Towards a dispersive determination of the pion transition form factor

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Abstract. We start with a brief motivation why the pion transition form factor is interesting and, in particular, how it is related to the high-precision standard-model calculation of the gyromagnetic ratio of the muon. Then we report on the current status of our ongoing project to calculate the pion transition form factor using dispersion theory. Finally we present and discuss a wish list of experimental data that would help to improve the input for our calculations and/or to cross-check our results.

1 Motivation

This presentation is closely related to the talks by B. Kubis and M. Procura; see also their contributions to these proceedings. The first parts of this presentation are based on [1] to which we refer for further details.

The pion transition form factor is defined by

\[
\int d^4x e^{iq_1 \cdot x} i \langle 0| T j_{\mu}(x) j_{\nu}(0) \pi^0(q_1 + q_2) \rangle = -\varepsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{\pi^0\gamma\gamma}^{\gamma\gamma}(q_1^2, q_2^2)
\]

where

\[
j_{\mu} = e \sum_f Q_f \bar{q}_f \gamma_{\mu} q_f
\]

denotes the electromagnetic current carried by the quarks and \(Q_f\) the electric charge of the quark of flavor \(f\) (in units of the proton charge \(e\)).

Experimentally the pion transition form factor \(F_{\pi^0\gamma\gamma}^{\gamma\gamma}(q_1^2, q_2^2)\) can be probed for various virtualities \(q_1^2\) and \(q_2^2\) of the electromagnetic currents; see also [2] and references therein. For the decay \(\pi^0 \to 2\gamma\) the quantity \(F_{\pi^0\gamma\gamma}^{\gamma\gamma}(0, 0)\) enters. The Dalitz decay \(\pi^0 \to \gamma e^+ e^-\) probes the singly virtual pion transition form factor \(F_{\pi^0\gamma\gamma}^{\gamma\gamma}(q_1^2, 0)\) for small timelike di-electron virtualities \(q_1^2\) satisfying \(4m_e^2 \leq q_1^2 \leq m_{\pi^0}^2\). Larger virtualities of this singly virtual pion transition form factor are probed, e.g., by the production process \(e^+ e^- \to \pi^0\gamma\); here \(s = q_1^2 \geq m_{\pi^0}^2\). The corresponding processes to explore the doubly virtual

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case in the timelike region are $\pi^0 \rightarrow 2e^+e^-$ and $e^+e^- \rightarrow \pi^0 \ell^+\ell^-$ with a lepton $\ell$. Finally, the doubly virtual pion transition form factor in the spacelike region $q_1^2, q_2^2 < 0$ is probed, e.g., by reactions $2e^- \rightarrow \pi^0 2e^-$ and $e^+e^- \rightarrow \pi^0 e^+e^-$ where the pion can be produced via the fusion of two virtual photons.

Like for any quantum-field theoretical quantity there are kinematical regions of the doubly virtual pion transition form factor that are not or not easily experimentally accessible, for instance if $q_1^2$ is spacelike and $q_2^2$ timelike. For the pion transition form factor there is an additional challenge: It is of order $e^2$, i.e. already small at the amplitude level. Observables where the pion transition form factor enters quadratically are therefore very much suppressed. As we will see below, dispersion theory offers a way to circumvent the direct determination of such small quantities.

