Optimal margin levels in Gaussian environments

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Abstract

The purpose of this project is to provide insight on the clearing process as part of a derivative trade, emphasize the importance of the initial margin requirements in centrally cleared trades and illustrate the effect of different interpretations of the regulatory constraints as conditions which the initial margin should satisfy.
1 Derivatives

What is a derivative?

As the name itself suggests, a derivative is a financial contract whose value is derived from the value of its underlying asset. The underlying assets are usually stocks, bonds, currencies and commodities. The derivatives’ values can also depend on indices, interest and exchange rates and even things like temperature and the amount of liters of rain.

The contract is signed by two parties, usually two financial institutions or a financial institution and its client. Essentially this contract means that the parties agreed that by a certain time (expiration date), certain amount of money will be exchanged. If the underlying is an actual asset, the contract then claims that the asset will be bought/sold for a certain price (strike price). At the time of exercise, the actual underlying asset does not necessarily have to be exchanged. The contract is considered fulfilled even if the buyer is given enough money to buy the asset at the current price. This makes trading derivatives more attractive than trading the asset itself.

The use of derivatives is not just limited to making financial profit. Derivatives can also be used by companies to lower risk in their business. For instance, a production company might benefit from agreeing with its supplier to buy a particular material at some point in the future, e.g. in one year’s time, for a certain price. That way, if the price of the material rises in a year, the production company might still be able to buy the goods for the previously agreed price. Similarly, if the supplier is from another country and expects a payment in another currency, then the production company might benefit from agreeing with a bank on a mutually acceptable exchange rate in a year’s time. That way, if the exchange rate becomes unfavorable, the production company might still be able to exchange the money in the supplier’s currency for the previously agreed exchange rate.

What types of derivatives exist?

The most commonly traded types of derivatives are forwards, futures, options and collateralized debt obligations (CDOs) and swaps. These types will now be briefly introduced. For a more detailed study, see Hull [1].
A forward contract is an obligatory agreement between two parties to buy/sell the underlying asset for the strike price at the expiration date. The forward contracts are mostly traded privately, between two parties.

A futures contract is almost the same, only it is traded between parties that do not necessarily know each other, so the regulatory third party is present. Moreover, the contract is only referred to by its delivery month. It is the regulatory party’s duty to specify the delivery period within this month and the holder of the short position can choose the exact time.

An option, in contrast to forwards and futures, gives its holder the right but not the obligation to sell/buy the underlying by the expiration date for the strike price. A call option gives its holder the buying rights and a put option gives them the selling rights. Based on the time of exercising, the options can be European, meaning that they can be exercised only precisely at the expiration date, and American, meaning that they can be exercised at any point up to (and including) the expiration date.

A CDO combines different types of debt like loans and mortgages into one instrument whose value relies on the promised repayment of the loan. Two types of CDOs are asset-backed commercial paper (corporate debt) and mortgage-backed securities (MBS). MBS had a major role in the global financial crisis of 2008 after the housing market crisis caused their values to drop.

A swap is a contract in which the parties agree to exchange one asset for a similar one. Most common swaps are interest rate swaps, in which one party usually has access to virtually limitless loans at a floating rate, while the other has access to a more stable interest rate. There are also credit default swaps (CDS), which are meant to protect their holders against corporate debt and MBSs. During the crisis, when the MBS market went down, there was no money left to pay off the holders of CDSs.

Moreover, there is a vast number of different, more complex derivatives which are often referred to as exotic. Some of the exotic options have payoffs that depend on the average value of the underlying during a period of time (e.g. Asian options), some depend on maximum values of the underlying during this time, some have exercise prices that depend on time itself and so on.
Who trades the derivatives?

Three groups of traders are usually spotted on the market: hedgers, speculators and arbitrageurs. **Hedgers** use derivatives to protect their investments from the risk of undesired price movements in the market. **Speculators** make predictions on the future market movements and use derivatives to place their bets. **Arbitrageurs** take offsetting positions in two or more instruments or markets to lock in a riskless profit.

Nearly all financial models are grounded on the assumption of no-arbitrage, i.e. the assumption that the riskless profit does not exist. So how come the arbitrageurs exist? First of all, it is not entirely true that the markets are free of arbitrage. Occasionally, arbitrage opportunities can occur, but they do not last very long. Arbitrageurs usually use advanced algorithms and extremely fast internet connection, which allow them to spot and act on these opportunities fast. Moreover, the arbitrage windows are not so wide - in order to make a noteable profit, one would need to invest a large amount of capital. As an example of such arbitrage opportunity, consider two markets that, for some reason, have a different price for the same asset. First the arbitrageurs will buy large amounts of the asset where it is cheaper, thus raising its demand and its price in this market. Then they will sell it on the market where the asset is more expensive, creating a higher supply which will lower the price. Eventually, the two prices will be equivalent. This shows that markets can, in a certain way, self-regulate and that the no-arbitrage is a reasonable assumption after all.

How are derivatives traded?

Trading derivatives, as well as all financial instruments, happens in three phases: **execution**, **clearing** and **settlement**. Execution implies mutual acceptance of the parties to sell/buy an asset in an obligatory transaction. Settlement is the actual exchange of money, or some other value, for the underlying asset. The entire processes following the execution and leading up to settlement is referred to as clearing. This includes actually recording the transaction, i.e. updating the parties’ accounts and preparing for the transfer of funds that will occur at the time of exercise.
Trading can be done on an exchange, such as NASDAQ and Chicago Board of Trade Derivatives, or over-the-counter (privately between the parties). Exchanges can be considered financial centers where the counterparties agree to trade standardized contracts for a specified price. The exchanges enhance liquidity of the market, meaning that they help trades happen more frequently and in large volume by centralizing trading in one place. Another responsibility is to standardize the products that are traded by defining expiration dates, strike prices, delivery locations and so on. The access to the exchanges is only allowed to a limited number of companies and individuals, who have to satisfy strict rules provided by the exchange. Among other things, the minimal amount of capital that an exchange participant should own is defined. This exclusivity helped create the brokerage business. For a fee, brokerage companies offer the services of helping individuals and companies outside the exchange to trade by simply passing on the ordered transactions to a direct participant.

Over-the-counter participants, on the other hand, are not subject to such strict regulations. Instead, they are free to negotiate any mutually acceptable terms of the contract and it is their responsibility of finding a way to enforce honoring of the contract. The lack of constraints makes the over-the-counter market more attractive and, when measured in terms of the total volume traded, about 95% of the derivative contracts are traded on this market (see Amadeo [2]). Of course, there is another side to not having constraints. For a long time, there was virtually no mechanism that would deal with member’s default, i.e. there was no proper regulatory figure set in place that would protect the surviving party in case its counterparty becomes bankrupt or tries to back out from a deal. What is more, this feature of over-the-counter market helped intensify the market crash in 2008. Having learned from this experience, the Dodd-Frank Wall Street Reform Act was created to prevent the repetition of such crisis. Among other requirements, it demanded that all standardized over-the-counter derivatives have to be cleared through a central counterparty.

The next section focuses on different types of clearing, while the section after it studies the risks related to the clearing process and the mechanisms for their neutralization. For further details on Dodd-Frank, see Amadeo [3] or Gregory [4, Chapter 4].
2 Clearing

The process of clearing stretches from the moment two parties agree on the details of the trade to the moment the agreed funds and assets are exchanged. Sometimes this process lasts for only a couple of days, e.g. stock trades are settled in 3 business days, government bonds in 1, but for derivatives this process can be much longer.

The longer the clearing period, the more uncertain is the state of the market at the expiry of the contract. One major risk affecting clearing is the counterparty risk, i.e. the risk that one of the parties will not be able to honor the terms of the agreement. This might be due to the unfavorable market movements to one of the parties and they are trying to back out, or even worse, they have bankrupted and do not have the sufficient funds to act according to the contract. Traditionally, this risk has been managed differently, depending whether the trade was done on an exchange or over-the-counter.

Clearing exchange-traded derivatives

Derivatives exchanges have the responsibility of surveilling the market and making sure that the risk of loss due to default of one or more participants is minimized. The type of clearing that an exchange performs is called the central clearing. It involves the exchange’s clearing house that acts as an intermediary for the exchange-traded derivatives. The clearing house will act as a buyer to the original seller in the trade and then as a seller to the original buyer of the trade. This way, if one of the party defaults, the other one will not be damaged.

Note that this does not fully eliminate the counterparty risk - it centralizes it. This way, the clearing house is exposed to such risk on both sides. Remember that the access to the exchange and its clearing house is only given to a chosen few individuals and companies that managed to satisfy rigorous conditions regarding their capital and creditworthiness. In a way, this can be considered as the first line of defense for the clearing house - it only deals with trusted parties. However, in finance, future is uncertain and the creditworthiness of a participant can easily change. This is why the clearing houses must use additional mechanisms to protect themselves from the exposure to the counterparty risk.
One very frequent regulatory requirement is that the clearing house should use the concept margins to minimize the counterparty exposure. A participant should post margins that will make up for their counterparty’s (clearing house) exposure to risk when dealing with this participant. The counterparty risk and the detailed mechanism of margins will be covered in the next section, but for now it is sufficient to be aware that there exists a concept of the initial margin, which is required of the participant upon the execution of trade and which should provide a cover in case of their default. This margin is posted on a so-called margin account, which is updated daily, according to the movements of the market.

