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On Smooth Knots and Tangent Lines

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Contents

1 Introduction

The basic problem in topology in general and knot theory in particular is determining whether two objects belong to the same equivalence class under some certain equivalence relation, which in knot theory is the *knot type*. Numerous more or less computable invariants have been developed for proving that two knots are different, usually algebraic, but proving that two knots are equivalent often has to be done by exhibiting an ambient isotopy. A few sufficient conditions for a knot to be trivial have been proven. A basic example is the following:

Theorem (Basic Unknotting Theorem)**.** Let $\gamma : S^1 \to \mathbb{R}^3$ be a topological knot. If there exists an embedding $\Gamma : D^2 \to R^3$ that restricts to γ on the boundary, then γ is trivial.

For proof, see a basic knot theory textbook like Rolfsen ([1], 2F7). A different and much less basic theorem due to Fary and Milnor is the following:

Theorem. Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a smooth knot and $\kappa : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+$ be the curvature. If $\oint_\gamma \kappa(s)\, ds \leqslant 4\pi$, then γ is trivial.

Despite the integral in the theorem being at least possible to estimate, this is not the main interest of the theorem. What is interesting is the way it formalises the intuitive idea that if a knot is non-trivial, it is quite crooked. In his paper [2], Milnor even defines a measure of crookedness.

In Section 3 of this paper we establish another geometric criterion for unknottedness inspired by convexity. Note first that if $C \subset R^3$ is a convex body and if γ is a simple closed curve on its boundary, then by the Jordan curve theorem γ bounds a disk in ∂C and hence it is trivial. Here we look at a version of codimension two convexity and prove the following theorem.

Theorem 1.1. Let $\gamma : [0,l] \to \mathbb{R}^3$ be a regular knot with non-zero curvature parametrised by arclength. Then, if the line $\gamma(t) + \lambda\dot{\gamma}(t)$ does not intersect the knot again for any t, the knot is trivial.

This result was communicated to the author by T. Ekholm who first learned about it from T. Tsuboi in 1999.

Another way that this theorem could be seen is as a formalisation of the idea that a non-trivial knot loops through itself, that is, we cannot go across the knot with an infinitely long pole without hitting something. At some point, the pole will hit an arch made by the knot itself.

One may wonder whether we can strengthen the result of Theorem 1.1. Perhaps we can allow a few tangent lines intersecting the knot again and still be sure that the knot is trivial. A conjecture on this question can be found in Section 4.

In Section 3, we will use Dehn's Lemma, whose proof we present in Section 2.

2 Background: Dehn's Lemma

In order to prove the theorem in Section 3, we will use a certain lemma from 3-dimensional piece-wise linear (PL) topology, namely the following:

Figure 1: A generic path.

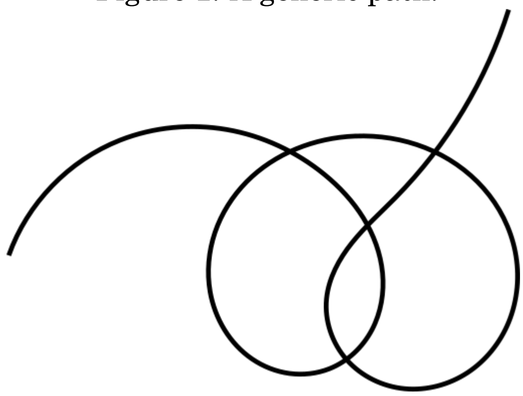
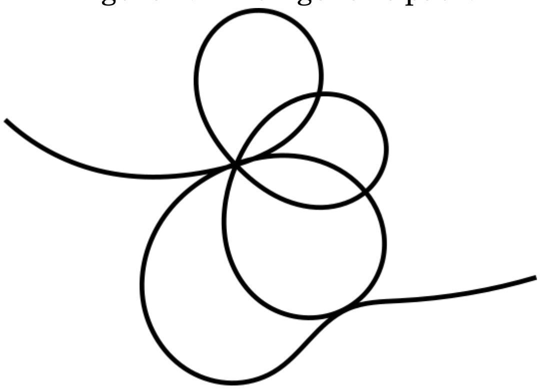


Figure 2: A non-generic path.



