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The Orbit Method and Geometric Quantisation

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ABSTRACT

In this thesis we study Kirillov's orbit method, a method for establishing a correspondence between unitary representations of Lie groups and geometric objects called *coadjoint orbits*. We give a comprehensible introduction and investigate the relation to its counterpart in mathematical physics known as geometric quantisation, a way of passing from a classical mechanical system to a quantum mechanical system by methods from symplectic geometry.

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1 Introduction

The orbit method (or the method of coadjoint orbits), first developed by Alexandre Kirillov in the early 1960's, is a method for finding unitary representations of Lie groups. Kirillov worked on the case of nilpotent groups and in the special case of nilpotent, simply connected groups the theory produces a perfect correspondence between the coadjoint orbits of the group and its irreducible, unitary representations. The theory was later extended to solvable groups by Bertram Kostant, Lajos Pukánszky, Jean-Marie Souriau and others. More recently, David Vogan and others has studied the case of reductive groups. The main purpose of the orbit method is not to explicitly find all unitary representations of a given Lie group, but rather to establish a correspondence between the so called *coadjoint orbits* of the group, which are geometric objects, and its unitary representations. Admittedly, it only works prefectly for certain classes of Lie groups (which is why it is called a method, rather than a theorem), but it has been conjectured that the orbit method may be used to find all representations needed to study automorphic forms (see Vogan [10]). This is but one of the reasons why it seems to be worthy of study.

Geometric quantisation[1] deals with defining a quantum theory given a particular classical theory. In its modern form, it was mainly developed by Kostant and Souriau. The theory of geometric quantisation can be seen as a physical counterpart of the orbit method and the main goal of this thesis is to illustrate this.

We give an introduction to the orbit method through a mathematical description of geometric quantisation. The reader is assumed to be familiar with differential geometry, however we recall some basic facts from the theory of Lie groups/algebras and their representations.

2 Lie Groups and Representations

2.1 Preliminaries

We assume that the reader is already somewhat familiar with Lie groups, however we will begin this section by recalling some basic concepts from the theory of Lie groups and Lie algebras. For further details on this subject, we recommend Hall [9].

Definition 2.1. *A* Lie group *is a smooth manifold G equipped with a group structure, such that the group operation*

$$G \times G \to G,\ (g, g') \mapsto gg'$$

and inversion

$$G \to G,\ g \mapsto g^{-1}$$

[1]Note that we have chosen the English spelling, in the literature one often sees the American spelling *quantization*.

are smooth maps.

A large class of Lie groups are the so called *matrix Lie groups*. These are the Lie groups G that are closed subgroups of $\mathrm{GL}(n, \mathbb{C})$. That is, G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ such that if $\{A_k\}$ is a sequence of linear maps in G which converges to A, then either $A \in G$ or $\det(A) = 0$. Many interesting Lie groups are matrix groups, such as the special linear group $\mathrm{SL}(n, \mathbb{C})$, the unitary and orthogonal groups $\mathrm{U}(n)$ and $\mathrm{O}(n)$ with their special subgroups $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$, the Lorentz group $\mathrm{O}(3; 1)$, the symplectic groups $\mathrm{Sp}(n, \mathbb{R})$, the Euclidean groups, the Pointcaré groups, and the Heisenberg group, to name a few examples.

Definition 2.2. *A* Lie group homomorphism *is a group homomorphism*

$$\varphi : G \to H$$

which is continuous.

Note that we could have defined a Lie group homomorphism as a *smooth* group homomorphism, but it turns out that a continuous group homomorphism from one Lie group to another is automatically smooth.

Lie groups inherit topological properties such as compactness, connectedness and simple connectedness from its underlying manifold.

In this thesis we will be particularly interested in the action of Lie groups on manifolds and the arising orbits. Let us recall some fundamental definitions.

Definition 2.3. *A* Lie group action *of a Lie group G on a manifold M is a continuous map*

$$G \times M \to M, \ (g, x) \mapsto g \cdot x$$

satisfying the following conditions:

1. *$e \cdot x = x$, where e is the identity element in G;*

2. *$g \cdot (h \cdot x) = (gh) \cdot x$.*

Definition 2.4. *Given a Lie group action of G on M, the* orbit *of an element $x \in M$ is the subset*

$$\mathcal{O}_x := G \cdot x.$$

The stabilising group *G_x is the subgroup*

$$G_x = \{g \in G : g \cdot x = x\},$$

and we have a diffeomorphism (under the condition that the action is smooth)

$$\mathcal{O}_x \cong G/G_x.$$

Definition 2.5. *A* (real or complex) Lie algebra *is a vector space* $\mathfrak{g}$ *(over* $\mathbb{R}$ *or* $\mathbb{C}$*), equipped with a bilinear map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

such that for all $X, Y, Z \in \mathfrak{g}$ *the following holds:*

1. $[X,Y] = -[Y,X]$ (skew symmetry)*;*
2. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$ (the Jacobi identity).

The map $[\cdot,\cdot]$ *is called the* bracket *on* $\mathfrak{g}$*.*

Two elements $X, Y \in \mathfrak{g}$ *commute* if $[X,Y] = 0$. In particular, if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$, then the Lie algebra $\mathfrak{g}$ is said to be *commutative*.

Definition 2.6. *A* subalgebra *of a Lie algebra* $\mathfrak{g}$ *is a subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *which is closed under the bracket of* $\mathfrak{g}$*.*

If $[X,H] \in \mathfrak{h}$ *for all* $X \in \mathfrak{g}$ *and* $H \in \mathfrak{h}$*, then* $\mathfrak{h}$ *is said to be an* ideal *in* $\mathfrak{g}$*.*

The center *of* $\mathfrak{g}$ *is the set of* $X \in \mathfrak{g}$ *which commute with all* $Y \in \mathfrak{g}$*.*

A Lie algebra is called irreducible if it has no nontrivial ideals. If a Lie algebra is irreducible and non-commutative it is called simple*.*

Definition 2.7. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be Lie algebras. A* Lie algebra homomorphism *is a linear map* $\phi : \mathfrak{g} \to \mathfrak{h}$ *such that for all* $X, Y \in \mathfrak{g}$

$$\phi([X,Y]) = [\phi(X), \phi(Y)]\,.$$

Definition 2.8. *Let* $\mathfrak{g}$ *be a Lie algebra and* $X \in \mathfrak{g}$ *and define a linear map*

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g},\ Y \mapsto \mathrm{ad}_X(Y) := [X,Y]\,.$$

Then the map $X \mapsto \mathrm{ad}_X$ *is called the* adjoint map*.*

Note that by the Jacobi identity, we have

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]\,,$$

that is, the mapping $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a Lie algebra homomorphism.

Definition 2.9. *On a Lie algebra* $\mathfrak{g}$ *we define the symmetric, bilinear form*

$$B(X,Y) = \mathrm{trace}(\mathrm{ad}_X \mathrm{ad}_Y).$$

The form B *is called the* Killing form*.*

Definition 2.10. *Let* $\mathfrak{g}$ *be a Lie algebra. We define the sequence of ideals called the* upper central series *as follows.*

$$\mathfrak{g}^j = \begin{cases} \mathfrak{g}, & \text{if } j = 0, \\ \{X \in \mathfrak{g} : X = [Y,Z]\,,\ Y \in \mathfrak{g},\ Z \in \mathfrak{g}^{j-1}\}, & \text{if } j > 0. \end{cases}$$

The Lie algebra $\mathfrak{g}$ is called nilpotent *if $\mathfrak{g}^j = 0$ for some j.*

Proposition 2.11. *If $\mathfrak{g}$ is a nilpotent Lie algebra, then the Killing form on $\mathfrak{g}$ is identically zero.*

Let us briefly recall the relationship between Lie algebras and Lie groups.

Definition 2.12. *Let G be a Lie group and T_eG the tangent space at $e \in G$. Let $X \in T_eG$ and let $\gamma_X(t)$ be the maximal integral curve of X. Then we define the* exponential map *by*

$$\exp(X) := \gamma_X(1).$$

We remark that if G is a matrix Lie group, then the exponential map is simply the usual matrix exponentiation

$$\exp(X) = e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Definition 2.13. *Let G be a Lie group. The* Lie algebra of G, *denoted* $\mathrm{Lie}(G)$ *or $\mathfrak{g}$, is the space of* left-invariant *vector fields on G, i.e. vector fields $X \in \Gamma(G)$ such that*

$$X_g = dl_g X_e,$$

where l_g is the left translation on G: $l_g(g') = gg'$. The bracket operation on $\mathfrak{g}$ is the usual commutator of vector fields, i.e.

$$[X, Y] = XY - YX.$$

Clearly, $\mathfrak{g}$ can be identified with the tangent space at the identity element of G. When G is a matrix Lie group, its Lie algebra can be defined as the space of matrices X such that

$$e^{tX} \in G,$$

for all $t \in \mathbb{R}$.

Theorem 2.14. *Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Let $\Phi : G \to H$ be a Lie group homomorphism. Then there exists a unique Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \phi\ } & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\ \Phi\ } & H \end{array}$$

Definition 2.15. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. For each $g \in G$, we define a linear map*

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g},\ \mathrm{Ad}_g(X) := gXg^{-1}.$$

The map

$$G \to \mathrm{GL}(\mathfrak{g}),\ g \mapsto \mathrm{Ad}_g$$

is called the Adjoint map*.*

The Adjoint map is a Lie group homomorphism and thus has an associated Lie algebra homomorphism which fits into a commutative diagram. This Lie algebra homomorphism is of course the adjoint map (definition 2.8). Hence we have the following commutative diagram.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mathrm{ad}\ } & \mathrm{End}(\mathfrak{g}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\ \mathrm{Ad}\ } & \mathrm{GL}(\mathfrak{g}) \end{array}$$

Definition 2.16. *Let $\mathfrak{g}$ be a Lie algebra and suppose there exists a compact Lie group K such that*

$$\mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}},$$

that is, $\mathfrak{g}$ is isomorphic to the complexification of the Lie algebra of K. Then $\mathfrak{g}$ is said to be reductive*.*

A reductive Lie algebra with trivial center is said to be semisimple*.*

There are several characterizations of semisimple Lie algebras. In addition to the defining property we have given, we state two more characterizations that we could have chosen as the definition of semisimplicity.

Proposition 2.17. *The following statements are equivalent.*

(i) *$\mathfrak{g}$ is semisimple (in the sense of definition 2.16);*

(ii) *the Killing form is non-degenerate on $\mathfrak{g}$;*

(iii) *$\mathfrak{g}$ decomposes as a Lie algebra direct sum*

$$\mathfrak{g} = \bigoplus_{j=1}^{n} \mathfrak{g}_{\mathrm{j}},$$

where each $\mathfrak{g}_{\mathrm{j}} \subset \mathfrak{g}$ is simple.

2.2 Representations of Lie Groups

In many ways, representations of Lie groups lies at the heart of this thesis. The orbit method is a method for finding representations of Lie groups and as we shall see, it begins with the consideration of orbits of a certain representation (the *coadjoint representation*) of a Lie group. Let us first recall some basic representation theory. For more details, see Hall [9], Kirillov [5] and Berndt [1].

Definition 2.18. *Let G be a Lie group. A* finite-dimensional representation *of G is a Lie group homomorphism*

$$\pi : G \to \mathrm{GL}(V),$$

where V is a finite-dimensional vector space with dimension at least 1.

A finite-dimensional representation *of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism*

$$\pi_* : \mathfrak{g} \to \mathfrak{gl}(V),$$

where $\mathfrak{gl}(V)$ is the space $\mathrm{End}(V)$ with the commutator bracket.

The vector space V is called the representation space *and a representation is said to be complex (real) if V is a complex (real) vector space.*

Example 2.1. *Let G be a Lie group and define for each element $g \in G$ a map*

$$C_g : G \to G,\ x \mapsto gxg^{-1}.$$

The differential of this map is the map $\mathrm{Ad}_g := (dC_g)_e \in \mathrm{End}(\mathfrak{g})$ (for a matrix Lie group this is simply the map from definition 2.15). The mapping

$$\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}),\ g \mapsto \mathrm{Ad}_g$$

is called the adjoint representation *and it is a representation of G with representation space $\mathfrak{g}$.*

Definition 2.19. *Let π be a Lie group representation acting on the representation space V. A subspace $W \subset V$ is called* invariant *if*

$$\pi(g)w \in W, \quad \textit{for all } w \in W.$$

A representation with no invariant nontrivial subspaces is called irreducible.

If V is a vector space and A is an operator on V, we denote the dual space by V^* and the dual operator by A^t.

Definition 2.20. *Suppose π is a Lie group representation acting on V. Then the* dual *(or contragredient) representation π^* acting on V^* is given by*

$$\pi^*(g) = \pi(g^{-1})^t.$$

If π_* is a Lie algebra representation acting on V, then the dual representation is given by

$$(\pi_*)^*(X) = -\pi_*(X)^t.$$

More generally, we consider infinite-dimensional representations of Lie groups.

Definition 2.21. *Let G be a Lie group, H a Hilbert space and $\mathcal{U}(H)$ the group of unitary operators on H. A* unitary representation *of G is a continuous homomorphism*

$$\pi : G \to \mathcal{U}(H),$$

where $\mathcal{U}(H)$ is equipped with the strong operator topology.

Recall that strong continuity of π means that for any $v \in H$, the function

$$g \mapsto \pi(g)v$$

is continuous. The vector $v \in H$ is called *smooth* if the function above is infinitely differentiable. The subspace of smooth vectors in H is denoted H^∞.

Theorem 2.22. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation and let $\mathfrak{g} = \mathrm{Lie}(G)$. Then*

1. *The subspace H^∞ is dense in H and stable with respect to all operators $\pi(g)$.*

2. *For any $X \in \mathfrak{g}$ the operator*

$$A = -i\pi_*(X) =: -i\frac{d}{dt}\pi(\exp(tX))\Big|_{t=0}$$

   with domain H^∞ has self-adjoint closure.

3. *If G is connected, then the representation π is completely determined by the representation π_* of $\mathfrak{g}$ in H^∞ defined above. In particular, for any $v \in H^\infty$ we have*

$$\pi(\exp(tX))v = e^{itA}v.$$

See the appendix about representation theory in Kirillov [5] for a proof.

