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The greedy walk on an inhomogeneous Poisson process

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Abstract

The greedy walk is a deterministic walk that always moves from its current position to the nearest not yet visited point. In this paper we consider the greedy walk on an inhomogeneous Poisson point process on the real line. We prove that the property of visiting all points of the point process satisfies a 0–1 law and determine explicit sufficient and necessary conditions on the mean measure of the point process for this to happen. Moreover, we provide precise results on threshold functions for the property of visiting all points.

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1 Introduction and main results

Consider a simple point process Π without accumulation points in a metric space (E, d). We think of Π either as an integer-valued measure or as a collection of points (the support of the measure). With the latter viewpoint in mind, we define the greedy walk on Π as follows. Let $S_0 \in E$ and $\Pi_0 = \Pi$. Define, for $n \geq 0$,

$$S_{n+1} = \arg\min\{d(S_n, X) : X \in \Pi_n\},$$
$$\Pi_{n+1} = \Pi_n \setminus \{S_{n+1}\}.$$

The set Π_n denotes the set of unvisited points of Π up until (and including) time n. Once the underlying environment Π is fixed, the process $(S_n)_{n=0}^{\infty}$ is deterministic (except possibly for ties which need to be broken, but these will almost surely not occur in our setting). A typical problem to study is whether all points of Π are eventually visited by the greedy walk. If this happens, we say that the walk is *recurrent*. Otherwise we say that it is *transient*.

The greedy walk has been studied before in the literature, with various choices of the underlying point process. When Π is a homogeneous Poisson process on $\mathbb{R}$, one can show, using a Borel–Cantelli–type argument, that the greedy walk does not visit all the points of the underlying point process, with probability 1. More precisely, the expected number of times the greedy walk starting from 0 changes sign is $1/2$ [4]. Rolla *et al.* [7] considered a related problem, in which each point in the process can be visited either once, with probability $1-p$, or twice, with probability p. For any $0 < p < 1$, they show that every point is eventually visited. Another modification of the greedy walk on $\mathbb{R}$

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is studied by Foss *et al.* [3]. The authors considered a dynamic version of the greedy walk, where the times and positions of new points arriving in the system are given by a Poisson process on the space-time half-plane. They show that the greedy walk still diverges to infinity in one direction and does not visit all points. In the survey paper [2], Bordenave *et al.* state several questions about the behaviour of the greedy walk on an inhomogeneous Poisson process in $\mathbb{R}^d$. We resolve here the problem for $d = 1$.

In this paper we define Π to be an inhomogeneous Poisson process on $\mathbb{R}$ (with the Euclidean metric) given by some non-atomic mean measure μ. For such a process, the number of points in disjoint measurable subsets of $\mathbb{R}$ are independent and

$$\mathbb{P}[\Pi(a,b) = k] = \frac{\mu(a,b)}{k!} e^{-\mu(a,b)}$$

for any $a < b$ and any $k \geq 0$, where, for any measurable $A \subseteq \mathbb{R}$, $\Pi(A) = \Pi A$ is the cardinality of the restriction of Π to the set A. This means that the number of points in any interval (a, b) is distributed like $\mathrm{Poi}(\mu(a, b))$. Sometimes, we assume that the mean measure μ is absolutely continuous and given in terms of a measurable intensity function $\lambda : \mathbb{R} \to [0, \infty)$, so that

$$\mu(A) = \int_A \lambda(x)\,\mathrm{d}x$$

for any measurable $A \subseteq \mathbb{R}$.

To avoid certain degenerate cases, we impose the following two conditions on the measure μ.

(i) $\mu(-\infty, 0) = \mu(0, \infty) = \infty$.

(ii) $\mu(A) < \infty$ for all bounded measurable $A \subseteq \mathbb{R}$.

Denote by $\mathcal{M}$ the set of all measures on $\mathbb{R}$ which satisfy (i) and (ii). If $\mu \in \mathcal{M}$ is given in terms of a intensity function λ, we abuse notation and write also $\lambda \in \mathcal{M}$. Note that the first condition is equivalent to $\Pi(-\infty, 0) = \Pi(0, \infty) = \infty$ with probability 1. The second condition is equivalent to $\Pi(A) < \infty$ with probability 1, for any bounded measurable $A \subseteq \mathbb{R}$, which implies that there are no accumulation points of the process. Indeed, if a process has accumulation points, it is possible that the $\arg\min$ in the definition of the greedy walk is not well-defined.

Throughout we let $S_0 = 0$ (note that $0 \notin \Pi$ with probability 1), so that the walk starts in the origin. The process $(S_n)_{n=0}^{\infty}$ will be referred to as `GWIPP`. If we want to emphasise the underlying point process, the underlying mean measure, or the underlying intensity function, we write `GWIPP`(Π), `GWIPP`(μ) or `GWIPP`(λ), respectively.

