Classifying Categories
The Jordan-Hölder and Krull-Schmidt-Remak Theorems for Abelian Categories

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Abstract

The Jordan-Hölder and Krull-Schmidt-Remak theorems classify finite groups, either as direct sums of indecomposables or by composition series. This thesis defines abelian categories and extends the aforementioned theorems to this context.

Contents

1 Introduction

Category theory was developed by Eilenberg and Mac Lane in the 1942-1945, as a part of their research into algebraic topology. One of their aims was to give an axiomatic account of relationships between collections of mathematical structures. This led to the definition of categories, functors and natural transformations, the concepts that unify all category theory,

Categories soon found use in module theory, group theory and many other disciplines. Nowadays, categories are used in most of mathematics, and has even been proposed as an alternative to axiomatic set theory as a foundation of mathematics.[Law66]

Due to their general nature, little can be said of an arbitrary category. Instead, mathematical theory must focus on a specific type of category, the choice of which is largely dependent on ones interests. In this work, the categories of choice are abelian categories. These categories were independently developed by Buchsbaum[Buc55] and Grothendieck[Gro57].

Grothendieck's work was especially groundbreaking, as he unified the cohomology theories for groups and for sheaves, which had similar properies but lacked a formal connection. This showed that abelian categories was the basis of general framework for cohomology theories, a powerful incentive for research.

Abelian categories are highly structured, possessing both a matrix calculus and various generalizations of the isomorphism theorems. This gives rise to a refined structure theory, which is the topic of this thesis. Of special interest here is the structure of subobjects to an object in an abelian category, since this structure contains a lot of information about the objects themselves.

The ultimate aim of a structure theory is to provide theorems that classify some collection of objects up to isomorphism. Here, two results pertaining to such theorems are presented. The first is the Jordan-Hölder theorem, which classifies objects by maximal chains of subobjects. The second is the Krull-Schmidt-Remak theorem, which gives a classification of objects by linearly independent components.

These theorems do not provide a universal classification theorem for all abelian categories. The problem with the Jordan-Hölder theorem is that not all objects in an abelian category has a maximal chain of subobjects, while the problem for the Krull-Schmidt-Remak theorem is that is requires that the endomorphisms of certain objects are of a particular form, which is not true for alla objects in an abelian category.

Fortunately, one can show that every object that can be classified using Jordan-Hölder can also be classified using Krull-Schmidt-Remak. The extent to which Krull-Schmidt-Remak can be extended is not discussed further.

The thesis is divided into four chapters, each divided into two sections. The first chapter covers the basics of category theory and defines subobjects and quotients in general categories. The aim is to set up the the coming chapters, and fix terminology etc.

The second chapter defines additive categories, and gives an account of the matrix calculus it contains. Then, abelian categories are defined and some fundamental properties are proven, so as to set up the third chapter, which further develops the theory. In the third chapter, the focus is on developing the theory of exact sequences, an important tool in the study of abelian categories, and to further deepen our understanding the subobject structure of abelian categories.

In the fourth and final chapter, the theory is used to prove the Jordan-Hölder and Krull-Schmidt-Remak theorems.

2 Preliminaries

Categories is a general framework for studying mathematical structures and how they relate to one another.

2.1 Basic Category Theory

This section covers the basics of category theory, in order to fix terminology and notation. Proofs and detailed examples are omitted. The interested reader should consider the introductory chapter in Leinster's book *Basic Category Theory*[Lei14].

Definition 2.1. A *category* $\mathcal{C}$ consists of a class of *objects* and a class of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for every object A and B in $\mathcal{C}$, subject to the following constraints.

  (i) For each f in $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and g in $\mathrm{Hom}_{\mathcal{C}}(B, C)$ there is a morphism $g \circ f$ in $\mathrm{Hom}_{\mathcal{C}}(A, C)$, called the *composition* of f and g.

 (ii) For all f in $\mathrm{Hom}_{\mathcal{C}}(A, B)$, g in $\mathrm{Hom}_{\mathcal{C}}(B, C)$ and h in $\mathrm{Hom}_{\mathcal{C}}(C, D)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In other words, the composition is associative.

(iii) For all objects A in $\mathcal{C}$, there is an morphism id_A in $\mathrm{Hom}_{\mathcal{C}}(A, A)$, called *the identity* on A, such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$ for all morphisms f and g.

The composition $g \circ f$ is written as gf most of the time, and one usually writes Hom instead of $\mathrm{Hom}_{\mathcal{C}}$.

A morphism in $\mathrm{Hom}(A, A)$ is called an *endomorphism*, and the collection of endomorphisms on A is denoted $\mathrm{End}(A)$. Composition turns $\mathrm{End}(A)$ into a monoid, with the identity as unit object. If f is a morphism in $\mathrm{End}(A)$, the morphism f^n is the endomorphism on A defined by

$$\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

By convention, $f^0 = \mathrm{id}_A$.

Definition 2.2. Let f be a morphism in $\mathrm{Hom}(A, B)$ in some category. Then A is called the *domain* of f, while B is called the *codomain* of f, written $f : A \to B$.

Example 2.3.   (i) **Set** is the category of all sets and functions under composition.

  (ii) **Vect**$_K$ is the category of all vector spaces and linear transformations (over a field K) under composition.

 (iii) R-**Mod** is the category of all R-modules and module morphisms (over a ring R) under composition.

 (iv) **Grp** is the category of all groups and group morphisms under composition.

The definition of category does not assume that the collections of objects and morphisms are sets. In some circumstances this can be problematic. For details, consider [ML98] or [Lei14].

From old categories, new ones arise.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ is the category such that

   (i) the objects of $\mathcal{C} \times \mathcal{D}$ are the pairs (A, B) with A from $\mathcal{C}$ and B from $\mathcal{D}$.

  (ii) the morphisms from (A, B) to (A', B') are pairs of morphisms (f, g) from $\mathcal{C}$ and $\mathcal{D}$, with $f : A \to A'$ and $g : B \to B'$.

 (iii) the identity morphisms $\mathrm{id}_{(A,B)}$ are $(\mathrm{id}_A, \mathrm{id}_B)$.

 (iv) the composition of morphisms (f, g) and (f', g') is $(f'f, g'g)$.

Definition 2.5. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ is the category such that

   (i) the objects of $\mathcal{C}^{op}$ are the objects in $\mathcal{C}$.

  (ii) for every morphism $f : A \to B$, there is a morphism $f^{op} : B \to A$.

 (iii) the identity morphisms id_A are id_A.

 (iv) the composition of morphisms $f^{op} : A \to B$ and $g^{op} : B \to C$ in $\mathcal{C}^{op}$ is the morphism $(fg)^{op} : A \to C$.

The opposite category is also called the *dual category*.

For any property of morphisms and objects in a category, there is a corresponding dual property in the dual category where the morphisms are reversed. So, if a property holds in a category, then the dual property holds in the dual category.

Since any category is the dual category of its dual category, this means that if a property holds for all categories, the dual property holds for all categories. In particular, for every theorem, there is a dual theorem that holds in the dual categories (see [ML98, p.33-35] for a more complete discussion of this).

Some caution is needed. If a property holds for a category, the dual property holds for the dual category. This does not mean that dual property hold for the original category! Since there is no guarantee that categories and dual categories have similar properties, this limits the scope of the duality principle.

In an arbitrary category, morphisms are not functions. Thus, there is no concept of surjective, injective or bijective morphisms. Instead, one uses a different terminology.

Definition 2.6. A morphism f is

(i) an *epimorphism* if $gf = hf$ implies $g = h$, for all morphisms g, h. Then f is called *epic*.

(ii) a *monomorphism* if $fg = fh$ implies $g = h$, for all morphisms g, h. Then f is called *monic*.

(iii) an *isomorphism* if there is a morphism g such that $fg = \text{id}_B$ and $gf = \text{id}_A$. Then, A and B are *isomorphic*, denoted $A \simeq B$.

An isomorphism from an object to itself is called an *automorphism*.

Note that epimorphisms and monomorphisms are dual concepts: a monomorphism in a category is an epimorphism in the dual category and vice versa. Isomorphisms are dual to themselves: isomorphisms are isomorphisms in the dual category as well.

Remark. Not all epimorphisms are surjective, nor are all monomorphisms injective. Also, bijective morphisms and isomorphisms do not coincide in general. See [IB68, p.3-7] and [Lei14, p.12] for details.

The following facts will be used liberally throughout the thesis.

Proposition 2.7. *Let $f : A \to B$ and $g : B \to C$ be morphisms in a category.*

 (i) If f and g are epic, so is gf.

 (ii) If gf is epic, so is g.

 (iii) If f and g are monic, so is gf.

 (iv) If gf is monic, so is f.

 (v) If f and g are isomorphisms, so if gf.

 (vi) All isomorphisms are epic and monic.

Remark. The converse of (vi) is not true: there are morphisms that are not isomorphisms, yet still epic and monic.[Lei14, p.12]

Maps between categories that preserve composition are called functors.

Definition 2.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor $F : \mathcal{C} \to \mathcal{D}$* is an assignment of objects in $\mathcal{C}$ to $\mathcal{D}$ and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ such that

(i) for all A and B in $\mathcal{C}$ and $f : A \to B$, we have $F(f) : F(A) \to F(B)$.

(ii) for all A in $\mathcal{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(iii) for all $f : A \to B$ and $g : B \to C$, we have $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

Definition 2.9. A *bifunctor* on a category $\mathcal{C}$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$.

Functors are assignments between categories, and can be composed pointwise on objects and morphisms. This composition has an identity and is associative. Hence, the collection of categories and functors behaves like a category.

Functors are maps between categories - natural transformations are maps between functors.

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and F and G functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* from F to G assigns a morphism

$$\eta_A : F(A) \to G(A)$$

in $\mathcal{D}$ for all A in $\mathcal{C}$, such a that if f is a morphism in $\mathcal{C}$ between A and B, the diagram

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\eta_A \downarrow & & \downarrow \eta_B \\
G(A) & \xrightarrow[G(f)]{} & G(B)
\end{array}
$$

commutes, i.e $\eta_B F(f) = G(f)\eta_A$. If η_A is an isomorphism for every A, then η is a *natural isomorphism*, denoted $F \simeq G$.

Just as with functors, natural transformations can be composed pointwise. Once again, the composition of two natural transformations is a natural transformation and the composition is associative.

Two objects in a category are isomorphic to each other if there are invertible morphisms between them. There is a natural analogue to this condition for categories.

Definition 2.11. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$.

Two categories are isomorphic if and only if they are isomorphic as objects in the category of categories. However, isomorphic categories rarely occur in practice. Instead, a weaker notion is used.

Definition 2.12. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$.

Definition 2.13. A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is

- *faithful* if the induced map from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(F(A), F(B))$ defined by mapping f to $F(f)$ is injective.

- *full* if the induced map from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(F(A), F(B))$ defined by mapping f to $F(f)$ is surjective.

- *dense* if all objects in $\mathcal{D}$ is isomorphic to $F(A)$ for some A in $\mathcal{C}$.

Proposition 2.14. *A functor is an equivalence if and only if it is faithful, full and dense.*

2.2 Subobjects and Quotients

This thesis is concerned with classifies objects in a category using subobjects. But how can one speak of subobjects without sets? The idea to define a subobject of an object as an equivalence class of morphisms.

Let $\mathrm{Mono}(A)$ denote the class of monomorphisms with codomain A. If i lies in $\mathrm{Mono}(A)$, the domain of i is denoted A_i.

Definition 2.15. Let A be an object in a category, and suppose that i and j are morphisms in $\mathrm{Mono}(A)$. Then i *contains* j via a morphism f, denoted $j \le i$, if there is a morphism f such that the diagram

$$A_i \xrightarrow{\ i\ } A$$

with $f: A_j \to A_i$ and $j: A_j \to A$

commutes, i.e if $if = j$.

If i and j contain each other, they are *equivalent*, denoted $i \sim j$. Otherwise, the containment is *proper*, denoted $j < i$.

Proposition 2.16. *Suppose that i and j are monomorphisms in $\mathrm{Mono}(A)$ and that i contains j via f. Then f is a monomorphism, and if f' satisfy $if' = j$, then $f = f'$.*

Proof. If $if' = j = if$, cancel i on both sides and obtain $f = f'$. That f is monic follows from (iv) in **Proposition 2.7**. $\qquad\square$

Proposition 2.17. *Let A be an object in a category and suppose that i and j are morphisms in $\mathrm{Mono}(A)$. Then i and j are equivalent if and only there is an isomorphism f such that i contains j via f.*

Proof. If there is such an isomorphism f, then $if = j$ and $i = jf^{-1}$, so i and j are equivalent.

If i and j are equivalent, there are morphisms f_1 and f_2 such that $i = jf_2$ and $j = if_1$. Thus,

$$\begin{cases} i = if_1f_2 \\ j = jf_2f_1 \end{cases} \Leftrightarrow \begin{cases} \mathrm{id}_{A_i} = f_1f_2 \\ \mathrm{id}_{A_i} = f_2f_1, \end{cases}$$

so f_1 and f_2 are isomorphisms. $\qquad\square$

A *preorder* is a transitive and reflexive relation on a class of objects. A *partial order* is preorder which is antisymmetric.

Any preorder $\leq$ induces an equivalence relation $\approx$ on its underlying set via $x \approx y$ if and only if $x \leq y$ and $y \leq x$. The set of equivalence classes of $\approx$ is partially ordered by comparing representatives using $\leq$.

Proposition 2.18. *The relation $\leq$ is a preorder on* $\mathrm{Mono}(A)$ *for every object A*

Proof. To show that any object is contained in itself, take f to be the identity.

For transitivity, suppose that i, j and k in $\mathrm{Mono}(A)$ satisfy $i \leq j$ and $j \leq k$. By assumption, there are morphisms f and g be such that $k = jf$ and $j = ig$. Then gf satisfies

$$igf = jf = k,$$

and so i is contained in k. $\qquad\square$

By definition, $\sim$ is the equivalence relation induced by the preorder $\leq$.

Definition 2.19. A *subobject* of an object A is an equivalence class of $\mathrm{Mono}(A)$ under the relation $\sim$. The class of subobjects of an object A is denoted $\mathcal{S}_A$.

By previous remarks, $\mathcal{S}_A$ is partially ordered by $\leq$.

The dual to a subobject is a quotient. As with subobjects, quotients are defined as equivalence classes of morphisms. If A is an object, let $\mathrm{Epi}(A)$ denote the class of epimorphisms out of A.

Definition 2.20. Let A be an object in a category, and suppose that p and q are epimorphisms in $\mathrm{Epi}(A)$. Then p *contains* q, denoted $q \leq p$, if there is a morphism f such that the diagram

commutes, i.e if $fp = q$. If p and q contain each other, they are *equivalent*, denoted $p \sim q$, otherwise it is *proper*, denoted $q < p$

Quotients are entirely dual to subobjects, so the following proofs are omitted.

Proposition 2.21. *Suppose that p and q are epimorphisms in* Epi(*A*) *and that i contains j via f. Then f is an epimorphism, and if f′ satisfy f′p = q, then f = f′.*

Proposition 2.22. *Let A be an object in a category and suppose that p and q are morphisms in* Epi(*A*). *Then p and q are equivalent if and only there is an isomorphism f such that p contains q via f.*

Proposition 2.23. *Let A be an object in a category. Then ≤ is a preorder on* Epi(*A*).

