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Volodymyr Mazorchuk · Kaiming Zhao

Graded simple Lie algebras and graded simple representations

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Abstract. Let Q be an abelian group and $\Bbbk$ a field. We prove that any Q-graded simple Lie algebra $\mathfrak{g}$ over $\Bbbk$ is isomorphic to a loop algebra in case $\Bbbk$ has a primitive root of unity of order $|Q|$, if Q is finite, or $\Bbbk$ is algebraically closed and $\dim \mathfrak{g} < |\Bbbk|$ (as cardinals). For Q-graded simple modules over any Q-graded Lie algebra $\mathfrak{g}$, we propose a similar construction of all Q-graded simple modules over any Q-graded Lie algebra over $\Bbbk$ starting from nonextendable gradings of simple $\mathfrak{g}$-modules. We prove that any Q-graded simple module over $\mathfrak{g}$ is isomorphic to a loop module in case $\Bbbk$ has a primitive root of unity of order $|Q|$ if Q is finite, or $\Bbbk$ is algebraically closed and $\dim \mathfrak{g} < |\Bbbk|$ as above. The isomorphism problem for simple graded modules constructed in this way remains open. For finite-dimensional Q-graded semisimple algebras we obtain a graded analogue of the Weyl Theorem.

1. Introduction

1.1. General overview

The present paper studies Q-graded simple Lie algebras for any abelian group Q and Q-graded simple modules over Q-graded Lie algebras over a field $\Bbbk$ under some mild restrictions.

Study of gradings on Lie algebras goes back at least as far as to the paper [17] which started a systematic approach to understanding of gradings by abelian groups on simple finite dimensional Lie algebras over algebraically closed fields of characteristic 0. In the past two decades, there was a significant interest to the study of gradings on simple Lie algebras by arbitrary groups, see the recent monograph [7] and references therein. In particular, there is an essentially complete classification of fine gradings (up to equivalence) on all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0, see [5,7,18]. Some properties

V. Mazorchuk (✉): Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden. e-mail: mazor@math.uu.se

K. Zhao: Department of Mathematics, Wilfrid Laurier University, Waterloo, ON N2L 3C5, Canada. e-mail: kzhao@wlu.ca

College of Mathematics and Information Science, Hebei Normal (Teachers) University, Shijiazhuang 050016, Hebei, People's Republic of China

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of simple $\mathbb{Z}_2$-graded Lie algebras were studied in [19]. A classification of various classes of $\mathbb{Z}^2$-graded Lie algebras with at most one-dimensional homogeneous spaces was obtained in [13–15]. A classification of simple Lie algebras with a $\mathbb{Z}^n$-grading such that all homogeneous spaces are one-dimensional was obtained in [9]. For a given abelian group Q, a classification of Q-gradings (up to isomorphism) on classical simple Lie algebras over an algebraically closed field of characteristic different from 2 was obtained in [2,5], see also [7]. In [1], one finds some characterizations of graded-central-simple algebras with split centroid, see Correspondence Theorem 7.1.1 in [1]. In general, it is difficult to find the centroid for a graded simple algebra. In the paper we will establish a correspondence theorem for graded-simple Lie algebras without assuming they are central or with split centroid.

When Q is finite, Billig and Lau described quasi-finite Q-graded-simple modules over Q-graded associative (or Lie) algebras in [3]. With respect to this, we will study Q-graded-simple modules over Q-graded Lie algebras without assuming qasifiniteness of modules or finiteness of Q.

1.2. Notation and setup

Throughout this paper, $\Bbbk$ denotes an arbitrary field with some restrictions in the context. If not explicitly stated otherwise, all vector spaces, algebras and tensor products are assumed to be over $\Bbbk$. As usual, we denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{C}$ the sets of integers, positive integers, nonnegative integers and complex numbers, respectively.

Let Q be an additive abelian group. A *Q-graded Lie algebra over* $\Bbbk$ is a Lie algebra $\mathfrak{g}$ over $\Bbbk$ endowed with a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha \quad \text{such that} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in Q.$$

Recall that a graded Lie algebra is called *graded simple* if $\mathfrak{g}$ is not commutative and does not contain any non-trivial graded ideal. Without loss of generality, for a Q-graded simple Lie algebra $\mathfrak{g}$ we will always assume that the elements $\alpha \in Q$ with $\mathfrak{g}_\alpha \neq 0$ generate Q. Gradings satisfying this condition will be called *minimal*.

Let Q' be another abelian group, $\mathfrak{g}$ a Q-graded Lie algebra over a field $\Bbbk$ and $\mathfrak{g}'$ be a Q'-graded Lie algebra over $\Bbbk$. We say that graded Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are *graded isomorphic* if there is a group isomorphism $\tau : Q \to Q'$ and a Lie algebra isomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}'$ such that $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}'_{\tau(\alpha)}$ for all $\alpha \in Q$.

A *Q-graded module V* over a Q-graded Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module endowed with a decomposition

$$V = \bigoplus_{\alpha\in Q} V_\alpha \quad \text{such that} \quad \mathfrak{g}_\alpha \cdot V_\beta \subset V_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in Q.$$

A graded module V is called *graded simple* if $\mathfrak{g}V \neq 0$ and V does not contain any non-trivial graded submodule.

Let $\mathfrak{g}$ be a Q-graded Lie algebra over $\Bbbk$ and

$$W' = \bigoplus_{\alpha \in Q} W_\alpha, \quad W = \bigoplus_{\alpha \in Q} W'_\alpha$$

be two Q-graded $\mathfrak{g}$-module. A $\mathfrak{g}$-module isomorphism $\sigma : W \to W'$ is called a *graded isomorphism of degree* $\beta \in Q$ if $\sigma(W_\alpha) = W'_{\alpha+\beta}$ for all $\alpha \in Q$. We say that W and W' are *graded isomorphic* if there is a graded isomorphism $\sigma : W \to W'$ of some degree $\beta \in Q$.

1.3. Results and structure of the paper

In Sect. 2, for a subgroup P of an abelian group Q and a simple Lie algebra $\mathfrak{a}$ with a fixed Q/P-grading, we recall the loop algebras which are Q-graded simple Lie algebras $\mathfrak{g}(Q, P, \mathfrak{a})$. If P is finite and $\Bbbk$ contains a primitive root of unity of order $|P|$, then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is a direct sum of $|P|$ ideals that are isomorphic to $\mathfrak{a}$. If char($\Bbbk$) does divide $|P|$, then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is not semisimple as an ungraded algebra.

In Sect. 3, we assume that Q is finite and $\Bbbk$ contains a primitive root of unity of order $|P|$. We prove that any Q-graded simple Lie algebra has to be of the form $\mathfrak{g}(Q, P, \mathfrak{a})$ for some simple Lie algebra $\mathfrak{a}$ with a Q/P-grading, see Corollary 12 (we actually prove a more general result in Theorem 11). Our proof is based on a detailed analysis of properties of the character group of Q and is directed towards showing existence of a non-graded simple ideal in the case when the underlying ungraded Lie algebra is not simple. Thanks to the recent classification of all gradings on finite dimensional simple Lie algebras, see [5,7,18], we thus obtain a full classification of all finite-dimensional Q-graded simple Lie algebras over any algebraically closed field of characteristic 0.

In Sect. 4 we establish a graded analogue of Schur's Lemma. It is frequently used in the remainder of the paper. In Sect. 5, for arbitrary additive abelian group Q, using Correspondence Theorem 7.1.1 from [1], we prove that any Q-graded simple Lie algebra $\mathfrak{g}$ over an algebraically closed field $\Bbbk$ with $\dim \mathfrak{g} < |\Bbbk|$ has to be of the form $\mathfrak{g}(Q, P, \mathfrak{a})$, for some simple Lie algebra $\mathfrak{a}$ with a Q/P-grading, see Theorem 15.

In Sect. 6, using our classification of Q-graded simple Lie algebras, we prove a graded analogue of the Weyl's Theorem, see Theorem 18. Namely, we show that any Q-graded finite dimensional module over a Q-graded semi-simple finite dimensional Lie algebra over an algebraically closed field of characteristic 0 is completely graded-reducible.

Finally, in the last section of the paper, we reduce classification of all Q-graded simple modules W over any Q-graded Lie algebras $\mathfrak{g}$ over $\Bbbk$ to the study of nonextendable gradings on simple $\mathfrak{g}$-modules. For any simple $\mathfrak{g}$-module V with a Q/P-grading, we first construct a Q-graded $\mathfrak{g}$-module $M(Q, P, V)$ which are called loop modules. We prove that any Q-graded simple module over $\mathfrak{g}$ is isomorphic to a loop module if $\Bbbk$ has a primitive root of unity of order $|Q|$ in the case of finite Q; or $\Bbbk$ is algebraically closed with $\dim \mathfrak{g} < |\Bbbk|$, see Theorems 24 and 31. We finish the paper with an open problem: find necessary and sufficient conditions for two graded simple $\mathfrak{g}$-modules $M(Q, P, V)$ and $M(Q, P', V')$ to be isomorphic.

2. Construction of graded simple Lie algebras

2.1. Construction

We will recall some concepts and results from [1] with different notation and discuss some properties on graded simple Lie algebras. In this section $\Bbbk$ is an arbitrary field.

Let Q be an abelian group and P a subgroup of Q. Assume we are given a simple Lie algebra $\mathfrak{a}$ over $\Bbbk$ with a fixed Q/P-grading

$$\mathfrak{a} = \bigoplus_{\bar{\alpha} \in Q/P} \mathfrak{a}_{\bar{\alpha}}.$$

Consider the group algebra $\Bbbk Q$ with the standard basis $\{t^\alpha : \alpha \in Q\}$ and multiplication $t^\alpha t^\beta = t^{\alpha+\beta}$ for all $\alpha, \beta \in Q$. Then we can form the Lie algebra $\mathfrak{a} \otimes \Bbbk Q$. For $x, y \in \mathfrak{a}$ and $\alpha, \beta \in Q$, we have

$$\left[x \otimes t^\alpha, y \otimes t^\beta\right] = [x, y] \otimes t^{\alpha+\beta}.$$

Define the Q-graded Lie algebra

$$\mathfrak{g}(Q, P, \mathfrak{a}) := \bigoplus_{\alpha \in Q} \mathfrak{g}(Q, P, \mathfrak{a})_\alpha, \quad \text{where} \quad \mathfrak{g}(Q, P, \mathfrak{a})_\alpha := \mathfrak{a}_{\bar{\alpha}} \otimes t^\alpha.$$

For example, $\mathfrak{g}(Q, Q, \mathfrak{a}) = \mathfrak{a} \otimes \Bbbk Q$ (with the obvious Q-grading) while $\mathfrak{g}(Q, \{0\}, \mathfrak{a}) = \mathfrak{a}$ (with the original Q-grading). These graded Lie algebras $\mathfrak{g}(Q, P, \mathfrak{a})$ are called loop algebras in [1]. From the definition it follows that $\dim \mathfrak{g}(Q, P, \mathfrak{a}) = \dim(\mathfrak{a})|P|$ if $\mathfrak{a}$ is finite-dimensional and P is finite. The following result is Lemma 5.1.1 in [1].