When dealing with classification problems in representation theory, we work with equivalence classes of representations rather than representations themselves. Here we are interested in unitary representations of Lie groups, so the following is the appropriate equivalence relation to work with. Two unitary representations π and $\tilde{\pi}$ of a Lie group G on representation spaces V and $\tilde{V}$ are *equivalent* if there exists a unitary transformation $\varphi : V \to \tilde{V}$ such that the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{\pi(g)} & V \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \tilde{V} & \xrightarrow{\tilde{\pi}(g)} & \tilde{V} \end{array}$$

Definition 2.23. *Given a Lie group G, we define its* unitary dual *$\hat{G}$ as the set of equivalence classes of unitary representations of G.*

Suppose we have a locally compact topological group G and a subgroup H for which we have a representation π_0 on some space $\mathcal{H}_0$. The representation induces a representation

$$\mathrm{Ind}_H^G \pi_0$$

of G on a space of functions $\phi : G \to \mathcal{H}_0$ satisfying a certain functional equation. To define this representation, we need some background from measure theory. Recall that the σ-algebra generated by the open sets of G is called the *Borel algebra.* For an element S of the Borel algebra and $g \in G$, the *left translate* of S is the set

$$gS = \{gs : s \in S\}$$

and the *right translate* of S is

$$Sg = \{sg : s \in S\}.$$

A measure μ is called *left-* (resp. *right-*) *invariant* if $\mu(gS) = \mu(S)$ (resp. $\mu(Sg) = \mu(S)$). If μ is both left- and right-invariant it is simply called *invariant.*

Theorem 2.24 (due to A. Haar)**.** *On any locally compact topological group G with countable basis for the topology, there exists a non-trivial left-invariant measure (and similarly a right-invariant measure). Up to multiplication by a positive constant this measure is unique and it is called a left- (resp. right-) Haar measure.*

Definition 2.25. *Let G be a locally compact topological group. Then the* modular function *of G is the continuous group homomorphism*

$$\Delta_G : G \to \mathbb{R}_+$$

such that for a right-Haar measure μ and every Borel set S

$$\mu(g^{-1}S) = \Delta_G(g)\mu(S).$$

By Haar's theorem, the modular function is well-defined. The group G is called *unimodular* if $\Delta_G \equiv 1$, or equivalently if the Haar measure is invariant.

Now, let G be a unimodular, connected, matrix group. Let H be closed subgroup and let K be a closed, unimodular subgroup such that the map

$$H \times K \to G,\ (h, k) \mapsto hk$$

is a homeomorphism. Let π_0 be a representation of H on the Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}$ be the space of functions $\phi : G \to \mathcal{H}_0$ satisfying the functional equation

$$\phi(hg) = \Delta_H(h)^{1/2}\pi_0(h)\phi(g),$$

for all $h \in H$ and $g \in G$, and the condition

$$\phi|_K \in L^2(K).$$

We define a norm for $\mathcal{H}$ by

$$|\phi|^2_{\mathcal{H}} := \int_K |\phi(k)|_K|^2_{\mathcal{H}_0} dk$$

and a scalar product by

$$\left\langle \phi, \tilde{\phi} \right\rangle := \left\langle \phi|_K, \tilde{\phi}|_K \right\rangle.$$

Definition 2.26. *The* representation of G induced by π_0 *is the representation*

$$\pi = \mathrm{Ind}_H^G \pi_0$$

on the space $\mathcal{H}$, given by

$$(\pi(g')\phi)(g) = \phi(gg').$$

We remark that if π_0 is unitary, then so is $\mathrm{Ind}_H^G \pi_0$.

This approach is certainly not the most general one, but for our purposes it shall suffice. The reader who wants more generality may consult Kirillov [5] or Berndt [1].

3 Mechanics and Quantisation

3.1 Classical Mechanics

Classical mechanics is a theory in physics which describes the motions of macroscopic objects. Historically, it began with what is now known as Newtonian mechanics which utilized calculus to describe the motion of bodies influenced by forces. Later more sophisticated methods led to reformulations known as Lagrangian, and Hamiltonian mechanics. In Lagrangian mechanics we have the manifold Q called the *configuration space* consisting of points q describing the position of a particle. In Hamiltonian mechanics we consider the so called *phase space*, which is the cotangent bundle T^*Q of the configuration space, consisting of the position q and momentum p of a particle. We shall see that the phase space is canonically a symplectic manifold. Smooth functions on the phase space are called *classical observables*. In a classical mechanical system, the *energy* of that system is the sum of the potential energy and the momentum. The energy of a system is given by the Hamiltonian $H(q, p)$, which is a function on the phase space, i.e. a classical observable. The dynamics of the system is given by Hamilton's equations

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i},\ \dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{1}$$

Example 3.1. *Suppose that the energy of a physical system is constant, that is*

$$H(q,p) = C.$$

Then by Hamilton's equations (1), $\dot{q} = 0$ and $\dot{p} = 0$ and thus $q(t) = q_0$ and $p(t) = 0$. This is interpreted, of course, as nothing is happening in the system.

Example 3.2. *In a system consisting of a particle with mass m (considered as a point in $\mathbb{R}^3$) on which a force field with potential $V(q)$ acts, the Hamiltonian is*

$$H(q,p) = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} + V(q).$$

By Hamilton's equations we have $\dot{q}_i = \frac{p_i}{m_i}$ and $\dot{p}_i = -\partial_i V(q) = F_i$. Differentiating $\dot{q}_i$ yields

$$\ddot{q}_i = \frac{\dot{p}_i}{m_i} = \frac{-\partial_i V(q)}{m_i} = \frac{F_i}{m_i}.$$

The second time derivative of the position is acceleration, denoted $\ddot{q} = a$. Rearranging the equation above, we have derived

$$F_i = ma_i,$$

which is the familiar equation first derived by I. Newton*, albeit through an entirely different reasoning.*

We can interpret this Hamiltonian formalism geometrically. The *Hamiltonian vector field* is the vector field

$$X_H = (\nabla_p H, -\nabla_q H)$$

on T^*Q. A curve $\gamma(t) = (q(t), p(t))$ is an integral curve for X_H if

$$\dot{\gamma} = (\dot{q}, \dot{p}) = (\nabla_p H, -\nabla_q H).$$

This is just the assertion that γ fulfills Hamilton's equations.

3.2 Symplectic Geometry

Another interpretation is the *symplectic interpretation.* Let us first recall some basic symplectic geometry.

Definition 3.1. *A* symplectic manifold *is a differentiable manifold M equipped with a differential form $\omega \in \Omega^2(M)$, which is closed and non-degenerate, i.e.*

1. *$d\omega = 0$;*
2. *If $\omega(X,Y) = 0$ for all $Y \in \Gamma(M)$, then $X = 0$.*

Note that since ω is a 2-form, it is anti-symmetric, that is, for all $X, Y \in \Gamma(M)$

$$\omega(X,Y) = -\omega(Y,X).$$

If we pick a basis on T_pM, ω_p is given by an invertible matrix

$$[\omega_p] = A(p) \in \mathrm{GL}(\dim M, \mathbb{R}).$$

The matrix $A(p)$ is skew-symmetric, that is

$$A(p)^t = -A(p),$$

so that we see

$$\det(A) = \det(-A^t) = (-1)^{\dim M} \det(A).$$

Hence, a symplectic manifold has even dimension.

On T^*Q we have the *canonical 1-form*

$$\alpha = \sum q_i dp_i$$

and the exterior derivative of α is called the *symplectic form*

$$\omega = d\alpha = \sum dq_i \wedge dp_i.$$

Note that $\omega \in \Omega^2(M)$ is closed and non-degenerate, so (T^*Q, ω) is a symplectic manifold. This is the sense in which T^*Q has a canonical symplectic structure.

We remark that this seemingly very special canonical form is in fact very general, this is summarized in a theorem due to G. Darboux

Theorem 3.2. *On a symplectic manifold (M, ω) there exists a neighborhood U and a C^∞ function*

$$\varphi : U \to \mathbb{R}^{2k}$$

such that

$$\omega|_U = \varphi^* \left(\sum_{j=1}^{k} dq_j \wedge dp_j \right).$$

Recall that given a vector field X we have the contraction with an r-form α (also known as the interior product), denoted

$$X \lrcorner \alpha.$$

This can be considered as a mapping

$$X \lrcorner : \Omega^r(M) \to \Omega^{r-1}(M).$$

A characteristic property of the Hamiltonian vector field X_H is that

$$X_H \lrcorner \omega = \sum \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i = dH.$$

Hence a curve γ on T^*Q satisfies Hamilton's equations if it is an integral curve to a vector field X such that

$$X \lrcorner \omega = dH. \tag{2}$$

Finally, we briefly consider the *Poisson interpretation.*

Definition 3.3. *Let f and g be two smooth functions with coordinates (q, p), then the* Poisson bracket *is defined as*

$$\{f, g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \tag{3}$$

Then Hamilton's equations become

$$\nabla_p H = \{q, H\},\ \nabla_q H = \{p, H\}.$$

Indeed, we have

$$\{q_j, H\} = \sum_{i=1}^{n} \frac{\partial q_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial q_j}{\partial p_i} \frac{\partial H}{\partial q_i} = \frac{\partial H}{\partial p_j}$$

and

$$\{p_j, H\} = \sum_{i=1}^{n} \frac{\partial p_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_j}{\partial p_i} \frac{\partial H}{\partial q_i} = -\frac{\partial H}{\partial q_j}.$$

We can also express the Poisson bracket in terms of Hamiltonian vector fields.

Definition 3.4. *Let $f, g \in C^\infty(M)$, and let X_f be the Hamiltonian vector field associated with f. Then the Poisson bracket is*

$$\{f, g\} := X_g(f).$$

Lemma 3.5. *The Poisson bracket can be expressed in terms of the symplectic form as*

$$\omega(X_f, X_g) = \{f, g\}.$$

Proof. We have

$$\{f, g\} = X_g(f) = df(X_g) = (X_f \lrcorner \omega)(X_g) = \omega(X_f, X_g).$$

□

One can furthermore show that

$$[X_f, X_g] \lrcorner \omega = -d\{f, g\},$$

(see Berndt [2]) from which it follows that

$$[X_f, X_g] = X_{\{g,f\}}. \tag{4}$$

We also note that the space $C^\infty(M)$ can be equipped with the structure of a Lie algebra.

Proposition 3.6. *The space $C^\infty(M)$ equipped with the Poisson bracket forms a Lie algebra.*

Proof. We know of course already that $C^\infty(M)$ is a vector space, so we only have to verify that the Poisson bracket is a Lie bracket. Bilinearity follows immediately from the definition of Hamiltonian vector fields. By lemma 3.5, skew-symmetry follows from skew-symmetry of ω. The Jacobi identity for the Poisson bracket can be written

$$d\omega(X_f, X_g, X_h) = 0$$

and thus follows from the fact that ω is closed. □

3.3 Quantisation

When the objects under consideration are small enough classical mechanics is no longer accurate. It then becomes necessary to introduce *quantum mechanics*. A state in a quantum mechanical system is represented by a one-dimensional subspace of a Hilbert space $\mathcal{H}$, and the observables are self-adjoint operators on $\mathcal{H}$.

In physics, *quantisation* is a way of assigning a self-adjoint operator on a Hilbert space to a classical observable. The operators on this Hilbert space become the observables in the associated quantum mechanical system. Quantisation is however a purely mathematical process. The inspiration for quantisation may come from physics, but there are also purely mathematical applications. In this thesis we will see how to use quantisation to find irreducible representations of Lie groups.

There are, of course, many ways to simply assign an operator to a function and so, some restrictions are in order. Dirac formulated conditions which guarantee that the assigned operators form a Hilbert space representation of the classical observables.

Definition 3.7 (The Dirac quantisation conditions)**.** *The assignment of an operator $\hat{a}$ to a classical observable $a : M \to \mathbb{R}$ should satisfy the following conditions.*

(Q1) The map $a \mapsto \hat{a}$ is linear;

(Q2) If a is constant, then $\hat{a}$ is the corresponding multiplication operator;

(Q3) $[\hat{a}_1, \hat{a}_2] = -ih\widehat{\{a_1, a_2\}}$,

where $[\cdot,\cdot]$ *is the commutator,* $\{\cdot,\cdot\}$ *is the Poisson bracket and* h *is Planck's constant.*

We remark that the fact that Planck's constant appears in $(Q3)$ only matters for physical applications. When we make use of quantisation to find representations of Lie groups we will assume that $h = 1$.

As an example we consider the Weyl quantisation. Let $a(q,p)$ be a function on the phase space T^*M. The Weyl quantisation of $a(q,p)$ is the operator

$$a(q,p) \overset{\text{Weyl}}{\longmapsto} a^W(q,D)f(q) = \frac{1}{(2\pi h)^n} \iint a\left(\frac{q+\xi}{2}, p\right) e^{i(q-\xi)p/h} f(\xi)d\xi dp.$$

If $a(q,p)$ is real-valued, the operator $a^W(q,D)$ is self-adjoint in the sense that if

$$\langle f, g\rangle_{L^2} = \int f\bar{g}dq,$$

then

$$\left\langle a^W(q,D)f, g\right\rangle_{L^2} = \left\langle f, a^W(q,D)g\right\rangle_{L^2}.$$

Let H be a Hamiltonian, then Weyl quantisation yields the corresponding self-adjoint operator H^W. In Schrödinger's picture of quantum mechanics the time-development of wave functions is given by $\psi(t) = U(t)\psi_0$, where $U(t)$ is the unitary operator

$$U(t) = e^{-\frac{it}{h}H^W}.$$

Taking the time-derivative yields *Schrödinger's equation*

$$ih\frac{\partial\psi}{\partial t} = H^W\psi.$$

This concludes our inspiring, though brief, tour through the realm of physics. There is a lot to be said about the difficulties that arise when one is trying to construct a physically feasible quantisation, but we will not concern ourselves with that here. Instead we turn our attention to geometric quantisation and its applications to pure mathematics.