As mentioned, our interest is to study the recurrence or transience of `GWIPP`. Since `GWIPP` is on the real line, it is recurrent if and only if it changes sign infinitely many times. As $|S_n|$ increases, it becomes more difficult for `GWIPP` to change sign. Intuitively speaking, recurrence is equivalent to the points of Π eventually being sparse enough that there are infinitely many "sufficiently long" empty intervals on both half-lines.

One of our main results is that recurrence (and consequently transience) satisfies a 0–1 law.

Theorem 1.1. *Let $\mu \in \mathcal{M}$. Then* `GWIPP`(μ) *is recurrent with probability* 0 *or* 1.

The proof of this and the following theorem, which provides an analytic condition (in terms of μ) for when `GWIPP` is recurrent, is an application of Campbell's theorem and the Borel–Cantelli lemma.

Theorem 1.2. *Let $\mu \in \mathcal{M}$. Then* GWIPP(μ) *is recurrent with probability 1 if and only if*

$$\int_0^\infty \exp(-\mu(x, 2x+R))\mu(\mathrm{d}x) = \infty \quad \textit{and} \quad \int_{-\infty}^0 \exp(-\mu(2x-R, x))\mu(\mathrm{d}x) = \infty,$$

for all $R \geq 0$.

It is a straightforward consequence of Theorem 1.2 that GWIPP on an underlying *homogeneous* Poisson process is transient.

Our next result is a coupling result between different measures. Intuitively, adding more points to an already transient process only makes it "more" transient, since it will be more difficult to find long empty intervals which allow $(S_n)_{n=1}^\infty$ to change sign. Conversely, removing points from an already recurrent process makes it "more" recurrent.

Lemma 1.3. *Let $\mu, \mu' \in \mathcal{M}$ and suppose there is some $K > 0$ such that $\mu'(A) \geq \mu(A)$ for all measurable $A \subseteq (-\infty, -K) \cup (K, \infty)$. If* GWIPP$(\mu)$ *is transient with probability* 1*, then* GWIPP(μ') *is transient with probability* 1*. Conversely, if* GWIPP(μ') *is recurrent with probability* 1*, then* GWIPP(μ) *is recurrent with probability* 1*.*

When dealing with properties exhibiting dichotomous behaviour, it is common to analyse the boundary or the phase transition between the two possible states. In our case, this amounts to finding a threshold measure or threshold density function for recurrence and transience. In the next proposition, we define a parametric family of density functions, and we are able to determine precisely the region on which the corresponding greedy walk is recurrent respectively transient. In particular, this parametric family exhibits a sharp threshold behaviour.

Moreover, Lemma 1.3 provides a tool to move outside this parametric family and determine the behaviour for other types of density functions. This can sometimes be easier to use than the integral condition in Theorem 1.2.

To state the proposition, we need some notation. We define the iterated logarithm $\log^{(n)}$, for $n \geq 1$, to be the function defined recursively by

$$\log^{(1)} t := \begin{cases} \log t & \text{if } t > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\log$ is the ordinary natural logarithm, and, for any $n \geq 2$,

$$\log^{(n)} t := \log^{(1)} \left(\log^{(n-1)} t \right).$$

Proposition 1.4. *Let*

$$\lambda(t) := \frac{1}{|t| \log 2} \sum_{i=2}^{n} a_i \log^{(i)} |t|.$$

where $n \in \{2, 3, 4, \dots\}$ and $a_i \geq 0$ for all $2 \leq i \leq n$. Then GWIPP(λ) *is transient with probability* 1 *if and only if*

- $a_2 > 1$*, or*
- $a_2 = 1, a_3 > 2$*, or*
- $a_2 = 1, a_3 = 2$*, and there exists some $m \geq 4$ such that $a_4 = 1, a_5 = 1, \dots, a_m = 1$ and $a_{m+1} > 1$.*

Moreover, if

$$\lambda(t) := \frac{1}{|t| \log 2} \left(\log^{(3)} |t| + \sum_{i=2}^{\infty} \log^{(i)} |t| \right),$$

then GWIPP(λ) *is recurrent.*

The remainder of this paper is outlined as follow. In Section 2 we prove mainly general results, including Theorem 1.1, Theorem 1.2 and Lemma 1.3. In Section 3 we concentrate on threshold results, i.e. Proposition 1.4 along with related results.

2 Proofs of general results

Throughout we will write $\Pi = \{X_i : i \in \mathbb{Z} \setminus \{0\}\}$, assuming as we may that

$$\cdots < X_{-2} < X_{-1} < 0 < X_1 < X_2 < \cdots .$$

For $k > 0$, let

$$A_k^R = \{\Pi(X_k, 2X_k + R) = 0\} = \{d(X_k, -R) < d(X_k, X_{k+1})\}$$

and

$$B_k^R = \{\Pi(2X_{-k} - R, X_{-k}) = 0\} = \{d(X_{-k}, R) < d(X_{-k}, X_{-k-1})\}.$$

The following lemma describes the connections between these events and recurrence of `GWIPP`.