A clearing house expects their participants to post margin on all their trades. Since the access to the exchange and its clearing house is limited to only a few carefully picked companies and individuals, the participants frequently handle not only their own transactions, but the transactions of the brokers that operate through them. It is not rare that some of these transactions offset. If a clearing house uses netting (and most of them do), then the participant will have to post margin only on the netted position, which is another reason for letting other companies to trade through them.

The relationship between a clearing house and its direct participants is a little bit different than the relationship between a direct participant and the traders who trade through them. The direct participants are asked to balance their margins daily. That is not the case for the traders who trade through them. Instead, a broker who helps a trader outside the exchange trade will set a maintenance margin. It is usually set to be around 75% of the initial margin requirement. If the market movements are favorable, the trader will receive money on the margin account from the broker. Any excess can be withdrawn or kept on the account so that it earns interest. If the market movements are unfavorable, but the updated amount on the margining account is above the maintenance margin, the trader does not have to post additional funds. If the amount on the margining account drops below this margin, the trader will receive a margin call and will be expected to fill the account up to the initial margin amount. This amount should be posted in cash and within a certain time period (can be up to 24h). Otherwise the broker will sell out the trader’s positions. Brokers are free to specify the level of margin requirements, but are limited from below, since they have to post the margin to the direct participant for the trades that go through them.
Besides margins, the clearing house participants are also required to post their contributions to the default fund, which should be used in case that one or more participants default. When a participant fails to post margin as instructed, the clearing house proclaims the participant’s default and manages it. During the process of default management, the losses are bound to occur. The order in which the funds are used to cover this loss is called the waterfall. First in line is the participant’s margin account. If that is not sufficient then the participant’s contribution in the default fund is used. After that, the other participants’ contribution to the default fund are used and if that is not sufficient to cover the losses either, the clearing houses equity is used.

**Clearing OTC derivatives**

There are mainly two ways when it comes to clearing the over-the-counter transactions - bilateral clearing and central clearing. Prior to the crash of 2008, about 25% of the over-the-counter transactions have been cleared centrally, while the remaining 75% was clear bilaterally (see Hull [5]). New regulations impose that all standardized trades between financial institutions must be cleared centrally. Some nonstandard transactions are still allowed to be traded bilaterally.

**Bilateral clearing** implies that the counterparties themselves are in charge of enforcing the realization of their contract. There are agreements that can be used as guidelines, such as the ISDA agreement [6] (ISDA stands for International Swaps and Derivatives Association). It defines what happens in case when one side fails to make margin payments, if there are any, as well as what happens if one side is unable to honor the terms of the contract. An annex to this agreement, known as the credit support annex (CSA) defines whether the margins should be used, which securities can be posted as margins, which haircuts will be applied etc. New regulations require bilaterally cleared transactions to be based on the ISDA agreement and that the initial and variation margin are included when the trade is done between two financial institutions. If the margins are included, the margin account has to be held by a third party custodian. The fact that the new regulations insist on having a third party custodian that oversees the bilateral clearing hints that soon probably all OTC derivatives will be cleared centrally.
Central clearing of the OTC derivatives resembles a lot to clearing the exchange-traded derivatives. Here the third party is called central counterparty (CCP) and is almost the same as the exchange’s clearing house. The main difference is in the type of the trades handled - the CCP clears less standard transactions than the exchange’s clearing house and therefore the calculations of the margin are more complicated. When presented with a transaction by the two trading parties, if a CCP accepts to clear it, it steps is an intermediary by entering into offsetting positions with the mentioned parties and becoming their counterparty. The parties are asked to post initial margins. If this is the only transaction either of the parties clear through this CCP, then they will have the same initial margin. More frequently, both of the parties have multiple transactions through the CCP. These transactions will be netted when calculating their margin requirements. Besides margins, just as the clearing houses, the CCPs also require the default fund contributions from their participants. The default waterfall is the same as in the case of clearing through a clearing house.

The ultimate merge

It has been mentioned that the bilateral way of clearing is starting to resemble central clearing. Moreover, there are virtually no differences between central clearing of OTC derivatives and the exchange-traded derivatives. The new regulations are affecting not only the handling of derivatives transactions, but the derivatives themselves. The OTC derivatives are becoming more and more standardized, while the exchange-traded derivatives are starting to increase in variety. It seems that the border between OTC market and exchanges blurs more and more in time. One might even expect that in the future there will no OTC market in the traditional sense and that all derivatives will be cleared centrally.

Critics might say that by replacing bilateral clearing with central one is only replacing ”too-big-to-fail” banks with ”too-big-to-fail” CCPs. Duffie and Zhu [7] argue against this attitude claiming that there might be a way to design the regulations involving CCPs and its participants in such a way so that it is impossible for the CCP to fail. Hull [8] argues that en CCPs are much easier to oversee and regulate. For further details on how the CCPs are supervised, see Rehlon and Nixon [9].
3 Managing Counterparty Risk

When a participant enters the trade with another party, they are facing the risk that the other party might not be able to abide to the terms agreed upon in the contract, mainly due to the market risk and the credit risk. A common name for the concept of the probability of the risk realization and the magnitude of its consequences is often referred to as the exposure (see Culp [10]). In centrally cleared transactions, the CCP is in the middle, as a buyer to the original seller and a seller to the original buyer. This creates exposure for the CCP from both sides.

The market exposure represents the exposure involving market movements that affect the value of the underlying in the trade. If the market movements are unfavorable to the participant, they are suffering a loss, while the CCP gains. Conversely, if the market movements are favorable to the participant, they profit and the CCP is at loss. Note that any profits of the CCP over one side of the original trade are implying losses over the other side. Moreover, note that the one participant of the clearing house is always at loss. If these losses are not carefully monitored, they might, over time, become great losses and pose a threat to the financial eco-system within the CCP.

It seems therefore reasonable that, if the participant has lost money during the trading day, they should post the amount they lost to the CCP. This way of controlling the market exposure is called marking to market. The amount that the participant should post is referred to as the variation margin. The CCP asks the participant to pay their variation margin to their margin account. The CCP will then forward the variation margin to the original profiting side of the transaction. Variation margin is supposed to be paid in cash only. Any excess amount can be withdrawn, or kept in order to earn interest, just as an ordinary bank account. Usually, the CCP will give its participants until the beginning of the next trading day to post the required variation margin. If a participant fails to do so, the CCP will pronounce the default of this participant and will start managing the participant’s default.

The other component of the counterparty exposure, credit exposure, lurks over the default management process. When a participant defaults, they are unable to pay the variation margin. The CCP exists so that the original counterparty does not suffer the consequences, so the CCP pays it.
The CCP is now directly exposed to the market risk, which is a motive to close out the participant’s positions as soon as possible. Most likely, the CCP will auction them to other participants. Because it is in a hurry to sell them, it can happen that the lower prices than what the positions are actually worth are accepted. The loss over selling the participant’s positions is called the replacement cost, since it replaces the defaulting participant as a holder of the positions with the participant(s) that bought them. The period from the moment the defaulted participant paid the variation margin for the last time until all of their positions are closed out is referred to as the liquidation period.

The amount of variation margin that the CCP pays during the liquidation period and the replacement cost are the defaulted participant’s responsibility. It seems fair that the participant should provide funds to cover the CCP’s exposure to these costs. It is, however, impossible for the defaulting participant to do so after declaring bankruptcy. It would be ideal if the participant could pay for the costs of their default in advance. Of course, the loss that the CCP is exposed to during the default management is not known with certainty at the beginning of the trade. However, one can refer to the exposure of this loss as the potential future exposure. An illustration of this uncertainty is shown in the Figure 1.

Figure 1: The future exposure illustration, taken from [11] can represent any situation from one trade to a portfolio of trades.
Principle 4 on credit risk in Principles for financial market infrastructures (PFMI) [12] defines potential future exposure as "the maximum exposure estimated to occur at a future point in time at a high level of statistical confidence". In relation to this, the current exposure is in the same document defined as "the larger of zero or the market value (or replacement cost) of a transaction or portfolio of transactions within a netting set with a counterparty that would be lost upon the default of the counterparty". There are lot of different ways to model and compute the replacement cost, but it not possible to think of all the possible scenarios. A general principal of risk management states that whether the risk should be managed or not depends not only on the impact of risk, but also on its probability (see Culp [10]). That essentially means that maybe not all of the extreme cases of exposure have to be considered. The participant’s default fund contribution can be used to cover unexpected exposure anyway.

As the result of this discussion, it is sensible to observe the potential future exposure with some set level of certainty. It is advised that the CCP should collect the initial margin after the execution of the trade, that should "meet an established single-tailed confidence level of at least 99 percent of the estimated distribution of future exposure" (see Principle 6 on margins in PFMI [12]). In addition, it is suggested that the initial margin is calculated separately for different products, portfolios and markets. The total initial margin requirement for one participant should be the sum of initial margins for all the products they clear.

The initial margin can be posted in cash or in the securities that the party owns. In the latter case, because the securities also depend on the market, i.e. are subject to market risk, they are not taken at their full value. The haircut refers to the percentage by which the current market value of a security is reduced, in order to determine its value for the purposes of margin. For example, short-term government debt securities, such as Treasury-bills receive a 10% haircut, meaning that they are accepted as collateral at 90% of their market value. Shares of a company can be marginable at up to 70% of their value, which means that they receive a 30% haircut etc. These haircuts are not specified by the PFMI [12] and are instead the CCP’s responsibility.
There is a variety of methodologies that are used to determine the initial margins. The most common are Value at Risk (VAR), which is described in detail in Hull [1, Chapter 16], and Standard Portfolio Analysis of Risk (SPAN), explained in [13], but some CCPs use more exotic methodologies for some products groups - e.g. NASDAQ uses the principal component analysis (PCA) for some of the instruments they clear ([14]). For further reading on comparison of the initial margin methodologies, see Khwaja [15].

