UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2018:3

Drinfeld centers

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Examensarbete i matematik, 30 hp
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April 2018

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Abstract

In the first part of this paper, we describe the structure of the center $\mathcal{Z}(R\text{-}\mathbf{Mod})$ of the category of left R-modules. Its natural structure as a ring is shown to be isomorphic to the subring $Z(R)$.

In the sections that follow, we present the basics of monoidal categories by regarding them as single-object bicategories. The Drinfeld center $\mathcal{Z}(\mathscr{C})$ of a monoidal category $\mathscr{C}$ is defined and its basic properties presented.

The second half of the paper is devoted to describing the structure of the Drinfeld center of the monoidal categories $\mathbf{Vect}_{\mathbb{C}}$ and $\mathbb{Z}_2\text{-}\mathbf{mod}$.

Contents

1 Basics of cateogries

In order to make this paper as self-contained as possible, we present the basics of category theory. This secion, and those similar to it, are of course highly skippable.

Definition 1.1. A category $\mathcal{C}$ consists of the following:

- a class $\mathrm{Ob}(\mathcal{C})$ of objects.

- for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class of morphisms or arrows, denoted by $\mathrm{Hom}(X, Y)$. In particular, for the pair (X, X), we require the existence of an identity morphism $\mathrm{id}_X : X \to X$.

- for every three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\varphi \in \mathrm{Hom}(X, Y)$ and $\psi \in \mathrm{Hom}(Y, Z)$, a binary operation $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z)$ with $(\varphi, \psi) \mapsto \psi \circ \varphi$, called the composition.

We require that the composition satisfies the following axioms:

i) if $\varphi \in \mathrm{Hom}(X, Y), \psi \in \mathrm{Hom}(Y, Z)$ and $\xi \in \mathrm{Hom}(Z, W)$ then $\xi \circ (\psi \circ \varphi) = (\xi \circ \psi) \circ \varphi$.

ii) if $\varphi \in \mathrm{Hom}(X, Y)$ and $\psi \in \mathrm{Hom}(Y, Z)$ then $\varphi \circ \mathrm{id}_X = \varphi$ and $\mathrm{id}_Z \circ \psi = \psi$.

Remark 1.2. The identity morphism id_X is unique for every object $X \in \mathrm{Ob}(\mathcal{C})$. If id'_X were another identity morphism, we would have

$$\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}'_X = \mathrm{id}'_X .$$

Definition 1.3. A category $\mathcal{C}$ is said to be small if both the class of objects and the morphism classes are sets, and not proper classes.

Definition 1.4. The terminal category, denoted by $\mathbf{1}$, is the category having a single object and a single morphism(the identity).

Definition 1.5. Let $\mathcal{C}$ be a category. A morphism $\varphi \in \mathrm{Hom}(X, Y)$ is called an isomorphism if if there exists $\psi \in \mathrm{Hom}(Y, X)$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

Definition 1.6. Let $\mathcal{C}$ be a category. Two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ are said to be isomorphic if there exists $\varphi \in \mathrm{Hom}(X, Y)$ which is an isomorphism.

Definition 1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then a functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following:

- a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.

- for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ such that

  i) for every $X \in \mathrm{Ob}(\mathcal{C})$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

  ii) for composable morphisms φ and ψ we have $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$.

Definition 1.8. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is full if each map

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is surjective.

Definition 1.9. Let $\mathcal{C}, \mathcal{A}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is faithful if each map

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is injective.

Definition 1.10. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is dense if each $Y \in \mathrm{Ob}(\mathcal{D})$ is isomorphic to an object $F(X)$ for some $X \in \mathrm{Ob}(\mathcal{C})$.

Definition 1.11. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \Rightarrow G$ is a map $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ such that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(\varphi)]{} & G(Y) \end{array}$$

commutes for every morphism $\varphi \in \mathrm{Hom}(X, Y)$.

Definition 1.12. A natural transformation η such that each component η_X is an isomorphism is called a natural isomorphism.

Definition 1.13. Let $\mathcal{C}, \mathcal{D}$ be categories. An equivalence of $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists another functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\eta : F \circ G \to \mathrm{id}_{\mathcal{D}}$ and $\mu : G \circ F \to \mathrm{id}_{\mathcal{C}}$.

Theorem 1.14. *Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then F is an equivalence if and only if F is full, faithful and dense.*

Example 1.15. *Let the categories $\mathcal{C}$ and $\mathcal{D}$ be defined as follows:*

Fix a positive integer n. The category $\mathcal{C}$ has a single object and its morphisms are $n \times n$ complex matrices, with composition given by matrix multiplication.

The category $\mathcal{D}$ has as objects complex vector spaces of dimension n, and its morphisms are linear maps. Composition is given by the usual composition of maps.

Fixing the standard basis of the vector space $\mathbb{C}^n$, we define a functor $F : \mathcal{C} \to \mathcal{D}$ by mapping the object of $\mathcal{C}$ to the vector space $\mathbb{C}^n$ and mapping each matrix to the linear map which in the standard basis is given by that matrix.

Then it is clear by linear algebra that F is a functor and that F is full and faithful. Moreover, F is dense since all objects of $\mathcal{D}$, being vector spaces of the same finite dimension, are isomorphic.

Since the functor F is full, faithful and dense it is an equivalence.

Definition 1.16. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Then the product category $\mathcal{C} \times \mathcal{D}$ consists of the following:

- pairs (X, Y) of objects, where $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.
- pairs $(\varphi, \psi) : (X, Y) \to (Z, W)$ of morphisms, where for $X, Z \in \mathrm{Ob}(\mathcal{C})$ and $Y, W \in \mathrm{Ob}(\mathcal{D})$, we have $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ and $\psi \in \mathrm{Hom}_{\mathcal{D}}(Y, W)$.
- componentwise composition $(\varphi_2, \psi_2) \circ_{\mathcal{C}\times\mathcal{D}} (\varphi_1, \psi_1) = (\varphi_2 \circ_{\mathcal{C}} \varphi_1, \psi_2 \circ_{\mathcal{D}} \psi_1)$.
- identity morphisms of the form $\mathrm{id}_{(X,Y)} = (\mathrm{id}_X, \mathrm{id}_Y)$.

Proposition 1.17. *Let $\mathcal{C}$ be a small category and $\mathbf{1}$ the terminal category. Then we have $\mathcal{C} \times \mathbf{1} \cong \mathcal{C}$ (as objects in the category of small categories).*

Proof. Consider the map $F : \mathcal{C} \to \mathcal{C} \times \mathbf{1}$ given by

$$X \mapsto (X, 1)$$

$$f \mapsto (f, \mathrm{id}).$$

Then we have $F(\mathrm{id}_X) = (\mathrm{id}_X, \mathrm{id}) = \mathrm{id}_{(X,1)} = \mathrm{id}_{F(X)}$ and

$$F(g \circ f) = (g \circ f, \mathrm{id}) = (g, \mathrm{id}) \circ (f, \mathrm{id}) = F(g) \circ F(f)$$

so F is a functor. It is obvious that we can construct the inverse functor $F^{-1} : \mathcal{C} \times \mathbf{1} \to \mathcal{C}$ by

$$(X, 1) \mapsto X$$

$$(f, \mathrm{id}) \mapsto f.$$

□

Definition 1.18. Let $\mathcal{C}$ be a category. We define the center of $\mathcal{C}$, denoted by $\mathcal{Z}(\mathcal{C})$ to be the class of natural transformations from the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself.

Example 1.19. *We recall that a monoid is a set with a binary associative operation with identity. Any monoid M may be regarded as a category. Define the category $\mathcal{M}$ as follows:*

- *$\mathcal{M}$ has one object, $\bullet$.*
- *Morphisms are all elements of M.*
- *Composition is given by the multiplication of M. The identity of M then acts as identity for the composition, and composition is associative since the multiplication of M is associative.*

It is clear that given such a category we can reconstruct our monoid, so the above is an equivalent definition.

Now consider the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$. Since $\mathcal{M}$ has only one object and the morphisms are elements of M, a natural transformation from the identity functor to itself consists of an element $z \in M$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{x} & M \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle z} \\ M & \xrightarrow[x]{} & M \end{array}$$

commutes for all $x \in M$. Since the composition is just multiplication in M, this is equivalent to $zx = xz$ for all $x \in M$. Now it is clear that $\mathcal{Z}(\mathcal{M}) = Z(M) = \{z \in M : zx = xz \ \forall x \in M\}$.

2 Categories of modules

In the following section, let R be a unital ring. Recall that the center of a ring R is the subring $Z(R) = \{z \in R : zr = rz \ \forall r \in R\}$.

Definition 2.1. A left R-module is an abelian group $(M, +)$ together with a binary operation $\cdot : R \times M \to M$ such that

i) $r \cdot (x + y) = r \cdot x + r \cdot y$

ii) $(r + s) \cdot x = r \cdot x + s \cdot x$

iii) $(rs) \cdot x = r \cdot (s \cdot x)$

iv) $1_R \cdot x = x$

for all $x, y \in M$ and $r, s \in R$. When necessary, we refer to the action of R on M as $\cdot_M$.

Definition 2.2. Let M, N be left R-modules. A homomorphism of R-modules is a map $\varphi : M \to N$ such that

$$\varphi(r \cdot_M x + s \cdot_M y) = r \cdot_N \varphi(x) + s \cdot_N \varphi(y)$$

for all $x, y \in M$ and $r, s \in R$.

For a fixed ring R the left R-modules together with module homomorphisms form a category, R-**Mod**.

2.1 The center of R-mod

Proposition 2.3. *$\mathcal{Z}(R$-**Mod**$)$ is a ring under componentwise addition and composition of homomorphisms.*

Proof. $\mathcal{Z}(R\text{-}\mathbf{Mod})$ consists of natural transformations from $\mathrm{id}_{R\text{-}\mathbf{Mod}}$ to itself, so $\eta \in \mathcal{Z}(R\text{-}\mathbf{Mod})$ maps every module M to an endomorphism η_M of M.

For any module, its endomorphisms carry the natural structure of a ring with pointwise addition and composition. This implies that the componenwise addition and composition of a family of endomorphisms yields another family. We check that these operations preserve naturality. Let $\varphi : M \to N$ be a homomorphism and let $x \in M$.

$$\begin{aligned}
\varphi((\eta_M + \mu_M)(x)) &= \varphi(\eta_M(x) + \mu_M(x)) \\
&= \varphi(\eta_M(x)) + \varphi(\mu_M(x)) \\
&= \eta_M(\varphi(x)) + \mu_M(\varphi(x)) \\
&= (\eta_M + \mu_M)(\varphi(x))
\end{aligned}$$

$$\begin{aligned}
(\varphi \circ (\eta_M \circ \mu_M))(x) &= ((\varphi \circ \eta_M) \circ \mu_M)(x) \\
&= ((\eta_M \circ \varphi) \circ \mu_M)(x) \\
&= (\eta_M \circ (\varphi \circ \mu_M))(x) \\
&= (\eta_M \circ (\mu_M \circ \varphi))(x) \\
&= ((\eta_M \circ \mu_M) \circ \varphi)(x)
\end{aligned}$$

□

Definition 2.4. For any $z \in Z(R)$, define a natural transformation $\eta^z : \mathrm{id}_{R\text{-}\mathbf{Mod}} \to \mathrm{id}_{R\text{-}\mathbf{Mod}}$ by $\eta^z_M(x) = z \cdot x,\ x \in M \in R\text{-}\mathbf{Mod}$.

Remark 2.5. The natural transformation η^z is indeed in the center of R-**Mod**. Since η^z_M is given by left multiplication with z, we get:

$$\begin{aligned}
\eta^z_N \circ \varphi(x) &= \eta^z_N(\varphi(x)) \\
&= z \cdot_N \varphi(x) \\
&= \varphi(z \cdot_M x) \\
&= \varphi(\eta^z_M(x)) \\
&= \varphi \circ \eta^z_M(x)
\end{aligned}$$

which is equivalent to commutativity of the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
{\scriptstyle \eta^z_M}\downarrow & & \downarrow{\scriptstyle \eta^z_N} \\
M & \xrightarrow{\varphi} & N
\end{array}$$

Definition 2.6. For any $z \in Z(R)$, define the endomorphism φ^z of R by $\varphi^z(x) = z \cdot x$.

Proposition 2.7. *The homomorphisms φ^z induced by $Z(R)$ form a subring $\mathrm{End}^Z_{R-}(R)$ of $\mathrm{End}_{R-}(R)$.*

Proof. The identity endomorphism is φ^1. Let $z, c \in Z(R)$. We observe that

i) $(\varphi^z + \varphi^c)(x) = (z + c) \cdot x$

ii) $(\varphi^z + \varphi^{-z})(x) = (z + (-z)) \cdot x = 0$

iii) $(\varphi^z(\varphi^c))(x) = \varphi^z(c \cdot x) = z \cdot c \cdot x = (zc) \cdot x$

and we have $z + c, -z, zc \in Z(R)$ since $Z(R)$ is a subring. □

Proposition 2.8. *The rings $Z(R)$ and $\mathrm{End}^Z_{R-}(R)$ are isomorphic.*

Proof. Consider the map $F : Z(R) \to \mathrm{End}^Z_{R-}(R)$ defined by $F(z) = \varphi^z$. Let $z, c \in Z(R)$. We have:

$$\begin{aligned} F(z + c)(x) &= \varphi^{z+c}(x) \\ &= (z + c)(x) \\ &= (\varphi^z + \varphi^c)(x) \\ &= (F(z) + F(c))(x) \end{aligned}$$

$$\begin{aligned} F(zc)(x) &= \varphi^{zc}(x) \\ &= (zc) \cdot x \\ &= (\varphi^z \circ \varphi^c)(x) \\ &= (F(z) \circ F(c))(x). \end{aligned}$$

We clearly have $F(1) = \varphi^1 = \mathrm{id}_R$ so F is a homomorphism. For injectivity, note that $F(z) = F(c) \implies \varphi^z = \varphi^c$ which implies $z = \varphi^z(1) = \varphi^c(1) = c$. For any φ^z we have $F(z) = \varphi^z$ so F is surjective, hence an isomorphism. □

Theorem 2.9. *The evaluation map $\varepsilon : \mathcal{Z}(R\text{-}\boldsymbol{Mod}) \to \mathrm{End}_{R-}(R)$ defined by $\eta \mapsto \eta_R$ induces an isomorphism between $\mathcal{Z}(R\text{-}\boldsymbol{Mod})$ and $Z(R)$.*

Lemma 2.10. *Let M be an R-module and let R be the regular module. Then, for any $x \in M$, there exists a unique homomorphism $\xi : R \to M$ such that $\xi(1) = x$.*

Proof of lemma 2.10. Consider the map $\xi(r) = r \cdot_M x$. Then ξ is a homomorphism:

$$\begin{aligned} \xi(r + s) &= (r + s) \cdot_M x \\ &= r \cdot_M x + s \cdot_M x \\ &= \xi(r) + \xi(s) \end{aligned}$$

$$\begin{aligned} \xi(rs) &= (rs) \cdot_M x \\ &= r \cdot_M (s \cdot_M x) \\ &= r \cdot_M \xi(s) \end{aligned}$$

Suppose $\psi : R \to M$ is another homomorphism with $\psi(1) = x$. Then

$$\psi(r) = r \cdot_M \psi(1) = r \cdot_M \xi(1) = \xi(r), \ \forall r \in R$$

so ξ is unique. ■

Corollary 2.11. *The rings* $\operatorname{End}_{R-}(R)$ *and* R^{op}*, are isomorphic.*

Proof. For every endomorphism φ in $\operatorname{End}_{R-}(R)$ we have $\varphi(r) = r \cdot \varphi(1)$ and hence φ is given by right multiplication with $\varphi(1)$. But from lemma 2.10 it follows that every element of R^{op}(wich is the same as an element of R) is given as the image of 1 under a unique endomorphism, so the map $f : \operatorname{End}_{R-}(R) \to R^{op}$ with $\varphi \mapsto \varphi(1)$ gives a clear bijection. Moreover, f is a homomorphism since

$$f(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1)$$

and

$$f(\varphi \circ \psi) = \varphi(\psi(1)) = \psi(1)\varphi(1) = f(\psi)f(\varphi).$$

■

Corollary 2.12. *The ring* $\operatorname{End}_{R-R}(R)$*, where* R *is the regular* R*-*R*-bimodule, is isomorphic to* $Z(R)$*.*

Proof. For every endomorphism φ , we have $r \cdot \varphi(1) = \varphi(r) = \varphi(1) \cdot r$, so φ is given by multiplication with the central element $\varphi(1)$. Again, by lemma 2.10, this induces the bijection $f : \operatorname{End}_{R-R}(R) \to Z(R)$ defined $\varphi \mapsto \varphi(1)$. That f is a homomorphism can be checked in the same way as in the previous corollary. ■

Proof of theorem 2.9. Consider $\varepsilon : \mathcal{Z}(R\text{-}\mathbf{Mod}) \to \operatorname{End}_{R-}(R)$ defined by $\eta \mapsto \eta_R$. For every $x \in R$ there exists a unique module homomorphism $\psi : R \to R$ with $\psi(1) = x$ by lemma 2.10. Then we have:

$$\begin{aligned} \eta_R(1) \cdot x &= \eta_R(1) \cdot \psi(1) \\ &= \psi(\eta_R(1)) \\ &= \eta_R(\psi(1)) \\ &= \psi(1) \cdot \eta_R(1) \\ &= x \cdot \eta_R(1). \end{aligned}$$

so $z = \eta_R(1) \in Z(R)$. Now, for any $r \in R$, we have $\eta_R(r) = r \cdot \eta_R(1) = \eta_R(1) \cdot r$. So we have $\eta_R = \varphi^z$.

By definition of ε, η^z is mapped to η^z_R. But by the above, $\eta_R = \varphi^z$ for any $\eta \in \mathcal{Z}(R\text{-}\mathbf{Mod})$. So in particular, $\varepsilon(\eta^z) = \eta^z_R = \varphi^z$. Thus, the map ε is surjective onto $\operatorname{End}^Z_{R-}(R)$, since $\eta^z \in \mathcal{Z}(R\text{-}\mathbf{Mod})$ exists for every $z \in Z(R)$.

If $\varepsilon(\eta) = \varepsilon(\mu)$, then $\eta_R = \mu_R$. By naturality of η and μ we get the following commutative diagrams:

$$\begin{CD} R @>\varphi>> M \\ @V{\eta_R=\mu_R}VV @VV{\eta_M}V \\ R @>\varphi>> M \end{CD} \qquad \begin{CD} R @>\varphi>> M \\ @V{\eta_R=\mu_R}VV @VV{\mu_M}V \\ R @>\varphi>> M \end{CD}$$

which imply

$$\eta_M \circ \varphi = \varphi \circ \eta_R = \varphi \circ \mu_R = \mu_M \circ \varphi$$

for every homomorphism $\varphi : R \to M$. By lemma 2.10 there is a unique homomorphism $\psi : R \to M$ with $\psi(1) = x$ for every $x \in M$. From this, we get:

$$\begin{aligned} \eta_M(x) &= \eta_M(\psi(1)) \\ &= \mu_M(\psi(1)) \\ &= \mu_M(x) \end{aligned}$$

so $\eta_M = \mu_M$ for any M and hence $\eta = \mu$, so ε is injective.

Since addition and composition are performed componentwise in $\mathcal{Z}(R\text{-}\mathbf{Mod})$ we get:

$$\begin{aligned} \varepsilon(\eta + \mu) &= (\eta + \mu)_R \\ &= \eta_R + \mu_R \\ &= \varepsilon(\eta) + \varepsilon(\mu) \end{aligned}$$

$$\begin{aligned} \varepsilon(\eta \circ \mu) &= (\eta \circ \mu)_R \\ &= \eta_R \circ \mu_R \\ &= \varepsilon(\eta_R) \circ \varepsilon(\mu_R). \end{aligned}$$

If we let id denote the identity natural transformation and let id_R denote the identity homomorphism, then $\varepsilon(\mathrm{id}) = \mathrm{id}_R$ so ε is a homomorphism of unital rings.

Lemma 2.13. *Every natural transformation $\eta \in \mathcal{Z}(R\text{-}\boldsymbol{Mod})$ is of the form $\eta = \eta^z$ for some $z \in Z(R)$.*

Proof of lemma 2.13. Let $x \in M$. By lemma 2.10 there is a unique homomorphism $\psi : R \to M$ with $\psi(1) = x$. We also know that $\eta_R = \varphi^z$ for some $z \in Z(R)$. Then, using naturality of η, we get:

$$\begin{aligned} \eta_M(x) &= \eta_M(\psi(1)) \\ &= \psi(\eta_R(1)) \\ &= \psi(z) \\ &= z \cdot \psi(1) \\ &= z \cdot x. \end{aligned}$$

so we see that the map η_M is actually η^z_M for every module M, so we have $\eta = \eta^z$. ■

Using lemma 2.13 we now see that the map $\varepsilon : \mathcal{Z}(R\text{-}\mathbf{Mod}) \to \mathrm{End}^{Z}_{R-}(R)$ is an isomorphism. By proposition 2.8 $\mathrm{End}^{Z}_{R-}(R)$ is isomorphic to $Z(R)$, so in conclusion we get

$$\mathcal{Z}(R\text{-}\mathbf{Mod}) \cong \mathrm{End}^{Z}_{R-}(R) \cong Z(R) \implies \mathcal{Z}(R\text{-}\mathbf{Mod}) \cong Z(R).$$

□

2.2 Tensor product of modules

Definition 2.14. Let R be a ring, let M be a right R-module, let N be a left R-module and let G be an abelian group. Then an R-balanced product is a map $\varphi : M \times N \to G$ satisfying the following:

i) $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$

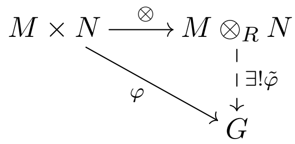
ii) $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$

iii) $\varphi(m \cdot_M r, n) = \varphi(m, r \cdot_N n)$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$.

Definition 2.15. Let R be a ring, let M be a right R-module, and let N be a left R-module. Then the tensor product over R, denoted by $M \otimes_R N$ is an abelian group together with a balanced product $\otimes : M \times N \to M \otimes_R N$ satisfying the following:

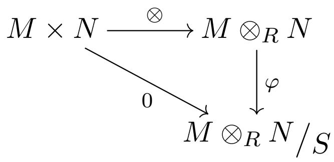
For every abelian group G and every balanced product $\varphi : M \times N \to G$ there exists a unique group homomorphism $\tilde{\varphi} : M \otimes_R N \to G$ such that $\tilde{\varphi} \circ \otimes = \varphi$. In a commutative diagram:



This condition is known as the universal property of the tensor product.