4 Prequantisation

Our goal is to construct unitary representations of Lie groups. For that we need first of all a representation space. Furthermore, we need to put restrictions on this representation space so that it becomes small enough. N. Woodhouse gives a nice discussion on what spaces are appropriate to consider in the chapter on prequantisation in [3]. We will construct representation spaces by first constructing line bundles over symplectic manifolds and consider smooth sections of these bundles.

The arising Hilbert spaces are far to big for our purposes, but we set that problem aside for the next section.

4.1 Vector Bundles

Definition 4.1. *A* vector bundle *of rank n over a manifold M is a triple*

$$E \xrightarrow{\pi} M$$

where E is called the total space, π is a (differentiable) projection map, and each fiber *$E_x = \pi^{-1}(\{x\})$, for $x \in M$, is an n-dimensional (real) vector space. Furthermore, E is required to be locally trivial, i.e. for each $x \in M$ there exists a neighborhood U and a diffeomorphism*

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

such that for every $y \in U$

$$\varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^n$$

is a vector space isomorphism. A pair (φ, U) of such a diffeomorphism and neighborhood is called a bundle chart*.*

In the definition above we can replace all $\mathbb{R}^n$ with $\mathbb{C}^n$ to get *complex vector bundles.* In the special case when $n = 1$, a vector bundle is called a *line bundle.* In this thesis a *line bundle* will mean a *complex* line bundle, unless stated otherwise.

Let $E \xrightarrow{\pi} M$ be a vector bundle of rank n, cover M with open sets $\{U_j\}_{j\in J}$ over which E is trivial and let

$$\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^n$$

be the local trivializations. Then on $U_j \cap U_k \neq \emptyset$ we have *transition maps*

$$\psi_{kj} : U_j \cap U_k \to \mathrm{GL}(n, \mathbb{R})$$

defined by the relation

$$\varphi_k \circ \varphi_j^{-1}(x, v) = (x, \psi_{kj}(x)v).$$

Given the transition maps, we can reconstruct the vector bundle. Indeed, we have the following theorem (see Jost [4]).

Theorem 4.2.

$$E = \bigcup_{j\in J} (U_j \times \mathbb{R}^n)/\sim$$

where the union is disjoint, and the equivalence relation $\sim$ is defined by

$$(x, v) \sim (y, w) \Leftrightarrow x = y \text{ and } w = \psi_{kj} v.$$

Again, we may exchange the $\mathbb{R}^n$ for $\mathbb{C}^n$ in the case of complex vector bundles.

Given vector spaces V and W, we can construct the dual space V^*, the direct sum $V \oplus W$, and the tensor product $V \otimes W$. By doing these constructions fiberwise, we can construct new vector bundles from existing ones. Note that the possibility of constructing the dual space and the tensor product gives us a bundle of homomorphisms, namely

$$\mathrm{Hom}(V, W) = W \otimes V^*.$$

A *section* of a vector bundle $E \xrightarrow{\pi} M$ is a map

$$s : M \to E$$

such that $\pi(s(x)) = x$, for all $x \in M$.

Remark 4.3. *Note that by a* section *we will always mean a* smooth section*, unless stated otherwise.*

The space of sections of E is denoted $\Gamma(E)$. We will use the special notation

$$\Gamma(M) := \Gamma(TM)$$

for the space of vector fields on M. We will also use the notation

$$\Omega^r(E) = \Gamma(E \otimes \Lambda^r(T^*M))$$

for E-valued r-forms and the special notation

$$\Omega^r(M) := \Omega^r(T^*M)$$

for differential r-forms on M. Recall that for any $X, Y \in \Gamma(M)$, we have the *Lie bracket*

$$[X, Y] := X \circ Y - Y \circ X.$$

For differential forms $\theta \in \Omega^r(M)$, we have the *Lie derivative*

$$\mathcal{L}_X \theta := X \lrcorner d\theta + d(X \lrcorner \theta).$$

The exterior derivative of an r-form θ can be written explicitly as

$$\begin{aligned} d\theta(X_0, \cdots, X_r) &= \sum_{i=0}^{r} (-1)^i X_i \theta(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_r) \\ &+ \sum_{0 \leqslant i < j \leqslant r} (-1)^{i+j} \theta([X_i, X_j], X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_r). \end{aligned}$$

In particular, for $\theta \in \Omega^2(M)$ we have the formula

$$\begin{aligned} d\theta(X,Y,Z) &= X\theta(Y,Z) - Y\theta(X,Z) + Z\theta(X,Y) \\ &\quad - \theta([X,Y],Z) + \theta([X,Z],Y) - \theta([Y,Z],X). \end{aligned} \tag{5}$$

On a vecor bundle we can define, loosely speaking, a way to relate the fibers with each other.

Definition 4.4. *A* connection ∇ *on a vector bundle* $E \overset{\pi}{\to} M$ *is a linear map*

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$$

such that for all $f \in C^\infty(M)$ *and* $s \in \Gamma(E)$

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

This extends to a map

$$\nabla : \Omega^r(E) \to \Omega^{r+1}(E),$$

such that for all $s \in \Gamma(E)$ and $\alpha \in \Omega^r(M)$

$$\nabla(s \otimes \alpha) = (\nabla s) \wedge \alpha + s \otimes d\alpha.$$

Using this extension we can define the curvature of a connection.

Definition 4.5. *The* curvature *of a connection* ∇ *is the operator*

$$\nabla^2 = \nabla \circ \nabla : \Gamma(E \otimes T^*M) \to \Omega^2(E).$$

The curvature ∇^2 is a 2-form with values in $\mathrm{End}(E)$, i.e. a section of the bundle

$$\Lambda^2(T^*M) \otimes \mathrm{End}(E).$$

If $X, Y \in \Gamma(M)$ and $s \in \Gamma(E)$, we may write

$$\nabla^2(X,Y)s = [\nabla_X, \nabla_Y]\, s - \nabla_{[X,Y]} s. \tag{6}$$

Line bundles is of special interest for us, so we will recall a few important properties, many of these (with slight conventional modifications) can be found in Woodhouse [3], but we give significantly more details here. On a line bundle $L \to M$, any local trivialization (U, φ) is determined by the section

$$\sigma = \varphi^{-1}(\cdot, 1).$$

In this trivialization, the *potential 1-form* $\theta \in \Omega^1(U)$ of the connection is characterized by

$$\nabla\sigma = \theta\sigma.$$

Every section on L is of the form

$$s = f\sigma,$$

where f is a complex-valued function. Hence the so called *covariant derivative* is given by

$$\nabla_X s := X \lrcorner \nabla s = (X(f) + X \lrcorner \theta f)\sigma.$$

If we identify a section $s = f\sigma$ with the function f we may write

$$\nabla = d + \theta.$$

Then since

$$(d+\theta)(df + f\theta) = d^2 f + \theta \wedge df + f d\theta + df \wedge \theta + \theta \wedge \theta = d\theta$$

the curvature of ∇ is

$$\nabla^2 = d\theta$$

and we can easily verify formula (6) in this case. Indeed, we have

$$\begin{aligned}
&[\nabla_X, \nabla_Y]\, s - \nabla_{[X,Y]} s \\
&= (X+\theta(X))(Y(f)+\theta(Y)f)\sigma - (Y+\theta(Y))(X(f)+\theta(X)f)\sigma - ([X,Y]\,(f)+\theta([X,Y])f)\sigma \\
&= (X(\theta(Y))f + \theta(Y)X(f) + \theta(X)Y(f) - Y(\theta(X))f + \theta(X)Y(f) + \theta(Y)X(f) - \theta([X,Y])f)\sigma \\
&\qquad = (X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]))s = d\theta(X,Y)s.
\end{aligned}$$

Given two trivializations (U_j, φ_j) and (U_k, φ_k) with corresponding sections σ_j and σ_k the transition function c_{jk} is the function such that

$$\sigma_k = c_{jk}\sigma_j$$

on $U_j \cap U_k$.

We notice that the endomorphism bundle of a line bundle $L \to M$ is trivial, i.e. $\mathrm{End}(L) \cong \mathbb{C} \times M$. Knowing this, together with the natural identification

$$\mathrm{End}(L) = L \otimes L^*$$

we see that L^* is an "inverse w.r.t. the tensor product". This fact can also be proven in terms of transition functions. Indeed, if the transition functions of L are φ_{ij}, then the transition functions of L^* are φ_{ij}^{-1} and the transition functions of $L \otimes L^*$ are

$$\varphi_{ij}\varphi_{ij}^{-1} = 1.$$

Given a curve $\gamma : [0,1] \to M$, we can lift γ to a *section along* γ, i.e. a section

$$s : [0,1] \to L$$

such that $\pi(s(t)) = \gamma(t)$. Suppose γ has tangent vector $\dot{\gamma}$. Then the covariant derivative is

$$\nabla_{\dot{\gamma}} s = \left(\frac{df}{dt} + \dot{\gamma} \lrcorner \theta \right) s,$$

(where $s = f\sigma$). If $\nabla_{\dot{\gamma}} s = 0$, then s is said to be *parallel* along γ. A parallel section along γ is uniquely determined by its initial value $s(0)$, and this section is called the *parallel transport* along γ.

Definition 4.6. *Let $\gamma : [0,1] \to M$ be a closed curve and suppose that s is parallel along γ. The* holonomy *of the connection ∇ is the complex number $\xi \in \mathbb{C}$ such that*

$$s(1) = \xi s(0).$$

The holonomy is independent of the parametrization of γ, the section s and the choice of base point $\gamma(0)$. Furthermore, if there exists a smooth surface $S \in M$ such that $\gamma = \partial S$, then

$$\xi = e^{i \int_S d\theta}.$$

Definition 4.7. *An* Hermitian structure *on a line bundle L is an Hermitian inner product $(\cdot, \cdot)$ on each fiber such that the function*

$$L \to \mathbb{C}, \ v \mapsto (v, v)$$

is smooth.

An Hermitian structure is said to be *compatible* with a connection ∇ if for all sections s and s', and for all (real) vector fields X, we have

$$X \lrcorner d(s, s') = (\nabla_X s, s') + (s, \nabla_X s').$$

If L has local trivializations (U_j, φ_j) such that the transition functions have unit modulus, i.e.

$$|c_{jk}| = 1$$

and the potential forms θ_j satisfy $\theta_j = -\bar{\theta}_j$, then L can be equipped with an Hermitian structure compatible with ∇ such that

$$(\sigma_j, \sigma_j) = 1$$

for all j. Indeed,

$$(\sigma_k, \sigma_k) = |c_{jk}|(\sigma_j, \sigma_j)$$

and

$$(\nabla_X \sigma_j, \sigma_j) + (\sigma_j, \nabla_X \sigma_j) = (X \lrcorner \theta_j + X \lrcorner \bar{\theta}_j) = 0 = X \lrcorner d(\sigma_j, \sigma_j).$$

The vector bundle $\mathrm{End}(L)$ has the subbundle $\mathrm{End}(L, (\cdot, \cdot))$ consisting of skew-Hermitian maps and we have

$$\mathrm{End}(L, (\cdot, \cdot)) \cong M \times i\mathbb{R}.$$

If the connection ∇ is compatible with an Hermitian structure, then the curvature 2-form of ∇ takes its values in $\mathrm{End}(L, (\cdot, \cdot))$. Hence, the 2-form

$$-i\nabla^2 \in \Omega^2(M)$$

is just a a real-valued 2-form.

4.2 The Prequantum Line Bundle

We will construct a special line bundle over a symplecic manifold (M, ω). This Hermitian line bundle will have a connection ∇ whose curvature 2-form is equal to $-i\omega$. Moreover the Hermitian structure of the line bundle and ∇ are to be compatible. This line bundle is called a *prequantum line bundle*. It is possible to formulate precisely not only when a prequantum line bundle exists, but also how to parametrize the inequivalent choices of such bundles. However, we begin by giving a seemingly slightly weaker condition.

Definition 4.8. *A symplectic manifold (M, ω) is said to be* quantisable*, if the form ω is integral. The integrality condition means that for any closed, oriented 2-surface $S \subset M$*

$$\int_S \omega \in 2\pi\mathbb{Z}.$$

To investigate what this means, we begin with a lemma due to H. Poincaré.

Lemma 4.9. *If $U \subset M$ is open and contractible, then every closed r-form on U is exact.*

Let us try to directly construct a prequantum line bundle. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover of M such that each U_j is contractible. Then on each U_j, there is a 1-form β_j, such that

$$\omega|_{U_j} = d\beta_j.$$

Also, on $U_j \cap U_k \neq \emptyset$ which are contractible we have

$$d(\beta_j - \beta_k) = \omega|_{U_j \cap U_k} - \omega|_{U_j \cap U_k} = 0,$$

so $\beta_j - \beta_k$ is closed and therefore exact on $U_j \cap U_k$. Thus there exists functions $f_{jk} : U_j \cap U_k \to \mathbb{R}$ such that

$$df_{jk} = \beta_j - \beta_k.$$

Note that $f_{jk} = -f_{kj}$. On contractible intersections $U_j \cap U_k \cap U_l \neq \emptyset$ we define a function a_{jkl} by

$$f_{jk} + f_{kl} + f_{lj} = 2\pi a_{jkl}.$$

However, we notice that

$$da_{jkl} = \frac{1}{2\pi} d(f_{jk} + f_{kl} + f_{lj}) = \frac{1}{2\pi}(\beta_j - \beta_k + \beta_k - \beta_l + \beta_l - \beta_j) = 0$$

so that a_{jkl} is actually constant. The statement that M is quantisable now simply means that $a_{jkl} \in \mathbb{Z}$ (to see why, we need to introduce Čech cohomology groups, see definition 4.13).