Theorem 2.1 (Dehn's Lemma, Papakyriakopoulos)**.** Let M be a 3-manifold with boundary and $f:(D^2,S^1)\to(M,\partial M)$ a map. Let $S(f)=\{x\in D \mid \exists y\in D: y\neq x \text{ and } f(x)=f(y)\}$. If $\overline{S(f)}\cap\partial D=\emptyset$, then there exists an embedding $g:(D^2,S^1)\to(M,\partial M)$ such that $g(\partial D)=f(\partial D)$.

The theorem can more clearly be stated otherwise in the following way:

Theorem. Let M be a 3-manifold with boundary and $f:(D^2,S^1)\to(M,\partial M)$ a map. If f is an embedding on a collar neighbourhood of S^1 in D^2, then there exists an embedding $g:(D^2,S^1)\to(M,\partial M)$ such that $g(\partial D)=f(\partial D)$.

This theorem was first stated by Max Dehn in 1910, but in 1929 his proof was found to have serious flaws. A correct proof was not found until 1957 when Papakyriakopoulos (a.k.a. "Papa") presented his tower construction. (Rolfsen [1], Chapter 4.)

An excellent text proving Dehn's Lemma is Appendix B in Rolfsen's book *Knots and Links* [1] based on a guest lecture by David Gillman. In this section, we are too going to present a proof, but it is largerly based on the one of Gillman, as well as lecture notes by Danny Calegari [3].

First, we start by discussing the moral reason for Dehn's Lemma to be true. This statement is in a way related to the statement saying that if we have a curve with self-intersections between two points on a surface, then there is also an embedded curve between them. To show this, we first put the curve in *general position*, that is only having simple transversal self-intersections. (See Figure 1 for an example and Figure 2 for a non-example.) All self-intersections now look like crosses. Exchanging every cross for two arcs that do not intersect, we obtain a 1-manifold that consist of one embedded curve and some embedded circles, thus proving the result. An idea would be doing the same thing in three dimensions. We have a disk D that is somehow mapped into M, perhaps with self-intersections. We put D in general position. We remove the self-intersections and hope that we will have a disk left when we are done. However, there are a few questions to answer first: What is a general position for a disk in a 3-manifold? How many types of self-intersections will we have to deal with? Can we do this in a way that indeed does leave us with an embedded disk?

The first question is partially answered by the following lemma:

Lemma 2.2 (General Position)**.** Let $\phi : M^2 \to N^3$ be an immersion of a compact 2-manifold with boundary inside a 3-manifold with boundary. It is arbitrarily close to an immersion $\tilde{\phi}$ such that $S(\tilde{\phi})$ is a union of finitely many transverse double arcs meeting in finitely many triple points.

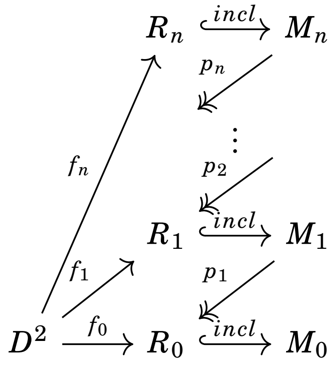
This lemma will be enough for our proof of Dehn's lemma as the maps that we will apply it to will be local embeddings, that is an embedding on a small enough neighbourhood around each point.

Proof Sketch. The result follows from Corollary 3.3 in Golubitsky-Guillemin [4]. We just need to figure out what a normal crossing looks like in this particular case. If $f(x_1) = \cdots = f(x_s) = y$, then we should have $d\phi^{(s)}(T_{(x_1,\dots,x_s)}M^s) + T_{(y,\dots,y)}\Delta N = T_{(y,\dots,y)}N^s$. The first term has dimension $2s$, the second has dimension 3 and the right hand side has dimension $3s$. Thus $s \leqslant 3$. Also, Lemma 3.6 in the same book says that the $d\phi(T_{x_i}M)$ are in general position as subspaces of T_yN, which immediately implies the lemma. □

The idea of Papakyriakopoulos was to reduce the number of times the disk intersects itself by constructing a double cover of the area around its image and then lifting the map to this double cover. This is done until the disk intersects itself a minimal amount of times. This is the famous *tower of double covers*. Then, pushing down the tower, one only needs to consider double intersections.

Dehn's Lemma is a theorem of PL topology. The proof uses that the area around the images of our disk can be described by finitely many simplices. On the other hand, we are going to use this theorem in a piecewise smooth setting. We will use that these two settings are essentially the same.

