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Introduction to rational billiards

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Abstract

In this paper we introduce and illustrate some properties of billiards, later we will focus on rational polygonial billiards and develop tools and prove a theorem regarding them. We will also see these tools are well suited to study other related problems. The last section will include some intuition and counterexamples regarding existance of billiards with strange properties.

Contents

1 Introduction

1.1 Historical background

In 1963 Yakov Sinai published a paper in which he proved that the Sinai billiard is chaotic and ergodic. In the Sinai billiard a particle bounces around in a square with a circle cut out from the center. The Sinai billiard was the first dynamical system which was proved chaotic and ergodic [4]. A system is chaotic if small changes in inital values have significantly different behavour and ergodic means that the particle pass through all available space. Since his publication the subject of billiards have continued to be studied and extended upon.

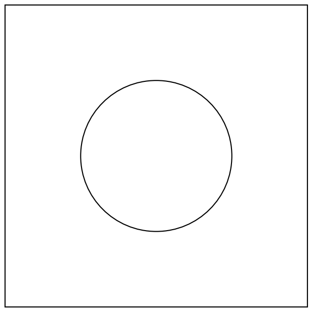

Figure 1: The Sinai billiard

1.2 Overview

The study of billiards is not a single confined theory or subject. The subject is too vast to be covered all at once and there are many different approaches to billiards and methods to study them. In his lectures *Billiard table as a mathematician's playground* A.B. Katok says the following: "In a more serious sense, the expression "playground" should be understood as "testing grounds": various questions, conjectures, methods of solution, etc. in the theory of dynamical systems are "tested" on various types of billiard problems."[3]. There are many known results concerning rational billiards. They are a subgroup of billiards which are reasonably easy to work with. As soon as we don't restrict ourselves we come across simple questions whose answer is not known, e.g. "Do all triangular billiards have periodic orbits?". It have been proven all triangles with degrees less than 100° have periodic trajectories [5].

1.3 Definitions

Definition 1.1. A two dimensional billiard is a domain bounded by a piecewise smooth closed curve in $\mathbb{R}^2$ and a point-mass. The point moves with unit speed along straight lines and when it hits the boundary its tangential velocity remains the the same and the normal velocity changes sign. In the case the ball hits the curve on a point where it is not smooth, the reflection is undefined and the motion terminates.

Lemma 1.2. *When a billiard trajectory is reflected in the boundary the law of reflection holds i.e the angle of incidence is equal to the angle of reflection.*

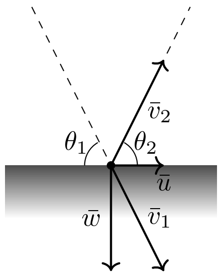

Figure 2: A reflection in a straight line.

Proof. Let $\bar{v}_1, \bar{v}_2$ be the velocity vector of the particle before and after reflection, $\bar{u} = (u_1, u_2)$ and $\bar{w} = (w_1, w_2)$ be the tangential and normal components respectively such that $\bar{v}_1 = \bar{u} + \bar{w}$ and $\bar{v}_2 = \bar{u} - \bar{w}$. Let θ_1 be the angle of incidence and θ_2 be the angle of reflection. Then $\cos\theta_1 = \frac{||\bar{u}||}{||\bar{v}_1||} = ||\bar{u}||$ and $\cos\theta_2 = \frac{||\bar{u}||}{||\bar{v}_2||} = ||\bar{u}||$ and beacause $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$ we can draw the conclusion $\theta_1 = \theta_2$. $\square$

The study of billiards revolve around the general behaviour of this motion. This definition can be generalized to n-dimensional, non-euclidian space.

A ratinal polygononial billiard is a billiard whose boundary is a polygon whose angles are rational multiples of π.

1.4 Motivation

Billiards can be used to model mechanical systems but is also of interest to quantum physics, e.g the phase space of a one-dimensional mechanical system with two elastic particles can be represented as a two-dimensional billiard.

I came in contact with billiards when i was presented a counterexample to a question I had about non-dense, non-periodic geodesics on polyhedral surfaces. In fact, a billiard and its trajectories can be seen as geodesics on a flat polyhedron. To further empathize billiards as a "mathematicians playground" we will see that some of the tools we will develop in this paper are indeed very well suited to study geodesics on polyhedrons.