Now let

$$c_{jk} = e^{if_{jk}},$$

so that

$$dc_{jk} = ic_{jk}(\beta_j - \beta_k)$$

and for $a_{jkl} \in \mathbb{Z}$ on $U_j \cap U_k \cap U_l \neq \emptyset$, we have

$$c_{jk}c_{kl}c_{lj} = e^{i(f_{jk}+f_{kl}+f_{lj})} = e^{2\pi i a_{jkl}} = 1.$$

Hence, c_{jk} are transition functions on some line bundle $\mathcal{B}$ over M, which we can reconstruct from these functions. Suppose now that M is simply connected. Take some point in M and choose a closed curve γ passing through this point. Since M is simply connected, there exists a surface S such that $\partial S = \gamma$. Assume that S is contained in U_j for some j (if it is not, we simply choose a subdivision of S so that each piece is contained in some U_j). Then

$$e^{i \oint_\gamma \beta_j} = e^{i \int_S \omega}$$

and we can use this to reconstruct a connection ∇ on $\mathcal{B}$ with curvature ω, since the holonomy of such a connection is

$$\xi = e^{i \int_S \omega}.$$

Furthermore, since $|c_{jk}| = 1$ for all j, k and the forms β_j are real, there exists an Hermitian structure on $\mathcal{B}$ which is compatible with ∇. The bundle $\mathcal{B}$ with this connection ∇ and compatible Hermitian structure is called a *prequantum line bundle*.

The integrality condition as we have stated it is a necessary condition for a prequantum line bundle to exist. However for non-simply connected manifolds, it is much harder to directly construct prequantum line bundles as there are different possibilities for the holonomy of ∇ around loops that cannot be contracted to points. Therefore, we reformulate the integrality condition in terms of cohomology groups which makes things slightly more abstract, but easier to deal with.

4.3 Cohomology

Let us very briefly recall some basic cohomology theory. We begin with *de Rham cohomology*, a cohomology theory based on differential forms. For practicality, we also introduce *Čech cohomology groups* which are based on intersection properties of open covers, as these are generally easier to compute.

4.3.1 de Rham Cohomology

Let M be a smooth manifold and define the *de Rham cochain complex* as

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(M) \to 0$$

where d is the exterior derivative. If we write

$$\cdots \to \Omega^{k-1}(M) \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \to \cdots$$

then the closed k-forms on M is $\ker(d_k)$ and the exact k-forms on M is $\mathrm{Im}(d_{k-1})$. Note that the property $d \circ d = 0$ implies that

$$\mathrm{Im}(d_{k-1}) \subseteq \ker(d_k).$$

Now we can define the de Rham cohomology groups.

Definition 4.10. *The* kth de Rham cohomology group *is defined as*

$$H^k(M) = \ker(d_k)/\mathrm{Im}(d_{k-1}).$$

de Rham cohomology is dual to simplicial homology in the following sense: Let $H_k(M)$ be the kth homology group and let $[c] \in H_k(M)$ be a cycle. Then $H^k(M)$ can be identified with the dual of $H_k(M)$ via the mapping

$$[c] \mapsto \int_c \alpha,$$

for $[\alpha] \in H^k(M)$. This is the content of *de Rham's theorem*. We will denote the subgroup of classes of k-forms $[\alpha]$ such that for any k-cycle c

$$\int_c \alpha \in \mathbb{Z}$$

by $H^k(M, \mathbb{Z})$. This subgroup is (by the *de Rham isomorphism*) isomorphic to the subgroup $\check{H}^k(U, \mathbb{Z})$ of the kth *Čech cohomology group* $\check{H}^k(U, \mathbb{R})$ described below (see Woodhouse [3] for further details).

4.3.2 The Čech Cohomology Groups

Let $U = \{U_i\}_{i\in I}$ be an open cover of M.

Definition 4.11. *A* k-cochain*, relative to* U*, is a collection* $f := \{f_{i_1,\cdots,i_{k+1}}\}$ *of functions* $f_{i_1,\cdots,i_{k+1}}$ *(of some specified type) defined on*

$$U_{i_1} \cap \cdots \cap U_{i_{k+1}}$$

such that

$$f_{i_1,\cdots,i_{k+1}} = \frac{1}{(k+1)!} \sum_{\sigma\in S_{k+1}} \mathrm{sign}(\sigma) f_{\sigma(i_1),\cdots,\sigma(i_{k+1})}$$

and such that f *contains only one function* $f_{i_1,\cdots,i_{k+1}}$ *for each ordered set of* $k+1$ *indices such that*

$$U_{i_1} \cap \cdots \cap U_{i_{k+1}} \neq \emptyset.$$

We define the *Čech coboundary operator*, acting on k-cochains, by

$$\delta f := \{(\delta f)_{i_1,\cdots,i_{k+2}}\},$$

where

$$(\delta f)_{i_1,\cdots,i_{k+1}} := (k+2) \sum_{m=1}^{k+2} (-1)^{m-1} f_{i_1\cdots,i_{m-1},i_{m+1},\cdots,i_{k+2}}|_{U_{i_1}\cap\cdots\cap U_{i_{k+2}}}.$$

Note that δ maps k-cochains to $(k+1)$-cochains and that $\delta^2 = 0$.

Definition 4.12. *A* k*-cochain* f *is called a* k-cocycle *if* $\delta f = 0$ *and called a* k-coboundary *if there exists a* $(k-1)$*-cochain* g *such that* $f = \delta g$*. The groups of* k*-cocycles and* k*-coboundaries are denoted by* $Z^k(U)$ *and* $C^k(U)$*, respectively.*

Definition 4.13. *The* kth Čech cohomology group *is the group*

$$\check{H}^k(U) := Z^k(U)/\delta C^{k-1}(U).$$

If the functions $f_{i_1,\cdots,i_{k+1}}$ are specified to be locally constant, taking values in some abelian group A, we denote the kth Čech cohomology group by

$$\check{H}^k(U, A).$$

4.4 The Integrality Condition

We are now ready to formulate the integrality condition.

Definition 4.14 (The Integrality Condition)**.** *Let* (M,ω) *be a symplectic manifold. Then* ω *is integral iff* $(2\pi)^{-1}\omega$ *determines a class in* $H^2(M,\mathbb{Z})$*.*

This definition allows us to formulate precisely when a prequantum line bundle exists.

Theorem 4.15. *Suppose* (M, ω) *is a symplectic manifold. Then there exists a line bundle* $\mathcal{B} \to M$ *with connection* ∇ *and compatible Hermitian structure such that the curvature is* $-i\omega$ *iff the integrality condition holds.*

In fact, even more can be said about the inequivalent choices of $\mathcal{B}$ and ∇ when M is not simply connected. These are parametrized by the group $\check{H}^1(M, S^1)$. See Woodhouse [3] (*proposition 8.3.1*).

Before we construct the prequantum Hilbert space, let us take a moment to reflect on what we have done so far. Given a symplectic manifold (M, ω) we denote by $\mathcal{P}(M)$ the collection of triples $(L, \nabla, (\cdot, \cdot))$ where L is a line bundle over M with connection ∇ and compatible Hermitian structure $(\cdot, \cdot)$. Then the curvature of ∇ yields a mapping

$$\mathcal{P}(M) \to \Omega^2(M), \ (L, \nabla, (\cdot, \cdot)) \mapsto -i\nabla^2.$$

What we are trying to achieve is to find an "inverse" to this mapping. However, the map is in general not injective and neither is every symplectic form in the image. The integrality condition has given us a (partial) solution to this problem. It tells us when a symplectic form lies in the image of this map.

4.5 The Prequantum Hilbert Space

Now we can construct a first attempt at a representation space, the so called *prequantum Hilbert space*, consisting of certain smooth sections of the prequantum line bundle.

Let (M, ω) be a symplectic manifold of dimension $2n$ and suppose ω satisfies the integrality condition. Then there exists a prequantum line bundle $\mathcal{B} \to M$ with connection ∇ and compatible Hermitian structure $(\cdot, \cdot)$. We will denote by $\Gamma_0(\mathcal{B})$ the space of compactly supported sections of $\mathcal{B}$, i.e.

$$\Gamma_0(\mathcal{B}) := \{s \in \mathcal{B} : \text{supp}(s) \text{ is compact}\}.$$

Consider, for $s_1, s_2 \in \Gamma_0(\mathcal{B})$, the integral

$$\int_M (s_1, s_2)\varepsilon_\omega,$$

where ε_ω is the *Liouville form*

$$\varepsilon_\omega := \frac{1}{n!}\omega^n = \frac{1}{n!}\omega \wedge \cdots \wedge \omega.$$

Since s_1 and s_2 have compact support, we have $(s_1, s_2) \in C_0^\infty(M)$ and hence the integral converges. We equip $\Gamma_0(\mathcal{B})$ with the inner product defined by

$$\langle s_1, s_2\rangle := \int_M (s_1, s_2)\varepsilon_\omega.$$

Definition 4.16. *Given a prequantum line bundle* $(\mathcal{B}, \nabla, (\cdot,\cdot))$*, the Hilbert space*

$$(\mathcal{L}^2_{\mathcal{B}}, \langle\cdot,\cdot\rangle) := \textit{the completion of } (\Gamma_0(\mathcal{B}), \langle\cdot,\cdot\rangle)$$

is called the prequantum Hilbert space*.*

Let $f \in C^\infty(M)$ (i.e. f is a classical observable if M is a phase space), then the operator $\hat{f}$ on $\mathcal{L}^2_{\mathcal{B}}$ corresponding to f is

$$s \mapsto \hat{f}s = i\nabla_{X_f}s + fs. \tag{7}$$

This is the so called *Kostant-Souriau prequantum operator*. It is an unbounded and, as we shall see, symmetric operator defined on a domain $\mathcal{D}(\hat{f}) \subset \mathcal{L}^2_{\mathcal{B}}$.

To see why this is an appropriate assignment of an operator to a function, let us check that it satisfies Dirac's quantisation conditions.

Lemma 4.17. *Let* $\mathcal{B} \to M$ *be a prequantum line bundle. The operator* $\hat{f}$ *satisfies Dirac's quantisation conditions if the map*

$$C^\infty(M) \to \mathrm{Hom}(\Gamma(\mathcal{B}), \Gamma(\mathcal{B})),\ f \mapsto \hat{f}$$

is linear and satisfies the condition

$$\left[\hat{f}, \hat{g}\right] = -i\widehat{\{f,g\}}, \tag{8}$$

for all $f, g, \in C^\infty(M)$*.*

Proof. Since we have assumed that $h = 1$, *(Q1)* and *(Q3)* are clearly satisfied. Since

$$X_f \lrcorner \omega = df$$

it is clear that $X_f = 0$ if f is constant. Hence, for f constant, the operator $\hat{f}$ is just multiplication by f, so *(Q2)* is satisfied. □

Now we can show that the assignment of operator as in (7) is appropriate.

Proposition 4.18. *The map*

$$C^\infty(M) \to \mathrm{Hom}(\Gamma(\mathcal{B}), \Gamma(\mathcal{B})),\ f \mapsto \hat{f} := i\nabla_{X_f} + f$$

satisfies Dirac's quantisation conditions.

Proof. We need to check that the conditions of lemma 4.17 are satisfied. It is clear that the map is linear. Let $f, g \in C^\infty(M)$ be arbitrary functions. We compute

$$\begin{aligned}\left[\hat{f}, \hat{g}\right] &= \left[i\nabla_{X_f} + f, i\nabla_{X_g} + g\right] \\ &= -\left[\nabla_{X_f}, \nabla_{X_g}\right] + i\left[\nabla_{X_f}, g\right] + i\left[f, \nabla_{X_g}\right] + [f, g] \\ &= -\left[\nabla_{X_f}, \nabla_{X_g}\right] + i\left[\nabla_{X_f}, g\right] + i\left[f, \nabla_{X_g}\right]\end{aligned}$$

Consider the expression $\left[\nabla_{X_f}, g\right]$. Let $s \in \Gamma(\mathcal{B})$ be an arbitrary section. Then

$$\begin{aligned}\left[\nabla_{X_f}, g\right] s &= \nabla_{X_f}(gs) - g\nabla_{X_f} s \\ &= g\nabla_{X_f} s + X_f(g)s - g\nabla_{X_f} s \\ &= X_f(g)s \\ &= \{g, f\}s.\end{aligned}$$

In the same manner we see that

$$\left[f, \nabla_{X_g}\right] = \{g, f\}.$$

Since the curvature of ∇ on $\mathcal{B}$ is $-i\omega$, we have for any $X, Y \in \Gamma(M)$

$$-i\omega(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

and hence

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} - i\omega(X, Y).$$

Utilizing this, we find

$$\begin{aligned}\left[\hat{f}, \hat{g}\right] &= \cdots = -\left[\nabla_{X_f}, \nabla_{X_g}\right] + i\left[\nabla_{X_f}, g\right] + i\left[f, \nabla_{X_g}\right] \\ &= -\nabla_{\left[X_f, X_g\right]} + i\omega(X_f, X_g) + 2i\{g, f\} \\ &= -\nabla_{X_{\{g,f\}}} + i\{f, g\} + 2i\{g, f\} \\ &= \nabla_{X_{\{f,g\}}} - i\{f, g\} \\ &= -i\widehat{\{f, g\}}.\end{aligned}$$

□

Moreover, we have the following proposition.

Proposition 4.19. *The map*

$$C^\infty(M) \to \operatorname{Hom}(\Gamma_0(\mathcal{B}), \Gamma_0(\mathcal{B})),\ f \mapsto \hat{f} := i\nabla_{X_f} + f$$

is symmetric, i.e.

$$\langle \hat{f}s_1, s_2\rangle - \langle s_1, \hat{f}s_2\rangle = 0,$$

for all $s_1, s_2 \in \Gamma_0(\mathcal{B})$.