Proof of Theorem. Rename $f = f_0$ and $M = M_0$. Recursively, we define 1) R_i as the regular neighbourhood of $\operatorname{Im} f_i$ in M_i consisting of all simplices in the 2nd *barycentric subdivision* which intersect $f_i(\operatorname{Int} D)$, 2) M_i to be a double cover of R_{i-1}, if it exists, and 3) $f_i : D^2 \to M_i$ to be the lift of f_{i-1} to the double cover. This lift exists since D^2 is simply connected. The definition of R_i is made in order to ensure that $f_i(\partial D) \subset \partial R_i$.



Lemma 2.3. This process stops, that is, there exists an n such that R_n has no double cover.

Proof. If $g: X \to Y$, let $\tilde{S}(g) \in X \times X$ be the subset of pairs (x, y) such that $g(x) = g(y)$. Then $S(g) = \mathrm{proj}_1(\tilde{S}(g))$. It is clear that $\tilde{S}(f_i) \subseteq \tilde{S}(f_{i-1})$. Suppose that $\tilde{S}(f_i) = \tilde{S}(f_{i-1})$. We claim that $f_i(D)$ and $f_{i-1}(D)$ are homeomorphic. The homeomorphism is given by the projection map $\rho_i : M_i \to R_{i-1}$ restricted to a map $f_i(D) \to f_{i-1}(D)$. It is a continuous map, which is surjective by the definition of f_i, and injective by the singular set equality condition. Since it is a bijective continuous map from a compact space to a Hausdorff space, it is indeed a homeomorphism. Now, we have the commutative diagram

$$\begin{array}{ccc} f_i(D) & \xhookrightarrow{incl} & M_i \\ \downarrow{\scriptstyle \rho_i} & & \downarrow{\scriptstyle \rho_i} \\ f_{i-1}(D) & \xhookrightarrow{incl} & R_{i-1}. \end{array}$$

This yields a commutative diagram

$$\begin{array}{ccc} \pi_1(f_i(D)) & \xrightarrow{incl_*} & \pi_1(M_i) \\ \downarrow{\scriptstyle \rho_{i*}} & & \downarrow{\scriptstyle \rho_{i*}} \\ \pi_1(f_{i-1}(D)) & \xrightarrow{incl_*} & \pi_1(R_{i-1}). \end{array}$$

The left homomorphism is an isomorphism as it arises from a homotopy equivalence which is even a homeomorphism. (See Hatcher [5], Proposition 1.18.) The lower homomorphism is an isomorphism as R_{i-1} is a regular neighbourhood of $f_{i-1}(D)$ and thus retracts onto $f_{i-1}(D)$. (See Hatcher [5], Proposition 1.17.) For the diagram to commute, the right homomorphism has to be an epimorphism. This is however not true since M_i is a double cover of R_{i-1} and thus $[\pi_1(R_{i-1}) : \rho_{i*}(\pi_1(M_i))] = 2$. Hence $\tilde{S}(f_i) \subsetneq \tilde{S}(f_{i-1})$.

By piece-wise linearity of f_i and f_{i-1}, we have that the property of being a singularity is constant on simplices for some finite simplicial subdivision. Since D and thus $D \times D$ consist of finitely many simplices, the number $\#(\tilde{S}(f_i)/\sim)$, where $(x_1, y_1) \sim (x_2, y_2)$ if (x_1, x_2) and (y_1, y_2) belong to the same simplices, is strictly decreasing and non-negative. Thus the process must stop. □

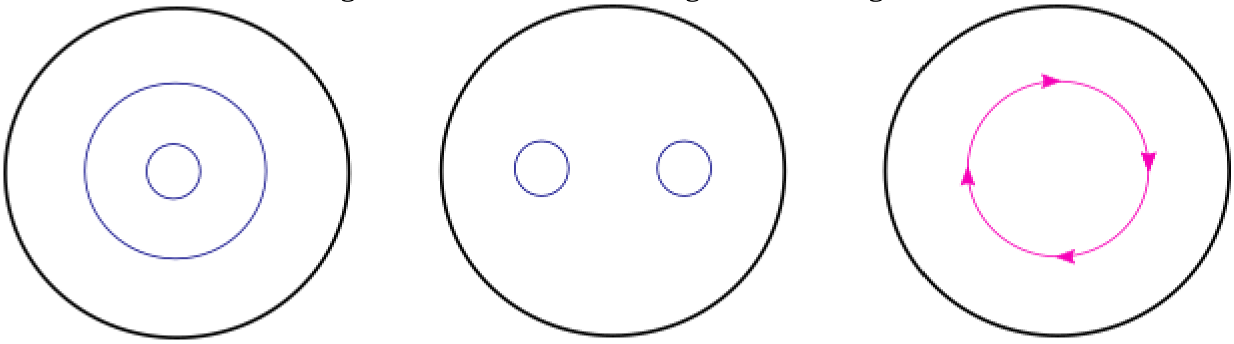
Lemma 2.4. If R_n has no double cover, then ∂R_n consists of finitely many spheres.