Lemma 1. *If Q is an abelian group, P a subgroup of Q and $\mathfrak{a}$ is a simple Lie algebra with a Q/P-grading, then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is a Q-graded simple Lie algebra.*

Making a parallel with affine Kac–Moody algebras [10,12], it is natural to divide these algebras into two classes. The algebras $\mathfrak{g}(Q, Q, \mathfrak{a})$ will be called *untwisted graded simple Lie algebra* while all other algebras will be called *twisted graded simple Lie algebras*.

2.2. Properties of $\mathfrak{g}(Q, P, \mathfrak{a})$ in the case of finite Q

Now we need to establish some properties of the graded simple Lie algebras $\mathfrak{g}(Q, P, \mathfrak{a})$ for finite groups Q. So in the rest of this section we assume that Q is finite and that $\Bbbk$ contains a primitive root of unity of order $|Q|$ (which implies that $\mathrm{char}(\Bbbk)$ does not divide $|Q|$).

Let $\hat{Q}$ denote the character group of Q, that is the group of all group homomorphisms $Q \to \Bbbk^*$ under the operation of pointwise multiplication. Note that $\hat{Q} \cong Q$ because of our assumption on char($\Bbbk$). For any $f \in \hat{Q}$, we define the associative algebra automorphism

$$\tau_f : \Bbbk Q \to \Bbbk Q \quad \text{via} \quad \tau_f(t^\alpha) = f(\alpha)t^\alpha \quad \text{for all } \alpha \in Q.$$

This induces the Lie algebra automorphism

$$\begin{aligned} \tau_f : \mathfrak{g}(Q, P, \mathfrak{a}) &\to \mathfrak{g}(Q, P, \mathfrak{a}) \\ x_{\bar{\alpha}} \otimes t^\alpha &\mapsto f(\alpha)x_{\bar{\alpha}} \otimes t^\alpha \end{aligned} \tag{1}$$

for all $\alpha \in Q$ and $x_{\bar{\alpha}} \in \mathfrak{a}_{\bar{\alpha}}$. Note that $\tau_{fg} = \tau_f \tau_g$ for all $f, g \in \hat{Q}$, in other words, $\hat{Q}$ acts on $\mathfrak{g}(Q, P, \mathfrak{a})$ via automorphisms τ_f. We will use the following:

Remark 2. If Q is a finite abelian group, P a subgroup of Q, $\hat{Q}$ the group of characters of Q over a field $\Bbbk$ such that char($\Bbbk$) does not divide $|Q|$ and $P^\perp := \{f \in \hat{Q} : f(\alpha) = 1 \text{ for all } \alpha \in P\}$, then $|\hat{Q}/P^\perp| = |P|$. Indeed, because of our assumption on $\Bbbk$, we know that $|\hat{P}| = |P|$. Therefore it is enough to prove that each character of P can be extended to a character of Q. The latter follows directly from the Frobenius reciprocity.

Lemma 3. *Assume that Q is finite and $\Bbbk$ contains a primitive root of unity of order $|Q|$. Then the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is a direct sum of $|P|$ ideals. Each of these ideals is Q/P-graded and, moreover, isomorphic to $\mathfrak{a}$ as Q/P-graded Lie algebras.*

Proof. For $\alpha \in Q$, set

$$\underline{t}^\alpha := t^\alpha \sum_{\beta \in P} t^\beta.$$

Then, for any $\alpha, \alpha' \in Q$, we have $\underline{t}^\alpha = \underline{t}^{\alpha'}$ if and only if $\alpha - \alpha' \in P$. Consider the vector space

$$I = \bigoplus_{\bar{\alpha} \in Q/P} \mathfrak{a}_{\bar{\alpha}} \otimes \underline{t}^\alpha, \tag{2}$$

which is well-defined because of the observation in the previous sentence. Since $\underline{t}^\alpha t^\beta = \underline{t}^{\alpha+\beta}$, for all $\alpha, \beta \in Q$, the space I is an ideal. Note that $[I, I] \neq 0$ since $\mathfrak{a}$ is a simple Lie algebra and char($\Bbbk$) does not divide $|Q|$ (and thus it does not divide $|P|$ either, which implies $\underline{t}^\alpha \underline{t}^\alpha = |P|\underline{t}^\alpha \neq 0$). It follows that $I \cong \mathfrak{a}$ as a Q/P-graded Lie algebra. Simplicity of $\mathfrak{a}$ even implies that I is a minimal ideal.

Define the *invariant subgroup* Inv(I) *of* I as

$$\mathrm{Inv}(I) = \left\{ f \in \hat{Q} \mid \tau_f(I) = I \right\},$$

which is a subgroup of $\hat{Q}$. Then the set $\mathbf{I} := \left\{ \tau^\alpha(I) \,:\, \alpha \in \hat{Q} \right\}$ consist of $|\hat{Q}/\mathrm{Inv}(I)|$ different minimal ideals of $\mathfrak{g}(Q, P, \mathfrak{a})$. From the definitions it follows that

$$\mathrm{Inv}(I) \subset P^\perp := \left\{ f \in \hat{Q} \mid f(\alpha) = 1 \text{ for all } \alpha \in P \right\}$$

and hence $|\hat{Q}/\mathrm{Inv}(I)| \geq |\hat{Q}/P^{\perp}| = |P|$, see Remark 2 for the latter equality. Comparing the number of non-zero homogeneous components in $\mathfrak{g}(Q, P, \mathfrak{a})$ and in the subspace

$$\bigoplus_{J \in \mathbf{I}} J \subset \mathfrak{g}(Q, P, \mathfrak{a}),$$

we deduce that these two algebras coincide. The statement of the lemma follows. □

The ideals in Lemma 3 are the only ones of the Lie algebra $\mathfrak{g}$.

Example 4. In case Q is finite and $\mathrm{char}(\Bbbk)$ does divide $|Q|$, the algebra $\mathfrak{g}(Q, P, \mathfrak{a})$ is not a direct sum of simple ideals in general. For example, let us consider the case that $\mathrm{char}(\Bbbk) = |Q|$, $Q = \mathbb{Z}_p$ and $P = 0$. Since $(t^{\bar{1}})^p - 1 = (t^{\bar{1}} - 1)^p$, we have that

$$\mathfrak{g}(Q, P, \mathfrak{a}) \simeq \mathfrak{a} \otimes \left(\Bbbk[x]/\langle x^p \rangle\right),$$

where $x = t^{\bar{1}} - 1$. The latter algebra has an abelian ideal $\mathfrak{a} \otimes x^{p-1}$ and a nilpotent ideal $\mathfrak{a} \otimes x\Bbbk[x]$. The latter is, in fact, a maximal ideal.

3. Graded simple Lie algebras: the case of finite Q

3.1. Preliminaries

In this section, for a finite abelian group Q, we will characterize all Q-graded simple Lie algebras $\mathfrak{g}$ (without assuming that $\mathfrak{g}$ is central or with split centroid) when $\Bbbk$ is an arbitrary field containing a primitive root of unity of order $|Q|$. That is, in this setup we establish an analogue of Correspondence Theorem 7.1.1 from [1]. We start with an arbitrary additive abelian group Q at this moment, not necessarily finite, and a Q-graded simple Lie algebra

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha.$$

Every $x \in \mathfrak{g}$ can be written in the form $x = \sum_{\alpha \in Q} x_\alpha$ where $x_\alpha \in \mathfrak{g}_\alpha$. In what follows the notation x_α, for some $\alpha \in Q$, always means $x_\alpha \in \mathfrak{g}_\alpha$. We define the *support* of x as

$$\mathrm{supp}(x) := \{\alpha \in Q \mid x_\alpha \neq 0\}.$$

Similarly, we can define $\mathrm{supp}(X)$ for any nonempty subset $X \subset \mathfrak{g}$. Recall that we have assumed that the grading on $\mathfrak{g}$ is *minimal* in the sense that $\mathrm{supp}(\mathfrak{g})$ generates Q.

Classification of graded simple Lie algebras for which the underlying Lie algebra $\mathfrak{g}$ is simple, reduces to classification of gradings on simple Lie algebras. Grading on simple Lie algebras are, to some extent, well-studied, see [6,7], and we will not study this problem in the present paper. Instead, now we assume that $\mathfrak{g}$ is not simple.

For $\alpha \in Q$, define $\pi_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$ as the projection with respect to the graded decomposition. Take a non-homogeneous non-zero proper ideal I of $\mathfrak{g}$. As we progress we may re-choose this I later in this section. Define the *size* of I as

$$\operatorname{size}(I) = \min\left\{|\operatorname{supp}(x)| \,:\, x \in I\backslash\{0\}\right\}.$$

Lemma 5.

(a) *For any nonzero* $x \in I$, *we have* $|\operatorname{supp}(x)| > 1$.
(b) *We have* $\pi_\alpha(I) = \mathfrak{g}_\alpha$, *for each* $\alpha \in Q$.

Proof. Claim (a) follows from the observation that the set

$$J = \{x \in I \,:\, |\operatorname{supp}(x)| \leq 1\}$$

is a nontrivial graded ideal of $\mathfrak{g}$ which has to be zero as $\mathfrak{g}$ is graded simple.