It seems that CCPs can get competitive over who has the margin methodology that provides the most attractive (lowest) initial margin. A concern arose that this might even turn in to a ‘race to the bottom’ (see [16]). However, the CCPs themselves are subject to strict regulations. ESMA (ESMA stands for European Securities and Markets Authority) provides a list of all the CCPs that are allowed to operate in the European Union and the corresponding regulatory bodies that oversee them (see [17]).
4 Defining the problem

Consider two financial institutions that want to trade a certain derivative. The trade is to be cleared centrally, so they choose a CCP and present it with their plan to trade. The CCP should determine the initial margin requirement for both parties. For simplicity, assume that this trade is the only one the two parties clear through this CCP, so their initial margin is the same.

The international standard for the CCPs, PFMI [12], states that the initial margin should "meet the established confidence level of at least 99% w.r.t. the estimated distribution of future exposure". Gregory [4, Chapter 6] explains that the initial margin should "based on an extreme but plausible move in the underlying portfolio value", while Heller and Vause [18] describe that the initial margin should "cover 99% of possible valuation changes over the liquidation period".

It seems that there are different interpretations of what was written in the PFMI, suggesting that the original formulation might be a bit vague. Moreover, notice how both Heller and Vause’s and Gregory’s interpretation focus only on the moves/changes in the valuation process. They do not mention the replacement cost, i.e. the eventual loss that would come from replacing the defaulted party as the holder of the positions with the buyer.

Let the "conditions that the initial margin is supposed to satisfy" be referred to as the acceptable margin conditions. Reading just the statement Heller made, (at least) two acceptable margin conditions come to mind:

- the probability acceptable margin condition suggests that the initial margin should, at each time point throughout the life of the trade, cover the valuation change over the liquidation process with probability of 0.99;

- the time acceptable margin condition suggests that the initial margin should cover the valuation change over the liquidation process at 99% of the time points throughout the life of the trade.

Because of the simplicity of not having to model the replacement costs, it is attractive to set a hypothetical trade valuation model and observe the eventual differences in the initial margins obtained by these two different acceptable margin conditions.
**Acceptable initial margins**

Let the following notation be introduced:

\[ T \quad \text{time to the expiry of the derivative} \]
\[ S(t) \quad \text{the value of the derivative at } t \in [0, T] \]
\[ \delta \quad \text{the liquidation period} \]
\[ a \quad \text{constant initial margin} \]

The process of valuation change over the liquidation period \( \Delta S^\delta \) can be defined as:

\[
\Delta S^\delta(t) = \begin{cases} 
S(t) - S(t - \delta), & \text{if } t - \delta > 0 \\
S(t) - S(0), & \text{otherwise} 
\end{cases}
\]  

(1)

The probability acceptable margin condition can be written as:

\[ P \left\{ |\Delta S^\delta(t)| \leq a \right\} \geq 0.99, \quad \forall t \in [0, T]. \]  

(2)

The time acceptable margin condition is not so easy to formulate without introducing some new concepts first. Let the set of time points in which the valuation change process is covered by the margin \( a \) be denoted by:

\[ B_a = \left\{ t \in [0, T] \left| |\Delta S^\delta(t)| \leq a \right. \right\}. \]

It is said that the set \( B_a \) should be larger or equal to 99% of the trade’s lifetime, so a measure of the set \( B_a \) is needed. If e.g. Lebesgue measure \( \lambda \) is used, then the following holds (see e.g. Nelson [19, Chapter 1]):

\[ \lambda ([0, T]) = T \]  

(3)

\[ \emptyset \subseteq B_a \subseteq [0, T] \implies 0 \leq \lambda(B_a) \leq \lambda ([0, T]). \]  

(4)
The inequality (4) can be transformed into:

\[ 0 \leq \frac{\lambda(B_a)}{T} \leq 1, \tag{5} \]

for the convenience of observing the covered time ratio.

The set \( B_a \) depends on \( a \) and a particular trajectory of the price change process \( \Delta S^d \). The measure of this set is a random variable and whether the ratio given in (5) is larger than 99% can only be described in terms of probability. If the e.g. 95%-confidence interval is used, then it is equivalent to:

\[ P \left\{ \frac{\lambda(B_a)}{T} \geq 0.99 \right\} \geq 0.95. \]

To avoid complications of adding the confidence level as yet another parameter to the model, one could use the expected value:

\[ E \left[ \frac{\lambda(B_a)}{T} \right] \geq 0.99. \tag{6} \]

Recall that \( \lambda(B_a) \) is the Lebesgue measure of the set \( B_a \) and so:

\[ \lambda(B_a) = \int_0^T \mathbf{1}_{\{t \in B_a\}}(t) \, dt. \]

Applying the expected value of this equation yields the following:

\[
E[\lambda(B_a)] = E \left[ \int_0^T \mathbf{1}_{\{t \in B_a\}}(t) \, dt \right] \\
= \int_0^T E \left[ \mathbf{1}_{\{t \in B_a\}}(t) \right] \, dt \\
= \int_0^T P \{ t \in B_a \} \, dt,
\]
where the transition of expectation through the integral sign was allowed by Fubini’s theorem (see Geiss [20, Chapter 3]).

This gives:

$$
E[\lambda(B_a)] = \int_0^T P \left\{ |\Delta S^\delta(t)| \leq a \right\} dt.
$$

(7)

Dividing the equation (7) by $T$ gives:

$$
E \left[ \frac{\lambda(B_a)}{T} \right] = \frac{1}{T} \int_0^T P \left\{ |\Delta S^\delta(t)| \leq a \right\} dt.
$$

(8)

Finally, applying the condition (6) to the equation (8) gives the time acceptable margin condition:

$$
\frac{1}{T} \int_0^T P \left\{ |\Delta S^\delta(t)| \leq a \right\} dt \geq 0.99.
$$

(9)

The equation (9) is equivalent to the following equation:

$$
\int_0^T P \left\{ |\Delta S^\delta(t)| \leq a \right\} dt \geq 0.99T,
$$

(10)

so the equations (9) and (10) will be used as time acceptable margin condition interchangeably.

Notice that in both equations (2) and (9), the probability of coverage:

$$
P \left\{ |\Delta S^\delta(t)| \leq a \right\}, \quad t \in [0, T]
$$

(11)

appears. In terms of this probability, the condition (2) states that for every $t \in [0, T]$ the probability of coverage (11) must be larger or equal to 0.99, while the condition (9) states that the integral over the lifetime of the trade $[0, T]$ w.r.t. the probability of coverage (11) must be larger or equal to 0.99.
Optimal margins

Assume that the chosen CCP is very competitive and that it would like to take the minimal initial margin that satisfies the acceptable margin condition. Let the minimal initial margin further be referred to as the **optimal margin**.

The **probability-wise optimal margin** \( \hat{a} \) is therefore the minimal initial margin that satisfies the probability acceptable margin condition (2). Similarly, the **time-wise optimal margin** \( a^* \) is the minimal initial margin that satisfies the time acceptable margin condition (9) (and equivalently, the condition (10)).

Recall that the probability acceptable margin condition (2) states that the probability of coverage (11) should be larger or equal to 0.99 for every \( t \in [0, T] \) and that the time acceptable margin condition (9) states that the integral over the lifetime of the trade \([0, T]\) w.r.t. the probability of coverage (11) must be larger or equal to 0.99.

Before determining the probability- and time-wise optimal margins, it seems useful to determine the minimal initial margin for which the probability of coverage (11) is larger or equal to 0.99. Since the probability of coverage (11) is the cumulative distribution function (CDF) of the random variable \(|\Delta S^\delta(t)|\) for a given \( t \in [0, T] \), it is an increasing function w.r.t. the argument \( a \). In other words, the higher the margin, the higher the probability of coverage and vice versa. This also means that, for a given \( t \in [0, T] \), the minimal initial margin is the initial margin for which the probability of coverage (11) is exactly 0.99.

Given that, in general, the random variable \(|\Delta S^\delta(t)|\) can have a different distribution depending on the time \( t \in [0, T] \), the above described minimal initial margin is a function of time. Let it be onwards referred to as the **minimal initial margin function**:\[ P\left\{ |\Delta S^\delta(t)| \leq a_{\min}(t) \right\} = 0.99, \quad t \in [0, T]. \] (12)
The probability-wise optimal margin can then be defined as:

\[ \hat{a} = \max_{t \in [0,T]} a_{\min}(t), \quad (13) \]

where \( a_{\min}(t) \) is defined by (12) for a given \( t \in [0,T] \).

To understand why, define the time point \( \hat{t} \) in the following way:

\[ \hat{t} = \arg \max_{t \in [0,T]} a_{\min}(t). \]

The probability-wise optimal margin \( \hat{a} \) given by (13) covers the valuation changes over the liquidation period at \( \hat{t} \) with probability of exactly 0.99 and at the rest of the \( t \in [0,T] \) this probability would be larger that 0.99. Therefore it is the minimal initial margin that satisfies the condition (2), i.e. the equation (13) indeed gives the probability-wise optimal margin.