Lemma 2.1. *With probability 1,*

$$\{\texttt{GWIPP} \textit{ recurrent}\} = \bigcap_{R \geq 0} \{A_k^R \textit{ i.o.}, B_k^R \textit{ i.o.}\}.$$

Proof. With probability 1, $\Pi(A) < \infty$ for any finite set A and for all n there is a unique point which is the closest unvisited point to S_n. On this event, the walk is well-defined and $|S_n| \to \infty$. Then, either $\{|S_n| \to \infty$ but $S_n S_{n+1} < 0$ i.o.$\}$ occurs (i.e. `GWIPP` is recurrent), or $\{S_n \to \infty\} \cup \{S_n \to -\infty\}$ occurs (i.e. `GWIPP` is transient).

Suppose first that `GWIPP` is recurrent. For all $R > 0$, there exists infinitely many n such that $S_n > 0$ and $S_{n+1} < -R$. For all such n it holds that $\Pi(S_n, 2S_n + R) = 0$ (otherwise $S_{n+1} > 0$), implying that A_k^R occurs infinitely often. Similarly B_k^R occurs infinitely often.

For the other direction, assume $\{A_k^R$ i.o.$\}$ and $\{B_k^R$ i.o.$\}$ occur for all $R \geq 0$. Choose any n, and without loss of generality assume that $S_n > 0$. Let $Y = \max\{X \in \Pi : X < \min_{0 \leq k \leq n} S_k\}$ be the rightmost point on the negative half-line that was not visited before time n. For $R > |Y|$, there exists $k > 0$ such that $X_k \geq S_n$ and A_k^R. Then $d(X_k, Y) < d(X_k, -R) < d(X_k, X_{k+1})$ and `GWIPP` visits Y on the negative half-line before visiting X_{k+1}. □

This characterisation suggests that the Borel–Cantelli lemmas will be useful. In particular, we use the extended Borel–Cantelli Lemma.

Lemma 2.2 (Extended Borel–Cantelli lemma, [5, Corollary 6.20])**.** *Let $\mathcal{F}_n$, $n \geq 0$, be a filtration and let $A_n \in \mathcal{F}_n$, $n \geq 1$. Then, with probability 1,*

$$\{A_n \textit{ i.o.}\} = \left\{ \sum_{n=1}^{\infty} \mathbb{P}[A_n \mid \mathcal{F}_{n-1}] = \infty \right\}.$$

The convergence or divergence of the associated random series will be determined using Campbell's theorem for sums of non-negative measurable functions, which provides a zero-one law for the convergence of a random series.

Theorem 2.3 (Campbell's theorem, [6, Section 3.2])**.** *Let Π be a Poisson process on S with mean measure μ and let $f : S \to [0, \infty]$ be a measurable function. Then the sum*

$$\sum_{X \in \Pi} f(X)$$

is convergent with probability 1 if and only if

$$\int_S \min\{f(x),1\}\mu(\mathrm{d}x) < \infty.$$

Moreover, the sum diverges with probability 1 if and only if the integral diverges.

We prove Theorem 1.1 and 1.2 together, as follows. We prove that GWIPP(μ) is recurrent with probability 1 if the integral conditions in Theorem 1.2 are true, and transient with probability 1 otherwise. This immediately implies Theorem 1.1.

Proof of Theorem 1.1 and 1.2. From Lemma 2.1 it follows that the sufficient and necessary conditions implying that the GWIPP is recurrent with probability 1, are the same as those implying that $\{A_k^R \text{ i.o.}\}$ and $\{B_k^R \text{ i.o.}\}$ occur with probability 1 for all $R \geq 0$.

Let $\mathcal{F}_k = \sigma(X_1, X_2, \ldots, X_k)$. Then $A_k^R \in \mathcal{F}_{k+1}$ for any $R \geq 0$, and

$$\mathbb{P}[A_k^R \mid \mathcal{F}_k] = \mathbb{P}[\Pi(X_k, 2X_k + R) = 0 \mid \mathcal{F}_k] = \exp(-\mu(X_k, 2X_k + R)),$$

where the final equality holds since $X_k \in \mathcal{F}_k$, $\Pi \cap (X_k, \infty)$ is independent of $\mathcal{F}_k$ and the number of points in a measurable set $A \subseteq \mathbb{R}$ is distributed like Poi$(\mu(A))$. Applying Theorem 2.3 with $f(x) = \exp(-\mu(x, 2x + R))$, we obtain

$$\sum_{k=1}^{\infty} \mathbb{P}[A_k^R \mid \mathcal{F}_k] = \sum_{k=1}^{\infty} \exp(-\mu(X_k, 2X_k + R)) = \infty$$

with probability 1 if and only if

$$\int_0^{\infty} \exp(-\mu(x, 2x + R))\mu(\mathrm{d}x) = \infty.$$

Moreover, Lemma 2.2 implies that $\sum_{k=1}^{\infty} \mathbb{P}[A_k^R \mid \mathcal{F}_k] = \infty$ a.s. if and only if $\mathbb{P}[A_k^R \text{ i.o.}] = 1$. Thus, the integral above diverges if and only if $\mathbb{P}[A_k^R \text{ i.o.}] = 1$.