Proof. This proof uses some beautiful arguments of algebraic topology.

First, we note that R_n is orientable, since all non-orientable manifolds have a double cover. It follows that ∂R_n is orientable, for example by using an induced orientation. Call the connected components of ∂R_n by $B_1, B_2, \ldots, B_k$. By the classification theorem of closed orientable surfaces and knowledge of their Euler characteristic, we have that $\chi(B_i) \geqslant -2$ with equality if and only if B_i is a sphere. It follows that $\chi(\partial R_n) \geqslant -2k$ with equality if and only if all B_i are spheres.

Consider $\tilde{R}_n$ the manifold obtained by sewing two copies of R_n along the boundary. We have that $\chi(\tilde{R}_n) = 2\chi(R_n) - \chi(\partial R_n)$ by the inclusion-exclusion principle and $\chi(\tilde{R}_n) = 0$ since $\tilde{R}_n$ is closed manifold of odd dimension. (Hatcher [5], Corollary 3.37.) Hence $\chi(\partial R_n) = 2\chi(\partial R_n)$. However, $\chi(R_n) = -r_0 + r_1 - r_2 + r_3$ with $r_i = \mathrm{rk}\, H_i(R_n)$. We have that $r_0 = 1$ as R_n is connected, $r_3 = 0$ as R_n has a boundary and $r_2 \geqslant k - 1$ as $\{[B_1], [B_2], \ldots, [B_{k-1}]\}$ is a set of linearly independent elements of $H_2(R_n)$. Finally, $H_1(R_n)$ is finite, as otherwise it would contain a $\mathbb{Z}$ component, which would make it possible to construct an epimorphism

Figure 3: The three building stones of $S(g)$.



$\pi_1(R_n) \to H_1(R_n) \to \mathbb{Z}_2$, which would give us a double cover of R_n. Thus $\chi(\partial R_n) = 2\chi(R_n) \leqslant -2k$. This finishes the proof. $\square$

Lemma 2.5. The conclusion of Dehn's Lemma holds in the top storey. More precisely, there exists an embedding $\tilde{f}_n : D \to R_n$.

Proof. By our definition of R_n, we have that $f_n(\partial D) \subset \partial R_n$. There are no singularities on the boundary of D, so f_n embeds ∂D into some sphere of ∂R_n. By Jordan's theorem this curve bounds a disk in ∂D. $\square$

Lemma 2.6. If the conclusion of Dehn's lemma holds for storey $j+1$, then it also holds for storey j.

Proof. Let $f : D \to M_{j+1}$ be an embedding. Then $g = p_{j+1} \circ f$ only has double critical points as p_{j+1} is a double cover. We also have that g locally is an embedding, that is, if we restrict ourselves to a small enough neighbourhood of a point in D, it will be embedded into R_j. If g now is in *general position*, $S(g)$ will be a one-manifold without boundary and $g|_{S(g)}$ will be a double cover of its image. Since D is compact, $S(g)$ will consist of finitely many circles. The idea now is to keep replacing g by new maps having fewer and fewer of these circles until they all disappear. Figure 3 shows the possible covers of **one** circle in $\operatorname{Im} g|_{S(g)}$, but in general, $S(g)$ will be some finite and disjoint union of these pictures. We proceed as follows:

1. If we have a pair of corresponding circles which are "concentric" (one lies inside the other), then we construct a new map by removing the annulus between the circles in the singularity diagram of Figure 3 and sewing one circle to the other. This gives a new map which is still a continuous and piece-wise linear map from the disk into R_j. An illustration of this case can be found in Figure 4.