As for the time-wise optimal margin, notice that the expected covered time ratio given by (8) increases with the increase of the margin level \( a \) - the higher the margin level, the higher the expected covered time ratio and vice versa. Therefore, the minimal margin that satisfies the condition (9) is the one that makes the expected covered time ratio (8) precisely equal to 0.99.

Formally, the time-wise optimal margin \( a^* \) is defined by:

\[ \frac{1}{T} \int_0^T P \left\{ |\Delta S^\delta(t)| \leq a^* \right\} \, dt = 0.99. \quad (14) \]

The main question that this project is set to answer is whether the probability-wise optimal margin \( \hat{a} \), defined given (13), and the time-wise optimal margin \( a^* \), defined given (14) are one and the same.

From the definitions of these optimal margins it is obvious that the probability of coverage (11), i.e. the distribution of the valuation change over the liquidation process \( \Delta S^\delta(t) \) plays an important role. Therefore the model for the valuation of the derivative and the model for its valuation change over the liquidation period will be set in the next section.
5 Valuation model

The valuation model for the derivative is assumed to have the following dynamics:

\[
\begin{cases}
    dS(t) = \sigma(t) \, dW(t) \\
    S(0) = s_0,
\end{cases}
\]

where \( \sigma(t) \) denotes the volatility function of the valuation process and \( W(t) \) denotes the Wiener process. Moreover, \( \sigma \) is assumed to be deterministic, \( t \in [0,T] \) and \( s_0 \in \mathbb{R} \). According to the proposition regarding the existence and uniqueness of an SDE (see e.g. Björk [21, Chapter 5]), the process \( S \) that solves the SDE given above exists, is unique and is a Markov process.

The integral form of this process is as follows:

\[
S(t) = s_0 + \int_0^t \sigma(\tau) \, dW(\tau), \quad t \in [0,T].
\]  (15)

The following proposition provides the distribution of \( S(t) \).

**Proposition 5.1.** Let \( \sigma \) be a deterministic, continuous function on \([0,t]\). If the stochastic integral \( \int_0^t \sigma(\tau) \, dW(\tau) \) is well defined, then it follows the \( \mathcal{N}\left(0, \int_0^t \sigma^2(\tau) \, d\tau\right) \) distribution.

**Proof.**

Observe the following sequence of partitions of time interval \([0,t]\):

for any \( n \in \mathbb{N} \): \( 0 = t_0^{[n]} < t_1^{[n]} < \ldots < t_n^{[n]} = t \), where \( t_i^{[n]} := t_n^i \).

Note that the mesh goes to zero as \( n \to \infty \).
Now the given stochastic integral can be defined as:

$$\int_0^t \sigma(\tau) \, dW(\tau) = \lim_{n \to \infty} \sum_{i=0}^{n-1} \sigma(t_{i+1}^{[n]}) \cdot \left[ W(t_{i+1}^{[n]}) - W(t_i^{[n]}) \right].$$

Notice that for each $n \in \mathbb{N}$ the sum in the limit is a Gaussian random variable, as the sum of increments of the Wiener process. The probabilistic limit of these sums, i.e. the given stochastic integral, is therefore also a Gaussian random variable.

The mean and the variance of this distribution are given by Ito Isometry (see Björk [21, Chapter 4]):

$$\mathbb{E} \left[ \int_0^t \sigma(\tau) \, dW(\tau) \right] = 0$$

and

$$\mathbb{E} \left[ \left( \int_0^t \sigma(\tau) \, dW(\tau) \right)^2 \right] = \int_0^t \mathbb{E} [\sigma^2(\tau)] \, d\tau = \int_0^t \sigma^2(\tau) \, d\tau.$$ 

It follows from the above that the process $S$ follows:

$$S(t) \sim \mathcal{N} \left( s_0, \int_0^t \sigma^2(\tau) \, d\tau \right).$$ \hspace{1cm} (16)

It is not hard to see that the process (15) has independent, normally distributed increments. The only thing preventing this process to be a Wiener process is the fact that it doesn’t start from 0. However, in e.g. Øksendal [22, Chapter 2]), the definition for a Wiener process does not insist on this condition as long as the conditions regarding continuity and increments are fulfilled.
Moreover:
\[
E[S(t)|\mathcal{F}_s] = E[S(t) - S(s) + S(s)|\mathcal{F}_s] \\
= E[S(t) - S(s)|\mathcal{F}_s] + E[S(s)|\mathcal{F}_s] \\
= E[S(t) - S(s)] + S(s),
\]
since \(S(t) - S(s)\) is independent of the observations up to the time point \(s\) and since \(S(s)\) is known if the observations up to the time point \(s\) are known. Because \(S(t)\) and \(S(s)\) have the same mean \(s_0\), the expectation on the right-hand side of the last equation will be equal to 0 and:
\[
E[S(t)|\mathcal{F}_s] = S(s),
\]
making the process \(S\) a martingale.

**Valuation change**

In previous section it was written that the valuation change process should be defined as given in (1). Under the assumed valuation model, the process of valuation change over the liquidation period is defined as follows:

\[
\Delta S^\delta(t) = \int_{\max\{0, t-\delta\}}^{t} \sigma(\tau) \, dW(\tau)
\] (17)

From (16) and (1), it can be shown that the distribution of the valuation change (over the liquidation period) process is:

\[
\Delta S^\delta(t) \sim \mathcal{N} \left( 0, \int_{\max\{0, t-\delta\}}^{t} \sigma^2(\tau) \, d\tau \right). \quad (18)
\]

Apart from being normally distributed, the process (17) is not particularly easy to work with. It does not inherit the nice properties of the price process (15). It obviously does not satisfy the Markovian property, since it generally does not depend solely on the previous observation.
It is also not a martingale, since:

\[ E[\Delta S^\delta(t) | F_s] = E[\Delta S^\delta(t) - \Delta S^\delta(s) + \Delta S^\delta(s) | F_s] \]
\[ = E[\Delta S^\delta(t) - \Delta S^\delta(s) | F_s] + E[\Delta S^\delta(s) | F_s] \]
\[ = E[S(t) - S(t - \delta) - S(s) + S(s - \delta) | F_s] + \Delta S^\delta(s) \]
\[ = E[S(t) - S(s) | F_s] - E[S(t - \delta) - S(s - \delta) | F_s] + \Delta S^\delta(s) \]

If one would take, e.g. \( 0 < s < t - \delta \), and note that \( S(t) - S(s) \) is independent of the observations \( F_s \) then:

\[ E[\Delta S^\delta(t) | F_s] = E[S(t) - S(s)] - E[S(t - \delta) - S(s) + S(s - \delta) | F_s] + \Delta S^\delta(s) \]
\[ = 0 - E[S(t - \delta) - S(s) | F_s] - E[S(s) - S(s - \delta) | F_s] + \Delta S^\delta(s) \]
\[ = -E[S(t - \delta) - S(s)] - (S(s) - S(s - \delta)) + \Delta S^\delta(s) \]
\[ = 0 - \Delta S^\delta(s) + \Delta S^\delta(s) \]
\[ = 0 \neq \Delta S^\delta(s). \]

Moreover, this process does not have independent increments either. For instance, if one would take any \( 0 \leq t_1 < t_2 \) and then define \( t_3 := t_2 + \frac{1}{2} \delta \) and take any \( t_4 > t_3 \), the time intervals \([t_1, t_2]\) and \([t_3, t_4]\) do not intersect. However, the increments \( \Delta S^\delta(t_2) - \Delta S^\delta(t_1) \) and \( \Delta S^\delta(t_4) - \Delta S^\delta(t_3) \) cover the time intervals \([\max\{0, t_1 - \delta\}, t_2]\) and \([\max\{0, t_3 - \delta\}, t_4]\) = \([\max\{0, t_2 - \frac{1}{2} \delta\}, t_4]\), which do overlap. Therefore, these increments do not represent independent variables.

Good news, however, is that it is sufficient that this process is normally distributed. Normal distribution is completely defined by its mean and variance. Notice from (18) that the variance depends on the function \( \sigma \). If some assumptions regarding this function are made, then the distribution will be entirely known and certain calculations regarding this process, such as calculating the probability of coverage (11), will be possible.
6 Constant volatility model

Let the volatility function $\sigma$ of the process (15) be constant throughout the lifetime of the derivative, i.e. let:

$$\sigma(t) = \sigma > 0, \quad \forall t \in [0, T].$$  \hspace{1cm} (19)

Under the assumption (19) the distribution of the valuation change over the liquidation period process (18) becomes:

$$\Delta S^\delta(t) \sim N \left( 0, \sigma^2 \cdot \int_{\max\{0,t-\delta\}}^{t} d\tau \right),$$ \hspace{1cm} (20)

which results in:

$$\Delta S^\delta(t) \sim \begin{cases} 
N(0, \sigma^2 t), & t \leq \delta \\
N(0, \sigma^2 \delta), & t > \delta 
\end{cases}$$ \hspace{1cm} (21)