Similarly, if the integral above converges, so does the sum

$$\sum_{k=1}^{\infty} \mathbb{P}[A_k^R \mid \mathcal{F}_k]$$

with probability 1, and then by Lemma 2.2, the event $\{A_k^R \text{ i.o.}\}$ does not occur with probability 1.

In the same way one can show that $\int_{-\infty}^{0} \exp(-\mu(2x - R, x))\mu(\mathrm{d}x) = \infty$ if and only if $\mathbb{P}[B_k^R \text{ i.o.}] = 1$; and, conversely, if the integral converges, then $\mathbb{P}[B_k^R \text{ i.o.}] = 0$.

In particular, the above proves that GWIPP is recurrent with probability 1 or 0, which is Theorem 1.1. □

Remark 2.4. We lose no generality by assuming that the greedy walk on Π starts from the origin, since recurrence/transience does not depend on the starting point. One explanation of this is that the distribution of the points in any finite interval around the origin does not influence the behaviour of the greedy walk far away from the origin. More precisely, suppose the walk starts from $a \in \mathbb{R}$, $a > 0$ (one can argue similarly for $a < 0$). One can show that the events $\{\Pi(X_k, 2X_k - a + R) = 0 \text{ i.o.}\}$ and $\{\Pi(2X_{-k} - a - R, X_{-k}) = 0 \text{ i.o.}\}$ occur for all $R \geq 0$ if and only if $\{A_k^R \text{ i.o.}\}$ and $\{B_k^R \text{ i.o.}\}$ occur for all $R \geq 0$. By Lemma 2.1, GWIPP(Π) is recurrent if these events occur.

A natural question is which conditions one needs to place on μ (or λ) so that $\{A_k^R \text{ i.o.}\}$ for all $R > 0$ if and only if $\{A_k^0 \text{ i.o.}\}$. The reason why this is not an unreasonable demand is that the events $\Pi(X_k, 2X_k + R) = 0$ and $\Pi(X_k, 2X_k) = 0$ should not be too different for large X_k, since the length of the interval $(2X_k, 2X_k + R)$ becomes negligible compared to the length of $(X_k, 2X_k)$ in the limit. However, the following example shows that some extra conditions need to be placed, and that, in general, $\{A_k^0 \text{ i.o.}\}$ does not imply that $\{A_k^R \text{ i.o.}\}$ for all $R > 0$.

Remark 2.5. Let

$$\lambda(t) = \sum_{n=1}^{\infty} a_n \mathbb{1}(2^n - 2 < t < 2^n - 1)$$

for some increasing sequence $(a_n)_{n=1}^{\infty}$.

If X equals the rightmost point in the interval $(2^n - 2, 2^n - 1)$, then $\Pi(X, 2X) = 0$ almost surely. This implies that X is always closer to 0 than to the leftmost point in $(2^{n+1} - 2, 2^{n+1} - 1)$. Hence, $\{A_n^0 \text{ i.o.}\}$ occurs with probability 1.

However, for $R = 3$ we have $\{A_n^3 \text{ i.o.}\} \subseteq \{\Pi(2^n - 2, 2^n - 1) = 0 \text{ i.o.}\}$. Choose now the sequence $(a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}[\Pi(2^n - 2, 2^n - 1) = 0] = \sum_{n=1}^{\infty} e^{-a_n} < \infty.$$

By the Borel–Cantelli lemma, the probability of $\{\Pi(2^n - 2, 2^n - 1) = 0 \text{ i.o}\}$ is 0, which implies that also $\mathbb{P}(A_n^3 \text{ i.o.}) = 0$. Therefore $\texttt{GWIPP}(\lambda)$ is transient even though A_n^0 occurs infinitely often with probability 1.

Denote by $\mathcal{M}_b \subseteq \mathcal{M}$ those measures $\mu \in \mathcal{M}$ with the property that for any $R \geq 0$, there exists some constant $C = C(R) > 0$, such that $\mu(x, x + R) < C$ and $\mu(-x - R, -x) < C$ for all $x \geq 0$. As the following lemma shows, this boundedness assumption disallows any examples of the type in Remark 2.5.