Proof. Since ∇ is compatible with the Hermitian structure $(\cdot,\cdot)$, we have

$$\begin{aligned}\langle \nabla_{X_f} s_1, s_2\rangle + \langle s_1, \nabla_{X_f} s_2\rangle &= \int_M \left[(\nabla_{X_f} s_1, s_2) + (s_1, \nabla_{X_f} s_2)\right]\varepsilon_\omega \\ &= \int_M X_f(s_1, s_2)\varepsilon_\omega.\end{aligned}$$

Since the Lie derivative has the property

$$\mathcal{L}_X(\alpha\wedge\beta) = (\mathcal{L}_X\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_X\beta)$$

for any differential forms α, β and any vector field X, and since

$$\mathcal{L}_{X_f}\omega = X_f \lrcorner d\omega + d(X_f\lrcorner\omega) = d(df) = 0,$$

we have

$$\mathcal{L}_{X_f}\varepsilon_\omega = 0.$$

Hence,

$$\mathcal{L}_{X_f}\left[(s_1,s_2)\varepsilon_\omega\right] = \left[\mathcal{L}_{X_f}(s_1,s_2)\right]\varepsilon_\omega + (s_1,s_2)\mathcal{L}_{X_f}\varepsilon_\omega = \left[\mathcal{L}_{X_f}(s_1,s_2)\right]\varepsilon_\omega = X_f(s_1,s_2)\varepsilon_\omega.$$

By our definition of the Lie derivative, we have for any differential $2n$-form θ on M that

$$\mathcal{L}_X\theta = X\lrcorner d\theta + d(X\lrcorner\theta) = d(X\lrcorner\theta),$$

since $d\theta = 0$. By Stokes theorem, we have

$$\int_M d\left(X_f\lrcorner(s_1,s_2)\varepsilon_\omega\right) = \int_{\partial M} X_f\lrcorner(s_1,s_2)\varepsilon_\omega = 0.$$

Putting things together, we find

$$\begin{aligned}\langle \nabla_{X_f} s_1, s_2\rangle + \langle s_1, \nabla_{X_f} s_2\rangle &= \int_M \left[(\nabla_{X_f} s_1, s_2) + (s_1, \nabla_{X_f} s_2)\right]\varepsilon_\omega \\ &= \int_M X_f(s_1,s_2)\varepsilon_\omega \\ &= \int_M \mathcal{L}_{X_f}\left[(s_1,s_2)\varepsilon_\omega\right] \\ &= \int_M d\left(X_f\lrcorner(s_1,s_2)\varepsilon_\omega\right) = 0.\end{aligned}$$

Now notice that

$$\langle fs_1, s_2\rangle - \langle s_1, fs_2\rangle = f(\langle s_1,s_2\rangle - \langle s_1,s_2\rangle) = 0,$$

so that

$$\langle (i\nabla_{X_f} + f)s_1, s_2\rangle - \langle s_1, (i\nabla_{X_f} + f)s_2\rangle = i(\langle \nabla_{X_f} s_1, s_2\rangle + \langle s_1, \nabla_{X_f} s_2\rangle) = 0.$$

This concludes the proof. □

Remark 4.20. *If the Hamiltonian vector field X_f is complete, i.e. its integral curves $\phi_{X_f}(t)$ is defined for all $t \in \mathbb{R}$, one can extend the operator $\hat{f}$ to a self-adjoint operator by* Stone's theorem. *See Hall* [8].

5 Quantisation

5.1 Polarization

To define polarizations we need a few preparatory definitions.

Definition 5.1. *Let V be a vector space with a symplectic form ω. A subspace L is said to be* Lagrangian *if*

1. *L is* isotropic*, i.e. $\omega(u, v) = 0$ for all $u, v \in L$;*
2. *L is* maximally isotropic*, i.e. for any isotropic L' such that $L \subset L' \subset V$ we have $L = L'$.*

A Lagrangian subspace $L \subset V$ always has dimension $\frac{1}{2}\dim V$.

We may extend this definition to immersed submanifolds as follows.

Definition 5.2. *Let (M, ω) be a symplectic manifold. An immersed submanifold N is said to be* Lagrangian *if for all $p \in M$, its tangent space T_pN is a Lagrangian subspace of T_pM (with the symplectic form ω_p).*

Definition 5.3. *A real* distribution *P on a manifold M is a subbundle $P \subset TM$ of the tangent bundle. A complex distribution is a subbundle $P \subset T^{\mathbb{C}}M$ of the complexified tangent bundle.*

Given a distribution P on M, then a, immersed submanifold $N \subset M$ is called an *integral manifold of P* if for every $p \in N$, we have $T_pN = P_p$. We will denote the space of integral manifolds of M by M/P. This is not necessarily a smooth manifold.

Now we define real polarizations.

Definition 5.4. *A (real) polarization on a symplectic manifold (M, ω) is a real distribution $\mathcal{P} \subset TM$ such that*

1. *each fiber $\mathcal{P}_p$ is a Lagrangian subspace of T_pM;*
2. *$\mathcal{P}$ is* integrable*, i.e. for any $X, Y \in \Gamma(\mathcal{P})$ we have $[X, Y] \in \Gamma(\mathcal{P})$.*

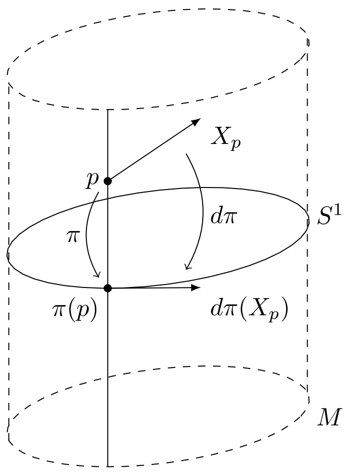

Figure 1: The symplectic manifold $M = T^*\mathbb{S}^1$, the canonical projection mapping π and its differential $d\pi$ acting on a vector.

Note that some authors call a distribution satisfying (2) in the definition above *involutive*. However, by the Frobenius theorem $\mathcal{P}$ is involutive if and only if it is integrable.

Example 5.1. *Consider the unit circle $\mathbb{S}^1$. Let $M = T^*\mathbb{S}^1$ and note that*

$$M \cong \mathbb{S}^1 \times \mathbb{R}.$$

Fix coordinates (θ, x) on M. Then the canonical symplectic form ω is given by

$$\omega = d\theta \wedge dx.$$

Let π be the projection mapping

$$\pi : T^*S^1 \to S^1.$$

Then M has a polarization $\mathcal{P}$ given by

$$\mathcal{P} := \ker(d\pi).$$

In coordinates, the map $d\pi$ is given by

$$d\pi : TM \to TS^1,\ X_p = x_1 \frac{\partial}{\partial\theta}|_p + x_2 \frac{\partial}{\partial x}|_p \mapsto d\pi(X_p) := x_1 \frac{\partial}{\partial\theta}|_p$$

which can be viewed geometrically as in figure 1. Hence the kernel of this map is

$$\ker(d\pi) = \left\{ x_1 \frac{\partial}{\partial\theta}|_p + x_2 \frac{\partial}{\partial x}|_p \in TM : x_1 = 0, x_2 \in \mathbb{R} \right\}.$$

Let us verify that this is indeed a polarization. Fix any $p = (\theta, x) \in M$. Then $\mathcal{P}_p$ is a Lagrangian subspace of T_pM. Indeed it is isotropic, since for any $X_p, Y_p \in T_pM$ we have

$$\omega_p(X_p, Y_p) = d\theta \wedge dx(x_2 \frac{\partial}{\partial x}|_p, y_2 \frac{\partial}{\partial x}|_p) = 0.$$

It is also maximally isotropic, since $\dim T_pM = 2$. *A section $X \in \Gamma(\mathcal{P})$ is given in coordinates by*

$$X = x_2 \frac{\partial}{\partial x},$$

where $x_2 \in C^\infty(M)$. Hence, for any $X, Y \in \Gamma(\mathcal{P})$, we have

$$[X, Y] = 0 \in \Gamma(\mathcal{P})$$

so that $\mathcal{P}$ is indeed a polarization.

This particular example is easy to visualize, but in fact for any manifold N the kernel of the differential of the projection map from the cotangent bundle T^*N is a polarization $\mathcal{P}$ of T^*N. This polarization is called the vertical polarization of T^*N *and in local coordinates*

$$(q_i, p_i) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$$

it is given by

$$\mathcal{P} = \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial p_i} : f_i \in C^\infty(U) \right\}.$$

Not every manifold admit a real polarization. As an example the 2-sphere S^2 has no real polarization since that would imply that it has a nowhere zero vector field, which contradicts the "hairy ball theorem". We will define the more general complex polarizations, but we need the following definition first.

Definition 5.5. *A complex distribution P of a symplectic manifold M is* integrable *if in some neighborhood U of each point, one has*

$$f_{k+1}, \cdots, f_l \in C^\infty(U),$$

where $l =$ rank P and $k = \dim M$, such that their differentials $df_{k+1}, \cdots, df_l$ are independent and annihilate all complex vector fields X such that $X_p \in P_p$ for all $p \in M$.

Definition 5.6. *A* complex polarization *of a symplectic manifold (M, ω) is a complex distribution $\mathcal{P}$ such that*

1. *each fiber $\mathcal{P}_p$ is a Lagrangian subspace of $T^{\mathbb{C}}M$;*
2. *for all $p \in M$, the space $\mathcal{P}_p \cap \bar{\mathcal{P}}_p \cap T_pM$ has constant dimension;*
3. *$\mathcal{P}$ is integrable.*

For any polarization $\mathcal{P}$ of M, we have two special distributions, namely

$$E := \mathcal{P} \oplus \bar{\mathcal{P}} \cap TM$$

and

$$D := \mathcal{P} \cap \bar{\mathcal{P}} \cap TM.$$

Clearly, D is an integrable distribution but E might not be. This calls for the definition of a particular class of polarizations.

Definition 5.7. *A polarization $\mathcal{P}$ of M is called* strongly integrable *if E is integrable.*

Two distinguished types of strongly integrable polarizations are real polarizations and the so called *Kähler polarizations*. We will define and direct our attention to Kähler polarizations in the next section. For more details on the more general case, see e.g. Hall [8].

5.2 The Right Hilbert Space

The key idea behind the construction of the Hilbert space suitable for geometric quantisation is to take the prequantum Hilbert space $\mathcal{L}^2_{\mathcal{B}}$, a polarization $\mathcal{P}$ of M and consider the sections in $\mathcal{L}^2_{\mathcal{B}}$ which are constant along the fibers of $\mathcal{P}$. However, for many polarizations difficulties arise. For example, no such sections may exist. In the general theory of geometric quantisation, this has been treated extensively (see e.g. Lerman [7]). We will consider a class of manifolds particularly well suited for geometric quantisation, namely *Kähler manifolds*.

Recall that a complex structure on a real vector space V is a linear transformation

$$J : V \to V$$

such that $J^2 = -I$. A complex structure J on a manifold M is a complex structure J_p on each T_pM, varying smoothly with p. This may be expressed as the complex distribution P which is spanned by the vector fields

$$X - iJX,\ X \in \Gamma(M)$$

is integrable.

Definition 5.8. *A* Kähler manifold *is a $2n$-dimensional manifold M with a symplectic structure ω and a complex structure J which are compatible in the sense that*

$$g(X, Y) := 2\omega(X, JY),\ X, Y \in \Gamma(M)$$

is a non-degenerate, positive-definite, symmetric tensor.

Any (complex) polarization $\mathcal{P}$ may be equipped with the Hermitian form

$$\langle X_p, Y_p \rangle := -4i\omega(X_p, \bar{Y}_p) \tag{9}$$

and $\mathcal{P}$ is said to be of *type* (r, s) if this Hermitian form is of sign (r, s), i.e. if T_pM has a basis so that the matrix A, where

$$\langle X_p, Y_p \rangle = (X_p)^t A Y_p,$$

is on the form

$$\begin{pmatrix} I_r & \\ & -I_s \end{pmatrix}.$$

Definition 5.9. *A polarization $\mathcal{P}$ of M is a* pseudo-Kähler polarization *if it is of type (r, s), with $r + s = n$. It is a* Kähler polarization *if it is of type $(n, 0)$, i.e. the Hermitian form (9) is positive-definite.*

A Kähler polarization on (M, ω) induces a complex structure J making (M, ω, J) a Kähler manifold. Conversely, any Kähler manifold carries two canonical Kähler polarizations $\mathcal{P}$ and $\bar{\mathcal{P}}$ spanned by

$$\frac{\partial}{\partial z^k}$$

and

$$\frac{\partial}{\partial \bar{z}^k},$$

respectively. These are called the *holomorphic polarization* and the *anti-holomorphic polarization.* If (U, φ) are local coordinates on (M, ω, J), we can find a function $f \in C^\infty(U, \mathbb{R})$ such that

$$\omega|_U = i\partial\bar{\partial} f,$$

and if

$$\theta = -i\partial f,$$

then θ and $\bar{\theta}$ are symplectic potentials adapted to $\mathcal{P}$ and $\bar{\mathcal{P}}$, respectively.

A polarization on (M, ω) singles out a certain class of sections of the prequantum line bundle $\mathcal{B}$.

Definition 5.10. *Let (M, ω) be a symplectic manifold, let $\mathcal{P}$ be a polarization of M, and let $(\mathcal{B}, \nabla, \langle \cdot, \cdot \rangle)$ be a prequantum line bundle. A section $s \in \Gamma(\mathcal{B})$ is said to be* polarized, *if*

$$\nabla_{\bar{X}} s = 0$$

for all complex vector fields X such that $X_p \in \mathcal{P}_p$ for all $p \in M$. We will denote the space of polarized sections by $\Gamma_{\mathcal{P}}(\mathcal{B})$.

Finally, we can construct the sought after Hilbert space. Given a quantizable, symplectic manifold (M, ω), a Kähler polarization $\mathcal{P}$ and a prequantum line bundle

$(\mathcal{B}, \nabla, \langle\cdot,\cdot\rangle)$ yielding the prequantum Hilbert space $\mathcal{L}^2_{\mathcal{B}}$, we define

$$\mathcal{H}_{\mathcal{P}} := \mathcal{L}^2_{\mathcal{B}} \cap \Gamma_{\mathcal{P}}(\mathcal{B}).$$

This is the Hilbert space we want. It consists of "square integrable", polarized sections of the prequantum line bundle. It is indeed a Hilbert space, since it is a closed subspace of a Hilbert space. For further details, see Hall [8].

An interesting special case is when M is compact, then $\mathcal{H}_{\mathcal{P}}$ is finite-dimensional.

We end this section by a brief investigation on which functions $f \in C^\infty(M)$ yield operators on $\mathcal{H}_{\mathcal{P}}$. Consider the space

$$C^\infty_{\mathcal{P}}(M) := \{f \in C^\infty(M) : \left[X_f, \bar{X}\right] \in \Gamma(\mathcal{P}), \text{ for all } X \in \Gamma(\mathcal{P})\}.$$

We'll start by showing that this space is closed under the Poisson bracket and hence forms a Lie algebra.