In general, form factors describe the deviation from a pointlike behavior, i.e. for a pointlike pion the normalized transition form factor $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)/F_{\pi^0\gamma^*\gamma^*}(0, 0)$ would be unity, independent of the virtualities $q_1^2$ and $q_2^2$ of the electromagnetic currents. In other words, form factors encode the information about the composite structure of an object. Thus, from the point of view of hadron physics, the pion transition form factor tells us something about the intrinsic (quark-gluon) structure of the neutral pion. Yet there is a broader interest in this particular form factor beyond the motivation to explore the composite structure of strongly interacting matter. One promising way to look for physics beyond the standard model is to compare experimental results and corresponding standard-model calculations for selected observables where one can achieve high precision both on the experimental side and on the theoretical (standard model) side. A clear (statistically significant) deviation between theory (standard model) and experiment would point towards an additional impact on the respective observable from physical effects beyond the standard model. For this endeavor the gyromagnetic ratio of the muon provides a very promising observable [3, 4]; see also the presentations by B. Kubis, M. Procura, M. Knecht, and F. Jegerlehner in these workshop proceedings. The pion transition form factor is a crucial ingredient to improve the standard-model prediction for this quantity. We note in passing that also the rare decay $\pi^0 \rightarrow e^+e^-$ is sensitive to the pion transition form factor and might have some potential to reveal effects from physics beyond the standard model; see [2, 5] and references therein.

At present, the largest uncertainty in the standard-model prediction for the gyromagnetic ratio of the muon resides in the so-called hadronic vacuum-polarization contribution depicted on the left hand side of figure 1. It turned out that a significant part of this contribution comes from hadronic physics below 2 GeV. Thus perturbative QCD cannot be used to calculate this contribution. Also the non-perturbative regime of the strong interaction is probed here. On the other hand, a phenomenological

![Figure 1.](image-url)
hadronic model might produce a number for the hadronic vacuum-polarization contribution but not a serious theory uncertainty estimate. Given that one wants to test the standard model and not the quality of a phenomenological hadronic model, one has to resort to methods firmly based on QCD or quantum field theory in general. Fortunately, the hadronic vacuum-polarization contribution can be directly related via dispersion theory to the cross section $e^+e^− \rightarrow \text{hadrons}$ [3]. Thus, instead of modeling this contribution with not determinable accuracy one can use data with experimentally known accuracy.

This interrelation between the hadronic vacuum-polarization contribution and hadronic data has triggered significant experimental activities as can be deduced from numerous presentations in these proceedings. As a consequence of these on-going activities it can be expected that the uncertainty in the standard-model prediction originating in the hadronic vacuum-polarization contribution will be significantly reduced in the near future. This development moves the focus to the next hadronic contribution to the gyromagnetic ratio of the muon: the hadronic light-by-light scattering contribution depicted in the middle of figure 1. So far, the size of this contribution has been determined by hybrids of hadronic and quark models and attaching a rather conservative and therefore big model uncertainty to the results [3, 4].

Given the ongoing experimental activities to improve on the direct determination of the gyromagnetic ratio of the muon and on the input for the hadronic vacuum-polarization contribution, it is high time to improve also on the standard-model prediction for the hadronic light-by-light scattering contribution including a reliable uncertainty estimate. In [6–9] a dispersive framework has been proposed to address the hadronic light-by-light scattering contribution. This framework is based on data and fundamental principles of quantum field theory, QED and QCD. It is expected that the numerically most important part of the hadronic light-by-light scattering contribution comes from processes where the hadronic blob in the middle diagram of figure 1 is replaced by the lowest-mass hadronic state, the pion. Such a process, the “pion-pole term”, is sketched by the right diagram of figure 1. Other important parts of the hadronic light-by-light scattering contribution come from processes where the hadronic blob in the middle diagram of figure 1 is replaced by an $\eta$- or $\eta'$-meson or by a pair of pions. These processes are discussed in the contributions of B. Kubis and M. Procura. The central quantity that needs to be determined for the calculation of the pion-pole term is just the doubly virtual pion transition form factor. Note that in the dispersive framework the pion in figure 1 is onshell while the photons in the loops can have arbitrary virtuality; see the corresponding discussion in [6]. This matches with the definition (1) of the pion transition form factor where the pion is an external, i.e. physical state.

2 Dispersion theory

The ambition is to determine via dispersion theory and experimental data the pion-pole term, which constitutes an important part of the hadronic light-by-light scattering contribution to the gyromagnetic ratio of the muon. The aim is a numerical value for this contribution and an associated reliable theory uncertainty estimate. Using dispersion theory provides the advantage that this framework and therefore also the uncertainty estimate is solely based on QED, QCD and fundamental principles of quantum field theory. In turn quantum field theory constitutes the basis of the standard model. Thus an uncertainty estimate based on this data-driven dispersive approach can truly be regarded as a standard-model uncertainty estimate.