Lemma 2.6. *Let $\mu \in \mathcal{M}_b$. Then $\texttt{GWIPP}(\mu)$ is recurrent with probability 1 if and only if*

$$\int_0^{\infty} \exp(-\mu(x, 2x))\mu(\mathrm{d}x) = \infty \quad \textit{and} \quad \int_{-\infty}^{0} \exp(-\mu(2x, x))\mu(\mathrm{d}x) = \infty.$$

Proof. Fix $R > 0$. We have

$$\begin{aligned} \exp(-C) \int_0^{\infty} \exp(-\mu(x, 2x))\mu(\mathrm{d}x) &\leq \int_0^{\infty} \exp(-\mu(x, 2x) - \mu(2x, 2x + R))\mu(\mathrm{d}x) \\ &= \int_0^{\infty} \exp(-\mu(x, 2x + R))\mu(\mathrm{d}x) \\ &\leq \int_0^{\infty} \exp(-\mu(x, 2x))\mu(\mathrm{d}x). \end{aligned}$$

The integral on the negative half-line can be similarly bounded. Therefore the integrals in the statement of the lemma diverge if and only if the corresponding integrals in Theorem 1.2 diverge. This proves the claim. □

For instance, if $\mu \in \mathcal{M}$ and the maps $x \mapsto \mu(0, x)$ and $x \mapsto \mu(-x, 0)$ from $[0, \infty)$ to $[0, \infty)$ are Lipschitz, then $\mu \in \mathcal{M}_b$. Also, $\overline{\lim}_{t \to \pm\infty} \lambda(t) < \infty$ implies that $\lambda \in \mathcal{M}_b$, which gives the following corollary.

Corollary 2.7. *Suppose $\lambda \in \mathcal{M}$ and $\overline{\lim}_{t \to \pm\infty} \lambda(t) < \infty$. Then $\texttt{GWIPP}(\lambda)$ is recurrent with probability 1 if and only if*

$$\int_0^{\infty} \exp\left(-\int_x^{2x} \lambda(t)\,\mathrm{d}t\right) \lambda(x)\,\mathrm{d}x = \infty \quad \textit{and} \quad \int_{-\infty}^{0} \exp\left(-\int_{2x}^{x} \lambda(t)\,\mathrm{d}t\right) \lambda(x)\,\mathrm{d}x = \infty.$$

Next we prove Lemma 1.3.

Proof of Lemma 1.3. Denote by Π the point process with mean measure μ and let $(S_n)_{n=0}^{\infty}$ be $\texttt{GWIPP}(\Pi)$. Similarly, denote by Π' the point process with mean measure μ' and let $(S'_n)_{n=0}^{\infty}$ be $\texttt{GWIPP}(\Pi')$. Denote the points of Π and Π' by

$$\cdots < X_{-2} < X_{-1} < 0 < X_1 < X_2 < \cdots \quad \text{and} \quad \cdots < X'_{-2} < X'_{-1} < 0 < X'_1 < X'_2 < \cdots$$

respectively. Since $\mu'(A) \geq \mu(A)$ for all measurable $A \subset (-\infty, -K) \cup (K, \infty)$, we can couple Π and Π' together so that $x \in ((-\infty, -K) \cup (K, \infty)) \cap \Pi$ implies that $x \in \Pi'$.

Assume $\texttt{GWIPP}(\Pi)$ is transient. Without loss of generality, we may assume that $S_n \to \infty$ as $n \to \infty$. Then there is some $M_0 \geq 1$ such that $S_{k+1} > S_k$ for all $k > M_0$, i.e. $(S_n)_{n=1}^{\infty}$ moves only to the right after time M_0. Assume moreover that M_0 is large enough that $S_{M_0} > K$, so that we are on the region where Π and Π' are coupled. Let $Y = \max\{X \in \Pi : X < \min_{0 \leq k \leq M_0} S_k\}$, that is, let Y be the rightmost point of Π that is never visited. Note that Y is well-defined because of the transience of $\texttt{GWIPP}(\Pi)$ and the assumption $S_n \to \infty$ as $n \to \infty$.

Then there are 3 cases: (i) $\texttt{GWIPP}(\Pi')$ never visits a point in $(-\infty, Y]$, so $S'_n \to \infty$ and $\texttt{GWIPP}(\Pi')$ is transient. (ii) $\texttt{GWIPP}(\Pi')$ visits a point in $(-\infty, Y]$ and never visits a point in $[S_{M_0}, \infty)$ after that, so $S'_n \to -\infty$ and $\texttt{GWIPP}(\Pi')$ is transient. (iii) $\texttt{GWIPP}(\Pi')$ visits a point in $(-\infty, Y]$ and visits a point in $[S_{M_0}, \infty)$ after that. We claim that $S'_{J+1} > S'_J$ for all large enough J, which implies that $\texttt{GWIPP}(\Pi')$ is transient. See Figure 1 for an illustration of this case and the argument which follows.