Proposition 2.16. *Elements of the form $x \otimes_R y$ with $x \in M, y \in N$ generate $M \otimes_R N$.*

Proof. Consider the subgroup $S \subset M \otimes_R N$ generated by elements of the form $x \otimes_R y$. Let π be the projection onto the quotient group $M \otimes_R N/S$. Note that the zero map $M \times N \to M \otimes_R N/S$ is a balanced product. By the universal property, there exists a unique group homomorphism φ which makes the diagram



commute. We clearly have $0 = 0 \circ \otimes$, but we also have $(\pi \circ \otimes)(x, y) = \pi(x \otimes y) = 0$ since $(x, y) \in S$. By the universal property, we have $\pi = 0$ which implies $S = M \otimes_R N$. □

Corollary 2.17. *Let M be a left R-module. Then the tensor product $R \otimes_R M$ is itself a left R-module with module structure given by $r \cdot (x \otimes y) = rx \otimes y$.*

Proof. Note that R is an $R-R-$bimodule. First we have

$$\begin{aligned} r \cdot \sum x_i \otimes y_i &= r \cdot \sum 1 \otimes x_i y_i \\ &= r \cdot (1 \otimes \sum x_i y_i) \\ &= r \otimes \sum x_i y_i \\ &= \sum r \otimes x_i y_i \\ &= \sum r x_i \otimes y_i \end{aligned}$$

so our scalar multiplication extends nicely to sums of elements of the form $x \otimes y$. Next we check the module axioms:

i)

$$\begin{aligned} r \cdot \left(\sum_{i=1}^{n} x_i \otimes y_i + \sum_{j=1}^{m} x_j \otimes y_j \right) &= \sum_{i=1}^{n} r x_i \otimes y_i + \sum_{j=1}^{m} r x_j \otimes y_j \\ &= r \cdot \sum_{i=1}^{n} x_i \otimes y_i + r \cdot \sum_{j=1}^{m} x_j \otimes y_j . \end{aligned}$$

ii)

$$\begin{aligned} (r+s) \cdot \sum_{i=1}^{n} x_i \otimes y_i &= (r+s) \cdot \sum_{i=1}^{n} 1 \otimes x_i y_i \\ &= (r+s) \cdot \left(1 \otimes \sum_{i=1}^{n} x_i y_i \right) \\ &= (r+s) \otimes \sum_{i=1}^{n} x_i y_i \\ &= r \otimes \sum_{i=1}^{n} x_i y_i + s \otimes \sum_{i=1}^{n} x_i y_i \\ &= \sum_{i=1}^{n} r x_i \otimes y_i + \sum_{i=1}^{n} s x_i \otimes y_i \\ &= r \cdot \sum_{i=1}^{n} x_i \otimes y_i + s \cdot \sum_{i=1}^{n} x_i \otimes y_i . \end{aligned}$$

iii)

$$\begin{aligned}(rs) \cdot \sum_{i=1}^{n} x_i \otimes y_i &= \sum_{i=1}^{n} (rs)x_i \otimes y_i \\ &= \sum_{i=1}^{n} r(sx_i) \otimes y_i \\ &= r \cdot \sum_{i=1}^{n} sx_i \otimes y_i \\ &= r \cdot (s \cdot \sum_{i=1}^{n} x_i \otimes y_i).\end{aligned}$$

iv)

$$1 \cdot \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} 1x_i \otimes y_i = \sum_{i=1}^{n} x_i \otimes y_i.$$

■

Proposition 2.18. *Let M be a left R-module. Then the left R-modules $R \otimes_R M$ and M are isomorphic.*

Proof. Consider the map $\varphi : R \otimes_R M \to M$ defined by $\sum_{i=1}^{n} x_i \otimes y_i \mapsto \sum_{i=1}^{n} x_i y_i$.

$$\begin{aligned}\varphi\left(\sum_{i=1}^{n} x_i \otimes y_i + \sum_{j=1}^{m} x_j \otimes y_j\right) &= \varphi\left(\sum_{i=1}^{n} 1 \otimes x_i y_i + \sum_{j=1}^{m} 1 \otimes x_j y_j\right) \\ &= \varphi\left(1 \otimes \sum_{i=1}^{n} x_i y_i + 1 \otimes \sum_{j=1}^{m} x_j y_j\right) \\ &= \varphi\left(1 \otimes \left(\sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{m} x_j y_j\right)\right) \\ &= 1 \cdot \left(\sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{m} x_j y_j\right) \\ &= \sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{m} x_j y_j \\ &= \varphi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) + \varphi\left(\sum_{j=1}^{m} x_j \otimes y_j\right).\end{aligned}$$

$$\begin{aligned}
\varphi\left(r\cdot\sum_{i=1}^{n}x_i\otimes y_i\right) &= \varphi\left(\sum_{i=1}^{n}rx_i\otimes y_i\right)\\
&= \sum_{i=1}^{n}rx_iy_i\\
&= r\cdot\sum_{i=1}^{n}x_iy_i\\
&= r\cdot\varphi\left(\sum_{i=1}^{n}x_i\otimes y_i\right)
\end{aligned}$$

so φ is a homomorphism. Suppose

$$\varphi\left(\sum_{i=1}^{n}x_i\otimes y_i\right)=\sum_{i=1}^{n}x_iy_i=\sum_{j=1}^{m}x'_jy'_j=\varphi\left(\sum_{j=1}^{m}x'_j\otimes y'_j\right).$$

Then we get:

$$\begin{aligned}
\sum_{i=1}^{n}x_i\otimes y_i &= \sum_{i=1}^{n}1\otimes x_iy_i\\
&= 1\otimes\sum_{i=1}^{n}x_iy_i\\
&= 1\otimes\sum_{j=1}^{m}x'_jy'_j\\
&= \sum_{j=1}^{m}1\otimes x'_jy'_j\\
&= \sum_{j=1}^{m}x'_j\otimes y'_j
\end{aligned}$$

so φ is injective. For every $m\in M$, we have $\varphi(1\otimes m)=m$, so φ is surjective, hence an isomorphism. □

Proposition 2.19. *Let M, N be left R-modules and let $\varphi : M\to N$ be a homomorphism. Then the map $\varphi_\otimes : R\otimes_R M\to R\otimes_R N$ defined by $x\otimes y\mapsto x\otimes\varphi(y)$ is a homomorphism.*

Proof.

$$\begin{aligned}
\varphi_{\otimes}\left(\sum_{i=1}^{n} x_i \otimes y_i\right) &= \varphi_{\otimes}\left(\sum_{i=1}^{n} 1 \otimes x_i y_i\right) \\
&= \varphi_{\otimes}\left(1 \otimes \sum_{i=1}^{n} x_i y_i\right) \\
&= 1 \otimes \varphi\left(\sum_{i=1}^{n} x_i y_i\right) \\
&= 1 \otimes \sum_{i=1}^{n} \varphi(x_i y_i) \\
&= 1 \otimes \sum_{i=1}^{n} x_i \varphi(y_i) \\
&= \sum_{i=1}^{n} 1 \otimes x_i \varphi(y_i) \\
&= \sum_{i=1}^{n} x_i \otimes \varphi(y_i)
\end{aligned}$$

$$\begin{aligned}
\varphi_{\otimes}\left(\sum_{i=1}^{n} x_i \otimes y_i + \sum_{j=1}^{m} x_j \otimes y_j\right) &= \varphi_{\otimes}\left(1 \otimes \left(\sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{m} x_j y_j\right)\right) \\
&= 1 \otimes \varphi\left(\sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{m} x_j y_j\right) \\
&= 1 \otimes \left(\sum_{i=1}^{n} x_i \varphi(y_i) + \sum_{j=1}^{m} x_j \varphi(y_j)\right) \\
&= \sum_{i=1}^{n} 1 \otimes x_i \varphi(y_i) + \sum_{j=1}^{m} 1 \otimes x_j \varphi(y_j) \\
&= \sum_{i=1}^{n} x_i \otimes \varphi(y_i) + \sum_{j=1}^{m} x_j \otimes \varphi(y_j) \\
&= \varphi_{\otimes}\left(\sum_{i=1}^{n} x_i \otimes y_i\right) + \varphi_{\otimes}\left(\sum_{j=1}^{m} x_j \otimes y_j\right) \\
\varphi_{\otimes}\left(r \sum_{i=1}^{n} x_i \otimes y_i\right) &= \varphi_{\otimes}\left(\sum_{i=1}^{n} r x_i \otimes y_i\right) \\
&= \sum_{i=1}^{n} r x_i \otimes \varphi(y_i) \\
&= r \cdot \sum_{i=1}^{n} x_i \otimes \varphi(y_i) \\
&= r \cdot \varphi_{\otimes}\left(\sum_{i=1}^{n} x_i \otimes y_i\right).
\end{aligned}$$

$\square$

Proposition 2.20. *Define F on R-**Mod** by*

i) $M \mapsto R \otimes_R M$ *for modules.*

ii) $\varphi : M \to N \mapsto \varphi_\otimes : R \otimes_R M \to R \otimes_R N$ *for homomorphisms.*

*Then F is an endofunctor of R-**Mod**.*

Proof. For every module M, we have:

$$F(\mathrm{id}_M) = \mathrm{id}_{M\otimes} = \mathrm{id}_{R\otimes_R M}\,.$$

For homomorphisms $\varphi : M \to N$ and $\psi : N \to L$ we have:

$$\begin{aligned}
F(\psi \circ \varphi)\left(\sum_{i=1}^{n} x_i \otimes y_i\right) &= (\psi \circ \varphi)_\otimes \left(\sum_{i=1}^{n} x_i \otimes y_i\right) \\
&= \sum_{i=1}^{n} x_i \otimes (\psi \circ \varphi)(y_i) \\
&= \sum_{i=1}^{n} x_i \otimes \psi(\varphi(y_i)) \\
&= \psi_\otimes \left(\sum_{i=1}^{n} x_i \otimes \varphi(y_i)\right) \\
&= \psi_\otimes \left(\varphi_\otimes \left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right) \\
&= F(\psi) \circ F(\varphi) \left(\sum_{i=1}^{n} x_i \otimes y_i\right).
\end{aligned}$$

$\square$

Theorem 2.21. *Let F be the functor defined in proposition 2.20. Then $F \cong \mathrm{id}_{R\text{-}\boldsymbol{Mod}}$.*

Proof. Consider $\eta : F \to \mathrm{id}_{R\text{-}\mathbf{Mod}}$ with components $\eta_M : R \otimes_R M \to M$ given by

$$\eta_M \left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} x_i y_i.$$

This is an isomorphism of modules by proposition 2.18. The diagram

$$\begin{array}{ccc}
R \otimes_R M & \xrightarrow{\varphi_\otimes} & R \otimes_R N \\
{\scriptstyle \eta_M}\big\downarrow & & \big\downarrow{\scriptstyle \eta_N} \\
M & \xrightarrow[\varphi]{} & N
\end{array}$$

commutes since

$$\begin{aligned}
\varphi\left(\eta_M\left(\sum_{i=1}^n x_i \otimes y_i\right)\right) &= \varphi\left(\sum_{i=1}^n x_i y_i\right) \\
&= \sum_{i=1}^n x_i \varphi(y_i) \\
&= \eta_N\left(\sum_{i=1}^n x_i \otimes \varphi(y_i)\right) \\
&= \eta_N\left(\varphi_\otimes\left(\sum_{i=1}^n x_i \otimes y_i\right)\right)
\end{aligned}$$

so η is a natural isomorphism. □

Theorem 2.22. *Bimodule endomorphisms of R are exactly the components at R of the natural transformations in $\mathcal{Z}(R\textbf{-Mod})$.*

Proof. Let $\varphi \in \mathrm{End}_{R-R}(R)$ be an endomorphism of bimodules. Then, for every $r \in R$, we have

$$\varphi(r) = r \cdot \varphi(1) \quad \text{and} \quad \varphi(r) = \varphi(1) \cdot r.$$

This implies that φ is given by multiplication with the element $z = \varphi(1) \in Z(R)$. We know that such endomorphisms are components of natural transformations from $\mathrm{id}_{R\textbf{-Mod}}$ itself at R. Moreover, by lemma 2.13, $\mathcal{Z}(R\textbf{-Mod})$ consists only of such natural transformations. □

2.2.1 Tensor product of bimodules

Proposition 2.23. *Let R, S, T be (unital) rings. Let ${}_R M_S$ be an R-S-bimodule and let ${}_S N_T$ be an S-T-bimodule. Then $M \otimes_S N$ is an R-T-bimodule, with scalar multiplication given by*

$$\begin{cases} r \cdot \sum_{i=1}^n m_i \otimes n_i = \sum_{i=1}^n r m_i \otimes n_i \\ \sum_{j=1}^m m_j \otimes n_j \cdot t = \sum_{j=1}^m m_j \otimes n_i t \end{cases} \quad \forall r \in R, t \in T$$

Proof.

$$r \cdot \left(\sum_{i=1}^{n} m_i \otimes n_i + \sum_{j=1}^{m} m_j \otimes n_j \right) = \sum_{i=1}^{n} r m_i \otimes n_i + \sum_{j=1}^{m} r m_j \otimes n_j$$
$$= r \cdot \sum_{i=1}^{n} m_i \otimes n_i + r \cdot \sum_{j=1}^{m} m_j \otimes n_j$$

$$(r + r') \sum_{i=1}^{n} m_i \otimes n_i = \sum_{i=1}^{n} (r + s) m_i \otimes n_i$$
$$= \sum_{i=1}^{n} (r m_i + s m_i) \otimes n_i$$
$$= \sum_{i=1}^{n} r m_i \otimes n_i + r' m_i \otimes n_i$$
$$= r \cdot \sum_{i=1}^{n} m_i \otimes n_i + r' \cdot \sum_{i=1}^{n} m_i \otimes n_i$$

$$(rr') \cdot \sum_{i=1}^{n} m_i \otimes n_i = \sum_{i=1}^{n} (rr') m_i \otimes n_i$$
$$= \sum_{i=1}^{n} r(r' m_i) \otimes n_i$$
$$= r \cdot \sum_{i=1}^{n} r' m_i \otimes n_i$$

$$1_R \cdot \sum_{i=1}^{n} m_i \otimes n_i = \sum_{i=1}^{n} 1_R m_i \otimes n_i$$
$$= \sum_{i=1}^{n} m_i \otimes n_i$$

so we have left R-module structure by using the left R-module strucutre of M. The right

T-module structure is checked similarly. Moreover,

$$\begin{aligned}\left(r \cdot \sum_{i=1}^{n} m_i \otimes n_i\right) \cdot t &= \left(\sum_{i=1}^{n} r m_i \otimes n_i\right) \cdot t \\ &= \sum_{i=1}^{n} r m_i \otimes n_i t \\ &= r \cdot \left(\sum_{i=1}^{n} m_i \otimes n_i t\right).\end{aligned}$$

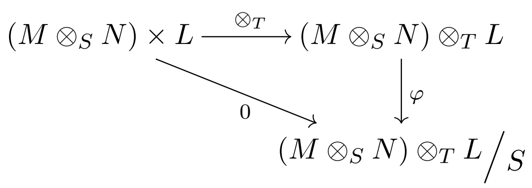
□

Proposition 2.24. *If R, S, T, U are (unital) rings and we have bimodules ${}_R M_S, {}_S N_T, {}_T L_U$, then*

$$(M \otimes_S N) \otimes_T L \cong M \otimes_S (N \otimes_T L).$$

Lemma 2.25. *Elements of the form $(m \otimes_S n) \otimes_T l$ generate $(M \otimes_S N) \otimes_T L$ and elements of the form $m \otimes_S (n \otimes_T l)$ generate $M \otimes_S (N \otimes_T L)$.*

Proof of lemma 2.25. Let S be the subgroup of $(M \otimes_S N) \otimes_T L$ generated by the elements $(m \otimes_S n) \otimes_T l$ and let π be the projection onto the quotient $(M \otimes_S N) \otimes_T L / S$. By the universal property of the tensor product, there exists a unique homomorphism φ which makes the following diagram commute:



Clearly, we have $0 = 0 \circ \otimes_T$. But since $((m \otimes_S n) \otimes_T l) \in S$ we have

$$\pi \otimes_T ((m \otimes n), l) = \pi((m \otimes_S n) \otimes_T l) = 0 \implies \pi = 0 \implies S = (M \otimes_S N) \otimes_T L.$$

The second statement can be proved in the same way. ■

Proof of proposition 2.24. Consider the map

$$f : (M \otimes_S N) \otimes_T L \to M \otimes_S (N \otimes_T L)$$

defined by

$$\sum_{i=1}^{n} (m_i \otimes_S n_i) \otimes_T l_i \mapsto \sum_{i=1}^{n} m_i \otimes_S (n_i \otimes_T l_i).$$

$$f\left(\sum_{i=1}^{n}(m_i \otimes_S n_i) \otimes_T l_i + \sum_{j=1}^{m}(m_j \otimes_S n_j) \otimes_T l_j\right) = \sum_{i=1}^{n} m_i \otimes_S (n_i \otimes_T l_i) + \sum_{j=1}^{m} m_j \otimes_S (n_j \otimes_T l_j)$$
$$= f\left(\sum_{i=1}^{n}(m_i \otimes_S n_i) \otimes_T l_i\right) + f\left(\sum_{j=1}^{m}(m_j \otimes_S n_j) \otimes_T l_j\right)$$

$$\begin{aligned} f\left(r \cdot \sum_{i=1}^{n}(m_i \otimes_S n_i) \otimes_T l_i \cdot u\right) &= f\left(\sum_{i=1}^{n} r(m_i \otimes_S n_i) \otimes_T l_i u\right) \\ &= f\left(\sum_{i=1}^{n}(rm_i \otimes_S n_i) \otimes_T l_i u\right) \\ &= \sum_{i=1}^{n} rm_i \otimes_S (n_i \otimes_T l_i u) \\ &= r \cdot \sum_{i=1}^{n} m_i \otimes_S (n_i \otimes_T l_i) \cdot u \\ &= r \cdot f\left(\sum_{i=1}^{n}(m_i \otimes_S n_i) \otimes_T l_i\right) \cdot u \end{aligned}$$

for any $r \in R, u \in U$ so f is a homomorphism of bimodules. Moreover, f is clearly invertible with inverse given by

$$\sum_{i=1}^{n} m_i \otimes_S (n_i \otimes_T l_i) \mapsto \sum_{i=1}^{n}(m_i \otimes_S n_i) \otimes l_i$$

so f is an isomorphism. □

Proposition 2.26. *Let $f :_R M_S \to_R M'_S$ and $g : {}_S N_T \to {}_S N'_T$ be bimodule homomorphisms. Then the map $f \otimes g :_R (M \otimes_S N)_T \to_R (M' \otimes_S N')_T$ defined by*

$$\sum_{i=1}^{n} m_i \otimes n_i \mapsto \sum_{i=1}^{n} f(m_i) \otimes g(n_i)$$

is a homomorphism of R-T-bimodules.

Proof.

$$
\begin{aligned}
f \otimes g \left(\sum_{i=1}^{n} m_i \otimes n_i + \sum_{j=1}^{m} m_j \otimes n_j \right) &= \sum_{i=1}^{n} f(m_i) \otimes g(n_i) + \sum_{j=1}^{m} f(m_j) \otimes g(n_j) \\
&= f \otimes g \left(\sum_{i=1}^{n} m_i \otimes n_i \right) + f \otimes g \left(\sum_{j=1}^{m} m_j \otimes n_j \right)
\end{aligned}
$$

$$
\begin{aligned}
f \otimes g \left(r \cdot \sum_{i=1}^{n} m_i \otimes n_i \cdot t \right) &= f \otimes g \left(\sum_{i=1}^{n} r m_i \otimes n_i t \right) \\
&= \sum_{i=1}^{n} f(r m_i) \otimes g(n_i t) \\
&= \sum_{i=1}^{n} r f(m_i) \otimes g(n_i) t \\
&= r \cdot \sum_{i=1}^{n} f(m_i) \otimes g(n_i) \cdot t \\
&= r \cdot f \otimes g \left(\sum_{i=1}^{n} m_i \otimes n_i \right) \cdot t.
\end{aligned}
$$

□

3 2-categories

If we consider a category where the morphism classes are themselves equipped with the structure of a category, we arrive at the notion of a 2-category. Much of this section follows from [1], with more details spelled out.

Definition 3.1. A 2-category $\mathcal{C}$ consists of the following:

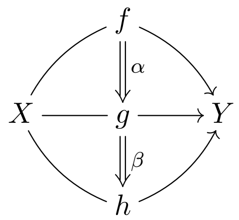
- a class $\mathrm{Ob}(\mathcal{C})$ of objects.

- for every pair of objects X, Y, a small category $\mathcal{C}(X, Y)$, also called the hom-category. The objects $f, g : X \to Y$ of this category are the morphisms from X to Y, called 1-morphisms. Its morphisms $\alpha : f \Rightarrow g$ are called 2-morphisms. The composition of this category is denoted by $\bullet$ and is also called vertical composition.

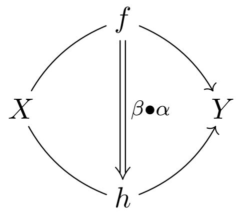
- for every object X, a functor I_X from the terminal category $\mathbf{1}$ to $\mathcal{C}(X, X)$. This functor maps the object of $\mathbf{1}$ to the identity 1-morphism $\mathrm{id}_X : X \to X$ and the morphism of $\mathbf{1}$ to the identity 2-morphism $\mathrm{id}_f : f \Rightarrow f$.

- for all objects X, Y, Z a bifunctor $\circ : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$. This functor is also called horizontal composition.

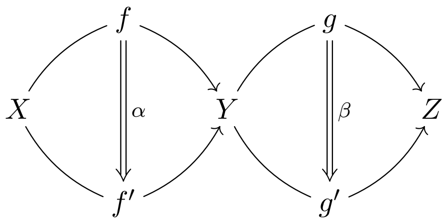
Example 3.2 (Vertical composition). *For objects X, Y, 1-morphisms f, g, h and 2-morphisms α, β as in*



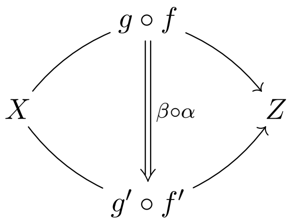
we get the 2-morphism $\beta \bullet \alpha : f \Rightarrow h$ as in



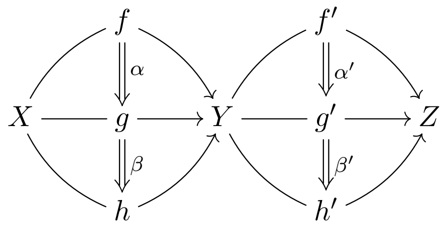
Example 3.3 (Horizontal composition). *For $f, f', g, g', \alpha, \beta$ as in*



we get the 2-morphism $\beta \circ \alpha : g \circ f \Rightarrow g' \circ f'$ as in



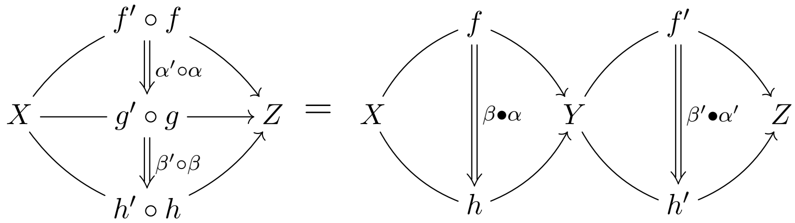
Remark 3.4 (Interchange law). Since $\circ$ is a functor, it commutes with the (vertical) composition of the hom-categories, so we have, for composable 2-morphisms as in



we have

$$(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = (\beta' \bullet \alpha') \circ (\beta \bullet \alpha).$$

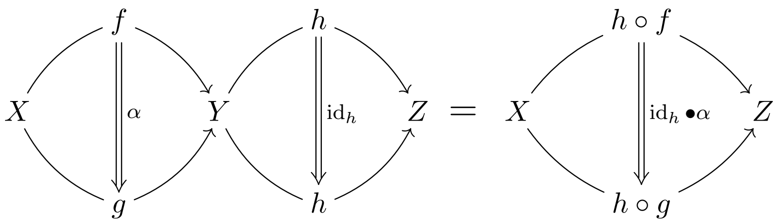
Or, in terms of diagrams:



The left diagram corresponds to the left hand side of the above equation and the right diagram corresponds to the right hand side of the above equation.