As seen in the section 4, the probability of coverage (11) is of great importance. Under the assumed distribution (21), this probability is equal to:

$$P \left\{ \left| \Delta S^\delta(t) \right| \leq x \right\} = P \left\{ -x \leq \Delta S^\delta(t) \leq x \right\}$$ \hspace{1cm} (22)

$$= P \left\{ -\frac{x}{\sigma \sqrt{t}} \leq \frac{\Delta S^\delta(t)}{\sigma \sqrt{t}} \leq \frac{x}{\sigma \sqrt{t}} \right\}$$ \hspace{1cm} (23)

$$= P \left\{ -\frac{x}{\sigma \sqrt{t}} \leq Z \leq \frac{x}{\sigma \sqrt{t}} \right\}, \quad Z \sim N(0,1)$$ \hspace{1cm} (24)

$$= P \left\{ Z \leq \frac{x}{\sigma \sqrt{t}} \right\} - P \left\{ Z \leq -\frac{x}{\sigma \sqrt{t}} \right\}$$ \hspace{1cm} (25)

$$= \Phi \left( \frac{x}{\sigma \sqrt{t}} \right) - \Phi \left( -\frac{x}{\sigma \sqrt{t}} \right)$$ \hspace{1cm} (26)

$$= \Phi \left( \frac{x}{\sigma \sqrt{t}} \right) - \left( 1 - \Phi \left( \frac{x}{\sigma \sqrt{t}} \right) \right)$$ \hspace{1cm} (27)

$$= 2\Phi \left( \frac{x}{\sigma \sqrt{t}} \right) - 1, \quad t \leq \delta,$$ \hspace{1cm} (28)
and:

\[ P \left\{ |\Delta S^\delta(t)| \leq x \right\} = 2\Phi \left( \frac{x}{\sigma \sqrt{\delta}} \right) - 1, \quad t > \delta, \tag{29} \]

where \( \Phi \) denotes the cumulative distribution function of the standard \( \mathcal{N}(0, 1) \) distribution.

Transforming (28) and (29), the minimal initial margin function will now be defined by:

\[ \Phi \left( \frac{a_{\text{min}}(t)}{\sigma \sqrt{\min\{t, \delta\}}} \right) = 0.995, \quad t \in [0, T]. \tag{30} \]

Define the following function:

\[ h(x) = \Phi(x) - 0.995, \quad x \in \mathbb{R}. \tag{31} \]

Figure 2: Function \( h(x) \) on the interval \([0, 5]\).
The cumulative distribution function $\Phi(x)$ is an increasing function w.r.t. $x$, so this holds true for the function $h$ as well. Moreover, notice that for $x = 0$ the function $h$ is negative, while for $x = 5$ it is positive (see Figure 2).

Recall the Intermediate Value theorem (see Adams [23], proof can be found in e.g. Rudin [24, Chapter 4]). The implication of this theorem is that there exists a zero of the function $h$ on the interval $[0, 5]$. Moreover, because the function $h$ is strictly increasing, this zero will be unique. Applied to the problem at hand, this means that for every $t \in [0, T]$ the fraction:

$$\frac{a_{\text{min}}(t)}{\sigma \sqrt{\min\{t, \delta\}}}$$

for which the condition (30) is satisfied is unique on $[0, 5]$. Therefore, the **minimal initial margin function** can be analytically expressed as:

$$a_{\text{min}}(t) = \sigma \cdot \sqrt{\min\{t, \delta\}} \cdot \Phi^{-1}(0.995), \quad t \in [0, T]. \quad (32)$$

![The minimal initial margin function $a_{\text{min}}(t)$ against a sample trajectory of the valuation change process](image)

Figure 3: The minimal initial margin function $a_{\text{min}}(t)$ assuming the constant volatility valuation model.
The Figure 3 shows an illustration of the minimal initial margin function $a_{\min}(t)$ defined by (32), under the assumption that the valuation change over the liquidation period is distributed according to (21).

To simulate a trajectory of the $\Delta S^\delta(t)$ process the following parameters were used: $\sigma = 0.3$, $\delta = \frac{5}{365}$, $T = 1$ and $\Delta t = \frac{1}{365}$. The last parameter denotes the time step that was used to discretize the given interval $[0, T]$, due to the computational inabilities to simulate time continuously. The idea to take the liquidation period $\delta$ to be 5 days was inspired by Gregory [4, Chapter 7].

![The minimal initial margin $a_{\min}(t)$ against a sample trajectory of the valuation change process](image)

Figure 4: Zoomed part of Figure 3 which shows $a_{\min}(t)$ for $t \leq \delta$ in the constant volatility valuation model.

The Figure 4 represents the zoomed part of the Figure 3 that captures the behavior of the minimal initial margin function for $t_i \leq \delta$, where the time steps are defined as $t_i = \frac{T}{\Delta t}$. It is important to notice that the time step $\Delta t$ is somewhat large, so there are only 6 time points $t_i$ that are smaller than $\delta$ and because of this, the minimal initial margin function $a_{\min}(t)$ does not look very smooth. In reality, however, the function $a_{\min}(t)$, defined as in (32) is continuous.
The **probability-wise optimal margin**, defined by (13), is in this case:

\[
\hat{a} = \sigma \cdot \sqrt{\delta} \cdot \Phi^{-1}(0.995).
\] (33)

The performance of the probability-wise optimal margin \(\hat{a}\), under the assumption that the process \(\Delta S^\delta(t)\) is distributed accord to (21), against the same trajectory shown in Figure 3 is now shown in Figure 5.

![The optimal margin against a sample trajectory of the valuation change process](image)

**Figure 5:** An illustration of performance of the probability-wise optimal margin \(\hat{a}\) under the constant volatility valuation model.
When determining the time-wise optimal margin $a^*$ consider the following:

$$E[\lambda(B_a)] = \int_0^T P\{|\Delta S^\delta(t)| \leq a\} \, dt$$

$$= \int_0^\delta P\{|\Delta S^\delta(t)| \leq a\} \, dt + \int_{\delta}^T P\{|\Delta S^\delta(t)| \leq a\} \, dt$$

$$= \int_0^\delta \left(2\Phi\left(\frac{a}{\sigma\sqrt{t}}\right) - 1\right) \, dt + \int_{\delta}^T \left(2\Phi\left(\frac{a}{\sigma\sqrt{\delta}}\right) - 1\right) \, dt$$

$$= 2\int_0^\delta \Phi\left(\frac{a}{\sigma\sqrt{t}}\right) \, dt - \delta + 2\int_{\delta}^T \Phi\left(\frac{a}{\sigma\sqrt{\delta}}\right) \, dt - (T - \delta)$$

$$= 2\int_0^\delta \Phi\left(\frac{a}{\sigma\sqrt{t}}\right) \, dt + 2\Phi\left(\frac{a}{\sigma\sqrt{\delta}}\right) (T - \delta) - T,$$

where the third equation in line holds because of the derivations (22)-(28) and (29) for the probability of coverage, while the last equation in line holds because the function in the second integral on the right-hand side did not depend on $t$.

However, the problem is that the first integral on the right-hand side:

$$\int_0^\delta \Phi\left(\frac{a}{\sigma\sqrt{t}}\right) \, dt \tag{34}$$

is hard (if not impossible) to calculate analytically. Using the change of variables e.g. $y = \frac{a}{\sigma\sqrt{t}}$, this integral transforms into the integral of the type:

$$2\frac{a^2}{\sigma^2} \int_{\frac{a}{\sigma\sqrt{\delta}}}^{+\infty} \frac{1}{y} \Phi(y) \, dy,$$

but even this integral cannot be found in e.g. Owen’s tables (see [25]) and the integration by parts is not of much help either.
Because of this, the \textbf{time-wise optimal margin} \( a^* \), which satisfies the condition:

\[
\frac{1}{T} \left( 2 \int_0^\delta \Phi \left( \frac{a^*}{\sigma \sqrt{t}} \right) \, dt + 2 \Phi \left( \frac{a^*}{\sigma \sqrt{\delta}} \right) (T - \delta) - T \right) = 0.99
\] (35)

cannot be analytically expressed.

Notice that if:

\[
\int_0^\delta \Phi \left( \frac{a^*}{\sigma \sqrt{t}} \right) \, dt = \delta \cdot \Phi \left( \frac{a^*}{\sigma \sqrt{\delta}} \right)
\] (36)

then the condition (35) turns to:

\[
2 \Phi \left( \frac{a^*}{\sigma \sqrt{\delta}} \right) - 1 = 0.99,
\]

making the time-wise optimal margin \( a^* \) equal to the probability-wise optimal margin \( \hat{a} \) given by (33).

However, since the integral (34) is unknown, there is no way to tell if (36) is true and whether \( \hat{a} = a^* \) for sure. On the other hand, the integral (34) is small, given that \( \delta \) is substantially smaller than \( T \). This is why it is expected that \( \hat{a} \) and \( a^* \) are very close for the constant volatility valuation model.

The integral (34) can still be estimated numerically. The idea is to:

\begin{itemize}
  \item cover the interval \([0, \delta]\) with a large number \( K \) of equidistant points \( 0 = \tau_0 < \tau_1 < \ldots < \tau_K = \delta \) where \( \tau_j = j \cdot \frac{\delta}{K} \);
  \item calculate the cumulative distribution function \( \Phi \) in points \( \frac{a^*}{\sigma \sqrt{\tau_j}} \), for each \( j = 0, 1, \ldots, K \);
  \item sum of the calculated CDF values; and
  \item multiply the sum by \( \frac{\delta}{K} \).
\end{itemize}
Mathematically formulated, the integral (34) can be estimated by the expression:

\[ \hat{I}(a, \sigma, \delta, K) := \frac{\delta}{K} \sum_{j=0}^{K} \Phi \left( \frac{a}{\sigma \sqrt{T_j}} \right), \quad (37) \]

thus making the estimate for the time-wise optimal margin \( a_{est}^* \) defined by:

\[ \frac{1}{T} \left( 2\hat{I}(a_{est}^*, \sigma, \delta, K) + 2\Phi \left( \frac{a_{est}^*}{\sigma \sqrt{\delta}} \right) (T - \delta) - T \right) = 0.99. \quad (38) \]

Under the assumption that \( \Delta S^\delta(t) \) is distributed according to (21), the comparison of performance of the estimated time-wise optimal margin \( a_{est}^* \) and the probability-wise optimal margin \( \hat{a} \) against the same sample trajectory used in Figure 5, is shown in Figure 6.