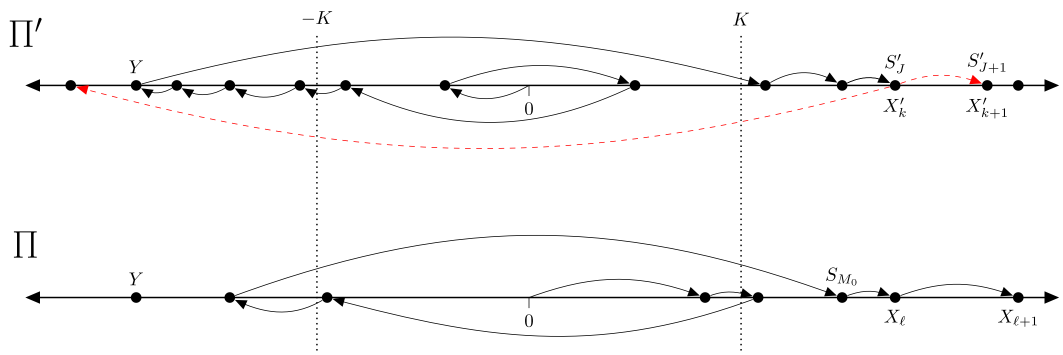

Figure 1: An illustration of final part of the proof of Lemma 1.3. Note that both the positive and negative axis have been rescaled logarithmically. The proof shows that $S'_{J+1} = X'_{k+1}$ is forced.

By assumption, for all large enough J there exists some $n < J$ such that $S'_n < Y$ and $S'_J > S_{M_0}$. Let $S'_J = X'_k$ and let ℓ be such that $X_\ell \leq X'_k < X_{\ell+1}$. Since $\texttt{GWIPP}(\Pi)$ only moves to the right after time M_0, we have $d(X_\ell, X_{\ell+1}) < d(X_\ell, Y)$. The coupling between Π and Π' on (K, ∞) implies that $X_\ell \leq S'_J < X'_{k+1} \leq X_{\ell+1}$. Hence $d(S'_J, X'_{k+1}) \leq d(X_\ell, X_{\ell+1}) < d(X_\ell, Y) \leq d(S'_J, Y)$. Hence $S'_{J+1} = X'_{k+1} > S'_J$, as claimed. Therefore $\texttt{GWIPP}(\Pi')$ is transient. ☐

3 Threshold results

In this section we study the threshold between transience and recurrence, proving Propositions 1.4–3.5 and related results. We focus on symmetric intensity functions of the form

$$\lambda_f(t) = \frac{\log f(|t|)}{|t| \log 2},$$

where $f : (0, \infty) \to [1, \infty)$ is a regularly varying function with non-negative index β, meaning that $\lim_{t\to\infty} f(at)/f(t) = a^\beta$ for any $a \geq 0$. If $\beta = 0$, then f is said to be slowly varying. For a thorough introduction to the theory of regular variation, we refer the reader to [1].

Let $\mathcal{M}_s$ be the set of all intensity functions $\lambda_f \in \mathcal{M}$ such that $f : (0, \infty) \to [1, \infty)$ is a regularly varying function with index $\beta \geq 0$. Since

$$\int_0^1 \lambda_f(x)\, \mathrm{d}x < \infty,$$

we necessarily have that $f(x) \to 1$ as $x \to 0$. (This also means that there is no issue with integrability near 0 in the results that follow.) Moreover, one can show that $\mathcal{M}_s \subset \mathcal{M}_b \subset \mathcal{M}$, so we may apply all results developed in Section 2. The intensity functions in $\mathcal{M}_s$ are symmetric about 0, so it suffices to only look at the positive half-line. However, the results in this section can easily be adapted to the case when λ is not assumed to be symmetric.

We use the following standard notation. If $f, g : \mathbb{R} \to \mathbb{R}$ are two functions and there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all large enough x, then we write $f(x) = O(g(x))$. If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, then we write $f(x) = \Theta(g(x))$.

Lemma 3.1 ([1, Theorem 1.5.2])**.** *If $A \subseteq (0, \infty)$ is a compact set and $f : (0, \infty) \to [0, \infty)$ is regularly varying with index β, then $f(ax)/f(x) \to a^\beta$ as $x \to \infty$, uniformly for all $a \in A$.*

Lemma 3.2. *Suppose $\lambda_f \in \mathcal{M}_s$. Then* GWIPP$(\lambda_f)$ *is recurrent with probability 1 if and only if*

$$\int_0^\infty \frac{\log f(x)}{x f(x)}\, \mathrm{d}x = \infty.$$

Proof. The set $[1, 2]$ is compact and f is regularly varying. It follows by Lemma 3.1, that

$$\int_x^{2x} \frac{\log f(t)}{t \log 2}\, \mathrm{d}t = \log f(x) + O(1)$$

as $x \to \infty$. Hence

$$\int_0^\infty \exp\left(-\int_x^{2x} \lambda_f(t)\, \mathrm{d}t\right) \lambda_f(x)\, \mathrm{d}x = \int_0^\infty \frac{\Theta(1)}{f(x)} \frac{\log f(x)}{x \log 2}\, \mathrm{d}x = \Theta(1) \int_0^\infty \frac{\log f(x)}{x f(x)}\, \mathrm{d}x.$$

The claim follows from Corollary 2.7. □

The next corollary states that if f is regularly varying with positive index, then we obtain a transitive process with probability 1.