Definition 2.24. A *quotient object* of an object A is an equivalence class of Epi(*A*) under the relation $\sim$. The class of quotients of A is denoted $\mathcal{Q}_A$.

Remark. Every object A has at least one subobject and quotient, represented by the identity morphism. This subobject is identified with A itself, so that one may speak of A as a subobject and quotient of itself.

As a subobject, it contains every subobject and as a quotient object it is contained in every quotient object. In other words, A is a lowest upper bound in $\mathcal{S}_A$ and a greatest lower bound in $\mathcal{Q}_A$.

Example 2.25. Every subgroup of the abelian group $\mathbb{Z}$ is of the form

$$n\mathbb{Z} = \{\cdots - 2n, -n, 0, n, 2n, \cdots\}$$

for some unique natural number n. The inclusions $j_n : n\mathbb{Z} \to \mathbb{Z}$ are defined by $j_n(x) = x$.

Each subobject of $\mathbb{Z}$ is an equivalence class of monomorphisms into $\mathbb{Z}$. Each class contains precisely one of the morphisms j_n. Moreover, j_m contains j_n if and only if there is a morphism $f : n\mathbb{Z} \to m\mathbb{Z}$ so that $j_m f = j_n$, i.e

$$m f(x) = nx$$

for all integers x in $\mathbb{Z}$. This happens only when m divides n, in which case f is defined via $f(x) = (n/m)x$. Hence,

$$j_n \leq j_m \Leftrightarrow m|n$$

and $\mathcal{S}_{\mathbb{Z}}$ is isomorphic as a partial order to $\mathbb{Z}$ under the reversed divisibility order.

What about quotients? Let $p : \mathbb{Z} \to G$ be a surjective group homomorphism. Then $G = \mathrm{im}(p) \simeq \mathbb{Z}/\ker(p)$.

The kernel of p is a subgroup of $\mathbb{Z}$, and so there is a natural number n such that $\ker(p) \simeq n\mathbb{Z}$. Consequently, G is isomorphic to $C_n = \mathbb{Z}/n\mathbb{Z}$, the cyclic group on n elements. Thus every quotient of $\mathbb{Z}$ is represented by a unique epimorphism $p_n : \mathbb{Z} \to C_n$, defined by $p_n(x) = x + n\mathbb{Z}$

Suppose that the quotient object $p_n : \mathbb{Z} \to C_n$ contains another quotient $p_m : \mathbb{Z} \to C_m$. Then there is an epimorphism $f : C_n \to C_m$, so that $f p_n = p_m$, i.e

$$f(p_n(x)) = p_m(x) \Leftrightarrow f(x + n\mathbb{Z}) = x + m\mathbb{Z}.$$

This holds if and only if m divides n, and so

$$p_m \leq p_n \Leftrightarrow m | n$$

and $\mathcal{Q}_{\mathbb{Z}}$ is isomorphic as a partial order to $\mathbb{Z}$ under the divisibility order.

Notice that $\mathcal{S}_{\mathbb{Z}}$ and $\mathcal{Q}_{\mathbb{Z}}$ are order isomorphic up to reversal of the order. This is not coincidental: it is a special case of **Proposition 3.27**.

3 Abelian Categories

Abelian categories are additive categories with additional structure.

3.1 Additive Categories

Additive categories can be seen as the most general type of category that retains a kind of matrix calculus.

Definition 3.1. An object A in a category $\mathcal{C}$ is

  (i) *initial* if for every object B in $\mathcal{C}$ there is exactly one morphism from A to B.

 (ii) *terminal* if for every object B in $\mathcal{C}$ there is exactly one morphism from B to A.

(iii) *null* if it is both initial and terminal.

Proposition 3.2. *Initial, terminal and null objects are unique up to a unique isomorpism.*

Proof. By definition, the only endomorphism on an initial object is the identity morphism. Let I and J be initial objects. Then, there are unique morphisms $f : I \to J$ and $g : J \to I$, and $fg = \mathrm{id}_J$ and $gf = \mathrm{id}_I$, so I and J are isomorphic.

The proofs for terminal and null objects are dual. $\qquad\square$

Example 3.3.   (i) In **Set** , the empty set is an initial object and singleton set is a terminal object. There is no null object.

 (ii) In **Grp** , the trivial group is a null object. Similarly, the zero module is a null object in R-**Mod**.

Definition 3.4. Let 0 be a null object in a category and A and B be objects in the same category. The *null morphism* between A and B is the unique morphism given by the composition of the morphisms $A \to 0$ and $0 \to B$.

Example 3.5. In **Grp** , the null morphism between two groups G and H is the morphism from G to H defined by mapping every element in G to 1_H.

Definition 3.6. A category is *preadditive* if it has a null object and every set of morphisms between two objects form an abelian group, such that composition is biadditive. That is,

$$f(g+h) = fg + fh \quad \text{and} \quad (f+g)h = fh + gh$$

for all morphisms.

In a preadditive category, the set of endomorphisms on an object is a ring, with morphism addition as addition and composition as the ring multiplication.

The endomorphism ring is a $\mathbb{Z}$-bimodule, via

$$nf = fn = \begin{cases} -f - \cdots - f & n \text{ times (if } n \text{ is negative.)} \\ f + \cdots + f & n \text{ times (if } n \text{ is positive.)} \\ 0 & \text{(if } n \text{ is 0.)} \end{cases}$$

Proposition 3.7. *In preadditive categories, the following are equivalent for an object A:*

(i) *A is initial.*

(ii) *A is terminal.*

(iii) id_A *is the additive identity in the endomorphism ring.*

(iv) *The endomorphism ring is trivial.*

Proof. See [ML98, p.194]. $\qquad\square$

When there is a null object in a category, the null morphism $0 : A \to B$ and the additive identity in $\text{Hom}(A, B)$ coincide, since 0 is the composition of the additive identity in $\text{Hom}(A, 0)$ and $\text{Hom}(0, A)$.

Definition 3.8. A *direct sum* of objects $A_1, \dots, A_n$ in a preadditive category is an object S along with morphisms

$$A_k \xrightarrow{i_k} S \xrightarrow{p_l} A_l$$

such that

$$\sum_{k=1}^{n} i_k p_k = \text{id}_S.$$

and

$$p_l i_k = \delta_{lk} = \begin{cases} \text{id}_{A_k} & \text{if } l = k \\ 0 & \text{otherwise.} \end{cases}$$

The morphisms i_k are called *injection morphisms*, while the morphisms p_l are called *projection morphisms*. The objects $A_1, \dots, A_n$ are called *direct summands* of S, and the

collection of objects $A_1, \ldots, A_n, S$ along with the morphisms is called a *direct sum system*.

A direct sum is *trivial* if every summand is isomorphic to either A or the zero object, and *nontrivial* otherwise.

Every direct summand is a subobject of the direct sum, and a proper one if and only if the direct sum is nontrivial.

Not all subobjects are direct summands. For example, $\mathbb{Z}$ cannot be written as a non-trivial direct sum, but has a lot of subobjects.

Direct sums are self-dual, since every direct sum system gives rise to a direct sum system in the dual category, by switching projection and injection morphisms.

Proposition 3.9. *Any direct sum in a preadditive category is unique up to isomorphism.*

Proof. Let S and S' be direct sums of $A_1, \ldots, A_n$, and p_k, i_k, p'_k and i'_k the corresponding injection and projection morphisms. Define f from S to S' by

$$f = \sum_{k=1}^{n} i'_k p_k$$

and g from S' to S by

$$g = \sum_{k=1}^{n} i_k p'_k.$$

Then

$$fg = \sum_{j=1}^{n} i'_j p_j \sum_{k=1}^{n} i_k p'_k = \sum_{j=1}^{n} \sum_{k=1}^{n} i'_j p_j i_k p'_k = \sum_{k=1}^{n} i'_k p'_k = \mathrm{id}_{S'}$$

and

$$gf = \sum_{j=1}^{n} i_j p'_j \sum_{k=1}^{n} i'_k p_k = \sum_{j=1}^{n} \sum_{k=1}^{n} i_j p'_j i'_k p_k = \sum_{k=1}^{n} i_k p_k = \mathrm{id}_{S}.$$

Hence, S and S' are isomorphic. $\qquad\square$

The above proposition allows us talk about *the* direct sum of $A_1, \ldots, A_n$, denoted $\bigoplus A_j$. Note that the isomorphism between two direct sums is not unique.

Definition 3.10. A category is *additive* if it is preadditive and every set of objects has a direct sum.

Example 3.11. If R is a ring, the category R-**Mod** is additive. The null object is the zero module, and direct sum is cartesian product.

Direct sums extends to morphisms.

Proposition 3.12. *Let $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_n$ be objects in an additive category with corresponding projection and injection morphisms p_k, i_k, p'_k and i'_k. Suppose f_j is a morphism from A_j to A'_j, for every $j = 1, \ldots, n$.*

Then there is a unique morphism, denoted $\bigoplus f_j$, from $\bigoplus A_j$ to $\bigoplus A'_j$, such that the diagram

$$
\begin{array}{ccc}
\bigoplus A_j & \xrightarrow{\ \oplus f_j\ } & \bigoplus A'_j \\
{\scriptstyle p_k}\big\downarrow & & \big\downarrow{\scriptstyle p'_k} \\
A_k & \xrightarrow[\ f_k\ ]{} & A'_k
\end{array}
$$

commutes for every k.

Proof. Let

$$
\bigoplus f_j = \sum_{j=1}^{n} i'_j f_j p_j.
$$

Then

$$
p'_k \left(\bigoplus f_j \right) = p'_k \left(\sum_{j=1}^{n} i'_j f_j p_j \right) = \sum_{j=1}^{n} p'_k i'_j f_j p_j = f_k p_k
$$

for all k. To prove uniqueness, suppose that $p'_j g = f_j p_j = p'_j g$ for all j. Then

$$
i'_j p'_j g = i'_j p'_j f
$$

for all j. Summing over j gives

$$
\sum_{j=1}^{n} i'_j p'_j g = \sum_{j=1}^{n} i'_j p'_j f \Rightarrow \left(\sum_{j=1}^{n} i'_j p'_j \right) g = \left(\sum_{j=1}^{n} i'_j p'_j \right) f \Rightarrow \mathrm{id}_{S'} g = \mathrm{id}_{S'} f \Rightarrow g = f.
$$

$\square$

Remark. There is an dual definition of f_j, where one replaces the projection morphisms with the injection morphisms in the opposite direction, resulting in the diagrams

$$
\begin{array}{ccc}
\bigoplus A_j & \xrightarrow{\ \oplus f_j\ } & \bigoplus A'_j \\
{\scriptstyle i_k}\big\uparrow & & \big\uparrow{\scriptstyle i'_k} \\
A_k & \xrightarrow[\ f_k\ ]{} & A'_k
\end{array}
$$

and equations

$$
\left(\bigoplus f_j \right) i_k = f_k i'_k.
$$

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The process of constructing morphisms between direct sums from morphisms between the summands can be inverted.

Definition 3.13. Suppose $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_m$ are objects in an additive category, and that f is a morphism from $\bigoplus A_j$ to $\bigoplus A'_k$. The *component* f_{jk} of f is the morphism from A_j to A_k defined by

$$f_{jk} = p'_k f i_j.$$

The *matrix* of f is the matrix

$$[f] = \begin{bmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm}. \end{bmatrix}$$

Example 3.14. Let A be the direct sum of objects $A_1, \ldots, A_n$ in an additive category. Then

$$[\mathrm{id}_A] = \begin{bmatrix} p_1 \, \mathrm{id}_A \, i_1 & \cdots & p_n \, \mathrm{id}_A \, i_1 \\ \vdots & \ddots & \vdots \\ p_1 \, \mathrm{id}_A \, i_n & \cdots & p_n \, \mathrm{id}_A \, i_n \end{bmatrix} = \begin{bmatrix} \mathrm{id}_{A_1} & 0 & \cdots & 0 \\ 0 & \mathrm{id}_{A_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{id}_{A_n} \end{bmatrix}.$$

Similarly, the matrices of the injection and projection morphisms are

$$[i_j] = \begin{bmatrix} 0 & \cdots & 0 & \mathrm{id}_{A_j} & 0 & \cdots & 0 \end{bmatrix}$$

and

$$[p_k] = \begin{bmatrix} 0 & \cdots & 0 & \mathrm{id}_{A_k} & 0 & \cdots 0 \end{bmatrix}^T.$$

Matrices of morphisms can be seen as elements of the abelian group

$$\prod_{j=1}^{n} \prod_{k=1}^{m} \mathrm{Hom}(A_j, A'_k)$$

with addition defined componentwise. The identity of this group given by the matrix whose entries are all zero morphism.

Proposition 3.15. *Let $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_m$ be objects in an additive category. Then*

$$\mathrm{Hom}\left(\bigoplus A_j, \bigoplus A'_k\right) \simeq \prod_{j=1}^{n} \prod_{k=1}^{m} \mathrm{Hom}(A_j, A'_k).$$

as abelian groups.

Proof. Define

$$\varphi : \mathrm{Hom}\left(\bigoplus A_j, \bigoplus A_k'\right) \to \prod_{j=1}^{n}\prod_{k=1}^{m} \mathrm{Hom}(A_j, A_k').$$

by mapping the morphism $f : \bigoplus A_j \to \bigoplus A_k'$ to its matrix

$$\varphi(f) = [f] = \begin{bmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm.} \end{bmatrix}$$

If $f = 0$, then $f_{jk} = 0$ for all j and k, and hence φ preserves the zero matrix. Moreover, if f and g are morphisms from $\bigoplus A_j$ to $\bigoplus A_k'$, then

$$(f + g)_{jk} = p_k'(f + g)i_j = p_k'f i_j + p_k'g i_j = f_{jk} + g_{jk},$$

so φ is a group morphism. Next, define the map

$$\psi : \prod_{j=1}^{n}\prod_{k=1}^{m} \mathrm{Hom}(A_j, A_k') \to \mathrm{Hom}\left(\bigoplus A_j, \bigoplus A_k'\right)$$

by

$$\psi\left(\begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nm,} \end{bmatrix}\right) = \sum_{j=1}^{n}\sum_{k=1}^{m} i_j'g_{jk}p_k.$$

Let f be a morphism from $\bigoplus A_j$ to $\bigoplus A_k'$. Then

$$\psi(\varphi(f)) = \psi\left(\begin{bmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm,} \end{bmatrix}\right) = \psi\left(\begin{bmatrix} p_1'f i_1 & \cdots & p_m'f i_1 \\ \vdots & \ddots & \vdots \\ p_1'f i_n & \cdots & p_m'f i_n, \end{bmatrix}\right) =$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} i_j'p_j'f i_k p_k = \left(\sum_{j=1}^{m} i_j'p_j'\right) f \left(\sum_{k=1}^{n} i_k p_k\right) = f.$$

Similarly, one can show that $\psi(\varphi([f_{jk}])) = [f_{jk}]$ for all matrices, which establishes that φ is an isomorphism. $\qquad\square$

The above proposition shows that every morphism in an additive can be viewed as a matrix. It turns out that composition can be transfered as well.

