolev norms as in [2]; consequently, our bounds do not match those available for (Maass and holomorphic) cusp forms.

3. Summary in Swedish

Denna avhandling består av fem artiklar. De fyra första artiklarna handlar om problem relaterade till *effektiv likafördelning* i *homogena rum* och den femte studerar Fourierkoefficienterna för *Eisensteinserier* på det hyperboliska övre halvplanet.

De homogena rum som studeras i avhandlingen är alla kvotrum på formen $\Gamma \backslash G$, där G är en halvenkel Liegrupp och Γ är en diskret delgrupp till G . Dessa rum kommer utrustade med en naturlig höger G -verkan: givet $\Gamma h \in \Gamma \backslash G$ och $g \in G$, definierar vi $\Gamma h \cdot g = \Gamma hg$. I *homogen dynamik* studeras asymptotiska egenskaper hos gruppverkan på $\Gamma \backslash G$ för olika delgrupper till G . Ofta är man särskilt intresserad av *statistiska* egenskaper (med avseende på ett naturligt mått på $\Gamma \backslash G$; i många fall är detta ett G -invariant mått som induceras från ett Haarmått på G) hos dessa banor.

I artiklarna I-IV låter vi $H < G$ vara antingen en *horosfärisk* eller *symmetrisk* delgrupp till G och $\{g_t\}_{t \in \mathbb{R}} < G$ en 1-parameterdelgrupp till G som (i viss mening) "expanderar" H . Målet i artiklarna är att ge en så exakt kvantitativ beskrivning som möjligt på hur jämnt utspridda, eller *likafördelade*, delmängder av $\Gamma \backslash G$ på formen xBg_t , där $x \in \Gamma \backslash G$ och B är en delmängd i H , är i $\Gamma \backslash G$ då $t \rightarrow \pm\infty$. Delmängderna på formen xBg_t kallas (g_t) -*translat* av delar av H -banan av x .

Artikel I studerar likafördelningen av translat av delar av horosfäriska banor i fallet då $G = \mathrm{SL}(2, \mathbb{C})$ och Γ är ett *gitter* i G , d.v.s. Γ har ändlig kovolum i G . Eftersom $\Gamma < G$ är ett gitter finns det ett unikt G -invariant mått $\mu_{\Gamma \backslash G}$ på $\Gamma \backslash G$ (som är inducerat från ett Haarmått på G). Vi väljer delgrupperna H och $\{g_t\}_{t \in \mathbb{R}}$ som ovan genom att göra följande definitioner:

$$H = N = \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}, \quad g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

För att mäta hur jämnt utspridd i $\Gamma \backslash G$ ett translat $x\{n_z : z \in B\}g_t$ är, där $B \subset \mathbb{C}$ är en delmängd med positivt Lebesguemått (som vi betecknar $dm(z)$), undersöker vi skillnaden mellan värdet av integraler över den med värdet av integraler över hela rummet $\Gamma \backslash G$ för olika val av funktioner f på $\Gamma \backslash G$ d.v.s vi undersöker hur stort följande uttryck:

$$\left| \frac{1}{m(B)} \int_B f(xn_z g_t) dm(z) - \int_{\Gamma \backslash G} f d\mu_{\Gamma \backslash G} \right|. \quad (3.1)$$

Huvudresultatet i artikel I säger att (3.1) avtar exponentiellt med avseende på t då $t \rightarrow -\infty$ för delmängder $B \subset \mathbb{C}$ och funktioner f som uppfyller vissa

tekniska krav. Dessutom gäller det att hastigheten hos detta exponentiella avtagande är densamma som för avtagandet hos *matriskoefficienterna* för den *unitära representationen* $(\rho, L^2(\Gamma \backslash G)_0)$. Här betecknar $L^2(\Gamma \backslash G)_0$ det ortogonala komplementet till konstantfunktionerna i $L^2(\Gamma \backslash G) = L^2(\Gamma \backslash G, \mu_{\Gamma \backslash G})$, och ρ betecknar *högertranslation*: $(\rho(g)f)(x) := f(xg)$ för alla $g \in G$, alla funktioner $f \in L^2(\Gamma \backslash G)$ och $\mu_{\Gamma \backslash G}$ -nästan alla $x \in \Gamma \backslash G$.