To prove claim (b), let $\bar{I} = \sum_{\alpha \in Q} \pi_\alpha(I)$. As $\mathfrak{g}_\alpha \pi_\beta(I) \subset \pi_{\alpha+\beta}(I)$, it follows that $\bar{I}$ is a nonzero graded ideal of $\mathfrak{g}$ which has to be $\mathfrak{g}$ itself since $\mathfrak{g}$ is graded simple. This completes the proof. □

3.2. Auxiliary lemmata

From now on in this section, we assume that $Q = Q_0 \times Q_1$ where Q_0 is finite, $\mathbb{k}$ contains a primitive root of unity of order $|Q_0|$ and any ideal of $\mathfrak{g}$ is Q_1-graded. Then we have the graded isomorphisms $\tau_f : \mathfrak{g} \to \mathfrak{g}$, for any $f \in \hat{Q}_0$ defined as in (1) (with the convention that $f(\alpha) = 1$ for all $\alpha \in Q_1$).

Lemma 6.

(a) *An ideal* J *of* $\mathfrak{g}$ *is* Q*-graded if and only if* $\tau_f(J) \subset J$, *for all* $f \in \hat{Q}_0$.
(b) *The center of* $\mathfrak{g}$ *is zero.*

Proof. Claim (a) is clear. Claim (b) follows from claim (a) since the center is an ideal and is invariant under all automorphisms including τ_f for $f \in \hat{Q}_0$. □

For convenience, we define

$$\operatorname{Inv}_0(I) = \left\{f \in \hat{Q}_0 \,|\, \tau_f(I) = I\right\}$$

and set

$$P_0 = \operatorname{Inv}_0(I)^\perp := \{\alpha \in Q_0 : f(\alpha) = 1, \text{ for all } f \in \operatorname{Inv}_0(I)\}.$$

From Remark 2 it follows that $|P_0| = |\hat{Q}_0/\operatorname{Inv}_0(I)|$. Then we know that $\operatorname{Inv}_0(I)$ is a proper subgroup of Q_0.

Now we can consider $\mathfrak{g}$ as a Q/P_0-graded Lie algebra. The homogeneous spaces in this graded Lie algebra are indexed by $\bar{\alpha} = \alpha + P_0$, where $\alpha \in Q$. Note that all these homogeneous components are eigenspaces for each τ_f, where $f \in \mathrm{Inv}_0(I)$. For $\alpha \in Q$, we thus have

$$\mathfrak{g}_{\bar{\alpha}} = \bigoplus_{\beta \in P_0} \mathfrak{g}_{\alpha+\beta}. \tag{3}$$

The decomposition

$$\mathfrak{g} = \bigoplus_{\bar{\alpha} \in Q/P_0} \mathfrak{g}_{\bar{\alpha}}$$

is the decomposition of $\mathfrak{g}$ into a direct sum of common eigenvectors with respect to the action of all τ_f, where $f \in \mathrm{Inv}_0(I)$. Since I is preserved by all such τ_f, we obtain

$$I_{\bar{\alpha}} = I \cap \mathfrak{g}_{\bar{\alpha}}.$$

For $\mathbf{I} := \left\{\tau_f(I) \,:\, f \in \hat{Q}_0\right\}$, we have that $|\mathbf{I}| = |\hat{Q}_0/\mathrm{Inv}_0(I)|$. Now we can prove the following statement.

Lemma 7. *Assume that $\mathfrak{g}$ is not simple. Then there exists a non-homogeneous non-zero proper ideal I of $\mathfrak{g}$ such that different elements in $\mathbf{I}$ have zero intersection.*

Proof. Suppose $J \cap J' \neq 0$ for some $J \neq J'$ in $\mathbf{I}$. We may assume that $I \cap \tau_f(I) = I_1 \neq 0$ for some $f \in \hat{Q}_0 \backslash \mathrm{Inv}_0(I)$. We see that I_1 is a non-homogeneous non-zero proper ideal of $\mathfrak{g}$, and $\mathrm{Inv}_0(I) \subseteq \mathrm{Inv}_0(I_1) \subset \hat{Q}_0$.

Let $r = \mathrm{size}(I)$ and $r_1 = \mathrm{size}(I_1)$. Note that $r \leq |P_0|$ and $r_1 \leq |P_0|$. Let I_2 be the subideal of I_1 generated by all $x \in I_1$ with $|\mathrm{supp}(x)| = r_1$. Take a nonzero $x \in I_1$ with $|\mathrm{supp}(x)| = r_1$. Then $x = \tau_f(y)$ for some $y \in I$. If $x \notin \Bbbk y$, then I contains a linear combination of x and y which has strictly smaller support. This means that $r < r_1$. Consequently, in the case $r = r_1$, the previous argument shows that $\tau_f^{-1}(x) \in I_1$, for any $x \in I_1$ with $|\mathrm{supp}(x)| = r_1$. This means that $\tau_f(I_2) \subset I_2$ and thus either $\mathrm{Inv}_0(I_2)$ properly contains $\mathrm{Inv}_0(I)$ or $r_1 > r$.

Now we change our original ideal I to I_2. In this way we either increase the size of the ideal or the cardinality of the invariant subgroup and start all over again. Since both the size and the cardinality of the invariant subgroup are uniformly bounded, the process will terminate in a finite number of steps resulting in a new ideal of $\mathfrak{g}$ which will have the property that different elements in the corresponding $\mathbf{I}$ have zero intersection. □

Since Q_0 is finite, we may further take a non-homogeneous non-zero proper ideal I of $\mathfrak{g}$ which has the property described in Lemma 7 and such that $\mathrm{Inv}_0(I)$ is minimal with respect to inclusion. Then I is a Q/P_0-graded Lie algebra

$$I = \bigoplus_{\bar{\alpha} \in Q/P_0} I_{\bar{\alpha}}.$$

From (3) it follows that $\mathrm{size}(I) \leq |P_0|$. Directly from the definitions we also have $\mathrm{Inv}_0(I) \subset Q_0$.

Lemma 8. *We have:*

(a) $[J, J'] = 0$ *for any* $J \neq J'$ *in* $\mathbf{I}$*;*
(b) $\mathfrak{g} = \bigoplus_{J \in \mathbf{I}} J$*;*
(c) I *is a simple Lie algebra.*

Proof. Claim (a) follows from the fact that $[J, J'] = J \cap J' = 0$.

To prove claim (b), we first note that $\sum_{J \in \mathbf{I}} J = \mathfrak{g}$ as the left hand side, being closed under the action of $\hat{Q}_0$, is a non-zero homogeneous ideal of $\mathfrak{g}$ (and hence coincides with $\mathfrak{g}$ as the latter is graded simple). Let us prove that this sum is direct. For $J \in \mathbf{I}$, consider

$$X_J = J \cap \sum_{J' \in \mathbf{I} \setminus \{J\}} J',$$

which is an ideal of $\mathfrak{g}$. We have $[X_J, \mathfrak{g}] = 0$. Hence $X_J = 0$ by Lemma 6(b). Claim (b) follows.

Finally, suppose I is not simple as a Lie algebra. If $[I, I] = 0$, then from (b) we have $[\mathfrak{g}, \mathfrak{g}] = 0$, which is impossible. So $[I, I] \neq 0$. We take a non-zero proper ideal I_1 of I. From (b) we have the direct sum

$$\bigoplus_{f \in \hat{Q}_0/\mathrm{Inv}_0(I)} \tau_f(I_1).$$

We see that $\mathrm{Inv}_0(I_1)$ is a subgroup of $\mathrm{Inv}_0(I)$. From the minimality of $\mathrm{Inv}_0(I)$ we deduce that $\mathrm{Inv}_0(I_1) = \mathrm{Inv}_0(I)$. Thus the above direct sum is a homogeneous non-zero proper ideal of $\mathfrak{g}$ which contradicts the graded simplicity of $\mathfrak{g}$. So I is a simple Lie algebra. This completes the proof. □

Because of Lemma 8, from now on we may assume that I is a non-homogeneous non-zero proper ideal of $\mathfrak{g}$, that is a simple Lie algebra.

Let $\alpha \in Q$ and $x \in I_{\bar{\alpha}} \setminus \{0\}$. Since

$$I_{\bar{\alpha}} = I \bigcap \bigoplus_{\beta \in P_0} \mathfrak{g}_{\alpha+\beta},$$

there are unique vectors $x_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$, where $\beta \in P_0$, such that

$$x = \sum_{\beta \in P_0} x_{\alpha+\beta}. \tag{4}$$

Applying τ_f, where $f \in \hat{Q}_0/\mathrm{Inv}_0(I)$, to both side of (4), we obtain $|P_0|$ identities:

$$\sum_{\beta \in P_0} f(\alpha+\beta) x_{\alpha+\beta} = \tau_f(x), \quad f \in \hat{Q}_0/\mathrm{Inv}_0(I).$$

From Lemma 8 (b), we have that $\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}_0(I)\}$ is a set of linearly independent elements. Since $|P_0| = |\hat{Q}_0/\mathrm{Inv}_0(I)|$, we see that $\{x_{\alpha+\beta} : \beta \in P_0\}$ is a set of linearly independent elements and

$$\mathrm{span}\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}_0(I)\} = \mathrm{span}\{x_{\alpha+\beta} : \beta \in P_0\}.$$

Thus $\{x_{\alpha+\beta} : \beta \in P_0\}$ can be uniquely determined from the above $|Q_0|$ identities in terms of $\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}_0(I)\}$. Therefore the coefficient matrix $\big(f(\alpha+\beta)\big)$, where $f \in \hat{Q}_0/\mathrm{Inv}_0(I)$ and $\beta \in P_0$, is invertible. The above argument yields the following linear algebra result:

Lemma 9. *Let $\alpha \in Q$ and*

$$x = \sum_{\beta \in P_0} x_{\alpha+\beta} \in I_{\bar{\alpha}} \setminus \{0\},$$

where $x_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$, for $\beta \in P_0$. Then $\{x_{\alpha+\beta} : \beta \in P_0\}$ is a set of linearly independent elements and each $x_{\alpha+\beta}$ can be uniquely expressed in terms of elements in $\{\tau_f(x) : f \in \hat{Q}_0/\mathrm{Inv}_0(I)\}$ and entries of the invertible matrix $\big(f(\alpha+\beta)\big)_{f,\beta}$, where $f \in \hat{Q}_0/\mathrm{Inv}_0(I)$ and $\beta \in P_0$. Consequently, $\mathrm{size}(I) = |P_0|$.

