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Homological Algebra for Quiver Representations

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HOMOLOGICAL ALGEBRA FOR QUIVER REPRESENTATIONS

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Abstract.
Quiver representations arise naturally in the study of representation theory for associative algebras. A particularly simple case to consider is that of representations of finite acyclic quivers, which is the object of study of this thesis.
In this thesis we give an explicit presentation of some elementary constructions in the category of quiver representations, which then allows us to formulate and discuss some basic homological algebra within that category. The tools of homological algebra can then be applied to study the canonical problem of representation theory: classification of the indecomposable representations.

Contents

1. Introduction

Quivers, and in particular their representations, owe their significance to their role as an important tool in study of representations of finite-dimensional algebras (and more generally also Artinian algebras), especially due to the fundamental developments of that field made by Peter Gabriel (resulting in Gabriel's famous theorem presented in [3]) and those made by Maurice Auslander and Idun Reiten, resulting in an entire field of study known as Auslander-Reiten theory (basic elements of which are introduced in the following text; for a more detailed and advanced text we refer to [2]).

The following text focuses on finite-dimensional representations of finite acyclic quivers, and hence treats them as an independent object of study rather than a tool used in study of other objects. The first part of the text introduces the category $\mathrm{Rep}_K Q$ of finite-dimensional representations of a given finite acyclic quiver together with its fundamental properties (in particular, $\mathrm{Rep}_K Q$ is a K-linear category), gives explicit descriptions of some essential categorical notions realized in quiver representations, such as direct sums and products, pullbacks and pushouts and also kernels and cokernels.

Second part introduces projective and injective representations, describes the fundamental family $\{P(i)\}_{i\in Q_0}$ of indecomposable projective representations, and similarly a family $\{I(i)\}_{i\in Q_0}$ of indecomposable injective representations, and then shows that those families generate the projective respectively injective representations of given quiver Q, that is, given a projective representation P, P can be decomposed as a direct sum with summands in said family, $P = \bigoplus_{i\in Q_0}(\bigoplus_{j=1}^{n_i} P(i))$, and similarly for injective representations. The extensive presentation of projective representations is followed up by a discussion of projective resolutions, concluding that every representation admits a projective resolution of length 1. Those facts then allow us to conclude that similar statements hold for injective representations of Q, when considered (due to a duality between $\mathrm{Rep}_K Q$ and $\mathrm{Rep}_K Q^{op}$ shown in the same part) as the projective representations of the opposite quiver Q^{op}. The second part concludes with a brief introduction to the Auslander-Reiten translation - a map $\tau : \mathrm{Rep}_K Q \to \mathrm{Rep}_K Q$ whose significance is revealed in the fourth and final part of the text.

The third part of the text gives a more concrete description of the relation between representation theory of quivers and that of finite-dimensional algebras, by presenting an equivalence of $\mathrm{Rep}_K Q$ and mod-KQ - the category of finite dimensional KQ-modules, where KQ denotes the path algebra of Q. It focuses more on homological algebra of those modules, introducing important notions such as almost split sequences and irreducible morphisms and their direct relation to the fundamental question of representation theory (here of finite-dimensional algebras, although for this text the path algebras of finite acyclic quivers, being a proper subset of finite-dimensional algebras, is the object of focus) - the classification of indecomposable representations.

This question is even more relevant to the fourth part of the text, which reveals some important properties of the aforementioned Auslander-Reiten translation, most notably it being a bijection between the set of indecomposable non-injective modules and that of indecomposable non-projective modules. It also introduces the Auslander-Reiten quiver Γ(mod-A) of mod-A (where A denotes a finite-dimensional algebra), which, using τ gives a visual representation of the structure of indecomposable modules of A, hence providing crucial information about the category mod-A itself.

Finally it should be noted that the first half of the text follows (roughly) chapters 1 and 2 of [7], whereas the latter half consists of parts of chapter IV of [1]. Hence the reader is referred to those texts for a more thorough exposition of the material presented here.

