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Classification of simple transitive 2-representations

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Abstract

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The representation theory of finitary 2-categories is a generalization of the classical representation theory of finite dimensional associative algebras. A key notion in classical representation theory is the notion of simple modules as those are in some sense the building blocks of all modules. A correct analogue of simple modules in the realm of 2-representations is the notion of simple transitive 2-representations since those also turn out to be building blocks of 2-representations.

This thesis is concerned with the classification of simple transitive 2-representations for a number of different interesting 2-categories. In Paper I we study simple transitive 2-representations of Soergel bimodules in Coxeter type $I_2(4)$ and show that all simple transitive 2-representations in this case are equivalent to cell 2-representations. In Paper II we classify simple transitive 2-representations for the quotient of the 2-category of Soergel bimodules over the coinvariant algebra which is associated to the two-sided cell that is the closest to the two-sided cell containing the identity element, in all Coxeter types but $I_2(12)$, $I_2(18)$ and $I_2(30)$. It turns out that, in most of the cases, simple transitive 2-representations are exhausted by cell 2-representations. However, in Coxeter types $I_2(2k)$, where $k \geq 3$, there exist simple transitive 2-representations which are not equivalent to cell 2-representations. In Paper III we show that for any complex polynomial $p(X)$ the set of irreducible, integer matrices which are annihilated by $p(X)$ is finite. Moreover, we study the set of irreducible, integral matrices satisfying $X^2 = nX$, for $n \geq 1$, and count its elements. In Paper IV we show that every simple transitive 2-representations of the 2-category of projective functors for a certain quotient of the quadratic dual of the preprojective algebra associated with a tree is equivalent to a cell 2-representation. Finally, in Paper V we study simple transitive 2-representations of certain 2-subcategories of the 2-categories of projective functors over star algebras. In the simplest case, which is associated with Dynkin type A_2, we show that simple transitive 2-representations are classified by cell 2-representations. However, in the general case we conjecture that there exist many more simple transitive 2-representations.

Keywords: 2-representation theory, 2-categories, Soergel bimodules, projective functors, cell 2-representations, simple transitive 2-representations

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J. Zimmermann. *Simple transitive* 2*-representations of Soergel bimodules in type* B_2. J. Pure Appl. Algebra **221** (2017), no. 3, 666–690.

II T. Kildetoft, M. Mackaay, V. Mazorchuk, and J. Zimmermann. *Simple transitive* 2*-representations of small quotients of Soergel bimodules*. arXiv:1605.01373v2, to appear in Trans. AMS.

III E. Thörnblad, J. Zimmermann. *Counting quasi-idempotent irreducible integral matrices*. Journal of Integer Sequences, Volume 21 (2018).

IV J. Zimmermann *Simple transitive* 2*-representations of some* 2*-categories of projective functors*. Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry, Aug 2017.

V J. Zimmermann, *Simple transitive* 2*-representations of left cell* 2*-subcategories of projective functors over star algebras*. arXiv:1805.05724, to appear in Communications in Algebra.

Additional papers

The following papers are not included in this thesis.

VI A. Pasquali, E. Thörnblad, and J. Zimmermann *Existence of symmetric maximal noncrossing collections of k-element sets*.
Manuscript.

VII C.-W. Chen, B. Frisk Dubsky, H. Jonsson, V. Mazorchuk, E. Persson Westin, X. Zhang, J. Zimmermann. *Extreme representations of semirings*, arXiv:1806.06501.

Contents

1. Introduction

Representation theory has its origins in the work of Burnside, Frobenius and Molien and their work on characters for finite groups in the 1890s. Their idea was to represent elements of a given abstract group by invertible matrices such that multiplication of matrices respects the group operation. This was mostly used to obtain structure theorems about finite groups whose study is still a very active area.

In the 1920s and 1930s Noether introduced the notion of module in the study of representations. She extended the theory of modules to associative algebras and studied representations of a group G via its group algebra $\Bbbk G$. Moreover, she introduced a one to one correspondence between modules over an algebra and representations of this algebra. Today, representation theory is an active field of research with many applications both inside mathematics itself and in other areas such as physics and chemistry. For more details about the history of representation theory we refer the reader to [5, 13].

One of the classical problems of representation theory is to classify all modules over a given group or algebra. Phrased in this generality, this is often a hopeless or at least very difficult problem to solve. Often one tries therefore to classify some families of modules, for example, simple modules which, in some sense, are the building blocks of all modules. Even this problem is far from being easy, but it can be solved in many examples.

This thesis deals with higher representation theory in the sense that in our setting we study representations of so-called 2-categories rather than groups or algebras. We represent 2-categories using functorial actions on additive or abelian categories. The idea to study such objects stems from the idea of categorification which has its origin in mathematical physics, see [3, 4]. The original idea of categorification was to upgrade set-theoretic notions to categorical ones in the hope to obtain more structure, in particular, via upgrading usual categories to 2-categories. In this way the classical representation theory, which can be considered as representation theory of usual categories, gets upgraded to higher representation theory, that is representation theory of 2-categories. The approach of categorification had its major breakthrough with the invention of Khovanov homology in [11]. The higher representation theoretic reformulation of Khovanov homology was given in [34] using categorical action of the Temperley-Lieb Category on blocks of the BGG category $\mathcal{O}$. A celebrated application of higher representation theory to the study of Broué's conjecture appeared in [2].

Motivated by these applications, Mazorchuk and Miemietz started in [19] a systematic study of 2-representation theory of 2-analogues of finite dimensional associative algebras, called *finitary* 2-*categories*. This resulted in the series [19, 20, 21, 23, 24, 22] of papers which also motivated further research in this area that appeared in [35, 36, 37, 14, 25, 26, 15, 16, 17, 18, 27, 28], see also references therein. The original motivation was to develop abstract 2-representation theory of finitary 2-categories with the principal emphasis on construction, classification and comparison of 2-representations.

It took some time before the "correct" 2-analogue of the notion of simple module appeared in [24]. This is the class of the so-called *simple transitive* 2-representations of finitary 2-categories. One of the results that was obtained in [24] is an analogue of the classical Jordan-Hölder theorem where the role of irreducible representations is played by simple transitive 2-representations. In particular, simple transitive 2-representations are, in some sense, building blocks for all 2-representations. This naturally motivated the problem of classification of simple transitive 2-representations for arbitrary finitary 2-categories. This problem was studied, for various special cases, in [24, 22, 37, 14, 25, 26, 15, 16, 17, 18, 27, 28]. It is also the main motivation for the present thesis, which studies it in many cases providing a complete classification for a number of interesting 2-categories and exhibiting a number of non-trivial new phenomena. Moreover, we develop a certain combinatorial tool used in the theory.

2. Algebras and their (1-)representations

In this section we want to make some of the above a bit more rigorous without losing ourselves in details. The interested reader can find some more details in, e.g., [6, 9, 10, 29].

2.1 Basics about algebras

Let $\mathbb{k}$ be an algebraically closed field, for all intense and purposes we may think of $\mathbb{C}$. An *algebra over* $\mathbb{k}$, or a $\mathbb{k}$-*algebra* is a $\mathbb{k}$-vector space A together with a bilinear, associative multiplication rule. That is, for any $a, b \in A$, we have that $a \cdot b = ab$ is again an element in A and for $a, b, c \in A$ and $\lambda \in \mathbb{k}$ we moreover have that

- $a(b + c) = ab + ac$
- $(a + b)c = ac + bc$
- $(\lambda a)b = \lambda(ab)$
- $a(\lambda b) = \lambda(ab)$
- $a(bc) = (ab)c.$

Additionally, we will require the existence of an element $1 \in A$ satisfying $1a = a1 = a$ for all $a \in A$, that is, we assume our algebras to be *unital*.

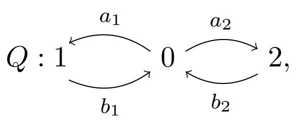
We say that an algebra A is *finite dimensional* if the underlying vector space is, else it is called *infinite dimensional*, and we denote the dimension of A by $\dim(A)$. Let A and B be $\mathbb{k}$-algebras. A $\mathbb{k}$-linear map $\varphi : A \to B$ is called an *algebra homomorphism* if it intertwines the algebra structure of A and B respectively, i.e., if for all $a_1, a_2 \in A$ we have $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ and $\varphi(1_A) = 1_B$.

Examples of algebras are:

- The field $\mathbb{k}$ is a $\mathbb{k}$-algebra where multiplication is simply the multiplication of the field.
- For each n, the space of $n \times n$-matrices $\mathrm{Mat}_n(\mathbb{k})$ is a $\mathbb{k}$-algebra where multiplication is given by matrix multiplication.