We will also need the following recognition result.

Lemma 10. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Q-graded simple Lie algebras with non-homogeneous non-zero proper ideals I and I', respectively, that are simple Lie algebras. If* $\mathrm{Inv}_0(I) = \mathrm{Inv}_0(I')$ *and $I \simeq I'$ as Q/P_0-graded Lie algebras, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic as Q-graded Lie algebras.*

Proof. From the discussion above we know that both I and I' are simple Lie algebras. Let $\varphi_0 : I \to I'$ be an isomorphism of Q/P_0-graded Lie algebras. We know that, for each $\alpha \in Q$, we have the decomposition

$$I_{\bar{\alpha}} \subset \bigoplus_{\beta \in P_0} \mathfrak{g}_{\alpha+\beta}.$$

For any $\bar{f} \in \hat{Q}_0/\mathrm{Inv}_0(I)$, set

$$\varphi_f := \tau_f \circ \varphi_0 \circ \tau_f^{-1} : \tau_f(I) \to \tau_f(I').$$

By taking the direct sum, we obtain an isomorphism of Q/P_0-graded Lie algebras as follows:

$$\Phi := \bigoplus_{\bar{f} \in \hat{Q}_0/\mathrm{Inv}_0(I)} \varphi_f : \mathfrak{g} = \bigoplus_{\bar{f} \in \hat{Q}_0/\mathrm{Inv}_0(I)} \tau_f(I) \longrightarrow \mathfrak{g}' = \bigoplus_{\bar{f} \in \hat{Q}_0/\mathrm{Inv}_0(I)} \tau_f(I').$$

The isomorphism Φ commutes with all τ_f by construction. Therefore, Φ is even an isomorphism of Q-graded Lie algebras. □

3.3. Classification

The following theorem is the main result of this section.

Theorem 11. *Let $Q = Q_0 \times Q_1$ be an additive abelian group where Q_0 is finite, $\mathbb{k}$ be an arbitrary field containing a primitive root of unity of order $|Q|$, and $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ be a Q-graded simple Lie algebra over $\mathbb{k}$. Assume that any ideal of $\mathfrak{g}$ is Q_1-graded. Then there exists a subgroup $P_0 \subset Q_0$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P_0-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P_0, \mathfrak{a})$.*

Proof. We may assume that $\mathfrak{g}$ is not simple. Using Lemma 8, fix a nontrivial non-graded ideal $\mathfrak{a}$ of $\mathfrak{g}$ that is a simple Lie algebra. Then the Q/P_0-grading for $\mathfrak{a}$ is given by

$$\mathfrak{a} = \bigoplus_{\bar{\alpha} \in Q/P_0} \mathfrak{a}_{\bar{\alpha}}, \tag{5}$$

where

$$\mathfrak{a}_{\bar{\alpha}} = \mathfrak{a} \bigcap \bigoplus_{\beta \in \alpha + P_0} \mathfrak{g}_\beta.$$

From the definition of P_0 it follows that $\mathrm{Inv}_0(\mathfrak{a}) = P_0^\perp$. Now the claim follows from Lemma 10 applied to the graded Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(Q, P_0, \mathfrak{a})$, where in both cases the distinguished Q/P_0-graded ideal is $\mathfrak{a}$. □

The following result is a direct consequence of Theorem 11.

Corollary 12. *Let Q be a finite additive abelian group and $\mathfrak{g}$ be a Q-graded simple Lie algebra over an arbitrary field $\mathbb{k}$ containing a primitive root of unity of order $|Q|$. Then there exists a subgroup $P \subset Q$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$.*

The isomorphism problem is dealt with by the following statement.

Theorem 13. *Let Q, Q' be finite abelian groups, $\mathfrak{g}(Q, P, \mathfrak{a})$ be a Q-graded simple Lie algebra over $\mathbb{k}$ such that $\mathbb{k}$ contains primitive roots of unity of orders $|Q|$ and $|Q'|$, and $\mathfrak{g}(Q', P', \mathfrak{a}')$ be a Q'-graded Lie algebra over $\mathbb{k}$ with minimal gradings. Then $\mathfrak{g}(Q, P, \mathfrak{a})$ is graded isomorphic to $\mathfrak{g}(Q', P', \mathfrak{a}')$ if and only if there is a group isomorphism $\tau : Q \to Q'$ such that $\tau(P) = P'$ and the simple Lie algebras $\mathfrak{a}$ and $\mathfrak{a}'$ are graded isomorphic.*

Proof. The direction ($\Leftarrow$) is clear, we consider the direction ($\Rightarrow$).

Suppose $\Phi : \mathfrak{g}(Q, P, \mathfrak{a}) \to \mathfrak{g}(Q', P', \mathfrak{a}')$ is a graded isomorphism of Lie algebras. Then there is a group isomorphism $\sigma : Q \to Q'$ such that $\Phi(\mathfrak{g}_\alpha) = \mathfrak{g}'_{\sigma(\alpha)}$, for all $\alpha \in Q$.

From Lemma 3, we know that $\mathfrak{a}$ and $\mathfrak{a}'$ are the simple subalgebras of $\mathfrak{g}(Q, P, \mathfrak{a})$ and $\mathfrak{g}(Q', P', \mathfrak{a}')$ and all simple subalgebras are isomorphic. There is $f \in \hat{Q}'$ such that $\tau_f(\mathfrak{a}') = \Phi(\mathfrak{a})$. We have

$$\sigma(P) = \sigma(\mathrm{Inv}_0(\mathfrak{a})^\perp) = \mathrm{Inv}_0(\Phi(\mathfrak{a}))^\perp = \mathrm{Inv}_0(\tau_f(\mathfrak{a}'))^\perp = P'.$$

Using restriction, one can easily see that the simple Lie algebras $\mathfrak{a}$ and $\mathfrak{a}'$ are Q/P-graded isomorphic. □

In the above correspondence theorems, the algebras $\mathfrak{g}$ and $\mathfrak{a}$ may not be central or with split centroid, which is quite different from the setup of Correspondence Theorem 7.1.1 in [1]. Note that characterization on Q-graded simple Lie algebras over a field $\Bbbk$ for a general additive abelian group Q in the general case has to be dealt with by different methods. In what follows, we approach this problem using a graded version of Schur's lemma and using Correspondence Theorem 7.1.1 in [1].

4. Graded Schur's lemma

In this section we prove a graded version of Schur's lemma which we will frequently use in the rest of the paper. This is a standard statement, but we could not find a proper reference for the generality we need.

Let Q be an abelian group, $\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha$ a Q-graded Lie algebra over a field $\Bbbk$ and $W = \bigoplus_{\bar{\alpha}\in Q} W_\alpha$ a Q-graded module over $\mathfrak{g}$. For $\alpha \in Q$, we call a module homomorphism $\sigma : W \to W$ *homogeneous of degree* α provided that $\sigma(W_\beta) \subset W_{\beta+\alpha}$.

Theorem 14. (Graded Schur's Lemma) *Let Q be an abelian group and $\mathfrak{g}$ a Q-graded Lie algebra over an algebraically closed field $\Bbbk$. Let W be a Q-graded simple module over $\mathfrak{g}$ with* $\dim W < |\Bbbk|$. *Then, for any fixed $\alpha \in Q$, any two degree α automorphisms of W differ by a scalar factor only.*

Proof. Let $\mathrm{End}_0(W)$ be the algebra of all homogeneous degree zero endomorphisms of W. It is enough to show that $\mathrm{End}_0(W) = \Bbbk$. The usual arguments give that $\mathrm{End}_0(W)$ is a division algebra over $\Bbbk$. Then W, viewed as an $\mathrm{End}_0(W)$-module, is a sum of copies of $\mathrm{End}_0(W)$. In particular,

$$\dim \mathrm{End}_0(W) \leq \dim W < |\Bbbk|.$$

Since $\Bbbk$ is algebraically closed, if $\mathrm{End}_0(W)$ were strictly larger than $\Bbbk$, then $\mathrm{End}_0(W)$ would contain some σ which is transcendental over $\Bbbk$. Then the fraction field $\Bbbk(\sigma)$ would be contained in $\mathrm{End}_0(W)$. However, we have the elements $\frac{1}{\sigma-a} \in \Bbbk(\sigma)$, where $a \in \Bbbk$, which are linearly independent. Therefore

$$\dim \mathrm{End}_0(W) \geq \dim \Bbbk(\sigma) \geq |\Bbbk|,$$

contradicting the fact that $\dim \mathrm{End}_0(W) < |\Bbbk|$. Thus we conclude that $\mathrm{End}_0(W) = \Bbbk$. □

5. Graded simple Lie algebras: the case of infinite Q

Theorem 15. *Let Q be an arbitrary additive abelian group and $\mathfrak{g}$ be a Q-graded simple Lie algebra over a algebraically closed field $\Bbbk$ such that* $\dim \mathfrak{g} < |\Bbbk|$. *Then there exists a subgroup $P \subset Q$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$.*

Proof. From Theorem 14, we know that $\mathfrak{g}$ is graded-central-simple (see Definition 4.3.1 in [1]). From Lemma 4.3.8 in [1], we know that $\mathfrak{g}$ has a split centroid. The statement now follows from Correspondence Theorem 7.1.1 in [1].

Combining Lemma 3 with Theorems 11 and 15, we obtain:

Corollary 16. *Let Q be a finite additive abelian group and $\mathfrak{g}$ a Q-graded simple Lie algebra over a field $\mathbb{k}$. Suppose that one of the following holds:*

(i) $\mathbb{k}$ *contains a primitive root of unity of order $|Q|$;*
(ii) $\mathbb{k}$ *is algebraically closed and* $\dim \mathfrak{g} < |\mathbb{k}|$.

Then there exists a subgroup $P \subset Q$ and a simple Lie algebra $\mathfrak{a}$ with a Q/P-grading such that $\mathfrak{g} \simeq \mathfrak{g}(Q, P, \mathfrak{a})$.

The following result is a generalization of [11, Main Theorem (a)] which follows directly from Theorem 15.