2. First definitions

Definition 2.1. A quiver $Q = (Q_0, Q_1, s, t)$ consists of

- Q_0 - a set of vertices
- Q_1 - a set of arrows
- $s : Q_1 \to Q_0$ - a map mapping an arrow to its starting point
- $t : Q_1 \to Q_0$ - a map mapping an arrow to its terminal point

Definition 2.2. M is a representation of Q over field K if and only if $M = (M_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$, where M_i is a K-vector space, and $\varphi_\alpha : M_i \to M_j$ is a K-linear map, provided that $s(\alpha) = i$, $t(\alpha) = j$

Throughout this text we assume the following:

- All representations considered are finite-dimensional, i.e. each M_i is finite-dimensional.
- All representations are over an algebraically closed field K (unless otherwise stated).

Definition 2.3. Let $M = (M_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1}$, $M' = (M'_i, \varphi'_\alpha)_{i\in Q_0,\alpha\in Q_1}$ be two representations of quiver Q. A morphism of representations $f : M \to M'$ is a collection of linear maps $(f_i)_{i\in Q_0}$ such that for each $\alpha \in Q_1$, the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"] & M'_j \end{tikzcd}$$

In particular, f is an isomorphism if each f_i is an isomorphism of vector spaces.
Finally, we denote the set of morphisms from M to M' by $\mathrm{Hom}(M, M')$

Remark 2.4. *Given representations M, N, P of quiver Q, and given morphisms $g \in \mathrm{Hom}(M, N)$, $f \in \mathrm{Hom}(N, P)$, let $f \circ g = (f_i \circ g_i)_{i\in Q_0}$. Clearly $f \circ g$ is in $\mathrm{Hom}(M, P)$ and thus representations over K of Q together with their morphisms constitute a category, denoted by $\mathrm{Rep}_K Q$.*

Proposition 2.5. *Given two representations of Q, $M, M' \in \mathrm{Rep}_K Q$, the set of morphism $\mathrm{Hom}(M, M')$ forms a vector space, i.e. we have $\mathrm{Hom}(M, M') \in \mathrm{Vect}_K$.*

Proof. Let $f, g \in \mathrm{Hom}(M, M')$ We know that $\forall i \in Q_0 : f_i, g_i \in Hom(M_i, M'_i) \in Vect_K$ and thus by setting $f + g = (f_i + g_i)_{i\in Q_0}$, $kg = (kg_i)_{i\in Q_0} (\forall k \in K)$ we can satisfy all the vector space properties, provided that our morphisms are well defined, which is checked as follows:

$$\varphi'_\alpha \circ (f_i + g_i) = \varphi'_\alpha \circ f_i + \varphi'_\alpha \circ g_i = f_j \circ \varphi_\alpha + g_j \circ \varphi_\alpha = (f_j + g_j) \circ \varphi_\alpha$$

and

$$\varphi'_\alpha \circ kf_i = k(\varphi'_\alpha \circ f_i) = k(f_j \circ \varphi_\alpha) = kf_j \circ \varphi_\alpha$$

□

Proposition 2.6. $0 = (0_i, 0_\alpha)$ *is a zero object of* $\mathrm{Rep}_K Q$ *i.e. it is both initial and terminal in* $\mathrm{Rep}_K Q$ *.*

Proof. Let $M = (M_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}_K Q$
0 is initial: $\forall i \in Q_0, 0_i$ is initial in Vect_K i.e. $\exists! \psi \in \mathrm{Hom}(0_i, M_i) = 0$ and $0 \circ \varphi_\alpha = 0 = 0 \circ 0$, so the morphism is unique and well-defined.
0 is terminal: $\forall i \in Q_0, 0_i$ is terminal in Vect_K i.e $\exists! \lambda \in \mathrm{Hom}(M_i, 0_i) = 0$ and $\varphi_\alpha \circ 0 = 0 = 0 \circ 0$, so the morphism is unique and well-defined. □

Definition 2.7. Let $M = (M_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}_K Q$ and let $M' = (M'_i, \varphi'_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}_K Q$. We define the direct sum of M and M' as $M \oplus M' = (M_i \oplus M'_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix})$

Proposition 2.8. *Let $M, M' \in \mathrm{Rep}_K Q$ then $M \oplus M'$ is:*
a) The product $M \prod M'$ of M and M' in $\mathrm{Rep}_K Q$
b) The coproduct $M \coprod M'$ of M and M' in $\mathrm{Rep}_K Q$