Example 3.5. *An intuitive and motivating example of a 2-category is* ***Cat****, where the objects are categories, the 1-morphisms are functors and the 2-morphisms are natural transformations.*

Remark 3.6. [1] This structure also enables us to horizontally compose 2-morphisms with 1-morphisms, by composing the 2-morphism with the with the identity 2-morphism on the 1-morphism. We denote the horizontal composition of 1- and 2-morphisms by juxtaposition. For 1-morphisms f, g, h and a 2-morphism α we get the following diagram for $h\alpha$:



It is clear that we also can compose in the opposite direction.

Next we define the 2-categorical notions corresponding to functors and natural transformations.

Definition 3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then a 2-functor $F : \mathcal{C} \to \mathcal{D}$ consists of a triple of maps; a map of objects, a map of 1-morphisms and a map of 2-morphisms, satisfying the following:

i) For every object X we have $F(\text{id}_X) = \text{id}_{F(X)}$ and for every 1-morphism f we have $F(\text{id}_f) = \text{id}_{F(f)}$.

ii) For composable 1-morphisms f, g we have $F(g \circ f) = F(g) \circ F(f)$.

iii) For horizontally composable 2-morphisms α, α', we have $F(\alpha' \circ \alpha) = F(\alpha') \circ F(\alpha)$.

iv) For vertically composable 2-morphisms β, β' we have, $F(\beta' \bullet \beta) = F(\beta') \bullet F(\beta)$.

Remark 3.8. Note that a 2-functor $F : \mathcal{C} \to \mathcal{D}$, when applied to the objects and 1-morphisms is an ordinary functor between the categories formed by the objects and 1-morphisms of $\mathcal{C}$ and $\mathcal{D}$, so we can think of a 2-functor as an extension of an ordinary functor, respecting the additional structure of a 2-category.

Definition 3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be 2-functors. Then a 2-natural transformation $\eta : F \to G$ is a map sending every object X of $\mathcal{C}$ to a 1-morphism $\eta_X : F(X) \to G(X)$ such that for 1-morphisms $f, g : X \to Y$ and every 2-morphism $\alpha : f \Rightarrow G$ the following holds

$$F(X) \underset{F(g)}{\overset{F(f)}{\Downarrow F(\alpha)}} F(Y) \xrightarrow{\eta_Y} G(Y) \quad = \quad F(X) \xrightarrow{\eta_X} G(X) \underset{G(g)}{\overset{G(f)}{\Downarrow G(\alpha)}} G(Y)$$

Remark 3.10. If we consider the identity 2-morphism on f the above diagram becomes

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{\eta_Y} G(Y) = F(X) \xrightarrow{\eta_X} G(X) \xrightarrow{G(f)} G(Y)$$

which is just the usual naturality square

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

for a natural transformations between the ordinary functors of F and G, so in the same way as with 2-functors, 2-natural transformations can be seen as extensions of ordinary natural transformations to the 2-categorical framework.

4 Bicategories

If we weaken the requirements on 2-categories, by instead of requiring associativity of the horizontal composition, require associativity up to a natural isomorphism, we arrive at the notion of a bicategory. This section essentially follows from [2], but with more details spelled out.

4.1 Basics

Definition 4.1. A bicategory $\mathcal{B}$ consists of the following:

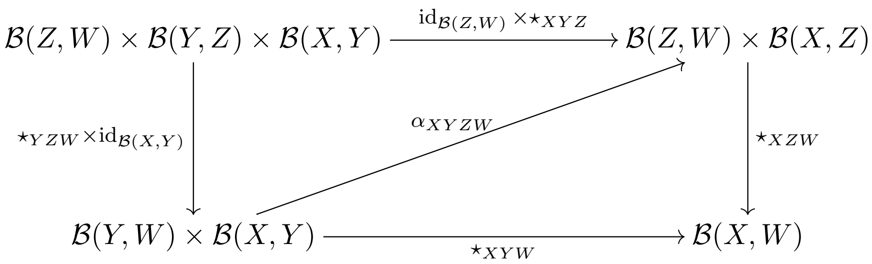
- a class $\mathrm{Ob}(\mathcal{B})$ of objects.
- for every pair X, Y of objects, a small hom-category $\mathcal{B}(X, Y)$. We denote its (vertical) composition by $\bullet$.
- for every object X, a functor I_X from $\mathbf{1}$ to $\mathcal{B}(X, X)$ as in the definition of a 2-category.
- for ordered triples of objects X, Y, Z, a bifunctor $\star : \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \to \mathcal{B}(X, Z)$. For notational convenience, we denote the horizontal composition of 1-morphisms by juxtaposition, so for 1-morphisms f, g and 2-morphisms α, β we get

$$\begin{aligned} \star : \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) &\to \mathcal{B}(X, Z) \\ (g, f) &\mapsto gf \\ (\beta, \alpha) &\mapsto \beta \star \alpha. \end{aligned}$$

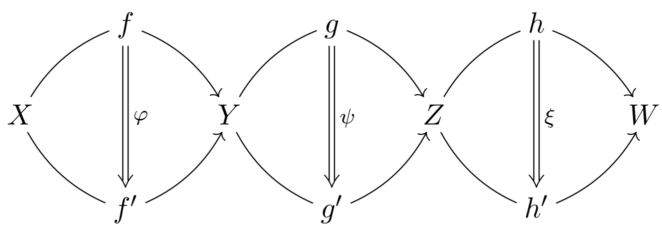
We might denote this specific bifunctor by $\star_{XYZ}$. Here we differ from the definition of a 2-category. We do not require associativity of $\star$, we only require it up to a natural isomorphism.

This is made precise in the following way:

- for objects X, Y, Z, W, a natural isomorphism α_{XYZW} as given in



called the associator. For 1- and 2-morphisms as given in



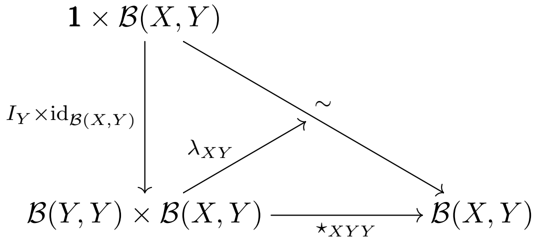
the naturality of α yields the commutative diagram

$$\begin{array}{ccc} (hg)f & \xrightarrow{(\xi\star\psi)\star\varphi} & (h'g')f' \\ \downarrow{\scriptstyle \alpha_{hgf}} & & \downarrow{\scriptstyle \alpha_{h'g'f'}} \\ h(gf) & \xrightarrow[\xi\star(\psi\star\varphi)]{} & h'(g'f') \end{array}$$

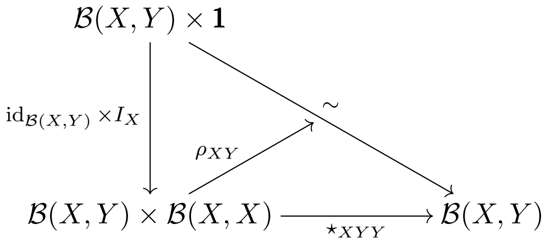
which means that for composable 1-morphisms h, g, f we have an invertible 2-morphism

$$\alpha_{hgf} : (hg)f \Rightarrow h(gf).$$

- for each pair X, Y of objects, natural isomorphisms λ_{XY} and ρ_{XY} as given in



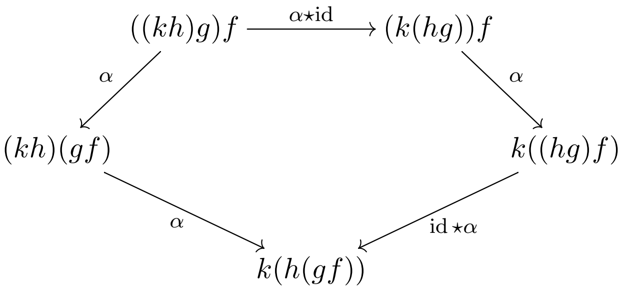
and

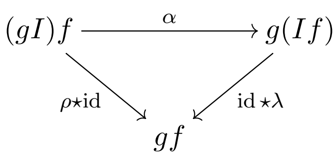


called left and right unitors, respectively. So for a 1-morphism $f \in \mathcal{B}(X, Y)$, we have invertible 2-morphisms

$$\lambda_f : \mathrm{id}_Y\, f \Rightarrow f$$
$$\rho_f : f\, \mathrm{id}_X \Rightarrow f.$$

Finally, we require the two following diagrams commute for composable 1-morphisms f, g, h, k.





Remark 4.2. It is clear that if the natural isomorphisms α, λ, ρ are all identities, in which case the composition is strictly associative, then the definition of bicategory coincides with that of a 2-category.

Definition 4.3. Let $\mathcal{B}$ be a bicategory. An internal equivalence in $\mathcal{B}$ consists if a pair of 1-morphisms as given in

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

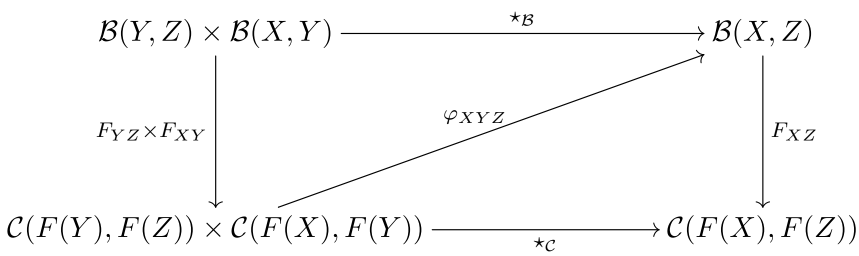
together with an isomorphism $\mathrm{id}_X \overset{\sim}{\Rightarrow} gf$ in the hom-category $\mathcal{B}(X,X)$ and an isomorphism $fg \overset{\sim}{\Rightarrow} \mathrm{id}_Y$ in the hom-category $\mathcal{B}(Y,Y)$. We say that X and Y are equivalent inside $\mathcal{B}$.

Definition 4.4. Let $\mathcal{B}$ and $\mathcal{C}$ be bicategories. A lax functor (F,φ) from $\mathcal{B}$ to $\mathcal{C}$ consists of the following:

- a map $F : \mathrm{Ob}\,(\mathcal{B}) \to \mathrm{Ob}\,(\mathcal{C})$ of objects
- for objects $X, Y \in \mathrm{Ob}\,(\mathcal{B})$, a functor of hom-categories

$$F_{XY} : \mathcal{B}(X,Y) \to \mathcal{C}(F(X),F(Y))$$

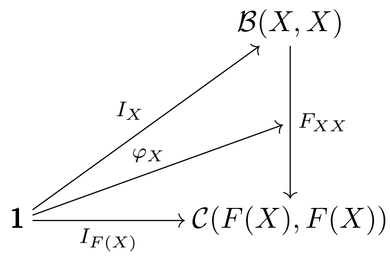
- for objects $X, Y, Z \in \mathrm{Ob}\,(\mathcal{B})$, a natural transformation φ_{XYZ} as given in



which, for composable 1-morphisms f, g gives the 2-morphism

$$\varphi_{gf} : F(g)F(f) \Rightarrow F(gf)$$

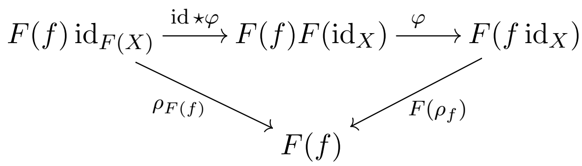
and a natural transformation φ_X as given in

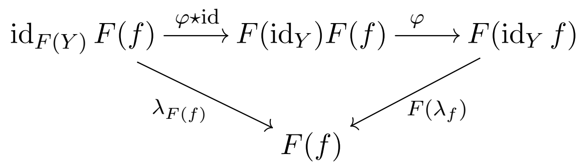


which gives the 2-morphism $\varphi_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$.

We require that the following diagrams commute for composable 1-morphisms f, g, h, denoting the associators in the categories $\mathcal{B}, \mathcal{C}$ by $\alpha_{\mathcal{B}}$ and $\alpha_{\mathcal{C}}$ respectively:

$$\begin{array}{ccccc}
(F(h)F(g))F(f) & \xrightarrow{\varphi \star \mathrm{id}} & F(hg)F(f) & \xrightarrow{\varphi} & F((hg)f) \\
\downarrow{\scriptstyle \alpha_{\mathcal{C}}} & & & & \downarrow{\scriptstyle F(\alpha_{\mathcal{B}})} \\
F(h)(F(g)F(f)) & \xrightarrow[\mathrm{id} \star \varphi]{} & F(h)F(gf) & \xrightarrow[\varphi]{} & F(h(gf))
\end{array}$$





Definition 4.5. If for some property of functors every functor F_{XY} has this property, we say that the lax functor F locally has this property. For example, a lax functor might be locally full.

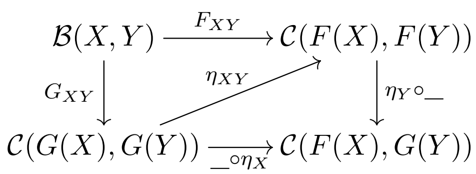
Definition 4.6. If (F, φ) is a lax functor such that all the natural transformations φ_{XYZ} and φ_X are natural isomorphisms, then (F, φ) is called a pseudofunctor.

Definition 4.7. If (F, φ) is a lax functor such that all the natural transformations φ_{XYZ} and φ_X are identities, then (F, φ) is called a strict 2-functor.

Definition 4.8. Let (F, φ) and (G, ψ) be lax functors from $\mathcal{B}$ to $\mathcal{C}$. Then a lax natural transformation η consists of the following:

- for each $X \in \mathrm{Ob}\,(\mathcal{B})$, a 1-morphism $\eta_X : F(X) \to G(X)$.

- natural transformations as given in:



so we have a 2-morphism

$$\eta_f : G(f)\eta_X \Rightarrow \eta_Y F(f).$$

Additionally, we require that the following diagrams commute for composable 1-morphisms f, g:

$$\begin{array}{ccccccc}
(G(g)G(f))\,\eta_X & \xrightarrow{\alpha_{\mathcal{C}}} & G(g)\,(G(f)\eta_X) & \xrightarrow{\mathrm{id}\star\eta_f} & G(g)\,(\eta_Y F(f)) & \xrightarrow{\alpha_{\mathcal{C}}^{-1}} & (G(g)\eta_Y)\,F(f) \\
 & & & & & & \downarrow{\scriptstyle \eta_g\star\mathrm{id}} \\
 & & & & & & (\eta_Z F(g))\,F(f) \\
{\scriptstyle \psi\star\mathrm{id}}\downarrow & & & & & & \downarrow{\scriptstyle \alpha_{\mathcal{C}}} \\
 & & & & & & \eta_Z\,(F(g)F(f)) \\
 & & & & & & \downarrow{\scriptstyle \mathrm{id}\star\varphi} \\
G(gf)\eta_X & & & \xrightarrow[\eta_{gf}]{} & & & \eta_Z F(gf)
\end{array}$$

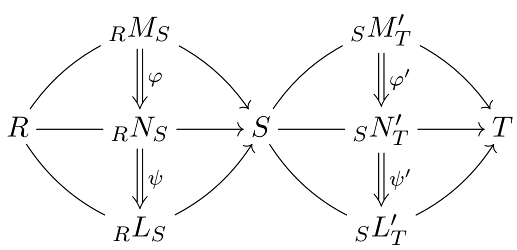
$$\begin{array}{ccccc}
\mathrm{id}_{G(X)}\,\eta_X & \xrightarrow{\lambda_{\mathcal{C}}} & \eta_X & \xrightarrow{\rho_{\mathcal{C}}^{-1}} & \eta_X\,\mathrm{id}_{F(X)} \\
{\scriptstyle \psi\star\mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}\star\varphi} \\
G(\mathrm{id}_X)\eta_X & & \xrightarrow[\eta_{\mathrm{id}_X}]{} & & \eta_X F(\mathrm{id}_X)
\end{array}$$

Definition 4.9. If η is a lax natural transformation such that the natural transformations η_{XY} are all natural isomorphisms, then η is called a pseudonatural transformation.

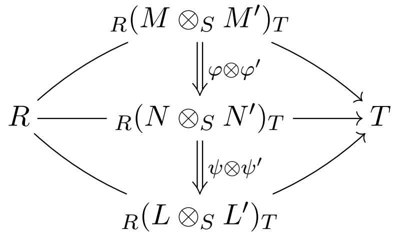
Definition 4.10. Let η and μ be lax natural transformations between the lax functors $(F,\varphi),(G,\psi)$ from $\mathcal{B}$ to $\mathcal{C}$. Then a modification $\Gamma:\eta\to\mu$ consists of 2-morphisms $\Gamma_X:\eta_X\Rightarrow\mu_X$ such that the following diagram commutes:

$$\begin{array}{ccc}
G(f)\eta_X & \xrightarrow{\mathrm{id}\star\Gamma_X} & G(f)\mu_X \\
{\scriptstyle \eta_f}\downarrow & & \downarrow{\scriptstyle \mu_f} \\
\eta_Y F(f) & \xrightarrow[\Gamma_Y\star\mathrm{id}]{} & \mu_Y F(f)
\end{array}$$

Example 4.11. *There is a bicategory* ***Bimod*** *whose objects are rings, 1-morphisms are bimodules and 2-morphisms are bimodule homomorphisms. Then a typical structure in* ***Bimod*** *would look like this:*



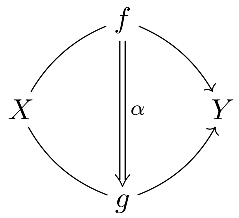
Composition of 1-morphisms is given by the bimodule tensor product and composition of 2-morphisms is just composition of bimodule homomorphisms. By proposition 2.23, the tensor product behaves nicely with respect to the bimodule structure, so the composite would look like:



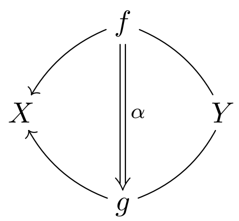
For larger composites, we have the required associativity up to isomorphism by proposition 2.24, that is

$$(M \otimes_S N) \otimes_T L \cong M \otimes_S (N \otimes_T L)\,.$$

Example 4.12. *If $\mathcal{B}$ is a bicategory, we may form a new bicategory B^{op} by reversing the 1-morphisms. So the diagram*



in $\mathcal{B}$ becomes the diagram

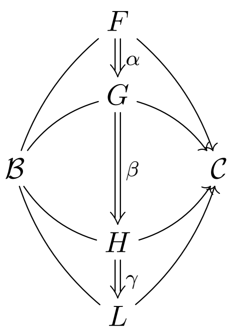


in $\mathcal{B}^{\mathrm{op}}$.

Example 4.13. *For bicategories $\mathcal{B}$ and $\mathcal{C}$, there is a functor bicategory* $\mathrm{Lax}(\mathcal{B},\mathcal{C})$*. Its objects are lax functors, the 1-morphisms are lax natural transformations and the 2-morphisms are modifications. It has a sub-bicategory $[\mathcal{B},\mathcal{C}]$ consisting of pseudofunctors, pseudonatural transformations, and modifications.*

Proposition 4.14. *Let $\mathcal{B}$ be a bicategory and $\mathcal{C}$ a 2-category. Then* $\mathrm{Lax}(\mathcal{B},\mathcal{C})$ *is a 2-category.*

Proof. Suppose we have lax functors and natural transformations as given in

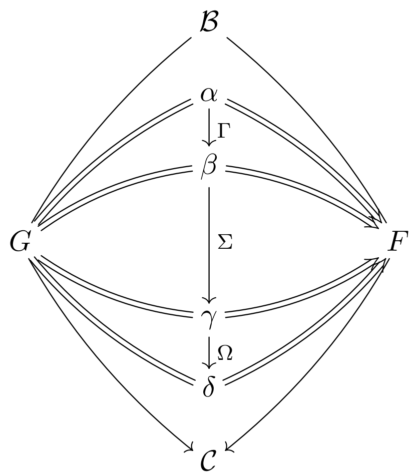


Then the composition of 1-morphisms in $\mathrm{Lax}(\mathcal{B}, \mathcal{C})$ is given by the composition of the transformations α, β, γ. But since these are transformations, this is just the componentwise composition. So for an object $X \in \mathrm{Ob}\,(\mathcal{B})$, we have 1-morphisms

$$F(X) \xrightarrow{\alpha_X} G(X) \xrightarrow{\beta_X} H(X) \xrightarrow{\gamma_X} L(X)$$

but these components are 1-morphisms of the 2-category $\mathcal{C}$, and this composition is associative, so we get $(\gamma\beta)\alpha = \gamma(\beta\alpha)$. By the same argument we have $\alpha\,\mathrm{id} = \alpha = \mathrm{id}\,\alpha$ for any transformation α.

Similarily, if we have lax functors, natural transformations and modifications as given in



□

we get 2-morphisms

$$\alpha_X \overset{\Gamma_X}{\Longrightarrow} \beta_X \overset{\Sigma_X}{\Longrightarrow} \gamma_X \overset{\Omega_X}{\Longrightarrow} \delta_X$$

which now are 2-morphisms of $\mathcal{C}$ and again this yields associativity. It is clear that we by the same argument have $\Gamma \star \mathrm{id} = \Gamma = \mathrm{id} \star \Gamma$ for any modification Γ.

4.2 Coherence

Definition 4.15. Let $\mathcal{B}$ and $\mathcal{C}$ be bicategories. A biequivalence of $\mathcal{B}$ and $\mathcal{C}$ consists of a pair of psuedofunctors

$$\mathcal{B} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

together with an internal equivalence $\mathrm{id}_{\mathcal{B}} \to GF$ in $[\mathcal{B}, \mathcal{B}]$ and an internal equivalence $FG \to \mathrm{id}_{\mathcal{C}}$ in $[\mathcal{C}, \mathcal{C}]$. It can be shown that a pseudofunctor $F : \mathcal{B} \to \mathcal{C}$ admits a biequivalence if and only if F is a local equivalence and if for every $Y \in \mathrm{Ob}\,(\mathcal{B})$ there exists an $X \in \mathrm{Ob}\,(\mathcal{C})$ such that $F(X)$ is internally equivalent to Y.

Example 4.16. *Let $\mathcal{C}$ be a category.*

We define the bicategory $\mathcal{X}$ as follows: it has only one object and only one 1-morphism, $\mathrm{id}_{\mathcal{C}}$*. Its 2-morphisms are natural transformations.*

We define the bicategory $\mathcal{Y}$ as follows: it has only one object. Its 1-morphisms are functors isomorphic to $\mathrm{id}_{\mathcal{C}}$*. Its 2-morphisms are natural transformations.*

Then we have a clear embedding $F : \mathcal{X} \to \mathcal{Y}$. Clearly, F is a pseudofunctor which is surjective on objects.