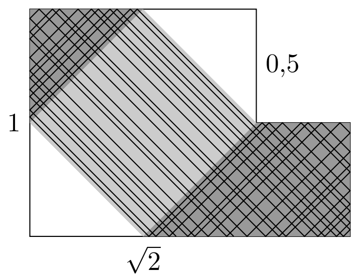

Figure 3: An example of a billiard table on which there exists non-periodic trajectories that are not dense on the billiard. The angle of which the trajectory hits the boundary is $\frac{\pi}{4}$.

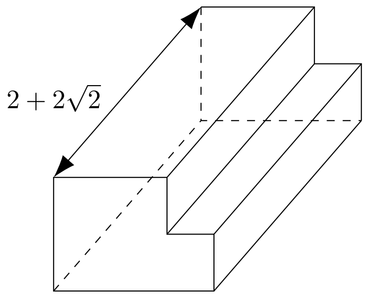

Figure 4: By elongating the L-shaped billiard table we can construct a polyhedron with geodesics with similar properties as the trajectories on the L-shaped billiard. Whenever we exit an L-shaped face we will spin around once and enter the oppocite L-face at the same point. The geodesic on the L-shaped faces are going to be similar to that of the trajectory on the L-shaped billiard, non-periodic and non-dense.

1.5 Circular billiards

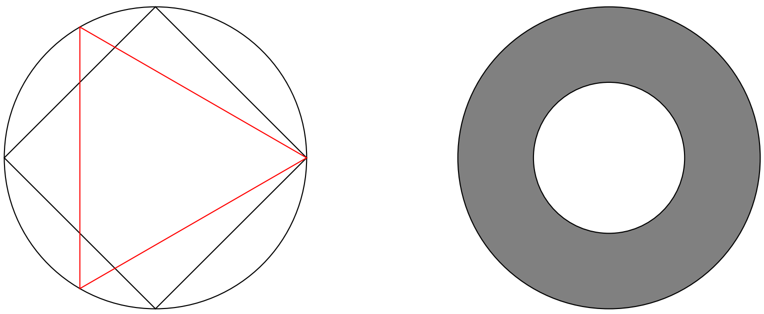

Figure 5: To the left a 4-periodic orbit in black and a 3-periodic orbit in red. To the right the closure of a non-periodic orbit in grey.

The circle billiard is perhaps the most simple. It has rotation symmetry and the trajectory always makes the same angle with the boundary at impact. If the angle at impact is α then the next impact point can be found by rotating the circle 2α. From this it is obvious that the trajectory is periodic if and only if $\alpha = \frac{p}{q}\pi$, $p, q \in \mathbb{Z}$ as the trajectory will return to a previous impact point after a while and the angle is the same.

Lemma 1.3. *If α is not π-rational then the points of impact will be dense on the circle.*

Proof. Let p be our first point of impact. Let d_1 be the length of the arc between two points of impact. Let x be the first point where the arclength between x and p is less than d_1. Let y be the point with arclength d_1 from x such that p is on the arc xy. The arclength between p and one of x and y is always going to be less than $d_1/2$, call this distance d_2. We can then apply the same argument with d_2 to find d_3 and so on, we note that $\lim_{n->\infty} d_n = 0$. Any interval I with length l will contain a point of impact as we can find a small enough inteval d_n such that $d_n < l$. □

Lemma 1.4. *The closure of a non-periodic orbit in a circular billiard is an annulus with inner radius $r \cdot \cos\alpha$, where r is the radius of the circle.*

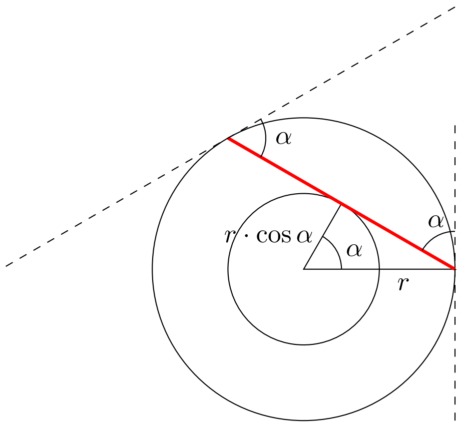

Figure 6: An in depth look at a piece of a trajectory (in red) between two points of impact. The dashed lines are tangents to the circle.