Corollary 17. *Let $Q = Q_0 \times Q_1$ be an additive abelian group where Q_0 is torsion subgroup of Q. Let $\mathfrak{g}$ be a finite dimensional Q-graded simple Lie algebra. Then $\mathfrak{g}$ is a Q_0-graded simple Lie algebra.*

The necessary and sufficient conditions for two graded-simple algebras $\mathfrak{g}(Q, P, \mathfrak{a})$ where $\mathfrak{a}$ is central-simple were given in Correspondence Theorem 7.1.1 in [1].

From Corollary 16, we actually obtain a full classification of all finite-dimensional Q-graded simple Lie algebras over any algebraically closed field of characteristic 0 due to the recent classification of all gradings on finite dimensional simple Lie algebras, see [5,7,18]. For a similar classification over an algebraically closed field of characteristic $p > 0$ it remains only to determine all gradings on finite dimensional simple Lie algebras. Some partial results in this direction can be found in [7], see also references therein.

6. Graded Weyl theorem

One consequence of our Theorem 15 is that any finite dimensional Q-graded simple Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic 0 is semi-simple after forgetting the grading (note that this property is not true in positive characteristic). This allows us to prove a graded version of the Weyl Theorem.

Theorem 18. (Graded Weyl Theorem) *Let Q be an abelian group and $\mathfrak{g}$ a finite dimensional Q-graded semi-simple Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic 0. Then any finite dimensional Q-graded module V over $\mathfrak{g}$ is completely reducible as a graded module, that is, V is a direct sum of Q-graded simple submodules of V.*

Proof. Since $\mathfrak{g}$ is finite-dimensional, the minimal grading of $\mathfrak{g}$ is by a finitely generated subgroup of Q. Since $\mathfrak{g}$ is finite dimensional, from Corollary 16, we know that $\mathfrak{g}$ is Q_0-graded simple, for a finite subgroup Q_0 of Q. From Lemma 3, it follows that, as an ungraded Lie algebra, $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra. (This can also be proved by looking at the radical of $\mathfrak{g}$ without using Corollary 16).

We need to show that any Q-graded submodule X of a Q-graded finite dimensional $\mathfrak{g}$-module W has a Q-graded complement. By Weyl Theorem, we have an ungraded $\mathfrak{g}$-submodule Y_1 such that $W = X \oplus Y_1$. By [6, Lemma 1.1] (see also [4, Theorem 2.3′]), there is a Q-graded submodule Y of W such that $W = X \oplus Y$. The theorem follows. □

7. Graded simple modules over graded Lie algebras

We will study graded simple modules over graded Lie algebras (not necessarily graded simple) in this section. Since we will use the graded analogue of Schur's Lemma, we will assume that the base field $\Bbbk$ is algebraically closed later in Sect. 7.3.

7.1. Construction

Let Q be an abelian group and P a subgroup of Q. Let, further, $\mathfrak{g}$ be a Q-graded Lie algebra over an arbitrary field $\Bbbk$. Note that in this section we even do not need to assume that the Q-grading of $\mathfrak{g}$ is minimal. Consider $\mathfrak{g} = \bigoplus_{\bar{\alpha}\in Q/P} \mathfrak{g}_{\bar{\alpha}}$ as a Q/P-graded Lie algebra with $\mathfrak{g}_{\bar{\alpha}} = \bigoplus_{\beta\in P} \mathfrak{g}_{\alpha+\beta}$. Let $V = \bigoplus_{\bar{\alpha}\in Q/P} V_{\bar{\alpha}}$ be a simple $\mathfrak{g}$-module with a fixed Q/P-grading.

Then we can form the $\mathfrak{g}$-module $V \otimes \Bbbk Q$ as follows: for $x \in \mathfrak{g}_\alpha$, $v \in V$ and $\beta \in Q$, define

$$x \cdot \left(v \otimes t^{\beta}\right) = (xv) \otimes t^{\alpha+\beta}.$$

Define the Q-graded $\mathfrak{g}$-module

$$M(Q, P, V) := \bigoplus_{\alpha\in Q} M(Q, P, V)_\alpha, \quad \text{where} \quad M(Q, P, V)_\alpha := V_{\bar{\alpha}} \otimes t^{\alpha}.$$

For example, $M(Q, Q, V) = V \otimes \Bbbk Q$ (with the obvious Q-grading) while $M(Q, \{0\}, V) = V$ (with the original Q-grading). From the definition it follows that $\dim M(Q, P, V) = \dim(V)|P|$, if V is finite-dimensional and P is finite. These graded modules are called loop modules in [8].

Unlike the algebra case, the Q-graded $\mathfrak{g}$-module $M(Q, P, V)$ is generally not graded simple. Even it is hard to see whether it is graded simple. We will first discuss this problem.

We say that a simple $\mathfrak{g}$-module V with a Q/P-grading is *grading extendable* if there is a subspace decomposition

$$V = \sum_{\alpha\in Q} X_\alpha \quad \text{with} \quad V_{\bar{\alpha}} = \sum_{\beta\in P} X_{\alpha+\beta}$$

for any $\alpha \in Q$ (here both sums are not necessarily direct) such that $\mathfrak{g}_\beta X_\alpha \subset X_{\alpha+\beta}$ for all $\beta \in Q$ and at least one $X_\alpha \neq V_{\bar{\alpha}}$. Note that the set $\{X_\alpha : \alpha \in Q\}$ is called a Q-covering of V in [3].

Now we can obtain some necessary and sufficient conditions for the Q-graded module $M(Q, P, V)$ to be Q-graded simple.

Lemma 19. *The Q-graded module $M(Q, P, V)$ is Q-graded simple if and only if the simple module V is not grading extendable.*

Proof. If V is grading extendable, there is a decomposition $V = \sum_{\alpha\in Q} X_\alpha$ with $V_{\bar{\alpha}} = \sum_{\beta\in P} X_{\alpha+\beta}$, for any $\alpha \in Q$, such that $\mathfrak{g}_\alpha X_\beta \subset X_{\alpha+\beta}$ and at least one $X_{\alpha+\beta} \neq V_{\bar{\alpha}}$. Then the module $M(Q, P, V)$ has a nonzero proper Q-graded submodule

$$\bigoplus_{\alpha\in Q} X_\alpha \otimes t^\alpha.$$

Thus $M(Q, P, V)$ is not Q-graded simple.

Now suppose that $M(Q, P, V)$ is not Q-graded simple. Consider the ideal I in $\mathbb{C}Q$ generated by $\{t^\alpha - 1 : \alpha \in P\}$ and let

$$N = M(Q, P, V)/(M(Q, P, V)) \cap (V \otimes I)).$$

The module N is naturally Q/P-graded and is, in fact, isomorphic to the Q/P-graded module V with the original grading. We have that

$$N = \bigoplus_{\bar{\alpha}\in Q/P} V_{\bar{\alpha}} \otimes t^{\bar{\alpha}},$$

where $\{t^{\bar{\alpha}} : \bar{\alpha} \in Q/P\}$ is a basis for the group algebra of Q/P. Let

$$\pi : M(Q, P, V) \twoheadrightarrow N,$$

be the canonical epimorphism.

Take a proper Q-graded submodule $X = \bigoplus_{\alpha\in Q} X_\alpha \otimes t^\alpha$ of $M(Q, P, V)$. Since $\pi(X) = N$, we have $V_{\bar{\alpha}} = \sum_{\beta\in P} X_{\alpha+\beta}$ for any $\alpha \in Q$ and, also, $\mathfrak{g}_\alpha X_\beta \subset X_{\alpha+\beta}$. Since W is proper, we have $X_\alpha \neq V_{\bar{\alpha}}$. Thus V is grading extendable. □

For any $f \in \hat{Q}$ and any $\mathfrak{g}$-module V, we define a new module V^f as follows: as a vector space, we set $V^f := V$, and the action of $\mathfrak{g}$ on V^f is given by

$$x_\alpha \circ v = f(\alpha) x_\alpha v, \quad \text{for all} \quad x_\alpha \in \mathfrak{g}_\alpha, \quad v \in V.$$

If W is a Q-graded $\mathfrak{g}$-module, for any $f \in \hat{Q}$, the linear map

$$\tau_f : W \to W^f; \quad \tau_f(w_\alpha) = f(\alpha) w_\alpha, \quad \forall \alpha \in Q, \quad w_\alpha \in W_\alpha,$$

is a Q-graded module isomorphism since

$$\tau_f(x_\alpha w) = f(\alpha) x_\alpha \tau_f(w) = x_\alpha \circ \tau_f(w)$$

for all $\alpha \in Q$, $x_\alpha \in \mathfrak{g}_\alpha$, $w \in W_\alpha$. But $\tau_f : W \to W$ is generally not a $\mathfrak{g}$-module homomorphism from W to W. However $\tau_f(N)$ is a submodule of W for any submodule N of W.

7.2. Classification of graded simple modules: the case of finite Q

In this section we assume that Q is a finite abelian group, the field $\Bbbk$ contains a primitive root of unity of order $|Q|$, and $\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha$ a Q-graded Lie algebra over $\Bbbk$. Although the major part of both results and methods for the study of Q-graded simple modules is rather different from those for the study of Q-graded simple Lie algebras in Sects. 3.2 and 3.3, there are some similarities. We will omit similar arguments in this section.

Lemma 20. *Assume that Q is finite and $\Bbbk$ contains a primitive root of unity of order $|Q|$. Then the module*

$$M(Q, P, V) = \bigoplus_{f\in\hat{Q}/P^\perp} V^f,$$

where each summand has a naturally induced Q/P-grading.

Proof. The proof is similar to that of Lemma 3.

We note that neither P nor V in Lemma 20 are uniquely determined in general, cf. Example 35.

Now let $W = \bigoplus_{\alpha\in Q} W_\alpha$ be a Q-graded simple $\mathfrak{g}$-module. As before, we define $\mathrm{supp}(v)$ for any $v = \sum_{\alpha\in Q} v_\alpha \in W$, where $v_\alpha \in W_\alpha$, and $\mathrm{size}(N)$ for any subset N of the module W.

We assume that W is not simple as a $\mathfrak{g}$-module. Let N be a proper nonzero submodule of W. Set $r := \mathrm{size}(N) > 1$ and define

$$R(N) := \mathrm{span}\{v \in N : \mathrm{size}(v) = r\}.$$

Then $R(N)$ is a non-homogeneous non-zero proper submodule of W. We will say that a submodule N of W is *pure* of size r if $\mathrm{size}(N) = r$ and, moreover, $R(N) = N$.