- For a finite group G we can define its (complex) *group algebra* $\mathbb{C}G$ as the vector space with basis $\{g \in G\}$, i.e., elements in $\mathbb{C}G$ are formal sums of scalar multiples of elements of G. The multiplication on base elements is given by the group operation and then extended by bilinearity to $\mathbb{C}G$.

Another class of $\Bbbk$-algebras are so-called *path algebras of quivers*. A (finite) *quiver* $Q = (Q_0, Q_1)$ consists of a finite set of vertices Q_0 and a finite set of arrows Q_1 between its vertices, e.g.,



(2.1)

where $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{a_1, a_2, b_1, b_2\}$. Here we say that the arrow $a_1 = (0, 1)$ has *source* 0 and *target* 1. A *path* in a quiver is a sequence of arrows $\alpha_1\alpha_2 \dots \alpha_n$ such that the source of α_{i-1} equals the target of α_i. Moreover, for each vertex $i \in Q_0$, there exists a trivial path $\varepsilon_i : i \to i$.

Now the path algebra $\Bbbk Q$ of a quiver Q has as underlying vector space the space generated by all paths of Q and multiplication is given by concatenation of paths whenever sensible, else the product is defined to be 0. As an algebra, we have that $\Bbbk Q$ is generated by all path of length at most one, i.e., all identities ε_i (those have length zero) and all arrows (which have length one as paths). For instance, in our example above, we have

$$\Bbbk Q = \Bbbk\langle \varepsilon_0, \varepsilon_1, \varepsilon_2, a_1, a_2, b_1, b_2\rangle.$$

If the quiver Q contains oriented cycles, $\Bbbk Q$ will not be a finite dimensional algebra unless we impose some *relations* such as, e.g., $b_1a_2 = b_2a_1 = 0$ in our example above. More precisely, we let $I = \langle R\rangle \subseteq \Bbbk Q$ be the ideal generated by the set of relations R. Then we can consider the quotient $A = \Bbbk Q/I$ which again is an algebra.

This is a general method to define an algebra A over $\Bbbk$. We can pick a set of generators $X = \{x_1, x_2, \dots\}$ and consider the free algebra $A = \Bbbk\langle X\rangle$ which consists of all *words* which we can build with elements in X. Then we can impose some relations R on A which we collect in an ideal $I := \langle R\rangle$, e.g., $I = \langle xy - yx \mid x, y \in X\rangle$, and consider A/I; the algebra generated by X subject to the relations R.

From now on let A be a finite dimensional $\Bbbk$-algebra.

2.2 (1-)Representation theory - some basics

The next important notion for us is that of a *representation* of A. Let M be a finite dimensional vector space over $\Bbbk$. A *representation* of A is an algebra homomorphism $\varphi : A \to \operatorname{End}(M)$, i.e., we interpret algebra elements as linear transformations on the vector space M and we do this in such a way that multiplication of elements in A corresponds to composition of linear transformations. This latter interpretation of a representation is what we mean by a *(left) A-module* structure on M or by an *action* of A on M. We assume all our modules to be left modules unless explicitly stated otherwise and if there is no risk of confusion, we will just write module instead of A-module. Moreover, we will use the terms representation and module interchangeably, as they are essentially the same, that is every representation can be expressed as a module and vice versa.

As in the case of algebras we have the notion of a homomorphism between modules. Let M, N be A-modules. Then a linear map $\varphi : M \to N$ is called a *module homomorphism*, provided that $\varphi(a \cdot m) = a\varphi(m)$, i.e., φ intertwines the action of A on M with the action of A on N.

Examples of modules are:

- For every algebra A, there exist two trivial modules, the smallest module $\{0\}$ and ${}_AA$, the *regular* module A where the action is given by left multiplication.
- The matrix algebra $\operatorname{Mat}_n(\mathbb{k})$ acts on $\mathbb{k}^n$ by usual matrix multiplication.
- Let $\mathbb{k}Q$ be the path algebra of the quiver from (2.1) and consider the algebra $A = \mathbb{k}Q/I$ where $I = \mathbb{k}\langle a_ib_ia_i, b_ia_ib_i, a_2b_1, a_1b_2 \mid i = 1, 2\rangle$. Then we have the following representation of A

$$P_0 : \mathbb{C} \underset{\mathrm{id}}{\overset{0}{\leftrightarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0. \tag{2.2}$$

  This diagram should be interpreted in the following way: A acts on the vector space $V = \mathbb{C} \oplus \mathbb{C}$ and the action of the generators of A is given by the following matrices

$$\varepsilon_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

  and all other generators act as 0. It is easily checked that this action respects the relations and hence is a well-defined A-action on V.

A *submodule* of a module M is a subspace N of M which is stable under the action of A. Every module M has two trivial submodules: the zero submodule $\{0\}$ and the module M itself.

Having defined submodules we can also take quotients of modules, that is, let M be an A-module and N a submodule of M, then the action of A induces a module structure on the quotient space M/N. This module is called the *quotient of M by N*. This construction enables us to obtain more examples of modules, e.g., every left ideal I of A is naturally a submodule of the regular module ${}_AA$ and, by the above, the quotient A/I is also an A-module.

To consider an A-module that you know and take the quotient of it by one of its submodules, is one of the two most commonly used ways to construct new A-modules. As any module is a quotient of a free module, up to isomorphism, all A-modules can be constructed in this way starting from the left regular A-module (and its direct sums). If the algebra A is given by generators and relations, we can also construct A-modules by defining the actions of the generators and checking that the relations are satisfied, cf. (2.2).

A non-zero module M is called *simple* if $\{0\}$ and M are its only submodules. Simple modules play an important role in representation theory of finite dimensional algebras since all modules can be "built-up" from simple modules. More precisely, the classical Jordan-Hölder theorem states that every finite dimensional module M has a finite chain

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{r-1} \subset M_r = M$$

of submodules such that all successive quotients M_i/M_{i-1} are simple.

Another important type of modules are *projective* modules. An A-module P is projective if the following is satisfied: if $\varphi : P \to N$ and $\rho : M \to N$ are module homorphisms, and ρ is surjective, then $\varphi = \rho \circ \psi$, for some homorphism $\psi : P \to M$. Observe that ψ is, in general, not unique. Examples of projective modules are, for instance, free modules and one can show that, for finite dimensional algebras, projective modules are precisely direct summands of free modules, up to isomorphism. Another example of a projective module is, if A is an algebra and $e \in A$ and idempotent, i.e., $e^2 = e$, then Ae is a projective A-module.

2.3 (1-)Categories

In this section we want to present a more abstract way of defining representations, which is the point of view which we take when we later generalize this concept to the setup of 2-categories.

A *(1)-category* $\mathcal{C}$ consists of a class of objects $\{\mathtt{i}, \mathtt{j}, \dots\}$ and morphisms between any pair of objects, denoted by $\mathrm{Hom}_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$ or $\mathcal{C}(\mathtt{i}, \mathtt{j})$. The imposed axioms are: existence of an associative composition law for morphisms (whenever composition makes sense) and existence of an identity morphism $\varepsilon_{\mathtt{i}}$, for every object $\mathtt{i}$. A category is called *small* if its class of objects and its class of morphisms are sets. Examples of categories are the category of sets, the categories of groups, rings, associative algebras, etc. Further, we can assign to each algebra A its module category $A\text{-mod}$ and the category $A\text{-proj}$ of projective A-modules. In both latter cases the categories are *skeletally small*, that is there exist small categories which are equivalent to $A\text{-mod}$ and $A\text{-proj}$, respectively.

Another example of a category is the category **Cat** of small categories. Here objects are small categories and morphisms between categories are so-called *functors*. A functor is a map F between two categories $\mathcal{C}$ and $\mathcal{D}$ which respects the categorical structure of $\mathcal{C}$ and $\mathcal{D}$, i.e., it maps objects to objects, morphisms to morphisms, identities to identities and intertwines the composition of morphisms.