Proof. The trajectory can not be closer to the center than $r \cdot \cos\alpha$. Assume there is an open ball B in the annulus with outer radius r and inner radius $r \cdot \cos\alpha$ which is not in the closure of our open trajectory. Take two points $a, b \in B$ such that their clockwise tangent with the circle with radius $r \cdot \cos\alpha$ does not coincide. Project the segment ab onto an arc on the outer circle by drawing the clockwise tangents to the smaller circle for each point on the segment ab. The intersection between this set of tangents will be two arcs, chose one of them. As seen in the previous proof there will be points that belong to the trajectory on this arc, thus there will be points on the segment ab that belong to the trajectory which is a contradiction. The closure must be the entire annulus. $\square$

1.6 Elliptic billiards

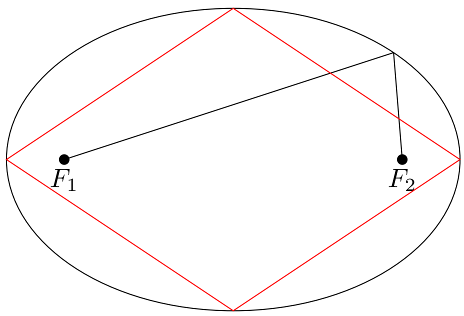

Figure 7: An ellipse with a 4-periodic trajectory in red. The two focus points F_1 and F_2 have the property that any trajectory staring in F_1 will be reflected and travel through F_2.

The behaviour of the trajectories in the ellipse can be categorized in three categories: focal, elliptic and hyperbolic.

If a trajectory travels through one of the focal points it will alternate travelling through the two focal points. Over time this motion will converge to the major axis. If a trajectory is traveling

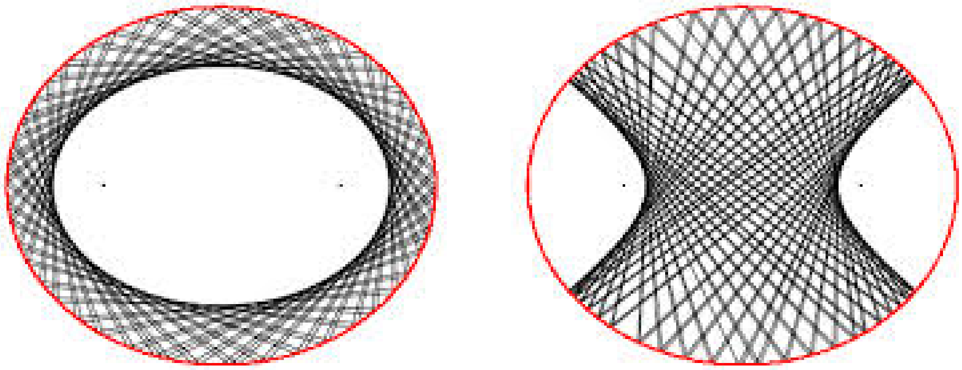

Figure 8: Two different kinds of trajectories in an elliptic billiard

clockwise with respect to the center then the clockwise arc length between two points of impact will always be shorter than half the circumference of the ellipse, two points of impact will always be on opposite sides of the major axis. This will make the points of impacts converge to the intersection between the ellipse and its major axis. The only periodical trajectory that travels through the focal points is the one that coincides with the major axis.

If the trajectory intersects the major axis between the focal points the trajectory will be bounded by a hyperbola with the same focal points, if the trajectory is non periodic the closure will be the entire region which is bounded by the ellipse and the hyperbola.

Similarly if the trajectory intersects the major axis outside of the segment between the focal points it will be bounded by a ellipse with the same focal points, again if it is non-periodic the closure will be the entire region between the two ellipses.

1.7 Rectangular billiards

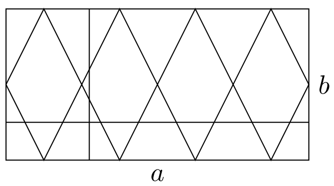

Figure 9: Three different periodic trajectories in a rectangle.