Two central aspects of relativistic local quantum field theory are (a) the probabilistic interpretation of scattering amplitudes, implied by quantum theory, and (b) the locality, which ensures the demand from the theory of relativity that information cannot propagate faster than the speed of light. Unitarity and analyticity of scattering amplitudes are directly related to these fundamental aspects (a) and (b), respectively.
Transitions from any initial to any final state can be encoded in the $S$-matrix. Demanding that the total probability to obtain an arbitrary final state must be 1 (probabilistic interpretation of quantum mechanics) leads to the unitarity condition

$$SS^\dagger = 1.$$ (3)

The $S$-matrix contains the possibility that no interaction has taken place. This trivial part can be split off by introducing the $T$-matrix via $S = 1 + iT$. The unitarity condition (3) provides a profound relation for $T$, the optical theorem:

$$2\text{Im}T = TT^\dagger.$$ (4)

Note that this is a matrix equation. For an initial state $A$ and a final state $B$ the relation (4) reads

$$2\text{Im}T_{A\to B} = \sum_X T_{A\to X} T_{X\to B}^\dagger.$$ (5)

The locality requirement of relativistic quantum field theory demands that scattering amplitudes consist of polynomials (from the local interaction vertices), propagators and integrals thereof. The poles of the propagators are related to physical states. Consequently scattering amplitudes are analytic except for poles and cuts, which have a one-to-one correspondence to physical intermediate states. Schematically this can be expressed via the dispersion relation

$$T(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds' \text{Im}T(s') \frac{1}{s' - s - i\epsilon}.$$ (6)

Thus, based on analyticity, one can obtain the whole amplitude from its imaginary part. On the other hand, based on unitarity, one can obtain the imaginary part of the amplitude from other amplitudes on account of (5).

One situation where such a framework is of particular importance is the case where $T_{A\to B}$ is very small. For instance, the pion transition form factor is doubly suppressed by the electromagnetic coupling constant $e$. Here one might identify $A$ with $\gamma^*$ and $B$ with $\gamma^*\pi^0$. Because of the overwhelming strength of the strong interaction the sum of intermediate states $X$ in (5) is dominated by hadrons. Thus, while the amplitude $T_{A\to B}$ is of order $e^2$, the amplitudes $T_{A\to X}$ and $T_{X\to B}$ are both of order $e$, i.e. less suppressed and therefore more easily accessible by experiment.

Of course, there are practical limitations for such a dispersive reconstruction of amplitudes. First of all, one formally needs to know $\text{Im}T$ in (6) up to infinitely large energies. This can never be achieved by experiment. Second, it seems that it is necessary to know/measure the amplitudes $T_{A\to X}$ and $T_{X\to B}$ for all intermediate states $X$. Both problems can be tamed by subtracted dispersion relations and by restricting the attention to not too large values of $|s|$ in (6). In principle, a systematic framework to determine hadronic scattering amplitudes at very low energies is chiral perturbation theory. However, it turned out that the energy regime covered by chiral perturbation theory is not large enough to provide a reliable calculational framework for the pion-pole term (right diagram in figure 1). The dispersive framework is in practice also restricted to low energies, but the results remain reliable in a regime that is larger than the one covered by chiral perturbation theory.

As already spelled out, one can utilize subtracted dispersion relations to suppress the influence of the largely unknown high-energy region and to restrict the sum over all intermediate states $X$ to a

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1In reality a scattering amplitude depends on several kinematical variables. For instance a partial-wave decomposition might boil it down to a one-variable function.