Proposition 5.11. *For any $f, g \in C^\infty_{\mathcal{P}}(M)$, we have $\{f, g\} \in C^\infty_{\mathcal{P}}(M)$.*

Proof. Let $X \in \Gamma(\mathcal{P})$. Then

$$\left[X, X_{\{f,g\}}\right] = [X, [X_g, X_f]] = [[X, X_f], X_g] + [[X_g, X], X_f],$$

and $[X_g, X], [X, X_f] \in \Gamma(\mathcal{P})$ since $f, g \in C^\infty_{\mathcal{P}}(M)$. Thus, since $\Gamma(\mathcal{P})$ is closed under the Lie bracket, the statement follows. □

Now we can modify proposition 4.18 into a "polarized" version.

Proposition 5.12. *The map*

$$C^\infty_{\mathcal{P}}(M) \to \mathrm{Hom}(\Gamma_{\mathcal{P}}(\mathcal{B}), \Gamma_{\mathcal{P}}(\mathcal{B})),\ f \mapsto \hat{f} = i\nabla_{X_f} + f$$

satisfy the relation

$$\left[\hat{f}, \hat{g}\right] = -i\{f, g\},$$

for all $f, g \in C^\infty_{\mathcal{P}}(M)$.

Proof. In proposition 4.18 we proved the statement for the Lie algebras $C^\infty(M)$ and $\mathrm{Hom}(\Gamma(\mathcal{B}), \Gamma(\mathcal{B}))$. The map we are considering now is simply the restriction to the subalgebra $C^\infty_{\mathcal{P}}(M)$, so all we need to show is that

$$\nabla_{\bar{X}}(\hat{f}s) = 0,$$

for all $X \in \Gamma(\mathcal{P})$, $f \in C^\infty_{\mathcal{P}}(M)$ and $s \in \Gamma_{\mathcal{P}}(\mathcal{B})$. We have

$$\nabla_{\bar{X}}(\hat{f}s) = \nabla_{\bar{X}}(i\nabla_{X_f}s + fs) = i\nabla_{\bar{X}}(\nabla_{X_f}s) + \nabla_{\bar{X}}(fs).$$

First consider the second term. We have

$$\nabla_{\bar{X}}(fs) = (\bar{X}(f)s + f\nabla_{\bar{X}}s) = \bar{X}(f)s,$$

since s is polarized. Next, since $-i\omega$ is the curvature of ∇, we have

$$\nabla_{\bar{X}}(\nabla_{X_f}s) = -i\omega(\bar{X}, X_f)s + \nabla_{X_f}(\nabla_{\bar{X}}s) + \nabla_{[\bar{X},X_f]}s = -i\omega(\bar{X}, X_f)s,$$

because s is polarized and $f \in C^\infty_{\mathcal{P}}(M)$. Finally we write

$$\begin{aligned} -i\omega(\bar{X}, X_f)s &= i(X_f \lrcorner \omega)(\bar{X})s \\ &= idf(\bar{X})s \\ &= i\bar{X}(f)s. \end{aligned}$$

Putting everything together we have

$$\nabla_{\bar{X}}(\hat{f}s) = i\nabla_{\bar{X}}(\nabla_{X_f}s) + \nabla_{\bar{X}}(fs) = -\bar{X}(f)s + \bar{X}(f)s = 0,$$

and we are done! □

Let us summarize what we have done so far. Given a symplectic manifold (M, ω) which is quantisable, that is, for which ω satisfies the integrality condition, we find a prequantum line bundle $\mathcal{B}$ with connection ∇ such that the curvature is $-i\omega$ and a compatible Hermitian structure. The endomorphisms of the sections of $\mathcal{B}$ forms a Lie algebra, and we have a homomorphism of Lie algebras

$$C^\infty(M) \to \operatorname{Hom}(\Gamma(\mathcal{B}), \Gamma(\mathcal{B})),\ f \mapsto \hat{f}.$$

The space of sections of $\mathcal{B}$ which are "square integrable", i.e.

$$\int_M (s, s)\varepsilon_\omega < \infty$$

forms a Hilbert space, the prequantum Hilbert space $\mathcal{L}^2_{\mathcal{B}}$, on which the $\hat{f}$:s act as symmetric operators. To make this space smaller, we introduce a polarization $\mathcal{P}$ on M. In particular we consider M which admit a Kähler polarization and consider the space of polarized sections, i.e. the sections $s \in \Gamma(\mathcal{B})$ for which

$$\nabla_{\bar{X}}s = 0.$$

We then take as our Hilbert space the subspace of $\mathcal{L}^2_{\mathcal{B}}$ consisting of polarized sections. The homomorphism $f \mapsto \hat{f}$ then restricts to a homomorphism of Lie algebras between the space of "polarization preserving" functions on M and the endomorphism space of the polarized sections.

6 The Orbit Method

So far we have given a description of geometric quantisation. Let us now attempt to illustrate the connection between geometric quantisation and the orbit method. We begin by introducing coadjoint orbits, following Berndt [1] but giving significantly more details. Then we consider how geometric quantisation can yield unitary representations and compare this consideration with (essentially) Kirillovs' original approach. Finally, we give two examples; the Heisenberg group which is nilpotent (thus Kirillov's original approach works perfectly) and SU(2) on which we apply geometric quantisation more directly.

6.1 Coadjoint Orbits

Recall the adjoint representation Ad of a Lie group G on its Lie algebra $\mathfrak{g}$ (as defined in example 2.1).

Definition 6.1. *The* coadjoint representation Ad^* *of a Lie group G is the dual representation of the adjoint representation* Ad*, and is given by*

$$\mathrm{Ad}^*(g) := \mathrm{Ad}(g^{-1})^t.$$

The representation space of Ad^* *is* $\mathfrak{g}^*$*, the dual of the Lie algebra* $\mathfrak{g}$.

For any linear functional $\alpha \in \mathfrak{g}^*$, let $< \alpha, \cdot >$ denote its value

$$< \alpha, X > := \alpha(X).$$

Then we have the following description of Ad^*, in terms of Ad

$$< \mathrm{Ad}^*(g)\alpha, X > = < \alpha, \mathrm{Ad}(g^{-1})X > .$$

If $\mathfrak{g}$ is semisimple, then the Killing form B is non-degenerate and we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the map

$$\alpha \mapsto X_\alpha,$$

where X_α is defined by

$$\alpha(Y) = B(X_\alpha, Y), \text{ for all } Y \in \mathfrak{g}.$$

In this case, the description of the coadjoint representation simplifies to

$$\alpha \mapsto \mathrm{Ad}(g^{-1})X_\alpha.$$

From now on, we will often use the abbreviation

$$g \cdot \alpha := \mathrm{Ad}^*(g)\alpha.$$

Definition 6.2. *Let $\alpha \in \mathfrak{g}^*$. The* coadjoint orbit $\mathcal{O}_\alpha$ *of α is the orbit of α as G acts on $\mathfrak{g}^*$ via the coadjoint representation, i.e.*

$$\mathcal{O}_\alpha := G \cdot \alpha.$$

Recall that the stabilizing group is

$$G_\alpha = \{g \in G : g \cdot \alpha = \alpha\}$$

and that we have the diffeomorphism

$$\mathcal{O}_\alpha \cong G/G_\alpha$$

so the coadjoint orbits are homogeneous spaces. If $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ via a non-degenerate bilinear form (as above) then

$$\mathcal{O}_\alpha = \{\mathrm{Ad}(g)X_\alpha : g \in G\}.$$

In particular, if G is a matrix Lie group and we identify $\mathfrak{g}^*$ with (the matrix space) $\mathfrak{g}$, then we have

$$\mathcal{O}_\alpha = \{g^{-1}X_\alpha g : g \in G\}.$$

It is an amazing fact that the coadjoint orbits are symplectic manifolds. We will prove this in the special case of semisimple matrix groups. We refer the reader to Kirillov [6] for a proof of the general case.

Lemma 6.3. *Let G be a Lie group, let $M = \mathcal{O}_\alpha$ be the coadjoint orbit of $\alpha \in \mathfrak{g}^*$ and let $\mathfrak{g}_\alpha$ be the Lie algebra of the stabilizing group. Then the tangent space of M at α is*

$$T_\alpha M \cong \mathfrak{g}/\mathfrak{g}_\alpha.$$

Proof. We have the diffeomorphism

$$M \cong G/G_\alpha.$$

Let

$$\pi : G \to G/G_\alpha$$

be the canonical projection. Notice that $\pi(e) = \alpha$ (where e is the neutral element in G), indeed the canonical projection corresponds to Ad^*. Hence we have a surjective map

$$d\pi_e : \mathfrak{g} \to T_\alpha(G/G_\alpha).$$

Consider the commutative diagram.

$$\begin{array}{ccccc} \ker(d\pi_e) & \hookrightarrow & \mathfrak{g} & \xrightarrow{d\pi_e} & T_\alpha(G/G_\alpha) \\ & & \downarrow & \nearrow_{\varphi} & \\ & & \mathfrak{g}/\ker(d\pi_e) & & \end{array}$$

By the first isomorphism theorem, the map φ is an isomorphism (of vector spaces) and we have

$$\mathfrak{g}/\ker(d\pi_e) \cong T_\alpha(G/G_\alpha).$$

The kernel of π consists of the elements in G such that $g \cdot \alpha = \alpha$, that is

$$\ker(\pi) = G_\alpha.$$

Thus, the kernel of $d\pi_e$ is

$$\ker(d\pi_e) = T_e(\ker(\pi)) = T_e G_\alpha = \mathfrak{g}_\alpha$$

and we have

$$T_\alpha M \cong T_\alpha(G/G_\alpha) \cong \mathfrak{g}/\mathfrak{g}_\alpha.$$

□

Lemma 6.4. *Let $M = \mathcal{O}_\alpha$. The so called* Kirillov-Kostant form

$$\omega_\alpha(X, Y) := \alpha([X, Y]),$$

with $X, Y \in \mathfrak{g}$, is a well-defined, skew-symmetric, bilinear form on $T_\alpha M$.

Proof. If ω_α is well-defined, it is obviously bilinear and skew-symmetric, since the Lie bracket is. By the isomorphism

$$T_\alpha M \cong \mathfrak{g}/\mathfrak{g}_\alpha$$

proven in the previous lemma, we can consider elements of $T_\alpha M$ as elements on the form $X + \mathfrak{g}_\alpha$, with $X \in \mathfrak{g}$. Any representative of the coset $X + \mathfrak{g}_\alpha$ is on the form $X + Z$, for some $Z \in \mathfrak{g}_\alpha$. By the equality

$$\alpha([X + Z, Y]) = \alpha([X, Y]) + \alpha([Z, Y])$$

it is enough to show that

$$[Z, Y] = 0, \text{ for all } Z \in \mathfrak{g}_\alpha,\ Y \in \mathfrak{g}$$

to show that ω_α is well-defined. Recall that

$$[Z, \cdot] = \mathrm{ad}_Z$$

and for all $Z \in \mathfrak{g}_\alpha$, we have

$$\exp(Z) \in G_\alpha.$$

Using the relationship between Ad, Ad*, ad and exp, we have for any $\beta \in \mathfrak{g}^*$

$$(\exp(\mathrm{ad}_Z))^t \beta = (\mathrm{Ad}(\exp(Z)))^t \beta = \mathrm{Ad}^*(\exp(Z)^{-1})\beta = \beta,$$

since of course also $\exp(Z)^{-1} \in G_\alpha$. Thus,

$$\exp(\mathrm{ad}_Z) = \mathrm{id}$$

and so $\mathrm{ad}_Z = 0$. In particular $\mathrm{ad}_Z(Y) = [Z, Y] = 0$ for any $Y \in \mathfrak{g}$. □

Now we will show that the coadjoint orbits are symplectic manifolds in the special case of semisimple matrix Lie groups.

Theorem 6.5. *Suppose G is a semisimple matrix Lie group. The coadjoint orbit $M = \mathcal{O}_\alpha$ is a symplectic manifold.*

Proof. Since M is a homogeneous space, it is enough to show that the form ω_α on $T_\alpha M$ is a symplectic form. By lemma 6.4, the form ω_α is a well-defined 2-form. Since G is semisimple, its Lie algebra $\mathfrak{g}$ has trivial center and thus ω_α is non-degenerate. All we need to show is that ω_α is closed. Denote the differential of Ad* by

$$d\mathrm{Ad}^* =: \mathrm{ad}^*.$$

This is the representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ corresponding to Ad*. Hence, the vector fields $X \in T_\alpha M$ are spanned by

$$\mathrm{ad}^*(X)\alpha$$

for $X \in \mathfrak{g}$. Thus, if we denote the value of α at X by $< \alpha, X >$, we have

$$X\alpha(Y) =< \mathrm{ad}^*(X)\alpha, Y >=< \alpha, -\mathrm{ad}_X Y >= \alpha([Y, X]).$$

Let us utilize this and formula (5) for the exterior derivative of a 2-form to compute

$$\begin{aligned} d\omega_\alpha(X,Y,Z) &= X\omega_\alpha(Y,Z) - Y\omega_\alpha(X,Z) + Z\omega_\alpha(X,Y) \\ &\quad - \omega_\alpha([X,Y],Z) + \omega_\alpha([X,Z],Y) - \omega_\alpha([Y,Z],X) \\ &= X\alpha([Y,Z]) - Y\alpha([X,Z]) + Z\alpha([X,Y]) \\ &\quad - \alpha([[X,Y],Z]) + \alpha([[X,Z],Y]) - \alpha([[Y,Z],X]) \\ &= \alpha([[Y,Z],X]) - \alpha([[X,Z],Y]) + \alpha([[X,Y],Z]) \\ &\quad - \alpha([[X,Y],Z]) + \alpha([[X,Z],Y]) - \alpha([[Y,Z],X]) \\ &= 0. \end{aligned}$$

□

Remark 6.6. *The Kirillov-Kostant form is sometimes written*

$$\omega_\alpha(\tilde{\alpha})(\xi_X(\tilde{\alpha}), \xi_Y(\tilde{\alpha})) = \tilde{\alpha}([X,Y]), \text{ for } X, Y \in \mathfrak{g},\ \tilde{\alpha} \in \mathcal{O}_\alpha, \tag{10}$$

where ξ_X is the vector field associated to X acting of smooth functions f on $\mathcal{O}_\alpha$ as a differential operator defined by

$$(\xi_X f)(\tilde{\alpha}) := \frac{d}{dt}(f(\mathrm{Ad}^*(\exp(tX))\tilde{\alpha})),$$

(see Berndt [1], section 8.2.4*). This is often conventient for computations.*

6.2 Unitary Representations from Coadjoint Orbits

Suppose we have a Lie group G, and have determined the coadjoint orbits $\mathcal{O}_\alpha$. Finding unitary representations of G now amounts to performing geometric quantisation on "admissible orbits". In this context, admissible of course means quantisable, i.e. the Kirillov-Kostant form ω_α must satisfy the integrality condition. Take an integral orbit $\mathcal{O}_\alpha$. Equip $\mathcal{O}_\alpha$ with a suitable polarization. Take the arising Hilbert space of square-integrable, polarized sections of a prequantum line bundle over $\mathcal{O}_\alpha$ as a representation space. The polarization preserving functions yield symmetric operators via the Kostant-Souriau prequantum operator, which is a homomorphism of Lie algebras. This will give us a symmetric representation of the Lie algebra $\mathfrak{g}$ and lifting this to G gives us a unitary representation of G.