The induced functor of the hom-categories is clearly dense, since it sends $\mathrm{id}_{\mathcal{C}}$ *to itself and every 1-morphism in $\mathcal{Y}$ is isomorphic to* $\mathrm{id}_{\mathcal{C}}$ *by construction. Moreover, it is faithful since it is an inclusion on 1-morphisms, and it is full since the 2-morphisms of $\mathcal{X}$ and $\mathcal{Y}$ are the same.*

So F is a local equivalence and hence a biequivalence.

The following result is a version of the Yoneda lemma for bicategories, which we state without proof.

Theorem 4.17 (Yoneda lemma for bicategories)**.** *Let $\mathcal{B}$ be a bicategory and let $F : \mathcal{B}^{\mathrm{op}} \to \boldsymbol{Cat}$ be a pseudofunctor. Then, for any $X \in \mathrm{Ob}\,(\mathcal{B})$, there is an equivalence of categories*

$$[\mathcal{B}^{\mathrm{op}}, \boldsymbol{Cat}]\,(\mathcal{B}(\_, X), F) \simeq F(X)$$

which is pseudonatural in X and in F.

From the Yoneda lemma it follows that there is an analogue of the usual Yoneda embedding. This means that we have a pseudofunctor

$$\mathcal{Y} : \mathcal{B} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Cat}]$$

which is locally full, faithful and dense. In other words, $\mathcal{Y}$ is a local equivalence.

Theorem 4.18. *Let $\mathcal{B}$ be a bicategory. Then $\mathcal{B}$ is biequivalent to a 2-category.*

Proof. Let $\mathcal{Y}$ be the Yoneda pseudofunctor and let $\mathcal{C}$ be the image of $\mathcal{Y}$ in $[\mathcal{B}^{\mathrm{op}}, \mathbf{Cat}]$. By this we mean that $\mathcal{C}$ is the sub-2-category of $[\mathcal{B}^{\mathrm{op}}, \mathbf{Cat}]$ whose objects are the objects in the image of $\mathcal{Y}$, with all 1- and 2-morphisms of $[\mathcal{B}^{\mathrm{op}}, \mathbf{Cat}]$. Then, seen as a psuedofunctor $\mathcal{Y} : \mathcal{B} \to \mathcal{C}$, we have that $\mathcal{Y}$ is surjective on objects by construction and a local equivalence, so it is a biequivalence. □

5 Monoidal categories

A monoidal category is usually defined as a category equipped with a tensor product. So for a category $\mathscr{C}$ we would define the tensor product as a bifunctor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$

obeying certain axioms.

However, thanks to the previous section, we can simply define a monoidal category as the hom-category of a bicategory with one single object.

Then, taking the tensor product as horizontal composition and the identity 1-morphism as the tensor unit, the associator and unitor isomorphisms together with their coherence axioms yield exactly the standard definition of monoidal category.

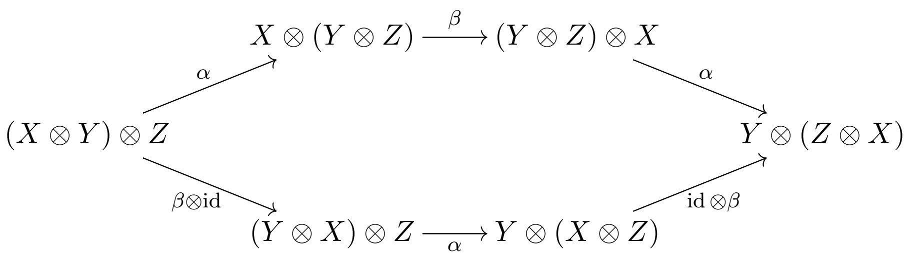
In the same way, we effortlessly get the definitions of a monoidal functor and a monoidal transformation from the definitions of lax functors and lax natural transformations in the previous section.

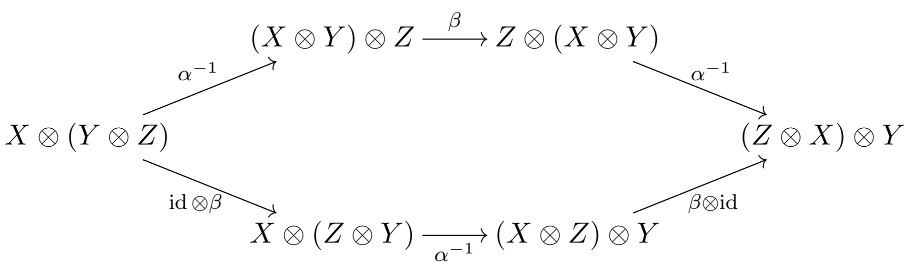
A monoidal category where the associator and unitors are all identities, is, unsurprisingly, called a strict monoidal category. This section essentially follows from [3], with more details spelled out.

Definition 5.1. Let $\mathscr{C}$ be a monoidal category. Then a braiding β of $\mathscr{C}$ is a natural isomorphism with components

$$\beta_{X,Y} : X \otimes Y \to Y \otimes X.$$

We require that the braiding satisfies the hexagon identities, given by the following commutative diagrams:





A monoidal category $\mathscr{C}$ together with chosen braiding is called a braided monoidal category.

Definition 5.2. A braided monoidal category is called symmetric if the braiding satisfies

$$\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X \otimes Y}$$

for every pair of objects.

5.1 The Drinfeld center

Definition 5.3. Let $\mathscr{C}$ be a monoidal category. Then we define its Drinfeld center $\mathcal{Z}(\mathscr{C})$ as the following monoidal category:

- objects are pairs $(X, \eta_{X,\_})$ where X is an object of $\mathscr{C}$ and $\eta_{X,\_}$ is a natural isomorphism

  $$\eta_{X,\_} : X \otimes \_ \to \_ \otimes X$$

  such that

  $$\eta_{X,Y \otimes Z} = (\mathrm{id}_Y \otimes \eta_{X,Z})(\eta_{X,Y} \otimes \mathrm{id}_Z).$$

  The naturality of $\eta_{X,\_}$ yields commutativity of the square

  $$\begin{CD} X \otimes Y @>{\mathrm{id}_X \otimes g}>> X \otimes Z \\ @V{\eta_{X,Y}}VV @VV{\eta_{X,Z}}V \\ Y \otimes X @>>{g \otimes \mathrm{id}_X}> Z \otimes X \end{CD}$$

  for any morphism $g : Y \to Z$.

- a morphism $f : (X, \eta_{X,\_}) \to (Y, \eta_{Y,\_})$ is a morphism $f : X \to Y$ in $\mathscr{C}$ such that

  $$(\mathrm{id}_Z \otimes f)\eta_{X,Z} = \eta_{Y,Z}(f \otimes \mathrm{id}_Z)$$

  for every $Z \in \mathscr{C}$. This is equivalent to the square

  $$\begin{CD} X \otimes Z @>{f \otimes \mathrm{id}_Z}>> Y \otimes Z \\ @V{\eta_{X,Z}}VV @VV{\eta_{Y,Z}}V \\ Z \otimes X @>>{\mathrm{id}_Z \otimes f}> Z \otimes Y \end{CD}$$

  commuting for every $Z \in \mathscr{C}$.

- the tensor product of $\mathcal{Z}(\mathscr{C})$ is given by

  $$(X, \eta_{X,\_}) \otimes (Y, \eta_{Y,\_}) = (X \otimes Y, \eta_{X \otimes Y,\_})$$

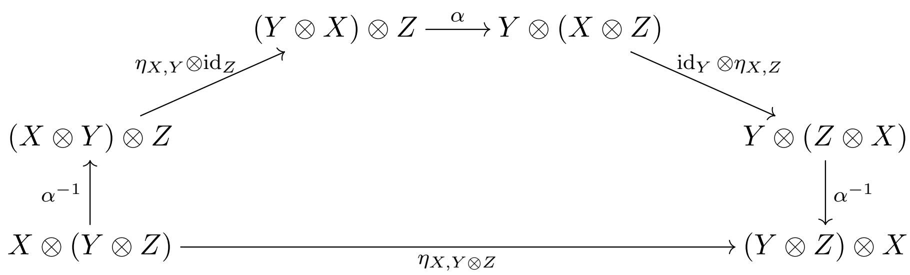
  where $\eta_{X \otimes Y,Z} : (X \otimes Y) \otimes Z \to Z \otimes (X \otimes Y)$ is given by

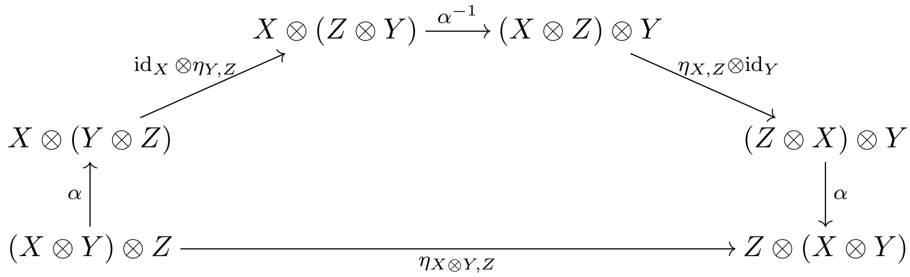
  $$\eta_{X \otimes Y,Z} = (\eta_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \eta_{Y,Z}).$$

Remark 5.4. The conditions

$$\eta_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \eta_{X,Z})(\eta_{X,Y} \otimes \mathrm{id}_Z)$$
$$\eta_{X\otimes Y,Z} = (\eta_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \eta_{Y,Z})$$

are is not quite correct. In the above definition, we have left out some associators necessary to make sense of the equations. With the associators spelled out, the conditions amount to commutativity of the following diagrams:





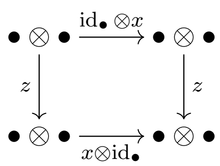
which in turn yield the accurate equations:

$$\eta_{X,Y\otimes Z} = \alpha^{-1}_{YZX}(\mathrm{id}_Y \otimes \eta_{X,Z})\alpha_{YXZ}(\eta_{X,Y} \otimes \mathrm{id}_Z)\alpha^{-1}_{XYZ}$$
$$\eta_{X\otimes Y,Z} = \alpha_{ZXY}(\eta_{X,Z} \otimes \mathrm{id}_Y)\alpha^{-1}_{XZY}(\mathrm{id}_X \otimes \eta_{Y,Z})\alpha_{XYZ}.$$

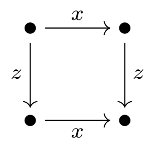
If, however, $\mathscr{C}$ is a strict monoidal category, then the previously stated conditions are just fine.

Example 5.5. *Recall the category defined in example 1.19, the categorical equivalent of a monoid. We can impose a tensor product on* $\mathcal{M}$ *by putting* $\bullet \otimes \bullet = \bullet$ *for the object of* $\mathcal{M}$ *and* $x \otimes y = xy$ *for the morphisms to get a strict monoidal category* $\mathscr{M}$*.*

Now we want to consider possible objects in the Drinfeld center $\mathcal{Z}(\mathscr{M})$*. They must be of the form* $(\bullet, \eta_{\bullet,\_})$*. Since* $\bullet$ *is the only object in* $\mathscr{M}$*,* $\eta_{\bullet,\_}$ *has only the component* $\eta_{\bullet,\bullet}$*. Denoting this component by* z*, we require that the diagram*



commutes for any $x \in M$. But this diagram is actually just the diagram



which we know commutes if and only if $z \in Z(M)$. Since z must also be an isomorphism, we require that z be invertible. Writing out the condition for a morphism $x : \bullet \to \bullet$ to be in $\mathcal{Z}(\mathscr{M})$ yields the same diagram as above, so we conclude that

$$\mathrm{Hom}_{\mathscr{M}}(\bullet, \bullet) = \mathrm{Hom}_{\mathcal{Z}(\mathscr{M})}(\bullet, \bullet).$$

So the Drinfeld center $\mathcal{Z}(\mathscr{M})$ consists of objects of the form $(\bullet, z)$ where $z \in M$ is invertible and central. Morphisms $(\bullet, z) \to (\bullet, c)$ are elements of M.

Proposition 5.6. *Let $\mathscr{C}$ be a strict monoidal category. Then its Drinfeld center $\mathcal{Z}(\mathscr{C})$ is a strict braided monoidal category, with braiding given by*

$$\eta_{X,Y} : (X, \eta_{X,\_}) \otimes (Y, \otimes \eta_{Y,\_}) \to (Y, \otimes \eta_{Y,\_}) \otimes (X, \eta_{X,\_}).$$

Proof. We check that $\eta_{X,Y}$ is indeed a morphism in $\mathcal{Z}(\mathscr{C})$. We have

$$\begin{aligned}(X, \eta_{X,\_}) \otimes (Y, \eta_{Y,\_}) &= (X \otimes Y, \eta_{X\otimes Y,\_})\\ (Y, \eta_{Y,\_}) \otimes (X, \eta_{X,\_}) &= (Y \otimes X, \eta_{Y\otimes X,\_})\end{aligned}$$

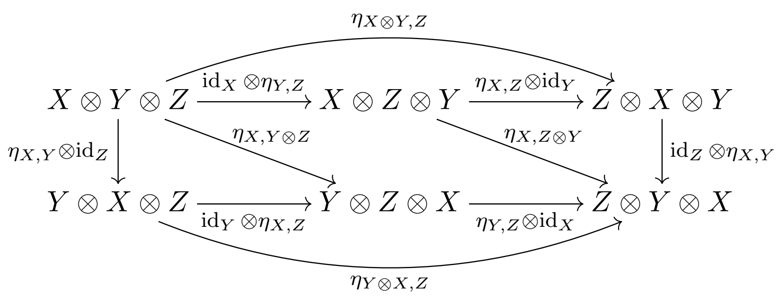
so $\eta_{X,Y}$ is a morphism between the correct objects of $\mathscr{C}$. The criterion for $\eta_{X,Y}$ being a morphism in $\mathcal{Z}(\mathscr{C})$ is

$$(\mathrm{id}_Z \otimes \eta_{X,Y})\eta_{X\otimes Y,Z} = \eta_{Y\otimes X,Z}(\eta_{X,Y} \otimes \mathrm{id}_Z).$$

We have

$$\begin{aligned}(\mathrm{id}_Z \otimes \eta_{X,Y})\eta_{X\otimes Y,Z} &= (\mathrm{id}_Z \otimes \eta_{X,Y})(\eta_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \eta_{Y,Z})\\ &= \eta_{X,Z\otimes Y}(\mathrm{id}_X \otimes \eta_{Y,Z})\\ &= (\eta_{Y,Z} \otimes \mathrm{id}_X)\eta_{X,Y\otimes Z}\\ &= (\eta_{Y,Z} \otimes \mathrm{id}_X)(\mathrm{id}_Y \otimes \eta_{X,Z})(\eta_{X,Y} \otimes \mathrm{id}_Z)\\ &= \eta_{Y\otimes X,Z}(\eta_{X,Y} \otimes \mathrm{id}_X).\end{aligned}$$

Equivalently, we can show that the diagram



commutes. It does, since the triangles commute by our conditions and the center parallelogram is just a naturality square of $\eta_{X,\_}$. Left to check are the hexagon identities. In a strict monoidal category, these are equivalent to the diagrams

$$
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{\eta_{X,Y\otimes Z}} & Y \otimes Z \otimes X \\
{\scriptstyle \eta_{X,Y}\otimes \mathrm{id}_Z}\downarrow & \nearrow{\scriptstyle \mathrm{id}_Y \otimes \eta_{X,Z}} & \\
Y \otimes X \otimes Z & &
\end{array}
\qquad
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{\eta_{X\otimes Y,Z}} & Z \otimes X \otimes Y \\
{\scriptstyle \mathrm{id}_X \otimes \eta_{Y,Z}}\downarrow & \nearrow{\scriptstyle \eta_{X,Z}\otimes \mathrm{id}_Y} & \\
X \otimes Z \otimes Y & &
\end{array}
$$

commuting, which they do by our definition of $\mathcal{Z}(\mathscr{C})$.

□

Remark 5.7. This result holds even for non-strict monoidal categories. That is, for any monoidal category $\mathscr{C}$, its Drinfeld center $\mathcal{Z}(\mathscr{C})$ is a braided monoidal category. The proof, however, is not given here.

6 The Drinfeld center of $\mathbf{Vect}_{\mathbb{C}}$

Now we consider the category $\mathbf{Vect}_{\mathbb{C}}$, consisting of finite-dimensional vector spaces over $\mathbb{C}$ and linear maps between them. We impose the structure of a monoidal category on $\mathbf{Vect}_{\mathbb{C}}$ using the usual tensor product of vector spaces.

Note that for vector spaces V and W, there is a canonical isomorphism $V \otimes W \to W \otimes V$, defined by $v \otimes w \mapsto w \otimes v$. For any pair of vector spaces in, let $\Phi_{V,W}$ denote this isomorphism.

Proposition 6.1. *Let V be a finite-dimensional complex vector space and let*

$$\Phi_{V,\_} : V \otimes \_ \to \_ \otimes V$$

have components given by the canonical isomorphism $\Phi_{V,W}$. Then the pair $(V, \Phi_{V,\_})$ is in $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$.

Proof. We need the square

$$
\begin{array}{ccc}
V \otimes W & \xrightarrow{\mathrm{id}_V \otimes F} & V \otimes X \\
{\scriptstyle \Phi_{V,W}}\downarrow & & \downarrow{\scriptstyle \Phi_{V,X}} \\
W \otimes V & \xrightarrow[F\otimes \mathrm{id}_V]{} & X \otimes V
\end{array}
$$

to commute for any linear map $F : W \to X$.

$$
\begin{aligned}
(F \otimes \mathrm{id}_V)\,(\Phi_{V,W}(v \otimes w)) &= (F \otimes \mathrm{id}_V)(w \otimes v) \\
&= F(w) \otimes v \\
&= \Phi_{V,X}\,(v \otimes F(w)) \\
&= \Phi_{V,X}\,((\mathrm{id}_V \otimes F)(v \otimes w))
\end{aligned}
$$

so we get a natural family of isomorphisms. Left to check is the condition

$$\Phi_{V,W\otimes X} = \alpha^{-1}_{WXV}(\mathrm{id}_W \otimes \Phi_{V,X})\alpha_{WVX}(\Phi_{V,W} \otimes \mathrm{id}_X)\alpha^{-1}_{VWX}.$$

$$\begin{aligned}
\alpha^{-1}(\mathrm{id}_W \otimes \Phi_{V,X})\alpha(\Phi_{V,W} \otimes \mathrm{id}_X)\alpha^{-1}(v \otimes (w \otimes x)) &= \alpha^{-1}(\mathrm{id}_W \otimes \Phi_{V,X})\alpha(\Phi_{V,W} \otimes \mathrm{id}_X)((v \otimes w) \otimes x) \\
&= \alpha^{-1}(\mathrm{id}_W \otimes \Phi_{V,X})\alpha((w \otimes v) \otimes x) \\
&= \alpha^{-1}(\mathrm{id}_W \otimes \Phi_{V,X})((w \otimes (v \otimes x)) \\
&= \alpha^{-1}(w \otimes (x \otimes v)) \\
&= (w \otimes x) \otimes v \\
&= \Phi_{V,W\otimes X}(v \otimes (w \otimes x)).
\end{aligned}$$

This condition is also easily checked by chasing the element $(v\otimes(w\otimes x))$ through the diagram below.

$$\begin{array}{ccccc}
 & (W \otimes V) \otimes X & \xrightarrow{\alpha} & W \otimes (V \otimes X) & \\
\nearrow^{\Phi_{V,W}\otimes \mathrm{id}_X} & & & & \searrow^{\mathrm{id}_W \otimes \Phi_{V,X}} \\
(V \otimes W) \otimes X & & & & W \otimes (X \otimes V) \\
\uparrow{\scriptstyle \alpha^{-1}} & & & & \downarrow{\scriptstyle \alpha^{-1}} \\
V \otimes (W \otimes X) & & \xrightarrow[\Phi_{V,W\otimes X}]{} & & (W \otimes X) \otimes V
\end{array}$$

□

To avoid notational clutter, from this point forward, we drop the indices of morphisms when domain and codomain are clear from context.