Corollary 3.3. *Let f be a regularly varying function with index $\beta > 0$. Then* GWIPP(λ_f) *is transient with probability 1.*

Proof. There exists a slowly varying function $\ell(x)$ such that $f(x) = x^\beta \ell(x)$ (see, e.g. [1, Theorem 1.4.1]). Then for $x > 0$,

$$\frac{\log f(x)}{x f(x)} = \frac{\log f(x)}{x^{1+\beta} \ell(x)} = \frac{L(x)}{x^{1+\beta}}, \tag{3.1}$$

where $L(x) = \frac{\log f(x)}{\ell(x)}$ is a slowly varying function (see, e.g. [1, Theorem 1.3.6]). The function on the right hand side of (3.1) is integrable on $(0, \infty)$ whenever $\beta > 0$, and by Lemma 3.2, GWIPP(λ_f) is transient. □

The intensity functions in Proposition 1.4 lie in $\mathcal{M}_s$ with f slowly varying, showing that a transition between recurrence and transience occurs inside the subclass of $\mathcal{M}_s$ for which f is slowly varying.

For notational convenience in the following proofs, we define the "power tower" recursively by $a \uparrow\uparrow 0 := 1$ and $a \uparrow\uparrow n := a^{a\uparrow\uparrow(n-1)}$ for any $a \in [0,\infty)$ and $n \geq 1$. Note that $\log^{(n)} t = 0$ for any $t \leq e \uparrow\uparrow (n-1)$.

Proof of Proposition 1.4. Let

$$f(t) = \max\left(\prod_{i=2}^{n} (\log^{(i-1)} t)^{a_i}, 1\right),$$

so that $\lambda(t) = \frac{\log f(t)}{t \log 2}$. Note that $\lambda \in \mathcal{M}_s$. Assume first that $a_2 > 0$. Then

$$\int_{e\uparrow\uparrow n}^{t} \frac{\log f(x)}{x f(x)}\,\mathrm{d}x = \int_{e\uparrow\uparrow n}^{t} \frac{\sum_{i=2}^{n} a_i \log^{(i)} x}{x\prod_{i=2}^{n}(\log^{(i-1)} x)^{a_i}}\,\mathrm{d}x = \Theta\left(\int_{e\uparrow\uparrow n}^{t} \frac{\log^{(2)} x}{x\prod_{i=2}^{n}(\log^{(i-1)} x)^{a_i}}\,\mathrm{d}x\right),$$

as $t \to \infty$, where the final integral is the leading order term of the sum. The final integral is convergent precisely when one of the conditions in the statement is satisfied. (This is seen by repeatedly using the change of variables $x \mapsto e^x$.) By Lemma 3.2, the statement follows. If $a_2 = 0$, then consider instead $\lambda'(t) := \lambda(t) + \frac{1}{2}\frac{\log^{(2)}|t|}{|t|\log 2}$ and use the above along with Lemma 1.3 to conclude that $\mathtt{GWIPP}(\lambda)$ is recurrent in this case. This completes the proof of the first part. The second part of the claim follows immediately from the more general Proposition 3.4. $\square$

In the next Proposition we consider a generalisation of Proposition 1.4.

Proposition 3.4. *Let $a_3 = 2$ and $a_2 = 1 = a_4 = a_5 = \ldots$ and let $g : (0,\infty) \to [1,\infty)$ be a non–decreasing slowly varying function satisfying $\log^{(1)} g(t) = O(\log^{(2)}|t|)$ and let*

$$\lambda(t) := \frac{1}{|t|\log 2}\left(\sum_{i=2}^{\infty} a_i \log^{(i)}|t| + \log^{(1)} g(|t|)\right).$$

For $n \geq 1$, let $b_n := g(e \uparrow\uparrow n)$. Then $\mathtt{GWIPP}(\lambda)$ is reccurent with probability 1 if and only if $\sum_{n=2}^{\infty} 1/b_n = \infty$.

Proof. For $t > 0$ we have $\lambda(t) = \frac{\log f(t)}{t\log 2}$ with

$$\log f(t) = \sum_{i=2}^{\infty} a_i \log^{(i)}(t) + \log g(t).$$

Because of our definition of the iterated logarithm, this implies that

$$f(t) = \prod_{i=1}^{\infty}\left(\max(1, (\log^{(i)} t)^{a_{i+1}})\right) g(t).$$

Since $b_{n-1} \le g(x)$ for any $e \uparrow\uparrow (n-1) \le x \le e \uparrow\uparrow n$, we obtain

$$\begin{aligned}
\int_e^\infty \frac{\log f(x)}{xf(x)}\,\mathrm{d}x &= \sum_{n=2}^\infty \int_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \frac{\sum_{i=2}^n a_i \log^{(i)} x + \log g(x)}{x \prod_{i=1}^{n-1} (\log^{(i)} x)^{a_{i+1}} g(x)}\,\mathrm{d}x \\
&= \Theta(1) \sum_{n=2}^\infty \int_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \frac{\log^{(2)} x}{x \prod_{i=1}^{n-1} (\log^{(i)} x)^{a_{i+1}} g(x)}\,\mathrm{d}x \\
&\le \Theta(1) \sum_{n=2}^\infty \int_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \frac{1}{x \prod_{i=1}^{n-1} (\log^{(i)} x) b_{n-1}}\,\mathrm{d}x \\
&= \Theta(1) \sum_{n=2}^\infty \frac{1}{b_{n-1}} \left[\log^{(n)} x\right]_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \\
&= \Theta(1) \sum_{n=2}^\infty \frac{1}{b_{n-1}}.
\end{aligned}$$