Proposition 3.16. *Let $A_1, \ldots, A_n, A_1', \ldots, A_m'$, and $A_1'', \ldots, A_p''$ be objects in an additive category, with morphisms*

$$f : \bigoplus A_j \to \bigoplus A_k' \quad and \quad g : \bigoplus A_k' \to \bigoplus A_l''.$$

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Then, the matrix of the composition

$$gf : \bigoplus A_j \to \bigoplus A_l''$$

is given by the matrix

$$[gf] = \begin{bmatrix} h_{11} & \cdots & h_{1p} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{np} \end{bmatrix}$$

where

$$h_{ij} = \sum_{k=1}^{m} g_{ki} f_{jk}$$

for all i and j.

Proof. By definition, $f_{ik} = p_k' f i_i$ and $g_{kj} = p_j'' g i_k'$. Thus,

$$(gf)_{ij} = p_j'' g f i_i = \left(p_j'' g \right) \left(\sum_{k=1}^{m} i_k' p_k' \right) (f i_i) = \sum_{k=1}^{m} p_j'' g i_k' p_k' f i_i = \sum_{k=1}^{m} g_{kj} f_{ik}.$$

$\square$

Not only is there a matrix calculus in additive categories, the direct sum is also functorial.

Proposition 3.17. *Let A_j, A_k' and A_l'' be three n-tuples of objects in an additive category, and $f_j : A_j \to A_j'$ and $f_j' : A_j' \to A_j''$ two n-tuples of morphisms. Then*

$$\left(\bigoplus f_j' \right) \circ \left(\bigoplus f_j' \right) = \bigoplus \left(f_j' \circ f_j \right)$$

and

$$\bigoplus \mathrm{id}_{A_j} = \mathrm{id}_{\bigoplus A_j}.$$

Proof. Straightforward calculation gives

$$\left(\bigoplus f_j' \right) \circ \left(\bigoplus f_j' \right) = \left(\sum_{j=1}^{n} i_j'' f_j' p_j' \right) \circ \left(\sum_{k=1}^{n} i_k' f_k p_k \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} i_j'' f_j' p_j' i_k' f_k p_k =$$

$$= \sum_{j=1}^{n} i_j'' (f_j' \circ f_j) p_j = \bigoplus \left(f_j' \circ f_j \right).$$

and

$$\bigoplus \mathrm{id}_{A_j} = \sum_{j=1}^{n} i_j p_j = \mathrm{id}_{\bigoplus A_j}.$$

$\square$

The above result shows that in an additive category $\mathcal{C}$, are functors

$$\oplus_n : \mathcal{C}^n \to \mathcal{C}$$

for every n, defined by

$$\oplus_n(A_1, \ldots, A_n) = \bigoplus_{i=1}^{n} A_i$$

and

$$\oplus_n(f_1, \ldots, f_n) = \bigoplus_{i=1}^{n} f_i.$$

Since the composition of two functors is a functor, iteration yields a myriad of functors that purports to be the direct sum of n. Even if one restricts oneself to iteration of the bifunctor $\oplus_2$, the number of different direct sum functors of n variables is

$$\frac{(2n)!}{(n+1)!n!},$$

each corresponding to a unique bracketing of n variables.

Can any sense be made of this? The answer is yes - one can show that the direct sum is a *monoidal product* on $\mathcal{C}$. Such categories are subject to a coherence theorem, which essentially states that it does not matter how one places the brackets in a direct sum.

A detailed treatment of these issues is beyond the scope of this thesis, and the reader is referred to [ML98, p.161-170]. From now on, all direct sums of the same objects and morphisms are treated as equal, and it is assumed that no problems can arise due to bracketing of direct summands.

3.2 Abelian Categories

Every morphism between two modules can be described uniquely by its kernel and image. It is desirable to find a similar decomposition for additive categories. For this to work, the concept of kernel and image must be redefined using morphisms.

It turns out that, even with a proper account of these concepts, a morphism in additive category does not necessarily have a kernel or an image. Additive categories that do are called abelian categories.

Definition 3.18. Let $\mathcal{C}$ be a category with a null object and null morphism 0. A *kernel* of an morphism $f : A \to B$ is a morphism $k : K \to A$ such that $fk = 0$, and for every

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$h : C \to A$ such that $hk = 0$, there is a unique $h' : C \to K$ such that $h = kh'$.
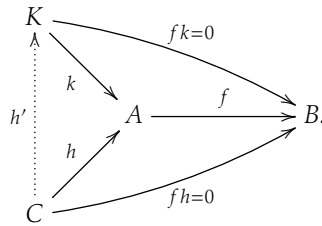
Example 3.19. Suppose that $f : A \to B$ is a morphism of abelian group, and let $k : K \to A$ be the inclusion of the preimage of identity.

Clearly, $fk = 0$. If $k' : K' \to A$ satisfies $fk' = 0$, then the image of k' is contained (as a set) in the image of k.

Since inclusions are injective, each g in the image of k has a unique preimage $k^{-1}(g)$, such that $k(k^{-1}(g)) = g$. Define $h : K' \to K$ by
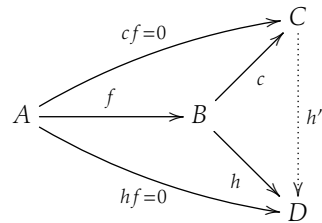
$$h(x) = k^{-1}(k'(x)).$$

Then

$$(kh)(x) = k(k^{-1}(k'(x))) = k'(x),$$

i.e $kh = k'$. Thus k is the kernel of f.

Definition 3.20. Let $\mathcal{C}$ be a category with a null object and null morphism 0. A *cokernel* of an morphism $f : A \to B$ is an object C and morphism $c : C \to C$ such that $cf = 0$, and for every $h : B \to D$ such that $ch = 0$, there is a unique $h' : C \to D$ such that $h = h'c$.

Example 3.21. Let $f : V \to W$ be a linear transformation, and let $c : V \to W/\mathrm{im}(f)$ be defined by $c(v) = v + \mathrm{im}(f)$. Then

$$cf(v) = c(f(v)) = f(v) + \mathrm{im}(f) = \mathrm{im}(f) = 0,$$

i.e the composition cf is 0.

Moreover, if c' from B to C' satisfies $c'f = 0$, define $h : W/\mathrm{im}(f) \to C'$ by

$$h(v + \mathrm{im}(f)) = c'(v).$$

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To show that this is well defined, suppose that v and w lie in the same equivalence class of the quotient $W/\text{im}(f)$. Then there is some u in V such that $f(u) = v - w$, and hence
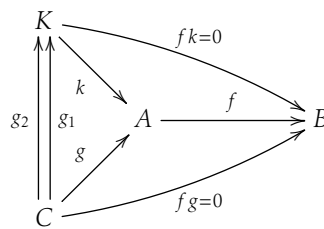
$$c'(v) - c'(w) = c'(v - w) = c'(f(u)) = 0$$

since $c'f = 0$ by assumption. Clearly $c' = hc$, and so c is the cokernel of f.

Remark. Kernels and cokernels are duals: the kernel of a morphism is the cokernel of the dual morphism and vice versa.

Proposition 3.22. *Kernels are monic and cokernels are epic.*

Proof. Let k be a kernel of a morphism f and suppose that $g = kg_1 = kg_2$. By definition $fg = 0$, and the diagram.

commutes. The uniqueness condition guarantees that $g_1 = g_2$ and so k is monic.

The proof that cokernels are epic is dual. $\qquad\square$

Proposition 3.23. *Kernels and cokernels are unique up to a unique isomorphism.*

Proof. Let k and k' be kernels of f. Then fk and fk' are both 0, and so there are morphisms h and h' such that $kh = k'$ and $k'h' = k$. Consequently,

$$k' = kh = k'h'h \quad \text{and} \quad k = k'h' = khh'$$

and since k' is monic one can cancel on both sides and find that h and h' are isomorphisms. The proof for cokernels is dual. $\qquad\square$

Since kernels and cokernels are unique, one speaks of *the* kernel and cokernel of a morphism $f : A \to B$, denoted $\ker(f)$ and $\text{cok}(f)$ respectively.

One way of thinking about the kernel of f is as the largest subobject of the domain that is mapped to zero by f. Dually, one can think of the cokernel as the smallest quotient that maps f to zero.

Definition 3.24. An additive category is *abelian* if

  (i) every morphism in the category has a kernel and cokernel.

 (ii) every monomorphism is a kernel and every epimorphism is a cokernel.

Since direct sums are self-dual, and kernels and monomorphisms are dual to cokernels and epimorphisms, respectively, the dual of an abelian category is also abelian. Hence, every theorem for general abelian categories has a dual theorem, obtained by reversing the morphisms and substituting monic for epic and kernel for cokernel, and vice versa.

Proposition 3.25. *Let f be a morphism in an abelian category. Then*

   (i) $\ker(f) = 0$ *if and only if f is monic.*

  (ii) *if g is a monomorphism, then* $\ker(gf) = \ker(f)$.

 (iii) $\mathrm{cok}(f) = 0$ *if and only if f is epic.*

 (iv) *if f is epic and g is a morphism, then* $\mathrm{cok}(gf) = \mathrm{cok}(g)$.

Proof. (i) Suppose f satisfy $\ker(f) = 0$ and that two morphisms g and h satisfy $fg = fh$. Let $l = g - h$. Then

$$fl = f(g - h) = fg - fh = 0.$$

Thus, there is a morphism h' such that $g - h = h'0 = 0$. Thus $g = h$, so f is monic.

Conversely, suppose that f is monic. Let k satisfy $fk = 0 = f0$. Since f is monic, cancellation yields $k = 0$, so $\ker(f) = 0$.

(ii) Suppose that g is monic. Then

$$gfk = 0 \Leftrightarrow gfk = g0 \Leftrightarrow fk = 0.$$

for all morphisms k, so $\ker(gf) = \ker(f)$.

The proofs for (iii) and (iv) are dual. $\qquad\square$

In abelian categories, kernel and cokernels induces maps between the class of subobjects and the class of quotients of an object.

Proposition 3.26. *Let A be an object in an abelian category.*

  (i) *If i and j in* $\mathrm{Mono}(A)$ *are equivalent, so are* $\mathrm{cok}(i)$ *and* $\mathrm{cok}(j)$.

 (ii) *If p and q in* $\mathrm{Epi}(A)$ *are equivalent, so are* $\ker(p)$ *and* $\ker(q)$.

Proof. If i and j are equivalent there is an isomorphism f such that $i = jf$, and hence

$$\mathrm{cok}(i) = \mathrm{cok}(jf) = \mathrm{cok}(j)$$

by **Proposition 2.7**. The second point is done similarly. $\qquad\square$

Define $\ker : Q_A \to S_A$ and $\mathrm{cok} : S_A \to Q_A$, by mapping p and i to $\ker(p)$ and $\mathrm{cok}(i)$ respectively.

Proposition 3.27. *The functions* ker *and* cok *are mutually inverse and order reversing.*

Proof. Let i and j be subobjects of A, such that $i \leq j$ via a morphism f. Let c_1 be the cokernel of i and c_2 be the cokernel of j. Then

$$c_2 i = c_2 j f = 0,$$

so there is a morphism g such that $g c_2 = c_1$. In other words, $\mathrm{cok}(j) \leq \mathrm{cok}(i)$. That ker is order reversing is proved similarly.

Let i be a monomorphism in $\mathrm{Mono}(A)$. Then it is the kernel of some map f. Let c be the cokernel of i and k the kernel of c.

By assumption $fi = 0$, and hence there is a map g such that $gc = f$. Also, $ci = 0$, so there is a map h_1 such that $kh_1 = i$. Finally,

$$fk = gck = 0,$$

so there is a map h_2 such that $ih_2 = k$.
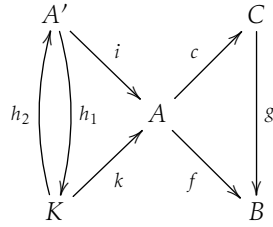
Thus i and k are equivalent and represent the same subobject, and thus

$$i = k = \ker(c) = \ker(\mathrm{cok}(i))$$

as subobjects. The other direction is proved is similarly. $\square$

The above proposition generalize **Example 2.25** - the subobject and quotient structure of an objects are mirror images of each other.

Proposition 3.28. *A morphism in an abelian category is an isomorphism if and only if it is monic and epic.*

Proof. Let f be a morphism that is both monic and epic.

Since f is monic, the kernel of f is zero. Hence, a cokernel of $\ker(f)$ is the identity. However, **Proposition 3.27** assures us that f is a cokernel of $\ker(f)$. Hence, there is a morphism g such that gf is the identity.

The exact same reasoning gives that the kernel of the cokernel of f is the identity, and that there exists a morphism h such that fh is the identity.

So f is both right and left invertible, and hence an isomorphism.

The other direction is (vi) in **Proposition 2.7**. $\square$

Cokernels can be extended to subobjects.

Proposition 3.29. *Let i, i', j and j' be monomorphisms, such that $i \sim i'$ and $j \sim j'$ via isomorphisms φ_i and φ_j, respectively. Suppose that the subobject represented by i and i' is contained in the subobject represented by j and j' via monomorphisms f and f'.*

Then the codomains of $\mathrm{cok}(f)$ and $\mathrm{cok}(f')$ are isomorphic.

Proof. Consider the diagram

$$
\begin{array}{ccc}
A_i & \xrightarrow{\ \varphi_i\ } & A_i' \\
\end{array}
$$

The assumptions that $i \sim i'$ and $j \sim j'$ imply that φ_i and φ_j are isomorphisms and the upper and lower triangle commute. The assumption that i is contained in j means that f and f' are monomorphisms and that the left and right triangle commute.

Hence,

$$i = i'\varphi_i = j'f'\varphi_i$$

and

$$i = jf = j'\varphi_j f.$$

Equating these expressions and cancelling j' yields $f'\varphi_i = \varphi_j f$. Let c and c' be the cokernels of f and f' respectively. Then

$$c'\varphi_j f = c'f'\varphi_i = 0\varphi_i = 0$$

and

$$c\varphi_j^{-1}f' = cf\varphi_i^{-1} = 0\varphi_i^{-1} = 0.$$

Hence there are morphisms h and h' so that the diagram

$$
\begin{array}{ccc}
A_i & \xrightarrow{\ f\ } A_j \xrightarrow{\ c\ } & C \\
\end{array}
$$

commutes. Thus,

$$
\begin{cases} h'c'\varphi_j = c \\ hc\varphi_j^{-1} = c' \end{cases}
\Leftrightarrow
\begin{cases} h'hc = c \\ hh'c' = c' \end{cases}
\Leftrightarrow
\begin{cases} h'h = \mathrm{id}_C \\ hh' = \mathrm{id}_{C'}. \end{cases}
$$

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$\square$

Definition 3.30. Let i and j be two subobjects with domains A_i and A_j, such that i is contained in j via a morphism f. The *quotient* of j by i, denoted A_j/A_i, is the codomain of the cokernel of f.

In abelian categories, all morphisms can be decomposed into monomorphisms and epimorphisms.

Proposition 3.31. *Let f be a morphism in an abelian category. Then $f = me$ for an epimorphism e and monomorphism m. Moreover, m is the kernel of the cokernel of f and e is the cokernel of the kernel of f.*

$$K \xrightarrow{\ k\ } A \xrightarrow{\overset{f}{\ e\ }} D \xrightarrow{\ m\ } B \xrightarrow{\ c\ } C$$

Proof. Let f be a morphism in an abelian category. Let c be the cokernel of f and let m to be the kernel of c.