The rectangular billiard is also a very simple one. We can break down this motion by analyzing its horizontal and vertical component individually. Because of the nature of reflections the horizontal component of the velocity will only be affected by reflection in the sides and the vertical in the top and bottom of the rectangle. When the particle hits the top or bottom the vertical velocity changes sign while the horizontal remains the same, if it hits either side the horizontal component will change sign and the vertical will be unaffected. Looking at periodic trajectories we see that a trajectory that have travelled $2b$ in the vertical direction will have the same vertical position and velocity as before. If we have travelled $2a$ in the horizontal direction we will have the same horizontal position and velocity.

Say that our starting point is (x_1, y_1) and our initial velocity have the same direction as the slope $y = kx$. After impact the velocity will be parallel to $y = -kx$. To find periodic trajectories we want to know if the equation $2b = 2ka \cdot t, t \in \mathbb{Q}$ have solutions. If we simplify we see that the equation have solutions iff $\frac{a}{b}k$ is a rational. It is not hard to see that the trajectory is periodic iff $\frac{a}{b}k$ is a rational number.

2 Triangular billiards

2.1 Periodic trajectories of acute triangles

A trajectory is periodic if there exist an h such that the particle have the same position and velocity at t and $t + h$ for almost all t. The period of a trajectory is the number of times it bounces before it returns to a previous visited point. In an acute triangle very easy to find periodic trajectories even if it is not rational. The periodic trajectory that connects the base points of the altitudes is called the Fagano billiard trajectory.

Lemma 2.1. *The triangle connecting the base points of the altitudes in an acute triangle is a 3-periodic trajectory.*

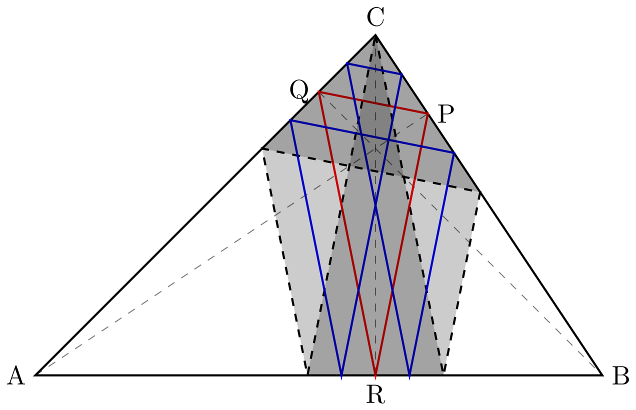

Figure 10: A triangle with a Fagano trajectory in red and a blue 6-periodic trajectory parallel to the Fagano trajectory. Illustrated in the picture is also a gray "strip" where more parallel trajectories like the blue one can be found.

Proof. BCR and BAP are similar because they share $\angle CBA$ and both have a right angle. Thus $\frac{BP}{BA} = \frac{BR}{BC} \iff \frac{BC}{BA} = \frac{BR}{BP}$. This means BRP and BAC are all similar. Similarly we can conclude that the triangles ABC, PBR, PQC, AQR are similar. Thus $\angle BAC = \angle RPB = \angle CPQ, \angle CBA = \angle PQC = \angle AQC, \angle ACB = \angle QRA = \angle BRP$. A particle following the red path will be reflected at Q, P, R as depicted in the picture and the triangle QRP is a billiard trajectory. □

Later we will see that all trajectory have a family of trajectories travelling parallel to them. For now we will simply observe this and note that the black dashed "boundary" of the gray strip is a terminating trajectory in both directions.

2.2 Periodic trajectories in right angeled triangles

For right triangles it the fagano trajectrory terminates but we can still find its family of trajectories.

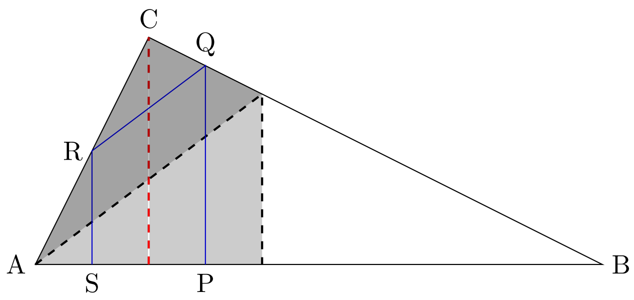

Figure 11: The degenerated Fagano trajectory in red and a blue 6-periodic trajectory. A gray "strip" of parallel trajectories is also illustrated.