For $\alpha \in Q$, define $\pi_\alpha : W \to W_\alpha$ as the projection with respect to the graded decomposition. The following lemma and its proof are similar to Lemmata 5 and 6.

Lemma 21.

(a) *For any nonzero $x \in N$ we have $|\mathrm{supp}(x)| > 1$.*
(b) *We have $\pi_\alpha(N) = W_\alpha$ for each $\alpha \in Q$.*
(c) *A submodule N of W is Q-graded if and only if $\tau_f(N) \subset N$ for all $f \in \hat{Q}$.*

As we did before, we define

$$\mathrm{Inv}(N) = \{f \in \hat{Q} \mid \tau_f(N) = N\},$$

and set

$$P = \mathrm{Inv}(N)^\perp := \{\alpha \in Q : f(\alpha) = 1, \text{ for all } f \in \mathrm{Inv}(N)\}.$$

We know that $|P| = |\hat{Q}/\mathrm{Inv}(N)|$. For $\mathbf{N} := \{\tau_f(N) : f \in \hat{Q}\}$, we have that $|\mathbf{N}| = |\hat{Q}/\mathrm{Inv}(N)|$.

Similarly to Lemma 7 and its proof, we have the following.

Lemma 22. *Assume that W is a Q-graded simple $\mathfrak{g}$-module that is not simple. Then W has a non-homogeneous non-zero proper submodule N so that different submodules in $\mathbf{N}$ have zero intersection.*

Now we have the following statement.

Lemma 23. *Assume that W is a Q-graded simple $\mathfrak{g}$-module that is not simple. Then W has a non-homogeneous simple submodule.*

Proof. Since Q is finite, we may take a non-zero proper submodule N of W having the property as in Lemma 22 and such that $\mathrm{Inv}(N)$ is minimal with respect to inclusion and, further, $\mathrm{size}(N)$ is maximal. Then N is a Q/P-graded submodule

$$N = \bigoplus_{\bar{\alpha}\in Q/P} N_{\bar{\alpha}}.$$

Note that $\mathrm{size}(N) = r > 1$. Take a nonzero $z = z_1 + z_2 + \cdots + z_r \in N_{\bar{\alpha}}$ where $z_i \in W_{\alpha+\alpha_i}, \alpha_i \in P$. Replace our N by the submodule of W generated by this z. Then we still have the property of Lemma 22, $\mathrm{Inv}(N)$ is minimal, $\mathrm{size}(N)$ is maximal, and, furthermore, N is pure.

Since $\mathrm{Inv}(N)$ is minimal, it follows that $\alpha_1, \alpha_2, \ldots, \alpha_r$ generate the group P. There are $f_1 = id, f_2, \ldots, f_r \in \hat{Q}$ such that the $r \times r$ matrix $(f_i(\alpha_j))$ is invertible. Then we have the direct sum

$$W = \bigoplus_{i=1}^{r} \tau_{f_i}(N) \tag{6}$$

since the submodule on the right hand side has size 1.

If N is not simple, we take a proper nonzero pure submodule N' of N. We see that $\mathrm{Inv}(N) = \mathrm{Inv}(N')$, and also

$$N' = \bigoplus_{\bar{\alpha}\in Q/P} N'_{\bar{\alpha}}.$$

Note that $\mathrm{size}(N') \geq r$. Since $\mathrm{size}(N)$ is maximal, we have $\mathrm{size}(N') = r$. We may assume that N' is generated by a nonzero element $z' = z'_1 + z'_2 + \cdots + z'_r \in N_{\bar{\alpha}}$ where $z'_i \in W_{\alpha'+\alpha'_i}$, for $\alpha'_i \in P$. Then $\alpha'_1, \alpha'_2, \ldots, \alpha'_r$ also generate the group P, and the $r \times r$ matrix $(f_i(\alpha'_j))$ is invertible. Similarly to the above, we have the direct sum

$$W = \bigoplus_{i=1}^{r} \tau_{f_i}(N'). \tag{7}$$

Comparing (6) with (7), we get

$$\bigoplus_{i=1}^{r} \tau_{f_i}(N) = \bigoplus_{i=1}^{r} \tau_{f_i}(N'). \tag{8}$$

As $N' \subsetneq N$, we have $\tau_{f_i}(N') \subsetneq \tau_{f_i}(N)$, for each i, and hence (8) is impossible. This implies that N is simple. □

Theorem 24. *Let Q be a finite additive abelian group and W be a Q-graded simple $\mathfrak{g}$-module over $\Bbbk$ such that* $\mathrm{char}(\Bbbk)$ *contains a primitive root of unity of order $|Q|$. Then there exist a subgroup $P \subset Q$, and a simple $\mathfrak{g}$-module V with a nonextendable Q/P-grading so that W is Q-graded isomorphic to $M(Q, P, V)$.*

Proof. From the previous lemma we know that W has a simple $\mathfrak{g}$-submodule N. We may assume that $\mathrm{Inv}(N)$ is minimal and $\mathrm{size}(N) = r$ is maximal. Let $P' = \mathrm{Inv}(N)^{\perp}$. Then N is Q/P'-graded

$$N = \bigoplus_{\bar{\alpha} \in Q/P'} N_{\bar{\alpha}}. \tag{9}$$

Take a nonzero $z = z_{\alpha} + z_{\alpha+\alpha_1} + \cdots z_{\alpha+\alpha_{r-1}} \in N$ where $z_{\gamma} \in W_{\gamma}$ and $\alpha_i \in P'$. We can define a linear map

$$\Lambda_{\alpha_i} : W \to W, \quad w_{\gamma} \mapsto w_{\gamma+\alpha_i},$$

where $w = w_{\gamma} + w_{\gamma+\alpha_1} + \cdots + w_{\gamma+\alpha_{r-1}} \in U(\mathfrak{g})_{\gamma-\alpha} z$, and $U(\mathfrak{g})_{\gamma-\alpha}$ is the homogeneous component of grading $\gamma - \alpha$ of the universal enveloping algebra. The map Λ_{α_i} is well-defined since $\mathrm{size}(N) = r$. Further Λ_{α_i} is a bijection. For any $y_{\gamma} \in U(\mathfrak{g})_{\gamma}$, from

$$y_{\gamma} w = y_{\gamma} w_{\beta} + y_{\gamma} w_{\beta+\alpha_1} + \cdots + y_{\gamma} w_{\beta+\alpha_{r-1}} \in N$$

we have that $\Lambda_{\alpha_i}(y_{\gamma} w_{\beta}) = y_{\gamma} \Lambda_{\alpha_i}(w_{\beta})$. Thus, Λ_{α_i} is a Q-graded $\mathfrak{g}$-module automorphism of W which is, moreover, homogeneous of degree α_i.

Let P be the subset consisting all $\alpha_i \in$ so that there is a graded isomorphism Λ_{α_i} of W of degree α_i. Then P is a subgroup of P'.

If $P \neq \{0, \alpha_1, \ldots, \alpha_{r-1}\}$, there is another $\alpha_r \in P$. Then $N' = \{x + \Lambda_{\alpha_r}(x) | x \in N\}$ is a simple $\mathfrak{g}$-submodule isomorphic to N. Also $\mathrm{Inv}(N) = \mathrm{Inv}(N)$ and $\mathrm{size}(N') > r$, which contradicts the choice of N. Thus $P = \{0, \alpha_1, \ldots, \alpha_{r-1}\}$, and N has the natural Q/P-grading

$$N = \bigoplus_{\bar{\bar{\alpha}} \in Q/P} N_{\bar{\bar{\alpha}}}. \tag{10}$$

Let $P^{\perp} = \{f \in \hat{Q} | f(P) = 1\}$. Then

$$W = \bigoplus_{f \in \hat{Q}/P^{\perp}} \tau_f(N). \tag{11}$$

From (9), (10), (11) we see that $N_{\bar{\bar{\alpha}}} = 0$ or $N_{\bar{\alpha}}$. For example $N_{\bar{\bar{\alpha}}} = 0$ if $\alpha \in P' \backslash P$.

From this we see that W is Q-graded to the submodule of $M(Q, P, N)$. This completes the proof. □

We illustrate our result by the following example.

Example 25. Let $\mathfrak{g}$ be the abelian Lie algebra with basis $g_{(0,0)}, g_{(1,0)}, g_{(0,1)}$ with the $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$-grading given by

$$\mathfrak{g}_{(i,j)} = \operatorname{span}\{g_{(i,j)}\} \quad \text{where} \quad g_{(1,1)} = 0.$$

Let W be the Q-graded simple $\mathfrak{g}$-module with basis $w_{(1,0)}, w_{(0,0)}$ and the action

$$g_{(i,j)} \cdot w_{(0,0)} = w_{(i,j)}, \quad g_{(i,j)} \cdot w_{(1,0)} = w_{(i+1,j)},$$

where $w_{(0,1)} = w_{(1,1)} = 0$. Let $V = \mathbb{k}v$ be the one dimensional $\mathfrak{g}$-module with the action

$$g_{(0,0)} \cdot v = v, \quad g_{(1,0)} \cdot v = v, \quad g_{(0,1)} \cdot v = 0.$$

We see that W is isomorphic to $M(Q, P, V)$ where $P = \mathbb{Z}_2 \times \{0\}$. In the proof of Theorem 24, $P' = Q$.

7.3. Classification of graded simple modules: the case of infinite Q

Let Q be an additive abelian group and $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ a Q-graded Lie algebra over an algebraically closed field $\mathbb{k}$. Let $W = \bigoplus_{\alpha \in Q} W_\alpha$ be a Q-graded simple $\mathfrak{g}$-module such that $\dim(W) < |\mathbb{k}|$.

We assume that W is not simple as a $\mathfrak{g}$-module. Let N be a proper nonzero submodule of W. Set $r := \operatorname{size}(N) > 1$. We may assume that N is pure of size r.