We can now use this new language to reformulate the definition of a representation of an algebra. To do so we first observe that we can associate a category to an algebra A in the following way. Consider the category $\mathcal{A}$ which consists of one element $\mathtt{i}$, a morphism $f_a : \mathtt{i} \to \mathtt{i}$, for each $a \in A$, and composition is given by multiplication, i.e., $f_b \circ f_a = f_{ba}$. Then a (1-)representation of $\mathcal{A}$ is a $\mathbb{k}$-linear functor $F : \mathcal{A} \to \mathbb{k}\text{-mod}$, where $\mathbb{k}\text{-mod}$ is the category of finite dimensional vector spaces over $\mathbb{k}$. It sends $\mathtt{i}$ to some vector space M and each morphism f_a to a linear map $\varphi_a : M \to M$ such that multiplication in A is intertwined with composition of linear maps. Therefore each such functor corresponds to exactly one A-module M and vice versa so this is really just another way of phrasing the definition of an A-module. Furthermore, it is easy to see that homomorphisms of A-modules correspond via this rephrasing to natural transformations of functors. Thus we can, for some small $\mathbb{k}$-linear category $\mathcal{C}$, define the category $\mathcal{C}\text{-mod}$ of $\mathcal{C}$-modules whose objects are $\mathbb{k}$-linear functors and whose morphisms are natural transformation.

The construction above can be further refined. We can associate to each algebra A the category $\hat{\mathcal{A}}$ whose objects are representatives of the isomorphism classes of indecomposable projective A-modules. Using $\hat{\mathcal{A}}$ one can then show that $A\text{-}\operatorname{mod} \simeq \hat{\mathcal{A}}^{\text{op}}\text{-}\operatorname{mod}$, i.e., the module ’ categories of A and $\hat{\mathcal{A}}^{\text{op}}$ are equivalent.

3. 2-Representation theory

In this section we turn our attention to the main subject of the thesis, namely 2-representations. Based on the notions which we described above we will introduce the main notions and results which are known and hint at some problems which are yet open.

Before we can turn our attention to the 2-world, we need one more category theoretical notion, namely that of *enriched* categories. Being enriched imposes an extra assumption on the morphism sets of a category, e.g., a category enriched over the category $\Bbbk\text{-mod}$ of $\Bbbk$-vector spaces is a category whose morphism spaces are $\Bbbk$-vector spaces and whose composition is bilinear. These are exactly the $\Bbbk$-linear categories mentioned above. When we are working with an enriched category, e.g., a $\Bbbk$-linear category, we will assume that our functor respects even this structure; in this example we call such functors $\Bbbk$-*linear*.

3.1 Some basics about 2-categories

For us a 2-*category* $\mathscr{C}$ is a category enriched over the category of small categories. In other words, $\mathscr{C}$ consists of

- objects, denoted by $\mathtt{i}, \mathtt{j}, \ldots$;
- small categories $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms which, in turn, consist of
  - 1-morphisms, denoted by $F, G, \ldots$;
  - 2-morphisms, denoted by $\alpha, \beta, \ldots$;

such that composition in $\mathscr{C}$ is bifunctorial and identity 1-morphisms for this composition exist. Note that we can compose 1-morphisms as we usually compose morphisms in a category, we denote this composition by $\circ$. However, for 2-morphisms we have two different compositions. The first type is so-called *horizontal* composition $\circ_h : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$, and the second one is *vertical*, denoted by $\circ_v$, which is composition inside $\mathscr{C}(\mathtt{i}, \mathtt{j})$. As a consequence of bifunctoriality of composition in $\mathscr{C}$ we have the following *interchange law* for any appropriately composable 2-morphisms $\alpha, \beta, \gamma, \delta$:

$$(\alpha \circ_h \beta) \circ_v (\gamma \circ_h \delta) = (\alpha \circ_v \gamma) \circ_h (\beta \circ_v \delta).$$

For every object $\mathtt{i}$ in $\mathscr{C}$, we denote the corresponding identity 1-morphism by $\mathbb{1}_{\mathtt{i}}$. For every 1-morphism F, the corresponding identity 2-morphism is denoted by id_F.

Some examples of 2-categories are:

- The 2-category **Cat** of small categories whose objects are small categories, 1-morphisms are functors and 2-morphisms are natural transformations.

- The 2-category $\mathfrak{A}_{\mathbb{k}}^{f}$ of *finitary* $\mathbb{k}$-*linear* categories, whose objects are categories equivalent to categories of projective modules over finite dimensional associative $\mathbb{k}$-algebras, 1-morphisms are additive $\mathbb{k}$-linear functors and 2-morphisms are natural transformations.

- The 2-category $\mathfrak{R}_{\mathbb{k}}$ of *finitary* $\mathbb{k}$-*linear abelian* categories, whose objects are categories equivalent to module categories of finite dimensional associative $\mathbb{k}$-algebras, 1-morphisms are right exact additive $\mathbb{k}$-linear functors and 2-morphisms are natural transformations.

As in the case of categories, we have an appropriate 2-analogue of the notion of a functors. Functors between 2-categories are called 2-*functors*. More precisely, a 2-functor $\mathbf{F}$ from a 2-category $\mathscr{C}$ to a 2-category $\mathscr{D}$ sends objects to objects, 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms respecting all appropriate compositions and identity morphisms.

For our purposes, the definition of a 2-category which we gave above is too general. We will have to make some finiteness assumptions which are motivated by the fact that the kind of 2-representation theory which Mazorchuk and Miemietz developed is build in strong analogy to the representation theory of finite dimensional algebras.

The first type of 2-categories which we study in this thesis consists of *finitary* 2-*categories*. A finitary 2-category is a 2-category $\mathscr{C}$ which has finitely many objects, finitary $\mathbb{k}$-linear morphism categories, and biadditive and $\mathbb{k}$-(bi)linear compositions.

The second and even more specialized type of 2-categories which is important for us is that of *fiat* 2-*categories*. A fiat 2-category is a finitary 2-category with a weak involution $*$. Moreover, it is required that the weak involution is such that, for any $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ and any $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, there exist the so-called *adjunction* 2-morphisms $\alpha : F \circ F^* \to \mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}} \to F^* \circ F$ such that $(\alpha \circ_h \mathrm{id}_F) \circ_v (\mathrm{id}_F \circ_h \beta) = \mathrm{id}_F$ and $(\mathrm{id}_{F^*} \circ_h \alpha) \circ_v (\beta \circ_h \mathrm{id}_{F^*}) = \mathrm{id}_{F^*}$.

3.2 Cells in 2-categories

One very useful tool for the study of finitary 2-categories is given by the combinatorial gadgets called *cells*. The cell structure of a 2-category is a combinatorial description of the composition of 1-morphisms inspired by Green's relations in semigroups. Let $\mathscr{C}$ be a finitary 2-category and F, G be indecomposable 1-morphisms in $\mathscr{C}$. We write $F \leq_{\mathcal{L}} G$ if there exists a 1-morphism H in $\mathscr{C}$ such that G is isomorphic to a direct summand of $H \circ F$. Similarly we can define $\geq_R$ and $\geq_J$ using composition from the right and both sides, respectively. These relations define partial pre-orders on the set of isomorphism classes of indecomposable 1-morphism of $\mathscr{C}$ and the equivalences classes corresponding to these pre-orders are called *left*, *right* and *two-sided cells*, respectively. For a 1-morphism F, we denote by $\mathcal{L}_F$, $\mathcal{R}_F$ and $\mathcal{J}_F$ the left, right and two-sided cells containing F, respectively.

A two-sided cell $\mathcal{J}$ is called *strongly regular* if, for any left cell $\mathcal{L}$ in $\mathcal{J}$ and any right cell $\mathcal{R}$ in $\mathcal{J}$, we have that $|\mathcal{L} \cap \mathcal{R}| = 1$. Moreover, we require that any two left cells in $\mathcal{J}$ are not comparable with respect to $\leq_{\mathcal{L}}$ and any two right cells in $\mathcal{J}$ are not comparable with respect to $\leq_{\mathcal{R}}$. A two-sided cell $\mathcal{J}$ is called *idempotent* if there exists $F, G, H \in \mathcal{J}$ such that H is isomorphic to a direct summand of $F \circ G$.

Cells will help us later to define some "natural" classes of 2-representations. For more details about this we refer the reader to Section 3.5.

In the next two sections we will introduce some of the protagonists of this thesis, namely the 2-categories of *Soergel bimodules* $\mathscr{S}_W$ corresponding to some finite Coxeter group (W, S) and the 2-category $\mathscr{C}_A$ of projective functors for some finite dimensional algebra A.

3.3 Soergel bimodules over the coinvariant algebra

Soergel bimodules originate from [31, 32] as a combinatorial tool to study BGG category $\mathcal{O}$. They have since been studied extensively, see [33, 7, 8] and references therein. They can be defined for any Coxeter system and one of their many features is that they categorify the Hecke algebra of the given Coxeter group; see [32, 7]. Soergel bimodules can be defined either over the polynomial algebra, reflecting the setup of [32], or over the coinvariant algebra, reflecting the setup of [31]. In order to fit our setup of finitary 2-categories we will work with Soergel bimodules over the coinvariant algebra. We will also consider Soergel bimodules as ungraded.

