UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2018:12

Uncountable categoricity

Henrik Jonasson

Examensarbete i matematik, 15 hp
Handledare: Vera Koponen
Examinator: Martin Herschend
Juni 2018

Department of Mathematics
Uppsala University

Uncountable categoricity

Henrik Jonasson

June 16, 2018

1 Introduction

Model theory is the study of the models of a mathematical theory T, in terms of the properties of T. Model theory also provides a variety of classifications of theories T, based on how their models behave.

One important such classification is κ-categoricity. A model is said to have *cardinality* (or synonymously, *power*) κ if the underlying set has cardinality (power) κ. A theory T is called *κ-categorical* (or *categorical in κ*) if T has a model of cardinality κ, and all models of T of cardinality κ are isomorphic - that is if T determines an essentially unique structure of size κ in which T is true. Model theory then asks about the consequence of such a property on the models of T.

The object of this text is to present the proof of one important consequence of κ-categoricity, namely *Morley's Categoricity Theorem*, which states that if T is a complete theory in a countable vocabulary, and T is categorical in some uncountable power κ, then T is categorical in every uncountable power.

The proof follows the one given in [1], but with a more thorough exposition of the arguments and an attempt to explain all the details, which will hopefully make the ideas of the proof more accessible.

It is worth mentioning that the proof in [1] is not the original proof presented by Morley in [4], but an easier one by Baldwin and Lachlan. Morley's result is an interesting fact in itself, but the original proof also introduced ideas about stable theories which were expanded by Shelah to develop a new field in model theory, namely stability theory.

There are actually not many theories occurring naturally in mathematics which are known to be categorical in some uncountable power, but the following are some examples of such theories, which by Morley's theorem are categorical in all uncountable powers:

- For any countable division ring K, the theory of vector spaces over K.
- The theory of torsion-free divisible Abelian groups.
- The theory of infinite Abelian groups where all elements have order p (where p is a prime).

- The theory of algebraically closed fields of characteristic 0 or p.
- The complete theory of the natural numbers with the successor function.

2 Background

We will recall some basic definitions in mathematical logic and model theory, and state theorems which will be needed in the proof of Morley's Theorem. The proofs belong in an under graduate course and can be found in an introduction on the subject, like [2], or in the basic chapters of a more advanced treatment, like [1].

We will work in first-order logic, which uses the logical symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$ $\leftrightarrow$, (,), $\exists$ and $\forall$, which are interpreted as usual, and a set of proof-rules, which can be said to be those we use in ordinary mathematics.

A *vocabulary* $\mathcal{V}$ is a set of *constant symbols*, *relation symbols*, and *function symbols*.

A $\mathcal{V}$-structure $\mathcal{M}$ is an underlying set M, called the *universe* or *domain* of $\mathcal{M}$, with an *interpretation* of every constant symbol c, relation symbol R (of arity k) and function symbol f (of arity l) as an element $c^{\mathcal{M}} \in M$, a relation $R \subset M^k$ and a function $f : M^l \to M$, respectively.

Introducing a set of variables $v_0, v_1, \ldots$ we can construct $\mathcal{V}$*-formulas* as follows:

- $\mathcal{V}$*-terms:* Every constant symbol in $\mathcal{V}$ and variable v_i is a $\mathcal{V}$-term. If $t_1, \ldots, t_k$ are $\mathcal{V}$-terms and $f \in \mathcal{V}$ a k-ary function symbol, then $f(t_1, \ldots, t_k)$ is a $\mathcal{V}$-term.
- *Atomic* $\mathcal{V}$*-formulas:* If t_1 and t_2 are $\mathcal{V}$-terms, then $t_1 \equiv t_2$ is an atomic formula. If R is a k-ary relation symbol and $t_1, \ldots, t_k$ are terms, then $R(t_1, \ldots, t_k)$ is an atomic formula.
- $\mathcal{V}$*-formulas:* Every atomic $\mathcal{V}$-formula is a $\mathcal{V}$-formula. If ϕ and ψ are formulas, and $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\neg\phi$ and $\phi\Box\psi$ are formulas. If ϕ is a formula and v a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

We write $\phi(v_1, \ldots, v_n)$ if the free variables of the formula ϕ are among $v_1, \ldots, v_n$. By $\phi(c)$ we mean the formula obtained by replacing every free occurrence of x in $\phi(x)$ by the constant symbol c. If a $\mathcal{V}$-formula contains no free variables, it is called a $\mathcal{V}$-sentence.

Now given a $\mathcal{V}$-structure $\mathcal{M}$ and a $\mathcal{V}$-sentence ϕ, we write $\mathcal{M} \models \phi$, or say that $\mathcal{M}$ is a model of ϕ, if ϕ is true in $\mathcal{M}$. Given a set Γ of sentences, we write that $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models \phi$ for each sentence $\phi \in \Gamma$. If $\mathcal{M} \models \Gamma$ implies that $\mathcal{M} \models \phi$, we write $\Gamma \models \phi$. If ϕ can be deduced from Γ using the proof rules of first-order logic, we write $\Gamma \vdash \phi$.

Theorem 1 (Soundness of first order logic). *If* $\Gamma \vdash \phi$*, then* $\Gamma \models \phi$*.*

Theorem 2 (Completeness of first order logic). *If* $\Gamma \models \phi$*, then* $\Gamma \vdash \phi$*.*

A set of $\mathcal{V}$-sentences Γ is said to be *inconsistent* if $\Gamma \vdash \phi \wedge \neg\phi$ for some sentence ϕ. Otherwise it is called *consistent.* Γ is *maximal consistent* if every set of sentences Σ such that $\Gamma \subset \Sigma$ is inconsistent. We call Γ *complete* if for every $\mathcal{V}$-formula ϕ, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

Theorem 3 (Model existence theorem). *A set of sentences* Γ *has a model if and only if* Γ *is consistent.*

Theorem 4 (Compactness theorem). *A set of sentences* Γ *has a model if and only if every finite subset of* Γ *has a model.*

If a set of sentences is consistent, it is called a *theory*, usually denoted T. The theory $Th(\mathcal{M})$ of a model $\mathcal{M}$ is the set of all sentences ϕ such that $\mathcal{M} \models \phi$.

Let $\mathcal{M}$ be a $\mathcal{V}$-structure and $\mathcal{N}$ a $\mathcal{W}$-structure. Suppose $\mathcal{V} \subset \mathcal{W}$, then we call $\mathcal{N}$ an *expansion* of $\mathcal{M}$ (or $\mathcal{M}$ a *reduct* of $\mathcal{N}$) if they have the same universe and the interpretation of every symbol in $\mathcal{V}$ is the same in both. Given a subset $A \subset M$ we denote the expansion of $\mathcal{M}$ to the vocabulary $\mathcal{V} \cup \{c_a : a \in A\}$ as $\mathcal{M}_A$ (sometimes $(\mathcal{M}, a)_{a \in A}$, or if A is enumerated by ordinals, $(\mathcal{M}, a_\alpha)_{\alpha<\kappa}$). If $\phi(v_1, \ldots, v_n)$ is a formula and $(a_1, \ldots, a_n) \in M$, then we by $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ mean that $\mathcal{M}_M \models \phi(c_{a_1}, \ldots, c_{a_n})$.

The theory $Th(\mathcal{M}_M)$ is called the *elementary diagram* of $\mathcal{M}$.

The following useful fact is called the "Lemma on constants", and can be found on page 43 in [3].

Lemma 5. *Let* T *be a* $\mathcal{V}$*-theory,* $\phi(v_1, \ldots, v_n)$ *a* $\mathcal{V}$*-formula and* $c_1, \ldots, c_n$ *a sequence of constants which do not occur in* $\mathcal{V}$*. If* $T \models \phi(c_1, \ldots, c_n)$*, then* $T \models (\forall x_1, \ldots, x_n)\phi(x_1, \ldots, x_n)$*.*

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{V}$-structures. Then they are *elementary equivalent*, denoted $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$.

They are said to be *isomorphic*, denoted $\mathcal{M} \cong \mathcal{N}$, if there is a bijective function $f : M \to N$ such that for every constant symbol c, relation symbol R (of arity k) and function symbol g (of arity l) in $\mathcal{V}$, we have that $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$, $(a_1, \ldots, a_k) \in R^{\mathcal{M}}$ if and only if $(f(a_1), \ldots, f(a_k)) \in R^{\mathcal{N}}$ and $f(g^{\mathcal{M}}(a_1, \ldots, a_l)) = g^{\mathcal{N}}(f(a_1), \ldots, f(a_l))$.

$\mathcal{M}$ is said to be a *substructure* of $\mathcal{N}$ (or $\mathcal{N}$ an *extension* of $\mathcal{M}$), denoted $\mathcal{M} \subset \mathcal{N}$ if $M \subset N$ and for every constant symbol c, relation symbol R (of arity k) and function symbol f (of arity l) in $\mathcal{V}$ we have that $c^{\mathcal{M}} = c^{\mathcal{N}}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^k$ and for all $(a_1, \ldots, a_l) \in M^l$, $f^{\mathcal{M}}(a_1, \ldots, a_l) = f^{\mathcal{N}}(a_1, \ldots, a_l)$. ($M{=}N$ and $\mathcal{M} = \mathcal{N}$ is allowed.)

$\mathcal{M}$ is said to be an *elementary substructure* of $\mathcal{N}$, denoted $\mathcal{M} \prec \mathcal{N}$, if $\mathcal{M} \subset \mathcal{N}$ and for every $\mathcal{V}$-formula $\phi(v_1, \ldots, v_n)$ and every $(a_1, \ldots, a_n) \in M^k$, we have that $\mathcal{M} \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \models \phi(a_1, \ldots, a_n)$. The following criterion is often useful:

Proposition 6 (The Tarski-Vaught criterion)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{V}$-structures such that $\mathcal{M} \subset \mathcal{N}$. Then $\mathcal{M} \prec \mathcal{N}$ if and only if for every $\mathcal{V}$-formula $\phi(x_1, \ldots, x_n, y)$ and every $(a_1, \ldots, a_n) \in M^n$ the following holds:*

$$\mathcal{N} \models \exists y \phi(a_1, \ldots, a_n, y) \Rightarrow \mathcal{N} \models \phi(a_1, \ldots, a_n, b) \textit{ for some } b \in M.$$

Recall that a *partial ordering* $\leq$ on a set M is an binary relation such that for all $x, y, z \in M$: $x \leq x$, $(x \leq y \wedge y \leq z) \Rightarrow x \leq z$ and $(x \leq y \wedge y \leq x) \Rightarrow x = y$. A *linear ordering* is a partial order such that $(\forall x, y \in M)(x \leq y \vee y \leq x)$. A *well ordering* is a linear ordering such that for every subset $X \subset M$ there is a least element with respect to $\leq$. The *well ordering principle*, which is equivalent to the axiom of choice, tells us that every set can be well ordered.

An *ordinal* is a set α such that for every $a \in \alpha$, $a \subset \alpha$ and α is well ordered by '$\in$' as a relation. The class of ordinals can be shown to be strictly well ordered by '$\in$' ($\alpha \notin \alpha$), and by $\alpha < \beta$ we mean that $\alpha \in \beta$. The successor of an ordinal α is the ordinal $\alpha \cup \{\alpha\}$. An ordinal β which is not a successor ordinal, i.e $(\forall \alpha \in \beta)(\alpha \cup \{\alpha\} \in \beta)$, is called a limit ordinal. It can be shown that all ordinals are either the empty set, a successor ordinal or a limit ordinal. Furthermore, induction can be carried out over ordinals similarly as over the natural numbers:

Proposition 7 (Transfinite induction)**.** *Let $P(\alpha)$ be a property of ordinals. Assume that for every ordinal β, if $P(\gamma)$ holds for each $\gamma < \beta$, then $P(\beta)$ holds. Then $P(\alpha)$ holds for every ordinal α.*

A *cardinal* $\kappa = \alpha$ is an ordinal such that for each $\beta < \alpha$, there is no injective function from α to β. For each set A, there is a unique cardinal such that there is a bijective function from A to κ. This κ is called *the cardinality of* A, and it is denoted $|A|$.

If either κ or λ is an infinite cardinal, then $\kappa + \lambda = max\{\kappa, \lambda\}$, and $\kappa \cdot \lambda = max\{\kappa, \lambda\}$.

The least cardinal larger than κ is denoted by κ^+. The least infinite cardinal is ω, and $\omega_1 = \omega^+$ denotes the least uncountable cardinal.

We define a cardinal to be *regular* if for every set $A \subset \kappa$, if $|A| < \kappa$, then $sup(A) < \kappa$.

Proposition 8. *Any infinite successor cardinal κ^+ is regular.*

The *power* of a $\mathcal{V}$-structure $\mathcal{M}$ is the cardinality of the universe M. As mentioned before, a theory T is *κ-categorical* if all models of T of power κ are isomorphic. We conclude the background with the following important theorems:

Theorem 9 (Upward Löwenheim-Skolem theorem)**.** *If a theory T has a model of infinite power, then T has models of arbitrarily large powers.*

If T is taken to be $Th(\mathcal{M}_M)$ for a model $\mathcal{M}$, the theorem yields that every model has arbitrarily large elementary extensions.