Proposition 6.2.

$$\mathrm{Hom}_{\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})}\left((V,\Phi),(W,\Phi)\right) = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{C}}}\left(V,W\right).$$

Proof. By definition, $\mathrm{Hom}_{\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})}\left((V,\Phi),(W,\Phi)\right)$ consists of linear maps $F : V \to W$ such that the diagram

$$\begin{array}{ccc}
V \otimes X & \xrightarrow{F\otimes \mathrm{id}} & W \otimes X \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
X \otimes V & \xrightarrow[\mathrm{id}\otimes F]{} & X \otimes W
\end{array}$$

for every X. We see that

$$\begin{aligned}
\Phi(F \otimes \mathrm{id})(v \otimes x) &= \Phi(F(v) \otimes x) \\
&= x \otimes F(v) \\
&= (\mathrm{id} \otimes F)(x \otimes v) \\
&= (\mathrm{id} \otimes F)\Phi(v \otimes x)
\end{aligned}$$

holds for any linear map F. □

Corollary 6.3. *If V and W are isomorphic as vector spaces, then (V, Φ) and (W, Φ) are isomorphic as objects of $\mathcal{Z}(\boldsymbol{Vect}_{\mathbb{C}})$.*

Proof. Any invertible linear map $G : V \to W$ is in $\mathrm{Hom}_{\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})}((V, \Phi), (W, \Phi))$ by proposition 6.2 and, similarly, its inverse is in $\mathrm{Hom}_{\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})}((W, \Phi), (V, \Phi))$, so G is an isomorphism in $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$. □

Proposition 6.4. *If $(\mathbb{C}, \Psi)$ is in $\mathcal{Z}(\boldsymbol{Vect}_{\mathbb{C}})$, then $\Psi = \Phi$.*

Proof. We consider the component of Ψ at the vector space V. Fix the basis 1 of $\mathbb{C}$ and the basis $\{v_i\}$ of V. Then $\{1 \otimes v_i\}$ is a basis of $\mathbb{C} \otimes V$ and $\{v_i \otimes 1\}$ is a basis of $V \otimes \mathbb{C}$. Put $n = \dim V$. Let $F : V \to V$ be some linear map. Then, the square

$$\begin{array}{ccc} \mathbb{C} \otimes V & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C} \otimes V \\ {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \Psi} \\ V \otimes \mathbb{C} & \xrightarrow[F \otimes \mathrm{id}]{} & V \otimes \mathbb{C} \end{array}$$

commutes so we have $(F \otimes \mathrm{id})\Psi = (\mathrm{id} \otimes F)\Psi$. Note that with respect to our bases, we have

$$[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = [F].$$

This means that, in terms of matrices, we have the equation $[F][\Psi] = [\Psi][F]$. Since this must hold for any linear map F, we see that $[\Psi]$ is a matrix that commutes with every other matrix. From linear algebra, we know that such a matrix must be a scalar multiple of the identity matrix. Now it follows that Ψ is a (non-zero) scalar multiple of the canonical isomorphism $\mathbb{C} \otimes V \to V \otimes \mathbb{C}$, say $\Psi = \lambda_V \Phi$ for some complex number λ_V. So now we know that $\Psi : \mathbb{C} \otimes V \to V \otimes \mathbb{C}$ is given by

$$1 \otimes v \mapsto \lambda_V v \otimes 1.$$

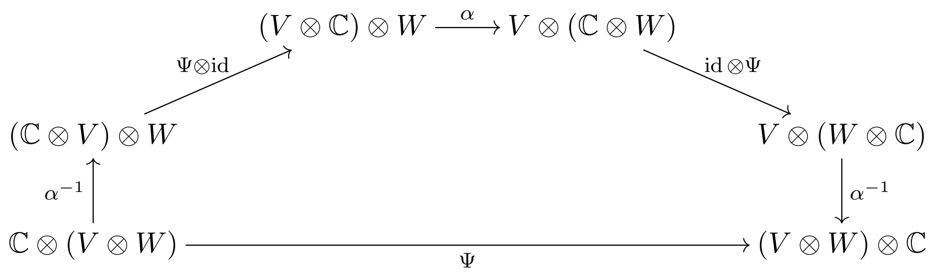
We note that the diagram

$$\begin{array}{ccc} \mathbb{C} \otimes V & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C} \otimes W \\ {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \Psi} \\ V \otimes \mathbb{C} & \xrightarrow[F \otimes \mathrm{id}]{} & W \otimes \mathbb{C} \end{array}$$

must commute for any vector space W and any linear map $F : V \to W$. So we must have

$$\begin{aligned} \lambda_V F(v) \otimes 1 &= (F \otimes \mathrm{id})(\lambda_V v \otimes 1) \\ &= (F \otimes \mathrm{id})\Psi(1 \otimes v) \\ &= \Psi(\mathrm{id} \otimes F)(1 \otimes v) \\ &= \Psi(1 \otimes F(v)) \\ &= \lambda_W F(v) \otimes 1 \end{aligned}$$

which implies $\lambda_V = \lambda_W$. So the constant λ_V is the same across every vector space. To reflect this, we drop the index and put $\lambda \coloneqq \lambda_V$. Moreover, since $(\mathbb{C}, \Psi)$ is in $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$, the diagram



commutes. This is equivalent to

$$\begin{aligned}
\lambda(v \otimes w) \otimes 1 &= \Psi(1 \otimes (v \otimes w)) \\
&= \alpha^{-1}(\mathrm{id} \otimes \Psi)\alpha(\Psi \otimes \mathrm{id})\alpha^{-1}(1 \otimes (v \otimes w)) \\
&= \alpha^{-1}(\mathrm{id} \otimes \Psi)\alpha(\Psi \otimes \mathrm{id})((1 \otimes v) \otimes w) \\
&= \alpha^{-1}(\mathrm{id} \otimes \Psi)\alpha((\lambda v \otimes 1) \otimes w) \\
&= \alpha^{-1}(\mathrm{id} \otimes \Psi)(\lambda v \otimes (1 \otimes w)) \\
&= \alpha^{-1}(\lambda(n)v \otimes (\lambda w \otimes 1)) \\
&= (\lambda v \otimes \lambda w) \otimes 1 \\
&= \lambda^2(v \otimes w) \otimes 1
\end{aligned}$$

which implies $\lambda^2 = \lambda$ and since $\lambda \neq 0$, we have $\lambda = 1$. □

Corollary 6.5. *If* (X, Ψ) *is in* $\mathcal{Z}(\boldsymbol{Vect}_{\mathbb{C}})$ *and* $\dim X = 1$, *then* $\Psi = \Phi$.

Proof. Follows immediately from the proof of the case $X = \mathbb{C}$. □

Now that we've established the behavior of objects of $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$ of the form $(\mathbb{C}, \Psi)$, we seek to generalize the previous arguments higher dimensions. Our first objects of study are elements of the form $\big(\mathbb{C}^2, \Psi\big)$. Throughout this section, we fix the standard basis of $\mathbb{C}^k$. If V is some vector space with a basis $\{v_1, \ldots, v_n\}$, we fix the basis

$$\{e_1 \otimes v_1, \ldots, e_1 \otimes v_n, \ldots, e_k \otimes v_1, \ldots, e_k \otimes v_n\}$$

of $\mathbb{C}^k \otimes V$ and the basis

$$\{v_1 \otimes e_1, \ldots, v_n \otimes e_1, \ldots, v_1 \otimes e_k, \ldots, v_n \otimes e_k\}$$

of $V \otimes \mathbb{C}^k$. To help us classify objects of the form $\big(\mathbb{C}^2, \Psi\big)$ in $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$, we have the following lemma.

Lemma 6.6. *If* $(\mathbb{C}^2, \Psi) \in \mathcal{Z}(\boldsymbol{Vect}_{\mathbb{C}})$*, then the component of* Ψ

$$\Psi_X : \mathbb{C}^2 \otimes X \to X \otimes \mathbb{C}^2$$

is given by the (invertible) matrix

$$[\Psi_X] = \begin{bmatrix} a_X I_{\dim X} & b_X I_{\dim X} \\ c_X I_{\dim X} & d_X I_{\dim X} \end{bmatrix}$$

with respect to the bases chosen above, for any vector space X*. In particular, the component of* Ψ *at the vector space* $\mathbb{C}$ *is given by an invertible* 2×2 *matrix* $\begin{bmatrix} a_{\mathbb{C}} & b_{\mathbb{C}} \\ c_{\mathbb{C}} & d_{\mathbb{C}} \end{bmatrix}$.

Proof. Consider the component of Ψ at X. Let $n = \dim X$ and let $F : X \to X$ be a linear map given by the $n \times n$ matrix $[F]$. Note that $\mathbb{C}^2 \otimes X$ and $X \otimes \mathbb{C}^2$ have dimension $2n$, so the linear isomorphism Ψ is given by a $2n \times 2n$ matrix $[\Psi]$. We write the matrix of Ψ as a 2×2 block matrix with blocks of size $n \times n$, so we have $[\Psi] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. By naturality of Ψ, the square

$$\begin{array}{ccc} \mathbb{C}^2 \otimes V & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}^2 \otimes V \\ \Psi \downarrow & & \downarrow \Psi \\ V \otimes \mathbb{C}^2 & \xrightarrow[F \otimes \mathrm{id}]{} & V \otimes \mathbb{C}^2 \end{array}$$

commutes. With respect to the chosen bases, we have

$$[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = \begin{bmatrix} [F] & 0 \\ 0 & [F] \end{bmatrix}.$$

Naturality then amounts to the matrix equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} [F] & 0 \\ 0 & [F] \end{bmatrix} = \begin{bmatrix} [F] & 0 \\ 0 & [F] \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

which yields the following relations among the blocks:

$$A[F] = [F]A, \quad B[F] = [F]B, \quad C[F] = [F]C, \quad D[F] = [F]D.$$

By the same argument as in the proof of proposition 6.4, we have

$$A = a_X I_n, \quad B = b_X I_n, \quad C = c_X I_n, \quad D = d_X I_n.$$

Now it is clear that

$$[\Psi] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_X I_n & b_X I_n \\ c_X I_n & d_X I_n \end{bmatrix}.$$

More explicitly, Ψ has the matrix

$$[\Psi] = \left[\begin{array}{ccc|ccc} a_X & 0 & \dots & b_X & 0 & \dots \\ 0 & a_X & & 0 & b_X & \\ \vdots & & \ddots & \vdots & & \ddots \\ \hline c_X & 0 & \dots & d_X & 0 & \dots \\ 0 & c_X & & 0 & d_X & \\ \vdots & & \ddots & \vdots & & \ddots \end{array}\right].$$

■

Proposition 6.7. *If $(\mathbb{C}^2, \Psi)$ is in $\mathcal{Z}(\textbf{Vect}_{\mathbb{C}})$ then $\Psi = \Phi$.*

Proof. By naturality, the square

$$\begin{array}{ccc} \mathbb{C}^2 \otimes V & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}^2 \otimes W \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ V \otimes \mathbb{C}^2 & \xrightarrow[F \otimes \mathrm{id}]{} & W \otimes \mathbb{C}^2 \end{array}$$

commutes for all vector spaces V, W and every linear map $F : V \to W$. Let $n = \dim V$ and $m = \dim W$. Fixing bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ of V and W respectively, we obtain bases for all involved vector spaces. Denote the component at V by Ψ_V and the component at W by Ψ_W. By lemma 6.6, we have

$$[\Psi_V] = \begin{bmatrix} a_V I_n & b_V I_n \\ c_V I_n & d_V I_n \end{bmatrix} \quad \text{and} \quad [\Psi_W] = \begin{bmatrix} a_W I_m & b_W I_m \\ c_W I_m & d_W I_m \end{bmatrix}.$$

Now we note the following:

(i) the linear map F is given by an $m \times n$ matrix $[F]$,

(ii) the linear maps $\mathrm{id} \otimes F$ and $F \otimes \mathrm{id}$ are given by $2m \times 2n$ matrices,

(iii) with respect to our bases we have

$$[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = \begin{bmatrix} [F] & 0_{m \times n} \\ 0_{m \times n} & [F] \end{bmatrix}.$$

Here $0_{m \times n}$ denotes a $m \times n$ block of zeroes.

Commutativity of the square now yields:

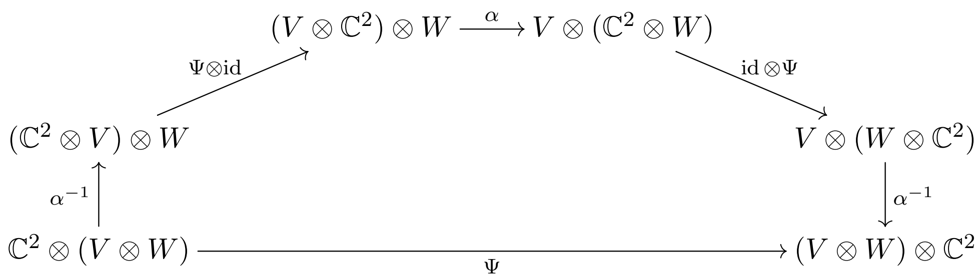
$$\begin{aligned} (F \otimes \mathrm{id})\Psi_V = \Psi_W(\mathrm{id} \otimes F) &\iff [F \otimes \mathrm{id}][\Psi_V] = [\Psi_W][\mathrm{id} \otimes F] \\ &\iff \begin{bmatrix} [F] & 0_{m \times n} \\ 0_{m \times n} & [F] \end{bmatrix} \begin{bmatrix} a_V I_n & b_V I_n \\ c_V I_n & d_V I_n \end{bmatrix} = \begin{bmatrix} a_W I_m & b_W I_m \\ c_W I_m & d_W I_m \end{bmatrix} \begin{bmatrix} [F] & 0_{m \times n} \\ 0_{m \times n} & [F] \end{bmatrix} \\ &\iff \begin{bmatrix} a_V[F] & b_V[F] \\ c_V[F] & d_V[F] \end{bmatrix} = \begin{bmatrix} a_W[F] & b_W[F] \\ c_W[F] & d_W[F] \end{bmatrix}. \end{aligned}$$

Since this must hold for every matrix $[F]$ this implies

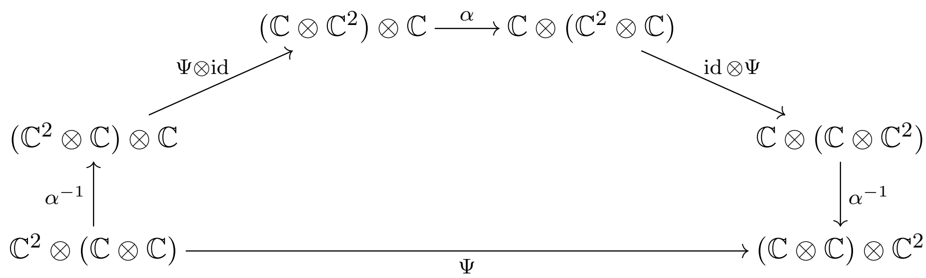
$$a_V = a_W, \quad b_V = b_W, \quad c_V = c_W, \quad d_V = d_W.$$

Since V and W are arbitrary vector spaces this shows that these numbers are invariant across all vector spaces, so we may drop the indices. This is an improvement upon the result of lemma 6.6, in which the numbers may depend on the vector space.

Since $(\mathbb{C}^2, \Psi) \in \mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$, the diagram



commutes. Putting $V = W = \mathbb{C}$, this diagram becomes



and now we observe that, with respect to the chosen bases, we have

$$[\alpha] = [\alpha^{-1}] = I_2 \quad \text{and} \quad [\Psi \otimes \mathrm{id}] = [\mathrm{id} \otimes \Psi] = [\Psi] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This means that the commutativity of the above diagram is equivalent to the matrix equation $[\Psi]^2 = [\Psi]$. Since $[\Psi]$ is invertible we get

$$\begin{aligned} [\Psi]^2 = [\Psi] &\iff [\Psi]^2[\Psi]^{-1} = [\Psi][\Psi]^{-1} \\ &\iff [\Psi] = I_2 \\ &\iff a = d = 1 \quad \text{and} \quad b = c = 0. \end{aligned}$$

This means that for any vector space, the matrix of $\Psi : \mathbb{C}^2 \otimes \_ \to \_ \otimes \mathbb{C}^2$ with respect to the chosen basis is the identity matrix. This shows that $\Psi = \Phi$. $\square$

Finally, we want to mimic the approach taken in the two-dimensional case in order to establish the following proposition.

Proposition 6.8. *If* $\left(\mathbb{C}^k, \Psi\right)$ *is in* $\mathcal{Z}(\textbf{\textit{Vect}}_{\mathbb{C}})$*, then* $\Psi = \Phi$*.*

Proof. First, we consider the component of Ψ at some vector space X. Let $n = \dim X$. By naturality, the square

$$\begin{array}{ccc} \mathbb{C}^k \otimes X & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}^k \otimes X \\ {\scriptstyle \Psi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ X \otimes \mathbb{C}^k & \xrightarrow[F \otimes \mathrm{id}]{} & X \otimes \mathbb{C}^k \end{array}$$

commutes. Writing $[\Psi]$ as a $k \times k$ block matrix with block size $n \times n$, we have

$$[\Psi] = \begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix}$$

and with respect to our bases we have

$$[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = \begin{bmatrix} [F] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [F] \end{bmatrix}.$$

Naturality now yields $A_{ij}[F] = [F]A_{ij}$ for all $i, j \in \{1, \dots, k\}$. This implies $A_{ij} = a_{ij}^X I_n$ as in the case $k = 2$. The square

$$\begin{array}{ccc} \mathbb{C}^k \otimes V & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}^k \otimes W \\ {\scriptstyle \Psi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ V \otimes \mathbb{C}^k & \xrightarrow[F \otimes \mathrm{id}]{} & W \otimes \mathbb{C}^k \end{array}$$

commutes for all vector spaces V, W and every linear map $F : V \to W$. The exact same technique as in the case $k = 2$ can be used to show that $a_{ij}^V = a_{ij}^W$ for all V, W.

Putting $V = W = \mathbb{C}$ and using the commutativity of the diagram

$$\begin{array}{ccccc} & (\mathbb{C} \otimes \mathbb{C}^k) \otimes \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C} \otimes (\mathbb{C}^k \otimes \mathbb{C}) & \\ {\scriptstyle \Psi \otimes \mathrm{id}}\nearrow & & & & \searrow{\scriptstyle \mathrm{id} \otimes \Psi} \\ (\mathbb{C}^k \otimes \mathbb{C}) \otimes \mathbb{C} & & & & \mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C}^k) \\ {\scriptstyle \alpha^{-1}}\big\uparrow & & & & \big\downarrow{\scriptstyle \alpha^{-1}} \\ \mathbb{C}^k \otimes (\mathbb{C} \otimes \mathbb{C}) & & \xrightarrow[\Psi]{\qquad\qquad} & & (\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C}^k \end{array}$$

we get that the component at a one-dimensional vector space is given by $[\Psi] = I_k$ which implies

$$a_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$$

so Ψ acts via the identity matrix and hence $\Psi = \Phi$. $\square$

Corollary 6.9. *If* (V, Ψ) *is in* $\mathcal{Z}(\textbf{\textit{Vect}}_{\mathbb{C}})$*, then* $\Psi = \Phi$*.*

Proof. The proof of proposition 6.8 applies without modification. □

6.1 Equivalence

Thanks to the results of the previous section, we can now completely describe the Drinfeld center of $\mathbf{Vect}_{\mathbb{C}}$.

- objects are pairs (V, Φ)
- morphisms from (V, Φ) to (W, Φ) are linear maps from V to W
- the tensor product is given by $(V, \Phi) \otimes (W, \Phi) = (V \otimes W, \Phi)$ and the tensor unit is $(\mathbb{C}, \Phi)$.

Theorem 6.10. $\textbf{\textit{Vect}}_{\mathbb{C}}$ *and* $\mathcal{Z}(\textbf{\textit{Vect}}_{\mathbb{C}})$ *are isomorphic as categories.*

Proof. Define a map $F : \mathbf{Vect}_{\mathbb{C}} \to \mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$ as follows:

- $V \mapsto (V, \Phi)$ for objects,
- $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(V, W) \ni f \mapsto f \in \mathrm{Hom}_{\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})}((V, \Phi), (W, \Phi))$ for morphisms.

It is clear from the results of the previous sections that F is a bijection on objects and on morphisms, hence an isomorphism of categories. □

Denoting the tensor product of $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$ by $\otimes_{\mathcal{Z}}$ to distinguish it from the usual tensor product in $\mathbf{Vect}_{\mathbb{C}}$, we note the following properties of F:

$$F(V) \otimes_{\mathcal{Z}} F(W) = (V, \Phi) \otimes_{\mathcal{Z}} (W, \Phi) = (V \otimes W, \Phi) = F(V \otimes W)$$
$$F(\mathbb{C}) = (\mathbb{C}, \Phi)$$

so F preserves the tensor product and the tensor unit.

Now it can be easily checked that the pair (F, id) satisfies the axioms of a monoidal functor. This fact together with theorem 6.10 shows that (F, id) is an isomorphism of monoidal categories.

7 Categories of group representations

Our next objects of study are suitable categories of group representations. The basics follow from [4].

Definition 7.1. Let K be a field, V a vector space over K and let G be a group. Then a representation of V is a homomorphism of groups

$$\rho : G \to \mathrm{GL}(V).$$

Definition 7.2. Let K be a field, V a vector space over K and let G be a group. Then V is a G-module if there exists a linear action of G on V, that is, a map $\varphi : G \times V \to V$ such that *i)* $\varphi(g,(v+w)) = \varphi(g,v) + \varphi(g,w)$

ii) $\rho(g,\lambda v) = \lambda\varphi(g,v)$

iii) $\varphi(gh,v) = \varphi(g,\varphi(h,v))$

iv) $\varphi(e,v) = v$.

Proposition 7.3. *The definitions 7.1 and 7.2 are equivalent.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Then, for any $g \in G$, $\rho(g)$ is an invertible linear map so defining the map $\varphi : G \times V \to V$ by $\varphi(g,v) = \rho(g)(v)$ defines a G-module structure on V. Let $\varphi : G \times V \to V$ be a G-module structure. Define the map $\rho : G \to \mathrm{GL}(V)$ by

$$g \mapsto \varphi(g, \_).$$

Then ρ is a homomorphism of groups:

$$\begin{aligned} \rho(gh) &= \varphi(gh, \_) \\ &= \varphi(g, \varphi(h, \_)) \\ &= \rho(g)\rho(h). \end{aligned}$$

□

Remark 7.4. Any field K viewed as a vector space over itself becomes a G-module for any group G with action given by

$$g(x) = x, \forall g \in G, x \in K.$$

This is called the trivial G-module.

Definition 7.5. Let V and W be G-modules. Then a G-homomorphism is a linear map $F : V \to W$ such that

$$F(g(v)) = g(F(v))$$

for every $g \in G$ and $v \in V$. In other words, the diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ {\scriptstyle g\cdot\_}\downarrow & & \downarrow{\scriptstyle g\cdot\_} \\ V & \xrightarrow[F]{} & W \end{array}$$

commutes.

Definition 7.6. Let V be a G-module and let $W \subset V$ be a linear subspace. Then W is a submodule of V if $w \in W \implies gw \in W$ for all $g \in G$.

Every G-module has two submodules: $\{0\}$ and itself. These are called trivial submodules. A submodule which is not trivial is called proper.

Proposition 7.7. *Let V, W be G-modules and let $F : V \to W$ be a G-homomorphism. Then* $\ker(F)$ *and* $\operatorname{im}(F)$ *are submodules of V and W, respectively.*

Proof. By linear algebra, we know that $\ker(F)$ and $\operatorname{im}(F)$ are subspaces. Left to check is that they are invariant under the action of G. Suppose $v \in \ker(F)$. Then

$$\begin{aligned} F(gv) &= g(F(v)) \\ &= g(0) \\ &= 0. \end{aligned}$$

Suppose $w \in \operatorname{im}(F)$. Then there exists $v_0 \in V$ such that $F(v_0) = w$. Then

$$\begin{aligned} F(gv_0) &= gF(v_0) \\ &= gw. \end{aligned}$$

□

Definition 7.8. A G-module which has no proper submodules is said to be simple. Note that any one-dimensional module is automatically simple.

Proposition 7.9. *Let V and W be G-modules. Then the tensor product $V \otimes W$ is a G-module with action given by*

$$g(v \otimes w) = g(v) \otimes g(w).$$

Proof. Clearly we can extend the action of g to sums of elements of the form $v \otimes w$, which yields the condition

$$g(v \otimes w + v' \otimes w') = g(v \otimes w) + g(v' \otimes w').$$

We then have

i)

$$\begin{aligned} g(\lambda(v \otimes w)) &= g(\lambda v \otimes w) \\ &= g(\lambda v) \otimes g(w) \\ &= \lambda g(v) \otimes g(w) \\ &= \lambda(g(v) \otimes g(w)) \end{aligned}$$

ii)

$$\begin{aligned} (gh)(v \otimes w)) &= (gh)(v) \otimes (gh)(w) \\ &= g(h(v)) \otimes g(h(w)) \\ &= g(h(v) \otimes h(w)) \end{aligned}$$

iii)

$$\begin{aligned} e(v \otimes w) &= e(v) \otimes e(w) \\ &= v \otimes w. \end{aligned}$$

□

Proposition 7.10. *Let V be a G-module and K the trivial G-module. Then there is an isomorphism of G-modules*

$$F : V \otimes K \xrightarrow{\sim} V.$$

Proof. Fix a basis $\{v_i\}$ and some $k \in K$. Then $\{v_i \otimes k\}$ is a basis of $V \otimes K$ and we know that defining F by $v_i \otimes k \mapsto v_i$ yields a linear isomorphism. But we also have

$$\begin{aligned} g(F(v \otimes k)) &= g(v) \\ &= F(g(v) \otimes k)) \\ &= F(g(v) \otimes g(k)) \\ &= F(g(v \otimes k)) \end{aligned}$$

so F is an isomorphism of G-modules. □

Remark 7.11. In a similar manner, it may be checked that the canonical isomorphism

$$F : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

is in fact a G-isomorphism, when the action of G on the tensor product is defined as in proposition 7.9.