Let (W, S) be a finite Coxeter system with a fixed geometric representation $\mathfrak{h}$ and let $C = C_W$ be the corresponding coinvariant algebra. For a simple reflection s, we denote by C^s the subalgebra of C consisting of all s-invariants.

Set $\hat{\theta}_e = \theta_e = C$. For $e \neq w \in W$ with a fixed reduced decomposition $w = s_1 s_2 \dots s_n$, consider the C-C-bimodule

$$\hat{\theta}_w = C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \cdots \otimes_{C^{s_n}} C,$$

Now, we define θ_w recursively as the unique indecomposable direct summand of $\hat{\theta}_w$ which is not isomorphic to $\theta_{w'}$ for any shorter $w' \in W$. The existence of such a summand is not a priori clear but was proved by Soergel, see [31, 32, 33]. The C-C-bimodule θ_w is the *(indecomposable) Soergel bimodule* associated to w. The bimodule θ_w does not depended, up to isomorphism, on the choice of a reduced expression for w even though $\hat{\theta}_w$ does.

Now let $\mathcal{A}$ be a small category equivalent to C-mod. Then the associated 2-category $\mathcal{S} = \mathcal{S}_W$ of Soergel bimodules has one object $\mathtt{i}$ which we identify with $\mathcal{A}$, 1-morphisms of $\mathcal{S}$ are endofunctors of $\mathcal{A}$ given by tensoring with direct sums of indecomposable Soergel bimodules and 2-morphisms are natural transformations. It was shown in [19, Section 7.1] that $\mathcal{S}$ is a fiat 2-category.

3.4 The 2-category of projective functors $\mathscr{C}_A$

Let A be an associative, basic, connected finite dimensional $\Bbbk$-algebra and $\mathcal{C}$ a small category equivalent to A-mod. Moreover, let $e_1, e_2, \dots, e_n$ be a complete list of pairwise orthogonal primitive idempotents in A. Then all projective indecomposable A-A-bimodules are, up to isomorphism, of the form $Ae_i \otimes e_jA$, for $1 \leq i, j \leq n$. Now the 2-category $\mathscr{C}_A$ of projective functors of A is defined as follows:

- It has one object $\mathtt{i}$ which we identify with $\mathcal{C}$.
- 1-morphisms are direct sums of endofunctors of $\mathcal{C}$ isomorphic to tensoring with projective A-A-bimodules and with the regular bimodule ${}_AA_A$.
- 2-morphisms are natural transformations.

In [19, Section 7.3] it was shown that $\mathscr{C}_A$ is a fiat 2-category if A is weakly symmetric, in other cases it is at least a finitary 2-category. Different choices of $\mathcal{C}$ result in biequivalent, in particular, 2-Morita equivalent 2-categories, cf. [23].

3.5 2-representations

As we have seen in Section 2.3, a (1-)representation can be viewed as a functor from a category $\mathcal{A}$ to the category $\Bbbk\text{-mod}$ of vector spaces. This is the correct point of view when we now want to upgrade this setup to 2-categories, however there are some subtleties that we have to take care of. The first question is what our "model" 2-category should be. In the classical setup we chose vector spaces because we understand the theory of vector spaces and linear transformations well or at least better than that of arbitrary abstract finite dimensional algebras or finite groups. In our 2-setup, we will mostly use the 2-category $\mathfrak{A}_{\Bbbk}^{f}$ of finitary $\Bbbk$-linear categories as our "model" 2-category but even $\mathfrak{R}_{\Bbbk}$ will play some role. Again this is because we think that we have a better understanding of those two 2-categories.

A *finitary* or *abelian* 2*-representation* of a 2-category $\mathscr{C}$ is a strict 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$ or $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$, respectively. All 2-representations of a given 2-category $\mathscr{C}$ form a 2-category where 1-morphisms are strong natural transformations and 2-morphisms are modifications; see [21, Subsection 2.3]. Unfortunately, this 2-category does not have all the nice properties that module (1-)categories of finite dimensional algebras have. For example it is, in general, not abelian.

For every finitary 2-category $\mathscr{C}$ and every object $\mathtt{i} \in \mathscr{C}$ we have the so-called *principal* 2*-representation* $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, \_)$, i.e., it sends each object $\mathtt{j} \in \mathscr{C}$ to the finitary category $\mathscr{C}(\mathtt{i}, \mathtt{j})$, the action of 1-morphisms is given as left composition, whenever that makes sense, and 2-morphisms act by left multiplication, again whenever this makes sense.

3.5.1 Cell 2-representations

One of the natural classes of 2-representations of a finitary 2-category is given by the so-called *cell* 2*-representations*. Morally speaking, the idea here is to pick a left cell $\mathcal{L}$ of a finitary 2-category $\mathscr{C}$ (right cells work in an analogous way for right 2-representations) and to consider all morphisms that are left equal or bigger than $\mathcal{L}$. This will certainly be stable under left multiplication with $\mathscr{C}$, if formulated correctly. Then we only need to factor out the "biggest" possible ideal to obtain something that is "as small as possible".

Let us make the above a bit more precise. Let $\mathscr{C}$ be a finitary 2-category and $\mathcal{L}$ be a left cell in $\mathscr{C}$. Then there exists an object $\mathtt{i} = \mathtt{i}_{\mathcal{L}} \in \mathscr{C}$ such that all 1-morphisms in $\mathcal{L}$ start from $\mathtt{i}$.

Consider the principal 2-representation $\mathbf{P}_{\mathtt{i}}$. For each $\mathtt{j} \in \mathscr{C}$ we denote by $\mathbf{N}(\mathtt{j})$ the additive closure in $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ of all indecomposable 1-morphisms F in $\mathbf{P}_{\mathtt{i}}(\mathtt{j}) = \mathscr{C}(\mathtt{i}, \mathtt{j})$ which satisfy $\mathcal{L} \leq_L F$. Observe that this makes sense since $\mathcal{L}$ is a left cell and thus all its elements are left equivalent. Now we can restrict the action of $\mathbf{P}_{\mathtt{i}}$ to $\mathbf{N}$ and thus turn $\mathbf{N}$ into a 2-representation of $\mathscr{C}$. It was shown in e.g., [24, Lemma 3] that there exists a unique maximal $\mathscr{C}$-invariant ideal $\mathbf{I}$ in $\mathbf{N}$. The associated quotient 2-representation $\mathbf{N}/\mathbf{I}$ is called the *cell 2-representation* corresponding to $\mathcal{L}$ and denoted by $\mathbf{C}_{\mathcal{L}}$. Note that this definition appears in [20]. For fiat 2-categories there is an alternative definition of cell 2-representations proposed in [19].

We note that cell 2-representation for different left cells can be equivalent. For example, the following result is proved in [24].

Theorem 1. *Let $\mathscr{C}$ be a fiat 2-category in which all two sided cells are strongly regular. Then for any left cells $\mathcal{L}$ and $\mathcal{L}'$ in $\mathscr{C}$ we have that $\mathbf{C}_{\mathcal{L}} \simeq \mathbf{C}_{\mathcal{L}'}$ if and only if $\mathcal{L}$ and $\mathcal{L}'$ belong to the same two-sided cell.*

3.5.2 Simple transitive 2-representations

As it turned out, cell 2-representations are, in general, not the correct analogue of simple modules, as was first shown in [24, Subsection 3.2]. In the same paper Mazorchuk and Miemietz gave the definition of *simple transitive 2-representation* that seems to be a correct 2-analogue of simple modules.

A 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is called *transitive* if, for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ and for any indecomposable $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{j})$, there exists a 1-morphism $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ such that Y is isomorphic to a summand of $\mathbf{M}(F)\ X$. A 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is called *simple* if it does not have any non-trivial $\mathscr{C}$-invariant ideals.

It follows by construction that every cell 2-representation is simple transitive. In many examples one can prove that, in fact, every simple transitive 2-representation is equivalent to a cell 2-representation, however, there are examples when this is not the case; see, e.g, [24, 14].

Moreover, in [1, Subsection 3.2] it was shown that every simple transitive 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ has an *apex*, which is defined as the unique two-sided cell that is maximal in the set of all two-sided cells whose elements are not annihilated by $\mathbf{M}$.