Theorem 10 (Downward Löwenheim-Skolem theorem)**.** *Let $\mathcal{M}$ be a $\mathcal{V}$-structure and $A \subset M$. Then there is a an elementary substructure $\mathcal{N} \prec \mathcal{M}$ such that $A \subset N$ and $|N| \leq \omega + |V| + |A|$*

3 Types

Definition. We say that a model $\mathcal{M}$ *realizes* a set $\Sigma(v_1, \ldots, v_n)$ of formulas $\sigma(v_1, \ldots, v_n)$ if there is an n-tuple $a_1, \ldots, a_n$ of M such that $\mathcal{M} \models \sigma(a_1, \ldots, a_n)$ for each $\sigma \in \Sigma$. If $\mathcal{M}$ does not realize Σ, then $\mathcal{M}$ is said to *omit* Σ.

Definition. A set $\Sigma(v_1, \ldots, v_n)$ of $\mathcal{V}$-formulas in the variables $v_1, \ldots, v_n$ is said to be a *type* if it is maximal consistent.

Note that given any model $\mathcal{M}$ and any n-tuple $a_1, \ldots, a_n \in M$, the set $\Sigma(v_1, \ldots, v_n)$ of all formulas $\sigma(v_1, \ldots, v_n)$ satisfied by $a_1, \ldots, a_n$ is a type. Because given any formula $\sigma(v_1, \ldots, v_n)$, either $a_1, \ldots, a_n$ satisfies or does not satisfy σ, hence $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$. That is, any Γ such that Σ is a proper subset of Γ must contain both σ and $\neg\sigma$ for some formula σ, hence it is inconsistent and Σ is maximal consistent. Furthermore, this is the unique type realized by $a_1, \ldots, a_n$, for if Σ' is another type realized by $a_1, \ldots, a_n$, then all $\sigma \in \Sigma'$ is also in Σ, and $\Sigma' \subset \Sigma$. As Σ' is maximal consistent and Σ is not inconsistent, Σ' can't be a proper subset of Σ and $\Sigma' = \Sigma$.

Definition. A formula $\sigma(v_1, \ldots, v_n)$ is *consistent* with a theory T if there is a model $\mathcal{M}$ of T which realizes σ.

Definition. Let $\Sigma(v_1, \ldots, v_n)$ be a set of formulas in a vocabulary $\mathcal{V}$. A $\mathcal{V}$-theory T is said to *locally realize* Σ if there is a $\mathcal{V}$-formula $\phi(v_1, \ldots, v_n)$ such that:

(1). ϕ is consistent with T.

(2). For all $\sigma \in \Sigma$, $T \models \phi \rightarrow \sigma$

That is, in every model of T, an n-tuple which satisfies ϕ realizes Σ.

T is said to *locally omit* Σ if T does not locally realize Σ, that is, for every formula $\phi(v_1, \ldots, v_n)$ which is consistent with T, there exists a $\sigma \in \Sigma$ such that $(\phi \wedge \neg\sigma)$ is consistent with T.

To limit the exposition on types, we state the following proposition without a proof; it can be read in [2] on page 271.

Proposition 11. *Let T be a complete theory and let $\mathcal{M} \models T$, then each type consistent with T is realized in some elementary extension of M.*

The following consequence of Proposition 11 will be of use later. Let $T = Th(\mathcal{M})$. Let $S_{\mathcal{M}}$ be the set of all types $\Sigma(x)$ consistent with $Th(\mathcal{M})$ and index them as $\Sigma(x)_\gamma$, $\gamma < \delta = |S_{\mathcal{M}}|$. By the proposition there is a $\mathcal{N}_0$ such that $\mathcal{M} \prec \mathcal{N}_0$ and $\Sigma(x)_0$ is realized in $\mathcal{N}_0$. By $\mathcal{N}_0$ being an elementary extension of $\mathcal{M}$, $Th(\mathcal{N}_0) = Th(\mathcal{M})$ hence $S_{\mathcal{M}} = S_{\mathcal{N}_0}$ and there is a model $\mathcal{N}_1$ such that $\mathcal{N}_0 \prec \mathcal{N}_1$ and $\Sigma(x)_1$ is realized in $\mathcal{N}_1$. Inductively, there is a model $\mathcal{N}_\delta$ such that $\mathcal{M} \prec \mathcal{N}_\delta$ and $\mathcal{N}_\delta$ realizes $\Sigma(x)_\gamma$ for all $\gamma < \delta$, hence realizes every type consistent with $Th(\mathcal{M})$.

Theorem 12 (Omitting types theorem)**.** *Let T be a consistent theory in a countable vocabulary $\mathcal{V}$ and $\Sigma(x_1, \ldots, x_n)$ be a set of formulas. If T locally omits Σ, then T has a countable model which omits Σ.*

Proof. To simplify notation, let $\Sigma(x)$ be a set of formulas in one variable. The proof is completely analogous in the general case $\Sigma(x_1, \ldots, x_n)$. Suppose T locally omits $\Sigma(x)$. Let $C = \{c_o, c_1, \ldots\}$ be a countable set of constant symbols not already in $\mathcal{V}$ and define $\mathcal{V}' = \mathcal{V} \cup C$, which will be countable. Hence there are countably many $\mathcal{V}'$-sentences, and we can enumerate them as $\phi_0, \phi_1, \ldots$. We will now construct an increasing sequence of theories

$$T = T_0 \subset T_1 \subset \ldots \subset T_m \subset \ldots$$

which satisfy:

(1). Each T_m is a consistent theory of $\mathcal{V}$ such that $T_m \setminus T$ is finite.

(2). Either $\phi_m \in T_{m+1}$ or $\neg\phi_m \in T_{m+1}$.

(3). If $\phi_m = (\exists x)\psi(x)$ and $\phi_m \in T_{m+1}$, then $\psi(c_p) \in T_{m+1}$, where c_p is the first constant not occuring in T_m or ϕ_m.

(4). There is a formula $\sigma(x) \in \Sigma(x)$ such that $(\neg\sigma(c_m)) \in T_{m+1}$.

Given a theory $T_m = T \cup \{\theta_1, \ldots, \theta_r\}$ a finite extension of T, we construct T_{m+1} as follows. Let $\theta = (\theta_1 \wedge \ldots \wedge \theta_r)$ and n be such that all (finitely many) constants from C occurring in θ is among $c_1, \ldots, c_n$. Then replacing each c_i in θ by x_i yields a $\mathcal{V}$-formula $\theta(x_1, \ldots, x_n)$ which is consistent with T, as any model of T_m is a model of T which realizes θ. As T locally omits $\Sigma(x)$, there is a $\sigma(x) \in \Sigma(x)$ such that $(\theta \wedge \neg\sigma)$ is consistent with T. By putting the sentence $\neg\sigma(c_m)$ in T_{m+1}, (4) holds. If ϕ_m is consistent with $T_m \cup \{\neg\sigma(c_m)\}$, put ϕ_m in T_{m+1}. Otherwise, put $\neg\phi_m$ in T_{m+1}. Then T_{m+1} is consistent, and also, (2) is assured. If $\phi_m = (\exists x)\psi(x)$ and ϕ_m is consistent with $T_m \cup \{\neg\sigma(c_m)\}$, put $\psi(c_p)$ into T_{m+1}. Then (3) holds. Finally, as T_{m+1} is obtained by adding a finite number of sentences consistent with T_m, (1) holds for T_{m+1}.

Let $T_\omega = \bigcup_{n<\omega} T_n$. By (1), T_ω is consistent, for if we assume otherwise, there would be some finite $\Delta \subset T_\omega$ such that Δ deduces a contradiction, but as we also have $\Delta \subset T_m$ for some m, this would contradict (1). Furthermore, (2) gives that T_ω is maximal consistent.

Now let $\mathcal{N}' = (\mathcal{N}, b_0, b_1, \ldots)$ be a countable model of T_ω, which exists by the Model existence theorem and the Downward Löwenheim-Skolem theorem. Let $\mathcal{M}' = (\mathcal{M}, b_0, b_1, \ldots)$ be the substructure of $\mathcal{N}'$ generated by the interpretation $\{b_0, b_1, \ldots\}$ of all constants. Given any $\mathcal{V}'$-function $f(x_1, \ldots, x_n)$ and

$(a_1, \ldots, a_n) \in \{b_0, b_1, \ldots\}^n$, if $f(a_1, \ldots, a_n) = y$, then the formula $(\exists y)(f(b_{a_1}, \ldots, b_{a_n}) = y)$ is true in $\mathcal{N}'$. As $\mathcal{N}' \models T_\omega$ and T_ω is maximal consistent, we must have that $(\exists y)(f(b_{a_1}, \ldots, b_{a_n}) = y) \in T_\omega$, hence $(\exists y)(f(b_{a_1}, \ldots, b_{a_n}) = y) \in T_m$ for some m and by (3), $\mathcal{N} \models f(b_{a_1}, \ldots, b_{a_n}) = b_p$ for some p. Hence we must have $y = b_p$ and $\{b_0, b_1, \ldots\}$ is closed under all functions, that is $M = \{b_0, b_1, \ldots\}$.

One can then prove that $\mathcal{M}' \models \phi \Leftrightarrow T_\omega \models \phi$ via induction over the complexity of sentences. The base case with atomic formulas is immediate, as $\mathcal{M}' \subset \mathcal{N}'$, and the interpretation of all constants of $\mathcal{V}'$ are in M. The induction step is straight forward and completely analogous to the one used in [2] in the proof of the model existence theorem.

So we get that $\mathcal{M}' \models T_\omega$. Let $\mathcal{M}$ be the reduct of $\mathcal{M}'$ to the vocabulary $\mathcal{V}$. We have that $\mathcal{M} \models T$, as $T \subset T_\omega$ was a theory of $\mathcal{V}$-sentences. By condition (4), there is for every element $m \in M$ a formula $\sigma \in \Sigma$ such that the $\mathcal{V}'$-sentence $\neg\sigma(c_m) \in T_{m+1} \subset T_\omega$. So as $\mathcal{M}'$ models the sentences of T_ω, there is no element of M which satisfies $\Sigma(x)$ in $\mathcal{M}$. Hence T has a countable model $\mathcal{M}$ which omits $\Sigma(x)$, as was to be shown. □

4 Atomic and saturated models

Definition. Consider a complete theory T. A formula $\phi(x_1, \ldots, x_n)$ is called *complete*, or *an atom*, (in T) if for every formula $\psi(x_1, \ldots, x_n)$, exactly one of

$$T \models (\forall x_1, \ldots, x_n)(\phi(x_1, \ldots, x_n) \to \psi(x_1 \ldots x_n)),$$
$$T \models (\forall x_1, \ldots, x_n)(\phi(x_1, \ldots, x_n) \to \neg\psi(x_1 \ldots x_n))$$

holds.

A formula $\psi(x_1, \ldots, x_n)$ is called *completable* (in T) if there exists an atom $\phi(x_1, \ldots, x_n)$ such that $T \models (\forall x_1, \ldots, x_n)(\phi(x_1, \ldots, x_n) \to \psi(x_1 \ldots x_n))$. Otherwise, ψ is called *incompletable*.

A theory T is said to be *atomic* if every $\mathcal{V}$-formula consistent with T is completable in T.

A model $\mathcal{M}$ is said to be an *atomic model* if every n-tuple $a_1, \ldots, a_n \in M$ satisfies an atom in $Th(\mathcal{M})$.

Definition. A model $\mathcal{M}$ is *κ-saturated* if for every subset $X \subset M$ with $|X| < \kappa$, the expansion $\mathcal{M}_X$ realizes every type $\Sigma(v)$ of the vocabulary $\mathcal{V} \cup \{c_a : a \in X\}$ which is consistent with $Th(\mathcal{M}_X)$. $\mathcal{M}$ is simply called *saturated* if it is $|M|$-saturated.

Lemma 13 (Back and forth lemma)**.** *Suppose that κ is infinite, $\mathcal{M}$ and $\mathcal{N}$ are both κ-saturated models and $\mathcal{M} \equiv \mathcal{N}$. Let a_ξ and b_ξ ($\xi < \kappa$) be sequences of elements of the underlying sets M and N of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then there* *are two sequences* $\overline{a}_\xi$ *and* $\overline{b}_\xi$ *such that:*

$$range(a_\xi : \xi < \kappa) \subset range(\overline{a}_\xi : \xi < \kappa)$$
$$range(b_\xi : \xi < \kappa) \subset range(\overline{b}_\xi : \xi < \kappa)$$
$$(\mathcal{M}, \overline{a}_\xi)_{\xi<\kappa} \equiv (\mathcal{N}, \overline{b}_\xi)_{\xi<\kappa}.$$

Proof. Write every ordinal $\xi < \kappa$ uniquely as $\xi = \omega \cdot \lambda + n$ where $n \in \omega$ and $\lambda < \kappa$. (That this can be done is a known set theoretical result, which can be shown by induction over ordinals.) ξ is called even if n is even, and odd if n is odd. We will construct two sequences $\overline{a}_\xi$ and $\overline{b}_\xi$ such that for each ordinal $\xi < \kappa$ the following hold:

(a). For all $\eta < \xi$, if $\eta = \omega \cdot \lambda + 2n$ is even, then $\overline{a}_\eta = a_{(\omega\cdot\lambda+n)}$

(b). For all $\eta < \xi$, if $\eta = \omega \cdot \lambda + (2n+1)$ is odd, then $\overline{b}_\eta = b_{(\omega\cdot\lambda+n)}$

(c). $(\mathcal{M}, \overline{a}_\eta)_{\eta<\xi} \equiv (\mathcal{N}, \overline{b}_\eta)_{\eta<\xi}$.