By definition, $cf = 0$, and since m is the kernel of c there is a morphism e such that $f = me$. By **Proposition 3.22**, e is epic and m is monic. Moreover,

$$e = \mathrm{cok}(\ker(e)) = \mathrm{cok}(\ker(me)) = \mathrm{cok}(\ker(f))$$

since m is monic. $\square$

The canonical decomposition transfers to morphisms.

Proposition 3.32. *Consider the commutative square*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ g\downarrow & & \downarrow h \\ A' & \xrightarrow[\ f'\ ]{} & B' \end{array}$$

in an abelian category, and let $f = me$ and $f' = m'e'$ be a canonical decomposition. Then there is a unique φ such that diagram

$$\begin{array}{ccccc} A & \xrightarrow{\ e\ } & D & \xrightarrow{\ m\ } & B \\ g\downarrow & & \varphi\downarrow & & \downarrow h \\ A' & \xrightarrow[\ e'\ ]{} & D' & \xrightarrow[\ m'\ ]{} & B' \end{array}$$

commutes.

Proof. Let f, f', g and h be given as above. By **Proposition 3.31**, there are decompositions $f = me$ and $f' = m'e'$. Let u be the kernel of f.

By definition $hfu = 0$, and thus $m'e'gu = 0$. Since m is monic, $e'gu = 0$, and since e is the cokernel of u, there is a unique morphism φ such that $e'g = \varphi e$.

$$
\begin{array}{ccccccc}
K & \xrightarrow{\ u\ } & A & \xrightarrow{\ e\ } & D & \xrightarrow{\ m\ } & B \\
 & & {\scriptstyle g}\downarrow & & {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle h} \\
 & & A' & \xrightarrow[\ e'\ ]{} & D' & \xrightarrow[\ m'\ ]{} & B'.
\end{array}
$$

with an overarching arrow f from A to B.

Moreover,

$$m'\varphi e = m'e'g = hme,$$

and since e is epic, $m'\varphi = hm$. $\qquad\qquad\square$

Proposition 3.33. *The canonical decomposition of a morphism in an abelian category is unique up to a unique isomorphism.*

Proof. Apply **Proposition 3.32** to the square

$$
\begin{array}{ccccccc}
A & \xrightarrow{\ e\ } & D & \xrightarrow{\ m\ } & B \\
{\scriptstyle \mathrm{id}_A}\downarrow & & {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[\ e'\ ]{} & D & \xrightarrow[\ m'\ ]{} & B
\end{array}
$$

to find the isomorphism φ. $\qquad\qquad\square$

Definition 3.34. Let f be a morphism in an abelian category, and $f = me$ its canonical decomposition. The *image* of f, denoted $\mathrm{im}(f)$ is the monomorphism m. The *coimage* of f, denoted $\mathrm{coim}(f)$, is the epimorphism e.

Since the coimage and image of a morphism are unique up to a unique isomorphism, they define a quotient and a subobject of A and B respectively.

Definition 3.35. A *span* into an object A in a category is a pair of morphisms with common codomain A. A *cospan* from an object B is a pair of morphisms with common domain B.

**Definition 3.36.** Let f and g be a span into A. A *pullback* of f and g is a cospan f' and g' such that $gf' = fg'$, with the property that if f'' and g'' is a cospan such that $gf'' = fg''$, then there is a unique morphism h such that $f'' = f'h$ and $g'' = g'h$.
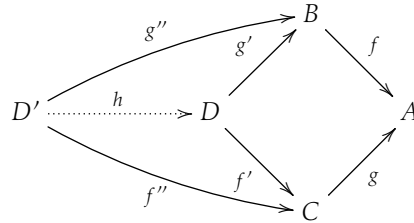
Definition 3.37. Let h and k be a cospan from A. A *pushout* of h and k is a span h' and k' such that $k'h = h'k$, with the property that if h'' and k'' is a span that satisfy $hk'' = kh''$, then there is a unique morphism p such that $ph' = h''$ and $pk' = k''$.
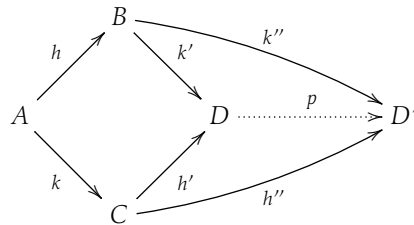
Pullbacks and pushout are dual to each other, as are span and cospans.

Proposition 3.38. *Pullbacks and pushouts are unique up to a unique isomorphism.*

Proof. Suppose that there are two pullbacks of the same span. Then there are unique maps h and h' such that the diagram
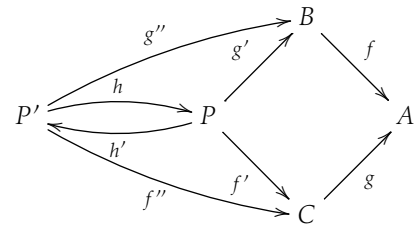
commutes. Hence $f' = f'hh'$ and $g' = ghh'$, and the diagram

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commutes. If one replaces $h'h$ by id_P, the diagram still commutes and thus the uniqueness criterion implies that $h'h = \mathrm{id}_P$. Similarly, $hh' = \mathrm{id}_{P'}$, and so h is an isomorphism. $\qquad\square$

Pullbacks and pushouts can be composed.

Proposition 3.39. *Suppose that the diagram*
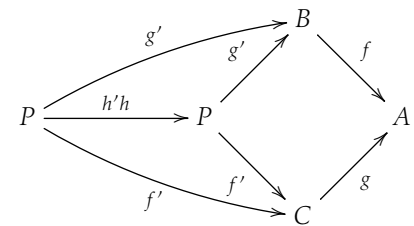
$$
\begin{array}{ccccc}
A & \xrightarrow{\ f\ } & C & \xrightarrow{\ g\ } & E \\
{\scriptstyle u}\big\downarrow & & {\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u''} \\
B & \xrightarrow[\ f'\ ]{} & D & \xrightarrow[\ g'\ ]{} & F.
\end{array}
$$

commute. Then

 (i) *if the two inner squares are pullback/pushouts, then so is the outer square.*

 (ii) *if the inner left-hand square is a pushout, the outer square is a pushout if and only if the inner right-hand square is a pushout.*

 (iii) *if the inner right-hand square is a pullback, the outer square is a pullback if and only if the inner left-hand square is a pullback.*

Proof. (i) Suppose that the two inner square are pushouts. Suppose that h and h' satisfies $h'u = hgf$.

Since the left-hand square is a pushout, there is a unique morphism φ so that $\varphi f' = h'$ and $\varphi u' = hg$. Since the right-hand square is a pushout as well, there is a unique ψ so that $\psi g' = \varphi$ and $\psi u'' = h$.

But then
$$
\psi g' f' = \varphi f' = h'.
$$

Since ψ is unique, this mean that the outer square is a pushout. The proof for pullbacks is dual.

 (ii) Suppose that the inner left-hand and the outer squares are pushouts. Let h and h' be such that $h'u' = hg$. Then
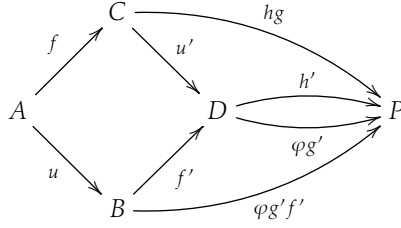
$$
hgf = h'u'f = h'f'u
$$

and since the outer square is a pushout, there is a unique φ so that $\varphi g'f' = h'f'$ and $\varphi u'' = h$. Then
$$
\varphi g'f'u = h'f'u = hgf
$$

and
$$
\varphi u''g = hg = \varphi g'u' = h'u'.
$$

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So there is a pushout

and uniqueness implies that $\varphi g' = h$, so the right-hand square is a pushout. The other implication is proved in (i).

(iii) Dual to (ii).

$\square$

Proposition 3.40. *In an abelian category, every span have a pullback and every cospan have a pushout.*

Proof. Let f and g be a span with domains B and C and common codomain A. Consider the direct sum system

$$B \underset{p}{\overset{i}{\rightleftarrows}} B \oplus C \underset{j}{\overset{q}{\rightleftarrows}} C$$

and let $h = fp - gq$. Let k be the kernel of h.

Since k is the kernel of h,

$$0 = hk = (fp - gq)k = fpk - gqk,$$

i.e $fpk = gqk$. Moreover, if g' and f' are such that $fg' = gf'$, let $h' = ig' - jf'$. Then

$$hh' = (fp - gq)(ig' - jf') = fg' - gf' = 0,$$

and since k is the kernel of h there is a unique map h'' such that $kh'' = ig' - jf'$.
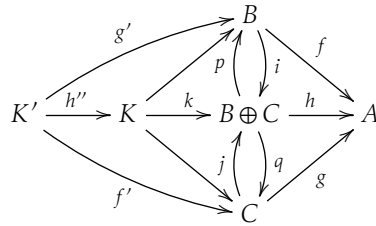
This yields $pkh'' = g'$ and $qkh'' = g'$, which shows that pk and qk is the pullback of f and g.

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For the pushout, suppose that f and g has common domain A and codomains B and C respectively. Let $h = if - jg$ and c be the cokernel of h. One can show, using a similar argument as above, that ci and cj is the pushout of f and g. $\square$

4 Structure Theory of Abelian Categories

The topic of chapter is the structure theory of abelian categories, a preparation for the proofs of the Jordan-Hölder and Krull-Schmidt-Remak theorems.

4.1 Exact Sequences

Exact sequences are the bread and butter of abelian categories.

Definition 4.1. A sequence of morphisms

$$\cdots A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} \cdots$$

in an abelian category is *exact at A_n* if $\mathrm{im}(d_n) = \ker(d_{n-1})$. A sequence is exact if it is exact at every object in the sequence.

Many properties of morphisms are characterized via exact sequences.

Proposition 4.2. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a sequence of morphisms. Then

(i) *$0 \to A \to B$ is exact if and only if f is monic.*

(ii) *$0 \to A \to B \to C$ is exact if and only if f is the kernel of g.*

(iii) *$A \to B \to 0$ is exact if and only if f is epic.*

(iv) *$A \to B \to C \to 0$ is exact if and only if g is the cokernel of f.*

(v) *$0 \to A \to B \to 0$ is exact if and only if f is an isomorphism.*

(vi) *$0 \to A \to B \to C \to 0$ is exact if and only if f is the kernel of g and g is the cokernel of f.*

Observe that if a sequence of morphisms is exact, the dual of that sequence is also exact.

Definition 4.3. An exact sequence of the form

$$0 \xrightarrow{\;0\;} A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \xrightarrow{\;0\;} 0$$

is called a *short exact sequence*.

The simplest examples of short exact sequences are of the form

$$0 \longrightarrow A \xrightarrow{\;i\;} A \oplus B \xrightarrow{\;p\;} B \longrightarrow 0$$

where p and i are the projection and injection maps. They are characterized thus.

Proposition 4.4 (Splitting lemma). *For all exact sequences*

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow 0$$

the following statements are equivalent.

  (i) *The middle object B is a direct sum of A and C, such that f is an injection morphism and g a projection morphism.*

 (ii) *There is a morphism l (called a* left split*) from B to A such that $lf = \mathrm{id}_A$.*

(iii) *There is a morphism r (called a* right split*) from C to B such that $gr = \mathrm{id}_C$*

Proof.   (i) In a direct sum system, injection morphisms and projection morphism is are right/left splits respectively.

 (ii) Suppose that an exact sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow 0$$

has a right split r such that $gr = \mathrm{id}_C$. Let $p = \mathrm{id}_B - rg$. By definition,

$$gp = g - grg = g - g = 0.$$

Since f is the kernel of g, there is a morphism l, such that $fl = p$, i.e $fl = \mathrm{id}_B - rg$.

Thus $fl + rg = \mathrm{id}_B$. Since $gr = \mathrm{id}_C$ by assumption, it suffices to prove that $lf = \mathrm{id}_A$ and $lr = 0$ to show that B is the direct sum of A and C. Yet

$$flf = (\mathrm{id}_B - rg)f = f - rgf = f - 0 = f,$$

and since f is monic, $lf = \mathrm{id}_A$. Also,

$$flr = (\mathrm{id}_B - rg)r = r - rgr = r - r = 0,$$

and since f is monic $lr = 0$. This shows that l, r, f and g form a direct sum system for $A \oplus C$.

(iii) Dual to (ii).

□

Remark. The splitting lemma implies that if

$$A \xrightarrow{\ f\ } A' \xrightarrow{\ g\ } A$$

is such that $h = gf$ is an automorphism, then A is a direct summand of A'. For if c is the cokernel of f, the sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } A' \xrightarrow{\ c\ } B \longrightarrow 0$$

is exact, and $h^{-1}g$, is a left split of f, i.e $A' \simeq A \oplus B$.

Pullbacks and pushouts in abelian categories can be described in terms of exact sequences.

Proposition 4.5. *Consider the diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\ f'\ } & C \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[f]{} & D.
\end{array}
$$

and the direct sum system

$$B \underset{p}{\overset{i}{\rightleftarrows}} B \oplus C \underset{j}{\overset{q}{\rightleftarrows}} C.$$

Let $t = jf' + ig'$ and $s = fp - gq$. Then

  (i) *the square commutes if and only if $st = 0$.*

  (ii) *the square is a pullback if and only if*

$$0 \longrightarrow A \xrightarrow{\ t\ } B \oplus C \xrightarrow{\ s\ } D$$

    is exact, i.e t is the kernel of s.

  (iii) *the square is a pushout if and only if*

$$A \xrightarrow{\ t\ } B \oplus C \xrightarrow{\ s\ } D \longrightarrow 0$$

    is exact, i.e s is the cokernel of t.

Proof. Note that $pt = g'$ and $qt = f'$, and $si = f$ and $sj = -g$.

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(i) The square is commutative if and only if

$$0 = fg' - g'f = sipt + sjqt = s(ip + jq)t = st.$$

(ii) Assume that the square is a pullback. Suppose that k is such that $sk = 0$. Then

$$0 = sk = s(ip + jq)k = sipk + sjqk = fpk - gqk,$$

i.e $fpk = gqk$. Since the square is a pullback, there is a unique morphism h such that $f'h = qk$ and $g'h = pk$. Hence $qth = qk$ and $pth = pk$, and so

$$th = (jq + ip)th = jqth + ipth = jqk + ipk = (jq + ip)k = k.$$

This shows that t is a kernel of s, so the sequence is exact.

Conversely, suppose that t is the kernel of s and that there are morphism f'' and g'' such that $gf'' = fg''$. Define $r = ig'' + jf''$. Then $pr = g''$, $qr = f''$, and

$$sr = (fp - gq)(ig'' + jf'') = fpig'' - gqjf'' = fg'' - gf'' = 0.$$

Since t is the kernel of s, there is a unique $m : U \to A$ such that $tm = r$, so $ptm = pr$ and $qtm = qr$, i.e $g'm = g''$ and $f'm = f''$. This shows that the square is a pullback.

(iii) Dual to the proof above.

$\square$

Proposition 4.6. *Consider the pullback*

in an abelian category. If f is monic, so is f', and if f is epic, so is f'.