För att visa resultaten i artikel I generaliserar vi en metod som först användes av Burger i [3] för att studera liknande frågor i fallet $G = \mathrm{SL}(2, \mathbb{R})$. Denna metod är av en representationsteoretisk karaktär och är baserad på identiteter för element i centret av den universella envelopperande algebran till en halvenkel Liealgebra. I artiklarna II-IV vidareutvecklar vi metoden i olika situationer. I artikel II studeras återigen den effektiva likafördelningen i $\Gamma \backslash G$ av expanderande translaterade banor för horosfäriska delgrupper, dock nu för generella halvenkla Liegrupper G och gitter $\Gamma < G$ som uppfyller ett visst tekniskt krav. Även i artikel III betrakas likafördelningsproblem i $\Gamma \backslash G$ för G och Γ som i artikel II, men nu för translaterade av *symmetriska* delgrupper till G . Artikel IV handlar om translaterade av banor av horosfäriska delgrupper i kvotrum $\Gamma \backslash \mathrm{SO}_0(n, 1)$, där $\mathrm{SO}_0(n, 1)$ är identitetskomponenten av $\mathrm{SO}(n, 1)$, och $\Gamma < \mathrm{SO}_0(n, 1)$ inte längre (nödvändigtvis) är ett gitter, utan *geometriskt ändligt*. Faktumet att $\Gamma \backslash \mathrm{SO}_0(n, 1)$ kan ha oändlig volym med avseende på de naturliga måtten inducerade från Haarmåtten på $\mathrm{SO}_0(n, 1)$ gör likafördelningspåståendena mer komplicerade; ofta involverar de flera mått utöver de som är relaterade till Haarmått. På senare tid har det funnits stort intresse för likafördelningsresultat i rum $\Gamma \backslash \mathrm{SO}_0(n, 1)$ med oändlig volym. Detta på grund av att de har tillämpningar i talteori, t.ex i räkneproblem för *Apolloniska cirkelpackningar* (se figur 2.1).

Artikel V handlar om *Eisensteinserier* på det hyperboliska övre halvplanet \mathbb{H} . Gruppen $G = \mathrm{PSL}(2, \mathbb{R})$ verkar på \mathbb{H} genom Möbiusavbildningar. Givet ett gitter $\Gamma < G$ med egenskapen att den hyperboliska ytan $\Gamma \backslash \mathbb{H}$ har $\kappa \geq 1$ stycken spetsar, så finns det en uppsättning Eisensteinserier $E_j(z, s)$, där $z \in \mathbb{H}$, $s \in \mathbb{C}$, och $j \in \{1, \dots, \kappa\}$. För varje $z \in \mathbb{H}$ är $s \mapsto E_j(z, s)$ en meromorf funktion på \mathbb{C} , och för s som är inte en pol så är $E_j(z, s)$ Γ -invariant: $E_j(\gamma \cdot z, s) = E_j(z, s)$ för alla $z \in \mathbb{H}$, $\gamma \in \Gamma$. Dessutom är Eisensteinserierna *Maass vågformer* (d.v.s. egenfunktioner till den hyperboliska Laplaceoperatör $\Delta = y^{-2}(\partial_x^2 + \partial_y^2)$): $-\Delta E_j(x + iy, s) = s(1 - s)E_j(x + iy, s)$. Som följd av detta har alla $E_j(z, s)$ en Fourierutveckling (1.3) i varje spets av $\Gamma \backslash \mathbb{H}$. Målet med artikel V är att hitta nya begränsningar för Fourierkoefficienterna i dessa utvecklingar för *godtyckliga* gitter Γ . För att göra detta använder vi *Rankin-Selberg*-metoden. Denna metod utvecklades ursprungligen för att studera liknande frågor för *cuspidala* funktioner, men tack vare en modifikation av Zagier [38] går den även att använda för Eisensteinserier. Genom att sedan omtolka Eisensteinserierna i termer av representationsteori kan vi använda en *analytisk fortsättning* av representationer som Bernstein och Reznikov gör i [2].

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References

- [1] T. A. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second Edition, Graduate Texts in Mathematics, Springer-Verlag, 1990.
- [2] J. Bernstein and A. Reznikov, *Analytic Continuation of Representations and Estimates of Automorphic Forms*, *Ann. of Math.*, Vol. 150 (1999), pp. 329-352.
- [3] M. Burger, *Horocycle Flow on Geometrically Finite Surfaces*, *Duke Math. J.*, Vol. 61, No. 3 (1990), pp. 779-803.
- [4] P. Deligne, *Formes Modulaires et Représentations l -adiques*, *Séminaire Bourbaki* 11 (1968-1969), pp. 139-172.
- [5] P. Deligne, *La Conjecture de Weil I*, *Publications Mathématiques de L'Institut des Hautes Scientifiques* 43 (1974), pp. 273-307.
- [6] P. Deligne and J. Serre, *Formes Modulaires de Poids 1*, *Annales scientifiques de l'École Normale Supérieure* (1974), pp. 507-530.
- [7] W. Duke, Z. Rudnick and P. Sarnak, *Density of Integer Points on Affine Symmetric Varieties*, *Duke Math J.* 71 (1993) pp. 143-180.
- [8] M. Einsiedler and T. Ward, *Ergodic Theory with a view towards Number Theory*, Graduate Texts in Mathematics Vol. 259, Springer, 2010.
- [9] A. Eskin and C. McMullen, *Mixing, Counting and Equidistribution in Lie Groups*, *Duke Math. J.* 71 (1993) pp. 181-209.
- [10] A. Good, *Cusp Forms and Eigenfunctions of the Laplacian*, *Math. Ann.* 255 (1981), pp. 523-548.
- [11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, Academic Press, 2007.
- [12] B. Hasselblatt and A. Katok, *Handbook of Dynamical Systems, Volume 1A*, Elsevier Science, North Holland, 2002.
- [13] D. Hejhal, *On the Uniform Equidistribution of Long Closed Horocycles in Loo-Keng Hua: A Great Mathematician of the Twentieth Century*, *Asian J. Math.* 4 (2000), No. 4, pp. 839-853.
- [14] R. Howe and C. C. Moore, *Asymptotic Properties of Unitary Representations*, *J. Func. Anal.* 32 (1979), Kluwer. Acad., pp. 72-96.
- [15] D. Kleinbock, N. Shah, and A. Starkov, *Dynamics of Subgroup Actions on Homogeneous Spaces of Lie Groups and Applications to Number Theory*, in *Handbook of Dynamical Systems, Volume 1A*, Elsevier Science, North Holland, 2002, pp. 813-930.
- [16] A. Knapp, *Representation Theory of Semisimple Lie Groups*, Princeton University Press, 1986.
- [17] A. Knapp, *Lie Groups Beyond an Introduction*, Second Edition, Birkhäuser, 2002.
- [18] A. Kontorovich, H. Oh, *Apollonian Circle Packings and Closed Horospheres on Hyperbolic 3-Manifolds*, *J. of AMS*, July 2011, Vol. 24, No. 3 , pp. 603-648.
- [19] J. Lee, *Introduction to Smooth Manifolds*, Second Edition, Graduate Texts in Mathematics, Vol. 218, Springer, 2012.