2. If we have a pair of corresponding circles that are not concentric, we choose a pair containing a circle whose bounded disk does not intersect the complementary annulus. The situation will look like in Figure 5. We remove this singularity by cutting across the intersection and swapping places between the disks bounded by the two circles in the singularity diagram of Figure 3. By changing to a nearby map we can now easily separate the two new feelers.

Figure 4: The case of concentric circles.

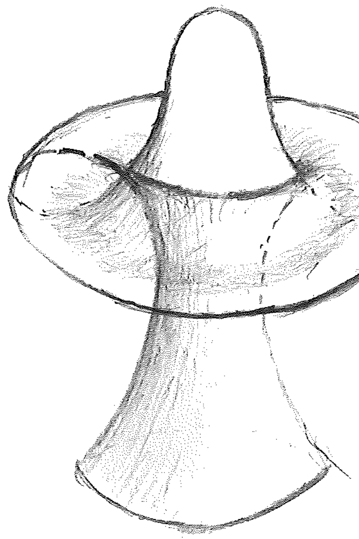


Figure 5: The case of non-concentric circles.

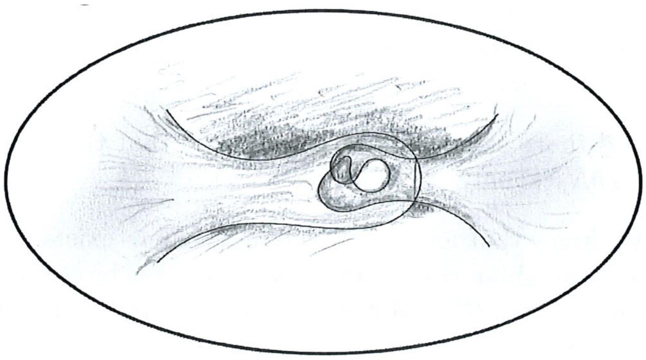
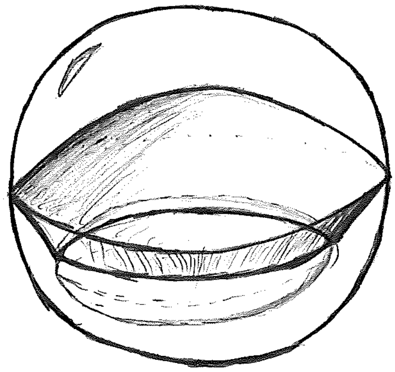


Figure 6: The case of one circle that covers its image twice.



3. The last case is if we have one circle that covers its image twice. While slightly difficult to imagine, it can happen. An example is drawn in Figure 6. It shows the 3-dimensional real projective space as a ball D^3 with antipodal points on the boundary identified. A disk is intersecting the boundary of the ball at the equator, which loops twice around itself. A hole was added in the illustration to make sure that the boundary of the disk maps to the boundary of the 3-manifold.

   In this case our disk looks a bit like a beret and our intuition says that we would solve the problem if we were to pull the beret slightly into the ball. However, considering the actual disk that we are embedding, the rights side of the bottom of beret top is in fact attached to the left side of the top of the beret bottom rather than the right side. What we do to solve this is that we cut out the disk bounded by the circle in the singularity diagram of Figure 3, turn it 180° and put it back again.

□

□

3 The Zero Intersection Case

The goal of this section is to prove the following theorem:

Theorem 3.1. Let $\gamma : [0, l] \to \mathbb{R}^3$ be a regular knot with non-zero curvature parametrised by arclength. Then, if the ray $\gamma(t) + \lambda\dot{\gamma}(t)$ with $\lambda > 0$ does not intersect the knot for any t, the knot is trivial.

The assumption about regularity is necessary to make sure that the tangent ray exists, while the assumption about non-zero curvature is simply making the proof easier. What we really want to avoid is points where derivatives of all orders are 0.

We are going to prove the theorem by showing that the knot in question is the boundary of an embedded disk. This embedded disk will be constructed using Dehn's Lemma. However, we will first need to show that there exists a tubular neighbourhood of γ having essentially the same properties as γ. That is, we want to construct a tubular neighbourhood such that each tangent ray of γ intersects the boundary of the tubular neighbourhood exactly once.