2Note that this is a schematic, oversimplified discussion of dispersion theory. For the full-fledged formalism we refer to [1] and references therein.
manageable amount. For simplicity we specify the scattering amplitude to some extent by studying a transition form factor $F_{BC}$ related to $\langle C|j|B \rangle$ where $j$ denotes a quark current. To make contact with the previous formulae one should identify schematically $A$ from (5) with $j$ and $B$ from (5) with $BC$. Furthermore we assume that the lowest-mass intermediate states $X$ are given by 2-pion states. Schematically a twice-subtracted dispersion relation is given by

$$F_{BC}(s) = F_{BC}(0) + F'_{BC}(0)s + \frac{s^2}{\pi} \int_{4m^2}^{\infty} ds' \frac{T_{BC\pi\pi}(s') F_{\pi\pi}(s')}{(s')^2 (s' - s - i\epsilon)} + \ldots.$$  

(7)

The general idea is to obtain the subtraction constants $F_{BC}(0)$, $F'_{BC}(0)$ from data and/or chiral perturbation theory and the amplitudes $T_{BC\pi\pi}$ and $F_{\pi\pi}$ from data or other dispersion relations. Note that the $s'$ integral starts at the threshold of the 2-pion states. The dots in (7) refer to other intermediate states different from 2-pion states. These additional states produce corresponding $s'$-integrals. It is crucial to understand that these additional states are heavy and therefore contribute only for large values of $s'$. We want to know the transition form factor $F_{BC}$ for small values of $|s|$. Then all contributions from large values of $s'$ are highly suppressed by $\sim (1/s')^3$. This should be compared to the unsubtracted dispersion relation (6) where there is only a $1/s'$ suppression of the high-energy part. Subtractions help to focus on the low-energy information. The price to pay are the subtraction constants that need to be obtained from other sources. In addition, the high-energy behavior of the amplitude is modified. Typically general constraints from scattering theory and asymptotic QCD are at odds with a polynomial growth of the scattering amplitudes. Thus especially for the uncertainty estimates it must be carefully monitored to which extent an incorrect high-energy behavior influences the results for observables.

In (7) all the quantities that we do not know so well — the contributions from other intermediate states and the amplitudes $T_{BC\pi\pi}(s')$ and $F_{\pi\pi}(s')$ for large values of $s'$ — are highly suppressed. In practice one can introduce a high-energy cutoff below which we trust our amplitudes $T_{BC\pi\pi}(s')$ and $F_{\pi\pi}(s')$. Here “trust” means that we can quantify the uncertainties of these amplitudes. Varying this cutoff in a reasonable range and exploring different high-energy extrapolations of the amplitudes allows one to quantify the overall uncertainty of the quantity of interest, namely $F_{BC}(s)$ for low values of $|s|$. In practice, $\sqrt{|s|}$ should be smaller than about 1 GeV, where, for instance, the 4-pion continuum starts to become noticeable.

### 3 First results

For the pion transition form factor one might schematically identify $B$ with $\gamma^*$ and $C$ with $\pi$ in (7). Thus one needs the amplitude $T_{\gamma^*\pi\pi}$ and the pion vector form factor $F_{\pi\pi}$. For both quantities one can set up again a dispersive framework. At low energies the relevant intermediate states are again 2-pion states. Thus the pion vector form factor can be determined from the 2-pion (p-wave) phase shift — and possibly a low-order polynomial accounting for not explicitly considered other intermediate states. These states cannot create a strongly varying function at low energies. Thus a low-order polynomial should be sufficient. The pion phase shifts have been determined from (dispersively improved) data by two different groups [10, 11]. The differences between the two analyses can be used as an uncertainty estimate for the input pion phase shift. The pion vector form factor is also experimentally very well known. It agrees fairly well with the dispersive reconstruction from the pion p-wave phase shift. The data can be used to pin down the low-order polynomial; see [12] for details.