3.6 Decategorification

One very useful tool for the study of simple transitive 2-representations $\mathbf{M}$ of a given finitary 2-category is what is known as the *Grothendieck decategorifica-*

tion of $\mathbf{M}$. To define what this is we first need to define the decategorification of $\mathscr{C}$.

The *decategorification* $[\mathscr{C}]$ of a finitary 2-category $\mathscr{C}$ is a (1-)category with the same objects as $\mathscr{C}$, where, for $\mathtt{i},\mathtt{j} \in [\mathscr{C}]$, the morphism space $[\mathscr{C}](\mathtt{i},\mathtt{j})$ is defined as the split Grothendieck group $[\mathscr{C}(\mathtt{i},\mathtt{j})]_{\oplus}$ of the additive category $\mathscr{C}(\mathtt{i},\mathtt{j})$. Composition in $[\mathscr{C}]$ is induced by composition in $\mathscr{C}$.

Now, let $\mathbf{M}$ be a 2-representation of $\mathscr{C}$, then the *decategorification* of $\mathbf{M}$ is the functor $[\mathbf{M}]$ from $[\mathscr{C}]$ to the category $\mathbf{Ab}$ of abelian groups which is defined as follows.

- For any object $\mathtt{i} \in [\mathscr{C}]$, the abelian group $[\mathbf{M}](\mathtt{i})$ is the split Grothendieck group of $\mathbf{M}(\mathtt{i})$.
- For any 1-morphism $F \in \mathscr{C}(\mathtt{i},\mathtt{j})$, the action of $[F] \in [\mathscr{C}]$ on $[\mathbf{M}](\mathtt{i})$ is induced by the functorial action of F on $\mathbf{M}(\mathtt{i})$.

This construction actually defines a functor from the category of additive 2-representations of $\mathscr{C}$ (where we forget the modifications) to the category of representations of $[\mathscr{C}]$ in $\mathbf{Ab}$.

Example 1.1. Let $\mathscr{C}$ be a finitary 2-category with one object denoted by $\mathtt{i}$. Then the decategorification of the principal 2-representation $\mathbf{P}_{\mathtt{i}}$ of $\mathscr{C}$ is, basically, the regular representation of the ring $[\mathscr{C}(\mathtt{i},\mathtt{i})]_{\oplus}$. This follows easily since the functor $[\mathbf{P}_{\mathtt{i}}]$ sends the only object $\mathtt{i}$ of $[\mathscr{C}]$ to $[\mathscr{C}(\mathtt{i},\mathtt{i})]_{\oplus}$ and the action of a 1-morphism $F \in \mathscr{C}(\mathtt{i},\mathtt{i})$ is given by left multiplication of elements in $[\mathscr{C}(\mathtt{i},\mathtt{i})]_{\oplus}$ since the action of F on $\mathbf{P}_{\mathtt{i}}(\mathtt{i}) = \mathscr{C}(\mathtt{i},\mathtt{i})$ is given by left composition.

Given a 2-representation $\mathbf{M}$ of $\mathscr{C}$ and F an indecomposable 1-morphism in $\mathscr{C}(\mathtt{i},\mathtt{i})$, we can already say a lot about $\mathbf{M}$ by considering the decategorified action of F on the indecomposable projectives in $\mathcal{M} := \coprod_{\mathtt{i}} \mathbf{M}(\mathtt{i})$. This action can be described by what we call the *matrix* $[F]$ *of the action of* F. The rows and columns of $[F]$ are indexed by isomorphism classes of indecomposable objects in $\mathcal{M}$ and the $X \times Y$-entry encodes the multiplicity of X as a direct summand of $\mathbf{M}(F)\,Y$. Thus $[F]$ is a non-negative, integral matrix.

3.7 Approach in the proofs

We have now collected all the necessary notions and definitions to explain the main idea which is used in most of the classification results for simple transitive 2-representations in this thesis.

Let $\mathscr{C}$ be a finitary 2-category with one object $\mathtt{i}$ and denote by F the sum of all indecomposable 1-morphisms in $\mathscr{C}$. Then one can easily convince oneself that $\mathbf{M}$ is transitive if and only if $[F]$ is a positive, integral matrix. Moreover, it turns out that in all our cases we have $[F]^2 = [F \circ F] = n[F]$, where n equals the dimension of the underlying algebra of $\mathbf{M}(\mathtt{i})$.

Now the first part of the proofs of our classification results consists of studying all possible positive integral matrices M satisfying $\mathsf{M}^2 = n\mathsf{M}$. In the next step we check which of the resulting matrices M can be written as a sum of m non-negative, integral matrices, where m is the number of isomorphisms classes of indecomposable 1-morphisms in $\mathscr{C}$. Moreover, we need to make sure that we can assign each summand to one of the 1-morphisms in such a way that the matrix multiplication respects the composition table of indecomposable 1-morphisms.

All of this limits the number of possible matrices in most cases significantly. The next step is then to find the simple transitive 2-representations which realize these matrices or to show that such 2-representations do not exist. In most cases the hope is that only the cell 2-representations will do the job. Else we need to construct new simple transitive 2-representations which is usually a very hard problem.

4. Summary of results

In this section we are going to briefly discuss the main ideas and results of the papers.

4.1 Paper I

In [24] Mazorchuk and Miemietz introduced the notion of *simple transitive* 2-representations and classified simple transitive 2-representations for fiat 2-categories with strongly regular two-sided cells; see [24, Theorem 18]. It turned out that for those categories all simple transitive 2-categories are equivalent to cell 2-representations. Their results include, among others, the 2-category $\mathscr{C}_B$ of projective functors for a weakly symmetric algebra B and the 2-category of Soergel bimodules of type A. Moreover, in [24] it was shown for Soergel bimodules in type B_2 that, if a simple transitive 2-representation decategorifies to a simple $D_{2\cdot 4}$-module V, then V has to be either the trivial or the sign module.

A natural question was whether one could classify simple transitive 2-representations for Soergel bimodule outside of type A. The first natural step outside of type A is type B_2, that is Soergel bimodules corresponding to the Weyl group $D_{2\cdot 4}$. Denote by $\mathscr{S}_{2\cdot 4}$ the 2-category of Soergel bimodules over the coinvariant algebra of the dihedral group $D_{2\cdot 4}$; as was defined in 3.3. Moreover, we denote the indecomposable Soergel bimodule corresponding to $w \in D_{2\cdot 4}$ by θ_w.

Recall that

$$\begin{aligned} D_{2\cdot 4} = \langle s, t \mid s^2 = t^2 = e, stst = tsts \rangle = \\ = \{e, s, t, st, ts, sts, tst, stst = tsts\} \end{aligned}$$

is the dihedral group of order 8. This group has five simple modules: the four 1-dimensional modules $V_{\varepsilon,\delta}$, where $\varepsilon, \delta \in \{-1, 1\}$, where s acts via ε and t via δ; and one 2-dimensional simple $D_{2\cdot 4}$-module which we denote by V_2.

The cell structure of $\mathscr{S}_{2\cdot 4}$ looks as follows (cf. [38, Section 4.4]):

θ_e

	$\mathcal{L}_s$	$\mathcal{L}_t$
$\mathcal{R}_s$	θ_s, θ_{sts}	θ_{st}
$\mathcal{R}_t$	θ_{ts}	θ_t, θ_{tst}

θ_{w_0}

Here the two-sided cells are

$$\mathcal{J}_1 = \{\theta_e\},\ \mathcal{J}_2 = \{\theta_s, \theta_t, \theta_{st}, \theta_{ts}, \theta_{sts}, \theta_{tst}\},\ \mathcal{J}_3 = \{\theta_{w_0} = \theta_{stst} = \theta_{tsts}\}.$$

We see that $|\mathcal{L}_s \cap \mathcal{R}_s| = 2 \neq 1$ and hence the two-sided cell $\mathcal{J}_2$ is not strongly regular and we cannot apply [24, Theorem 18].

The main result of Paper I is the following.

Theorem 2. *Each simple transitive* 2*-representation of the* 2*-category of Soergel bimodules over the coinvariant algebra of type* B_2 *is equivalent to a cell* 2*-representation.*

In other words we could show that the results of [24, Theorem 18] generalize to our case. Moreover, we showed that

$$\begin{aligned} [\mathbf{C}_{\mathcal{L}_e}]^{\mathbb{C}} &\simeq V_{-1,-1}, & [\mathbf{C}_{\mathcal{L}_s}]^{\mathbb{C}} &\simeq V_2 \oplus V_{1,-1}, \\ [\mathbf{C}_{\mathcal{L}_{w_0}}]^{\mathbb{C}} &\simeq V_{1,1}, & [\mathbf{C}_{\mathcal{L}_t}]^{\mathbb{C}} &\simeq V_2 \oplus V_{-1,1}. \end{aligned}$$

This shows that $\mathbf{C}_{\mathcal{L}_s}$ and $\mathbf{C}_{\mathcal{L}_t}$ are not equivalent even though they belong to the same two-sided cell. This was something that cannot happen in the strongly regular case where all cell 2-representations of left cells belonging to the same two-sided cell are equivalent.