We will use the following notation: $N_\varepsilon(\gamma)$ is the set of points at the distance of $\leqslant \varepsilon$ to γ. The Tubular Neighbourhood Theorem states that this is a manifold when ε is small enough. Moreover, $p : N_\varepsilon(\gamma) \to \gamma$ taking a point to its nearest point on γ is a well-defined submersion, and $p^{-1}(x)$ is a disk of radius ε. Thus, we can choose some parametrisation $\varphi : \mathbb{R} \times [0,\varepsilon] \times \mathbb{R} \to N_\varepsilon(\gamma)$ such that $\varphi(t,r,\theta)$ is at a distance r from γ, φ is 1-periodic in t and 2π-periodic in θ, and $p \circ \varphi(t,r,\theta) = \gamma(t)$.

Lemma 3.2. For all t_0, there exists some $\tilde{\varepsilon}(t_0) > 0$, such that the ray $\mu_{t_0}(\lambda) = \gamma(t_0) + \lambda\dot{\gamma}(t_0)$ (with $\lambda \geqslant 0$) intersects $\partial N_\delta(\gamma)$ only once for all $\delta < \tilde{\varepsilon}(t_0)$.

Proof. Suppose for a contradiction to the lemma statement that there exist arbitrarily small $\tilde{\delta}_n$ such that μ_{t_0} intersects $\partial N_{\tilde{\delta}_n}(\gamma)$ more than once. Then there must exist arbitrarily small δ_n such that μ_{t_0} is tangent to $\partial N_{\delta_n}(\gamma)$ at some point $P_n = (t_n, \delta_n, \theta_n)$. This is because in the case of intersection rather than tangency, there exists some segment of the tangent line with the endpoints lying on the tube and the interior lying inside the tube. Then taking a tubular neighbourhood with the radius being the minimal distance between the segment and the knot will provide us with a point of tangency between the tangent line (in particular that segment) and the boundary of the tubular neighbourhood.

By compactness of $N_\varepsilon(\gamma)$ implying that the t_n stay inside a bounded region, the P_n must contain a limit point. This limit point has vanishing distance to γ and thus lies on γ. It also lies on μ_{t_0}. Hence the limit point is $\gamma(t_0)$. We assume from now on that the P_n converge to $\gamma(t_0)$. We also have that $(t_n) \to t_0$, since a limit point must exist but can be nothing but t_0.

By tangency, $\dot{\gamma}(t_0) \in T_{P_n}\partial N_{\delta_n}(\gamma) = \operatorname{span}\{\dot{\gamma}(t_n), \frac{\partial}{\partial\theta}|_{P_n}\}$. Consider $\beta_n = \frac{\dot{\gamma}(t_n)-\dot{\gamma}(t_0)}{t_n - t_0}$. We have that $\langle \ddot{\gamma}(t_n), \beta_n\rangle = 0$ for all n as $\operatorname{span}(\ddot{\gamma}(t_n)) = \operatorname{span}(\frac{\partial}{\partial r}|_{P_n})$ and $\beta_n \in \operatorname{span}\{\dot{\gamma}(t_n), \frac{\partial}{\partial\theta}|_{P_n}\}$. Proceeding to the limit gives $|\ddot{\gamma}(t_0)| = 0$, which is impossible since we assumed that γ has non-vanishing curvature. □

Lemma 3.3. There exists an $\varepsilon > 0$ such that that for all t, the ray $\mu_t(\lambda) = \gamma(t) + \lambda\dot{\gamma}(t)$ (with $\lambda \geqslant 0$) intersects $\partial N_\varepsilon(\gamma)$ only once.

Proof. First, we will show that we can choose a common ε for a neighbourhood of each point t_0.