We construct such a sequence by induction.

For the base case $\xi = 0$, (a), (b) and (c) are trivially satisfied.

For the inductive step, assume that we have found $\overline{a}_\eta$ and $\overline{b}_\eta$ $(\eta < \xi)$ such that (a), (b) and (c) hold for ξ. We then find $\overline{a}_\xi$ and $\overline{b}_\xi$ as follows. If $\xi = \omega\cdot\lambda+2n$ is even, then let $\overline{a}_\xi = a_{(\omega\cdot\lambda+n)}$, which satisfies (a). Now let $\Sigma(v)$ be the type of $\overline{a}_\xi$ in $(\mathcal{M}, \overline{a}_\eta)_{\eta<\xi}$. By induction hypothesis, $(\mathcal{M}, \overline{a}_\eta)_{\eta<\xi} \equiv (\mathcal{N}, \overline{b}_\eta)_{\eta<\xi}$, hence $\Sigma(v)$ is consistent with $Th(\mathcal{N}, \overline{b}_\eta)_{\eta<\xi} = Th(\mathcal{M}, \overline{a}_\eta)_{\eta<\xi}$. By $\xi < \kappa$ and $\mathcal{N}$ being κ-saturated, there is an element b in $(\mathcal{N}, \overline{b}_\eta)_{\eta<\xi}$ which satisfies $\Sigma(v)$. Let $\overline{b}_\xi = b$; as ξ is even, (b) still holds.

Now consider any "new" sentence γ containing $\overline{a}_\xi$ in the vocabulary of the expansion $(\mathcal{M}, \overline{a}_\eta)_{\eta\leq\xi}$. This can be written as $\gamma = \sigma(\overline{a}_\xi)$ for some formula σ in the original vocabulary of $(\mathcal{M}, \overline{a}_\eta)_{\eta<\xi}$. Now as $\overline{a}_\xi$ and $\overline{b}_\xi$ satisfy the same type $\Sigma(v)$, we get that $\gamma = \sigma(\overline{b}_\xi)$ is true in $(\mathcal{N}, \overline{b}_\eta)_{\eta\leq\xi}$ if and only if $\gamma = \sigma(\overline{a}_\xi)$ is true in $(\mathcal{M}, \overline{a}_\eta)_{\eta\leq\xi}$. Hence $(\mathcal{M}, \overline{a}_\eta)_{\eta\leq\xi} \equiv (\mathcal{N}, \overline{b}_\eta)_{\eta\leq\xi}$ and (c) holds. If $\xi = \omega\cdot\lambda+(2n+1)$ is odd, let $\overline{b}_\xi = b_{(\omega\cdot\lambda+n)}$ and find $\overline{a}_\xi$ in the same way.

As $a_\xi = a_{(\omega\cdot\lambda+n)} = \overline{a}_{(\omega\cdot\lambda+2n)}$ for all $\xi < \kappa$, we have that range$(a_\xi : \xi < \kappa) \subset$ range$(\overline{a}_\xi : \xi < \kappa)$. Similarly range$(b_\xi : \xi < \kappa) \subset$ range$(\overline{b}_\xi : \xi < \kappa)$. $(\mathcal{M}, \overline{a}_\xi)_{\xi<\kappa} \equiv (\mathcal{N}, \overline{b}_\xi)_{\xi<\kappa}$ follows immediately as (c) holds for all $\xi < \kappa$, and the lemma is proved □

Theorem 14 (Uniqueness of saturated models)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be elementary equivalent, saturated models of power κ. Then $\mathcal{M} \cong \mathcal{N}$.*

Proof. Let a_ξ and b_ξ $(\xi < \kappa)$ be an enumeration of the underlying sets M and N of $\mathcal{M}$ and $\mathcal{N}$, respectively. By Lemma 13, there are $\overline{a}_\xi$ and $\overline{b}_\xi$ such that

$$(\mathcal{M}, \overline{a}_\xi)_{\xi<\kappa} \equiv (\mathcal{N}, \overline{b}_\xi)_{\xi<\kappa}$$

and $\overline{a}_\xi$ and $\overline{b}_\xi$ contains the enumerations a_ξ and b_ξ. Hence any $a \in M$ equals $\overline{a}_\xi$ for some ξ. Defining the function g by sending a to $\overline{b}_\xi$ yields that $\mathcal{M} \cong \mathcal{N}$. □

5 Elementary chains

An *elementary chain* is a chain of models $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \ldots \prec \mathcal{M}_\beta \prec \ldots$ where $\beta < \alpha$ and $\mathcal{M}_\gamma \prec \mathcal{M}_\beta$ whenever $\gamma < \beta < \alpha$. They will be used in the proof of the following:

Theorem 15. *Let $\mathcal{M} = (M, V, \ldots)$ be a model, where V is a unary relation such that $\omega \leq |V| < |M|$. Then there are two models $\mathcal{N} = (N, W, \ldots)$ and $\mathcal{O} = (O, W, \ldots)$ such that $\mathcal{N} \prec \mathcal{M}, \mathcal{N} \prec \mathcal{O}$, and $|N| = \omega, |O| = \omega_1$, where the interpretation of W is the same in both $\mathcal{N}$ and $\mathcal{O}$.*

Proof. By the Löwenheim Skolem theorem, we can assume that $|M| = |V|^+$. By Proposition 8, $|M|$ is regular. By the well-ordering principle, there is a well ordering $\leq$ of M of type $|M|$. Let $a_1, \ldots, a_n \in M$ and $\psi(x, y, v_1, \ldots, v_n)$ be an arbitrary formula of the vocabulary of $\mathcal{M}$. Assume that in the model $(\mathcal{M}, \leq)$ there are arbitrarily large $a \in M$ such that for some $b \in V$, $\mathcal{M} \models \psi(b, a, a_1, \ldots, a_n)$.

That $|M|$ is regular yields that there is a fixed $b \in V$ such that there are arbitrarily large $a \in M$ such that $\mathcal{M} \models \psi(b, a, a_1, \ldots, a_n)$, for if we assume towards a contradiction that there is no such b, there would for each $b \in V$ be an $a_b \in M$ such that for each $a \in M$, $\mathcal{M} \models \psi(b, a, a_1, \ldots, a_n) \Leftrightarrow a \leq a_b$. Let $A = \{a_b : b \in V\}$. For each $c \in M$, there is by assumption an element a_c such that $c \leq a_c$ and for some $b \in V$, $\mathcal{M} \models \psi(b, a_c, a_1, \ldots, a_n)$. Hence there is for each $c \in M$ some $a_b \in A$ such that $c \leq a_b$. Mapping A to the corresponding ordinals of a_b in $|M|$ yields a subset $A' \subset |M|$ such that $|A'| = |V| < |M|$, but $sup(A') = |M|$, contradicting that $|M|$ is regular.

Let the relation $V(x)$ be true if $x \in V$, then we have shown that the following sentence holds in $(\mathcal{M}, \leq)$:

$$(1)\ (\forall v_1, \ldots, v_n)[(\forall z \exists y, x)(z \leq y \wedge V(x) \wedge \psi(x, y, v_1, \ldots, v_n) \\ \rightarrow (\exists x \forall z \exists y)(z \leq y \wedge V(x) \wedge \psi(x, y, v_1, \ldots, v_n))].$$

Let us prove that:

(2) Every countable model $(\mathcal{N}_0, \leq_0) \equiv (\mathcal{M}, \leq)$ has a countable proper elementary extension $(\mathcal{N}_1, \leq_1)$ such that the interpretation W of V in $\mathcal{N}_0$ and $\mathcal{N}_1$ is the same.

The Löwenheim Skolem theorem immediately gives a model $(\mathcal{N}_0, \leq_0) \prec (\mathcal{M}, \leq)$ of power ω. So if (2) holds, we can construct a ω_1-termed elementary chain of countable models $(\mathcal{N}_\xi, \leq_\xi)$, $\xi < \omega_1$ such that $(\mathcal{N}_\xi, \leq_\xi) \equiv (\mathcal{M}, \leq)$ for each ξ, the interpretation of V in each N_ξ is W, and $N_{\xi+1} - N_\xi \neq \emptyset$.

Thus in the model $(\mathcal{O}', \leq') = \bigcup_{\xi < \omega_1} (\mathcal{N}_\xi, \leq_\xi)$, V is interpreted as W, and the underlying set O has power ω_1 (as each union "adds" a non-zero, countable number of elements to the underlying set, which yields ω_1 elements in total). Furthermore $(\mathcal{O}', \leq') \equiv (\mathcal{M}, \leq)$, so the reduct of $(\mathcal{N}_0, \leq_0)$ and $(\mathcal{O}', \leq')$ to the original vocabulary $\mathcal{V}$ are the desired models $\mathcal{N} = (N, W, \ldots)$ and $\mathcal{O} = (O, W, \ldots)$.

To prove (2), let $\mathcal{N}_0 = (N_0, W, \ldots)$ be a model such that $(\mathcal{N}_0, \leq_0) \equiv (\mathcal{M}, \leq)$. Note that $\leq_0$ need not be a well ordering on $\mathcal{N}_0$. Let $\mathcal{V}'$ be the expansion of $\mathcal{V}$ by adding $\{\leq, c\}$ and $\{c_b : b \in N_0\}$. Let T be the union of the elementary diagram of $(\mathcal{N}_0, \leq_0)$ and the set of sentences $\{c_b < c : b \in N_0\}$. Let $\Sigma(x)$ be the set of formulas $\{V(x)\} \cup \{x \not\equiv c_b : b \in W\}$.

We look for a countable model $\mathcal{N}'$ of T which omits $\Sigma(x)$, for if $(\mathcal{N}_0', \leq_0')$ is the reduct of $\mathcal{N}'$ to the vocabulary of $(\mathcal{N}_0, \leq_0)$, then we will have that

$$(\mathcal{N}_0', \leq_0') \succ (\mathcal{N}_0, \leq_0)$$

as $\mathcal{N}'$ models the elementary diagram of $(\mathcal{N}_0, \leq_0)$. Furthermore, by it modelling $\{c_b < c : b \in N_0\}$ there is an element $b \in N_0' - N_0$, and by omitting $\{V(x)\} \cup \{x \not\equiv c_b : b \in W\}$, every element which satisfies $V(x)$ must be interpreted as an element in the set W of N_0, that is the interpretation of V is the same in both $\mathcal{N}_0$ and $\mathcal{N}_1$, and (2) will be proved.

To find such a $\mathcal{N}'$, first note that T is a consistent theory. Because given any finite subset Δ of T, only a finite number of the sentences $\{c_b < c\}$ lie in Δ, hence there is an interpretation of c in $\mathcal{N}_0$ such that $c > c_b$ for all c_b, and $(\mathcal{N}_0, \leq_0)$ models Δ . The Compactness theorem then yields that T has a model, hence by the Model existence theorem T is a consistent theory.

We then prove that T locally omits Σ, as then the Omitting types theorem (Theorem 12) will yield the existence of such a model $\mathcal{N}'$. Let $\mathcal{N}_0^*$ be the expanded model $(\mathcal{N}_0, \leq, b)_{b \in N_0}$ and let $\psi(c)$ be a $\mathcal{V}'$-sentence and $\psi(y)$ the corresponding formula.

Assume that $T \cup \{\psi(c)\}$ is consistent. Given any $b \in N_0$, as $c_b < c \in T$ we have that $T \cup \{\psi(c) \wedge c_b < c\}$ is consistent, and there exists a model $\mathcal{O}^*$ of $T \cup \{\psi(c) \wedge c_b < c\}$. Thus $\mathcal{O}^*$ models the sentence $(\exists y)(\psi(y) \wedge c_b < y)$, and as T contained the elementary diagram of $(\mathcal{N}_0, \leq)$, the restriction of $\mathcal{O}^*$ to the vocabulary of $\mathcal{N}_0^*$ is an elementary extension of $\mathcal{N}_0^*$, and it follows that $\mathcal{N}_0^* \models (\exists y)(\psi(y) \wedge c_b < y)$.