Figure 6: An illustration comparing the probability-wise optimal margin \( \hat{a} \) and the estimated time-wise optimal margin \( a_{est}^* \) in the constant volatility valuation model.
The estimated time-wise optimal margin $a_{est}^*$ was computed in Matlab, using its built-in function `fzero` applied to the optimization function $g$ defined in the following way:

$$g(a) = \hat{E}\left[ \frac{\lambda(B_a)}{T} \right](a) - 0.99,$$  \hspace{1cm} (39)

along with the initial guess set to the maximum value of the simulated trajectory. The estimated expected covered time ratio was calculated as:

$$\hat{E}\left[ \frac{\lambda(B_a)}{T} \right](a) = \frac{1}{T} \left( 2\hat{I}(a, \sigma, \delta, K) + 2\Phi\left( \frac{a}{\sigma\sqrt{\delta}} \right) (T - \delta) \right),$$

where the $\hat{I}(a, \sigma, \delta, K)$ is given by (37) and $K = 10000$. 
7 Time dependent volatility model

When dealing with derivatives with an expiration date, practice has shown their value very often becomes more volatile as the expiration date approaches. To incorporate this behavior in the model one would define the volatility of the price process \( \sigma \) as an increasing function of time. For simplicity reasons, assume that \( \sigma \) is such linear function:

\[
\sigma(t) = \alpha t + \beta, \quad \forall t \in [0, T],
\]

where because of the assumed monotonicity, the coefficients are positive, i.e. \( \alpha > 0 \) and \( \beta > 0 \).

As long as the function \( \sigma \) is deterministic and Lipschitz continuous, Proposition 5.1 holds and the valuation change process continues to have the distribution given in (18). Under the assumption (40) the variance at time points will be:

\[
\text{Var}[\Delta S^\delta(t)] = \int_0^t (\alpha \tau + \beta)^2 \, d\tau = \frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t, \quad t \leq \delta
\]

and for time points \( t > \delta \):

\[
\text{Var}[\Delta S^\delta(t)] = \int_{t-\delta}^t (\alpha \tau + \beta)^2 \, d\tau
\]

\[
= \frac{\alpha^2}{3} \left( t^3 - (t - \delta)^3 \right) + \alpha \beta \left( t^2 - (t - \delta)^2 \right) + \beta^2 \left( t - (t - \delta) \right).
\]

The latter expression can be simplified to:

\[
\text{Var}[\Delta S^\delta(t)] = \frac{\alpha^2}{3} \delta \left( 3t^2 - 3t\delta + \delta^2 \right) + \alpha \beta \delta \left( 2t - \delta \right) + \beta^2 \delta, \quad t > \delta.
\]
Under the assumed distribution given by (18), with variance being (41) and (42), the probability of coverage (11) equals to:

\[ P \left\{ |\Delta S^\delta(t)| \leq a \right\} = 2\Phi \left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t}} \right) - 1, \quad t \leq \delta, \quad (43) \]

or:

\[ 2\Phi \left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta}} \right) - 1, \quad t > \delta. \quad (44) \]

Given (43) and (44), the minimal initial margin function is equivalent to:

\[ \Phi \left( \frac{a_{\text{min}}(t)}{\sqrt{\frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t}} \right) = 0.995, \quad t \leq \delta, \quad (45) \]

and:

\[ \Phi \left( \frac{a_{\text{min}}(t)}{\sqrt{\frac{\alpha^2 t^3}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta}} \right) = 0.995, \quad t > \delta. \quad (46) \]

The discussion about the zero of the function \( h \) given by (31) still holds, so the arguments of the function \( \Phi(x) \) that give the equations (45) and (46) do exist and are unique on \([0, 5]\). Moreover, the minimal initial margin function under the model (18), with variance being (41) and (42), can be defined as:

\[ a_{\text{min}}(t) = \begin{cases} \sqrt{\frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t} \cdot \Phi^{-1}(0.995), & t \leq \delta \\ \sqrt{\frac{\alpha^2 t^3}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta} \cdot \Phi^{-1}(0.995), & t > \delta \end{cases} \quad (47) \]
Figure 7 shows an illustration of the minimal initial margin function \( a_{\text{min}}(t) \), under the model (18) and the assumption that the volatility is of the form (40). Note how, because of the assumption (40), the trajectory becomes more volatile in time.

Figure 7: The minimal initial margin function \( a_{\text{min}}(t) \) for the time dependent volatility valuation model.

To simulate a trajectory of the process \( \Delta S^{\delta}(t) \) the following parameters were used: \( \alpha = 2, \beta = 0.8, \delta = \frac{5}{365}, T = 1 \) and \( \Delta t = \frac{1}{365} \). The variance was calculated using the formula (40) and the given parameters \( \alpha \) and \( \beta \).
Figure 8 enlarges the behavior of the minimal initial margin function for $t \leq \delta$ under the assumption that the process $\Delta S^\delta(t)$ is distributed according to (18), with variance being (41) for $t \leq \delta$ and (42) for $t > \delta$.

Figure 8: Zoomed in part of the Figure 7 that shows the behavior of $a_{min}(t)$ for $t \leq \delta$ in the time dependent volatility valuation model.
The probability-wise optimal margin $\hat{a}$, according to (13), should be the maximum value of the minimal initial margin function given by (47). From Figure 7 it can be clearly seen that the minimal initial margin function $a_{\text{min}}(t)$ is increasing with time.

Therefore, the **probability-wise optimal margin** $\hat{a}$ is equal to:

$$\hat{a} = \sqrt{\frac{\alpha^2}{3} \delta (3T^2 - 3T \delta + \delta^2) + \alpha \beta \delta (2T - \delta) + \beta^2 \delta \cdot \Phi^{-1}(0.995)}.$$  

Figure 9 illustrates the performance of the optimal margin $\hat{a}$ against the same trajectory shown in Figure 7, under the assumption that the process $\Delta S^\delta(t)$ is distributed according to (18), with variance being (41) and (42).

![The optimal margin against a sample trajectory of the valuation change process](image)

Figure 9: An illustration of performance of the probability-wise optimal margin $\hat{a}$ under the time dependent volatility valuation model.
When determining the time-wise optimal margin \( a^* \) consider the following:

\[
\mathbf{E}[\lambda(B_a)] = \int_0^T P\{ |\Delta S^\delta(t) \leq a \} \, dt
\]  

(49)

\[
= \int_0^\delta P\{ |\Delta S^\delta(t) \leq a \} \, dt + \int_\delta^T P\{ |\Delta S^\delta(t) \leq a \} \, dt
\]  

(50)

\[
= \int_0^\delta \left( 2\Phi\left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t}} \right) - 1 \right) \, dt
\]  

(51)

\[
+ \int_\delta^T \left( 2\Phi\left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta}} \right) - 1 \right) \, dt
\]  

(52)

\[
= 2 \int_0^\delta \Phi\left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t}} \right) \, dt - \delta
\]  

(53)

\[
+ 2 \int_\delta^T \Phi\left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta}} \right) \, dt - (T - \delta)
\]  

(54)

\[
= 2 \int_0^\delta \Phi\left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} + \alpha \beta t^2 + \beta^2 t}} \right) \, dt
\]  

(55)

\[
+ 2 \int_\delta^T \Phi\left( \frac{a}{\sqrt{\frac{\alpha^2 t^3}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta}} \right) \, dt - T,
\]  

(56)

where the third equation in line holds because of the derivations (43) and (44) for the probability of coverage (11).
The **time-wise optimal margin** $a^*$ satisfies the following condition:

$$
\frac{2}{T} \int_0^\delta \Phi \left( \frac{a}{\sqrt{\alpha^2 t^3 + \alpha\beta t^2 + \beta^2 t}} \right) dt \\
+ \frac{2}{T} \int_\delta^T \Phi \left( \frac{a}{\sqrt{\alpha^2 t^3 + \alpha\beta t^2 + \beta^2 t}} \right) dt - 1 = 0.99.
$$

(57)

(58)

In contrast to the constant volatility case, the expression (57)-(58) has both integrals dependent on time. These integrals are not easy (if not impossible) to calculate analytically. Instead, these integrals would be estimated numerically.

The first integral:

$$
\int_0^\delta \Phi \left( \frac{a}{\sqrt{\alpha^2 t^3 + \alpha\beta t^2 + \beta^2 t}} \right) dt
$$

will be estimated using the following expression:

$$
\hat{I}_1(a, \alpha, \beta, \delta, K) := \frac{\delta}{K} \sum_{j=0}^K \Phi \left( \frac{a}{\sqrt{\alpha^2 \tau_j^3 + \alpha\beta \tau_j^2 + \beta^2 \tau_j}} \right),
$$

(59)

where once again the time points $\tau_j$ are defined as: $\tau_j = j \cdot \frac{\delta}{K}$ for $j = 0, 1, \ldots, K$. 