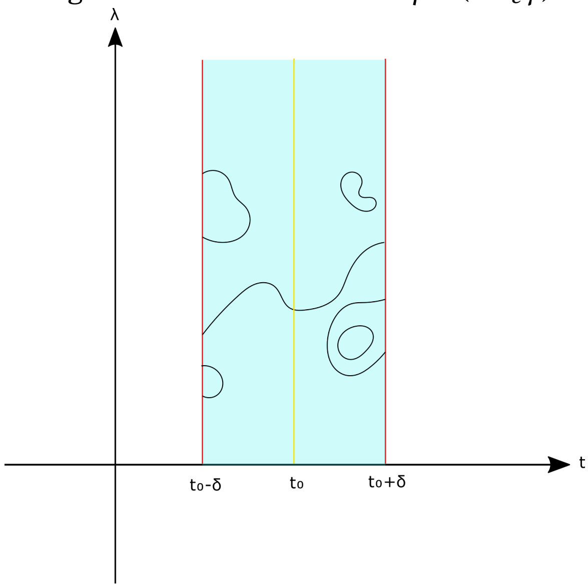
Consider the map $\mu(t,\lambda) = \mu_t(\lambda)$. It is clearly smooth in both t and λ. The idea is to use the transversality theorem to study $\mu^{-1}(\partial N_\varepsilon\gamma)$. We will show that this set is going to be a compact submanifold near t_0, that is a compact submanifold of $[t_0-\delta, t_0+\delta] \times (0,\infty)$. Together with the fact that $\#(\mu^{-1}(\partial N_\varepsilon\gamma)|_{\{t\}}) \geqslant 1$ with equality at $t = t_0$, and knowing that $\mu^{-1}(\partial N_\varepsilon\gamma)|_{[t_0-\delta,t_0+\delta]\times(0,\infty)}$ consists of finitely many circles and arcs with endpoints on $\{t_0 \pm \delta\} \times (0,\infty)$, we can immediately deduce that $\#(\mu^{-1}(\partial N_\varepsilon\gamma)|_{\{t\}}) = 1$ on a neighbourhood of t_0. See Figure 7 for an illustration of this subset.

We have:

$$D\mu = \begin{pmatrix}\dot{\gamma}(t) + \lambda\ddot{\gamma}(t) & \dot{\gamma}(t)\end{pmatrix} \sim \begin{pmatrix}\lambda\ddot{\gamma}(t) & \dot{\gamma}(t)\end{pmatrix}.$$

The map μ is transversal to $\partial N_\varepsilon\gamma$ if and only if $\operatorname{Im} D_{(t,\lambda)}\mu + T_{\mu(t,\lambda)}\partial N_\varepsilon\gamma = T_{\mu(t,\lambda)}\mathbb{R}^3$ whenever $\mu(t,\lambda) \in \partial N_\varepsilon\gamma$. We can suppose that this is true for (t_0,λ_0) such that $\mu(t_0,\lambda_0) \in \partial N_\varepsilon\gamma$, since by the proof of Lemma 3.2 we cannot have arbitrarily small ε_k such that $\gamma(t_0) + \lambda_0(\varepsilon_k)\dot{\gamma}(t_0)$ is tangent to $\partial N_{\varepsilon_k}\gamma$. Now, suppose that there are t_n arbitrarily near t_0 such that the intersections at (t_n,λ_n) for some λ_n are not transversal. Then $\dot{\gamma}(t_n) \in T_{\mu(t_n,\lambda_n)}\partial N_\varepsilon\gamma$ for all

Figure 7: The submanifold $\mu^{-1}(\partial N_\varepsilon\gamma)$.



these n. The $\mu(t_n,\lambda_n)$ have a limit point in $\partial N_\varepsilon\gamma$, so we can assume, by passing to a subsequence, that they converge. Also, since $\varepsilon \leqslant \lambda_n \leqslant \mathrm{diam}(\gamma)+2\varepsilon$, the λ_n must have a limit point, which cannot be other than λ_0. Hence $\langle\dot\gamma(t_0), \frac{\partial}{\partial r}(\mu(t_0,\lambda_0))\rangle = \lim_{n\to\infty}\langle\dot\gamma(t_n), \frac{\partial}{\partial r}(\mu(t_n,\lambda_n))\rangle = \lim_{n\to\infty} 0 = 0$, contradicting transversality. Thus $\mu|_{[t_0-\delta,t_0+\delta]} \pitchfork \partial N_\varepsilon\gamma$ for small δ.

Moreover, $\partial\mu|_{[t_0-\delta,t_0+\delta]} \pitchfork \partial N_\varepsilon\gamma$ as this is the same as $\dot\gamma(t_0\pm\delta)+T_{\mu(t_0\pm\delta,\lambda)}\partial N_\varepsilon\gamma = T_{\mu(t\pm\delta,\lambda)}\mathbb{R}^3$, which we have already shown holds for small enough δ. Thus $\mu^{-1}(\partial N_\varepsilon\gamma)$ is a submanifold with boundary of $[t_0-\delta,t_0+\delta]\times(0,\infty)$. Moreover, it is contained in $[t_0-\delta,t_0+\delta]\times[\varepsilon,\mathrm{diam}(\gamma)+2\varepsilon]$ and is thus compact. This completes the proof that ε fulfils the condition of Lemma 3.2 on a neighbourhood of t_0.