Proof. Suppose that f is monic. Let h and h' be morphisms from P' to P, such that $f'h = f'h'$. Then $gf'h = gf'h'$, and since the diagram commutes $fg'h = fg'h'$. Since f is monic, $g'h = g'h'$. Since the diagram is a pullback, the uniqueness property guarantees that $h = h'$.

Suppose that f is epic and consider the direct sum system

In the proof of **Proposition 3.40**, it was shown that if k is the kernel of $h = fp - gq$, then $f' = qk$ and $g' = pk$.

Suppose that $uh = 0$ for some morphism u. Then

$$0 = uh = uhi = u(fp - gq)i = ufpi = uf$$

and since f is epic, $u = 0$. Thus, h is epic as well. Thus, the sequence

$$0 \longrightarrow P \xrightarrow{\ k\ } B \oplus C \xrightarrow{\ h\ } A \longrightarrow 0$$

is exact, i.e h is the cokernel of k. Suppose that $uf' = 0$ for some morphism u. Then

$$0 = uf' = uqk$$

and hence there is morphism u' such that $uq = u'h$. Thus

$$0 = uqi = u'hi = u'(fp - gq)i = u'fpi = u'f.$$

Since f is epic, u' is 0, and
$$uq = u'h = 0.$$
Since q is epic $u = 0$, which shows that f' is epic. $\qquad\square$

The *nine lemma* is a generalization of the isomorphism theorems. The following proof is due to Popescu [Pop73] and [Fre64].

Proposition 4.7. *Consider the commutative diagram*

such that the bottom row is exact. The square is a pullback if and only if the sequence

$$0 \longrightarrow A \xrightarrow{\ g\ } C \xrightarrow{\ hf'\ } E$$

is exact, i.e g is the kernel of hf.

Proof. Suppose that the square is a pullback. Since the diagram is commutative and the bottom row is exact,
$$hf'g = hg'f = 0.$$
Let s be a morphism such that $hf's = 0$. Since the bottom row is exact, g' is the kernel of h. Since $hf's = 0$ by assumption, there is a unique morphism t so that $f's = g't$.

Moreover, since the square is a pullback, there is a unique r so that $gr = s$. Thus, g is the kernel of hf'.

Conversely, suppose that g is the kernel of hf'.

Let s and t be morphisms such that $f's = g't$. Since the diagram is commutative,

$$hf's = hg't = 0,$$

and since g is the kernel of hf', there is a unique morphism r so that $s = gr$.
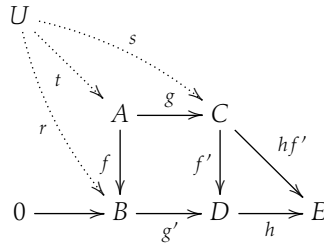
The diagram is commutative, so

$$g'fr = f'gr = f's = g't$$

and since g' is monic, cancellation yields $t = fr$. This shows that the square is a pullback. $\square$

Proposition 4.8. *Consider the commutative diagram*
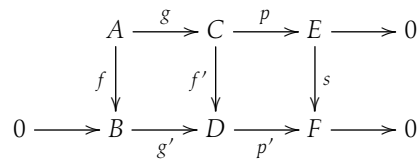
such that the right square is a pullback and the rows are exact. Then s is monic. If f' is epic then s is an isomorphism.

Proof. Let r be such that $sr = 0$. Let u and v be the pullback of p and r.

Since the right square is a pullback and the bottom row is exact, **Proposition 4.7** implies that g is the kernel of $p'f'$. Moreover,

$$p'f'v = spv = sru = 0.$$

Hence, there is a map t so that $gt = v$.

$$
\begin{array}{ccc}
U & \overset{u}{\dashrightarrow} & K \\
& & \downarrow r \\
A \overset{g}{\longrightarrow} C \overset{p}{\longrightarrow} & E & \longrightarrow 0 \\
\end{array}
$$

Thus,

$$ru = pv = pgt = 0$$

since p is the cokernel of g. Moreover, the morphism p is epic, and thus u is epic, so $r = 0$, which show that s is monic.

If f' is epic, the composition $p'f' = sp$ is epic, and so s is epic. Since s is always monic, s is an isomorphism. $\qquad\square$

Proposition 4.9. *Consider the commutative diagram*

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow A & \overset{h}{\longrightarrow} & D & \overset{k}{\longrightarrow} & G \\
f\downarrow & & f'\downarrow & & f''\downarrow \\
0 \longrightarrow B & \overset{h'}{\longrightarrow} & E & \overset{k'}{\longrightarrow} & H \\
g\downarrow & & g'\downarrow & & \\
0 \longrightarrow C & \overset{h''}{\longrightarrow} & F & & \\
\downarrow & & & & \\
0 & & & &
\end{array}
$$

with exact columns and exact middle row. Then the upper row is exact if and only if the bottom row is exact (i.e h'' is monic).

Proof. Suppose that the upper row is exact. The right column is exact, the diagram commutes and f'' is monic, so

$$h = \ker(k) = \ker(f''k) = \ker(f'k').$$

Hence, **Proposition 4.7** implies that the square

$$
\begin{array}{ccc}
A & \xrightarrow{\;h\;} & D \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f''} \\
B & \xrightarrow[\;h'\;]{} & E
\end{array}
$$

is a pullback. Thus, the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C & \longrightarrow & 0 \\
 & & {\scriptstyle h}\big\downarrow & & {\scriptstyle h'}\big\downarrow & & {\scriptstyle h''}\big\downarrow & & \\
0 & \longrightarrow & D & \xrightarrow[\;f'\;]{} & E & \xrightarrow[\;g'\;]{} & F & &
\end{array}
$$

has exact rows, with the right square being a pullback diagram. Thus h'' is monic and the bottom row is exact.

Conversely, suppose that the bottom row is exact, i.e h'' is monic. Then,

$$
f = \ker(g) = \ker(h''g) = \ker(g'h'),
$$

so the sequence

$$
0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g'h'\;} F
$$

is exact. Thus, the top right square is a pullback. Let r be a morphism such that $kr = 0$. Then

$$
k'f'r = f''kr = 0,
$$

and since the middle row is exact, h' is the kernel of k', and and there is a unique morphism t such that $h't = f't$.

Since the top right square is a pullback, there is a unique morphism s so that $hs = r$, which show that h is the kernel of k and the top row is exact. $\qquad\square$

Proposition 4.10 (Nine lemma). *Consider the commutative diagram*

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & D & \longrightarrow & G & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & H & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

with exact columns and exact middle row. Then the top row is exact if and only if the bottom row is exact.

Proof. Direct application of **Proposition 4.9** and its dual yields the conclusion. $\square$

The strength of the nine lemma is evident in ease of which it proves the second isomorphism theorem.

Proposition 4.11 (Second isomorphism theorem). *Suppose that i and j represent subobjects of A, so that i contains j via a morphism $f : A_j \to A_i$. Then exists there a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_i & \xrightarrow{\ i\ } & A & \xrightarrow{\ c\ } & A/A_i & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} & & \\
0 & \longrightarrow & A_i/A_j & \xrightarrow{\ k\ } & A/A_j & \xrightarrow{\ p\ } & (A/A_j)/(A_i/A_j) & \longrightarrow & 0
\end{array}
$$

such that the rows are exact, u'' is an isomorphism, and u and u' are the cokernels of f and j respectively. Moreover, the morphisms k and p are unique with this property.

Proof. Let $u = \mathrm{cok}(f)$, $u' = \mathrm{cok}(j)$ and $c = \mathrm{cok}(i)$.

By assumption, $j = if$. Hence

$$u'if = uj = 0,$$

and since u is the cokernel of f, there is a unique morphism k so that $u'i = ku$. Let p be the cokernel of k. Then $u'i = ku$ implies that

$$pu'i = pku = 0,$$

and hence there is a unique morphism u'' that satisfies $u''p = pu'$. Thus, the diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_j & \overset{f}{\longrightarrow} & A_i & \overset{u}{\longrightarrow} & A_i/A_j & \longrightarrow & 0 \\
& & \text{id}\downarrow & & i\downarrow & & k\downarrow & & \\
0 & \longrightarrow & A_j & \underset{j}{\longrightarrow} & A & \overset{u'}{\longrightarrow} & A/A_j & \longrightarrow & 0 \\
& & \downarrow & & c\downarrow & & p\downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & A/A_i & \underset{u''}{\longrightarrow} & (A/A_j)/(A_i/A_j) & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

is commutative. By assumption, the columns and the middle and upper rows are exact. Hence the lowest is exact as well, and u'' is an isomorphism. $\qquad\square$

4.2 The Subobject Lattice

Subobjects and quotients were defined for general categories in **Section 2.2**. It is time to return to topic in the case of abelian categories. But first, some order theory is required.

Until further notice, $\leq$ refers to an arbitrary partial order on some underlying set.

Definition 4.12. Let x and y be objects in a partial order.

A *greatest lower bound* of x and y is an element z such that $z \leq x$ and $z \leq y$, with the property that if w satisfy $w \leq x$ and $w \leq y$, then $w \leq z$.

Dually, a *lowest upper bound* of x and y is an element z so that $x \leq z$ and $y \leq z$, with the property that if w satisfy $x \leq w$ and $y \leq w$, then $z \leq w$.

Definition 4.13. A *lattice* is a partial order in which every pair of elements have a greatest lower bound and lowest upper bound.

The greatest lower bound and lowest upper bound of two elements x and y are unique if they exist, and are denoted $x \wedge y$ and $x \vee y$, respectively. The symbols $\wedge$ and $\vee$ are known as *meet* and *join*, respectively.

Example 4.14. The power set of a set is a lattice, where the meet is intersection and the join is union.

Lattices have been studied extensively by Birkhoff in [Bir67]. In the introduction, proofs can be found of the following three propositions.

Proposition 4.15. *In any lattice, the meet and join satisfies*

(i) $x \wedge x = x$ and $x \vee x = x$.

(ii) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.

(iv) $x \wedge (x \vee y) = x \vee (x \wedge y) = x$

for all lattice elements x,y and z. Moreover, $x \leq y$ is equivalent to each of the conditions

$$x \wedge y = x \quad and \quad x \vee y = y.$$

Proposition 4.16. *For all elements x,y and z in a lattice, if $y \leq z$, then*

$$x \wedge y \leq x \wedge z \quad and \quad x \vee y \leq x \vee z.$$

Proposition 4.17 (Modular inequality)**.** *For all elements x,y and z in a lattice, if $x \leq z$, then*

$$x \vee (y \wedge z) \leq (x \vee y) \wedge z.$$

Remark. In a lattice, the meet and the join can be seen as binary operations. Indeed, lattices can be characterized as a set with two binary operations that satisfy (i)-(iv) of **Proposition 4.15.**[Bir67, p.10]

Definition 4.18. A lattice is *modular* if $x \leq z$ implies that

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

for all lattice elements x, y and z.

The term modular lattice comes from module theory: the set of submodules of a module is a modular lattice.

Definition 4.19. An element $\top$ in a lattice is called a *top* if $x \leq \top$ for every element x in the lattice. An element $\bot$ is a *bot* if $\bot \leq x$ for every x in the lattice. A lattice with a top and bot is called *bounded*.

Example 4.20. The lattice of subsets of a set S has both a top and bot, given by $\top = S$ and $\bot = \emptyset$.

The top and bot are unique if they exist. Every element x in such lattices satisfy $\top \wedge x = x$, $\top \vee x = \top$, $\bot \vee x = x$ and $\bot \wedge x = \bot$.

Definition 4.21. A *complement* of an element x in a bounded lattice is an element c such that $x \wedge c = \bot$ and $x \vee c = \top$.

Example 4.22. In the lattice of subsets of a set S, every subset X has a unique complement given by $S \setminus X$.

Example 4.23. Complements are in general not unique. In the bounded lattice
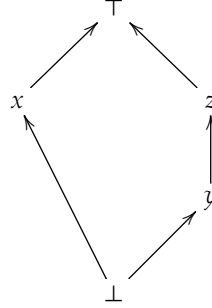
both y and z are complements of x.

Complements provide a neat characterization of modular lattices using intervals.

Definition 4.24. Let x and y be objects in a lattice such that $x \leq y$. The *interval* from x to y, denoted $[x, y]$, is the set of objects z such that $x \leq z$ and $z \leq y$.

Note that intervals are always bounded lattices, and so one can speak of a complement in an interval.

Proposition 4.25. *A lattice is modular if and only if every interval I has the property that if an element c in I has two complements a and b such that $a \leq b$, then $a = b$.*

Proof. Suppose that the lattice is modular and that an object c has two complements a and b, such that $a \leq b$. Then

$$a = a \vee \bot = a \vee (c \wedge b) = (a \vee c) \wedge b = \top \wedge b = b$$

via modularity.

Conversely, suppose that an object c has the property that all its comparable complements are equal. Let $a_1 = a \vee (c \wedge b)$ and $a_2 = (a \vee c) \wedge b$.

By the modular inequality, $a_1 \leq a_2$. Then

$$a_1 \wedge c = ((c \wedge b) \vee a) \wedge c \geq (c \wedge b) \vee (a \wedge c) = c \wedge b$$

since $a \leq b$. Also, $a_1 \leq b$, so $a_1 \wedge c \leq b \wedge c$, and so $a_1 \wedge c = c \wedge b$. Furthermore,

$$a_2 \wedge c = ((a \vee c) \wedge b) \wedge c = (a \vee c) \wedge (b \wedge c) = b \wedge c = c \wedge b.$$

On the other hand,

$$a_2 \vee c = ((a \vee c) \wedge b) \vee c \leq (a \vee c) \wedge (b \vee c) = a \vee c$$

and since $a \leq a_2$, we have $a_2 \vee c \leq a \vee c$, whence $a \vee c = a_2 \vee c$. Finally,

$$a_1 \vee c = a \vee (c \wedge b) \vee c = (c \wedge b) \vee (a \vee c) = a \vee c.$$

This shows that a_1 and a_2 are complements of c in the interval $[b \wedge c, a \vee c]$, and so by assumption they are equal, i.e

$$a \vee (c \wedge b) = (a \vee c) \wedge b.$$

$\square$

From now on, $\leq$ refers to the containment relation on subobjects and quotients.

Proposition 4.26. *Let A be an object of an abelian category. Then the class of subobjects of A, partially ordered by containment, is a lattice.*

Proof. If i and j are subobjects of A, they form a span. Let i' and j' be the pullback of this span.

$$\begin{array}{ccc} D & \xrightarrow{\;j'\;} & A_i \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ A_j & \xrightarrow{\;j\;} & A \end{array}$$

Let

$$k = j'i = i'j.$$

By **Proposition 4.6**, the morphisms i' and j' are monic, and hence k is. Thus, k represents a subobject of A, which is contained in i and j.

Suppose that k' is an object that is contained in both i and j. Then there are morphisms i'' and j'' so that

$$k' = ji'' = ij''$$

and since the square is a pullback, there exists a unique morphism h so that $i'' = hi'$ and $j'' = hj''$. Thus, k' is contained in k, which show that k is the greatest lower bound of i and j.

Moreover, the morphisms i' and j' forms a cospan, which has a pushout i'' and j''

$$\begin{array}{ccc} A_i \wedge A_j & \xrightarrow{\;j'\;} & A_i \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i''} \\ A_j & \xrightarrow{\;j''\;} & P. \end{array}$$

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The dual of **Proposition 4.6** ensures that i'' and j'' are monic. Moreover, the definition of the pullback, yields $ii' = jj'$, and hence there is unique morphism p such that $pi'' = i$ and $pj'' = j$. Thus, p is a subobject of A that contains both i and j.