Lemma 2.2. *A trajectory perpendicular to the hypotenuse in a right angeled triangle is periodic as long as it hits the other two sides before returning.*

Proof. Assume a triangle ABC and that $\angle ACB = 90°$ and that a trajectory leaves the hypotenuse perpendicular at a point P. Without loss of generality we can assume it reflects in CB then in AC in point Q and R respectively. That means $\angle CQR = \angle PQB$ and $\angle QRC = \angle ARS$. Triangle QBP is similar to ABC which makes QRC similar to ABC which makes ARS similar to ABC so $\angle RSA = 90°$ which means $PQRSRQ$ is a periodic trajectory. $\square$

3 The Zemlyakov-Katok construction

3.1 Rational angles

Let P be an n-gon with edges $e_1, e_2, ..., e_n$ counter-clockwise. Whenever the particle is reflected in an edge its speed vector undergoes a transformation by an element in $O(2)$. Denote reflection in the edge e_i by $r_i \in O(2)$. $O(2)$ is the orthogonal group of dimension 2 i.e. rotations and reflections in the plane. Let $\Gamma \subset O(2)$ be the set generated by the e_i's. Γ is the set of possible transformations of the velocity we may have during any trajectory.

Theorem 3.1. Γ *is finite if and only if the angles of the polygon* P *is rational.*

Proof. Assume Γ is finite. $r_k r_{k+1}$ is a rotation by $2\angle e_k e_{k+1}$ clockwise $k \in 1, ..., n-1$. Because Γ is finite there is some $m \in \mathbb{N}$ such that $(r_k r_{k+1})^{l+m} = (r_k r_{k+1})^l$ which means $(r_k r_{k+1})^m = 1$ (identity). Thus there is an m such that $m \cdot 2\angle e_k e_{k+1} = n \cdot 2\pi$ and $\angle e_k e_{k+1}$ is rational. Replace e_k and e_{k+1} with e_n and e_1 and repeat this argument.

Assume $\angle e_k e_{k+1} = \alpha_k$ and $\angle e_n e_1 = \alpha_n$ is rational for all $k \in \{1, 2, ..., n-1\}$. Extend e_1 and let it be our x-axis. All of the e_i's will be parallel or make a rational angle with the x-axis, denote this angle $\frac{p_i}{q_i}\pi$. Let $Q = LCM(q_1, q_2, ..., q_n)$. The matrix representatition for reflection in the edges will be the same as for reflections in the the dihedral group of order Q. Thus $\Gamma \subseteq D_Q$ and Γ is finite.

□

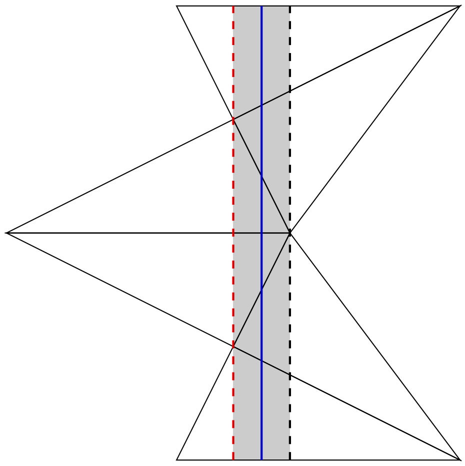

Figure 12: An example of an unfolding. The purpose of the gray strip from previous pictures become apparant here.

3.2 Unfoldning

Instead of reflecting the particle when we can reflect the table instead in a process called unfolding. This makes us able to study trajectories as straight lines instead.

Assume we have a rational polygon P that we want to unfold. Whenever we unfold P in an edge e_i we transform P with some element $r_i \in \Gamma$ and then translate it with some translation c_i. So what we are doing when we unfold is transforming P with some $\rho_i = r_i + c_i$, an elemet in the group of Euclidian isometries. Because Γ is finite we our constructed surface will consist of translations of all possible combinations of transformations of P by Γ. Instead of continuing this unfolding process we can associate sides such that lines will keep their direction. Thus we achieve a finite flat area on which geodesic flow corresponds to billiard flow.