Proposition 7.12 (Maschke's theorem)**.** *Let G be a finite group and K a field such that* $\mathrm{char}(K) \nmid |G|$*. Then any finite-dimensional G-module can be written as a direct sum of simple modules.*

Proof. If V is simple, we are done. If not, we use induction on the dimension of V. Since one-dimensional modules are automatically simple, it suffices to show that any submodule of V has a submodule complement.

Let Y be a submodule of V. Let X be a complement of Y, that is, a subspace such that $V = X \oplus Y$. Note that X is not necessarily a submodule.Let $P_0 : V \to V$ be a linear projection onto Y.

Now, define the linear endomorphism $P = \frac{1}{|G|} \sum\limits_{g \in G} g^{-1} P_0 g$.

Then, for any $y \in Y$, we have

$$\begin{aligned} P(y) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} P_0 g(y) \\ &= \frac{1}{|G|} \sum_{g \in G} g^{-1} P_0(g(y)) \\ &= \frac{1}{|G|} \sum_{g \in G} g^{-1} g(y) \\ &= \frac{1}{|G|} \sum_{g \in G} y \\ &= y \end{aligned}$$

which follows from the fact that Y is a submodule and $P_0|_Y = \mathrm{id}_Y$. Moreover, we have $\mathrm{im}(P) = Y$, so P is also a linear projection. It follows that we have a decomposition $V = Z \oplus Y$, where now $Z = \ker(P)$ and $Y = \mathrm{im}(P)$. Next, we note that P is a G-homomorphism. Indeed,

$$\begin{aligned} P(h(v)) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} P_0 g(h(v)) \\ &= \frac{1}{|G|} \sum_{g \in G} hh^{-1} g^{-1} P g h(v) \\ &= \frac{1}{|G|} h \sum_{g \in G} (gh)^{-1} P(gh)(v) \\ &= hP(v). \end{aligned}$$

Now Z is a submodule since it is the kernel of a G-homomorphism, so we have a submodule complement Z of Y, and we are done. □

In what follows, we assume that $K = \mathbb{C}$.

Proposition 7.13 (Schur's lemma)**.** *Let V and W be simple G-modules. Then the following hold:*

i) Every G-homomorphism $F : V \to W$ is either zero or an isomorphism.

ii) If $V = W$, the only non-zero G-homomorphisms are scalar multiples of the identity.

Proof. *i)* Suppose $F : V \to W$ is non-zero. Since $\ker(F)$ is a submodule, we have either $\ker(F) = 0$ or $\ker(F) = V$ since V is simple. Since F is non-zero, we have $\ker(F) = 0$ which implies that F is injective.

Similarily, since $\mathrm{im}(F)$ is a submodule, we have either $\mathrm{im}(F) = 0$ or $\mathrm{im}(F) = W$ since W is simple. Since F-is non-zero we have $\mathrm{im}(F) = W$, so F is surjective. So F is bijective, so it is an isomorphism.

ii) Suppose $F : V \to V$ is non-zero. Since our base field is $\mathbb{C}$, the map F has an eigenvalue, λ, with corresponding eigenvector v. Put $F' = F - \lambda\,\mathrm{id}_V$. Then $F'(v) = 0$.

Since $\ker(F)$ is either 0 or V, it must be V since it contains v, so $F' = 0$ which implies $F = \lambda\,\mathrm{id}_V$.

□

Corollary 7.14. *Let V and W be simple G-modules. Then*

$$\dim \mathrm{Hom}_{G\text{-}\boldsymbol{mod}}(V, W) = \begin{cases} 1 & \textit{if} \quad V \cong W \\ 0 & \textit{if} \quad V \ncong W \end{cases}$$

Proof. If $V \ncong W$, then the only G-homomorphism between them is the zero map by Schur's lemma. Suppose $V \cong W$ and let $F, G : V \to W$ be G-isomorphisms. Then $G^{-1}F : V \to V$ is an endomorphism of V, so we have $G^{-1}F = \lambda\,\mathrm{id}_V$. Then

$$\begin{aligned} G^{-1}F = \lambda\,\mathrm{id}_V &\implies G(G^{-1}F) = G(\lambda\,\mathrm{id}_V) \\ &\implies \left(GG^{-1}\right) F = \lambda G\,\mathrm{id}_V \\ &\implies \mathrm{id}_W F = \lambda G\,\mathrm{id}_V \\ &\implies F = \lambda G. \end{aligned}$$

□

7.1 Drinfeld centers

For a fixed group G and a field K, the finite-dimensional G-modules over K and their G-homomorphisms form a category, G-**mod**.

The basic properties of G-modules established thus far make it clear that we can impose the structure of a monoidal category on G-**mod**, by taking the tensor product as the usual tensor product of K-vector spaces, with the addition of defining action of G on this tensor product as in proposition 7.9. The tensor unit will be the trivial G-module K and the associator and unitor isomorphisms well be the canonical isomorphisms we are familiar with from the monoidal category $\mathbf{Vect}_{\mathbb{C}}$.

Definition 7.15. Let V and W be finite dimensional G-modules. For any $z \in Z(G)$, define

$$\Phi_z : V \otimes W \to W \otimes V$$

by

$$v \otimes w \mapsto z(w) \otimes v.$$

Since action of z is an invertible linear operator, this yields a linear isomorphism.

Proposition 7.16. *Let V and W be finite dimensional G-modules. Then any isomorphism*

$$\Phi_z : V \otimes W \to W \otimes V$$

is a G-isomorphism.

Proof. We immediately have

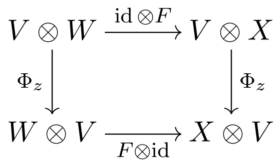
$$\begin{aligned}\Phi_z\left(g(v \otimes w)\right) &= \Phi_z(g(v) \otimes g(w)) \\ &= (zg)(w) \otimes g(v) \\ &= (gz)(w) \otimes g(v) \\ &= g\left(\Phi_z(v \otimes w)\right)\end{aligned}$$

□

Remark 7.17. Since for any group we have $e \in Z(G)$, the above definition contains the canonical isomorphism we are familiar with as a special case, namely $\Phi = \Phi_e$.

Proposition 7.18. *Let V be a finite dimensional G-module and $z \in Z(G)$. Then the pair (V, Φ_z) is in the Drinfeld center $\mathcal{Z}\left(G\textbf{-mod}\right)$.*

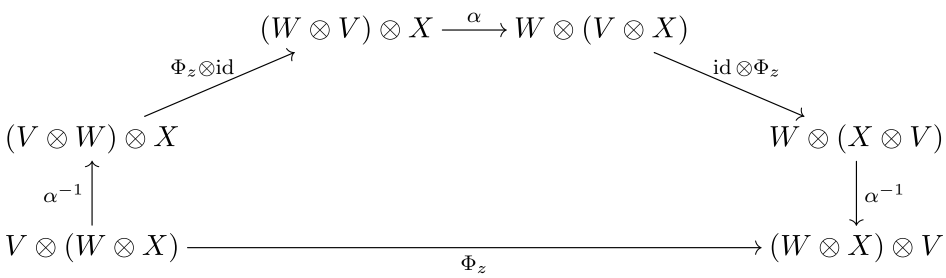
Proof. Let W and X be G-modules and $F : W \to X$ some G-homomorphism. We need the diagram



to commute, which is verified by

$$\begin{aligned}(F \otimes \mathrm{id})(\Phi_z(v \otimes w)) &= (F \otimes \mathrm{id})(zw \otimes v) \\ &= F(zw) \otimes v \\ &= zF(w) \otimes v \\ &= \Phi_z(v \otimes F(w)) \\ &= \Phi_z(\mathrm{id} \otimes F)(v \otimes w).\end{aligned}$$

Moreover, chasing the element $v \otimes (w \otimes x)$ through the diagram



shows that it is commutative. □

Proposition 7.19. *Let V, W be finite-dimensional G-modules. Then*

$$\mathrm{Hom}_{\mathcal{Z}(G\textbf{-mod})}\left((V, \Phi_z), (W, \Phi_z)\right) = \mathrm{Hom}_{G\textbf{-mod}}\left(V, W\right).$$

Proof. Let $F : V \to W$ be a G-homomorphism. We need the diagram

$$\begin{array}{ccc} V \otimes X & \xrightarrow{F \otimes \mathrm{id}} & W \otimes X \\ \Phi_z \downarrow & & \downarrow \Phi_z \\ X \otimes V & \xrightarrow[\mathrm{id} \otimes F]{} & X \otimes W \end{array}$$

to commute for any G-module X. We have

$$\begin{aligned} (\mathrm{id} \otimes F)(\Phi_z(v \otimes x)) &= (\mathrm{id} \otimes F)(zx \otimes v) \\ &= zx \otimes F(v) \\ &= \Phi_z(F(v) \otimes x) \\ &= \Phi_z(F \otimes \mathrm{id})(v \otimes x). \end{aligned}$$

□

Definition 7.20. Let V, W, X be G-modules. Denote by

$$\varepsilon : (V \oplus W) \otimes X \to (V \otimes X) \oplus (W \otimes X)$$

the canonical isomorphism defined by

$$(v, w) \otimes x \mapsto (v \otimes x, w \otimes x).$$

Definition 7.21. Let V, W, X be G-modules. Denote by

$$\delta : X \otimes (V \oplus W) \to (X \otimes V) \oplus (X \otimes W)$$

the canonical isomorphism defined by

$$x \otimes (v, w) \mapsto (x \otimes v, x \otimes w).$$

Definition 7.22. Let (V, Ψ) and (W, Θ) be in $\mathcal{Z}(\mathbf{Vect}_{\mathbb{C}})$ so that we for any module X have isomorphisms

$$\Psi : V \otimes X \to X \otimes V \quad \text{and} \quad W \otimes X \to X \otimes W.$$

Define $\Psi \boxplus \Theta$ by

$$\Psi \boxplus \Theta := \delta^{-1} (\Psi \oplus \Theta) \varepsilon : (V \oplus W) \otimes X \to X \otimes (V \oplus W).$$

Proposition 7.23. *If (V, Ψ) and (W, Θ) are in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$, then*

$$(V, \Psi) \oplus (W, \Theta) := (V \oplus W, \Psi \boxplus \Theta)$$

is in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$.

Proof. Let $F : X \to Y$ be a G-homomorphism and cosider the following diagram:

$$
\begin{array}{ccc}
(V \oplus W) \otimes X & \xrightarrow{\mathrm{id} \otimes F} & (V \oplus W) \otimes Y \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
(V \otimes X) \oplus (W \otimes X) & \xrightarrow{(\mathrm{id} \otimes F) \oplus (\mathrm{id} \otimes F)} & (V \otimes Y) \oplus (W \otimes Y) \\
\downarrow \Psi \oplus \Theta & & \downarrow \Psi \oplus \Theta \\
(X \otimes V) \oplus (X \otimes W) & \xrightarrow{(F \otimes \mathrm{id}) \oplus (F \otimes \mathrm{id})} & (Y \otimes V) \oplus (Y \otimes W) \\
\downarrow \delta^{-1} & & \downarrow \delta^{-1} \\
X \otimes (V \oplus W) & \xrightarrow[F \otimes \mathrm{id}]{} & Y \otimes (V \oplus W)
\end{array}
$$

It is clear that the perimeter is a naturality square of $\Psi \boxplus \Theta$. Moreover, the middle square is just the sum of naturality squares of Ψ and Θ, which commute by assumption. Hence, the middle square commutes.

In the top square, we have

$$
\begin{aligned}
\varepsilon(\mathrm{id} \otimes F)((v, w) \otimes x) &= \varepsilon((v, w) \otimes F(x)) \\
&= (v \otimes F(x), w \otimes F(x)) \\
&= ((\mathrm{id} \otimes F) \oplus (\mathrm{id} \otimes F))\,(v \otimes x, w \otimes x) \\
&= ((\mathrm{id} \otimes F) \oplus (\mathrm{id} \otimes F))\,\varepsilon((v, w) \otimes x)
\end{aligned}
$$

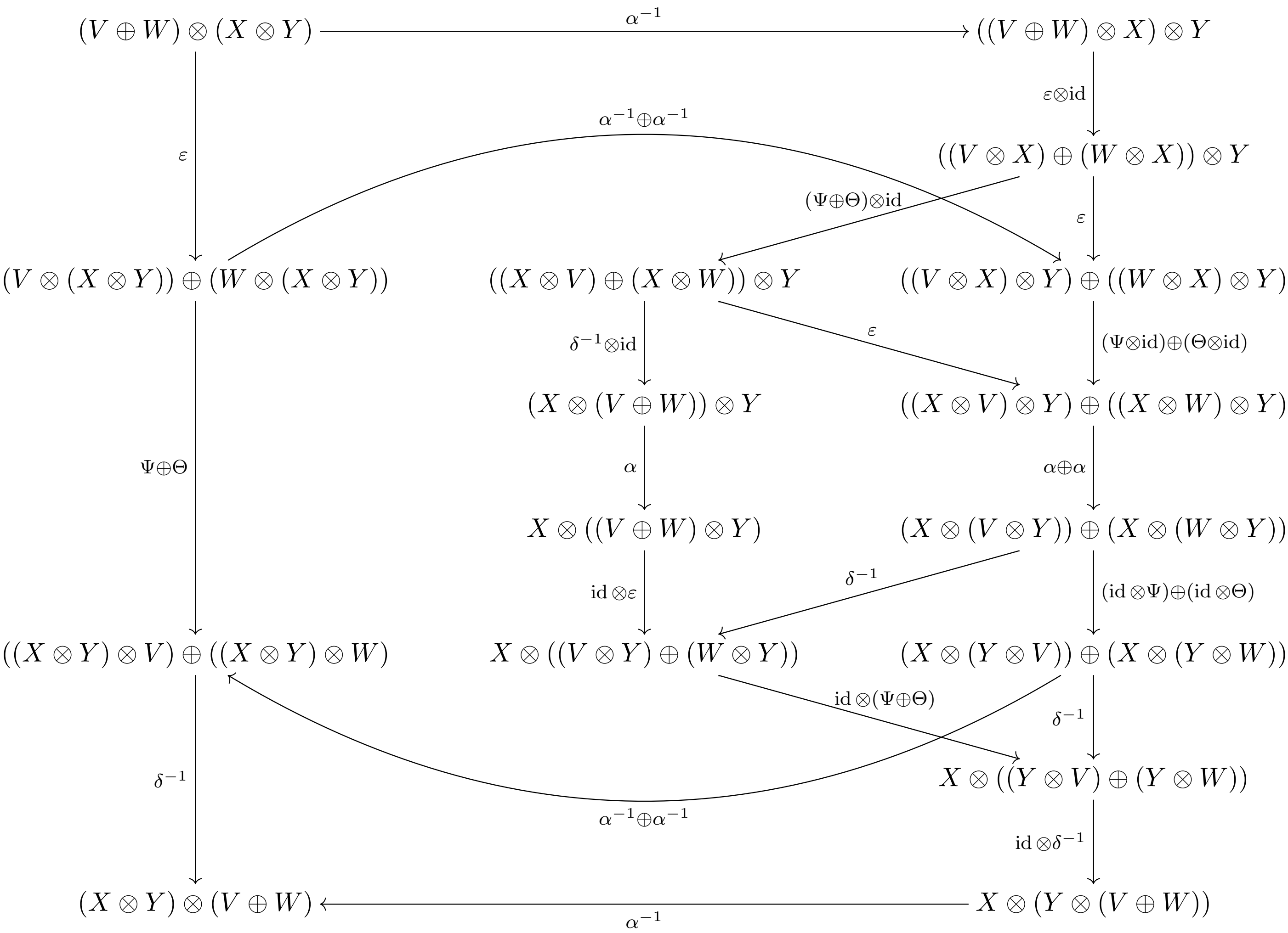
so the top square commutes.

In the bottom square, we have

$$
\begin{aligned}
\delta^{-1}\left((F \otimes \mathrm{id}) \oplus (F \otimes \mathrm{id})\right)((x \otimes v, x \otimes w)) &= \delta^{-1}((F(x) \otimes v, F(x) \otimes w)) \\
&= F(x) \otimes (v, w) \\
&= (F \otimes \mathrm{id})(x \otimes (v, w)) \\
&= (F \otimes \mathrm{id})\delta^{-1}((x \otimes v, x \otimes w))
\end{aligned}
$$

so the bottom square commutes. This shows that the perimeter commutes so $\Psi \boxplus \Theta$ is a natural family of isomorphisms.

Next, we consider the diagram



Now the diagram given by the curved arrows together with the first and third columns is just the sum of diagrams which commute by the assumption that (V, Ψ) and (W, Θ) are in $\mathcal{Z}(G\textbf{-mod})$, so it commutes.

Next, consider the diagram given by the top arrow, the first and third columns and the top curved arrow $\alpha^{-1} \oplus \alpha^{-1}$.

$$\begin{aligned}
\varepsilon(\varepsilon \otimes \mathrm{id})\alpha^{-1}((v, w) \otimes (x \otimes y)) &= \varepsilon(\varepsilon \otimes \mathrm{id})(((v, w) \otimes x) \otimes y) \\
&= \varepsilon((v \otimes x, w \otimes x), y) \\
&= ((v \otimes x) \otimes y, (w \otimes x) \otimes y) \\
&= (\alpha^{-1} \oplus \alpha^{-1})((v \otimes (x \otimes y), w \otimes (x \otimes y))) \\
&= (\alpha^{-1} \oplus \alpha^{-1})\varepsilon((v, w) \otimes (x \otimes y))
\end{aligned}$$

so this diagram commutes. The corresponding diagram on the bottom is shown to be commutative in the same way. These observations together show that the outer perimeter commutes.

Next, we consider the diagram where the third column branches out into the second. This diagram is divided into three parts.

In the top part, we have

$$\begin{aligned}((\Psi \otimes \mathrm{id}) \oplus (\Theta \otimes \mathrm{id}))\varepsilon((v \otimes x, w \otimes x) \otimes y) &= ((\Psi \otimes \mathrm{id}) \oplus (\Theta \otimes \mathrm{id}))((v \otimes x) \otimes y, (w \otimes x) \otimes y) \\ &= (\Psi(v \otimes x) \otimes y, \Theta(w \otimes x) \otimes y) \\ &= \varepsilon((\Psi(v \otimes x), \Theta(w \otimes x)) \otimes y) \\ &= \varepsilon((\Psi \oplus \Theta) \otimes \mathrm{id})((v \otimes x, w \otimes x) \otimes y)\end{aligned}$$

so the top part commutes. The bottom part is shown to be commutative in the same way.

In the middle part, we have

$$\begin{aligned}\delta^{-1}(\alpha \oplus \alpha)\varepsilon((x \otimes v, x \otimes w) \otimes y) &= \delta^{-1}(\alpha \oplus \alpha)((x \otimes v) \otimes y, (x \otimes w) \otimes y) \\ &= \delta^{-1}(x \otimes (v \otimes y), x \otimes (w \otimes y)) \\ &= x \otimes (v \otimes y, w \otimes y) \\ &= (\mathrm{id} \otimes \varepsilon)(x \otimes ((v, w) \otimes y)) \\ &= (\mathrm{id} \otimes \varepsilon)\alpha((x \otimes (v, w)) \otimes y) \\ &= (\mathrm{id} \otimes \varepsilon)\alpha(\delta^{-1} \otimes \mathrm{id})((x \otimes v, x \otimes w) \otimes y)\end{aligned}$$

so the middle part commutes. Now it follows that the whole branch commutes with the third column.

This implies, that if we follow the perimeter but instead follow the branch into the inner perimeter, the diagram still commutes. This diagram commuting verifies that $(V \oplus W, \Psi \boxplus \Theta)$ is in $\mathcal{Z}(G\textbf{-mod})$. □

8 The Drinfeld center of $\mathbb{Z}_2$-mod

We write $G = \mathbb{Z}_2 = \{e, s\}$ with the relation $s^2 = e$. It can be shown that $\mathbb{Z}_2$ has two simple modules. The first one is the trivial module $\mathbb{C}_{\mathbf{triv}}$. The second one is the sign module $\mathbb{C}_{\mathbf{sign}}$ where s acts as multiplication with -1. By Maschke's theorem, any module V is isomorphic to a direct sum of simples, that is $V \cong \mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$. Fixing the standard basis, we have a basis

$$e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}$$

of $\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$.

Proposition 8.1. *Any G-homomorphism $F : \mathbb{C}_{\mathbf{triv}}^{\oplus m_1} \oplus \mathbb{C}_{\mathbf{triv}}^{\oplus n_1} \to \mathbb{C}_{\mathbf{triv}}^{\oplus m_2} \oplus \mathbb{C}_{\mathbf{triv}}^{\oplus n_2}$ is given by a block matrix of the form*

$$\begin{bmatrix} A & 0_{m_2 \times n_1} \\ 0_{n_2 \times m_1} & B \end{bmatrix}$$

where A is an arbitrary $m_2 \times m_1$ matrix and B is an arbitrary $n_2 \times n_1$ matrix.