The amplitude $T_{\gamma^*\pi\pi}$ depends on several variables. The 3-pion invariant mass agrees with the virtuality of the photon. The variation of the amplitude in this variable is sensitive to the 3-pion
correlations. In contrast, the 2-pion invariant masses (Mandelstam variables) depend on the 2-pion correlations, i.e. on the pion phase shifts. As already discussed, the pion phase shifts are under control [10, 11]. With this input dispersion theory can be used to determine the dependence of $T_{\gamma 3\pi}$ on the 2-pion invariant masses for any photon virtuality [1, 13]. Physically this accounts for pion rescattering and cross-channel rescattering. What remains to be determined is the dependence of the amplitude $T_{\gamma 3\pi}$ on the photon virtuality, i.e. on the genuine 3-pion correlations. Here “genuine” means the correlations not caused by a succession of cross-channel 2-pion correlations.

![Figure 2. Fit to the cross section $e^+e^- \rightarrow 3\pi$. Figure taken from [1]. There is a tension between different data sets. This causes a significant theory uncertainty in our calculations.](image)

Starting from an amplitude that depends on several variables we have reduced the problem to the determination of a one-parameter function that accounts for the genuine 3-pion correlations. This can be achieved by fitting to a one-variable observable, namely to the energy dependence of the total cross section for the reaction $e^+e^- \rightarrow 3\pi$. Physically it is known that 3 pions with the quantum numbers of a photon are strongly correlated to the $\omega$ and $\phi$ mesons. Thus we parametrize our one-parameter fit function accordingly by dispersively improved Breit-Wigner functions and a polynomial. The quality of our fit can be inspected in figure 2.

![Figure 3. Postdiction for the cross section $e^+e^- \rightarrow \pi^0\gamma$ (probes the timelike pion transition form factor). Figure taken from [1]. Theory uncertainties from different data sets for $e^+e^- \rightarrow 3\pi$ (main uncertainty), different pion phase shifts, other intermediate states than 2\pi neglected.](image)

The present status of our dispersive calculation of the pion transition form factor is that we have determined the singly virtual transition form factor in the timelike and spacelike region for momenta be-
the final aim of our endeavor is to determine with a reliable (and hopefully small) uncertainty estimate the pion-pole term as one important part of the hadronic light-by-light scattering contribution to the standard-model prediction for the gyromagnetic ratio of the muon. Clearly the quality of our results hinges on the quality of the experimental input that we use. We will spend the rest of this presentation to spell out a wish list with interesting observables that could help to improve the input for the dispersive calculations and/or to further scrutinize our framework.

4 Interesting observables

It would improve the input for our dispersive framework [7], if (1) remaining ambiguities in the data for the cross section $e^+e^- \rightarrow 3\pi$ could be resolved, (2) a Dalitz plot for the decay $\omega \rightarrow 3\pi$ could be provided with the same accuracy as already achieved for $\phi \rightarrow 3\pi$, (3) the cross section for the reaction $\gamma\pi \rightarrow 2\pi$ could be provided.

Cross-checks of our approach can be obtained by improved determinations of the transition form factors $\omega \rightarrow \pi^0 e^+ e^-$ and $\phi \rightarrow \pi^0 e^+ e^-$ in the respective full kinematically accessible Dalitz decay regions.

4.1 Resolve ambiguities in the cross section $e^+e^- \rightarrow 3\pi$

As a matter of fact the various data sets for the cross section of the reaction $e^+e^- \rightarrow 3\pi$ are not fully consistent with each other. Fits to different data sets lead to significant uncertainties for our determination of the singly virtual pion transition form factor in the timelike and spacelike region. Other uncertainties are for instance caused by the high-energy extrapolations of the scattering amplitudes. A first impression of the differences induced by the ambiguities in the cross section $e^+e^- \rightarrow 3\pi$ can be deduced from figure 2 where the full line shows a fit to one set of data while the dashed line shows the fit to the data base that essentially contains all data. For instance, the inlay of the $\phi$ region (right

![Figure 4. Prediction for the space-like pion transition form factor. Figure taken from [1]. Theory uncertainties from different data sets for $e^+e^- \rightarrow 3\pi$ (main uncertainty), different pion phase shifts, other intermediate states than $2\pi$ neglected.](image-url)
one of the two inlays) shows visible differences at the $\phi$ peak. For a more detailed discussion of all the uncertainties we refer to [1].