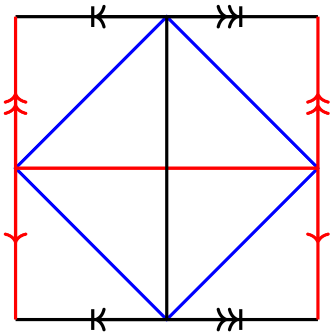

Figure 13: An unfolding of a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. It turns out it have the same geometry as a flat torus.

4 Translation surfaces

The The Zemlyakov-Katok contruction is an example of a translation surface. A translation surface is constructed by gluing parallel sides of polygons in the plane. We require edges to have opposite inward normals, otherwise direction will not be preserved. On these surfaces we can study trajectories as geodesic flows.

It may be tempting to try to define what a reflection at a vertex would look like as the geodesic flow is defined for all points in this particular unfoldning. This is not in general posible, at vertices we may have cone points of negative curvature. The angle at these points is $2\pi n, n \in \{2, 3, ...\}$ and the geodesic flow is not defined at these points.

4.1 The first return map

Let I be a closed horizontal segment and ϕ_t a vertical flow on a translation surface S. The first return map to I is a map

$$f : I \to I$$

$$p \mapsto \phi_{t_0}(p)$$

where $t_0 \neq 0$ is the smallest t for which $\phi_t(p) \in I$. But this map is not defined for all points.

We want to divide I into a finite amount of open subintervals I_j where the first return map is defined and continous. Points for which the first return map is not defined are points where the vertical flow hits a cone point or does not return to I at all. The number of vertical trajectories that can hit a cone point is small, a cone point with angle $2\pi n$ only have n vertical trajectories hitting it forward in time, it have n trajectories hitting it backward in time. The first return map have discontinuities at points where the vertical flow hits the endpoints of the interval I, two points close to each other may find themselves on opposite sides of a point whose vertical flow hits the boundary of I. Now we can divide I into a finite amount of segments I_j where the points stay together under the vertical flow until their first return to I.

Lemma 4.1. *The first return map is defined for all I_j's.*

Proof. Assume there is some $p \in I_k$ for which the first return map is not defined. Because we have removed all cone points this means that the vertical flow from p doesn't return to I. This must be true for all points in I_k as we have ensured, by removing all points of discontinuity, they must travel together. The area swept by I_k under the vertical flow is finite as the area of our translation surface is finite and it must overlap with itself. That is there is some $0 < t_1 < t_2$ such that $\phi_{t_1}(I_k) \cap \phi_{t_2}(I_k) \neq \emptyset$ chose $p \in \phi_{t_1}(I_k) \cap \phi_{t_2}(I_k)$, then there is p_1, p_2 such that $\phi_{t_1}(p_1) = \phi_{t_2}(p_2) = p$. Now consider $\phi_{-t_1}(p) = \phi_{t_2 - t_1}(p_2) = p_1$. This contradict our inital statement that there is no point in I_k that return to I_k. □

4.2 Boundaries of geodesics

With the help of the first return map we shall prove what have have been illustrated in previous figures.

Theorem 4.2. *Every closed geodesic is part of a family of closed geodesics of the same lenght contained in an open cylinder whose boundary is a union of saddle connections.*

Note: a saddle connection is a trajectory hitting singular points forward and backward in time.

Proof. Assume there is a point p with a closed geodesic passing through it. Choose as the vertical direction the direction of the geodesic and choose an appropriate horizontal interval I such that $P \in I$. The geodesic through p will have some period $n \in \mathbb{N}$ with respect to the first return map of I, i.e. how many times it returns to I before returning to p. Let B be the set of points for which the first return map is not defined and points that hit the endpoints of I. The set $f^{-1}(B) \cup f^{-2}(B) \cup ... \cup f^{-n}(B)$ divides I into open intervals I_k that travel together until at least the n-th time they reach I. If $p \in I_a$ then all points in I_a have period n. I_a is bounded by points s, t that forward in time hit cone points. If s, t does not hit a cone point backwards in time they must be periodic as they travel together with p, which is a contradidction. s, t must hit cone points backwards in time and are saddle connections. If s hits different cone points s_1, s_2 forward and backward there is a geodesic parallel to the periodic geodesics that goes between s_1, s_2 or goes from s_1 to a new cone point s_3, in the latter case we can again say that there is a geodesic parallel to the periodic ones between s_3, s_2 or between s_3 and a new s_4. The amount of cone points is finite so this cannot happen an infinite amount of times. □

Theorem 4.3. *Given a non-periodic vertical trajectory $\phi_t(p)$ in a translation surface S, the closure of $\{\phi_t(p)|t \in \mathbb{R}\}$ is a subsurface of S whose boundary is a union of saddle connections.*

Proof. Assume there is a boundary point q, if the closure A of $\{\phi_t(p)|t \in \mathbb{R}\}$ contains no boundary points then $A = S$ and the theorem holds. It is possible to construct a horizontal segment I such that $A \cap I = q$ because q is a boundary point of the closed set A.