Now assume that $\psi(y)$ is satisfied by arbitrarily large elements of $\mathcal{N}_0^*$. Every finite subset of $T \cup \{\psi(c)\}$ is contained in some subset $\Delta \cup \{\psi(c)\}$, where Δ is the union of the elementary diagram of $\mathcal{N}_0$ and $\{c_b < c : b \in B\}$, where B is a suitable finite subset of N_0. Now as $\psi(y)$ is satisfied by arbitrarily large elements, there is some y such that $c_b < y$ for all $b \in B$ and $\mathcal{N}_0^* \models \psi(y)$. By interpreting c as this y, we get that $\mathcal{N}_0^* \models \Delta \cup \{\psi(c)\}$, so every finite subset of $T \cup \{\psi(c)\}$ is consistent, and by compactness $T \cup \{\psi(c)\}$ is consistent. We have thus shown that:

(3) $\psi(c)$ is consistent with $T \ \Leftrightarrow \mathcal{N}_0^* \models (\forall x \exists y)(x \leq y \wedge \psi(y))$.

Now let $\theta(x, c)$ be any formula consistent with T. That is, there is a model of T which realizes $\theta(x, c)$. But in this model, $(\exists x)\theta(x, c)$ also holds. Hence the sentence $(\exists x)\theta(x, c)$ is consistent with T. For any x which satisfies the formula $\theta(x, c)$, x is either in W or not. Hence at least one of the following holds:

(4) $(\exists x)(\theta(x, c) \wedge \neg V(x))$ is consistent with T

or

$$(5)\ (\exists x)(\theta(x,c) \wedge V(x)) \text{ is consistent with } T.$$

If (4) holds, $\sigma(x) = V(x)$ is a formula in $\Sigma(x)$ such that $\theta(x,c) \wedge \neg\sigma(x)$ is consistent with T, and T locally omits $\Sigma(x)$. Suppose (5) holds. Then by letting $\psi(c) = (\exists x)(\theta(x,c) \wedge V(x))$, (3) yields that:

$$\mathcal{N}_0^* \models (\forall z \exists xy)(z \leq y \wedge (\theta(x,y) \wedge V(x))).$$

By (1) and the fact that $(\mathcal{N}_0, \leq_0)$ and $(\mathcal{M}, \leq)$ are elementary equivalent, the above yields that:

$$\mathcal{N}_0^* \models (\exists x \forall z \exists y)(z \leq y \wedge (\theta(x,y) \wedge V(x)).$$

So for some $b \in W$, where c_b is the corresponding constant:

$$\mathcal{N}_0^* \models (\forall z \exists y)(z \leq y \wedge (\theta(c_b,y) \wedge V(c_b))$$

which gives that:

$$\mathcal{N}_0^* \models (\forall z \exists y)(z \leq y \wedge (\exists x)(\theta(x,y) \wedge x \equiv c_b)).$$

So by (3), the sentence $(\exists x)(\theta(x,c) \wedge x \equiv c_b)$ is consistent with T. But as $\sigma(x) = (x \not\equiv c_b) \in \Sigma(x)$, we have found a $\sigma \in \Sigma$ such that $\theta(x,c) \wedge \neg\sigma(x)$ is consistent with T. Thus T locally omits $\Sigma(x)$. We have thus shown the existence of two models $\mathcal{N} = (N, W, \ldots)$ and $\mathcal{O} = (O, W, \ldots)$ such that $\mathcal{N} \prec \mathcal{M}, \mathcal{N} \prec \mathcal{O}$; $|N| = \omega, |O| = \omega_1$ and W is interpreted the same in both. □

6 Skolem functions and Indiscernibles

Given a vocabulary $\mathcal{V}$, the *Skolem expansion of* $\mathcal{V}$ is obtained by adding a new function symbol F_ψ corresponding to each formula ψ of the form $\psi(x_1, \ldots, x_n) = (\exists x)\phi(x, x_1, \ldots, x_n)$, where F_ψ has the same arity as ψ. The Skolem expansion is denoted by $\mathcal{V}^*$ and the F_ψ are called the *Skolem functions*.

Now for each Skolem function F_ψ, let $y_1, \ldots, y_n$ be variables not occurring in ψ and construct the following $\mathcal{V}^*$-sentence σ_ψ:

$$(\forall(y_1, \ldots, y_n)(\psi(y_1, \ldots, y_n) \rightarrow \phi(F_\psi(y_1, \ldots, y_n), y_1, \ldots, y_n))$$

That is, if ψ is true, the Skolem function of ψ gives the element which ψ says exists. The *Skolem theory* $\Sigma_\mathcal{V}$ of $\mathcal{V}$ is the theory with the σ_ψ above as axioms.

Given a $\mathcal{V}$-structure $\mathcal{M}$ and an expansion $\mathcal{M}^*$ to the vocabulary $\mathcal{V}^*$, $\mathcal{M}^*$ is called the *Skolem expansion of* $\mathcal{M}$ if $\mathcal{M}^* \models \Sigma_\mathcal{V}$. Furthermore, given a theory T, the *Skolem expansion of* T is the deductive closure of $T \cup \Sigma_\mathcal{V}$.

Given a subset $X \subset M$ of a Skolem expansion $\mathcal{M}^*$, the *Skolem hull* $H(X)$ is the smallest set $X \subset H(X) \subset M$ which is closed under all the Skolem functions. The corresponding substructure of $\mathcal{M}^*$ is denoted $\mathcal{H}(X)$.

A $\mathcal{V}$-theory T is said to have *built-in Skolem functions* if there already is a term t_ψ in $\mathcal{V}$ of arity n for each formula ψ of the form $\psi(x_1, \ldots, x_n) = (\exists x)\phi(x, x_1, \ldots, x_n)$ such that:

$$T \vdash (\forall(y_1, \ldots, y_n)(\psi(y_1, \ldots, y_n) \to \phi(t_\psi(y_1, \ldots, y_n), y_1, \ldots, y_n))$$

Proposition 16. *Every $\mathcal{V}$-structure $\mathcal{M}$ has a Skolem expansion $\mathcal{M}^*$*

Proof. Let $\mathcal{M}$ be a $\mathcal{V}$-structure and well-order M. Given a $\mathcal{V}$-formula of the form $\psi(x_1, \ldots, x_n) = (\exists x)\phi(x, x_1, \ldots, x_n)$, we let the interpretation G_ψ of the Skolem function F_ψ in the expansion $\mathcal{M}^* = (\mathcal{M}, \{F_\psi : \psi = (\exists x)\phi\})$ be as follows. Given any $a_1, \ldots, a_n \in M$, if $\mathcal{M} \models \psi(a_1, \ldots, a_n)$, let $G_\psi(a_1, \ldots, a_n)$ be the least element a such that $\mathcal{M} \models \phi(a, a_1, \ldots, a_n)$ (well-defined in a well ordering). If $\mathcal{M} \not\models \psi(a_1, \ldots, a_n)$, let $G_\psi(a_1, \ldots, a_n)$ be arbitrary. It follows immediately that $\mathcal{M}^*$ models the axioms of $\Sigma_\mathcal{V}$, hence $\mathcal{M}^*$ is a Skolem expansion of $\mathcal{M}$. □

Proposition 17. *Let T be a $\mathcal{V}$-theory. Then there exists an expansion $\overline{\mathcal{V}}$ of $\mathcal{V}$ and an extension $\overline{T}$ of T such that $\overline{T}$ has built-in Skolem functions. Furthermore, every model of T has a Skolem expansion which models $\overline{T}$.*

Proof. Given the vocabulary $\mathcal{V}_0 = \mathcal{V}$, define a sequence of expansions $\mathcal{V}_n$ by $\mathcal{V}_{n+1} = (\mathcal{V}_n)^*$. Let $\overline{\mathcal{V}} = \bigcup_n \mathcal{V}_n$.

Now define a $\overline{\mathcal{V}}$ -theory $\overline{T}$ by having the set $T \cup \bigcup_n \Sigma_{\mathcal{V}_n}$ as axioms. Any $\overline{\mathcal{V}}$-formula of the form $\psi = (\exists x)\phi$ contains a finite number of symbols, hence it is a $\mathcal{V}_n$-formula for some n and there is a corresponding Skolem function $F_\psi \in \mathcal{V}_{n+1} \subset \overline{\mathcal{V}}$. Now as $\overline{T} \models \Sigma_{\mathcal{V}_n}$, F_ψ is a built-in Skolem function of ψ in $\overline{T}$.

Also, note that by Proposition 16, if $\mathcal{M}_n$ is a model of $T_n = T \cup \bigcup_{k<n} \Sigma_{\mathcal{V}_k}$, then there is a Skolem expansion $\mathcal{M}_{n+1} = (\mathcal{M}_n)^*$ which models $T_{n+1} = T_n \cup \bigcup_{k \leq n} \Sigma_{\mathcal{V}_k}$. Hence there is inductively an expansion $\overline{\mathcal{M}}$ of $\mathcal{M}$ which models $\overline{T}$. □

Note that if $\mathcal{V}_n$ in the proof is countable, then there are at most countably many $\psi = (\exists x)\phi(x)$ in $\mathcal{V}_n$, and the Skolem expansion $(\mathcal{V}_n)^*$ of $\mathcal{V}_n$ is countable. Hence if $\mathcal{V}$ is a countable vocabulary, the expansion $\overline{\mathcal{V}}$ above is countable.

Given a $\mathcal{V}$-structure $\mathcal{M}$, we now define a set of *indiscernibles in $\mathcal{M}$* to be a subset $X \subset M$ such that X is strictly and linearly ordered by $<$ ($<$ is not necessarily in $\mathcal{V}$) and for every pair of increasing sequences $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ in X, $(\mathcal{M}, x_1, \ldots, x_n) \equiv (\mathcal{M}, y_1, \ldots, y_n)$. That is, a set X is indiscernible in $\mathcal{M}$ if no $\mathcal{V}$-formula can distinguish increasing sequences of elements in X.

The proof of the following lemma relies on Ramsey's theorem, but we will not delve into that theory in this text. The proof can be found in [1], on page 148.

Lemma 18. *Let $\mathcal{V}' = \mathcal{V} \cup \{c_n : n \in \omega\}$ where each c_n is a new constant symbol. Let T be a $\mathcal{V}$-theory with infinite models. Then the following set T' of*

$\mathcal{V}'$*-sentences is consistent:*

$$T' = T \cup \{\phi(c_{i_1}, \dots, c_{i_n}) \leftrightarrow \phi(c_{j_1}, \dots, c_{j_n}) : \phi(v_1, \dots, v_n) \textit{ is a } \mathcal{V}\textit{-formula}, \\ n \in \omega \textit{ and } i_1 < \dots < i_n,\ j_1 < \dots < j_n\} \cup \{\neg c_1 \equiv c_2\}$$

Theorem 19. *Let T be a $\mathcal{V}$-theory with infinite models and let $\langle X, < \rangle$ be any linearly ordered set. Then there is a model $\mathcal{M}$ of T such that $X \subset M$ and X is a set of indiscernibles in $\mathcal{M}$.*

Proof. Expand $\mathcal{V}$ to $\mathcal{V}' = \mathcal{V} \cup \{c_x : x \in X\}$. Note that every finite subset of X can be embedded in $\langle \omega, < \rangle$, hence given a finite number $x_1, \dots, x_n$ of elements from X, they can be said to be the interpretation of some $c_{x_i}, c_{y_i} \in \{c_n : n \in \omega\}$. Define T' as follows

$$T' = T \cup \{\phi(c_{x_1}, \dots, c_{x_n}) \leftrightarrow \phi(c_{y_1}, \dots, c_{y_n}) : \phi(v_1, \dots, v_n) \text{ is a } \mathcal{V}\text{-formula}, \\ n \in \omega \text{ and } x_1 < \dots < x_n,\ y_1 < \dots < y_n\} \cup \{\neg c_{x_1} \equiv c_{x_2} : x_1, x_2 \in X\}$$

and note that every finite subset Δ of T' only contains the sentence $\phi(c_{x_1}, \dots, c_{x_n}) \leftrightarrow \phi(c_{y_1}, \dots, c_{y_n})$ for finitely many formulas ϕ and finitely many sequences $x_1 < \dots < x_n$, $y_1 < \dots < y_n$. Hence Δ can be interpreted as a subset of some T'' of the form given in Lemma 18, and Δ is consistent. So by compactness, T' is consistent. Now let $\mathcal{M}'$ be a model of T', and $\mathcal{M}$ the reduct of $\mathcal{M}'$ to $\mathcal{V}$, which must be a model of the $\mathcal{V}$-theory T. If we in $\mathcal{M}'$ interpret each c_x by the element x, T' gives that for each formula $\phi(v_1, \dots, v_n)$ in $\mathcal{V}$ and sequences $x_1 < \dots < x_n$, $y_1 < \dots < y_n$, $\mathcal{M}' \models \phi(x_1, \dots, x_n)$ if and only if $\mathcal{M}' \models \phi(y_1, \dots, y_n)$. And as $\phi(v_1, \dots, v_n)$ was in $\mathcal{V}$, this holds in $\mathcal{M}$, so $(\mathcal{M}, x_1, \dots, x_n) \equiv (\mathcal{M}, y_1, \dots, y_n)$ and X is a set of indiscernibles in $\mathcal{M}$. □

We make the following observation. Let $\mathcal{V}$ be a countable vocabulary and T a $\mathcal{V}$-theory with infinite models. Proposition 17 gives us a countable vocabulary $\overline{\mathcal{V}}$ and a $\overline{\mathcal{V}}$-theory $\overline{T}$ with built-in Skolem functions. By Theorem 19, given any linearly ordered set $\langle X, < \rangle$, we can find a model $\mathcal{M}$ of $\overline{T}$ where X is a set of indiscernibles. Now the Skolem expansion $\mathcal{M}^*$ is essentially the same as $\mathcal{M}$ if we interpret the Skolem functions as the built-in Skolem functions in $\overline{T}$. Hence the substructure generated by X is the Skolem hull $\mathcal{H}(X)$ of X, and by the Tarski-Vaught criterion (Proposition 6), it is an elementary substructure of $\mathcal{M}$. So by setting $\overline{\mathcal{M}} = \mathcal{H}(X)$, this is also a model of $\overline{T}$ in which X is a set of indiscernibles.