Some uncertainties of the $e^+e^- \rightarrow 3\pi$ cross section data and maybe discrepancies between the data sets might be caused by the extrapolation of the actual 3-pion measurements to the full $4\pi$ angular coverage. In that context we note that the dispersive framework of [13] predicts the angular distribution of the three pions for a given collision energy of the $e^+e^-$ system, see also figure 5 below. Such information might help for cross-checking the experimental acceptance corrections and for the extrapolation to the full $4\pi$ angle.

4.2 Dalitz plot $\omega \rightarrow 3\pi$

The amplitude $T_{\gamma^*3\pi}$ discussed in section 3 is also probed in the decays of vector mesons $V$ to three pions. Here the virtuality of the photon (better to say: of the electromagnetic current) is fixed to the respective mass of the vector meson. If one utilizes a once subtracted dispersion relation, then one can predict the Dalitz decay distribution of the decay $V \rightarrow 3\pi$; see the left hand side of figure 5. The subtraction constant is fixed by the corresponding decay width. For the decay $\phi \rightarrow 3\pi$ it turns out that the data have such an excellent quality — see the top right panel of figure 5 — that a better description of the data can be achieved by a twice subtracted dispersion relation [13]. The two subtraction constants are determined by a combined fit to the decay width and the distribution itself. As we will argue now, we would like to know the values of these subtraction constant pair also for the decay $\omega \rightarrow 3\pi$ and for the reaction $\gamma\pi \rightarrow 2\pi$.

![Figure 5](image)

*Figure 5.* Top, left: Dispersive calculation of the Dalitz decay distribution for $\phi \rightarrow 3\pi$. Figure taken from [13]. Top, right: KLOE data for $\phi \rightarrow 3\pi$ with comparable quality. Figure taken from [14]. Bottom, left: Dispersive calculation of the Dalitz decay distribution for $\omega \rightarrow 3\pi$. Figure taken from [13]. Bottom, right: Missing, i.e. no data with comparable quality for $\omega \rightarrow 3\pi$.

In [1] we have determined the amplitude $T_{\gamma^*3\pi}$ for arbitrary photon virtualities from a once subtracted dispersion relation. Since the considered intermediate states, cf. (5), are 2-pion states, the
variable in the dispersion integral is one of the 2-pion invariant masses, not the 3-pion invariant mass. However, the subtraction constant depends on the 3-pion invariant mass, i.e. on the photon virtuality. As described in the previous section we have related this dependence of the subtraction constant to the cross section of the reaction $e^+e^- \rightarrow 3\pi$.

If one could determine the amplitude $T_{\gamma'\rightarrow 3\pi}$ from a twice instead of a once subtracted dispersion relation, then the unknown high-energy parts would be further suppressed, cf. the discussion around (7). In turn this would reduce our theory uncertainty. But then we would need to get an idea about the dependence of the second subtraction constant on the photon virtuality. So far, we only know the value at one point, where the virtuality coincides with the $\phi$ mass. As discussed above, this information comes from the high-quality Dalitz decay distribution $\phi \rightarrow 3\pi$. If we could deduce the corresponding second subtraction constant also from a comparable high-quality Dalitz decay distribution $\omega \rightarrow 3\pi$ and from the cross section of the reaction $\gamma\pi \rightarrow 2\pi$, then one could interpolate the dependence of this second subtraction constant on the photon virtuality from zero up to the $\phi$ mass.