Assume that q does not hit a singular point. q will be the endpoint of one of the I_j's on which the first return map is defined and continous. Thus $f(q)$ is defined and q must return to I. But A is invariant under the vertical flow so $f(q) \in A$ thus $f(q) = q$ and the vertical trajectory from q is periodic. By theorem 4.2 we then know $q \in C$ where C is some open cylinder but $\{\phi_t(p)\} \cap C = \emptyset$ as $\{\phi_t(p)\}$ wolud be periodic otherwise. This means $\{\phi_t(p)\}$ is not approaching q wich is a contradiction and thus q must hit a singular point. We can then repeat the same argument to se that q hits a singular point backwards in time. □

5 A discussion of tetrahedrons

Translation surfaces and the theorems we proved concerning closed and open geodesics are not restricted to the study of billiards. In this chapter we will use a what we proved i the previous chapter together with linear transformations of the translation surfaces.

Theorem 5.1. *The closure of any open geodesic on a tetrahedron with each face congruent covers the whole tetrahedron.*

Proof. Assume for some tetrahedron there is a open geodesic $\phi_t(p)$ whose closure does not cover the entire tetrahedron. We can unfold this tetrahedron to create a translation surface in the shape of a parallelogram. For any given flow in the paralellogram there is a linear transormation that preserve direction and relative distance of the flow that also transform the parallelogram into a square. Using theorem 4.3 we know the closure of $\{\phi_t(p)|t \in \mathbb{R}\}$ will be bounded by saddle connections. Looking at the renormalization in we realize that the slope of any saddle connection, with respect to a orthonormal coordinate system, is rational. But geodesics with

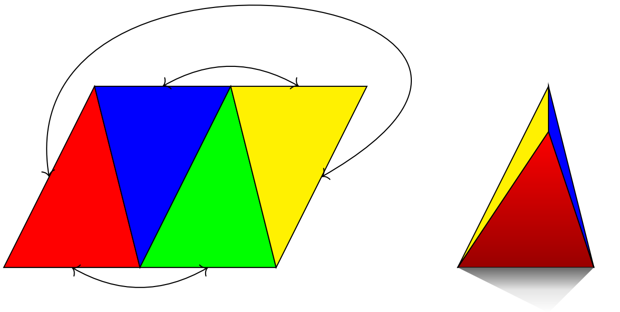

Figure 14: Given any triangle we can construct a tetrahedron such that all faces of the tetrahedron are congruent to the triangle.

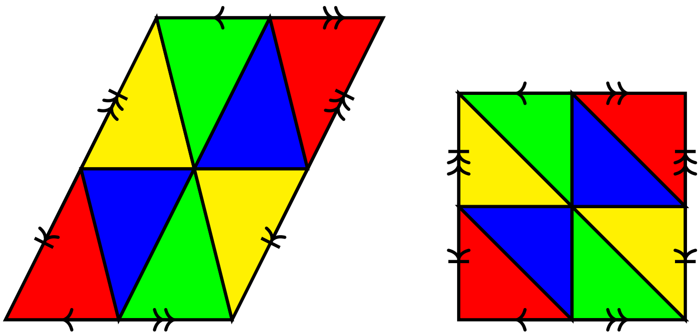

Figure 15: To the left is a translation surface of the tetrahedron. To the right is a linear transformation of said translation surface. This have the geometry of a flat torus.

rational slope on a flat torus is clearly periodic. This is a contradiction as $\phi_t(p)$ will have the same slope as its saddle connections and is not periodic. $\square$

The technique of linearly transforming the translation surface is very useful indeed. When we study billiard trajectories and geodesics in the translation surface we study them one at a time. Linearly transforming the translation surface is not going to affect the behavour of the geodesics. If it is periodic it will remain periodic, if it is open and is dense on the surface it will reamin so.