Proof. Consider a linear map

$$F : \mathbb{C}_{\mathbf{triv}}^{\oplus m_1} \oplus \mathbb{C}_{\mathbf{triv}}^{\oplus n_1} \to \mathbb{C}_{\mathbf{triv}}^{\oplus m_2} \oplus \mathbb{C}_{\mathbf{triv}}^{\oplus n_2}$$

It is given by an $(m_2 + n_2) \times (m_1 + n_1)$ matrix. We write the matrix of F as a block matrix

$$[F] = \begin{bmatrix} [F_{11}]_{m_2 \times m_1} & [F_{12}]_{m_2 \times n_1} \\ [F_{21}]_{n_2 \times m_1} & [F_{22}]_{n_2 \times n_1} \end{bmatrix}$$

where F_{ij} denotes the component of F mapping the j:th summand to the i:th summand. In other words, a linear map F as above corresponds to four linear maps

$$\begin{aligned} F_{11} &: \mathbb{C}_{\mathbf{triv}}^{\oplus m_1} \to \mathbb{C}_{\mathbf{triv}}^{\oplus m_2} \\ F_{12} &: \mathbb{C}_{\mathbf{sign}}^{\oplus n_1} \to \mathbb{C}_{\mathbf{triv}}^{\oplus m_2} \\ F_{21} &: \mathbb{C}_{\mathbf{triv}}^{\oplus m_1} \to \mathbb{C}_{\mathbf{sign}}^{\oplus n_2} \\ F_{22} &: \mathbb{C}_{\mathbf{sign}}^{\oplus n_1} \to \mathbb{C}_{\mathbf{sign}}^{\oplus n_2}. \end{aligned}$$

It is clear that F is a G-homomorphism if and only if all of its components are. By additivity of the Hom-functor and Schur's lemma, we have

$$\begin{aligned} \dim \mathrm{Hom}_{G\textbf{-mod}}\left(\mathbb{C}_{\mathbf{triv}}^{\oplus m}, \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) &= m \dim \mathrm{Hom}_{G\textbf{-mod}}\left(\mathbb{C}_{\mathbf{triv}}, \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \\ &= mn \dim \mathrm{Hom}_{G\textbf{-mod}}\left(\mathbb{C}_{\mathbf{triv}}, \mathbb{C}_{\mathbf{sign}}\right) \\ &= 0 \end{aligned}$$

for any m, n and

$$\begin{aligned} \dim \mathrm{Hom}\left(\mathbb{C}_{\mathbf{sign}}^{\oplus m'}, \mathbb{C}_{\mathbf{triv}}^{\oplus n'}\right) &= m' \dim \mathrm{Hom}\left(\mathbb{C}_{\mathbf{sign}}, \mathbb{C}_{\mathbf{triv}}^{\oplus n'}\right) \\ &= m'n' \dim \mathrm{Hom}\left(\mathbb{C}_{\mathbf{sign}}, \mathbb{C}_{\mathbf{triv}}\right) \\ &= 0 \end{aligned}$$

for any m', n'. This shows that $F_{ij} = 0$ whenever $i \neq j$, so a G-homomorphism

$$F : \mathbb{C}_{\mathbf{triv}}^{\oplus m_1} \oplus \mathbb{C}_{\mathbf{triv}}^{\oplus n_1} \to \mathbb{C}_{\mathbf{triv}}^{\oplus m_2} \oplus \mathbb{C}_{\mathbf{triv}}^{\oplus n_2}$$

is given by a matrix

$$[F] = \begin{bmatrix} [F_{11}] & 0 \\ 0 & [F_{22}] \end{bmatrix}.$$

The element $s \in \mathbb{Z}_2$ acts as the identity on $\mathbb{C}_{\mathbf{triv}}$, and as multiplication with -1 on $\mathbb{C}_{\mathbf{sign}}$. This means that on $\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$, the action of s is the linear extension of the map defined on the basis by

$$s(e_i) = \begin{cases} e_i & \text{if} \quad 1 \leq i \leq m \\ -e_i & \text{if} \quad m+1 \leq i \leq m+n \end{cases}$$

so we have

$$[s] = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}.$$

Denoting the matrix of the action of s on the different modules by $[s]_1$ and $[s]_2$, the condition of F being a G-homomorphism now amounts to the equation $[s]_2[F] = [F][s]_1$. Using our previous observation we see that

$$\begin{aligned} [s]_2[F] = [F][s]_1 &\iff \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{n_2} \end{bmatrix} \begin{bmatrix} [F_{11}] & 0 \\ 0 & [F_{22}] \end{bmatrix} = \begin{bmatrix} [F_{11}] & 0 \\ 0 & [F_{22}] \end{bmatrix} \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{n_1} \end{bmatrix} \\ &\iff \begin{bmatrix} I_{m_2}[F_{11}] & 0 \\ 0 & -I_{n_2}[F_{22}] \end{bmatrix} = \begin{bmatrix} [F_{11}]I_{m_1} & 0 \\ 0 & -[F_{22}]I_{n_1} \end{bmatrix}. \end{aligned}$$

This holds trivially for any matrices $[F_{ii}]$. □

Definition 8.2. Let V and W be finite-dimensional G-modules. Define the map $\Phi_e : V \otimes W \to W \otimes V$ by

$$v \otimes w \mapsto e(w) \otimes v.$$

This is a G-isomorphism by proposition 7.16 and (V, Φ_e) is in $\mathcal{Z}(G\text{-}\mathbf{mod})$ by proposition 7.18.

Definition 8.3. Let V, W be finite-dimensional G-modules. Define the map $\Phi_s : V \otimes W \to W \otimes V$ by

$$v \otimes w \mapsto s(w) \otimes v.$$

This is a G-isomorphism by proposition 7.16 and (V, Φ_e) is in $\mathcal{Z}(G\text{-}\mathbf{mod})$ by proposition 7.18.

Lemma 8.4. *Let $(\mathbb{C}_{triv}, \Psi)$ be in $\mathcal{Z}(G\text{-}\mathbf{mod})$ and let X be a one-dimensional module. Then the component*

$$\Psi_X : \mathbb{C}_{triv} \otimes X \to X \otimes \mathbb{C}_{triv}$$

has the form $\Psi_X = \lambda_X \Phi_{eX}$ for some nonzero $\lambda_X \in \mathbb{C}$. Moreover,

$$\lambda_X = \begin{cases} \lambda_{\mathbb{C}_{triv}} & \text{if } X \cong \mathbb{C}_{triv} \\ \lambda_{\mathbb{C}_{sign}} & \text{if } X \cong \mathbb{C}_{sign} \end{cases}.$$

Proof. Fix the basis 1 of $\mathbb{C}$ and the basis x of X, so that we have bases $1 \otimes x$ and $x \otimes 1$ of the respective tensor products.

Since $\Psi_X : \mathbb{C}_{\mathbf{triv}} \otimes X \to X \otimes \mathbb{C}_{\mathbf{triv}}$ is a linear isomorphism by assumption, we have

$$\Psi_X(1 \otimes x) = \lambda_X(x \otimes 1) = \lambda_X \Phi_{eX}(1 \otimes x)$$

so the first statement follows by linearity. Suppose $X \cong \mathbb{C}_{\mathbf{triv}}$. By naturality the square

$$\begin{array}{ccc} \mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}} & \xrightarrow{\mathrm{id} \otimes \varphi} & \mathbb{C}_{\mathbf{triv}} \otimes X \\ {\scriptstyle \Psi_{\mathbb{C}_{\mathbf{triv}}}} \downarrow & & \downarrow {\scriptstyle \Psi_X} \\ \mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}} & \xrightarrow[\varphi \otimes \mathrm{id}]{} & X \otimes \mathbb{C}_{\mathbf{triv}} \end{array}$$

commutes for any G-homomorphism. In particular, it commutes for any isomorphism φ. We then have

$$\begin{aligned} \lambda_{\mathbb{C}_{\mathbf{triv}}} \varphi(1) \otimes 1 &= (\varphi \otimes \mathrm{id}) \left(\lambda_{\mathbb{C}_{\mathbf{triv}}}(1 \otimes 1)\right) \\ &= (\varphi \otimes \mathrm{id})(\lambda_{\mathbb{C}_{\mathbf{triv}}} \Phi_{e\mathbb{C}_{\mathbf{triv}}}(1 \otimes 1)) \\ &= (\varphi \otimes \mathrm{id}) \Psi_{\mathbb{C}_{\mathbf{triv}}}(1 \otimes 1) \\ &= \Psi_X(\mathrm{id} \otimes \varphi)(1 \otimes 1) \\ &= \Psi_X(1 \otimes \varphi(1)) \\ &= \lambda_X \Phi_{eX}(1 \otimes \varphi(1)) \\ &= \lambda_X \varphi(1) \otimes 1. \end{aligned}$$

This implies $\lambda_{\mathbb{C}_{\mathbf{triv}}} = \lambda_X$. The case $X \cong \mathbb{C}_{\mathbf{sign}}$ can be proved in the same way. ■

Proposition 8.5. *If $(\mathbb{C}_{\boldsymbol{triv}}, \Psi)$ is in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$, then $\Psi = \Phi_e$ or $\Psi = \Phi_s$.*

Proof. The component of Ψ at some module $V = \mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$ is an isomorphism

$$\Psi : \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \to \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\mathbf{triv}}.$$

Fix bases $\{1 \otimes e_i\}$ and $\{e_i \otimes 1\}$ of the respective tensor products. We write the matrix of Ψ with respect to these bases as a block matrix

$$[\Psi] = \begin{bmatrix} A_{m\times m} & B_{m\times n} \\ C_{n\times m} & D_{n\times n} \end{bmatrix}.$$

The action of the element $s \in \mathbb{Z}_2$ on the chosen bases is given by

$$\begin{aligned} s(1 \otimes e_i) = s(1) \otimes s(e_i) = 1 \otimes s(e_i) \\ s(e_i \otimes 1) = s(e_i) \otimes s(1) = s(e_i) \otimes 1 \end{aligned}$$

since s acts as the identity in the trivial module. This means that in our bases we have

$$[s] = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}.$$

Now, Ψ being a G-homomorphism amounts to the matrix equation $[s][\Psi] = [\Psi][s]$. This equation yields

$$[s][\Psi] = [\Psi][s] \iff \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}$$
$$\iff \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = \begin{bmatrix} A & -B \\ C & -D \end{bmatrix}$$

so we must have $B = -B$ and $C = -C$ which implies $B = C = 0$. This shows that $[\Psi] = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. By proposition 8.1, any endomorphism of the module $\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$ is given by a matrix $[F] = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}$ where F_{11} is an arbitrary $m \times m$ matrix and F_{22} is an arbitrary $n \times n$ matrix. By naturality of Ψ the square

$$\begin{array}{ccc} \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \\ \Psi \downarrow & & \downarrow \Psi \\ \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\mathbf{triv}} & \xrightarrow[F \otimes \mathrm{id}]{} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\mathbf{triv}} \end{array}$$

commutes for any endomorphism F. With respect to our bases we have $[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = [F]$. Commutativity of the diagram amounts to the equation $[\Psi][F] = [F][\Psi]$. We have

$$[\Psi][F] = [F][\Psi] \iff \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$
$$\iff \begin{bmatrix} AF_{11} & 0 \\ 0 & DF_{22} \end{bmatrix} = \begin{bmatrix} F_{11}A & 0 \\ 0 & F_{22}D \end{bmatrix}$$
$$\iff AF_{11} = F_{11}A \quad \text{and} \quad DF_{22} = F_{22}D$$

and since F_{ii} is arbitrary we have $A = a_V I_m$ and $D = d_V I_n$.

Now let $V_1 = \mathbb{C}_{\mathbf{triv}}^{\oplus m_1} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n_1}$ and $V_2 = \mathbb{C}_{\mathbf{triv}}^{\oplus m_2} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n_2}$. By proposition 8.1 a G-homomorphism $F : V_1 \to V_2$ is given by a matrix of the form

$$[F] = \begin{bmatrix} F_{11 m_2 \times m_1} & 0_{m_2 \times n_1} \\ 0_{n_2 \times m_1} & F_{22 n_2 \times n_1} \end{bmatrix}.$$

The commutativity of the naturality square

$$\begin{array}{ccc} \mathbb{C}_{\mathbf{triv}} \otimes V_1 & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}_{\mathbf{triv}} \otimes V_2 \\ \Psi_{V_1} \downarrow & & \downarrow \Psi_{V_2} \\ V_1 \otimes \mathbb{C}_{\mathbf{triv}} & \xrightarrow[F \otimes \mathrm{id}]{} & V_2 \otimes \mathbb{C}_{\mathbf{triv}} \end{array}$$

is equivalent to $[\Psi_{V_2}][F] = [F][\Psi_{V_1}]$ since we have $[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = [F]$ with respect to our chosen bases. Then we have

$$
\begin{aligned}
[\Psi_{V_2}][F] = [F][\Psi_{V_1}] &\iff \begin{bmatrix} a_{V_2} I_{m_2} & 0 \\ 0 & d_{V_2} I_{n_2} \end{bmatrix} \begin{bmatrix} F_{11 m_2 \times m_1} & 0 \\ 0 & F_{22 n_2 \times n_1} \end{bmatrix} = \begin{bmatrix} F^{11}_{m_2 \times m_1} & 0 \\ 0 & F^{22}_{n_2 \times n_1} \end{bmatrix} \begin{bmatrix} a_{V_1} I_{m_1} & 0 \\ 0 & d_{V_1} I_{n_1} \end{bmatrix} \\
&\iff \begin{bmatrix} a_{V_2} F_{11} & 0 \\ 0 & d_{V_2} F_{22} \end{bmatrix} = \begin{bmatrix} a_{V_1} F_{11} & 0 \\ 0 & d_{V_1} F_{22} \end{bmatrix}
\end{aligned}
$$

which imples $a_{V_1} = a_{V_2}$ and $d_{V_1} = d_{V_2}$ so we may drop the indices. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) & \xrightarrow{\alpha^{-1}} & (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{triv}} \\
 & & \downarrow{\scriptstyle \Psi \otimes \mathrm{id}} \\
 & & (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{triv}} \\
{\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \alpha} \\
 & & \mathbb{C}_{\mathbf{triv}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}) \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes \Psi} \\
(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{triv}} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{triv}} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}})
\end{array}
$$

Since $[\Psi_{\mathbb{C}_{\mathbf{triv}}}] = [a]$ and $(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) \cong \mathbb{C}_{\mathbf{triv}}$, the left arrow is given by a by lemma 8.4. With respect to our bases, we have $[\alpha] = [\alpha^{-1}] = [1]$, so commutativity is equivalent to $a^2 = a$. This implies $a = 1$ since $a = 0$ would contradict Ψ being an isomorphism.

Now we consider the diagram

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbf{triv}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) & \xrightarrow{\alpha^{-1}} & (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{sign}} \\
 & & \downarrow{\scriptstyle \Psi \otimes \mathrm{id}} \\
 & & (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{sign}} \\
{\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \alpha} \\
 & & \mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}}) \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes \Psi} \\
(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{triv}} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}})
\end{array}
$$

We observe that $\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}} \cong \mathbb{C}_{\mathbf{triv}}$, so the left arrow is given by $a = 1$ by lemma 8.4. Commutativity is then equivalent to $d^2 = 1 \implies d = \pm 1$. Clearly, $d = 1$ corresponds to the isomorphism Φ_e, since then we have

$$
[\Psi] = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix} = I_{m+n}.
$$

If $d = -1$, then we have

$$[\Psi] = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} = [s]$$

which implies

$$\Psi(1 \otimes e_i) = \begin{cases} e_i \otimes 1 & \text{if} \quad 1 \le i \le m \\ -e_i \otimes 1 & \text{if} \quad m+1 \le i \le m+n \end{cases}$$
$$= \Psi_s(1 \otimes e_i)$$

so $\Psi = \Phi_s$. □

Proposition 8.6. *If $(\mathbb{C}_{\textit{sign}}, \Psi)$ is in $\mathcal{Z}(G\text{-}\textbf{\textit{mod}})$, then $\Psi = \Phi_e$ or $\Psi = \Phi_s$.*

Proof. We aim to mimic the proof of the previous proposition, so consider the component of Ψ at some direct sum of simples:

$$\Psi : \mathbb{C}_{\mathbf{sign}} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \to \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\mathbf{sign}}.$$

Fix bases $\{1 \otimes e_i\}$ and $\{e_i \otimes 1\}$ of the respective tensor products. Again we write

$$[\Psi] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Now, since we have $s(1) = -1$ in $\mathbb{C}_{\mathbf{sign}}$, we differ from the previous case and get

$$[s] = \begin{bmatrix} -I_m & 0 \\ 0 & I_n \end{bmatrix}.$$

The isomorphism Ψ commuting with the group action again yields $[\Psi] = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. Using proposition 8.1 now gives $A = a_V I_m$ and $D = d_V I_n$ just like before and naturality then implies $a_{V_1} = a_{V_2}$ and $d_{V_1} = d_{V_2}$ for any modules V_1 and V_2. Now we consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) & \xrightarrow{\alpha^{-1}} & (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{triv}} \\
\Big\downarrow \Psi & & \Big\downarrow \Psi\otimes\mathrm{id} \\
 & & (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{triv}} \\
 & & \Big\downarrow \alpha \\
 & & \mathbb{C}_{\mathbf{triv}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}) \\
 & & \Big\downarrow \mathrm{id}\otimes\Psi \\
(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{triv}} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{triv}} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}})
\end{array}$$

which shows $a = 1$ and the diagram

$$\begin{array}{ccc}
\mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) & \xrightarrow{\alpha^{-1}} & (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{sign}} \\
 & & \downarrow \Psi \otimes \mathrm{id} \\
 & & (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{sign}} \\
\Psi \downarrow & & \downarrow \alpha \\
 & & \mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \\
 & & \downarrow \mathrm{id} \otimes \Psi \\
(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{sign}} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{sign}} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}})
\end{array}$$

then shows that $d = \pm 1$. If $d = 1$ then $\Psi = \Phi_e$ and if $d = -1$ then $\Psi = \Phi_s$. □

Proposition 8.7. *Let $(V_1, \Psi_1), \ldots, (V_k, \Psi_k)$ be objects in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$ where all V_i are simple modules, so that $V_i \in \{\mathbb{C}_{\boldsymbol{triv}}, \mathbb{C}_{\boldsymbol{sign}}\}$ and $\Psi_i \in \{\Phi_e, \Phi_s\}$ for all i. Let V_i be the linear span of the vector v_i. Then*

$$\bigoplus_{i=1}^{k} (V_i, \Psi_i) = (V, \Psi)$$

where $V = \bigoplus_{i=1}^{k} V_i$ and $\Psi : V \otimes X \to X \otimes V$ is defined by

$$(v_1, \ldots, v_k) \otimes x \mapsto x \otimes (c_1 v_1, \ldots, c_k v_k)$$

where

$$c_i = \begin{cases} 1 & \text{if} \quad \Psi_i = \Phi_e \\ -1 & \text{if} \quad \Psi_i = \Phi_s \end{cases}$$

Proof. The fact that $V = \bigoplus_{i=1}^{k} V_i$ is clear. What this proposition aims to prove is that the addition of natural isomorphisms used in the Drinfeld center addition of proposition 7.23 is well-behaved. We proceed by induction. If $k = 2$ then

$$(V_1, \Psi_1) \oplus (V_2, \Psi_2) = (V_1 \oplus V_2, \delta^{-1}(\Psi_1 \oplus \Psi_2)\varepsilon)$$

by definition. We have

$$\begin{aligned}
\delta^{-1}(\Psi_1 \oplus \Psi_2)\varepsilon((v_1, v_2) \otimes x) &= \delta^{-1}(\Psi_1 \oplus \Psi_2)((v_1 \otimes x, v_2 \otimes x)) \\
&= \delta^{-1}((\Psi_1(v_1 \otimes x), \Psi_2(v_2 \otimes x))) \\
&= \delta^{-1}((x \otimes c_1 v_1, x \otimes c_2 v_2)) \\
&= x \otimes ((c_1 v_1, c_2 v_2)) \\
&= \Psi((v_1, v_2) \otimes x).
\end{aligned}$$

Next, we have $(V, \Psi) \oplus (V_{k+1}, \Psi_{k+1}) = (V \oplus V_{k+1}, \delta^{-1}(\Psi \oplus \Psi_{k+1})\varepsilon)$. Let Ψ' denote the isomorphism in the formulation.

$$\begin{aligned}
\delta^{-1}(\Psi \oplus \Psi_{k+1})\varepsilon(((v_1, \dots, v_k), v_{k+1}) \otimes x) &= \delta^{-1}(\Psi \oplus \Psi_{k+1})(((v_1, \dots, v_k) \otimes x, v_{k+1} \otimes x)) \\
&= \delta^{-1}((x \otimes (c_1 v_1, \dots, c_k v_k), x \otimes c_{k+1} v_{k+1})) \\
&= x \otimes ((c_1 v_1, \dots, c_k v_k), c_{k+1} v_{k+1}) \\
&= x \otimes (c_1 v_1, \dots, c_k v_k, c_{k+1} v_{k+1}) \\
&= \Psi'((v_1, \dots, v_{k+1}) \otimes x).
\end{aligned}$$

□

Lemma 8.8. *Let (V, Ψ) be in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$ and let $W \cong \mathbb{C}_{\boldsymbol{triv}}^{\oplus m} \oplus \mathbb{C}_{\boldsymbol{sign}}^{\oplus n}$ be some module. Then, for any fixed basis of V, there exists a basis of W such that with respect to these bases, we have*

$$[\Psi_{\mathbb{C}_{\boldsymbol{triv}}^{\oplus m} \oplus \mathbb{C}_{\boldsymbol{sign}}^{\oplus n}}] = [\Psi_W].$$

Proof. Let $\{v_i\}_{i=1}^k$ be a basis of V and let $\varphi : \mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n} \to W$ be a G-isomorphism and . In particular, it is a linear isomorphism so the set $\{\varphi(e_i)\}$ forms a basis of W. Fixing this basis of W, we have $[\varphi] = I_{m+n}$. By naturality, the square

$$\begin{array}{ccc}
V \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) & \xrightarrow{\mathrm{id} \otimes \varphi} & V \otimes W \\
\Big\downarrow{\scriptstyle \Psi_{\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}}} & & \Big\downarrow{\scriptstyle \Psi_W} \\
\left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes V & \xrightarrow[\varphi \otimes \mathrm{id}]{} & W \otimes V
\end{array}$$

commutes and since in our bases, we have

$$[\mathrm{id} \otimes \varphi] = [\varphi \otimes \mathrm{id}] = \begin{bmatrix} [\varphi] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [\varphi] \end{bmatrix} = I_{k(m+n)}$$

the result follows. ■

Proposition 8.9. *If $(\mathbb{C}_{\boldsymbol{triv}}^{\oplus k}, \Psi)$ is in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$, then $(\mathbb{C}_{\boldsymbol{triv}}^{\oplus k}, \Psi)$ is isomorphic to*

$$(\mathbb{C}_{\boldsymbol{triv}}, \Phi_e)^{\oplus k_1} \oplus (\mathbb{C}_{\boldsymbol{triv}}, \Phi_s)^{\oplus k_2}$$

for some k_1, k_2 such that $k_1 + k_2 = k$.