Moreover, we showed that, for Soergel bimodules in Coxeter type $I_2(n)$, we have that all simple transitive 2-representations of rank one decategorify to either the sign or the trivial module. Further, we could prove that there are no simple transitive 2-representations of rank 2.

4.2 Paper II

Paper II is a continuation of Paper I and studies simple transitive 2-representations of the 2-category of Soergel bimodules in type $I_2(n)$, where $n \geq 5$. In Paper I we studied the case $n = 4$, the case $n = 3$ begin covered by the results of [24]. Before Paper II there were only two examples of a 2-category whose simple transitive 2-representations were not classified by cell 2-representations: see [24, Example 3.2] and [14]. The former is easy and quite artificial. The latter is technically very involved and served as our main technical tool for construction of 2-representations.

For a long time, the main difficulty with the case of Soergel bimodules in Coxeter type $I_2(n)$, for $n > 5$, was that they were computationally quite involved and long. More precisely, in the last part of the approach described in 3.7 we had to show that the action of 1-morphisms on a simple transitive 2-representation maps any non-zero object to a projective object. The proofs of this fact which were given before Paper II were quite lengthy and computationally involved. Thus a classification of simple transitive 2-representations for Soergel bimodules in type $I_2(n)$ for $n \geq 5$ seemed unrealistic. One of the crucial results of Paper II is therefore the following.

Theorem 3. *Let* $\mathbf{M}$ *be a simple transitive* 2*-representation of a fiat* 2*-category* $\mathscr{C}$ *with apex* $\mathcal{I}$*. Then, for every* 1*-morphism* $F \in \mathcal{I}$ *and object* $X \in \overline{\mathbf{M}}(\mathtt{i})$*, the object* $F\ X$ *is projective. Moreover,* $\overline{\mathbf{M}}(F)$ *is a projective functor.*

With the help of this theorem we managed to classify all simple transitive 2-representations of the 2-category of Soergel bimodules in Coxeter type $I_2(n)$, where $n \geq 3$, for $n \notin \{12, 18, 30\}$. We could show the following:

Theorem 4. *Let* $\mathscr{S}_n$ *be the* 2*-category of Soergel bimodules of Coxeter type* $I_2(n)$ *over the coinvariant algebra.*

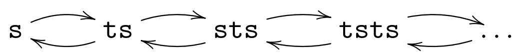
(*i*) *If* $n = 4$ *or* $n > 1$ *and* n *is odd, then every simple transitive* 2*-representation of* $\mathscr{S}_n$ *is equivalent to a cell* 2*-representation.*

(*ii*) *If* $n > 4$ *and even, then, apart from the cell* 2*-representations, there are two extra equivalence classes of simple transitive* 2*-representations which can be constructed explicitly. If* $n \neq 12, 18, 30$*, these are all simple transitive* 2*-representations.*

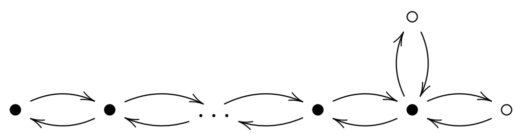
Observe that the case $n = 4$ was already dealt with in Paper I. The construction of the two non-cell 2-representations follows the ideas of [14] and is briefly described below.

As described in Section 3.7, the proof of this theorem is a case-by-case analysis done in two steps. At first we classify all the matrices which can possibly represent the action of indecomposable Soergel bimodules corresponding to simple reflections. Then we classify all simple transitive 2-representations which realize the matrices described in the first step. In the case of $n \in \{12, 18, 30\}$ there are in each case matrices for which we cannot construct simple transitive 2-representations which realize these matrices nor can we argue why such simple transitive 2-representation cannot exist. In a subsequent paper, Mackaay and Tubbenhauer [18] managed to construct the missing simple transitive 2-representations. Additionally they could show that, under the additional assumption of gradeability, their simple transitive 2-representations are, up to equivalence, all.

In the case when n is even, we observed that the algebra underlying $\overline{\mathbf{C}_{\mathcal{L}_s}}(\mathtt{i})$ is the path algebra of the double quiver $\mathtt{L}$ of Dynkin type A_{n-1}



modulo the relations that any path of the form $\mathtt{v} \to \mathtt{u} \to \mathtt{w}$ is zero if $v \neq w$ and all paths of the form $\mathtt{v} \to \mathtt{u} \to \mathtt{v}$ coincide. Using ideas and results from [14], we constructed a 2-representation $\mathbf{L}$ equivalent to $\mathbf{C}_{\mathcal{L}_s}$ with an involutive automorphism $\mathbf{\Phi}$. This automorphism swaps the projective P_x with $P_{x'w_0}$, for all vertices x in the quiver above, were x' is the element obtained by swapping s and t in the reduced expression of x. Moreover, we could, again, using ideas from [14], define a simple transitive 2-representation $\mathbf{N}_s^{(n)}$ on the orbit category $\mathbf{L}(\mathtt{i})^{\mathbf{\Phi}}$. Similarly, one constructs $\mathbf{N}_t^{(n)}$ starting from $\mathcal{L}_t$. The underlying algebra of $\mathbf{N}_s^{(n)}(\mathtt{i})$ is the path algebra of the double quiver of Dynkin type D



where we mod out by the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. Here the vertices $\bullet$ correspond to pairs $\{P_x, P_{x'w_0}\}$, where $x \in \mathtt{L}$ is such that $x \neq x'w_0$, while the two vertices $\circ$ correspond to the unique $x \in \mathtt{L}$ for which $x'w_0 = x$. Comparing decategorifications one can show that $\mathbf{C}_{\mathcal{L}_s}, \mathbf{C}_{\mathcal{L}_t}, \mathbf{N}_s^{(n)}, \mathbf{N}_t^{(n)}$ are all pairwise non-equivalent, for $n > 4$.

Moreover, Paper II deals with certain quotients of Soergel bimodules of finite Coxeter systems (W, S) with rank higher than two. The first observation here is that, for all $s \in S$, we have that all the corresponding irreducible Soergel bimodules θ_s lie in the same two-sided cell $\mathcal{J}$. Moreover, $\mathcal{J}$ is the unique minimal two-sided cell, with respect to the two-sided order, which differs from the cell corresponding to θ_e. Thus we can consider the maximal ideal $\mathscr{J}$ of $\mathscr{S}$ which does not contain id_F for any $F \in \mathcal{J}$. Then we define the *small quotient* $\underline{\mathscr{S}}$ of $\mathscr{S}$ to be the quotient 2-category $\mathscr{S}/\mathscr{J}$. By construction, this is a $\mathcal{J}$-simple fiat 2-category.

Our main result here is the following.

Theorem 5. *Let $\underline{\mathscr{S}}$ be the small quotient of the 2-category of Soergel bimodules for a finite Coxeter group of rank greater than two and $\mathbf{M}$ a simple transitive 2-representation of $\underline{\mathscr{S}}$. Then $\mathbf{M}$ is equivalent to a cell 2-representation.*

4.3 Paper III

In this paper we study two problems which are not directly related to 2-representation theory but rather have combinatorial flavor. However, these problems are motivated by the methods which were applied in many classification results for simple transitive 2-representations of various 2-categories.

The first result which we obtain is

Theorem 6. *Let $p \in \mathbb{C}[x]$. Then the set*

$$\bigcup_{k>0} \{A \in \mathrm{Mat}_{k\times k}(\mathbb{Z}_{\geq 0}) \mid A \text{ is irreducible and } p(A) = 0\}$$

is finite.

The motivation for this questions stems from the fact that in many cases when we study simple transitive 2-representations for some finitary 2-category $\mathscr{C}$ we often know a polynomial which has to annihilate some 1-morphism of $\mathscr{C}$. For example, in the case of Soergel bimodules corresponding to a Weyl group W we know that the complexified Grothendieck group of $\mathscr{S}_W$ is isomorphic to $\mathbb{C}[W]$ and hence a 2-representation of $\mathscr{S}_W$ decategorifies to a representation of $\mathbb{C}[W]$. Then the matrix M of $F = \bigoplus_{s\in S} \theta_s$ is annihilated by the product p of the characteristic polynomials of the element F taken over all simple $\mathbb{C}[W]$ modules. Moreover M has to be an irreducible, integral matrix. So a strategy which we applied in e.g., [38, 12] was to classify all such matrices which were annihilated by p. In these cases it turned out that the number of possible matrices was not only finite but small enough for a case by case analysis.