This concludes this short introduction. The subject of billiards is indeed a very rich subject and here we have only seen a fraction of a subgroup of the polygonial ones.

6 Some thoughts on billiards

In this section I will discuss questions, ideas and approaches which didn't really work out or that I had to abandon.

6.1 Moving forward and backwards in time

When we have analyzed trajectories we have propagated them both forward and backwards in time. A natural question rises, "Are there trajctories such that the behaviour forwards in time is different from that backwards in time?". First we need to answer what we mean by "different behaviur", as an example there are many trajectories that terminate in one direction (they hit a non-smooth piece of the boundary) but not in the other. If we don't take these trajectories in consideration, are there billiards with trajectories such that they leave a region and never return? Here I will give a convoluted example of a billiard with such trajectories.

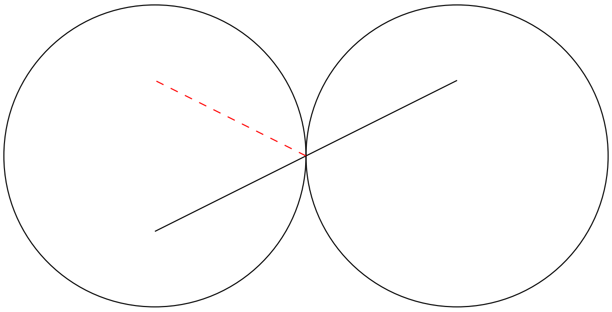

Figure 16: A billiard consisting of two tangent circles. Instead of reflecting trajectories hitting their common point we can instead let it continue in the other circle.

If we have a non-periodic trajectory starting in the left circle and it hits the circles common point we will never return to the circle we was in. This convoluted example is not really that intersting, it is not a proper billiard and the measure of the trajectories that hit their common point is zero. I have not found an eaxmple of this behavour can occur in proper billiards.

6.2 Generalising the L-shaped billiard

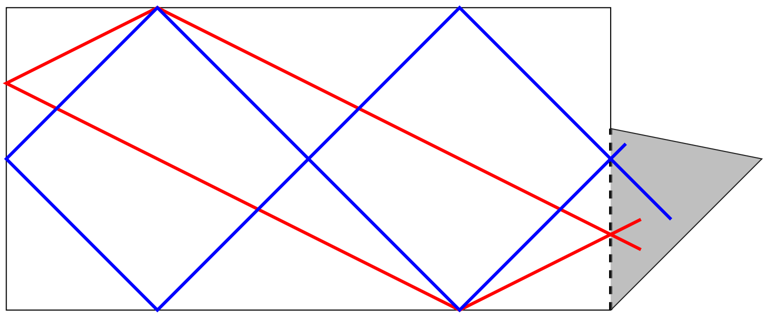

Figure 17: From the perspective of the gray triangle the red and blue trajectories behave as if they were reflections in the dashed line.

In the introduction I gave an example of a billiard with non-periodic trajectories that was not dense on the billiard. Using that billiard as an inspiration it may seem we can construct more billiards with that same property. From Figure 17 it may seem that we can, starting from any rational triangular billiard P, easily construct a billiard with non-periodic trajectories whose closure is not dense on the billiard. Imagine we have a non-periodic trajectory in the gray triangle, it seems plausible that by taking all posible directions the trajectory have during the motion and designing a special rectangle that have periodic trajectories in the same directions we could use it 'simulate' reflections by attaching it to one of the sides of the triangle. It turns out it is impossible to construct a rectangle for a general triangle.

The direction the trajectory have is dependent on the starting velocity and Γ. Γ consist of rotations of rational multiples of π. We have seen that the periodic trajectories in a rectangle have a slope of k where k is a rational multiple of $\frac{b}{a}$ where a is the base and b the height. For any two angles that differ by a rational multiple of π it is not always possible to find two different k_1, k_2 such that the lines with slope k_1, k_2 differ by that angle.

Related to this question is if we can for any given a trajectory on a billiard elongate the billiard in some way to get a polyhedra, as in the introduction, with the same behaviour. In general this is not possible again due to the dynamics of the rectangle.

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