Using instead the bound $b_n \ge g(x)$ for any $e \uparrow\uparrow (n-1) \le x \le e \uparrow\uparrow n$, we arrive at

$$\sum_{n=2}^\infty \frac{1}{b_n} \le \int_e^\infty \frac{\log f(x)}{xf(x)}\,\mathrm{d}x \le \Theta(1) \sum_{n=2}^\infty \frac{1}{b_{n-1}}.$$

Applying Lemma 3.2 completes the proof. □

The following result provides a useful tool for investigating the behaviour of a given intensity function. The idea behind the proof is essentially to find a suitable intensity function for comparison, and apply Lemma 1.3 and Proposition 1.4.

Proposition 3.5. *Let $\lambda \in \mathcal{M}$. Let $a_3 = 2$ and $a_2 = 1 = a_4 = a_5 = \dots$. If there exists some $n \ge 2$ such that*

$$\varliminf_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} > 1 \quad \textit{or} \quad \varliminf_{t\to-\infty} \frac{|t|\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} |t|}{a_n \log^{(n)} |t|} > 1$$

then $\mathtt{GWIPP}(\lambda)$ *is transient with probability* 1*. If there exists some $n \ge 2$ such that*

$$\varlimsup_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} < 1 \quad \textit{and} \quad \varlimsup_{t\to-\infty} \frac{|t|\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} |t|}{a_n \log^{(n)} |t|} < 1$$

then $\mathtt{GWIPP}(\lambda)$ *is recurrent with probability* 1*.*

Proof. Suppose first that

$$\varliminf_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} > 1$$

for some $n \ge 2$. (Let n be minimal with this property.) Let

$$a := \frac{1}{2}\left(1 + \varliminf_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t}\right) > 1$$

and define

$$\lambda'(t) := \begin{cases} \lambda(t), & t \le 0 \\ \frac{1}{|t|\log 2}\left(\sum_{i=2}^{n-1} a_i \log^{(i)} |t| + a a_n \log^{(n)} |t|\right), & t > 0. \end{cases}$$

Proposition 1.4 implies that $\mathtt{GWIPP}(\lambda')$ is transient. (In Proposition 1.4 we assumed that the intensity function be symmetric, but this does not change the evaluation of the integral on the positive half-axis.) Since $\lambda(t) \geq \lambda'(t)$ for all t large enough, Lemma 1.3 implies that $\mathtt{GWIPP}(\lambda)$ is transient.

Now suppose the second condition holds for some $n \geq 2$. If

$$\lambda'(t) := \frac{1}{|t| \log 2} \left(\sum_{i=2}^{n} a_i \log^{(i)} |t| \right),$$

then $\lambda(t) < \lambda'(t)$ for all sufficiently large t. By Proposition 1.4, $\mathtt{GWIPP}(\lambda')$ is recurrent, and Lemma 1.3 implies that $\mathtt{GWIPP}(\lambda)$ is recurrent. □

Proposition 3.5 does not answer what happens if, say,

$$\lim_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} = 1 \quad \text{and} \quad \lim_{t\to-\infty} \frac{|t|\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} |t|}{a_n \log^{(n)} |t|} = 1$$

for all $n \geq 2$. As seen in Proposition 3.4, both recurrence and transience are possible in this case.

References

[1] Bingham, N. H., Goldie, C. M. and Teugels, J. L. *Regular Variation.* Cambridge University Press, Cambridge, 1989. MR-1015093

[2] Bordenave, C., Foss, S. and Last, G. On the greedy walk problem. *Queueing Syst.* **68**, (2011), 333–338. MR-2834204

[3] Foss, S., Rolla, L. T. and Sidoravicius, V. Greedy walk on the real line. *Ann. Prob.* **43**, (2015), 1399–1418. MR-3342666

[4] Gabrysch, K. Distribution of the smallest visited point in a greedy walk on the line. *J. Appl. Prob.* **53**, (2016), 880–887. MR-3570100

[5] Kallenberg, O. *Foundations of Modern Probability.* Springer-Verlag, New York, 1997. MR-1464694

[6] Kingman, J. F. C. *Poisson Processes.* Oxford University Press, New York, 1993. MR-1207584

[7] Rolla, L. T., Sidoravicius, V. and Tournier, L. Greedy clearing of persistent Poissonian dust. *Stochastic Process. Appl.* **124**, (2014), 3496–3506. MR-3231630

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