Now, as γ is compact, covering it with such neighbourhoods of each $t_0 \in \mathbb{R}/\mathbb{Z}$, we can extract a finite subcover. Choosing the minimal ε gives an ε that works for the entire knot. □

Now that this technical lemma is out of the way, we can proceed to the topological part of the proof.

Proof of Theorem 3.1. Let ε be as in Lemma 3.3. Let R be a number such that $|\gamma(t)|<R$ and $M = \overline{B(0,R+1)} - (N_\varepsilon\gamma)^o$. This is a 3-manifold with boundary $\partial B(0,R+1)\cup\partial N_\varepsilon\gamma$. Choose any point $p\in\partial B(0,R)$. Let $a_t = a_{t,\varepsilon}$ be the intersection between μ_t and $\partial N_\varepsilon\gamma$ and let b_t be the intersection between μ_t and $\partial B(0,R)$. It is not too hard to show that b_t is smooth in t as b_t is of the form $\gamma(t)+\lambda_t\dot\gamma(t)$ such that $\langle\gamma(t)+\lambda_t\dot\gamma(t)\rangle = R^2$, which reduces to λ_t being the solution of a quadratic equation with a clearly positive discriminant, yielding the result that both λ_t and b_t are smooth. Not too differently, a_t is also smooth as it is smooth locally by the proof of Lemma 3.3. Up to reducing ε, we also have that a_t is an embedding as $a_{t,\varepsilon}$ is a homotopy, $a_{t,0}$ is an embedding, and embeddings form a stable class (Guillemin-Pollack [6], p. 35). We note that $a(t)=a_t$ is also a knot and that it must have the same knot type as γ.

We construct a map $\phi : D^2 \to M$ by:

$$\phi(r,\theta) = \begin{cases} \frac{2rp+(1-2r)b_{\theta/2\pi}}{||2rp+(1-2r)b_{\theta/2\pi}||} & \text{if } r<1/2 \\ (2r-1)a_{\theta/2\pi}+(2-2r)b_{\theta/2\pi} & \text{if } 1/2<r<1. \end{cases}$$

This is clearly a continuous and piece-wise smooth map from the disk into M such that ∂D is mapped to ∂M. It is clear that this map is an embedding on a collar neighbourhood of ∂D^2. It is easy to slightly perturb ϕ to a map $\tilde{\phi}$ which is smooth. By Whitehead [7], there is a unique PL manifold K and a piecewise differentiable homeomorphism $\psi : K \to M$ for any smooth manifold M. In particular, this holds for our M and for D^2. (For the latter, we choose $v : N \to D^2$ as our triangulation.) Now, $\psi^{-1}\tilde{\phi} \circ v : N \to K$ is a map. By Calegari ([3], Subsection 1.2), there is a simplicial map $\varphi : N \to K$ homotopic to it. By Dehn's Lemma, there exists an embedding $\tilde{\varphi}$ of N into K. But this is good enough! The map $\psi \circ \tilde{\varphi} \circ v^{-1}$ maps the boundary of D onto something isotopic to γ and D is embedded into M. Hence γ is isotopic to a knot which is the boundary of a disk. By the Basic Unknotting Theorem, γ is the trivial knot. □

4 When the Tangent Lines Do Intersect Again

In this section, we present a conjecture regarding what happens if some, but not so many, tangent lines do intersect the knot again.

Definition 4.1. Let the *tangency number* $\mathfrak{t}(\gamma)$ be the number of $t \in [0,1)$ such that $\gamma(t) + \lambda\dot{\gamma}(t) = \gamma(s)$ has a solution (λ, s) with $\lambda \neq 0$. Let the *minimal tangency number* $\mathfrak{T}(\gamma)$ be the minimal $\mathfrak{t}(K)$ over all knots K of the same class as γ.

We pose the following conjecture:

Conjecture 4.2. Let γ be a non-trivial knot. Then $\mathfrak{T}(\gamma) \geqslant 6$, with equality if γ is a trefoil knot.

It is not too hard to show that for a generic knot γ, $\mathfrak{t}(\gamma)$ is even. Intuitively, the 6 comes from the fact that the knot loops through itself three times and we get one intersection with the tangent line at each side of the loop.

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