Unfortunately there exists no $\omega \rightarrow 3\pi$ Dalitz plot with an accuracy comparable to the achievements for $\phi \rightarrow 3\pi$. Actually until recently there was no experimental indication at all that the pions in the decay $\omega \rightarrow 3\pi$ show any final-state interaction. Experimental data could not distinguish between pure phase space and pion rescattering. In [15] first steps beyond pure phase space are documented. Following [13] a parameter $\alpha$ has been introduced to parametrize for the Dalitz plot of $\omega \rightarrow 3\pi$ the main deviation from pure phase space. A vanishing $\alpha$ would signal the absence/smallness of pion rescattering. This possibility has been ruled out by [15]; see figure 6. The experimental value of $\alpha$ is definitely different from zero and lies in the ballpark of recent theory predictions. The dispersive prediction is labeled by “Bonn” in figure 6. The future experimental challenge is to produce an $\alpha$ value with a better accuracy than the one obtained by dispersion theory. Only with such a quality the parameters of a twice subtracted dispersion relation could be pinned down.

![Figure 6. $\omega \rightarrow 3\pi$ — first steps beyond pure phase space. Experimental verification (yellow band) of a Dalitz plot shape beyond pure phase space, i.e. $\alpha \neq 0$. Also shown is a comparison to recent theory predictions for $\alpha$. “Uppsala” refers to [16], “Bonn” to [13], and “JPAC” to [17]. Figure taken from [15].](image)

### 4.3 Cross section $\gamma + \pi \rightarrow 2\pi$

As already noted a remaining piece of information should come from the cross section of the reaction $\gamma + \pi \rightarrow 2\pi$. This could be measured with a pion beam by a Primakoff reaction, for instance by the COMPASS experiment at CERN. Actually such a measurement is interesting for several reasons. For low-momentum pions the interaction strength of $\gamma 3\pi$ is dictated by the chiral anomaly. Needless to say that it is very important to experimentally check clean predictions from QCD. Interestingly one can avoid the experimental challenge to extrapolate data to low pion momenta, i.e. to the reaction threshold [18]. Using dispersion theory and pion phase-shift data, the pion-pion interactions in the reaction $\gamma + \pi \rightarrow 2\pi$ are under control for all energies where additional intermediate states like $4\pi$ can be safely neglected. The only unknown quantity is a subtraction constant which can be identified with the chiral anomaly, up to chiral corrections and uncertainties from the neglected states. As shown in
figure 7 one can use the whole energy range up to the ρ-meson mass to pin down the value of the anomaly.

![Figure 7](image_url)  
*Figure 7. Prediction of the cross section $\gamma + \pi \rightarrow 2\pi$ based on dispersion theory. Solid line: prediction from anomaly; dashed line: size of anomaly scaled up by about 30%. Figure taken from [18].*

Concerning our amplitude $T_{\gamma^* 3\pi}$ we use the chiral anomaly to pin down a subtraction constant. A better procedure would be to use the corresponding physical quantity that includes chiral corrections caused by the non-vanishing light quark masses. With data from the reaction $\gamma + \pi \rightarrow 2\pi$ or maybe from lattice QCD [19] one could improve on this aspect. With higher-quality data at hand one could even use a twice subtracted dispersion relation and obtain additional knowledge and lower theory uncertainties as pointed out in the previous subsection.

### 4.4 Transition form factor $\omega \rightarrow \pi^0 \ell^+ \ell^-$

The decay of a vector meson to $\pi^0 \ell^+ \ell^-$, where $\ell$ denotes a lepton, probes a transition form factor that has the same quantum numbers as the pion transition form factor. Consequently the same dispersive framework can be used to determine the form factors of these electromagnetic transitions from a vector meson to a pion [12]. Experimentally one has to determine the differential decay width as a function of the dilepton mass.

For the decay $\omega \rightarrow \pi^0 \mu^+ \mu^-$ the dispersive results deviate significantly from the data obtained by the NA60 collaboration [20, 21]. On the other hand, recent data for the decay $\omega \rightarrow \pi^0 e^+ e^-$ obtained by the A2 collaboration [22] lie right on top of the dispersive prediction; see figure 8.