Proof. Fix the standard basis $\{e_i\}$ of $\mathbb{C}_{\mathbf{triv}}^{\oplus k}$ and the standard basis $\{v_i\}$ of $\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$. Then fix bases

$$\{e_1 \otimes v_1, \dots, e_1 \otimes v_{m+n}, \dots, e_k \otimes v_1, \dots, e_k \otimes v_{m+n}\}$$
$$\{v_1 \otimes e_1, \dots, v_1 \otimes e_k, \dots, v_{m+n} \otimes e_1, \dots, v_{m+n} \otimes e_k\}$$

of the respective tensor products. We write $[\Psi]$ as a $k \times k$ block matrix with block size $(m+n) \times (m+n)$, so that

$$[\Psi] = \begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix}.$$

In the module V, we have

$$s(v_i) = \begin{cases} v_i & \text{if} \quad 1 \leq i \leq m \\ -v_i & \text{if} \quad m+1 \leq i \leq m+n \end{cases}$$

so that we have

$$[s]_V = \begin{bmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{bmatrix}.$$

In other words, we have

$$s(e_i \otimes v_j) = e_i \otimes s(v_j) \quad \text{and} \quad s(v_j \otimes e_i) = s(v_j) \otimes e_i$$

since s acts as the identity in the trivial module. Then it is clear that the matrix of s in the tensor products is the $k \times k$ block matrix with blocks $[s]_V$ on the diagonal and zeroes everywhere else. Writing this out we have

$$[s]_{\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes V} = [s]_{V \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}} = \begin{bmatrix} [s]_V & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [s]_V \end{bmatrix}.$$

Since Ψ is a G-isomorphism by assumption, it commutes with the action of the group, so we have $\Psi(s(x \otimes y)) = s\Psi(x \otimes y)$. In terms of matrices, this is equivalent to the equation $[\Psi][s] = [s][\Psi]$. Then

$$\begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix} \begin{bmatrix} [s]_V & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [s]_V \end{bmatrix} = \begin{bmatrix} [s]_V & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [s]_V \end{bmatrix} \begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix}$$

which holds if and only if

$$\begin{bmatrix} A_{11}[s]_V & \dots & A_{1k}[s]_V \\ \vdots & \ddots & \vdots \\ A_{k1}[s]_V & \dots & A_{kk}[s]_V \end{bmatrix} = \begin{bmatrix} [s]_V A_{11} & \dots & [s]_V A_{1k} \\ \vdots & \ddots & \vdots \\ [s]_V A_{k1} & \dots & [s]_V A_{kk} \end{bmatrix}.$$

This shows that every matrix A_{ij} must commute with $[s]_V$. We write A_{ij} as a block matrix

$$A_{ij} = \begin{bmatrix} B_{m\times m} & C_{m\times n} \\ D_{n\times m} & E_{n\times n} \end{bmatrix}.$$

Writing out the condition $A_{ij}[s]_V = [s]_V A_{ij}$ we get

$$\begin{bmatrix} B_{m\times m} & C_{m\times n} \\ D_{n\times m} & E_{n\times n} \end{bmatrix} \begin{bmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{bmatrix} = \begin{bmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{bmatrix} \begin{bmatrix} B_{m\times m} & C_{m\times n} \\ D_{n\times m} & E_{n\times n} \end{bmatrix}$$

which holds if and only if

$$\begin{bmatrix} B & -C \\ D & -E \end{bmatrix} = \begin{bmatrix} B & C \\ -D & -E \end{bmatrix}.$$

This implies $C = -C$ and $D = -D$ so we must have $C = D = 0$. Consider the component of Ψ at the module $V = \mathbb{C}_{\textbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\textbf{sign}}^{\oplus n}$. By assumption, the diagram

$$\begin{array}{ccc} \mathbb{C}_{\textbf{triv}}^{\oplus k} \otimes \left(\mathbb{C}_{\textbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\textbf{sign}}^{\oplus n}\right) & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}_{\textbf{triv}}^{\oplus k} \otimes \left(\mathbb{C}_{\textbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\textbf{sign}}^{\oplus n}\right) \\ \Psi \big\downarrow & & \big\downarrow \Psi \\ \left(\mathbb{C}_{\textbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\textbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\textbf{triv}}^{\oplus k} & \xrightarrow[F \otimes \mathrm{id}]{} & \left(\mathbb{C}_{\textbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\textbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\textbf{triv}}^{\oplus k} \end{array}$$

commutes for any G-homomorphism $F : V \to V$. With respect to our bases, we know that the matrices of the homomorphisms $\mathrm{id} \otimes F$ and $F \otimes \mathrm{id}$ are given by

$$[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = \begin{bmatrix} [F] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [F] \end{bmatrix}.$$

It is now clear that the diagram commutes if and only if this matrix commutes with the matrix of Ψ. This condition is equivalent to $A_{ij}[F] = [F]A_{ij}$ for all i, j. By our previous observation the matrices A_{ij} have the form

$$A_{ij} = \begin{bmatrix} B_{ij} & 0 \\ 0 & B'_{ij} \end{bmatrix}$$

and by proposition 8.1 the matrix $[F]$ has the form

$$[F] = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}.$$

This means that our condition is equivalent to

$$\begin{bmatrix} B_{ij} & 0 \\ 0 & B'_{ij} \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} B_{ij} & 0 \\ 0 & B'_{ij} \end{bmatrix}$$

which holds if and only if

$$B_{ij}F_{11} = F_{11}B_{ij} \quad \text{and} \quad B'_{ij}F_{22} = F_{22}B'_{ij}.$$

Since the matrices F_{ii} are arbitrary, this implies that $B_{ij} = \lambda_V I_m$ and $B'_{ij} = \mu_V I_n$, where $\lambda_V, \mu_V \in \mathbb{C}$. Let $W = \mathbb{C}_{\mathbf{triv}}^{\oplus x} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus y}$ be another module. By assumption, the diagram

$$\begin{array}{ccc}
\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) & \xrightarrow{\mathrm{id} \otimes F} & \mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus x} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus y}\right) \\
\Psi \downarrow & & \downarrow \Psi \\
\left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k} & \xrightarrow[F \otimes \mathrm{id}]{} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus x} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus y}\right) \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}
\end{array}$$

commutes for any G-homomorphism $F : V \to W$. A similar argument now shows that we must have $A_{ij}^W [F] = [F] A_{ij}^V$ for all i, j. This is equivalent to

$$\begin{bmatrix} \lambda_{ij}^W I_x & 0 \\ 0 & \mu_{ij}^W I_y \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} \lambda_{ij}^V I_m & 0 \\ 0 & \mu_{ij}^V I_n \end{bmatrix}$$

which holds if and only if $\lambda_W F_{11} = \lambda_V F_{11}$ and $\mu_W F_{22} = \mu_V F_{22}$. This implies $\lambda_{ij}^V = \lambda_{ij}^W$ and $\mu_{ij}^V = \mu_{ij}^W$. Now, we consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) & \xrightarrow{\alpha^{-1}} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{triv}}\right) \otimes \mathbb{C}_{\mathbf{triv}} \\
\Big\downarrow & & \downarrow \Psi \otimes \mathrm{id} \\
 & & \left(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}\right) \otimes \mathbb{C}_{\mathbf{triv}} \\
\Psi & & \downarrow \alpha \\
 & & \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{triv}}\right) \\
\Big\downarrow & & \downarrow \mathrm{id} \otimes \Psi \\
(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}\right)
\end{array}$$

Since $\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}} \cong \mathbb{C}_{\mathbf{triv}}$, there exists a suitable basis of $\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}$ such that $[\Psi_{\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}}] = [\Psi_{\mathbb{C}_{\mathbf{triv}}}]$ by lemma 8.8. We also note that

$$[\mathrm{id}_{\mathbb{C}_{\mathbf{triv}}} \otimes \Psi_{\mathbb{C}_{\mathbf{triv}}}] = [\Psi_{\mathbb{C}_{\mathbf{triv}}} \otimes \mathrm{id}_{\mathbb{C}_{\mathbf{triv}}}] = [\Psi_{\mathbb{C}_{\mathbf{triv}}}].$$

Since the associators are given by identity matrices, commutativity of the diagram is equivalent to $[\Psi_{\mathbb{C}_{\mathbf{triv}}}]^2 = [\Psi_{\mathbb{C}_{\mathbf{triv}}}]$ and since $[\Psi_{\mathbb{C}_{\mathbf{triv}}}]$ is invertible it follows that $[\Psi_{\mathbb{C}_{\mathbf{triv}}}] = I_k$. This is equivalent to

$$\lambda_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

If we instead consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) & \xrightarrow{\alpha^{-1}} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{sign}}\right) \otimes \mathbb{C}_{\mathbf{sign}} \\
 & & \downarrow{\scriptstyle \Psi \otimes \mathrm{id}} \\
 & & \left(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}\right) \otimes \mathbb{C}_{\mathbf{sign}} \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \alpha} \\
 & & \mathbb{C}_{\mathbf{sign}} \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{sign}}\right) \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes \Psi} \\
(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{sign}} \otimes \left(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}\right)
\end{array}
$$

Since $\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}} \cong \mathbb{C}_{\mathbf{triv}}$ we may assume that the left arrow is given by I_k, by lemma 8.8. Commutativity of the diagram is then equivalent to $[\Psi_{\mathbb{C}_{\mathbf{sign}}}]^2 = I_k$.

Such a matrix must have eigenvalues ± 1 and be diagonalizable. This means that there is a basis of $\mathbb{C}_{\mathbf{sign}}^{\oplus k}$ such that

$$[\Psi_{\mathbb{C}_{\mathbf{sign}}}] = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

with respect to this basis, where $p + q = k$. At first sight, it seems like this change of basis could affect the arguments made thus far, but we are saved by the following observations:

- The matrices representing the associators, their inverses and $[\Psi_{\mathbb{C}_{\mathbf{triv}}}]$ are identity matrices. The identity matrix is invariant under any change of basis.

- The form of the module $\mathbb{C}_{\mathbf{triv}}^{\oplus k}$ is not affected by a change of basis.

So, we can safely make this change of basis, which shows that we may assume

$$\mu_{ij} = \begin{cases} \pm 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases}$$

and moreover,

$$\mu_{ii} = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq p \\ -1 & \text{if} \quad p+1 \leq i \leq k \end{cases} .$$

This shows that $\Psi : \mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes (\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}) \to (\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n})\mathbb{C}_{\mathbf{triv}}^{\oplus k}$ is defined by

$$\begin{array}{lll} e_i \otimes v_j \mapsto v_j \otimes e_i & \text{if} & 1 \leq i \leq p \\ e_i \otimes v_j \mapsto -v_j \otimes e_i & \text{if} & p+1 \leq i \leq k = p+q \end{array}$$

which corresponds to p copies of $(\mathbb{C}_{\mathbf{triv}}, \Phi_e)$ and q copies of $(\mathbb{C}_{\mathbf{triv}}, \Phi_s)$. □

Proposition 8.10. *If* $(\mathbb{C}_{\boldsymbol{sign}}^{\oplus k}, \Psi)$ *is in* $\mathcal{Z}(G\text{-}\boldsymbol{mod})$*, then* $(\mathbb{C}_{\boldsymbol{sign}}^{\oplus k}, \Psi)$ *is isomorphic to*

$$(\mathbb{C}_{\boldsymbol{triv}}, \Phi_e)^{\oplus k_1} \oplus (\mathbb{C}_{\boldsymbol{triv}}, \Phi_s)^{\oplus k_2}$$

for some k_1, k_2 *such that* $k_1 + k_2 = k$.

Proof. We use the same setup as in the proof of proposition 8.9. We start by considering the component of Ψ at the module $V = \mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$. The first point where we differ is the action of the group element s on the tensor products. We have

$$s(e_i \otimes v_j) = -e_i \otimes s(v_j) \quad \text{and} \quad s(v_j \otimes e_i) = -s(v_j) \otimes e_i$$

since s acts as -1 in the sign module. Like before, we have

$$[s]_V = \begin{bmatrix} I_m & 0_{m \times n} \\ 0_{n \times m} & -I_n \end{bmatrix}$$

but now we get a different matrix representing the action of s in the tensor products, namely:

$$[s]_{\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes V} = [s]_{V \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k}} = \begin{bmatrix} -[s]_V & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -[s]_V \end{bmatrix}.$$

If we write

$$[\Psi] = \begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix}.$$

The assumption that Ψ be a G-homomorphism again yields $[s]_V A_{ij} = A_{ij}[s]_V$ for all i, j. Just like in the previous proposition, this implies that the matrices A_{ij} have the form

$$A_{ij} = \begin{bmatrix} B_{ij} & 0 \\ 0 & B'_{ij} \end{bmatrix}.$$

Moreover, it follows in the same exact way that

$$B_{ij} = \lambda_{ij}^V I_m \quad \text{and} \quad B'_{ij} = \mu_{ij}^V I_n, \quad \lambda_V, \mu_V \in \mathbb{C}.$$

Next, naturality shows that if W is some other module, then $\lambda_V = \lambda_W$ and $\mu_V = \mu_W$. This means that so far, we know that the matrix representing the component of Ψ at V is given by

$$[\Psi] = \begin{bmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{bmatrix}$$

where the matrices A_{ij} have the form

$$A_{ij} = \begin{bmatrix} \lambda_{ij} I_m & 0 \\ 0 & \mu_{ij} I_n \end{bmatrix}.$$

Now, we consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes (\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) & \xrightarrow{\alpha^{-1}} & \left(\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{triv}}\right) \otimes \mathbb{C}_{\mathbf{triv}} \\
 & & \downarrow \Psi \otimes \mathrm{id} \\
 & & \left(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k}\right) \otimes \mathbb{C}_{\mathbf{triv}} \\
\Psi \downarrow & & \downarrow \alpha \\
 & & \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{triv}}\right) \\
 & & \downarrow \mathrm{id} \otimes \Psi \\
(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{triv}}) \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{triv}} \otimes \left(\mathbb{C}_{\mathbf{triv}} \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k}\right)
\end{array}$$

By the same arguments as in the proof of proposition 8.9, commutativity yields $[\Psi_{\mathbb{C}_{\mathbf{triv}}}]^2 = [\Psi_{\mathbb{C}_{\mathbf{triv}}}]$ which implies $[\Psi_{\mathbb{C}_{\mathbf{triv}}}] = I_k$. This is equivalent to

$$\lambda_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

Now, we consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes (\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) & \xrightarrow{\alpha^{-1}} & \left(\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{sign}}\right) \otimes \mathbb{C}_{\mathbf{sign}} \\
 & & \downarrow \Psi \otimes \mathrm{id} \\
 & & \left(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k}\right) \otimes \mathbb{C}_{\mathbf{sign}} \\
\Psi \downarrow & & \downarrow \alpha \\
 & & \mathbb{C}_{\mathbf{sign}} \otimes \left(\mathbb{C}_{\mathbf{sign}}^{\oplus k} \otimes \mathbb{C}_{\mathbf{sign}}\right) \\
 & & \downarrow \mathrm{id} \otimes \Psi \\
(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}) \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k} & \xleftarrow[\alpha^{-1}]{} & \mathbb{C}_{\mathbf{sign}} \otimes \left(\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus k}\right)
\end{array}$$

Since $\mathbb{C}_{\mathbf{sign}} \otimes \mathbb{C}_{\mathbf{sign}} \cong \mathbb{C}_{\mathbf{triv}}$ we may assume that the left arrow is given by I_k, by lemma 8.8. Commutativity of the diagram is then equivalent to $[\Psi_{\mathbb{C}_{\mathbf{sign}}}]^2 = I_k$. Now we use the same argument as in the proof of proposition 8.9 to arrive at the conclusion. □

Proposition 8.11. *If* $(\mathbb{C}_{\textit{triv}}^{\oplus k} \oplus \mathbb{C}_{\textit{sign}}^{\oplus l}, \Psi)$ *is in* $\mathcal{Z}(G\text{-}\boldsymbol{mod})$*, then* $(\mathbb{C}_{\textit{triv}}^{\oplus k} \oplus \mathbb{C}_{\textit{sign}}^{\oplus l}, \Psi)$ *is isomorphic to* $(\mathbb{C}_{\textit{triv}}^{\oplus k}, \Psi_1) \oplus (\mathbb{C}_{\textit{sign}}^{\oplus l}, \Psi_2)$.

Proof. Considering the component of Ψ at $V = \mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$ we get the commutative diagram

$$\begin{array}{ccc}
\left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) & \xrightarrow{\ \mathrm{id} \otimes F\ } & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \\
\Psi \big\downarrow & & \big\downarrow \Psi \\
\left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) & \xrightarrow[F \otimes \mathrm{id}]{} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right)
\end{array}$$

We write $[\Psi]$ as a $(k+l) \times (k+l)$ block matrix with block size $(m+n) \times (m+n)$. So we have

$$[\Psi] = \begin{bmatrix} A_{11} & \dots & A_{1(k+l)} \\ \vdots & \ddots & \vdots \\ A_{(k+l)1} & \dots & A_{(k+l)(k+l)} \end{bmatrix}$$

Let $\{u_i\}$ denote the standard basis of $\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}$ and $\{v_i\}$ the standard basis of $\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}$. Fix the basis

$$\{u_1 \otimes v_1, \dots, u_1 \otimes v_{m+n}, \dots, u_{k+l} \otimes v_1, \dots, u_{k+l} \otimes v_{m+n}\}$$

of the module $\left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right)$ and the basis

$$\{v_1 \otimes u_1, \dots, v_{m+n} \otimes u_1, \dots, v_{m+n} \otimes u_1, \dots, v_{m+n} \otimes u_{k+l}\}$$

of the module $\left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right)$. Now we note that

$$s(u_i) = \begin{cases} u_i & \text{if} \quad 1 \le i \le k \\ -u_i & \text{if} \quad k+1 \le i \le k+l \end{cases}, \quad s(v_i) = \begin{cases} v_i & \text{if} \quad 1 \le i \le m \\ -v_i & \text{if} \quad m+1 \le i \le m+n \end{cases}.$$

Letting $E = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}$ we get the matrices for the action of s in $U \otimes V$ and in $V \otimes U$ as $(k+l) \times (k+l)$ block matrices with blocks of the form E or $-E$ on the diagonal and zeroes everywhere else. Clearly, there are $k+l$ such blocks. Then, the first k blocks are E and the last l blocks are $-E$.

$$[s]_{U \otimes V} = \begin{bmatrix} E & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -E \end{bmatrix} = [s]_{V \otimes U}.$$

Now Ψ being a G-homomorphism is equivalent to $[\Psi][s]_{U \otimes V} = [s]_{V \otimes U}[\Psi]$. Writing the condition out we have

$$\begin{bmatrix} A_{11} & \dots & A_{1(k+l)} \\ \vdots & \ddots & \vdots \\ A_{(k+l)1} & \dots & A_{(k+l)(k+l)} \end{bmatrix} \begin{bmatrix} E & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -E \end{bmatrix} = \begin{bmatrix} E & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -E \end{bmatrix} \begin{bmatrix} A_{11} & \dots & A_{1(k+l)} \\ \vdots & \ddots & \vdots \\ A_{(k+l)1} & \dots & A_{(k+l)(k+l)} \end{bmatrix}$$

which holds if and only if

$$\begin{bmatrix} A_{11}E & \dots & A_{1k}E & -A_{1(k+1)}E & \dots & -A_{1(k+l)}E \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{k+l}1E & \dots & A_{k+l}kE & -A_{(k+l)(k+1)}E & \dots & -A_{(k+l)(k+l)}E \end{bmatrix} = \begin{bmatrix} EA_{11} & \dots & EA_{1(k+l)} \\ \vdots & & \vdots \\ EA_{k1} & \dots & EA_{k(k+l)} \\ -EA_{(k+1)1} & \dots & -EA_{(k+1)(k+l)} \\ \vdots & & \vdots \\ -EA_{(k+l)1} & \dots & -EA_{(k+l)(k+l)} \end{bmatrix}.$$

This means that we have four possibly different conditions on the matrices A_{ij}.

1) If $i \leq k, j \leq k$, we require $EA_{ij} = A_{ij}E$. Writing $A_{ij} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ we get

$$\begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} \iff \begin{bmatrix} B_{11} & B_{12} \\ -B_{21} & -B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & -B_{12} \\ B_{21} & -B_{22} \end{bmatrix}$$

which implies $B_{12} = B_{21} = 0$. The same holds if $i > k, j > k$.

2) If $i \leq k, j > k$, we require $EA_{ij} = -A_{ij}E$. The condition becomes

$$\begin{bmatrix} B_{11} & B_{12} \\ -B_{21} & -B_{22} \end{bmatrix} = \begin{bmatrix} -B_{11} & B_{12} \\ -B_{21} & B_{22} \end{bmatrix}$$

which implies $B_{11} = B_{22} = 0$. The same holds if $i > k, j \leq k$.

So Ψ is given by a matrix of the form

$$[\Psi] = \begin{bmatrix} A_{11} & \dots & A_{1(k+l)} \\ \vdots & \ddots & \vdots \\ A_{(k+l)1} & \dots & A_{(k+l)(k+l)} \end{bmatrix}$$

where the blocks A_{ij} have either the form $\begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ or $\begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}$, according to the cases above. We return to the naturality square

$$\begin{array}{ccc} \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) & \xrightarrow{\mathrm{id} \otimes F} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \\ \Psi \downarrow & & \downarrow \Psi \\ \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) & \xrightarrow[F \otimes \mathrm{id}]{} & \left(\mathbb{C}_{\mathbf{triv}}^{\oplus m} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus n}\right) \otimes \left(\mathbb{C}_{\mathbf{triv}}^{\oplus k} \oplus \mathbb{C}_{\mathbf{sign}}^{\oplus l}\right) \end{array}$$

and just like before

$$[\mathrm{id} \otimes F] = [F \otimes \mathrm{id}] = \begin{bmatrix} [F] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [F] \end{bmatrix}.$$

By the commutativity of the diagram, we get $A_{ij}[F] = [F]A_{ij}$ for all i, j. Note that $[F] = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix}$ by proposition 8.1. If A_{ij} has the form $A_{ij} = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ then we have

$$\begin{aligned} A_{ij}[F] = [F]A_{ij} &\iff \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \\ &\iff \begin{bmatrix} B_{11}F_{11} & 0 \\ 0 & B_{22}F_{22} \end{bmatrix} = \begin{bmatrix} F_{11}B_{11} & 0 \\ 0 & F_{22}B_{22} \end{bmatrix} \end{aligned}$$

which implies $B_{11} = a_{ij}^V I_m$ and $B_{22} = d_{ij}^V I_n$ since the matrices F_{ii} are arbitrary. If A_{ij} has the form $A_{ij} = \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}$ then we have

$$\begin{aligned} A_{ij}[F] = [F]A_{ij} &\iff \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} \\ &\iff \begin{bmatrix} 0 & B_{12}F_{22} \\ B_{21}F_{11} & 0 \end{bmatrix} = \begin{bmatrix} 0 & F_{11}B_{12} \\ F_{22}B_{21} & 0 \end{bmatrix} \end{aligned}$$

which holds if and only if

$$B_{12}F_{22} = F_{11}B_{12} \quad \text{and} \quad B_{21}F_{11} = F_{22}B_{21}.$$

Since the matrices F_{ii} are arbitrary, this implies $B_{12} = B_{21} = 0$. Using all of this, we may write Ψ as a 2×2 block matrix $[\Psi] = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ where all entries are block matrices with block size $(m+n) \times (m+n)$, where M is a $k \times k$ block matrix and N is an $l \times l$ block matrix. We see that M corresponds to a map $\mathbb{C}_{\mathbf{triv}}^{\oplus k} \otimes V \to V \otimes \mathbb{C}_{\mathbf{triv}}^{\oplus k}$ and N to a map $\mathbb{C}_{\mathbf{sign}}^{\oplus l} \otimes V \to V \otimes \mathbb{C}_{\mathbf{sign}}^{\oplus l}$, so we are done. □

Proposition 8.12. *Let (V, Ψ) and (W, Θ) where V and W are simple modules be in $\mathcal{Z}(G\text{-}\boldsymbol{mod})$. Then*

$$\mathrm{Hom}_{\mathcal{Z}(G\text{-}\boldsymbol{mod})}\left((V, \Psi), (W, \Theta)\right) = \begin{cases} \mathrm{Hom}_{G\text{-}\boldsymbol{mod}}(V, W) & \text{if} \quad V = W \quad \text{and} \quad \Psi = \Theta \\ 0 & \text{otherwise} \end{cases}$$

Proof. This follows immediately from Schur's lemma and proposition 7.19. □

We summarize the results of this section in the following theorem:

Theorem 8.13. *Each object of the Drinfeld center $\mathcal{Z}(G\text{-}\boldsymbol{mod})$ is isomorphic to a direct sum of the following objects, with some multiplicites:*

$$(\mathbb{C}_{\boldsymbol{triv}}, \Phi_e), \quad (\mathbb{C}_{\boldsymbol{triv}}, \Phi_s), \quad (\mathbb{C}_{\boldsymbol{sign}}, \Phi_e) \quad \text{and} \quad (\mathbb{C}_{\boldsymbol{sign}}, \Phi_s).$$

Morphisms are given by proposition 8.12.

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