Theorem 20. *Let $\mathcal{V}$ be a countable vocabulary and T a $\mathcal{V}$-theory with infinite models. Then for every infinite cardinal κ, T has a model $\mathcal{M}$ of power κ such that for every subset $N \subset M$, the expanded model $\mathcal{M}_N$ realizes at most $|N| + \omega$ types in the expanded vocabulary $\mathcal{V} \cup \{c_b : b \in N\}$*

Proof. As in the observation above, we extend T to a theory $\overline{T}$ which has built-in Skolem functions in an expanded vocabulary $\overline{\mathcal{V}}$. Let $(X, <)$ be a well-ordered set of order type κ. Then there is a model $\overline{\mathcal{M}} = \mathcal{H}(X)$ of $\overline{T}$ in which X is a set of indiscernibles. Note that $\overline{\mathcal{M}}$ is of power κ, because as $\overline{\mathcal{V}}$ is countable, there are at most countably many functions and the closure $H(X)$ of X under these functions thus has at most $\kappa \cdot \omega = \kappa$ elements.

Let $N \subset M = H(X)$ and choose for each $b \in N$ a term $t(v_1, \ldots, v_n)$ and $(y_1, \ldots, y_n) \in X^n$ such that $b = t(y_1, \ldots, y_n)$. This is called a *standard representation of* b. Let Y be the set of all $y \in X$ which appear in one of these standard representations. As there are finitely many y in each standard representation, we have $|Y| \leq |N|$ if $|N| \geq \omega$, and $|Y| \leq \omega$ if N is countable, hence $|Y| \leq |N| + \omega$.

Call two sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of elements in X equivalent over Y if for all $k \leq n$ and all $z \in Y$, we have $x_k, y_k \neq z$, and $x_k < z$ if and only if $y_k < z$.

Form the expanded vocabulary $\mathcal{V}' = \mathcal{V} \cup \{c_z : z \in Y\}$. Assume that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are equivalent over Y. Any $\mathcal{V}'$-formula $\phi(v_1, \ldots, v_n)$ can be written as $\phi(v_1, \ldots, v_n) = \psi(v_1, \ldots, v_n, c_{z_1}, \ldots, c_{z_m})$ with some suitable $\mathcal{V}$-formula $\psi(v_1, \ldots, v_n, w_1, \ldots, w_m)$, where $c_{z_1}, \ldots, c_{z_m}$ are the constants appearing in ϕ which are not in $\mathcal{V}$. As $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are equivalent over Y, the elements $z_1, \ldots, z_m$ can be "ordered in between" the sequence $x_1 < \ldots < x_n$ in the same way as in $y_1 < \ldots < y_n$, which gives two new sequences $x'_1 < \ldots < x'_{n+m}$ and $y'_1 < \ldots < y'_{n+m}$ of elements in X. Assume that $x_1 < \ldots < x_n < z_1 < \ldots < z_n$ - if the ordering is otherwise, rearrange the free variables $v_1, \ldots, v_n, w_1, \ldots, w_n$ so their appearance in the formula $\psi(v_1, \ldots, v_n, w_1, \ldots, w_n)$ matches the order of $x'_1 < \ldots < x'_{n+m}$ and $y'_1 < \ldots < y'_{n+m}$. As X is a set of indiscernibles, $\overline{\mathcal{M}} \models \psi(x_1, \ldots, x_n, z_1, \ldots, z_m) \Leftrightarrow \overline{\mathcal{M}} \models \psi(y_1, \ldots, y_n, z_1, \ldots, z_m)$. Hence $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ satisfy the same formulas $\phi(v_1, \ldots, v_n)$ in the expanded model $(\overline{\mathcal{M}}, z)_{z \in Y}$.

It follows that for any term $t(v_1, \ldots, v_n)$ of $\mathcal{V}'$, the two elements $t(x_1, \ldots, x_n)$ and $t(y_1, \ldots, y_n)$ realize the same type in the model $(\overline{\mathcal{M}}, z)_{z \in Y}$. Note that as $b = t(z_1, \ldots, z_n)$ for some t and $z_1, \ldots, z_n \in Y$, any type in the vocabulary of $(\overline{\mathcal{M}}, b)_{b \in N}$ can be written in the vocabulary of $(\overline{\mathcal{M}}, z)_{z \in Y}$, so the two elements $t(x_1, \ldots, x_n)$ and $t(y_1, \ldots, y_n)$ also realize the same types in $(\overline{\mathcal{M}}, b)_{b \in N}$. Let $\mathcal{M}$ be the reduct of $\overline{\mathcal{M}}$ to $\mathcal{V}$. Then $t(x_1, \ldots, x_n)$ and $t(y_1, \ldots, y_n)$ realizes the same type in $(\mathcal{M}, b)_{b \in N}$ whenever $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are equivalent over Y.

Let $x' =$ "least $z \in Y$ such that $x < z$" if such a $z \in Y$ exists, otherwise let $x' = \infty$. It follows immediately that two n-tuples $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are equivalent over Y if and only if $x'_1 = y'_1, \ldots, x'_n = y'_n$. Thus there is at most $|Y| + \omega$ non-equivalent n-tuples over Y, as this is the number of all possible $\{x'_1, \ldots, x'_n\} \subset Y$. Hence for a term t, evaluating t for all possible n-tuples not in Y gives a set whose elements realizes at most $|Y| + \omega$ different types in $(\overline{\mathcal{M}}, z)_{z \in Y}$. Call this set A_t.

Note that the vocabulary of $(\overline{\mathcal{M}}, z)_{z \in Y}$ has cardinality $|Y| + \omega$, so there are at most $|Y| + \omega$ different terms t. Hence the elements of $A = \bigcup_t A_t$ realizes at most $|Y| + \omega$ different types in $(\overline{\mathcal{M}}, z)_{z \in Y}$.

Every element of $M = H(X)$ is equal to some term $t(x_1, \ldots, x_n)$ in $\overline{\mathcal{M}}$, and if some $x_i \in Y$, we can replace x_i by a corresponding constant z_i to get equality to a term in the model $(\overline{\mathcal{M}}, z)_{z \in Y}$. Continuing like this, we get that our element is equal to a term $t'(x'_1, \ldots, x'_l)$ in $(\overline{\mathcal{M}}, z)_{z \in Y}$ where $x'_1, \ldots, x'_l \notin Y$. Hence $M = A$ and the model $(\overline{\mathcal{M}}, z)_{z \in Y}$ realizes at most $(|Y| + \omega) \leq (|N| + \omega)$ different types, hence $(\mathcal{M}, b)_{b \in N}$ realizes at most $(|N| + \omega)$ different types. $\square$

7 Morley's Categoricity Theorem

Theorem 21 (Morley's theorem)**.** *Suppose $\mathcal{V}$ is countable and T is a complete $\mathcal{V}$-theory. If T is categorical in some uncountable power, then T is categorical in every uncountable power.*

The following lemma shows that it will be enough to prove that if T is categorical in some uncountable power, then every model of T of power ω_1 is saturated. This will be done in the end of the section, using Lemmas 24, 25 and 29.

Lemma 22. *Suppose $\mathcal{V}$ is countable and T is a complete $\mathcal{V}$-theory, such that every model of T of power ω_1 is saturated. Then every uncountable model of T is saturated, and it follows that T is categorical in every uncountable power.*

Proof. If T has a model of some infinite power, then T has models of any infinite power. Given any infinite power κ, any two models $\mathcal{M}, \mathcal{N}$ of T of the same power κ are elementary equivalent. This follow from the completeness of T, as for every $\mathcal{V}$-sentence ϕ, $T \models \phi$ or $T \models \neg\phi$. Hence if $\mathcal{M} \models \phi$, we must have that $T \models \phi$, as $T \models \neg\phi$ would give that $\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi \wedge \neg\phi$, a contradiction. Now $\mathcal{N} \models T$ yields that $\mathcal{N} \models \phi$, and $Th(\mathcal{M}) \subset Th(\mathcal{N})$. Analogously, $Th(\mathcal{N}) \subset Th(\mathcal{M})$ and $Th(\mathcal{M}) = Th(\mathcal{N})$.

If we show that every uncountable model of T is saturated, the uniqueness of saturated models (Theorem 14) yields that the models are isomorphic, and hence T is κ-categorical.

Given any $\kappa > \omega_1$, assume towards a contradiction that T has a model $\mathcal{M}$ of power κ which is not saturated. Then there is a subset $X \subset M$ of power $|X| < \kappa$ and a set $\Sigma(v)$ of formulas in the expanded vocabulary $\mathcal{V}_X$ consistent with $Th(\mathcal{M})$ such that $\Sigma(v)$ is not satisfiable in $\mathcal{M}_X$, but every finite subset of Σ is satisfied in $\mathcal{M}_X$. Since $\mathcal{V}$ is countable, $|\mathcal{V}_X| = |\mathcal{V}| + |X| < \kappa$, hence $|\Sigma| < \kappa$ and we can choose a subset $U \subset M$ such that $|U| = |\Sigma|$, i.e. there is a bijection $\phi : U \to \Sigma$.

Let $U(u)$ be the relation which is true if $u \in U$. The function ϕ associates a formula ϕ_u in the vocabulary $\mathcal{V}_X$ to each element $u \in U$. Define the relation $R(v_1, v_2)$ to be true for all pairs $(u, x) \in M \times M$ such that $u \in U$, $x \in X$ and the constant c_x occurs in the formula ϕ_u. Thus for every u, there are only finitely many $x \in M$ such that $R(u, x)$. Let $S(v_1, v_2)$ be the relation which is true for all $(u, v) \in M \times M$ such that $u \in U$ and $\mathcal{M}_X \models \phi_u(v)$. Expand $\mathcal{M}$ to $\mathcal{M}' = (\mathcal{M}, U, R, S)$.

The idea of the proof is that sentences that hold in a given model will hold in elementary extensions and substructures of that model. It follows by the definition of S that the formula

$$(1)\ (\forall v)(S(u, v) \leftrightarrow \phi_u(v))$$

is satisfied by each $u \in U$ in the model $\mathcal{M}'_X$. Now define the sentence

$$(2)\ (\forall u_1, \ldots, u_n)[U(u_1) \wedge \ldots \wedge U(u_n) \rightarrow (\exists v)(S(u_1, v) \wedge \ldots \wedge S(u_n, v)].$$

This holds in $\mathcal{M}'$, as every finite subset of Σ is satisfiable in $\mathcal{M}'$, i.e. $(\forall u_1, \ldots, u_n)$ $[U(u_1) \wedge \ldots \wedge U(u_n) \rightarrow (\exists v)(\phi_{u_1}(v) \wedge \ldots \wedge \phi_{u_n}(v)]$, which by (1) is equivalent to (2). Also, the sentence

$$(3)\ \neg(\exists v \forall u)(U(u) \rightarrow S(u, v))$$

holds in $\mathcal{M}'$ as Σ is not realized by any $v \in M$, i.e $\neg(\exists v \forall u)(U(u) \rightarrow \phi_u(v))$ which by (1) is equivalent to (3).

By Theorem 15, since $|U| < |M|$, there exists two models

$$\mathcal{N}' = (N, U', R', S')\ ,\ \text{and}\ \mathcal{O}' = (O, U', R'', S'')$$

such that $\mathcal{N}' \prec \mathcal{M}', \mathcal{N}' \prec \mathcal{O}'$, $|N| = \omega, |O| = \omega_1$ and the interpretation U' of U is the same. $U' \subset N$ must be countable. Furthermore, if we put $X' = X \cap N$, it holds for the expanded models that:

$$(4)\ \mathcal{N}'_{X'} \prec \mathcal{M}'_{X'}$$

and

$$(5)\ \mathcal{N}'_{X'} \prec \mathcal{O}'_{X'}.$$

(As X' is a subset of N, and hence of M and O too.)