Suppose that s is a subobject which contains i and j, i.e there are morphisms f_1 and f_2 such that $i = sf_1$ and $j = sf_2$. Then

$$sf_1 i' = ii' = jj' = sf_2 j'$$

and since s is monic, $f_1 i' = f_2 j'$. Since the square is a pushout, there is a morphism h such that $f_1 = hi''$ and $f_2 = hj''$. Composing by s on the left yields

$$i = sf_1 = shi''$$

and

$$j = sf_2 = shj''.$$

By the uniqueness criterion of pushouts, $sh = p$, and thus p is a subobject of s. $\qquad\square$

Henceforth, the class of subobjects of an object is refered to as the *subobject lattice*. The meet of two subobjects is called the *intersection*, and the join is called their *product*.

Remark. By duality, the greatest lower bound of two quotients p and q of an object is given by the pushout of those quotients, and by **Proposition 3.27**, there are unique subobjects i and j such that $p = \text{cok}(i)$ and $q = \text{cok}(j)$.

Moreover, since the map cok is bijective and order reversing, the cokernel of $i \vee j$ is the greatest lower bound of $\text{cok}(i)$ and $\text{cok}(j)$. This means that the pushout of p and q is given by the cokernel of $i \vee j$, i.e

$$
\begin{array}{ccc}
A & \xrightarrow{\ q\ } & A/A_j \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\
A/A_i & \xrightarrow{\ q'\ } & A/(A_i \vee A_j).
\end{array}
$$

is a pushout.

Proposition 4.27. *Suppose that i and j are subobjects of an object A. Then there exists a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_i \wedge A_j & \xrightarrow{\ j'\ } & A_j & \xrightarrow{\ p\ } & A_j/A_i \wedge A_j & \longrightarrow & 0 \\
& & {\scriptstyle i'}\downarrow & & {\scriptstyle j''}\downarrow & & {\scriptstyle u}\downarrow & & \\
0 & \longrightarrow & A_i & \xrightarrow[\ i''\ ]{} & A_i \vee A_j & \xrightarrow[\ q\ ]{} & (A_i \vee A_j)/A_i & \longrightarrow & 0
\end{array}
$$

with exact rows such that u is an isomorphism.

Proof. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i \wedge A_j & \xrightarrow{\ j'\ } & A_j & \xrightarrow{\ q\ } & A_j/A_i \wedge A_j & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle k'} & & \\
0 & \longrightarrow & A_i & \xrightarrow{\ i\ } & A & \xrightarrow{\ q'\ } & A/A_i & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle c'} & & \\
0 & \longrightarrow & A_i/(A_i \wedge A_j) & \xrightarrow{\ k\ } & A/A_j & \xrightarrow{\ c\ } & C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

where i, i', j and j' are defined as in **Proposition 4.26**, and p, p', q and q' are the cokernels of i', j, j' and i respectively.

The morphism k exists and is unique, since

$$p'ii' = p'jj' = 0$$

and p is the cokernel of i', and that k' exist is proved similarly. The morphism c' is the cokernel of k', and the morphism c exists and is unique, since

$$c'q'j = c'k'q = 0$$

and p' is the cokernel of j. By definition, the columns and the two top rows are exact, and so the bottom one is also exact. Then,

$$c = \operatorname{cok}(k) = \operatorname{cok}(kp)$$

since p is epic, and by the dual of **Proposition 4.7** the bottom-right square is a pushout.

Hence, C is isomorphic to $A/(A_i \vee A_j)$, and so the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i \wedge A_j & \overset{j'}{\longrightarrow} & A_j & \overset{q}{\longrightarrow} & A_j/A_i \wedge A_j & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k'} & & \\
0 & \longrightarrow & A_i & \overset{i}{\longrightarrow} & A & \overset{q'}{\longrightarrow} & A/A_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle c'} & & \\
0 & \longrightarrow & A_i/(A_i \wedge A_j) & \overset{k}{\longrightarrow} & A/A_j & \overset{c}{\longrightarrow} & A/(A_i \vee A_j) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

is commutative with exact rows. To finish the proof, let $A = A_i \vee A_j$. $\qquad\square$

Morphisms between two objects can be extended to morphisms between the corresponding subobject lattices.

Definition 4.28. Let morphism $f : A \to B$ be a morphism. The *preimage* $f^{-1}(i)$ of a monomorphism $i : B_i \to B$ is the monomorphism i' in the pullback

$$
\begin{array}{ccc}
A' & \overset{f'}{\longrightarrow} & B_i \\
\downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\
A & \underset{f}{\longrightarrow} & B.
\end{array}
$$

The *image* of a monomorphism i into A is the image of fi.

Proposition 4.29. *Let $f : A \to B$ be a morphism.*

(i) *If i and i' are monomorphisms into A such that $i \le i'$, then $f(i) \le f(i')$.*

(ii) *If j and j' are monomorphisms into B such that $j \le j'$, then $f^{-1}(j) \le f^{-1}(j')$.*

Proof. (i) Since i is contained in i', there is a morphism φ' such that $i = i'\varphi$. In other words, the diagram

$$
\begin{array}{ccc}
A_i & \overset{fi}{\longrightarrow} & B \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} \\
A_j & \underset{fi'}{\longrightarrow} & B
\end{array}
$$

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commutes. By **Proposition 3.32** there is a commutative diagram

$$
\begin{array}{ccccc}
A_i & \xrightarrow{\ e\ } & \operatorname{im}(fi) & \xrightarrow{\ m\ } & B \\
{\scriptstyle \varphi}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
A_j & \xrightarrow[\ e'\ ]{} & \operatorname{im}(fi') & \xrightarrow[\ m'\ ]{} & B.
\end{array}
$$

Hence $f(i)$ is contained in $f(i')$.

(ii) Consider the pullback diagrams

$$
\begin{array}{ccc}
A_j & \xrightarrow{\ f'\ } & B_j \\
{\scriptstyle f^{-1}(j)}\downarrow & & \downarrow{\scriptstyle j} \\
A & \xrightarrow[\ f\ ]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A'_j & \xrightarrow{\ f''\ } & B'_j \\
{\scriptstyle f^{-1}(j')}\downarrow & & \downarrow{\scriptstyle j'} \\
A & \xrightarrow[\ f\ ]{} & B.
\end{array}
$$

Since j is contained in j', there is a morphism $\varphi : B_j \to B'_j$ such that $j = j'\varphi$.

Also, $jf' = ff^{-1}(j)$, and so $j'(\varphi f') = ff^{-1}(j)$.

By the universal property of pullbacks, there is a unique morphism h so that $f''h = \varphi f'$ and $f^{-1}(j')h = f^{-1}(j)$. Hence, $f^{-1}(j)$ is contained in $f^{-1}(j')$.

$\square$

Since $i \sim j$ if and only if $i \le j$ and $j \le i$, any morphism $f : A \to B$ induce order preserving maps

$$
f(-) : \mathcal{S}_A \to \mathcal{S}_B \quad \text{and} \quad f^{-1}(-) : \mathcal{S}_B \to \mathcal{S}_A.
$$

Proposition 4.30. *Let $f : A \to B$ be a morphism. Then*

(i) *any subobject i of A satisfies $f^{-1}(f(i)) \le i$.*

(ii) *any subobject j of B satisfies $f(f^{-1}(j)) \le j$.*

(iii) *the morphism f is monic if and only if $f^{-1}(f(i)) = i$ for all subobjects i of A.*

(iv) *the morphism f is epic if and only if $f(f^{-1}(j)) = j$ for all subobjects j of B.*

Proof. (i) Write fi as $f(i)e$, where $f(i)$ is the image of fi and e is the coimage of fi. Then $i' = f^{-1}(f(i))$ is defined via the pullback diagram

$$
\begin{array}{ccc}
A'_i & \xrightarrow{\ f'\ } & B_k \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle f(i)} \\
A & \xrightarrow[\ f\ ]{} & B.
\end{array}
$$

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Since $f(i)e = fi$ and the diagram is a pullback, there is a map h such that $ih = i'$ and $f'h = e$. Thus

$$i \leq i' = f^{-1}(f(i)).$$

(ii) The subobject $j' = f^{-1}(j)$ is defined via the pullback

$$
\begin{array}{ccc}
A'_j & \xrightarrow{\ f'\ } & B_j \\
{\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\
A & \xrightarrow{\ f\ } & B.
\end{array}
$$

The image of j' is defined as the image m of fj'. Let e be the coimage of fj'. Then $fj' = me$, and if k' is the kernel of $fj' = jf'$, then

$$e = \text{cok}(\text{ker}(fj')) = \text{cok}(k).$$

Then

$$0 = fj'k = jf'k$$

and since j is monic, $f'k = 0$. Thus, there is a morphism h such that $f' = he$.

$$
\begin{array}{ccccc}
K & \xrightarrow{\ k\ } & A'_j & \xrightarrow{\quad f'\quad} & B_j \\
 & & \downarrow{\scriptstyle j'} \ \ {\scriptstyle e}\searrow \quad {\scriptstyle h}\nearrow & & \downarrow{\scriptstyle j} \\
 & & & B'_j & \\
 & & & \searrow{\scriptstyle m} & \\
 & & A & \xrightarrow{\quad f\quad} & B.
\end{array}
$$

Moreover,

$$me = jf' = jhe,$$

and since e is epic, $m = jh$ and

$$j \leq m = f(f^{-1}(j)).$$

(iii) Suppose that f is monic. Then fi is monic, and so $f(i) = fi$. The preimage i'' of i' is given by the pullback

$$
\begin{array}{ccc}
A''_j & \xrightarrow{\ f'\ } & A_i \\
{\scriptstyle i''}\downarrow & & \downarrow{\scriptstyle fi} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

whence $fi'' = fif'$. Since f is monic, $i'' = if'$, and so $i'' \leq i$. In conjuntion with (i), one finds that $f^{-1}(f(i)) = i$ as subobjects.

For the converse, let k be the kernel of f. Then $fk = 0$, and so $f(k) = 0$. Hence

$$f^{-1}((f(k)) = f^{-1}(0) = 0.$$

Thus $k = 0$, and f is monic.

(iv) If f is epic, the morphism $f' = he$ from (ii) is epic and hence h is. Since h is always monic, h is an isomorphism and hence $f(f^{-1}(j)) = j$ as subobjects.

For the converse, note that $f^{-1}(B) = A$ for all morphisms f. Thus $f(f^{-1}(B)) = \operatorname{im}(f)$, and so $f(f^{-1}(k)) = k$ implies that $\operatorname{im}(f) = B$, so f is epic.

$\square$

The consequences of this is that

(i) if f is monic, then $f(-)$ is injective and $f^{-1}(-)$ is surjective.

(ii) if f is epic, then $f(-)$ is surjective and $f^{-1}(-)$ is injective.

(iii) if f is an isomorphism, f and f^{-1} are lattice isomorphism.

Proposition 4.31. *Let A be an object in an abelian category, and i a subobject of A. Then there is a lattice isomorphism from $\mathcal{S}_{A_i}$ to the interval $[0,i]$, and a lattice isomorphism from $\mathcal{S}_{A/A_i}$ to $[i, A]$.*

Proof. The morphism i defines an injective map $i(-)$ from $\mathcal{S}_{A_i}$ to $\mathcal{S}_A$, given by $i(f) = if$ for all subobjects f of A_i. A subobject j lies in $[0,i]$ if and only if j is contained in i, if and only if there is a monomorphism f such that

$$j = if = i(f).$$

Hence, the image of $i(-)$ is the interval $[0,i]$, and $\mathcal{S}_{A_i}$ and $[0,i]$ are lattice isomorphic.

For the other part, let $c : A \to A/A_i$ be the cokernel of i. Then $c^{-1}(-)$ is a injective map from $\mathcal{S}_{A/A_i}$ to $\mathcal{S}_A$, since c is surjective.

To show that $\mathcal{S}_{A/A_i}$ is lattice isomorphic to $[i, A]$, it suffices to show that the image of $c^{-1}(-)$ is $[i, A]$, i.e that a subobject of A is the left hand side of a pullback of a subobject of A/A_i precisely when the subobject contains i.

Suppose that j is a subobject of A/A_i. The subobject $j' = c^{-1}(j)$ is defined by the pullback

$$
\begin{array}{ccc}
A'_j & \xrightarrow{\ c'\ } & A_j \\
{\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\
A & \xrightarrow[\ c\ ]{} & A/A_i
\end{array}
$$

and since $ic = 0 = j0$, the universal property of pullbacks implies that there is a morphism h such that $j'h = i$, i.e $i \le j' = c^{-1}(j)$.

Conversely, suppose that j is a subobject of A such that $i \le j$ via a morphism f. Using a similar argument as in the proof of **Proposition 4.11**, one can construct an exact, commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i & \xrightarrow{\ f\ } & A_j & \xrightarrow{\ e\ } & A_j/A_i & \longrightarrow & 0 \\
 & & {\scriptstyle\text{id}}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle m}\downarrow & & \\
0 & \longrightarrow & A_i & \xrightarrow{\ i\ } & A & \xrightarrow{\ c\ } & A/A_i & \longrightarrow & 0 \\
 & & {\scriptstyle j}\downarrow & & {\scriptstyle c'}\downarrow & & {\scriptstyle c''}\downarrow & & \\
0 & \longrightarrow & A/A_j & \xrightarrow{\ u\ } & A/A_j & \longrightarrow & & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Since the diagram is exact and u is an isomorphism,

$$j = \ker(c') = \ker(uc') = \ker(c''c),$$

i.e the sequence

$$0 \longrightarrow A_j \xrightarrow{\ j\ } A \xrightarrow{\ c''c\ } A/A_j$$

is exact. By **Proposition 4.7**, the square

$$
\begin{array}{ccc}
A_j & \xrightarrow{\ e\ } & A_j/A_i \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle m} \\
A & \xrightarrow[\ c\ ]{} & A/A_i
\end{array}
$$

is a pullback, so j lies in the image of $c^{-1}(-)$. $\qquad\qquad\square$

The above proposition shows that any interval $[i, j]$ in the subobject lattice of an object is isomorphic to the subobject lattice of A_j/A_i. This is key to proving that the subobject lattice of an object in an abelian category is modular.

But first, the time has come to return to complements.

Proposition 4.32. *Suppose that i and j are subobjects of A such that $i \wedge j = 0$ and $i \vee j = A$. Then there are morphisms p and q so that i, j, p and q forms a direct sum system.*

Proof. Consider the direct sum system

$$A_i \underset{p_1}{\overset{i_1}{\rightleftarrows}} A_i \oplus A_j \underset{i_2}{\overset{p_2}{\rightleftarrows}} A_j.$$

Let $h = ip_1 + jp_2$. The image of h is $i \vee j$ (see the proof of **Proposition 4.26**). Calculations yields $hi_1 = i$ and $hi_2 = j$.

By assumption, $i \vee j = A$, and so h is epic.