Fix a nonzero element $v = v_\beta + v_{\beta+\alpha_1} + \cdots v_{\beta+\alpha_{r-1}} \in N$ where $v_\gamma \in W_\gamma$, and $\alpha_1, \alpha_2, \ldots, \alpha_{r-1} \in Q$. We replace our N by the submodule of W generated by v. Then N is still pure of size r. We can define a linear map

$$\Lambda_{N,\alpha_i} : W \to W, \quad w_\gamma \mapsto w_{\gamma+\alpha_i}, \tag{12}$$

where $w = w_\gamma + w_{\gamma+\alpha_1} + \cdots + w_{\gamma+\alpha_{r-1}} \in U(\mathfrak{g})_{\gamma-\beta} v$, and $U(\mathfrak{g})_{\gamma-\beta}$ is the homogeneous component of grading $\gamma - \beta$ of the universal enveloping algebra.

For any $y_\gamma \in U(\mathfrak{g})_\gamma$, from

$$y_\gamma w = y_\gamma w_\beta + y_\gamma w_{\beta+\alpha_1} + \cdots + y_\gamma w_{\beta+\alpha_{r-1}} \in N$$

we have that $\Lambda_{N,\alpha_i}(y_\gamma w_\beta) = y_\gamma \Lambda_{N,\alpha_i}(w_\beta)$. Thus, Λ_{N,α_i} is a Q-graded $\mathfrak{g}$-module automorphism of W which is, moreover, homogeneous of degree α_i. From Theorem 14 it follows that Λ_{N,α_i} does not depend on the choice of N up to a scalar multiple. We thus simplify the notation Λ_{N,α_i} to Λ_{α_i}. Set

$$P' = \{\alpha \in Q : \text{there is a degree } \alpha \text{ module isomorphism of } W\},$$
$$D' = \operatorname{span}\{\Lambda_\alpha : \alpha \in P'\}.$$

The following result is trivial.

Lemma 26.

(a) *The set P' is a nonzero subgroup of Q.*

(b) *The vector space* D' *has the structure of an associative* P'*-graded division algebra induced by composition of endomorphisms where* D' *is naturally* Q*-graded with* $\deg(\Lambda_\alpha) = \alpha$.

Now we need the following lemma.

Lemma 27. *The* Q*-graded associative* P'*-graded division algebra* D' *has a maximal* Q*-graded commutative subalgebra* D.

Proof. For any ascending chain of Q-graded commutative subalgebras of D': $A_1 \subset A_2, \subset \cdots \subset A_n \subset \cdots$, the union $\cup_{i=1}^{\infty} A_i$ is also a Q-graded commutative subalgebras of D'. Therefore D' has a maximal Q-graded commutative subalgebra D by Zorn's lemma. □

Let D be a maximal Q-graded commutative subalgebra of D' and let $P = \mathrm{supp}(D)$ which is a subgroup of Q. We remark that, unlike the algebra case, here, in general, $D \neq D'$, see Examples 35.

From Lemma 1.2.9(i) in [16], we have the following:

Lemma 28. *The* P*-graded associative division algebra* D *is isomorphic to the group algebra* $\Bbbk P$.

Now, we have elements Λ_α, for $\alpha \in P$, satisfying

$$\Lambda_\alpha \Lambda_\beta = \Lambda_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in P.$$

Let $\mathbf{\Lambda} = \{\Lambda_\alpha : \alpha \in P\}$ and V' be the span of the set

$$\left\{v_\beta - \Lambda_\alpha(v_\beta) : \beta \in Q, v_\beta \in W_\beta \text{ and } \alpha \in P\right\}.$$

Then V' is a submodule of W.

Lemma 29. *The submodule* V' *of* W *is a proper submodule of* W.

Proof. Let $\{\beta_j : j \in B\}$ be a set of representatives for cosets in Q/P, where B is an index set. Let $\left\{x_{\beta_j}^{(k)} : k \in B_j\right\}$ be a basis of W_{β_j}, where B_j is an index set. Then the set

$$\left\{\mathbf{\Lambda}\left(x_{\beta_j}^{(k)}\right) \,:\, j \in B, k \in B_j\right\}$$

is a basis for W. For $j \in B$ and $k \in B_j$, set

$$S_{jk} := \left\{x_{\beta_j}^{(k)} - \Lambda\left(x_{\beta_j}^{(k)}\right) \,:\, \Lambda \in \mathbf{\Lambda}\right\}$$

and

$$S = \bigcup_{j \in B} \bigcup_{k \in B_j} S_{jk}.$$

Comparing supports of involved elements, it is easy to see that the sum

$$\mathcal{I} = \sum_{j \in B} \sum_{k \in B_j} \operatorname{span}(S_{jk})$$

is direct. Define the linear map

$$\sigma : W \to \Bbbk \text{ such that } \Lambda\left(x_{\beta_j}^{(k)}\right) \mapsto 1 \text{ for all } \Lambda \in \mathbf{\Lambda}. \tag{13}$$

Clearly σ is onto and $\sigma(V') = 0$. Therefore $V' \neq W$ and thus V' is a proper submodule of W. By construction, this submodule is pure of size two. □

Lemma 30. *The submodule V' is a proper maximal submodule of W.*

Proof. Clearly, V' is a pure submodule of W of size 2. Assume that V' is not maximal. Then there is a proper submodule V'' of W which properly contains V'. This means that there exist different elements $\beta_0, \beta_0+\gamma_1, \beta_0+\gamma_2, \dots, \beta_0+\gamma_r$ in Q and non-zero elements

$$x_{\beta_0} \in \mathfrak{g}_{\beta_0}, x_{\beta_0+\gamma_1} \in \mathfrak{g}_{\beta_0+\gamma_1}, x_{\beta_0+\gamma_2} \in \mathfrak{g}_{\beta_0+\gamma_2}, \dots, x_{\beta_0+\gamma_r} \in \mathfrak{g}_{\beta_0+\gamma_r}$$

such that

$$x := x_{\beta_0} + x_{\beta_0+\gamma_1} + x_{\beta_0+\gamma_2} + \cdots + x_{\beta_0+\gamma_r} \in V''\backslash V'. \tag{14}$$

Assume that r is minimal possible for elements in $V''\backslash V'$. Note that $r > 0$ as $V'' \neq W$. If any γ_i or any difference $\gamma_j - \gamma_i$ for $i \neq j$ is in P, then we can use definition of $V''\backslash V'$ and subtract from x an element in V' getting a new element in $V''\backslash V'$ with strictly smaller support. Therefore neither γ_i nor any difference $\gamma_j - \gamma_i$ for $i \neq j$ is in P. In particular, it follows that $V''\backslash V'$ does not contain any element whose support would be a proper subset of the support of x. Note that any partial sum of x in (14) is not in $V''\backslash V'$ for otherwise either this sum or its complement to x would be an element in $V''\backslash V'$ with strictly smaller support.

Now we fix all these γ_i. Next we claim that, for a fixed $x_\beta \in W_\beta$ for which an element x of the form (14) in $V''\backslash V'$ exists (replacing β_0 with β), the elements

$$x_{\beta+\gamma_1} \in \mathfrak{g}_{\beta+\gamma_1}, x_{\beta+\gamma_2} \in \mathfrak{g}_{\beta+\gamma_2}, \dots, x_{\beta+\gamma_r} \in \mathfrak{g}_{\beta+\gamma_r}$$

leading to such x are uniquely determined. Indeed, otherwise the non-zero difference between two such elements of the form (14) would have a strictly smaller support and hence would belong to V' because of the minimality of r. This is, however, impossible by the previous paragraph.

By the above arguments, we have the $\mathfrak{g}$-module isomorphisms $\Lambda_{\gamma_i} : W \to W$ mapping x_β to $x_{\beta+\gamma_i}$ defined as in (12). We denote by V_1 the span of all elements of the form (14) where $\beta \in Q$ is arbitrary. From this construction and the above properties it follows that V_1 is a pure submodule of size $r+1$ that is contained in V''.

Now each x of the form (14) can be uniquely written as

$$x = x_\beta + \Lambda_{\gamma_1}(x_\beta) + \cdots + \Lambda_{\gamma_r}(x_\beta)$$

and all γ_i are in P' defined before Lemma 26. For any $\alpha \in P$, we have $x - \Lambda_\alpha(x) \in V'$, $\Lambda_\alpha(x) = x - (x - \Lambda_\alpha(x)) \in V'' \setminus V'$ and hence

$$\Lambda_\alpha(x) = \Lambda_\alpha(x_\beta) + \Lambda_\alpha \Lambda_{\gamma_1}(x_\beta) + \cdots + \Lambda_\alpha \Lambda_{\gamma_2}(x_\beta) \in V_1.$$

At the same time, we have

$$y = \Lambda_\alpha(x_\beta) + \Lambda_{\gamma_1}\Lambda_\alpha(x_\beta) + \cdots + \Lambda_{\gamma_r}\Lambda_\alpha(x_\beta) \in V_1,$$

as this element has the form (14) with x_β replaced by $\Lambda_\alpha(x_\beta)$.

We have $\Lambda_\alpha(x) - y \in V_1$. If we would have $\Lambda_\alpha(x) - y \neq 0$, then $\Lambda_{\alpha_j}(x) - y \notin V_1$ since V_1 does not contain elements with such support. Hence $\Lambda_\alpha(x) - y = 0$, i.e., $\Lambda_\alpha \Lambda_{\gamma_i} = \Lambda_{\gamma_i}\Lambda_\alpha$ for all $\alpha \in P$ and all $i = 1, 2, \ldots, n$. Since D is the maximal graded commutative subalgebra of D', thus at least one $\gamma_i \in P$, a contradiction. The claim follows. □

Now we have the following:

Theorem 31. *Let Q be an additive abelian group and $\mathfrak{g}$ be a Q-graded Lie algebra over an algebraically closed field $\Bbbk$. Let W be a graded simple $\mathfrak{g}$-module such that* $\dim(W) < |\Bbbk|$. *Then there is a subgroup $P \subset Q$ and a simple $\mathfrak{g}$-module V with a Q/P-grading such that W is Q-graded isomorphic to $M(Q, P, V)$.*

Proof. From Lemma 30 we have that the module $V = W/V'$ is a simple $\mathfrak{g}$-module with a Q/P-grading. It is easy to verify that the Q-graded canonical map

$$\begin{aligned} W &\to M(Q, P, V), \\ v_\alpha &\mapsto (v_\alpha + V') \otimes t^\alpha \end{aligned}$$

is a degree 0 injective homomorphism of $\mathfrak{g}$-modules. Thus W is Q-graded isomorphic to $M(Q, P, V)$. □

We note that a special case of Theorem 31 was obtained in [6] with a totally different approach. We remark that neither P nor V in Theorem 31 are uniquely determined. Further, $M(Q, P, V)$ might have non-trivial graded automorphisms, see Lemma 33 below.