By (4) and (5), the sentences (2) and (3) hold in $\mathcal{O}'$. If for each $u \in U'$ we have that the formula (1) is satisfied in $\mathcal{O}'_{X'}$, the sentences (2) and (3) will be equivalent to $(\forall u_1, \ldots, u_n)$ $[U(u_1) \wedge \ldots \wedge U(u_n) \rightarrow (\exists v)(\phi_{u_1}(v) \wedge \ldots \wedge \phi_{u_n}(v)]$ and $\neg(\exists v \forall u)(U(u) \rightarrow \phi_u(v))$ respectively, i.e. there is a set

$$\Sigma'(v) = \{\phi_a : a \in U'\}$$

which is not satisfied in the model $\mathcal{O}'_{X'}$, but every finite subset of $\Sigma'(v)$ is.

For each $u \in U' \subset N$, the set $\{x \in M : R(u, x)\}$ is finite, say $|\{x \in M : R(u, x)\}| = n$. The following formula states that "there are exactly n different elements x such that $R(y, x)$ is true":

$$\begin{aligned}\psi(y) =&(\exists x_1, \ldots, x_n)[(\bigwedge_{i \neq j}(x_i \not\equiv x_j) \wedge (R(y, x_1) \wedge \ldots \wedge R(y, x_n))] \wedge \\ &(\forall x_1, \ldots, x_n, x_{n+1})[R(y, x_1) \wedge \ldots \wedge R(y, x_{n+1}) \rightarrow (\bigvee_{i \neq j} x_i \equiv x_j)].\end{aligned}$$

Now as $\mathcal{M}'_{X'} \models \psi(u)$ and $u \in U' \subset N$, (4) gives that $\mathcal{N}'_{X'} \models \psi(u)$. Hence $|\{x \in N : R(u, x)\}| = |\{x \in M : R(u, x)\}|$. Note that by $N \subset M$ we have that $\{x \in N : R(u, x)\} \subset \{x \in M : R(u, x)\}$, hence by the equal cardinality of the sets we get that

$$\{x \in M : R(u, x)\} = \{x \in N : R'(u, x)\} \subset X'.$$

This yields that for each $u \in U'$, all constants c_x appearing in the corresponding formula ϕ_u lie in the vocabulary $\mathcal{V}'_X$ of $\mathcal{N}'_{X'}$. So by (4) and (5) we get that each $u \in U'$ actually satisfies (1) in $\mathcal{O}'_{X'}$.

By (4) and (5), $\mathcal{O}$ is a model of T of power ω_1. By above, there is a subset $X' \subset O$ such that $|X'| < \omega_1$ (as $X' \subset N$ and N is countable), but there is a set of formulas $\Sigma'(v)$ which is not satisfied in $\mathcal{O}'_{X'}$. Thus $\mathcal{O}$ is not saturated, contradicting the hypothesis of the lemma.

So every uncountable model $\mathcal{M}$ of T must be saturated. □

The proof of Morley's theorem relies heavily on the notion of a stable theory.

Definition. A theory is T is *stable in power* κ if for every model $\mathcal{M}$ of T and every set $X \subset M$ of power κ, the expansion $\mathcal{M}_X$ realizes no more than κ types in a single variable v.

Lemma 23. *Let T be a theory in $\mathcal{V}$ which is stable in power ω. Then T is stable in every infinite power.*

Proof. Assume towards a contradiction that there is a $\kappa > \omega$ such that T is not stable in power κ. That is assume T has a model $\mathcal{M}$ with a subset $X \subset M$ such that $|X| = \kappa$ and the expanded model $\mathcal{M}_X$ realizes more than κ types, that is at least κ^+ types. We can assume without loss of generality that $\mathcal{M}$ has power κ^+, as otherwise the Löwenheim Skolem theorem will give an elementary substructure of power κ^+. This gives that $\mathcal{M}_X$ can't realize more than κ^+ types (as each element in M can realize at most one type), hence $\mathcal{M}_X$ realizes exactly κ^+ types.

Since the vocabulary $\mathcal{V}$ is countable, the vocabulary $\mathcal{V}_X = \mathcal{V} \cup \{c_x : x \in X\}$ has power κ. Thus the set Σ of all formulas $\phi(v)$ of $\mathcal{V}_X$ has power κ. Given a subset $U \subset M$ of power κ, we can find a one to one function from U to Σ, that is, an enumeration $\phi_u(v)$ of all formulas in Σ, where $u \in U$.

As in the previous lemma, we define the relations $U(u)$ if $u \in U$, $R(u, x)$ if $u \in U$ and c_x occurs in the formula ϕ_u, and $S(u, v)$ if $\mathcal{M}_X \models \phi_u(v)$.

Furthermore, for each of the κ^+ different types realized by $\mathcal{M}_X$, choose one element which satisfies this type. Let $V \subset M$ be the subset of power $|V| = \kappa^+$ containing these elements, and define the relation $V(v)$ if $v \in V$. As an element can't satisfy two different types, our construction yield that two distinct elements in V will realize different types in $\mathcal{M}_X$. As M and V have the same power, there is a one to one function G from M into V. Expand $\mathcal{M}$ to the model

$$\mathcal{M}' = (\mathcal{M}, U, V, R, S, G)$$

As in the previous lemma, in the model $\mathcal{M}'_X$ all $u \in U$ satisfies the formula

$$(1)\ (\forall v)(S(uv) \leftrightarrow \phi_u(v)).$$

We can write that "two distinct elements of V realize different types" and "G is a bijection from the underlying set M of the model into V" as the following sentences, which will hold in $\mathcal{M}'$:

$$(2)\ (\forall v, w)[(v \not\equiv w \wedge V(v) \wedge V(w)) \rightarrow (\exists u)(U(u) \wedge \neg(S(u, v) \wedge S(u, w))]$$

$$(3)\ (\forall v, w)[(v \not\equiv w) \rightarrow (V(G(v)) \wedge G(v) \not\equiv G(w))].$$

Using Theorem 15, that $|U| < |M|$ gives the existence of two models

$$\mathcal{N}' = (N, U', V', R', S', G')\ ,\ \text{and}\ \mathcal{O}' = (O, U', V'', R'', S'', G'')$$

such that $\mathcal{N}' \prec \mathcal{M}', \mathcal{N}' \prec \mathcal{O}'$, $|N| = \omega, |O| = \omega_1$ and the interpretation of U' is the same in both $\mathcal{N}'$ and $\mathcal{O}'$. The sentences (2) and (3) must now hold in $\mathcal{N}$ and $\mathcal{O}$. (3) gives that G'' is a bijection between O and V'', hence

$$(4)\ |V''| = |O| = \omega_1.$$

Now for all $u \in U'$, if $x \in X$ and $R(u, x)$ holds, then we as in the previous lemma have that $x \in X' = X \cap N$. Thus every formula $\phi_a(v)$ is a formula in $\mathcal{V}_{X'}$, and (1) is satisfied by each $u \in U'$ in the models $\mathcal{N}'_{X'}$ and $\mathcal{O}'_{X'}$.

This gives that $(\exists u)(U'(u) \wedge \neg(S''(u, v) \wedge S''(u, w)) \Leftrightarrow (\exists u)(U'(u) \wedge \neg(\phi_u(v) \wedge \phi_u(w))$ in $\mathcal{O}'_{X'}$, hence (2) says that any two distinct elements of V'' realize different formulas, and they must realize different types in the model $\mathcal{O}'_{X'}$. So by (4), $\mathcal{O}'_{X'}$ realizes uncountably many types. But as $X' \subset N$, $|X'|$ is a countable set, which contradicts that T is a ω-stable theory.

Thus T is stable in every infinite power. □

Lemma 24. *If a theory T is categorical in some uncountable power κ, then T is stable in power ω.*

Proof. Assume towards a contradiction that T is not stable in power ω. Then there is a model $\mathcal{M}$ and a subset $X \subset M$ of power ω such that the simple expansion $\mathcal{M}_X$ realizes more than ω types in a single variable, i.e. at least ω_1 types. By the Löwenheim-Skolem theorem, we can assume that $|M| = \omega_1$, so $\mathcal{M}_X$ realizes at most, hence exactly, ω_1 types. Using Theorem 20, T has a model $\mathcal{N}$ of power κ with the property that for all $Y \subset N$, the expanded model $\mathcal{N}_Y$ realizes at most $|Y| + \omega$ types. As T is κ-categorical, this property holds for all models of power κ. By the Löwenheim-Skolem theorem, we get that there is an elementary extension $\mathcal{M}' \succ \mathcal{M}$ of power κ. Then by letting $Y = X$, $\mathcal{M}'_X$ realizes at most $|X| + \omega = \omega$ types, while $\mathcal{M}_X$ relizes ω_1 types. This is a contradiction, as every element of M realizes the same type in both $\mathcal{M}_X$ and $\mathcal{M}'_X$ □

Lemma 25. *Suppose T is a theory which has infinite models and is stable in power ω, then for every regular cardinal $\kappa > \omega$, T has an κ-saturated model of every power $\beta \geq \kappa$.*

Proof. Let $\mathcal{M}$ be a model of T of power $\beta \geq \kappa$, and $T_{\mathcal{M}}$ the complete theory of the expansion $\mathcal{M}_M$. By Proposition 11 there is a model $\mathcal{N}_M$ such that $\mathcal{M}_M \prec \mathcal{N}_M$ and every type consistent with $T_{\mathcal{M}}$ is realized in $\mathcal{N}_M$.

By Lemma 23 , T is stable in power β. Hence $\mathcal{N}_M$ realizes at most $|M| = \beta$ types. Let $A \subset N$ be a set containing for each type realized in $\mathcal{N}_M$ one element which realizes that type. Let $A \cup M = B \subset N$ and note that $|B| = \beta$.

The Löwenheim-Skolem theorem gives the existence of an elementary submodel $\mathcal{M}'_M \prec \mathcal{N}_M$ such that $B \subset M'$ and $|M'| = \beta$. By $A \subset M'$, every type consistent with $\mathcal{N}_M$, hence every type consistent with $T_{\mathcal{M}}$, is realized in $\mathcal{M}'_M$.

By $M \subset M'$ we immediately get $\mathcal{M} \subset \mathcal{M}'_M$. Given any formula $\phi(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n) \in M^n$, $\mathcal{M}_M \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N}_M \models \phi(a_1, \ldots, a_n)$ by $\mathcal{M}_M \prec \mathcal{N}_M$. As $M \subset M'$ and $\mathcal{M}'_M \prec \mathcal{N}_M$, we also have that $\mathcal{M}'_M \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N}_M \models \phi(a_1, \ldots, a_n)$, hence $\mathcal{M}_M \prec \mathcal{M}'_M$.

So we have shown the existence of an elementary extension $\mathcal{M}'_M \succ \mathcal{M}_M$ such that $|M'| = \beta$ and every type consistent with $T_{\mathcal{M}}$ is realized in $\mathcal{M}'_M$. For reference, call this result (1).

We use (1) κ times to construct an elementary chain $\mathcal{M}_\gamma$, $\gamma < \kappa$, such that each $\mathcal{M}_\gamma$ is a model of T of power β; also (1) gives that any type consistent with the theory of $\mathcal{M}_\gamma$ is realized in $\mathcal{M}_{\gamma+1}$. If γ is a limit ordinal, we simply define $\mathcal{M}_\gamma = \bigcup_{\delta<\gamma} \mathcal{M}_\delta$ Let $\mathcal{N}$ be the union $\mathcal{N} = \bigcup_{\delta<\kappa} \mathcal{M}_\delta$. As $\beta \geq \kappa$, $\mathcal{N}$ is of power β.

We now show that $\mathcal{N}$ is κ-saturated. Let X be any set $X \subset N$ of power $|X| < \kappa$. Well-order N and let X' be the corresponding ordinals of X. Since κ is regular it follows that $sup(X') < \kappa$, hence there exists a $\gamma < \kappa$ such that $X \subset M_\gamma$. Now every type $\Sigma(v)$ consistent with the complete theory of $(\mathcal{M}_\gamma, x)_{x\in X}$ is realized in $(\mathcal{M}_{\gamma+1}, x)_{x\in X}$, and hence realized in $(\mathcal{N}, x)_{x\in X}$. But as $\mathcal{M}_\gamma \prec \mathcal{N}$, the two models have the same complete theories, hence every type $\Sigma(v)$ consistent with the complete theory of $(\mathcal{N}, x)_{x\in X}$ is realized in $(\mathcal{N}, x)_{x\in X}$. So $\mathcal{N}$ is κ-saturated. □

The following two lemmas will be needed to prove Lemma 29.

Lemma 26. *Let T be a theory in a countable vocabulary $\mathcal{V}$ which is stable in power ω. Then for every model $\mathcal{M}$ of T and every subset $X \subset M$, the complete theory T_X of the expanded model $\mathcal{M}_X$ is atomic.*

Proof. Assume towards a contradiction that T_X is not atomic. Then there is a smallest positive integer n such that there is an incompletable formula $\psi(v_1, \ldots, v_n)$ in n variables, consistent with T_X.

We only need to show a contradiction in the simplest case when $n = 1$, for if we assume that $n > 1$, there exists an expansion of $\mathcal{M}_X$ and a theory which has an incomplete formula in one variable, hence the contradiciton will follow.