In the case of the 2-category $\mathscr{C}_A$ we can consider $F = \bigoplus_{i,j} F_{ij}$ which we know satisfies $F \circ F = F^{\bigoplus \dim(A)}$. Thus the matrix M of the action of F for any simple transitive 2-representation $\mathbf{M}$ satisfies $\mathsf{M}^2 = \dim(A)\mathsf{M}$, i.e. it is a quasi-idempotent. Moreover, it has to irreducible as $\mathbf{M}$ is simple transitive and integral since it records multiplicities. This is the motivation for the second question which we studied in this paper. Since the answer to the first question was positive, we know that there are only finitely many matrices corresponding to the action of F and hence a natural questions was if we can describe them better and maybe even count how many there are.

In the setup which motivates these questions we do not have one preferred ordering of basis vectors which means that it is natural to consider the sets

$$K_n = \bigcup_{k>0} \{\mathsf{M} \in \mathrm{Mat}_{k\times k}(\mathbb{Z}_{\geq 0}) \mid \mathsf{M}^2 = n\mathsf{M}, \mathsf{M} \text{ is irreducible}\}$$
$$L_n = K_n / \cong$$

where we say that $\mathsf{M} \cong \mathsf{N}$ if there exists a permutation matrix P such that $P^{-1}\mathsf{M}P = \mathsf{N}$.

We show that the sets K_n and L_n can be described in terms of (generalized) composition and (generalized) partitions of n, respectively. Unfortunately, we cannot attain a closed formula for either of the cardinalities of K_n or L_n but the description via compositions/partitions enabled us to write some simple code to obtain the value for $1 \leq n \leq 10000$. Moreover, an analysis of the asymptotics of K_n and L_n shows that K_n behaves asymptotically like the sequence of generalized compositions and L_n like the sequence of generalized partitions. Both of these sequences are well-studied with known asymptotic behavior.

4.4 Paper IV

In Paper IV we study simple transitive 2-representations of the 2-category $\mathscr{C}_A$ of projective functors for a certain infinite family of finite dimensional algebras. As mentioned in Chapter 1, we know that simple transitive 2-representations for $\mathscr{C}_A$ are classified by cell 2-representations for finite dimensional self-injective algebras; see [24, 22].

A similar result was obtained for other classes of finite dimensional algebras in [27, 28] and in [25]. In [28] its was shown that all simple transitive 2-representations of $\mathscr{C}_A$ are equivalent to cell 2-representations in case of directed algebras having a non-zero projective-injective module. In Paper IV we extend the method developed in [28] to not necessarily directed algebras which still have a non-zero projective-injective module. Our algebras are certain quotients of quadratic duals of preprojective algebras associated to trees, cf. [30]. We show that also in this case simple transitive 2-representations are equivalent to cell 2-representations.

We consider the following algebras. Let $T = (V, E)$ be a tree and $L \subseteq V$ the set of its leaves. Then consider the double quiver $Q = Q_T = (V, \hat{E})$ of T, i.e., every edge $e = \{i, j\}$ is replaced by two arrows (i, j) and (j, i) in Q_T. Lastly, let $S \subseteq L$ be a, possibly empty, subset of marked leaves. We want to define a certain quotient $A_{T,S}$ of the path algebra $\Bbbk Q$ of Q. For this let $\mathcal{I}$ be the ideal of $\Bbbk Q$ generated by the following relations:

- For all pairwise distinct vertices $v_1, v_2, v_3 \in V$ such that there are arrows $a_1, a_2 \in \hat{E}$ with
$$v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} v_3,$$
we set $a_2 a_1 = 0$.

- For all pairwise distinct vertices $v_1, v_2, v_3 \in V$ such that there exist arrows $a_1, a_2, b_1, b_2 \in \hat{E}$ with
$$v_1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} v_2 \underset{b_2}{\overset{a_2}{\rightleftarrows}} v_3,$$
we set $a_1 b_1 = b_2 a_2$.

- For $v \in V$ and $s \in S$ such that there are arrows $a, b \in \hat{E}$ with
$$v \underset{b}{\overset{a}{\rightleftarrows}} s,$$
we set $ab = 0$.

The algebra $A_{T,S}$, which we will denote simply by A, is now defined as the quotient of $\Bbbk Q$ by the ideal $\mathcal{I}$.

The main result of Paper IV is the following.

Theorem 7. *Let* $\mathbf{M}$ *be a simple transitive* 2*-representation of* $\mathscr{C}_A$*, then* $\mathbf{M}$ *is equivalent to a cell* 2*-representation.*

Note that if $S = \varnothing$, then A is weakly symmetric and hence is covered by [24].

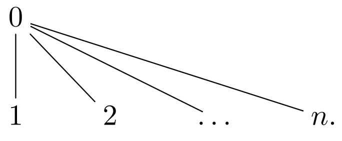
Shortly after this paper appeared on the arXiv, Mazorchuk, Miemietz and Zhang [26] proved that Theorem 7 holds for all finite dimensional associative algebra. Their approach is completely different and use two significant new ideas: a new version of the category of complexes, called the category of pyramids, and a non-trivial embedding of the $\mathscr{C}_A$ into a bigger 2-category which has some partial adjunction morphisms.

4.5 Paper V

As mentioned above, we know that, for every associative, finite dimensional algebra all simple transitive 2-representations of $\mathscr{C}_A$ are equivalent to cell 2-

representations; see [26, Theorem 12]. In Paper V we propose to study a 2-subcategory $\mathscr{C}$ of $\mathscr{C}_A$ in which we only consider the identity and one of the left cells with no restrictions on the right cells. In this way we hope to break the "left-right symmetry" of $\mathscr{C}_A$ and thus to produce a 2-category with a completely different behavior.

We study the 2-category $\mathscr{C}$ for so-called *star algebras*. A *star* S_n is a graph on $n+1$ vertices which looks as follows:



Let $\overline{S_n}$ denote the double quiver of S_n (in which $0 \text{---} i$ is replaced by $0 \underset{a_i}{\overset{b_i}{\rightleftarrows}} i$) and consider the path algebra $\Bbbk\overline{S_n}$ of $\overline{S_n}$. We then define Λ_n as the quotient of $\Bbbk\overline{S_n}$ by certain relations. In case $n = 1$ we require that

- $b_1a_1b_1 = a_1b_1a_1 = 0.$

If $n > 1$, we require that

- for all $1 \leq i, j \leq n$ we have $b_ia_i = b_ja_j$;
- for all $1 \leq i < j \leq n$ we have $a_jb_i = a_ib_j = 0$.

Note that these relations imply that, for all $1 \leq i \leq n$, we always have $a_ib_ia_i = b_ia_ib_i = 0$.

Let F_{ij} denote the projective Λ_n-Λ_n bimodule $\Lambda_n\varepsilon_i \otimes \varepsilon_j\Lambda_n$. Now we can define $\mathscr{C}_n$ to be the 2-full 2-subcategory of $\mathscr{C}_{\Lambda_n}$ given by the additive closure of $\mathbb{1}_{\mathtt{i}}, F_{00}, F_{10}, \dots, F_{n0}$.

We show that in the case $n = 1$ all simple transitive 2-representations of $\mathscr{C}_n$ are equivalent to cell 2-representations. However, for $n > 1$, at the moment we cannot classify all simple transitive 2-representations of $\mathscr{C}_n$. What we can prove is the following.

Theorem 8. *Let* $\mathbf{M}$ *be a simple transitive* 2*-representation of* $\mathscr{C}$ *of rank* r *with apex* $\mathcal{L}$*. Then there exists an ordering of the isomorphism classes of indecomposable objects in* $\mathbf{M}(\mathtt{i})$ *and a surjective function*

$$\varphi : \{1, 2, \dots, n\} \to \{2, 3, \dots, r\}$$

such that

$$\mathsf{M}_0 = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathsf{M}_i = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 2 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad (4.1)$$

where the index of the non-zero row of M_i *is* $\varphi(i)$.