39
The second integral:

\[ \int_{\delta}^{T} \Phi \left( \frac{a}{\sqrt{\frac{\alpha^2}{3} \delta (3t^2 - 3t\delta + \delta^2) + \alpha \beta \delta (2t - \delta) + \beta^2 \delta}} \right) \, dt \]

will be estimated using the expression:

\[ \hat{I}_2(a, \alpha, \beta, \delta, T, M) := \frac{T - \delta}{M} \]

\[ \cdot \sum_{l=0}^{L} \Phi \left( \frac{a}{\sqrt{\frac{\alpha^2}{3} \delta (3s_l^2 - 3s_l\delta + \delta^2) + \alpha \beta \delta (2s_l - \delta) + \beta^2 \delta}} \right), \]

where the points \( s_l \) are defined as \( s_l = \delta + l \cdot \frac{T - \delta}{M} \), where \( l = 0, 1, \ldots, M \).

The estimated time-wise optimal margin \( a_{est}^* \) is equal to:

\[ \frac{2}{T} \hat{I}_1(a_{est}^*, \alpha, \beta, \delta, K) + \frac{2}{T} \hat{I}_2(a_{est}^*, \alpha, \beta, \delta, T, M) - 1 = 0.99, \]

where \( \hat{I}_1(a_{est}^*, \alpha, \beta, \delta, K) \) is defined by (59), while \( \hat{I}_2(a_{est}^*, \alpha, \beta, \delta, T, M) \) is defined by (60)-(61).

Figure 10 shows the comparison of performance of the estimated time-wise optimal margin \( a_{est}^* \) and the probability-wise optimal margin \( \hat{a} \), given by (48) against the same sample trajectory used in Figure 9, under the assumed time dependent volatility valuation model.

In this case, the margins \( \hat{a} \) and \( a_{est}^* \) are not equal. The estimated time-wise optimal margin \( a_{est}^* \) is around 25% lower than the probability-wise optimal margin \( \hat{a} \).

This means that if the valuation change is modeled by (18), with variance equal to (41) for \( t \leq \delta \) and (42) for \( t > \delta \), the CCP can set lower initial margin for the parties using the time acceptable margin condition and thus be more attractive than the CCP which sets the initial margin according to the probability acceptable margin condition.
Figure 10: An illustration comparing the probability-wise optimal margin \( \hat{a} \) and the estimated time-wise optimal margin \( a_{est}^* \) in the time dependent volatility valuation model.

The estimated time-wise optimal margin \( a_{est}^* \) was again computed in Matlab, using its \texttt{fzero} function applied to the optimization function \( g \) defined by (39), along with the initial guess set to the minimum value of the simulated trajectory (although any value in \((0, 1]\) will work).

The estimated expected covered time ratio was calculated as:

\[
\hat{E} \left[ \frac{\lambda(B_a)}{T} \right] (a) = \frac{2}{T} \hat{I}_1(a_{est}^*, \alpha, \beta, \delta, K) + \frac{2}{T} \hat{I}_2(a_{est}^*, \alpha, \beta, \delta, T, M) - 1,
\]

where the \( \hat{I}_1(a, \alpha, \beta, \delta, K) \) is given by (59) and \( K = 10000 \), while the \( \hat{I}_2(a, \alpha, \beta, \delta, T, M) \) is given by (60)-(61) and \( M = 10000 \).
8 Problem extension

Continue observing the time dependent volatility valuation model, where the process of valuation change over the liquidation period $\Delta S^\delta(t)$ is distributed according to (18), with variance equal to (41) for $t \leq \delta$ and (42) for $t > \delta$.

If instead of a constant optimal margin $a$, observed in the previous two sections, one would search for such continuous optimal margin $a(t)$, then the probability-wise optimal margin $\hat{a}(t)$ would be the minimal function $a(t)$ that would satisfy the condition:

$$P\left\{ |\Delta S^\delta(t)| \leq a(t) \right\} \geq 0.99, \quad \forall t \in [0, T].$$

The time-wise optimal margin $a^*(t)$ would be the minimal function $a(t)$ that would satisfy:

$$\frac{1}{T} \int_0^T P\left\{ |\Delta S^\delta(t)| \leq a(t) \right\} dt \geq 0.99.\quad (62)$$

Notice that now the space of the possible optimal margins is a space of continuous mappings from $[0, T]$ to $\mathbb{R}^+$ and not just $\mathbb{R}^+$. Therefore, it is not entirely clear what is meant by "the minimal margin". An optimization problem, i.e. a minimal criteria should be defined in order to be able to discuss optimal functions.

Let:

- the probability-wise optimal margin $\hat{a}(t)$ be defined by:

$$\hat{a}(t) = \min_{a(t)} \frac{1}{T} \int_0^T a(t) \, dt \quad s.t. \quad P\left\{ |\Delta S^\delta(t)| \leq a(t) \right\} \geq 0.99, \quad \forall t \in [0, T]$$

(62)

- the time-wise optimal margin $a^*(t)$ be defined by:

$$a^*(t) = \min_{a(t)} \frac{1}{T} \int_0^T a(t) \, dt \quad s.t. \quad \frac{1}{T} \int_0^T P\left\{ |\Delta S^\delta(t)| \leq a(t) \right\} dt \geq 0.99$$

(63)
Even though the problem is now properly defined, solving it on the entire space of continuous mappings might a bit too ambitious. One could focus on e.g. **two-dimensional initial margins**, i.e. margins of the form:

\[
a(t) = \begin{cases} 
  a_1, & t \leq \tau \\
  a_2, & t > \tau 
\end{cases}
\]  

(64)

where \( \tau \in [0, T] \) is given. The practical meaning of the margin (64) is that the initial margin requirement up to the time \( \tau \) is \( a_1 \), whereas after the time \( \tau \), the initial margin requirement is \( a_2 \).

Moreover, the minimizing criteria given in (62) and (63) is now in the form of a weighted average and it will be further referred to as the **cost of setting the choosing the margin** \((a_1, a_2)\):

\[
\frac{1}{T} \int_{0}^{T} a(t) \, dt = a_1 \frac{T}{T} + a_2 \frac{T - \tau}{T}.
\]

(65)

This is just one of many ways to model the cost and set the minimization problem. This particular optimization criteria was chosen because of its simplicity.

It can be shown that the **probability-wise optimal margin** \( \hat{a} \) would, for \( \tau \leq \delta \), take the form of:

\[
\hat{a}(t) = \begin{cases} 
  \sqrt{\frac{\alpha^2 \tau^3}{3} + \alpha \beta \tau^2 + \beta^2 \tau \cdot \Phi^{-1}(0.995)}, & t \leq \tau \\
  \sqrt{\frac{\alpha^2 \delta}{3} \left(3T^2 - 3T \delta + \delta^2\right) + \alpha \beta \delta (2T - \delta) + \beta^2 \delta \cdot \Phi^{-1}(0.995)}, & t > \tau 
\end{cases}
\]

(66)

and for \( \tau > \delta \):

\[
\hat{a}(t) = \begin{cases} 
  \sqrt{\frac{\alpha^2 \delta}{3} \left(3 \tau^2 - 3 \tau \delta + \delta^2\right) + \alpha \beta \delta (2 \tau - \delta) + \beta^2 \delta \cdot \Phi^{-1}(0.995)}, & t \leq \tau \\
  \sqrt{\frac{\alpha^2 \delta}{3} \left(3T^2 - 3T \delta + \delta^2\right) + \alpha \beta \delta (2T - \delta) + \beta^2 \delta \cdot \Phi^{-1}(0.995)}, & t > \tau 
\end{cases}
\]

(67)
When trying to determine the time-wise optimal margin of the form (64) the expected covered time is equal to:

\[ E \left[ \lambda(B_{a_1,a_2}) \right] = \int_0^T P \left\{ \left| \Delta S^\delta(t) \right| \leq a(t) \right\} \, dt \]

\[
= 2 \int_0^\tau \Phi \left( \frac{a_1}{\sqrt{\alpha^2 t^3 + \alpha \beta t^2 + \beta^2 t}} \right) \, dt - \tau \\
+ 2 \int_\delta^\tau \Phi \left( \frac{a_2}{\sqrt{\alpha^2 t^3 + \alpha \beta t^2 + \beta^2 t}} \right) \, dt - (\delta - \tau) \\
+ 2 \int_\delta^T \Phi \left( \frac{a_2}{\sqrt{\frac{\alpha^2}{3} (3t^2 - 3t \delta + \delta^2) + \alpha \beta (2t - \delta) + \beta^2}} \right) \, dt - (T - \delta),
\]

for the case when the given change time is \( \tau \leq \delta \). For the expected covered time ratio, the last equation needs to be divided by \( T \) on each side. The derivation in fairly similar to (49)-(56). For the case when the given change time is \( \tau > \delta \), the expected covered time is of the form:

\[
E \left[ \lambda(B_{a_1,a_2}) \right] = 2 \int_0^\delta \Phi \left( \frac{a_1}{\sqrt{\alpha^2 t^3 + \alpha \beta t^2 + \beta^2 t}} \right) \, dt - \delta \\
+ 2 \int_\delta^\tau \Phi \left( \frac{a_1}{\sqrt{\frac{\alpha^2}{3} (3t^2 - 3t \delta + \delta^2) + \alpha \beta (2t - \delta) + \beta^2}} \right) \, dt - (\tau - \delta) \\
+ 2 \int_\delta^T \Phi \left( \frac{a_2}{\sqrt{\frac{\alpha^2}{3} (3t^2 - 3t \delta + \delta^2) + \alpha \beta (2t - \delta) + \beta^2}} \right) \, dt - (T - \tau).
\]

All these integrals are very hard (if not impossible) to calculate, so instead the expected covered time will be numerically estimated.
The most commonly used algorithm to approximate the expected values in finance is the Monte Carlo method. The idea behind it is to obtain a large number of samples from a known distribution and then take the mean as a representative value. One would therefore need to simulate a large number of realizations of the covered time ratio and take the mean as an approximation of the expected value. The problem with this approach is that the probability distribution of the covered time ratio is not analytically known.