To get this expansion, note that if $n > 1$, then the formula $\exists v_n \psi(v_1, \ldots, v_n)$ has n-1 variables, and thus an atom $\phi(v_1, \ldots, v_{n-1})$. Add new constant symbols to get the vocabulary $\mathcal{V}'_X = \mathcal{V}_X \cup \{c_1, \ldots, c_{n-1}\}$, and define the $\mathcal{V}'_X$-theory:

$$T'_X = T_X \cup \{\phi(c_1, \ldots, c_{n-1})\}$$

Note that any sentence δ in $\mathcal{V}'_X$ containing some of $c_1, \ldots, c_{n-1}$ can be written as $\delta = \gamma(c_1, \ldots, c_{n-1})$ for some formula $\gamma(v_1, \ldots, v_{n-1})$. As $\phi(v_1, \ldots, v_{n-1})$ is atomic, either $T_X \models \phi(v_1, \ldots, v_{n-1}) \to \gamma(v_1, \ldots, v_{n-1})$ or $T_X \models \phi(v_1, \ldots, v_{n-1})$

$\to \neg\gamma(v_1, \ldots, v_{n-1})$, hence either δ or $\neg\delta$ is a consequence of the sentence $\phi(c_1, \ldots, c_{n-1})$, and T'_X is a complete theory.

We now claim that the formula $\psi(c_1, \ldots, c_{n-1}, v_n)$ has no atoms in T'_X. For if there is an atom $\theta(c_1, \ldots, c_{n-1}, v_n)$ of $\psi(c_1, \ldots, c_{n-1}, v_n)$, then:

$$T'_X \models \theta(c_1, \ldots, c_{n-1}, v_n) \to \psi(c_1, \ldots, c_{n-1}, v_n)$$

Note that $\{c_1, \ldots, c_n\}$ was not in the original vocabulary $\mathcal{V}_X$. By Lemma 5 the above yields that $T'_X \models (\forall v_1, \ldots, v_{n-1})(\theta(v_1, \ldots, v_n) \to \psi(v_1, \ldots, v_n))$. Hence as $T_X \models \phi(v_1, \ldots, v_{n-1}) \to \exists v_n \psi(v_1, \ldots, v_n)$, we get that:

$$T_X \models \phi(v_1, \ldots, v_{n-1}) \wedge \theta(v_1, \ldots, v_n) \to \psi(v_1, \ldots, v_n)$$

As $\phi(v_1, \ldots, v_{n-1}) \wedge \theta(v_1, \ldots, v_n) \to \phi(v_1, \ldots, v_{n-1})$, $\phi \wedge \theta$ is an atom of T_X. But then the above contradicts that $\psi(v_1, \ldots, v_n)$ has no atoms. Now let $a_1, \ldots, a_{n-1}$ be elements such that $\mathcal{M}_X \models \phi(a_1, \ldots, a_{n-1})$, then $\mathcal{M}' = (\mathcal{M}_X, a_1, \ldots, a_{n-1})$ is a model of T'_X, and the formula $\psi(c_1, \ldots, c_{n-1}, v) = \psi(v)$ has no atoms in the complete theory of $\mathcal{M}'$.

So assume that $n = 1$ and that $\psi(v)$ is consistent with and incompletable in T_X. $\psi(v)$ is not an atom (as an atom is trivially completable), hence there is some formula $\gamma(v_1, \ldots, v_n)$ such that $T_X \not\models \psi \to \gamma$ and $T_X \not\models \psi \to \neg\gamma$. Thus the sentences $\psi_0(v) = \psi(v) \wedge \gamma(v)$ and $\psi_1 = \psi(v) \wedge \neg\gamma(v)$ are formulas consistent with T_X such that:

$$T_X \models \psi_0(v) \to \psi(v),\ T_X \models \psi_1(v) \to \psi(v), \text{ and } T_X \models \neg(\psi_0(v) \wedge \psi_1(v))$$

Furthermore, they are not atomic in T_X; for if say $\psi_0(v)$ was an atom, then $T_X \models \psi_0 \to \psi$ contradicts that $\psi(v)$ is incompletable.

Thus we can repeat the above argument for $\psi_0(v)$ to get formulas $\psi_{00}(v)$ and $\psi_{01}(v)$ with the same properties as above, and so on infinitely. We will get infinite sequences of zeroes and ones, each which corresponds to a consistent set of non-atomic formulas. As there are 2^ω such sequences, we get 2^ω sets of such sentences. Given any two different sequences, let "s" denote all initial elements which are identical. If A and B are the corresponding sets of sentences, $\psi_{s0} \in A$ and $\psi_{s1} \in B$. As $T_X \models \neg(\psi_{s0} \wedge \psi_{s1})$ we get that the two sets can not realized by the same element. Hence if we extend each set to a type, we get 2^ω different types.

Let Y be the set of all constants $x \in X$ which occur in some formula in our tree of formulas. There is a finite number of sequences of a given length n. If the tier $\Sigma_n(v)$ is the set of all $\psi_s(v)$ in the tree such that s has length n, there can only be a finite number of constants appearing in the formulas of $\Sigma_n(v)$. As $\bigcup_{n<\omega} \Sigma_n(v)$ are all formulas appearing in the tree, we get that Y contains at most ω elements.

Let T_Y be the complete theory of $\mathcal{M}_Y$. Then T_Y has 2^ω different types in $\mathcal{V}$. By Proposition 11, T_Y has a model $\mathcal{N}_Y$ which realizes all 2^ω types, so T is not ω stable, a contradiction. Thus T_X must be atomic. $\square$

Lemma 27. *Let T be a theory such that for every model $\mathcal{M}$ of T and every subset $X \subset M$, the complete theory of the expanded model $\mathcal{M}_X$ is atomic. Then for every model $\mathcal{M}$ of T and every subset $X \subset M$, the complete theory of $\mathcal{M}_X$ has an atomic model.*

Proof. Given the model $\mathcal{M}_X$ in the vocabulary $\mathcal{V}$, where $||\mathcal{V}|| = |\mathcal{V}| + \omega = \alpha$, we will construct a sequence a_β, $\beta < \alpha \cdot \omega$ of elements of M such that:

(1) For all $\beta < \alpha \cdot \omega$, a_β realizes an atom in the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<\beta}$.

(2) For all $n < \omega$ and every atom $\phi(v)$ in the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<\alpha\cdot n}$, there exists $\delta < \alpha \cdot (n+1)$ such that a_δ realizes $\phi(v)$

By then letting $\mathcal{N}_X$ be the submodel of $\mathcal{M}_X$ with underlying set $N = \{a_\beta : \beta < \alpha \cdot \omega\}$, we will in fact get an elementary submodel of $\mathcal{M}_X$ which is atomic.

We first construct the sequence a_β via induction, over $m < \omega$.

For the base case $m = 0$, (1) is trivially satisfied for all $\beta < \alpha \cdot m$ and so is (2) for each $n < m$. For the induction step, suppose we have a sequence a_β, $\beta < \alpha \cdot m$ such that (1) holds for all $\beta < \alpha \cdot m$ and (2) holds for all $n < m$. Given the model $(\mathcal{M}_X, a_\gamma)_{\gamma<\alpha\cdot m}$, its vocabulary has cardinality α, hence the set of all atomic formulas $\phi(v)$ for the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<\alpha\cdot m}$ can be ordered as $\phi_\delta(v)$, $\delta < \alpha$. We choose for $a_{\alpha\cdot m}$ an element in M which satisfies the atom $\phi_0(v)$ in $(\mathcal{M}_X, a_\gamma)_{\gamma<\alpha\cdot m}$. By the assumption of the lemma, the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<(\alpha\cdot m)+1}$ is atomic. Hence we can find an atom $\phi'_1(v)$ in the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<(\alpha\cdot m)+1}$ for our formula $\phi_1(v)$. (Although $\phi_1(v)$ is an atom for the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<\alpha\cdot m}$, it does not need to be one of $(\mathcal{M}_X, a_\gamma)_{\gamma<(\alpha\cdot m)+1}$.) Choose $a_{(\alpha\cdot m)+1}$ to be an element satisfying $\phi'_1(v)$.

Continuing in this manner, we get elements $a_{(\alpha\cdot m)+\delta}$, $\delta < \alpha$ such that $a_{(\alpha\cdot m)+\delta}$ realizes an atom $\phi'_\delta(v)$ in the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<(\alpha\cdot m)+\delta}$, hence (1) holds for all $\beta < \alpha \cdot (m+1)$. And given any atom $\phi(v) = \phi_\delta(v)$ of $(\mathcal{M}_X, a_\gamma)_{\gamma<(\alpha\cdot m)}$, that $\phi'_\delta(v)$ is an atom of $\phi_\delta(v)$ in $(\mathcal{M}_X, a_\gamma)_{\gamma<(\alpha\cdot m)+\delta}$ yields that $\phi_\delta(v)$ is realized by the element $a_{(\alpha\cdot m)+\delta} \in M$, hence (2) holds for $n = m$. By the induction principle, (1) holds for all $\beta < \alpha \cdot \omega$ and (2) for all $n < \omega$.

To show that $\mathcal{N}_X \prec \mathcal{M}_X$, we use the Tarski-Vaught criterion. Recall that $N = \{a_\beta : \beta < \alpha \cdot \omega\}$, hence given a formula $\phi(x_1, \ldots, x_n, v)$ and any tuple $(a_1, \ldots, a_n) \in N^n$, that $\mathcal{M}_X \models \exists v \phi(a_1, \ldots, a_n, v)$ implies that the formula $\phi(v) = \phi(a_1, \ldots, a_n, v)$ is consistent with the complete theory of $(\mathcal{M}_X, a_\gamma)_{\gamma<\alpha\cdot\omega}$.

There is an $n < \omega$ such that for all constants a_β not in X appearing in $\phi(v)$, $\beta < \alpha \cdot n$. Hence $\phi(v)$ is also consistent with the complete theory of $(\mathcal{M}_X, a_\beta)_{\beta<\alpha\cdot n}$, which is atomic by hypothesis, and there is an atom $\phi'(v)$ of $\phi(v)$ in this theory. By (2), there is an element a_δ, $\delta < \alpha \cdot (n+1)$ which satisfies $\phi'(v)$, hence satisfies $\phi(v)$. As a_δ lies in N, we get that $\mathcal{M}_X \models \phi(b)$ for some $b \in N$, and the Tarski-Vaught test thus yields that $\mathcal{N}_X \prec \mathcal{M}_X$.

Finally, we prove by induction that for all $\beta < \alpha \cdot \omega$, the following hold.

(3) For all $\delta_1, \dots, \delta_n < \beta$, the n-tuple $a_{\delta_1}, \dots, a_{\delta_n}$ satisfies an atom in $\mathcal{M}_X$.

That is, every n-tuple of elements in N satisfies an atom in the complete theory of the elementary substructure $\mathcal{N}_X \prec \mathcal{M}_X$, which gives that $\mathcal{N}_X$ is atomic.

(3) is trivially true for the base case $\beta = 0$. Suppose (3) holds for all $\beta < \gamma$. If γ is a limit ordinal, then (3) holds for γ. Let $\gamma = \eta + 1$ be a successor ordinal. If a_η is not in the n-tuple, (3) holds by the induction hypothesis, so we only need to consider $(n+1)$-tuples $a_{\delta_1}, \dots, a_{\delta_n}, a_\eta$. By (1), a_η realizes an atom $\phi(v)$ in the complete theory of $(\mathcal{M}_X, a_\beta)_{\beta<\eta}$. If $c_{\lambda_1}, \dots, c_{\lambda_n}$ are the constant symbols not in X appearing in $\phi(v)$, then $\phi(v)$ can be written as $\phi(v) = \phi(v, c_{\lambda_1}, \dots, c_{\lambda_n})$, where $\phi(v, u_1, \dots, u_m)$ is a formula in the theory of $\mathcal{M}_X$. Also, given the induction hypothesis, the $(n+m)$-tuple $a_{\delta_1}, \dots, a_{\delta_n}, a_{\lambda_1}, \dots, a_{\lambda_n}$ satisfies an atom $\theta(v_1, \dots, v_n, u_1, \dots, u_m)$ in $\mathcal{M}_X$. By $\phi(v)$ being an atom of $(\mathcal{M}_X, a_\beta)_{\beta<\eta}$, in that model one of the following holds for an arbitrary formula $\psi(v)$:

$$(4)\ (\forall v)(\phi(v) \to \psi(v)) \text{ ,or } (\forall v)(\phi(v) \to \neg\psi(v))$$

Given any formula $\psi(v, v_1, \dots, v_n, u_1, \dots, u_m)$ in $\mathcal{V}_X$, $(\forall v)(\phi(v, u_1, \dots, u_m) \to \psi)$ is a formula in the variables $v_1, \dots, v_n, u_1, \dots, u_m$, hence as $\theta(v_1, \dots, v_n, u_1, \dots, u_m)$ is an atom, it follows that either

$$\mathcal{M}_X \models \theta(v_1, \dots, v_n, u_1, \dots, u_m) \to (\forall v)(\phi(v, u_1, \dots, u_m) \to \psi)$$

or

$$\mathcal{M}_X \models \theta(v_1, \dots, v_n, u_1, \dots, u_m) \to \neg(\forall v)(\phi(v, u_1, \dots, u_m) \to \psi)$$