Observe that, for each $1 \leq r \leq n$, these are the matrices of the cell 2-representation of $\mathscr{C}_r$. Therefore we conjecture, for $n > 1$, the following:

Conjecture 8.1. *Equivalence classes of simple transitive* 2*-representations of* $\mathscr{C}_n$ *are in bijection with set partitions of* $\{1, 2, \ldots, n\}$.

The idea is that one can maybe lift the cell 2-representation of $\mathscr{C}_r$ to $\mathscr{C}_n$, for $1 \leq r \leq n$. However, we cannot provide a precise construction of such lift at the moment.

5. Summary in Swedish - Sammanfattning på svenska

Den här sammanfattningen består av tre delar. Den första inledande delen som på ett informellt sätt ska ge en känsla för vad representationsteori handlar om och som förutsätter väldigt lite förkunskap. Den andra delen ger en kort introduktion till området av 2-representationer och 2-kategorier, samt en kortfattad historisk bakgrund; här krävs det lite större matematisk förkunskap. Den sista delen sammanfattar avhandlingens huvudresultat.

5.1 Bakgrund - den klassiska världen

I abstrakt algebra studerar man, som namnet antyder, abstrakta algebraiska sturkturer så som grupper, ringar, kroppar och algebror. Med en algebraisk struktur menar vi här en mängd S av element och ett antal operationer på mängden som uppfyller vissa räkneregler. En grupp är till exempel en mängd G tillsammans med en binär operation på G som är associativ. Vidare ska G innehålla ett neutralt element med avseende på operationen och till varje element ska det finnas en invers. En algebra däremot är ett vektorrum över en kropp med en bilinjär multiplikation.

För alla dessa strukturer finns det exempel som vi är vana vid. Mängden av alla heltal tillsammans med addition av heltal är till exempel. en grupp, de reella talen tillsammans med vanlig addition och multiplikation en kropp, och mängden av alla polynom över en kropp tillsammans med punktvis addition och multiplikation en algebra. Det finns dock många fler exempel för varje sådan struktur. Fördelen med abstrakta studier av exempelvis grupper är att om man kan visa en viss egenskap hos en grupp enbart med hjälp av axiomen vet man att samma egenskap finns i varje konkret exempel.

Den stora nackdelen med detta abstrakta synsätt är att man ofta inte har en bra intutiv förståelse för strukturerna som man har för t.ex. heltalen. Det var därför man i slutet på 1800-talet började skapa det som vi idag kallar för *representationsteori*. Idén här är att man istället för att studera abstrakta grupper representera gruppelementen som linjära transformationer på ett vektorrum. Fördelen är att linjära transformationer är välstuderade objekt. De har en geometrisk tolking och många andra goda egenskaper, tack vare vilka vi har en bättre intuition och förståelse för dem. Med hjälp av denna intuitiva förståelse kan man ofta gissa vilka egenskaper gruppen borde ha och hur man skulle kunna visa det.

En *representation* av en grupp G är en linjär avbildning $\varphi : G \to \mathrm{Hom}(V, V)$, där V är ett vektorrum och $\mathrm{Hom}(V, V)$ betecknar gruppen av alla linjära avbildningar från V till V. Dessutom har vi att multiplikation av element i G stämmer med sammansättning av de motsvarande linjära operatorerna. Fördelen med att representera gruppelement som linjära operatorer är att vi på så vis kan använda oss av linjär algebra och matristeori för att studera många problem. Man säger att gruppen G verkar på V och kallar V för en G-modul. Representationsbegreppet kan även definieras för andra (algebraiska) strukturer som algebror och kategorier.

Numera är representationsteori ett mycket aktivt forskningområde i sin egen rätt med många intressanta och öppna problem. Ett typiskt sådant problem är att beskriva alla representationer som en given struktur har. I de allra flesta fall är en klassifikation av alla representationer för svår. I stället fokuserar man ofta på vissa klasser av representationer. En sådan klass utgörs av så kallade *irreducibla* representationer som i någon mening utgör en mängd av "grundbyggstenar" som alla representationer är uppbyggda av.

5.2 Bakgrund - 2-världen

Den här avhandlingen handlar om representationsteori, dock inte för grupper eller algebror, utan för så kallade finitära 2-kategorier. En finitär 2-kategori är en typ av 2-kategori som uppfyller några ändlighetskrav, så som till exempel att det bara finns ett ändligt antal objekt. Motivation för definitionen av finitära 2-kategorier härstammar från algebror av ändlig dimension och i synnerhet kategorin $A\text{-proj}$ för en ändligdimensionell algebra A.

För att studera 2-kategorier använder vi oss av samma idé som ovan, det vill säga vi representerar dem med hjälp av en "modellkategori". Här finns det inte en lika självklar modell som vektorrummen ovan. Beroende på situationen finns det olika kategorier att välja bland som abelska eller finitära kategorier. En 2-representation är sedan en (strikt) 2-funktor från vår 2-kategori till 2-kategorin av exempelvis finitära kategorier.

2-representationsteori är fortfarande en väldigt ny teori. Den har sitt ursprung i en serie av sex artiklar av Mazorchuk och Miemietz. I dessa artiklar definieras och visas många begrepp och resultat som har motsvarigheter i den klassiska representationsteorin. Vi är dock långt ifrån att ha en lika bra förståelse för 2-representationsteori som vi har för den klassiska världen.

Avhandlingens fokus ligger på studiet av "enkla" 2-representationer som är motsvarigheten till irreducibla representationer och som, i en lite svagare men analog mening, är grundbyggstenar till 2-representationer. De här "enkla" 2-representationerna kallas för *enkla transitiva* 2-representationer.

5.3 Sammanfattning av avhandlingens resultat

Den här avhandlingen består av fem artiklar. Fyrar av dem handlar om klassifikation av enkla transitiva 2-representationer av olika finitära 2-kategorier, och en om ett relaterat kombinatorisk problem.

I Artikel I studerar vi enkla transitiva 2-representationer av Soergelbimdouler av Dynkintyp B_2 (vilket är samma som Coxetertyp $I_2(4)$). Vårt huvudresultat är att alla enkla transitiva 2-representationer i det här fallet är ekvivalenta med cell-2-representationer. Vi kunde också visa att de cell-2-representationer som motsvarar till olika vänsterceller i den stora tvåsidiga cellen inte är ekvivalenta, vilket det inte hade funnits några tidigare exempel för.

I Artikel II klassificerar vi enkla transitiva 2-representationer av så kallade *små kvoter* av Soergelbimoduler för alla ändliga Coxetertyper förutom typ $I_2(12), I_2(18)$ och $I_2(30)$. För nästan alla Coxetertyper kan vi visa att alla enkla transitiva 2-representationer är ekvivalenta med cell-2-representationer, förutom för typ $I_2(2k)$ där det finns fler enkla transitiva 2-representationer. Vi konstruerar två nya enkla transitiva 2-representationer för varje k och kan visa att dessa två tillsammans med cell-2-representationerna klassificerar alla enkla transitiva 2-representationer ifall $k \notin \{6, 9, 15\}$.

I Artikel III studerar vi två kombinatoriska problem som motiveras av problem som ofta dyker upp i bevisen för klassifikationsresultat för enkla transitiva 2-representationer. Den första frågan är om mängden av alla irreducibla matriser med heltalskoefficienter som annihileras av ett givet polynom är ändlig. Vi visar att mängden är ändlig för alla polynom och analyserar sedan talföljden som ges av antalet irreducibla heltalsmatriser som annihileras av polynomet $X^2 - nX$.

I Artikel IV klassificerar vi enkla transitiva 2-representationer för 2-kategorin av projektiva funktorer över vissa typer av algebror. Vi visar att alla enkla transitiva 2-representationer är ekvivalenta med cell-2-representationer i det här fallet.

Den sista artikeln handlar också om klassifikationsproblem för enkla transitiva 2-representationer. Den här gången för vissa 2-delkategorier av 2-kategorin av projektiva funktorer för det vi kallar för stjärnalgebror. I det enklaste fallet, när algebran är tvådimensionell, kan vi återigen visa att alla enkla transitiva 2-representationer är ekvivalenta med cell-2-representationer. Däremot visar vi några numeriska resultat i fallet att dimensionen är större än två. Det låter oss förmoda att det finns fler enkla transitiva 2-representationer, som inte är ekvivalenta till cell-2-representationer. I nuläget kan vi dock inte konstruera någon sådan.

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We have a tendency to define ourselves in opposition to stuff. But try to also express your passion for things you love. Be demonstrative and generous in your praise of those you admire. Send thank-you cards and give standing ovations. Be pro-stuff, not just anti-stuff.

- Tim Minchin

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