If one just looks at the various data sets and their error bars in figure 8 — without the theory lines that might misguide the eye — then one might still get the impression that the data are not in complete contradiction to each other, though the trends differ. Yet it is remarkable that no hadronic theory has been able to fully explain the NA60 data and according to [23] part of these data disagree with general information obtained from QCD and quantum field theory. Thus it would be very illuminating to have additional data that could help to resolve the tension between NA60 on the one side and the dispersive calculation and A2 on the other side.

From a dispersive point of view the input for the transition form factor probed by $\omega \rightarrow \pi^0 \ell^+ \ell^-$ comes from the amplitude $\omega 3\pi$; the reader may just replace $B \rightarrow \omega$ and $C \rightarrow \pi$ in (7). The amplitude $\omega 3\pi$ can also be determined dispersively [13], but it would be good to have an experimental cross-check on this input for the transition form factor. Thus we repeat our call from subsection 4.2 for a high-quality Dalitz plot for the decay $\omega \rightarrow 3\pi$. The situation is somewhat different if one replaces $\omega$ by $\phi$, as we will discuss next.

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3The chiral-anomaly prediction is based on the chiral limit.
We note in passing that in principle the cross section for the reaction $e^+e^- \to \pi^0\omega$ probes the very same transition form factor, albeit in a kinematically different regime. For the energies relevant for this reaction one cannot expect that the relevant intermediate states, cf. (5), are just the 2-pion states. Thus this reaction cannot be used to scrutinize the present dispersive framework, which applies to reaction energies below 1 GeV, not above.

### 4.5 Transition form factor $\phi \to \pi^0 e^+e^-$ in $\rho$-peak region

Another closely related transition form factor is probed by the decay $\phi \to \pi^0\ell^+\ell^-$. A dispersive prediction has been provided in [12], see figure 9. The input for the dispersive calculation is provided by the amplitude $\phi 3\pi$, which can also be determined dispersively. The latter has been cross-checked successfully against high-quality data on the Dalitz plot $\phi \to 3\pi$ [13]. Thus a very firm prediction for $\phi \to \pi^0\ell^+\ell^-$ has been obtained. Of course, it would be interesting to see if one finds for the $\phi$ a deviation between data and theory that resembles the deviation between the NA60 data and theory for the $\omega$ case.

In contrast to the decay $\omega \to \pi^0\ell^+\ell^-$ there is a resonance in the region that is kinematically allowed for the $\phi$ decay — the $\rho$ meson. Of course, most discriminative power between various theory approaches is achieved around the resonance peak, where the differential decay rate turns large. Recently first data on the differential decay width for $\phi \to \pi^0 e^+ e^-$ have been published by the KLOE-2 collaboration [24] and the corresponding transition form factor has been extracted; see the right hand side of figure 9. So far, the data are restricted to the region below the $\rho$-meson peak and do not point to a significant disagreement between theory and experiment. But given the puzzle for the $\omega$ transition form factor it would be highly desirable to explore the full kinematically accessible range of the $\phi \to \pi^0 e^+ e^-$ decay.

All the data called for in the present section would help to cross-check and/or improve the precision of our dispersive framework. Dispersion theory combined with high-quality data has many applications but one concrete ongoing project is to obtain a numerical value with a reliable uncertainty estimate for the pion-pole part of the hadronic light-by-light scattering contribution to the gyromagnetic ratio of the muon.

![Figure 8](https://doi.org/10.1051/epjconf/201816600013)  
*Figure 8. The electromagnetic form factor of the transition $\omega \to \pi^0$ from various experiments and theory approaches. The figure is adopted from the recent work [22] of the A2 collaboration and “This work” refers to it. The magenta lines show the dispersive calculation from [12].*
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Left: Prediction for the electromagnetic form factor of the transition $\phi$ to $\pi^0$ based on dispersion theory [12]. The full calculation is shown by the yellow band. The complete kinematically allowed region for the decay $\phi \rightarrow \pi^0 e^+ e^-$ is shown. Right: Extraction of the same form factor from KLOE-2 data in the low-energy region. The yellow band from the plot on the left hand side translates here to the middle curve. Figure taken from [24].}
\end{figure}

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\textbf{References}