If one would simulate a large number of trajectories of the valuation change process and for each of the trajectories calculate the covered time ratio, then it would be possible to take the average over the derived samples of the covered time ratio and use it as an estimate of its expected value.

In order to set the algorithm for calculating the covered time ratio, recall that the valuation change process is only obtained in discrete time points $t_j = j\Delta t$, $j = 0, 1, \ldots, \frac{T}{\Delta t}$, where $\Delta t$ is the chosen time discretization step. The valuation change process can become:

- uncovered in between discrete time points, making the adjacent time points on the opposite sides of the margin level;
- uncovered an covered again in between discrete time points, making the adjacent time points on the same side of the margin level.

Both of the situations make it hard to compute the true value of the covered time ratio for a simulated trajectory of the process $\Delta S^{d}(t)$.

Unfortunately, apart from making the time step smaller, there is really no way to go around the latter issue. One should note that this also happens in reality. If the exchanges report prices e.g. every 15 seconds, the process can become uncovered and covered again also during these 15 seconds. However, then the process was uncovered less than 15 seconds, which is arguably negligible compared to the entire lifetime of the trade $[0, T]$.

The former issue might not be entirely solvable, but there might be an engineered improvement.
Let the following terminology be introduced:

- if the process satisfies:
  \[ |\Delta S^\delta(t_j)| < a(t_j), \quad |\Delta S^\delta(t_{j+1})| \geq a(t_{j+1}), \]
  then the moment in which the process becomes uncovered, the **exit time** \( t_{\text{exit}} \) belongs to \((t_j, t_{j+1})\);

- similarly, if the process satisfies:
  \[ |\Delta S^\delta(t_j)| \geq a(t_j), \quad |\Delta S^\delta(t_{j+1})| < a(t_{j+1}), \]
  then the moment in which the process returns to the covered area, the **return time** \( t_{\text{return}} \) belongs to the interval \([t_j, t_{j+1})\).

The exit times and return times can be approximated by e.g. linear interpolation:

\[
 t_{\text{exit}} = t_j + \frac{\text{sgn}(\Delta S^\delta(t_{j+1})) \cdot a(t_{j+1}) - \Delta S^\delta(t_j)}{|\Delta S^\delta(t_{j+1}) - \Delta S^\delta(t_j)|} \cdot (t_{j+1} - t_j). \tag{68}
\]

\[
 t_{\text{return}} = t_j + \frac{\Delta S^\delta(t_j) - \text{sgn}(\Delta S^\delta(t_j)) \cdot a(t_{j+1})}{|\Delta S^\delta(t_{j+1}) - \Delta S^\delta(t_j)|} \cdot (t_{j+1} - t_j). \tag{69}
\]

The \( \text{sgn} \) terms serve the purpose of handling both the case when the process hits the positive and the negative margin. Moreover, the interpolation method will handle the cases of the exit time or return time coinciding with the observed time points. This has a zero probability of happening in real-life, mainly due to the fact that a random process with a continuous probability distribution has a probability 0 of taking a particular real value, i.e:

\[
P\left\{ \Delta S^\delta(t) = x \right\} = 0, \quad x \in \mathbb{R}. \tag{70}
\]

In computer simulations this might happen, due to the real numbers being represented in the system with finite number of decimals.
Finally, the total covered time ratio can be approximated by:

$$\hat{\lambda}(B_{a_1,a_2}) \frac{1}{T} = 1 - \frac{1}{T} \sum_{q=1}^{Q} (t^q_{\text{return}} - t^q_{\text{exit}}), \quad (71)$$

where $Q$ is the number of times that the process has become uncovered.

Figure 11 shows an example process with $\tau = \frac{1}{2} \cdot T$, $a_1 = 3$, $a_2 = 6$, unrelated to the process $\Delta S^8(t)$, suitable for understanding the described algorithm for estimating the covered time ratio.

![An example process for illustrating the covered time ratio estimate.](image)

Figure 11: An example process for illustrating the covered time ratio estimate.
For the process shown in Figure 11:

- \( t_{exit1} = 0.75, \ return_1 = 1.17 \)
- \( t_{exit2} = 2.5, \ return_2 = 3.17 \)
- \( t_{exit3} = 7.2, \ return_3 = 8.5 \)
- \( t_{exit4} = 10.86, \ return_4 = 11.14 \)
- \( t_{exit5} = 13.75, \ return_5 = 14.25 \)
- \( t_{exit6} = 15.75, \ return_6 = 16 \)

The process was uncovered for approximately:

\[
(1.17 - 0.75) + (3.17 - 2.5) + \ldots + (16 - 15.75) = 3.42
\]
units of time.

Therefore, an estimate for the covered time ratio of this process is:

\[
\frac{\hat{\lambda}(B_{a1,a2})}{T} = 1 - \frac{3.42}{16} = 0.79.
\]

This seems reasonable.

The described algorithm will further be used to estimate the covered time ratios. However, the minimization problem given by (63) is hard to implement and very computationally heavy, so this section will instead be concluded with an experiment. The rest is up to the interested reader.
One can now perform the following experiment:

- take $\alpha = 2$, $\beta = 0.8$, $\delta = \frac{5}{365}$, $T = 1$ and $\Delta t = \frac{1}{365}$
- simulate $N = 10^5$ trajectories of the process $\Delta S^\delta(t)$ (one of them is shown in Figure 12)
- take $\tau = 0.7T$
- take 10 points in $[0.2, 0.5]$ as candidates for $a_{est1}^*$ (see Figure 12)
- take 10 points in $[0.5, 0.8]$ as candidates for $a_{est2}^*$ (see Figure 12)
- for each combination $(a_1, a_2)$ and for each trajectory $i$, compute the estimated covered time ratio $\hat{\lambda}(B_{a_1,a_2})i/T$ defined by (71)
- for each combination $(a_1, a_2)$ estimate the expected covered time ratio by:

$$\hat{E}\left[\frac{\lambda(B_{a_1,a_2})}{T}\right] = \frac{1}{N} \sum_{i=1}^{N} \frac{\hat{\lambda}(B_{a_1,a_2})i}{T}$$  

(72)
The combination \((a_1, a_2)\) that has the estimated expected ratio larger than 0.99 and has the lowest cost defined by (65) is the \textbf{estimated time-wise optimal margin} \((a^*_{est1}, a^*_{est2})\).

The performance of the time-wise optimal margin \((a^*_{est1}, a^*_{est2})\) acquired by performing the described approach is compared to the performance of the probability-wise optimal margin (defined by (67), since \(\tau > \delta\)) in Figure 13.

![Figure 13: An illustration comparing the two-level probability-wise optimal margin \(\hat{a}\) and the estimated two-level time-wise optimal margin \(a^*_{est}\) in the time dependent volatility valuation model.](image)

The Figure 13 shows that the time optimality conditions perform better than the probability optimal conditions - even the estimated time-wise optimal margin \((a^*_{est1}, a^*_{est2})\), chosen from the "hand-picked" 100 samples is lower than the probability-wise optimal margin \((\hat{a}_1, \hat{a}_2)\).
9 Conclusion

Using only Heller and Vause’s statement about initial margins (see [18]), there exist at least two different interpretations of the conditions that the initial margin is supposed to satisfy - the probability acceptable margin condition given by (2) and the time acceptable margin condition given by (9).

In the space of constant initial margins, the performance of the minimal margin that satisfied the probability condition (probability-wise optimal margin) and the minimal margin that satisfied the time condition (time-wise optimal margin) was compared under the considered valuation model given in (15). Under the assumption that the volatility of the valuation process is constant, the optimal margins for two interpretations were somewhat similar, while under the assumption that the volatility of the valuation process is time dependent the time-wise optimal margin was significantly lower.

For the time dependent volatility, it made sense to search for the optimal margin in the space of time dependent initial margins. However, since multidimensional optimization problems were known to be hard to solve, the space was narrowed to the class of two-dimensional margins. Even the two dimensional case was computationally intensive, so instead the experiment was observed. The experiment showed the consistent conclusion - the time-wise optimal margin is lower than the probability-wise optimal margin. The next step would be to look for the optimal initial margin by optimizing over the switch time $\tau$ as well. This would be done by solving the three dimensional problem.

Outside of Heller and Vause’s framework it would be interesting to try and model the replacement cost and then try to find the optimal margin that would cover not just the valuation change over the liquidation period, but the entire potential future exposure.

For now, the CCPs might be better off using the proven, standard methods for determining initial margins, but further investigation on the subject is strongly advised. Even a slightly better method would be advantageous in a competitive environment such as the financial industry, and the time acceptable margin condition definitely shows potential.
References