Now we want to consider graded isomorphisms between $\mathfrak{g}$-modules of the form $M(Q, P, V)$. By construction, we have $M(Q, P, V)^f = M(Q, P, V^f)$ for any $f \in \hat{Q}$. Note that V and V^f are not isomorphic as $\mathfrak{g}$-modules in general. Our next observation is the following.

Lemma 32. *In the situation above, there is a degree zero isomorphism of graded $\mathfrak{g}$-modules $M(Q, P, V)$ and $M(Q, P, V)^f$.*

Proof. Define $\Psi : M(Q, P, V)^f \to M(Q, P, V)$ by sending $v \mapsto \frac{1}{f(\alpha)} v$ for any $\alpha \in Q$ and $v \in M(Q, P, V)_\alpha$. Then, using the definitions, for all $\alpha, \beta \in Q$, $x_\beta \in \mathfrak{g}_\beta$ and $v \in M(Q, P, V)_\alpha$, we have

$$\Psi(x_\beta \circ v) = \Psi(f(\beta) x_\beta v) = \frac{f(\beta)}{f(\alpha + \beta)} x_\beta v = \frac{1}{f(\alpha)} x_\beta v = x_\beta \Psi(v).$$

As Ψ is obviously bijective, the claim follows. □

Lemma 33. *Let $\mathfrak{g}$ be a Q-graded Lie algebra and P subgroup of Q. Let, further, V be a simple $\mathfrak{g}$-module of dimension smaller than $|\mathbb{k}|$ with a nonextendable grading over Q/P. Assume $\alpha \in Q$. Then there is a degree α graded automorphism $\tau : M(Q,P,V) \to M(Q,P,V)$ if and only if there is $f \in \hat{Q}$ and a degree $\bar{\alpha} \in Q/P$ isomorphism $\mu : V^f \to V$ of Q/P-graded $\mathfrak{g}$-modules.*

Proof. To prove necessity, write τ in the form

$$\begin{aligned}\tau : M(Q,P,V) &\to M(Q,P,V),\\ v_{\bar{\beta}} \otimes t^{\beta} &\mapsto \mu_{\beta}(v_{\bar{\beta}}) \otimes t^{\beta+\alpha},\end{aligned}$$

where $\beta \in Q$ and $\mu_\beta : V_{\bar{\beta}} \to V_{\bar{\beta}+\bar{\alpha}}$ is a linear isomorphism of vector spaces. For any $\kappa \in P$, the map $\tau \circ \Lambda_\kappa \circ \tau^{-1} : M(Q,P,V) \to M(Q,P,V))$ is a $\mathfrak{g}$-module automorphism of degree κ. By Graded Schur's Lemma, there exists $a_\kappa \in \mathbb{C}^*$ such that $\tau\circ\Lambda_\kappa\circ\tau^{-1} = a_\kappa\Lambda_\kappa$, that is, $\tau\circ\Lambda_\kappa = a_\kappa\Lambda_\kappa\circ\tau$. This implies that $\mu_{\beta+\kappa} = a_\kappa\mu_\beta$ for all $\beta \in Q$ and $\kappa \in P$ (note that a_κ does not depend on β).

Since $\mathbb{C}^*$ is a divisible and hence injective abelian group, we can extend the map $\kappa \to a_\kappa$ to a character f of Q. Consider the isomorphism Ψ constructed in the previous lemma. We have the graded $\mathfrak{g}$-module isomorphism

$$\begin{aligned}\tau\Psi : M(Q,P,V^f) &\to M(Q,P,V),\\ v_{\bar{\beta}} \otimes t^{\beta} &\mapsto f(-\beta)\mu_{\beta}(v_{\bar{\beta}}) \otimes t^{\beta+\alpha}.\end{aligned}$$

From the definitions it follows that for all $\beta \in Q$ and $\kappa \in P$ we have

$$f(-\beta-\kappa)\mu_{\beta+\kappa} = f(-\beta)\mu_\beta.$$

Then we have a vector space automorphism $\mu : V^f \to V$ of degree $\bar{\alpha}$ given by $\mu(v_{\bar{\beta}}) = f(-\beta)\mu_\beta(v_{\bar{\beta}})$, for every $v_{\bar{\beta}} \in V_{\bar{\beta}} \;\left(= V^f_{\bar{\beta}}\right)$. For any $x_\gamma \in \mathfrak{g}_\gamma$ and $v_{\bar{\beta}} \in V_{\bar{\beta}}$, we have

$$\begin{aligned}\mu\left(x_\gamma \circ v_{\bar{\beta}}\right) \otimes t^{\gamma+\beta+\alpha} &= \tau\Psi\left(x_\gamma \circ v_{\overline{\beta}} \otimes t^{\beta+\gamma}\right)\\ &= \tau\Psi\left(x_\gamma \circ \left(v_{\bar{\beta}} \otimes t^{\beta}\right)\right)\\ &= x_\gamma\tau\Psi\left(v_{\bar{\beta}} \otimes t^{\beta}\right)\\ &= x_\gamma\left(\mu\left(v_{\bar{\beta}}\right) \otimes t^{\beta+\alpha}\right)\\ &= x_\gamma\mu\left(v_{\bar{\beta}}\right) \otimes t^{\gamma+\beta+\alpha}.\end{aligned}$$

Therefore $\mu(x_\gamma \circ v_{\bar{\beta}}) = x_\gamma\mu(v_{\bar{\beta}})$, and thus μ is an isomorphism of Q/P-graded $\mathfrak{g}$-modules and it has degree $\bar{\alpha} \in Q/P$ by construction.

Now let us prove sufficiency. Suppose $f \in \hat{Q}$ and $\mu : V^f \to V$ is a degree $\bar{\alpha} \in Q/P$ isomorphism of Q/P-graded $\mathfrak{g}$-modules. Then

$$f(\gamma)\mu(x_\gamma v_{\bar{\beta}}) = \mu(x_\gamma \circ v_{\bar{\beta}}) = x_\gamma\mu(v_{\bar{\beta}})$$

and this implies that

$$\begin{aligned}\tau : M(Q,P,V) &\to M(Q,P,V),\\ v_{\bar{\beta}} \otimes t^{\beta} &\mapsto f(\beta)\mu\left(v_{\bar{\beta}}\right) \otimes t^{\beta+\alpha},\end{aligned}$$

where $\beta \in Q$ and $v_{\bar{\beta}} \in V_{\bar{\beta}}$, is a degree α automorphism of the Q-graded $\mathfrak{g}$-module $M(Q, P, V)$. □

The following example, suggested by A. Elduque and M. Kochetov, shows that there are different choices for P in our Theorem 31.

Example 34. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with the standard basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$. Define on $\mathfrak{sl}_2$ the following Q-grading: the element h has degree $(1, 0)$, the element $e + f$ has degree $(0, 1)$ and the element $e - f$ has degree $(1, 1)$. This grading can be extended to the algebra of all 2×2 matrices by setting the degree of the identity matrix to be $(0, 0)$. Let W be the vector space of all 2×2 matrices considered as an $\mathfrak{sl}_2$-module via left multiplication. Then W becomes a Q-graded simple $\mathfrak{sl}_2$-module. Now one can check that $W = M(Q, P, V)$ for the following three choices of P and a fine grading on the natural $\mathfrak{sl}_2$-module $V = \mathbb{C}^2$:

$$P = \{0\} \times \mathbb{Z}_2, \quad \deg \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (0, 0) \text{ and } \deg \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1, 0),$$

$$P = \mathbb{Z}_2 \times \{0\}, \quad \deg \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0, 0) \text{ and } \deg \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0, 1),$$

$$P = \{(0, 0), (1, 1)\}, \quad \deg \begin{pmatrix} \mathbf{i} \\ 1 \end{pmatrix} = (0, 0) \text{ and } \deg \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} = (1, 0),$$

where $\mathbf{i}^2 = -1$ and $(1, 0) = (0, 1)$ in Q/P in the last case.

Here is an interesting example with different flavor in the case when $\mathfrak{g}$ is not simple.

Example 35. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ with the natural $\mathbb{Z}_2$-grading given by

$$\mathfrak{g}_{\bar{0}} = \{(x, x) \mid x \in \mathfrak{sl}_2(\mathbb{C})\} \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \{(x, -x) \mid x \in \mathfrak{sl}_2(\mathbb{C})\}.$$

We fix the standard triangular decomposition in each $\mathfrak{sl}_2(\mathbb{C})$ and consider highest weight $\mathfrak{g}$-modules $L(h_1, h_2)$ with respect to this decomposition, where h_i, $i = 1, 2$, gives the highest weight for the i-th copy of $\mathfrak{sl}_2(\mathbb{C})$. Then $L(h_1, h_2)$ admits a $\mathbb{Z}_2$-grading if and only if $h_1 = h_2$, in which case it is isomorphic to $M(\mathbb{Z}_2, \{0\}, L(h_1, h_2))$. If $h_1 \neq h_2$, then $L(h_1, h_2) \oplus L(h_2, h_1)$ has the natural $\mathbb{Z}_2$-grading making it a graded simple $\mathfrak{g}$-module isomorphic to both $M(\mathbb{Z}_2, \mathbb{Z}_2, L(h_1, h_2))$ and $M(\mathbb{Z}_2, \mathbb{Z}_2, L(h_2, h_1))$. □

Our Classification Theorem 31 reduces construction of Q-graded simple modules over a Q-graded Lie algebra $\mathfrak{g}$ to classification of non-extendable Q/P-grading on all simple $\mathfrak{g}$-modules, for any subgroup P of Q. Some results in this direction can be found in [6]. We note that [6, Theorem 8] determined the necessary and sufficient conditions for two finite dimensional $M(Q, P, V)$ to be graded isomorphic if $\mathfrak{g}$ is a finite dimensional simple Lie algebra. We complete the paper with the following question:

Problem 36. Find necessary and sufficient conditions for two graded simple $\mathfrak{g}$-modules $M(Q, P, V)$ and $M(Q, P', V')$ to be isomorphic.

Further studies on graded simple modules are going on in the recent paper [8].

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