- [20] M. Lee and H. Oh, Effective Circle Count for Apollonian Circle Packings and Closed Horospheres, *Geom. Funct. Anal.*, Vol. 23 (2013), pp. 580-621.
- [21] G. A. Margulis, Formes Quadratiques Indéfinies et Flots Unipotents sur les Espaces Homogènes, *C. R. Acad. Sci. Paris Ser. I* 304 (1987), pp. 247-253.
- [22] G. A. Margulis, Discrete Subgroups and Ergodic Theory, in *Number Theory, Trace Formulas and Discrete Groups*, Academic Press (1989), pp. 377-398.
- [23] G. A. Margulis, On Some Aspects of the Theory of Anosov Systems, with a survey, "Periodic Orbits of Hyperbolic Flows" by Richard Sharp, *Springer Monographs in Mathematics*, Springer, Berlin, 2004.
- [24] A. Mohammadi and H. Oh, Matrix Coefficients, Counting and Primes for Orbits of Geometrically Finite Groups, *J. Eur. Math. Soc.*, Vol. 17 (2015), pp. 837-897.
- [25] H. Oh, Harmonic Analysis, Ergodic Theory and Counting for Thin Groups, *Thin groups and Superstrong Approximation*, MSRI Publications Vol. 61, 2013.
- [26] R. A. Rankin, Contributions to the Theory of Ramanujan's Function $\tau(n)$ and Similar Arithmetical functions, I, *Proc. Cambridge Philos. Soc.* 35 (1939), pp. 351-372.
- [27] M. Ratner, On Measure Rigidity of Unipotent Subgroups of Semisimple Groups, *Acta Math.* 165 (1990), pp. 229-309.
- [28] M. Ratner, On Raghunathan's Measure Conjecture, *Ann. of Math.* 134 (1991), pp. 545-607.
- [29] P. Sarnak, Asymptotic Behavior of Periodic Orbits of the Horocycle Flow and Eisenstein Series, *Comm. Pure Appl. Math.* 34, No. 6 (1981), pp. 719-739.
- [30] P. Sarnak, *Some Applications of Modular Forms*, Cambridge Tracts in Math. 99, Cambridge University Press, 1990.
- [31] P. Sarnak, Integrals of Products of Eigenfunctions, *Int. Math. Res. Notes* (1994), pp. 251-260.
- [32] A. Selberg, Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, *Arch. Math. Naturvid.* 43 (1940), pp. 47-50.
- [33] G. Shimura, *Automorphic Functions and Number Theory*, Lecture Notes in Mathematics, Springer, 1968.
- [34] A. Strömbergsson, On the Deviation of Ergodic Averages for Horocycle Flows, *Journal of Modern Dynamics*, 7 (2013), pp. 291-328.
- [35] A. Strömbergsson, On the Uniform Equidistribution of Long Closed Horocycles, *Duke Math. J.* 123 (2004), pp. 507-547.
- [36] A. Södergren, On the Uniform Equidistribution of Closed Horospheres in Hyperbolic Manifolds, *Proc. Lond. Math. Soc. (3)* Vol. 105 No. 2 (2012), pp. 225-280.
- [37] D. Zagier, Eisenstein Series and the Riemann Zeta Function in Automorphic Forms, *Representation Theory and Arithmetic (Bombay, 1979)*, Tata Inst. Fund. Res. Studies in Math. 10, Tata Inst. Fundamental Res., Bombay (1981), pp. 275-301.
- [38] D. Zagier, The Rankin-Selberg Method for Automorphic Functions Which Are Not of Rapid Decay, *J. Fac. Sci. Univ. of Tokyo, Sect. 1A* 28 (1981), pp. 415-437.