If G is not semisimple, it is not as obvious how to translate between geometric quantisation and the orbit method. For the more general case we take the viewpoint of induced representations.

We give an algorithmic description of the orbit method from two points of view.

The Orbit Method from the viewpoint of Geometric Quantisation

To simplify this description we restrict ourselves to connected, simply connected, compact Lie groups.
Step 1: Take a semisimple Lie group G with Lie algebra $\mathfrak{g}$. Find the coadjoint orbits $\mathcal{O}_\alpha$ and determine the Kirillov-Kostant form ω_α on $\mathcal{O}_\alpha$.
Step 2: Determine which of the coadjoint orbits $\mathcal{O}_\alpha$ are quantisable.
Step 3: Choose a (Kähler) polarization $\mathcal{P}$ of $\mathcal{O}_\alpha$.
Step 4: Construct a prequantum line bunde $\mathcal{B}$ over $\mathcal{O}_\alpha$ and determine the Hilbert space $\mathcal{H}_\mathcal{P}$ of square-integrable, polarized sections of $\mathcal{B}$.
Step 5: Determine the space $C^\infty_\mathcal{P}(\mathcal{O}_\alpha)$ of polarization preserving functions on $\mathcal{O}_\alpha$ and assign to each $f \in C^\infty_\mathcal{P}(\mathcal{O}_\alpha)$ an operator

$$\hat{f} = i\nabla_{X_f} + f.$$

Step 6: For $n = \dim \mathfrak{g}$, find a collection

$$f_1, \cdots, f_n \in C^\infty_\mathcal{P}(\mathcal{O}_\alpha)$$

such that $\{\hat{f}_j\}$ is a linearly independent set of operators satisfying

$$\left[\hat{f}_i, \hat{f}_j\right] = \sum_{k=1}^{n} c_{ijk} \hat{f}_k,$$

where c_{ijk} are the structure constants of $\mathfrak{g}$. Then $\{\hat{f}_j\}$ gives a symmetric representation π_* of $\mathfrak{g}$ on $\mathcal{H}_\mathcal{P}$.
Step 7: Lift π_* to a unitary representation π of G on $\mathcal{H}_\mathcal{P}$.

The Orbit Method from the viewpoint of Induced Representations

For the more general case we take the point of view of induced representations. This is essentially Kirillov's original approach.
Step 1: For each orbit $\mathcal{O}_\alpha \subset \mathfrak{g}^*$, find a real subalgebra $\mathfrak{n}$ such that

$$< \alpha, [X, Y] >= 0 \tag{11}$$

for all $X, Y \in \mathfrak{n}$. Equivalently (and importantly), the mapping

$$X \mapsto 2\pi i < \alpha, X > \tag{12}$$

defines a one-dimensional representation of $\mathfrak{n}$.
Step 2: If $\mathfrak{n}$ satisfies the condition

$$\dim(\mathfrak{g}/\mathfrak{n}) = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}_\alpha), \tag{13}$$

where $\mathfrak{g}_\alpha$ is the Lie algebra of the stabilizing group, then we say that $\mathfrak{n}$ is a *real algebraic polarization of* α. Analogously, we define a *complex algebraic polarization* by extending α to a linear functional on

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$$

and consider complex subalgebras $\mathfrak{n} \subset \mathfrak{g}^{\mathbb{C}}$ satisfying, as in the real case, the conditions given by (11) and (13). Given an algebraic polarization $\mathfrak{n}$, we say it is *admissible* if for any $g \in G_\alpha$, we have

$$\mathrm{Ad}(g)X \in \mathfrak{n}, \text{ for all } X \in \mathfrak{n}.$$

These algebraic polarizations are directly related to the polarizations we defined earlier in the sense that there is a bijection between the set of G-invariant polarizations of $\mathcal{O}_\alpha$ and the set of admissible algebraic polarizations of α (see Kirillov [5], theorem 5 and theorem 5').

Step 3: Suppose furthermore that for the admissible, complex algebraic polarization $\mathfrak{n}$, we have that

$$\mathfrak{m}^{\mathbb{C}} := \mathfrak{n} + \bar{\mathfrak{n}}$$

is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ such that we have closed subgroups Q and M of G, with Lie algebras $\mathfrak{q}$ and $\mathfrak{m}$, satisfying the relations

$$G_\alpha \subset Q = G_\alpha \mathring{Q} \subset M, \tag{14}$$
$$\mathfrak{n} \cap \bar{\mathfrak{n}} = \mathfrak{q}^{\mathbb{C}}, \tag{15}$$
$$\mathfrak{n} + \bar{\mathfrak{n}} = \mathfrak{m}^{\mathbb{C}}. \tag{16}$$

Step 4: From this point of view, the integrality condition comes out as the following.

Proposition 6.7. *The coadjoint orbit $\mathcal{O}_\alpha$ is quantisable if the one-dimensional representation ρ_α given by (12) integrates to a unitary character χ_Q of Q, i.e.*

$$d\chi_Q = \rho_\alpha.$$

This gives us a unitary representation

$$\pi = \mathrm{Ind}_Q^G \chi_Q$$

of G on the completion of the space of $\varphi \in C^\infty(G)$, such that

$$\varphi(hg) = \Delta_Q(h)^{1/2}\chi_Q(h)\varphi(g), \text{ for all } g \in G, \ h \in G_\alpha$$

and

$$X(\varphi) = 0, \text{ for all } X \in \mathfrak{u},$$

where $\mathfrak{u}$ is defined by the relation

$$\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{u}$$

and Δ_Q is the modular function as defined in definition 2.25.

6.3 Examples

6.3.1 The Heisenberg Group

It has been known since the 1960's that the orbit method produces a perfect correspondence between the set of coadjoint orbits of a connected, simply connected, *nilpotent* Lie group G and its unitary dual $\hat{G}$. Therefore, an appropriate first example of an application of the orbit method to find unitary representations of some group is the *Heisenberg group*, defined as

$$G := \mathrm{Heis}(\mathbb{R}) := \left\{ (a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\},$$

with Lie algebra

$$\mathfrak{g} = \left\{ xE_{12} + zE_{13} + yE_{23} = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Coadjoint orbits of $G = \mathrm{Heis}(\mathbb{R})$

Using the trace form

$$\langle X, Y \rangle := \mathrm{trace}(XY)$$

we can identify $\mathfrak{g}^*$ with the space of strictly lower triangular matrices. The linear functional $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ given by

$$\alpha(xE_{12} + zE_{13} + yE_{23}) = x\alpha_1 + y\alpha_2 + z\alpha_3, \text{ with } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

is represented by

$$[\alpha] = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_1 & 0 & 0 \\ \alpha_3 & \alpha_2 & 0 \end{pmatrix}.$$

In this description of $\mathfrak{g}^*$ the coadjoint representation is given by

$$\mathrm{Ad}^*(g)\alpha = p(g^{-1}\,[\alpha]\,g),$$

where $p(X)$ denotes the strictly lower triangular part of X. Hence, we have

$$[\mathrm{Ad}^*((a,b,c))\alpha] = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_1 + b\alpha_3 & 0 & 0 \\ \alpha_3 & \alpha_2 - a\alpha_3 & 0 \end{pmatrix}. \tag{17}$$

From this we see that we have two families of coadjoint orbits. If $\alpha_3 = 0$, then the stabilizer $G_\alpha = G$ and the orbit $\mathcal{O}_\alpha$ is the point $(\alpha_1, \alpha_2, 0)$ in $\mathbb{R}^3$. If $\alpha_3 \neq 0$, then

$$\left[\mathrm{Ad}^*\left(\left(a, -\frac{\alpha_1}{\alpha_3}, \frac{\alpha_2}{\alpha_3}\right)\right)\alpha\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}$$

and thus the orbit is a plane in $\mathbb{R}^3$, given by

$$\mathcal{O}_\alpha = \{(x, y, \alpha_3) : x, y \in \mathbb{R}\}.$$

It is well-known that the unitary dual of the Heisenberg group can be thought of as the disjoint union

$$\hat{G} = \mathbb{R}^2 \cup (\mathbb{R}\backslash\{0\}),$$

(see Berndt [1], *section 7.4, example 7.11*). Thus, we have a perfect correspondence between the set of coadjoint orbits of G and the unitary dual $\hat{G}$.

On the coadjoint orbits $\mathcal{O}_\alpha$, with $\alpha_3 \neq 0$ all multiples of

$$\omega_0 = dx \wedge dy$$

are symplectic forms, let us determine which one is the Kirillov-Kostant form. Let

$$X = x_1 E_{12} + x_2 E_{23} + x_3 E_{13},\ Y = y_1 E_{12} + y_2 E_{23} + y_3 E_{13} \in \mathfrak{g}.$$

Then we have

$$[X, Y] = (x_1 y_2 - y_1 x_2) E_{13}$$

so by formula (10) we have

$$\omega_\alpha(\tilde{\alpha})(\xi_X(\tilde{\alpha}), \xi_Y(\tilde{\alpha})) = \tilde{\alpha}([X,Y]) = (x_1 y_2 - y_1 x_2)\tilde{\alpha}_3. \tag{18}$$

Recall that the vector fields ξ are given by

$$(\xi_X f)(\tilde{\alpha}) = \frac{d}{dt}(f(\mathrm{Ad}^*(\exp(tX))\tilde{\alpha})).$$

It is easy to compute that

$$\begin{aligned} \exp(tE_{12}) &= I_3 + tE_{12}, \\ \exp(tE_{23}) &= I_3 + tE_{23}, \\ \exp(tE_{13}) &= I_3 + tE_{13} \end{aligned}$$

and by (17) we have

$$\begin{aligned} \mathrm{Ad}^*(\exp(tE_{12}))\tilde{\alpha} &= (\tilde{\alpha}_1, \tilde{\alpha}_2 - t\tilde{\alpha}_3, \tilde{\alpha}_3), \\ \mathrm{Ad}^*(\exp(tE_{23}))\tilde{\alpha} &= (\tilde{\alpha}_1 + t\tilde{\alpha}_3, \tilde{\alpha}_2, \tilde{\alpha}_3), \\ \mathrm{Ad}^*(\exp(tE_{13}))\tilde{\alpha} &= (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3). \end{aligned}$$

Hence, we find that

$$\begin{aligned} \xi_{E_{12}}(\tilde{\alpha}) &= -\tilde{\alpha}_3 \frac{\partial}{\partial y}, \\ \xi_{E_{23}}(\tilde{\alpha}) &= \tilde{\alpha}_3 \frac{\partial}{\partial x}, \\ \xi_{E_{13}}(\tilde{\alpha}) &= 0 \end{aligned}$$

and thus

$$\xi_X(\tilde{\alpha}) = x_1 \xi_{E_{23}}(\tilde{\alpha}) + x_2 \xi_{E_{23}}(\tilde{\alpha}) + x_3 \xi_{E_{13}}(\tilde{\alpha}) = x_2 \tilde{\alpha}_3 \frac{\partial}{\partial x} - x_1 \tilde{\alpha}_3 \frac{\partial}{\partial y}.$$

Plugging these vector fields into the symplectic form ω_0 yield

$$\omega_0(\xi_X(\tilde{\alpha}), \xi_Y(\tilde{\alpha})) = (x_2\tilde{\alpha}_3(-y_1\tilde{\alpha}_3) - y_2\tilde{\alpha}_3(-x_1\tilde{\alpha}_3)) = (x_1y_2 - y_1x_2)\tilde{\alpha}_3^2.$$

By comparing with (18), we see that the Kirillov-Kostant form on $\mathcal{O}_\alpha \cong \mathbb{R}^2$ is

$$\omega_\alpha = \frac{1}{\alpha_3} dx \wedge dy.$$

An interesting feature of the coadjoint orbits is that none of the orbits contain a closed 2-surface, so *all of them are integral.*

Constructing representations

First, for the zero-dimensional orbits corresponding to $\alpha_3 = 0$ there is not much to do. We have $\alpha = (\alpha_1, \alpha_2, 0)$ so the condition

$$\alpha([X, Y]) = 0$$

holds for all $X, Y \in \mathfrak{g}$ and furthermore, the stabilizer $G_\alpha = G$ so the subgroup Q containing G_α must also be all of G. Hence the one-dimensional representation

$$X \mapsto 2\pi i\alpha(X) = 2\pi i(\alpha_1 x_1 + \alpha_2 x_2)$$

integrates to a unitary representation π of G, given by

$$\pi(a, b, c) = e^{2\pi i(\alpha_1 a + \alpha_2 b)}.$$

Now, for each orbit $\mathcal{O}_\alpha$ corresponding to $\alpha_3 \neq 0$, we want to find an algebraic polarization. One such polarization is obviously given by

$$\mathfrak{n} = \operatorname{span}\{E_{23}, E_{13}\}.$$

Indeed, since

$$[E_{23}, E_{23}] = [E_{13}, E_{23}] = [E_{13}, E_{13}] = 0$$

we have

$$\alpha([X, Y]) = 0, \text{ for all } X, Y \in \mathfrak{n}.$$

The stabilizer G_α is the group

$$G_\alpha = \{(0, 0, c) : c \in \mathbb{R}\}$$

and the one-dimensional representation

$$X \mapsto 2\pi i\alpha(X) = 2\pi i\alpha_3 x_3$$

integrates to a unitary character χ_Q of the subgroup Q with Lie algebra $\mathfrak{n}$. Since G is nilpotent, it is unimodular (it is well known that nilpotent groups are unimodular, so we state this without proof). On the stabilizer, the unitary character χ_Q has the form

$$\chi_Q((0, 0, c)) = e^{2\pi i\alpha_3 c}.$$

This leads us to consider functions satisfying

$$\varphi(hg) = \chi_Q(h)\varphi(g), \text{ for all } h \in G_\alpha, g \in G,$$

which takes the form

$$\varphi((a, b, c)) = e^{2\pi i\alpha_3 c}F(a, b),$$

where $F : \mathbb{R}^2 \to \mathbb{C}$ is a smooth function satisfying the polarization condition given by $\mathfrak{n}$, realized as the condition

$$-\frac{\partial}{\partial b}\varphi = -e^{2\pi i\alpha_3 c}\frac{\partial}{\partial b}F(a, b) = 0.$$

Hence, our representation space is functions on the form

$$\varphi((a,b,c)) = e^{2\pi i \alpha_3 c} f(a),$$

for some smooth function f. We recover the so called *Schrödinger representation*

$$f(t) \mapsto \pi_{\alpha_3}((a,b,c))f(t) := e^{2\pi i \alpha_3 (c+bt)} f(t+a)$$

and from the *Stone-von Neumann theorem*, we know that any representation obtained via some other polarization is equivalent to this representation.