Let k satisfy $hk = 0$. Then $ip_1 k + jp_2 k = 0$, which imply that

$$ip_1 k = -jp_2 k = j(-p_2 k).$$

Since $i \wedge j = 0$, the pullback of i and j is 0, and thus $p_1 k = 0$ and $p_2 k = 0$. Thus

$$i_1 p_1 k = i_2 p_2 k = 0,$$

and so

$$0 = i_1 p_1 k + i_2 p_2 k = (i_1 p_1 + i_2 p_2)k = k.$$

Hence h is monic, and an isomorphism. Let $p = p_1 h^{-1}$ and $q = p_2 h^{-1}$. Then

$$\mathrm{id}_A = hh^{-1} = i(p_1 h^{-1}) + j(p_2 h^{-1}) = ip + qp.$$

Moreover,

$$pi = p_1 h^{-1} hi_1 = p_1 i_1 = \mathrm{id}_{A_1}$$

and similarly $qj = p_2 i_2 = \mathrm{id}_{A_2}$. Finally, $iq = i_1 p_2 = 0$ and $jp = i_2 p_1 = 0$, and so i, j, p and q form a direct sum system. $\qquad\square$

Proposition 4.33. *Let i be a subobject of A. Suppose that j and j' are complements of i in the subobject lattice, and $j \leq j'$. Then $j = j'$.*

Proof. Suppose that $i : A_i \to A$ has complements $j : A_j \to A$ and $j' : A'_j \to A$, such that $j \leq j'$ via a monomorphism $f : A_j \to A'_j$ such that $j'f = j$. By **Proposition 4.32**, there is are direct sum systems

$$A_j \underset{j}{\overset{p}{\rightleftarrows}} A \underset{q}{\overset{i}{\rightleftarrows}} A_i$$

and

$$A'_j \underset{j'}{\overset{p'}{\rightleftarrows}} A \underset{q'}{\overset{i}{\rightleftarrows}} A_i.$$

Hence

$$jp + ip = \mathrm{id}_A = j'q' + ip'.$$

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Substituting $j'f$ for j multiplying from the left by q' yields

$$q'j'fq + q'ip = q'j'q' + q'ip'$$

and so $fq = q'$. Since q' is epic, **Proposition 2.7** ensures that f is epic, and hence f is an isomorphism, i.e $j = j'$ as subobjects. $\square$

Proposition 4.34. *The subobject lattice of an object A is modular.*

Proof. Let i and j be subobjects of A and consider the interval $[i, j]$ in $\mathcal{S}_A$. Suppose that k is a subobject in $[i, j]$ with complements c and c' such that $c \leq c'$.

The interval $[i, j]$ is lattice isomorphic to $\mathcal{S}_{A_j/A_i}$, and since complements are preserved under isomorphism, the complements c and c' are equal. Thus $\mathcal{S}_A$ is modular. $\square$

5 Classification Theorems

The Jordan-Hölder theorem asserts that if an object has a maximal chain of subobjects, then all maximal chains of subobjects are equivalent. However, the theorem does not assert that every object in an abelian category has such a chain, and in fact many do not.

The Krull-Schmidt-Remak theorem has a similar problem. Fortunately, one can show that if the Jordan-Hölder theorem applies, then so does the Krull-Schmidt-Remak theorem. At the end of the thesis, an example is given where the Krull-Schmidt-Remak theorem applies in a context outside abelian categories.

5.1 The Jordan-Hölder Theorem

The Jordan-Hölder theorem classifies object up to isomorphism, using maximal chain of subobjects.

Definition 5.1. An nonzero object in an abelian category is *simple* if every subobject of it is equivalent to either zero or the identity.

Proposition 5.2. *Let i and j be two subobjects of an object A and suppose that i is contained in j. Then, the following are equivalent.*

(i) i is properly contained in j.

(ii) A_j/A_i is nonzero.

(iii) $\mathcal{S}_{A_j/A_i}$ has at least two members.

Proof. If i and j are equivalent via a morphism f, then f is an isomorphism and $\mathrm{cok}(f) = 0$, i.e $A_j/A_i = 0$. Conversely, if $\mathrm{cok}(f) = 0$, then f is epic. Since f always monic, it is both monic and epic, and hence an isomorphism. Consequently, i and j are equivalent.

The third equivalence follows from that $[i, j]$ is isomorphic to the subobject lattice of A_j/A_i, which has one member if and only if i and j are equal. $\qquad\square$

Remark. An object is simple if and only if its subobject lattice contains exactly two subobjects.

Proposition 5.3. *Suppose that i and j are subobjects of an object in an abelian category such that i is properly contained in j. Then there is a subobject k such that*

$$i < k < j$$

if and only if A_j/A_i is not simple.

Proof. Let A be an object with subobjects i and j. Then A_j/A_i is not simple if and only if there is a chain of subobjects

$$0 < q < A_j/A_i.$$

Since the subobject lattice $\mathcal{S}_{A_j/A_i}$ is isomorphic to $[i, j]$, this holds if and only if there is a subobject k of A such that $i < k < j$. $\square$

Remark. It is clear that if the subobject $i < j$ are such that A_j/A_i is simple, and

$$i \leq k_1 \leq \cdots \leq k_n < j$$

for subobjects $k_1, \ldots, k_n$, then k_l is equal to precisely one i and j for all l. Moreover, if $k_l = j$ for some l, then $k_{l+1} = j$.

Definition 5.4. A *filtration* of an object A in an abelian category is a sequence

$$0 = i_0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-1} \leq i_n = A$$

of subobjects of A. The quotients i_{j+1}/i_j are called *factors*. A filtration is *proper* if every containment relation is proper.

Remark. Clearly, a filtration is proper if and only if all factors are nonzero.

Definition 5.5. A *refinement* of a filtration

$$0 = i_0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-1} \leq i_n = A.$$

is an filtration

$$0 = j_0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-1} \leq j_m = A.$$

such that i_k is equal to a unique j_l for all k.

Remark. In group theory, filtrations are called normal series, and Jordan-Hölder series are called composition series.

Definition 5.6. A filtration

$$0 = i_0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-1} \leq i_n = A.$$

of an object A in an abelian category is *maximal* if

$$i_j \leq k \leq i_{j+1} \Rightarrow k = i_j \text{ or } k = i_{j+1}$$

for all j. A maximal filtration that is proper is a *Jordan-Hölder series*.

Proposition 5.7. *A filtration is maximal if and only if all its factors are simple or zero and Jordan-Hölder if and only if every factor is simple.*

Proof. Apply **Proposition 5.3** and **Proposition 5.2** inductively. □

Proposition 5.8. *The nonzero factors of a Jordan-Hölder series are the same as those of each of its refinements.*

Proof. Let A be an object and

$$0 = i_0 < i_1 < i_2 < \cdots < i_{n-1} < i_n = A.$$

a Jordan-Hölder series. Let

$$0 = j_0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-1} \leq j_m = A.$$

be a refinement of i_k. For every k there is a number p_k so that $i_k = j_{p_k}$. For each k, consider the subsequence

$$i_k = j_{p_k} \leq j_{p_k+1} \leq \cdots \leq j_{p_{k+1}-1} \leq j_{p_{k+1}} = i_{k+1}.$$

This lies in the interval $[i_k, i_{k+1}]$. Since i_{k+1}/i_k is simple, this interval contains only two members. So either $j_{p_k+l} = i_k$ or $j_{p_k+l} = i_{k+1}$ for all l, and so the quotient j_{p_k+l+1}/j_{p_k+l} is either 0 or i_{k+1}/i_k. T

he latter case can only happen once, since the sequence is increasing, which shows that a refinement of i_k does not contain any new, nonzero factors. □

Definition 5.9. Two filtrations

$$0 = i_0 \leq i_1 \leq \cdots \leq i_n \leq A$$

and

$$0 = j_0 \leq j_1 \leq \cdots \leq j_m \leq A,$$

with factors are *equivalent* if $n = m$ and there is a bijection σ such that i_{k+1}/i_k and $j_{\sigma(k)+1}/j_{\sigma(k)}$ for all k.

Example 5.10. Let C_n denote the cyclic group on n elements. The filtrations

$$0 = C_1 \leq C_2 \leq C_6 \leq C_{12}$$

and

$$0 = C_1 \leq C_2 \leq C_4 \leq C_{12},$$

have factors C_2, C_2 and C_3 in different orders. Hence, they are equivalent.

Note that two filtrations can be equivalent, despite the subobjects in the filtration being different.

The Jordan-Hölder theorem states that in an abelian category, all Jordan-Hölder series are equivalent.

Proposition 5.11 (Zassenhaus' lemma). *Let i, i', j and j' be subobjects of A, such that $i' \leq i$ and $j' \leq j$. Then*

$$\frac{(i \wedge j) \vee i'}{(i \wedge j') \vee i'} = \frac{(i \wedge j) \vee j'}{(i' \wedge j) \vee j'}$$

as quotients of A.

Proof. Let $s = i \wedge j$ and $t = i' \vee (i \wedge j')$. Since $j' \leq j$, the subobject $i \wedge j$ contains $i \wedge j'$, and hence $(i \wedge j) \vee i'$ contains $(i \wedge j')$. Therefore,

$$s \vee t = (i \wedge j) \vee (i' \vee (i \wedge j')) = ((i \wedge j) \vee i') \vee (i \wedge j') = (i \wedge j) \vee i' = i' \vee (i \wedge j).$$

On the other hand, $i \wedge j'$ is contained in $i \wedge j$, and

$$t \wedge s = ((i \wedge j') \vee i') \wedge (i \wedge j) = (i \wedge j') \vee (i' \wedge (i \wedge j)) = (i \wedge j') \vee ((i' \wedge i) \wedge j) = (i \wedge j') \vee (i' \wedge j)$$

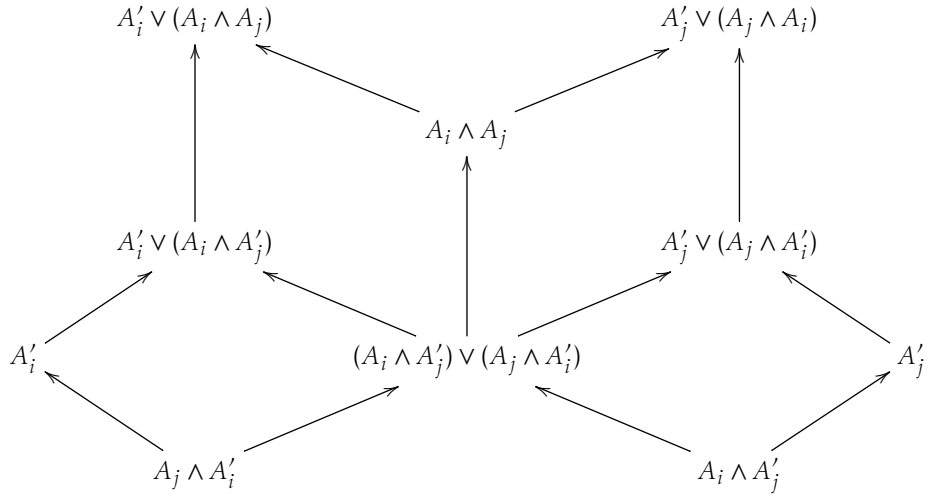via the modular law. The third isomorphisms theorem yields

$$\frac{i' \vee (i \wedge j)}{i' \vee (i \wedge j')} = \frac{s \vee t}{t} = \frac{s}{s \wedge t} = \frac{i \wedge j}{(i \wedge j') \vee (i' \wedge j)}.$$

Switching i for j and i' for j' in the above argument, yields

$$\frac{j' \vee (j \wedge i)}{j' \vee (j \wedge i')} = \frac{j \wedge i}{(j \wedge i') \vee (j' \wedge i)} = \frac{i \wedge j}{(i \wedge j') \vee (i' \wedge j)} = \frac{i' \vee (i \wedge j)}{i' \vee (i \wedge j')}.$$

$\square$

Remark. Zassenhaus' lemma is sometimes called *butterfly lemma*, due to the shape of the diagram of subobjects involved:

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Zassenhaus' lemma gives a smooth proof of the following theorem.

Proposition 5.12 (Schreier's refinement theorem). *Any two filtrations of an object in an abelian category have refinements that are equivalent to each other.*

Proof. Let A be an object in an abelian category with filtrations

$$0 = i_0 \leq i_1 \leq \cdots \leq i_{m-1} \leq i_m = A$$

and

$$0 = j_0 \leq j_1 \leq \cdots \leq j_{n-1} \leq j_n = A.$$

Every number $0 \leq a \leq nm - 1$ can be written uniquely as $a = ns + t$ for $0 \leq s \leq m - 1$ and $0 \leq t \leq n - 1$, and as $a = mp + k$ for unique numbers $0 \leq p \leq n - 1$ and $0 \leq k \leq m - 1$. Moreover, the map

$$\sigma : \{0, \dots, nm - 1\} \to \{0, \dots, nm - 1\}$$

defined by $\sigma(ns + t) = mt + s$ is bijective.

Define the filtrations

$$0 = k_0 \leq k_1 \leq \cdots \leq k_{nm-1} \leq p_{nm} = A$$

and

$$0 = p_0 \leq p_1 \leq \cdots \leq p_{nm-1} \leq p_{nm} = A$$

by

$$k_{nr+s} = i_r \vee (i_{r+1} \wedge j_s)$$

and

$$p_{ms+r} = j_s \vee (j_{s+1} \wedge i_r)$$

for $0 \leq s \leq n - 1$ and $0 \leq r \leq m - 1$, and set $k_{mn} = p_{nm} = A$.

Then, $k_{nr} = i_r$ and $p_{ms} = j_s$ for all r and s. In particular, $k_0 = 0$ and $p_0 = 0$.

Moreover, $i_{r+1} \wedge j_s$ is contained in i_{r+1} for all s. Thus

$$i_r = k_{nr} \leq k_{nr+1} \leq \cdots \leq k_{nr+(n-1)} \leq k_{n(r+1)} = i_{r+1}$$

for all r. Similarly, $j_{s+1} \wedge i_r$ is contained in j_{s+1} for all r, and

$$j_s = p_{ms} \leq p_{ms+1} \leq \cdots \leq p_{ms+(m-1)} \leq p_{m(s+1)} = j_{s+1}.$$

In other words, k_{nr+s} and p_{m+_r} are refinements of i_k and j_l.

Let $i' = i_r$, $i = i_{r+1}$, $j' = j_s$ and $j = j_{s+1}$. Then $i' \leq i$ and $j' \leq j$. Moreover,

$$k_{nr+s} = i' \vee (i \wedge j'), \quad k_{nr+s+1} = i' \vee (i \wedge j)$$

and

$$p_{ms+r} = j' \vee (j \wedge i'), \quad p_{ms+r+1} = j' \vee (j \wedge i)$$

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Zassenhaus' lemma yields

$$\frac{k_{nr+s+1}}{k_{nr+s}} = \frac{i' \vee (i \wedge j)}{i' \vee (i \wedge j')} = \frac{j' \vee (j \wedge i)}{j' \vee (j \wedge i')} = \frac{p_{ms+r+1}}{p_{ms+r}} = \frac{p_{\sigma(nr+s)}}{p_{\sigma(nr+s)+1}}.$$

Since σ is a bijection, this shows that the refinements k_{nr+s} and p_{ms+r} are equivalent. □

Proposition 5.13 (Jordan-Hölder theorem). *All Jordan-Hölder series of an object are equivalent.*

Proof. By Schreier's refinement theorem, all Jordan-Hölder series have refinements that are equivalent. These refinements have the same nonzero factors as the original Jordan-Hölder series, by **Proposition 5.8**, and since they are all equivalent to each other they all have the same nonzero factor.