Suppose we are in the second case and assume towards a contradiction that $(\exists v)(\phi(v, u_1, \dots, u_m) \to \psi)$. Let $v = b$ be such an element. By setting $(v_1, \dots, v_n, u_1, \dots, u_n) = (a_{\delta_1}, \dots, a_{\delta_n}, a_{\lambda_1}, \dots, a_{\lambda_n})$, our choice of b yields that

$$(\mathcal{M}_X, a_\beta)_{\beta<\eta} \models \phi(b, c_{\lambda_1}, \dots, c_{\lambda_n}) \to \psi(b, c_{\delta_1}, \dots, c_{\delta_n}, c_{\lambda_1}, \dots, c_{\lambda_n}).$$

Now as $\phi(v, c_{\lambda_1}, \dots, c_{\lambda_n})$ is our atom $\phi(v)$, (4) gives that we must have

$$(\forall v)(\phi(v, c_{\lambda_1}, \dots, c_{\lambda_n}) \to \psi(v, c_{\delta_1}, \dots, c_{\delta_n}, c_{\lambda_1}, \dots, c_{\lambda_n})).$$

But by the choice of θ, $(a_{\delta_1}, \dots, a_{\delta_n}, a_{\lambda_1}, \dots, a_{\lambda_n})$ satisfies θ, and this contradicts that $\mathcal{M}_X \models \theta(v_1, \dots, v_n, u_1, \dots, u_m) \to \neg(\forall v)(\phi(v, u_1, \dots, u_m) \to \psi)$. Hence we can rewrite the second case above as:

$$\mathcal{M}_X \models \theta(v_1, \dots, v_n, u_1, \dots, u_m) \to (\forall v)(\phi(v, u_1, \dots, u_m) \to \neg\psi).$$

Now $\Theta = \theta(v_1, \dots, v_n, u_1, \dots, u_m) \wedge \phi(v, u_1, \dots, u_m)$ is an atom in the theory of $\mathcal{M}_X$, as $\Theta \to \theta$. Note that $\psi(v, v_1, \dots, v_n, u_1, \dots, u_m) = \psi(v, v_1, \dots, v_n)$ is allowed, and for any such ψ, we still have that either $\Theta \to \psi$ or $\Theta \to \neg\psi$. Define $\Theta'(v, v_1, \dots, v_n) = (\exists u_1, \dots, u_m)\Theta(v, v_1, \dots, v_n, u_1, \dots, u_m)$. Thus for all $(v, v_1, \dots, v_n)$ such that $\Theta'(v, v_1, \dots, v_n)$ is true, we can pick $(u_1, \dots, u_m) =$

$(d_1, \ldots, d_m)$ such that $\Theta(v, v_1, \ldots, v_n, d_1, \ldots, d_m)$ is true, hence either $\Theta'(v, v_1, \ldots, v_n) \to \psi(v, v_1, \ldots, v_n)$ or $\Theta'(v, v_1, \ldots, v_n) \to \neg\psi(v, v_1, \ldots, v_n)$, so Θ' is an atom in $\mathcal{M}_X$.

Finally, $a_{\delta_1}, \ldots, a_{\delta_n}, a_\eta$ satisfies Θ', so (3) holds for all $\beta \leq \gamma$, and by induction, it holds for all $\beta < \alpha \cdot \omega$. □

The previous two lemmas immediatly yield the following corollary.

Corollary 28. *Let T be a theory in a countable vocabulary such that T is stable in power ω. Then for any model $\mathcal{M}$ of T and every subset $X \subset M$ the complete theory of $\mathcal{M}_X$ has an atomic model.*

Lemma 29. *Suppose T is a complete ω-stable $\mathcal{V}$-theory and $\mathcal{M}$ is an uncountable model of T. Then there is a proper elementary extension $\mathcal{N} \succ \mathcal{M}$ such that every countable set $\Gamma(v)$ of formulas of $\mathcal{V}_M$ which is realized in $\mathcal{N}_M$ is realized in $\mathcal{M}_M$.*

Proof. We first prove the existence of an atomic model $\mathcal{N}$ which is a proper elementary extension of $\mathcal{M}$.

We need a formula $\psi(v)$ such that:

(1) The set $A = \{b \in M : \mathcal{M}_M \models \psi(b)\}$ is uncountable
(2) For any formula $\phi(v)$ of $\mathcal{V}_M$ exactly one of the sets
$B = \{b \in M : \mathcal{M}_M \models \psi(b) \wedge \phi(b)\}$,
$C = \{b \in M : \mathcal{M}_M \models \psi(b) \wedge \neg\phi(b)\}$
is countable

Suppose that such a formula does not exist. Then for every formula $\psi(v)$ which is satisfied by ucountably many elements in $\mathcal{M}$, (2) does not hold. Thus there is a formula $\phi(v)$ such that both B and C are uncountable. Let $\psi_0 = \psi \wedge \phi$ and $\psi_1 = \psi \wedge \neg\phi$. Then the following holds:

$$\mathcal{M}_M \models \psi_0 \to \psi,\ \mathcal{M}_M \models \psi_1 \to \psi,\ \mathcal{M}_M \models \neg(\psi_0 \wedge \psi_1).$$

Note that both ψ_0 and ψ_1 are satisfied by uncountably many elements, hence we can repeat the argument above to get ψ_{00}, ψ_{01} and so on. As in the proof of Lemma 26, this yields 2^ω different sequences, hence 2^ω different sets realized by different elements, which can be extended to 2^ω different types consistent with $Th(\mathcal{M}_Y)$ for some countable $Y \subset M$. There is furthermore a model $\mathcal{N}_Y$ which realizes all these types, but this contradicts the assumption that T is ω-stable. As there exist formulas which are satisfied by uncountable many elements, for example '$v = v$', there must be a formula $\psi(v)$ satisfying (1) and (2).

Now if c is a new constant symbol, let Δ be the set of all sentences $\phi(c)$ in the vocabulary $\mathcal{V}_M \cup \{c\}$ such that for the formula $\phi(v)$, all but countably many elements which satisfy $\psi(v)$ satisfy $\phi(v)$. Note that given any finite subset $\Delta' \subset \Delta$, the set of all elements which satisfy $\psi(v)$ but not $\phi(v)$ for some $\phi \in \Delta'$ is a union of finitely many countable sets, hence it is a countable set. Thus there are uncountably many elements which satisfy $\psi(v)$ and $\phi(v)$ for every $\phi \in \Delta'$, and we can always find an interpretation of c such that the expansion of $\mathcal{M}_M$ models Δ'. Hence Δ' is consistent, and by compactness, so is Δ.

Note that $Th(\mathcal{M}_M)$ trivially is in Δ, as a sentence γ such that $\mathcal{M}_M \models \gamma$ is trivially satisfied by all elements as a formula $\gamma(v)$, hence the set of all $b \in M$ such that $\mathcal{M}_M \models \psi(b) \wedge \neg\gamma(b)$ is empty, and $\gamma(c) = \gamma$ is in Δ. So $\gamma \in \Delta$ or $\neg\gamma \in \Delta$ for each $\mathcal{V}_M$-sentence γ. Now each sentence γ containing the new constant symbol c can be written as $\gamma = \phi(c)$, where ϕ is a formula of $\mathcal{V}_M$. If C is countable for $\phi(v)$, $\phi(c) \in \Delta$. If B is countable for $\phi(v)$, C is countable for $\neg\phi(v)$ and $\neg\phi(c) \in \Delta$. So by exactly one of B or C being countable for every $\phi(v)$, $\gamma \in \Delta$ or $\neg\gamma \in \Delta$.

As Δ is consistent, the Model existence theorem gives some $\mathcal{V}_M \cup \{c\}$-structure $\mathcal{O}_{M\cup\{c\}}$ which models Δ. As $\gamma \in \Delta$ or $\neg\gamma \in \Delta$, for each $\mathcal{V}_M \cup \{c\}$-sentence γ, Δ is the complete theory of $\mathcal{O}_{M\cup\{c\}}$. Note that $\mathcal{O} \models T$, as $T \subset Th(\mathcal{M}_M) \subset \Delta$, so by Corollary 28, there is an atomic model $\mathcal{N}_{M\cup\{c\}}$ of Δ.

Let $\mathcal{N}$ be the reduct of $\mathcal{N}_{M\cup\{c\}}$ to $\mathcal{V}$. By $Th(\mathcal{M}) \subset \Delta$ we get that $\mathcal{M} \prec \mathcal{N}$. Furthermore, $\mathcal{N}$ is a proper extension of $\mathcal{M}$, as for each $a \in M$, all but countably elements satisfy the formula $\phi(v) = (\neg a \equiv v)$, hence $\phi(c) \in \Delta$ and $\Delta \models \neg c \equiv a$.

Now we show that every countable set $\Gamma(v)$ of formulas which is realized in this $\mathcal{N}_M$ is in fact realized in $\mathcal{M}_M$. Suppose $b \in N$ satisfies $\Gamma(v)$. By $\mathcal{N}_{M\cup\{c\}}$ being atomic, b satisfies an atomic formula $\phi(c, v)$. Now as $\mathcal{N}_{M\cup\{c\}}$ models the sentence $(\exists v)\phi(c, v)$, it is consistent with the complete theory Δ, and must be a consequence of Δ. Also, $\Delta \models \phi(c, v) \to \gamma(v)$ for all $\gamma(v) \in \Gamma(v)$, for given such a formula $\gamma(v)$, that $\phi(c, v)$ is an atom implies that either $\phi(c, v) \to \gamma(v)$ or $\phi(c, v) \to \neg\gamma(v)$ is a consequence of the complete theory Δ, but as b satisfies both $\phi(c, v)$ and $\gamma(v)$, the second implication is impossible.

Furthermore, since every consequence of Δ can be derived in a finite sequence from some finite subset of Δ, and $\Gamma(v)$ is countable, there is a countable $\Delta_0 \subset \Delta$ such that

$$\Delta_0 \models (\exists v)\phi(c, v), \text{ and } \Delta_0 \models \phi(c, v) \to \gamma(v)$$

for all $\gamma(v) \in \Gamma(v)$.

By our construction, each $\delta(v)$ such that $\delta(c) \in \Delta_0$ is satisfied by all but countably many of the uncountably many elements which satisfy $\psi(v)$ in $\mathcal{M}_M$, hence that Δ_0 is countable implies that only countably many elements in M do not satisfy all such $\delta(v)$. So there is a $c_0 \in M$ that satisfy all $\delta(v)$, which gives that $(\mathcal{M}_M, c_0)$ models Δ_0. Thus $\mathcal{M}_M$ models the existence of a $d_0 \in M$ such that $\phi(c_0, d_0)$ holds, hence d_0 satisfies every $\gamma(v) \in \Gamma(v)$, and $\Gamma(v)$ is realized in $\mathcal{M}_M$. □

We finally give the proof of Morley's theorem.

Proof. (Morley's Theorem) Let T be categorical in the uncountable power κ. By Lemma 22 it suffices to show that every model of T of power ω_1 is saturated. Let $\mathcal{M}$ be any model of T of power ω_1. By Lemma 24, T is ω-stable. Thus we can use Lemma 29 κ times, taking unions at limit ordinals, to obtain an elementary extension $\mathcal{N} \succ \mathcal{M}$ of power κ such that every countable set $\Gamma(v)$ of formulas of $\mathcal{V}_M$ which is realized in $\mathcal{N}_M$ is realized in $\mathcal{M}_M$. By Lemma 25, that T is ω-stable and $\kappa \geq \omega_1$ implies that T has a ω_1-saturated model of power κ. As T is κ-categorical, $\mathcal{N}$ must be isomorphic to this model, hence ω_1-saturated. Now, given any subset $Y \subset M$ with $|Y| < \omega_1$ and any set $\Gamma(v)$ of formulas consistent with $Th(\mathcal{M}_Y)$, and hence consistent with $Th(\mathcal{N}_Y)$, $\Gamma(v)$ is realized in $\mathcal{N}_Y$. But as $|\mathcal{V}_Y| \leq \omega$, $|\Gamma(v)|$ is countable, which implies that $\Gamma(v)$ is realized in $\mathcal{M}_Y$, hence $\mathcal{M}$ is ω_1-saturated. $\square$

References

[1] C.C Chang and H.J. Keisler, *Model Theory, Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Company, 2nd edition, 1973.

[2] S. Hedman, *A First Corse in Logic: An Introduction to Model Theory, Proof Theory, Computability, and Complexity*, Oxford University press, 1st edition, 2004.

[3] W. Hodges, *Model Theory*, Cambridge University press, 1st edition, 1993.

[4] M. Morley, Categoricity in power, *Transactions of the American Mathematical Society*, vol. 114, pp.514-538, 1965.