6.3.2 SU(2)

Here we will illustrate how the orbit method works in practice by using it to find the unitary representations of SU(2). This is by no means new, as the representation theory of SU(2) is already well understood. However, this serves as a good illustration of the power of the orbit method, since the group SU(2) is particularly nice to work with. Indeed this group is semisimple and compact but non-abelian, therefore it has irreducible representations of dimension greater than one.

Coadjoint orbits of SU(2)

We begin by realising SU(2) as a matrix group in the usual way

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

We will take a few well-known facts about the relation between SU(2) and SO(3) for granted. We collect them in the following lemma:

Lemma 6.8. *The Lie group* SU(2) *is a double cover of the Lie group* SO(3)*, i.e. there exists a 2-1 Lie group homomorphism*

$$\Psi : \mathrm{SU}(2) \to \mathrm{SO}(3).$$

The kernel of Ψ *is* $\{I, -I\}$*, so we have*

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{I, -I\}.$$

The Lie algebras $\mathfrak{su}(2)$ *and* $\mathfrak{so}(3)$ *are isomorphic and furthermore, via the isomorphism*

$$\begin{pmatrix} 0 & -x_2 & x_3 \\ x_2 & 0 & -x_1 \\ -x_3 & x_1 & 0 \end{pmatrix} \mapsto (x_1, x_2, x_3),$$

we have

$$(\mathfrak{so}(3), [\cdot,\cdot]) \cong (\mathbb{R}^3, \times),$$

where $\times$ *denotes the usual vector product in* $\mathbb{R}^3$.

Since SU(2) is semisimple, we may also identify

$$\mathfrak{su}(2)^* \cong \mathfrak{so}(3)$$

via the inner product

$$\langle X, Y\rangle := -\frac{\mathrm{trace}(XY)}{2}.$$

Using this, we see that the coadjoint orbits of SU(2) are

$$\mathcal{O}_\alpha = \{\Psi(g^{-1})x_\alpha\Psi(g) : g \in \mathrm{SU}(2)\},$$

where x_α is the point in $\mathbb{R}^3$ corresponding to α. This correspondence is given as follows. The form α is identified with $X_\alpha \in \mathfrak{so}(3)$ via

$$\alpha(Y) = -\frac{\mathrm{trace}(X_\alpha Y)}{2}, \text{ for all } Y \in \mathfrak{so}(3)$$

and this X_α is identified with x_α via the isomorphism $\mathfrak{so}(3) \to \mathbb{R}^3$.

Since $\Psi(g) \in \mathrm{SO}(3)$ for all $g \in \mathrm{SU}(2)$, we see that the coadjoint orbits are the 2-spheres $\mathbb{S}^2_r \subset \mathbb{R}^3$ with radii $r > 0$ and centres at the origin.

A symplectic structure on $\mathbb{S}^2_r$ is given by the Kirillov-Kostant form on $\mathbb{S}^2_r$, given by

$$(\omega_r)_x(u, v) := \frac{\langle x, u \times v\rangle}{r^2},\ x \in \mathbb{S}^2_r,\ u, v \in T_x\mathbb{S}^2_r.$$

This is just a scaled version of the usual area form on $\mathbb{S}^2_r$

$$dA_x = x \lrcorner (dx \wedge dy \wedge dz) = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

The orbit $\mathbb{S}^2_r$ is quantisable if

$$\int_{\mathbb{S}^2_r} \omega_r \in 2\pi\mathbb{Z}.$$

Indeed, the only closed 2-surface in $\mathbb{S}^2_r$ is the whole of $\mathbb{S}^2_r$ itself. Utilizing that

$$d\omega_r = \frac{3}{r^2} dx \wedge dy \wedge dz$$

we compute

$$\int_{\mathbb{S}^2_r} \omega_r = \int_{\partial B^3_r} \omega_r = \int_{B^3_r} d\omega_r = \frac{3}{r^2}\int_{B^3_r} dx \wedge dy \wedge dz = 4\pi r.$$

Hence, $\mathbb{S}^2_r$ is quantisable if $r \in \frac{1}{2}\mathbb{Z}$.

Kähler structure on $\mathbb{S}^2$

Our aim is to find a Kähler polarization of the coadjoint orbits, which we know are the 2-spheres of half-integer radius. Let us start by finding a Kähler structure on a general 2-sphere $\mathbb{S}^2$ with radius r.

Let $p \in \mathbb{S}^2$ be the "north pole" of the sphere and consider the stereographic projection

$$U_+ := \mathbb{S}^2 \backslash \{p\} \to \mathbb{C},\ (x_1, x_2, x_3) \mapsto z := \frac{x_1 + ix_2}{1 - x_3}.$$

This and the corresponding map given by removing the antipodal point $\bar{p}$

$$U_- := \mathbb{S}^2 \backslash \{\bar{p}\} \to \mathbb{C}$$

forms an atlas on $\mathbb{S}^2$. In these coordinates we may express ω_r as

$$\omega_r = \frac{i}{r(1 + |z|^2)^2} dz \wedge d\bar{z}.$$

Now consider the real function

$$K(|z|^2) := \frac{1}{r} \log\left(1 + |z|^2\right).$$

Note that

$$\bar{\partial} K = \frac{z}{r(1 + |z|^2)} d\bar{z},$$

so that we have

$$i\partial\bar{\partial} K = \frac{i}{r(1 + |z|^2)^2} dz \wedge d\bar{z} = \omega_r.$$

This shows that $\mathbb{S}^2$ is a Kähler manifold and hence admits the holomorphic and anti-holomorphic polarizations spanned by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, respectively.

The Hilbert space

Equip the coadjoint orbit $\mathbb{S}^2_r$, with $r \in \frac{1}{2}\mathbb{Z}$ with the atlas

$$(U_\pm, z_\pm)$$

given by stereographic projection. On $U_+ \cap U_-$ we have a homeomorphism given by

$$\mathbb{C} \cong U_+ \backslash \{\bar{p}\} \to U_- \backslash \{p\} \cong \mathbb{C},\ z \mapsto \frac{1}{z}.$$

As in theorem 4.2, we construct a line bundle over $\mathbb{S}^2_r$ with transition functions

$$\begin{aligned} c_{++}(p) &= 1 \\ c_{--}(p) &= 1 \\ c_{+-}(p) &= \frac{1}{z^{2r}}. \end{aligned}$$

We get the space

$$\mathcal{B} = \left[(U_+ \times \mathbb{C}) \cup (U_- \times \mathbb{C})\right] / \sim \cong (\mathbb{C}^2 \cup \mathbb{C}^2)/ \sim,$$

where the equivalence relation $\sim$ is defined by

$$(p, z) \sim (p', z')$$

iff $p = p'$, $z = z'$ for $p, p' \in U_\pm$, and $p' = \frac{1}{p}$, $z' = \frac{1}{z^{2r}}$ for $p \in U_+$, $p' \in U_-$. Let us consider the sections of this bundle polarized by the anti-holomorphic polarization $\bar{\mathcal{P}}$. We can identify the space of polarized sections $\Gamma_{\bar{\mathcal{P}}}(\mathcal{B})$ with the space of holomorphic complex valued functions on $\mathbb{S}^2_r$. Indeed, the condition

$$\nabla_{\bar{X}} s = \nabla_{\bar{X}}(f\sigma) = 0$$

simply becomes

$$\frac{\partial}{\partial \bar{z}} f = 0$$

because the potential 1-form corresponding to ∇ vanishes along $\bar{\mathcal{P}}$. Furthermore, on the intersections $U_+ \cap U_-$ we have

$$(p, f(p)) \mapsto \left(\frac{1}{p}, f\left(\frac{1}{p}\right)\right) \sim \left(p, p^{2r} f\left(\frac{1}{p}\right)\right)$$

and since $p^{2r} f\left(\frac{1}{p}\right)$ is holomorphic, it must have the form

$$\sum_{k=1}^{2r} a_k p^k.$$

Hence, the Hilbert space of square-integrable, polarized sections of the prequantum line bundle over $\mathbb{S}^2_r$ is the space $P(\mathbb{C})$ of complex polynomials of degree $\leqslant 2r$, with $r \in \frac{1}{2}\mathbb{Z}$.

Unitary representations of SU(2)

The curvature of the prequantum line bundle $\mathcal{B}$ is ω. The potential 1-form θ for the connection ∇ is

$$\theta = -i\partial K = \frac{i\bar{z}}{r(1+|z|^2)}dz.$$

Given some real-valued $f \in C^\infty(\mathbb{S}^2)$, the corresponding Hamiltonian vector field (in stereographic coordinates) is

$$X_f = ir\left(1+|z|^2\right)^2\left(\frac{\partial f}{\partial \bar{z}}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \bar{z}}\right).$$

Plugging X_f into the potential 1-form θ yields

$$\theta(X_f) = -\bar{z}(1+|z|^2)\frac{\partial f}{\partial \bar{z}}.$$

Hence, for an arbitrary section $s = g\sigma$ we have

$$\nabla_{X_f} s = \left(ir\left(1+|z|^2\right)^2\left(\frac{\partial f}{\partial \bar{z}}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}}\right) - \bar{z}(1+|z|^2)\frac{\partial f}{\partial \bar{z}}g\right)\sigma$$

Thus the Kostant-Souriau operator

$$\hat{f} = i\nabla_{X_f} + f$$

becomes in stereographic coordinates

$$\hat{f} = -r\left(1+|z|^2\right)^2\left(\frac{\partial f}{\partial \bar{z}}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \bar{z}}\right) - i\bar{z}(1+|z|^2)\frac{\partial f}{\partial \bar{z}} + f.$$

We now want to find functions f_1, f_2 and f_3 such that

$$\left[\hat{f}_i, \hat{f}_j\right] = c_{ijk}\hat{f}_k, \tag{19}$$

where c_{ijk} are the structure constants of $\mathfrak{su}(2)$. Using that the structure constants of $\mathfrak{su}(2)$ are

$$c_{123} = 2,\ c_{231} = 2,\ c_{312} = 2,$$

and the relation

$$\left[\hat{f}_i, \hat{f}_j\right] = -i\widehat{\{f_i, f_j\}},$$

we find that a condition equivalent to (19) is

$$f_k = \frac{r(1+|z|^2)^2}{2}\left(\frac{\partial f_i}{\partial \bar{z}}\frac{\partial f_j}{\partial z} - \frac{\partial f_i}{\partial z}\frac{\partial f_j}{\partial \bar{z}}\right).$$

A triple of such functions is

$$f_1 = \frac{i(z+\bar{z})}{r(1+|z|^2)},\ f_2 = \frac{z-\bar{z}}{r(1+|z|^2)},\ f_3 = \frac{-i(|z|^2-1)}{r(1+|z|^2)}.$$

We know that the Hilbert space is the space $P(\mathbb{C})$ of complex polynomials with degree less than $2r$. The homogeneous polynomials $P_n(\mathbb{C})$ of degree $n \leqslant 2r$ form an invariant subspace of $P(\mathbb{C})$. Hence, $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3$ form a representation of $\mathfrak{su}(2)$ in $P_n(\mathbb{C})$. Lifting this to the group level gives us all irreducible, unitary representations of SU(2).

References

[1] R. Berndt, *Representations of Linear Groups*, Vieweg, 2007.

[2] R. Berndt, *An Introduction to Symplectic Geometry*, American Mathematical Society, 2001.

[3] N. M. J. Woodhouse, *Geometric Quantization*, Clarendon press, 1997.

[4] J. Jost, *Riemannian Geometry and Geometric Analysis*, Springer, 2005.

[5] A. A. Kirillov, *Lectures on the Orbit Method*, American mathematical society, 2004.

[6] A. A. Kirillov, *Elements of the Theory of Representations*, Springer, 1976.

[7] E. Lerman, *Geometric quantization; a crash course*, 2012. arXiv:1206.2334 [math.SG].

[8] B. C. Hall, *Quantum Theory for Mathematicians*, Springer, 2013.

[9] B. C. Hall, *Lie Groups, Lie Algebras and Representations*, Springer, 2015.

[10] D. A. Vogan, *Review of "Lectures on the Orbit Method" by A. A. Kirillov*, 2005. Bull. Amer. Math. Soc. 42 (2005).