Hence, all Jordan-Hölder series have the same nonzero factors, and since all factors of a Jordan-Hölder series are nonzero they are equivalent. □

Since all Jordan-Hölder series are equivalent, they have the same length.

Definition 5.14. An object is of *finite length* if it has a Jordan-Hölder series. The *length* of an object of finite length is the number of objects in any of its Jordan-Hölder series.

Example 5.15. A vector space over a field K is of finite length if and only if it is isomorphic to K^n for some n, in which case it has length n. A Jordan-Hölder series for K^n is given by

$$0 < K < K^2 < \cdots < K^{n-1} < K^n.$$

Example 5.16. The Jordan-Hölder theorem is a generalization of the fundamental theorem of arithmetic.

Let C_n denote the cyclic group on n elements. Then C_n has finite length, since it is finite, and abelian. Moreover, every subgroup of C_n is cyclic, and if C_m is a subgroup of C_n then m divides n and the quotient C_n/C_m is isomorphic to $C_{n/m}$. Finally, C_n is simple if and only if n is prime.

Consider a Jordan-Hölder series

$$0 = C_{n_0} < C_{n_1} < \cdots < C_{n_{m+1}} < C_{n_m} = C_n.$$

The factors are simple cyclic groups of order $n_k/n_{k+1} = p_k$, where p_k is prime. Iteration yields $n = p_0 p_1 \cdots p_{m-1}$. Hence, the factors of a Jordan-Hölder series of the cyclic group C_n correspond to a prime factorizations of n.

Since two cyclic groups are isomorphic if and only if they have the same order, two Jordan-Hölder series of C_n are equivalent if and only if they correspond to the same prime factorization of n. Thus, the Jordan-Hölder theorem implies that prime factorization of any natural number is unique up to permutation.

Remark. Not all objects in an abelian category have a Jordan-Hölder series. Indeed, any filtration

$$0 = 0\mathbb{Z} < p_{n-1}\mathbb{Z} < p_{n-2}\mathbb{Z} < \cdots < p_1\mathbb{Z} < \mathbb{Z}$$

of $\mathbb{Z}$ has a proper refinement

$$0 = 0\mathbb{Z} < 2p_{n-1}\mathbb{Z} < p_{n-1}\mathbb{Z} < p_{n-2}\mathbb{Z} < \cdots < p_1\mathbb{Z} < \mathbb{Z}.$$

5.2 The Krull-Schmidt-Remak Theorem

The Krull-Schmidt-Remak theorem does not apply to all abelian categories, but only those where the objects have a particular form. Here, it is assumed that they are Artinian.

Definition 5.17. An object A in a category is *Artinian* if there is no infinite, descending sequence of proper subobjects of A. A category is Artinian if each of its objects are Artinian.

Remark. Most of the ideas in this section is from Atiyah [Ati56], who proved the Krull-Schmidt-Remak theorem in slightly more general circumstances.

Example 5.18. A vector space is Artinian if and only if it is has finite dimension. The group of integers is not Artinian.

Definition 5.19. An object in an additive category is *decomposable* if it is a direct sum of nonzero subobjects of itself. Otherwise, it is called *indecomposable*.

Example 5.20. The only indecomposable vector spaces have dimension one.

All simple objects are indecomposable. In general, the reverse is not true. For example, $\mathbb{Z}$ has many proper subobjects, but is indecomposable.

Every endomorphism generates chains of subobjects.

Proposition 5.21. *Let f be an endomorphism of an object A in an abelian category. Then* $\ker(f^n)$ *is contained in* $\ker(f^{n+1})$ *and* $\mathrm{im}(f^{n+1})$ *is contained in* $\mathrm{im}(f^n)$ *for all n.*

Proof. Let k_n and k_{n+1} be the kernel of f^n and f^{n+1} respectively. Then

$$k_n f^{n+1} = k_n f^n f = 0,$$

so there is a morphism h such that $k_n = k_{n+1}h$. Hence $\ker(f^n)$ is contained in $\ker(f^{n+1})$.

For the second containment, consider the commuting square

$$
\begin{array}{ccc}
A & \xrightarrow{f^{n+1}} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow{f^n} & A.
\end{array}
$$

and let $f^n = m_n e_n$ and $f^{n+1} = m_{n+1} e_{n+1}$ be the canonical decomposition of f^n and f^{n+1}. By **Proposition 3.32**, there is a morphism φ so that the diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{e_{n+1}} & \operatorname{im}(f^{n+1}) & \xrightarrow{m_{n+1}} & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow{e_n} & \operatorname{im}(f^n) & \xrightarrow{m_n} & A
\end{array}
$$

commutes. In particular, $m_{n+1} = m_n \varphi$, so $\operatorname{im}(f^{n+1})$ is contained in $\operatorname{im}(f^n)$. $\square$

Proposition 5.22 (Fitting's lemma). *Suppose that A is an Artinian object and f is an endomorphism on A. Then there is a positive integer n such that*

$$A = \operatorname{im}(f^n) \oplus \ker(f^n).$$

Proof. By **Proposition 5.21**, any endomorphism on A gives rise to a sequence

$$\cdots \leq \operatorname{im}(f^n) \leq \cdots \leq \operatorname{im}(f^2) \leq \operatorname{im}(f)$$

of subobjects of A. The subobject morphisms $m_n : \operatorname{im}(f^n) \to A$ are defined via the canonical decomposition $f^n = m_n e_n$, and the containment morphisms i_{n+1} are defined by the diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{e_{n+1}} & \operatorname{im}(f^{n+1}) & \xrightarrow{m_{n+1}} & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_{n+1}} & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow{e_n} & \operatorname{im}(f^n) & \xrightarrow{m_n} & A
\end{array}
$$

from **Proposition 5.21**. Similarly, the commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{f^n} & A \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} \\
A & \xrightarrow{f^{n+1}} & A
\end{array}
$$

yields a decomposition

$$
\begin{array}{ccccc}
A & \xrightarrow{e_n} & \operatorname{im}(f^n) & \xrightarrow{m_n} & A \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle f} \\
A & \xrightarrow{e_{n+1}} & \operatorname{im}(f^{n+1}) & \xrightarrow{m_{n+1}} & A.
\end{array}
$$

This gives the identities $e_{n+1} = p_{n+1} e_n$ and $m_{n+1} = m_n i_{n+1}$. Applying these inductively yields

$$m_n e_n m_n = f^n m_n$$

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and by repe

$$m_n(i_{n+1}\cdots i_{2n}p_{2n}\cdots p_{n+1}) = m_{2n}p_{2n}\cdots p_{n+1} = f m_{2n-1}p_{2n-1}\cdots p_{n+1} = \cdots = f^n m_n.$$

Equating the two expressions yields

$$m_n e_n m_n = m_n i_{n+1}\cdots i_{2n}p_{2n}\cdots p_{n+1}.$$

and since m_n is monic, one has

$$e_n m_n = i_{n+1}\cdots i_{2n}p_{2n}\cdots p_{n+1}.$$

Suppose that A is Artinian. Then for large enough n, all morphisms i_n and p_n are isomorphisms, and hence $\sigma_n = e_n m_n$ is an isomorphism as well.

Hence, the morphism $m_n\sigma_n^{-1}$ is a right split for the short exact sequence

$$0 \longrightarrow \ker(f^n) \xrightarrow{k_n} A \xrightarrow{e_n} \operatorname{im}(f^n) \longrightarrow 0,$$

since

$$e_n m_n \sigma_n^{-1} = \sigma_n \sigma_n^{-1} = \operatorname{id}_{\operatorname{im}(f^n)}.$$

By the splitting lemma, A is isomorphic to $\ker(f^n)\oplus\operatorname{im}(f^n)$. □

Definition 5.23. An endomorphism f in a preadditive category is *nilpotent* if $f^n = 0$ for some n.

Note that in abelian categories, $f^n = 0$ if and only if $\ker(f^n)$ is the domain of f.

Proposition 5.24. *Any endomorphism on an Artinian object which is indecomposable in an abelian category is either nilpotent or an automorphism.*

Proof. Let A be such an object and f an automorphism. By Fitting's lemma, $A = \operatorname{im}(f^n)\oplus\ker(f^n)$ for some n. Since A is indecomposable, either $\operatorname{im}(f^n) = 0$ or $\ker(f^n) = 0$. In the first case, $\ker(f^n) = A$ and f is nilpotent.

In the latter case, $\ker(f^i) = 0$ for all positive integers i, since the kernels form an ascending sequence of subobjects of A. Similarly, $\operatorname{im}(f^i) = A$ for all positive integers i, since the images form a descending sequence of subobjects. Thus $\operatorname{im}(f) = A$ and $\ker(f) = 0$, and f is an automorphism. □

Proposition 5.25. *If $f_1, \ldots, f_n$ is a sequence of endomorphisms in an abelian category on an Artinian object which is indecomposable and*

$$\sum_{i=1}^{n} f_i$$

is an automorphism, then f_i is an automorphism for some i.

Proof. Induction on n. If $n = 2$,

$$f_1 + f_2 = g$$

for some automorphism g. Multiplying by g^{-1} yields

$$g^{-1}f_1 + g^{-1}f_2 = \mathrm{id}_A.$$

Since g is an automorphism, it suffices to show that either $f_1' = g^{-1}f_1$ or $f_2' = g^{-1}f_2$ is an automorphism.

By Fitting's Lemma, f_1' and f_2' are either automorphisms or nilpotent. Suppose that both are nilpotent. Then there is some m such that $f_1'^m = 0$ and $f_2'^m = 0$.

If $f_1' + f_2' = \mathrm{id}_A$, then $f_2' = \mathrm{id} - f_1'$, and so

$$f_1'f_2' = f_1'(\mathrm{id} - f_1') = f_1' - f_1'^2 = (\mathrm{id} - f_1')f_1' = f_2'f_1'.$$

Since f_1' and f_2' commute, the binomial theorem can be applied.

$$(f_1' + f_2')^{2m} = \mathrm{id}_A^m = \mathrm{id}_A \Leftrightarrow \mathrm{id}_A = \sum_{k=0}^{2m} \binom{2m}{k} f_1'^{2m-k} f_2'^k = 0.$$

Since A is nonzero, this contradicts **Proposition 3.7** and so either f_1' or f_2' is an automorphism.

For the induction step, suppose that the statement holds for all sums of length p and that

$$\sum_{i=1}^{p+1} f_i = f_{p+1} + \sum_{i=1}^{p} f_i$$

is an automorphism. Then either f_{p+1} or $\sum_{i=1}^{p} f_i$ is an automorphism, and the induction hypothesis gives the result. $\qquad\square$

Proposition 5.26. *Any object in an Artinian category can be written as a direct sum of indecomposable subobjects.*

Proof. An object is *bad* if it cannot be written as a direct sum of indecomposable objects. The task is to show that if A is bad, it cannot be Artinian, by constructing an infinite sequence $A_0, A_1, A_2, \ldots,$ of bad objects, such that $A_0 = A$ and A_{n+1} is are nontrivial direct summands of A_n for all n.

Suppose that A is bad, and let $A_0 = A$. Assume that objects $A_0, \ldots, A_p$ have been chosen so that every A_i is bad, and A_{i+1} is a nontrivial direct summand of A_i.

Since A_p is bad, it is not indecomposable, and so there is a nontrivial direct sum

$$A_p = B \oplus C$$

of A_p. Suppose neither B nor C is bad. Write B and C as sums

$$B = \bigoplus_{j=1}^{n} B_j \quad \text{and} \quad C = \bigoplus_{k=1}^{m} C_k$$

of indecomposable objects. Then

$$A_p = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus C_1 \oplus \cdots \oplus C_m$$

is the direct sum of indecomposable objects, which contradicts the assumption that A_p was bad. So at least one of B and C is bad. Let A_{p+1} be one of them. Then A_{p+1} is bad and a nontrivial direct summand of A_p, which completes the induction. □

Proposition 5.27 (Krull-Schmidt-Remak). *Let A be an object in an Artinian category, with decompositions*

$$A = \bigoplus_{j=1}^{n} A_j \quad \text{and} \quad A = \bigoplus_{k=1}^{m} A_k'$$

into indecomposable subobjects. Then $n = m$, and there is a permutation σ such that $A_j \simeq A_{\sigma(j)}'$ for all j.

Proof. Suppose A has two decompositions

$$A = \bigoplus_{j=1}^{n} A_j \quad \text{and} \quad A = \bigoplus_{k=1}^{m} A_k'.$$

where A_j and A_k' are indecomposable. Without loss of generality, $n \leq m$.

When $n = 1$, the object A is indecomposable and the theorem holds. Suppose that the theorem holds for some $p \geq 2$, and that

$$A = \bigoplus_{j=1}^{p+1} A_j \quad \text{and} \quad A = \bigoplus_{k=1}^{m} A_k'.$$

for some indecomposables, with $m \geq p + 1$.

Let i_j, i_k', p_j and p_k' be the injection and projection morphisms defining the direct sums. Define $f_s = p_1 i_s' p_s' i_1$ for $s = 1, \ldots, m$. Then

$$\sum_{s=1}^{m} f_s = p_1 \left(\sum_{s=1}^{m} i_s' p_s' \right) i_1 = p_1 (\mathrm{id}_A) i_1 = \mathrm{id}_{A_1}.$$

By **Proposition 5.25**, there is some f_s which is an isomorphism. Decomposing f_s gives yields the diagram

$$A_1 \xrightarrow{i_1' p_1} A_s' \xrightarrow{i_s p_s'} A_1$$

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and by the remark after **Proposition 4.4**, A_1 is a direct summand of A_s'. But A_s' is indecomposable, so A_1 and A_s' are isomorphic. Let

$$B = \bigoplus_{j=2}^{p+1} A_j \quad \text{and} \quad B' = \bigoplus_{k=1, k \neq s}^{m} A_k'.$$

By definition

$$A \simeq A_1 \oplus B \simeq A_s' \oplus B'.$$

Let p, i, i' and p' denote the projection and injection morphisms corresponding to B and B', and φ the isomorphism from $A_1 \oplus B$ to $A_1 \oplus B'$. If $f = p'\varphi i$ from B to B', the diagram

commutes. Moreover, f is an isomorphism with inverse $g = p\varphi^{-1} i'$, since

$$fg = p'\varphi i p \varphi^{-1} i' = \mathrm{id}_{B'}$$

and

$$gf = p\varphi^{-1} i' p' \varphi i = \mathrm{id}_B.$$

Thus, B and B' are isomorphic, and the induction hypothesis applies. $\qquad\square$

Remark. If an object has a Jordan-Hölder series it must be Artinian, since no proper sequence of subobjects of an object can be longer than a Jordan-Hölder series. Thus the Krull-Schmidt-Remak theorem holds in all abelian categories of finite length.

The Krull-Schmidt-Remak theorem can be extended to other categories than abelian categories of finite length. For example, let R be a commutative, Noetherian and complete local ring, and A an R algebra which is finitely generated as a module over R. Then the Krull-Schmidt-Remak theorem holds for the class of finitely generated left A-modules.[Rei03, p.88]

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