Proof. For both parts we use the fact that the notions of finite product and finite coproduct coincide in Vect_K, i.e for $V, W \in \mathrm{Vect}_K$, we have $V \oplus W \simeq V \prod W \simeq V \coprod W$
First let $N = (N_i, \gamma_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}_K Q$
a) Let $f \in \mathrm{Hom}(N, M), f' \in \mathrm{Hom}(N, M')$.
Let $\pi = (\pi_i)_{i\in Q_0} : M \oplus M' \to M$, where $\pi_i = \begin{bmatrix} 1 & 0 \end{bmatrix}_i$, (i.e define π pointwise as projections from the direct sum onto the left summand), $g = (g_i)_{i\in Q_0} = \begin{bmatrix} f_i \\ f'_i \end{bmatrix} : N \to M \oplus M'$ (i.e define g pointwise as morphisms induced by product property of $M \oplus M'$). The following diagram then commutes for all $i \in Q_0$ due to product property of $M \oplus M'$:

$$\begin{tikzcd}
 & N_i \arrow[dl, "f_i"'] \arrow[d, dashed, "g_i"] \arrow[dr, "f'_i"] & \\
M_i & (M \oplus M')_i \arrow[l, "\pi_i"] \arrow[r, "\pi'_i"'] & M'_i
\end{tikzcd}$$

Moreover, the choice of g is unique as g_i is the unique morphism making said diagram commute. So all that needs to be checked is that π and g are well-defined, i.e for all $\alpha \in Q_1$ these diagrams commute:

$$\begin{tikzcd}
(M \oplus M')_i \arrow[r, "\begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{pmatrix}"] \arrow[d, "\pi_i"'] & (M \oplus M')_j \arrow[d, "\pi_j"] \\
M_i \arrow[r, "\varphi_\alpha"] & M_j
\end{tikzcd}$$

$$\begin{tikzcd}
N_i \arrow[r, "\gamma_\alpha"] \arrow[d, "g_i"'] & N_j \arrow[d, "g_j"] \\
(M \oplus M')_i \arrow[r, "\begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{pmatrix}"] & (M \oplus M')_j
\end{tikzcd}$$

And indeed we have

$$\varphi_\alpha \circ \pi_i = \varphi_\alpha \circ \begin{bmatrix} 1 & 0 \end{bmatrix}_i = \begin{bmatrix} \varphi_\alpha & 0 \end{bmatrix}_j = \begin{bmatrix} 1 & 0 \end{bmatrix}_j \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} = \pi_j \circ \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix}$$

and also

$$\begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \circ g_i = \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \circ \begin{bmatrix} f_i \\ f'_i \end{bmatrix} = \begin{bmatrix} \varphi_\alpha \circ f_i \\ \varphi'_\alpha \circ f'_i \end{bmatrix} = \begin{bmatrix} f_j \circ \gamma_\alpha \\ f'_j \circ \gamma_\alpha \end{bmatrix}$$

as both f and f' are well-defined morphisms, and finally $\begin{bmatrix} f_j \circ \gamma_\alpha \\ f'_j \circ \gamma_\alpha \end{bmatrix} = g_j \circ \gamma_\alpha$

b) Let $f \in \text{Hom}(M, N), f' \in \text{Hom}(M', N)$. Similarly to a), let $\iota = (\iota_i)_{i \in Q_0}$, where $\iota_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_i$ (i.e define $\iota : M \to M \oplus M'$ pointwise as inclusions of the left summand into the direct sum), $g = (g_i)_{i \in Q_0} : M \oplus M' \to N$, where $g_i \simeq \begin{bmatrix} f_i & f'_i \end{bmatrix}$ (i.e define g pointwise as morphisms induced by coproduct property of $M \oplus M'$. Just like in the case of product, our desired diagram now commutes pointwise for all $i \in Q_0$ and it also does so uniquely, all due to direct sum being the coproduct in Vect_K, and yet again all we need to check is that g and ι are well-defined. For ι we have:

$$\begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \circ \iota_i = \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix}_i = \begin{bmatrix} \varphi_\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_j \circ \varphi_\alpha$$

and for g we proceed as follows:

$$\gamma_\alpha \circ \begin{bmatrix} f_i & f'_i \end{bmatrix} = \begin{bmatrix} \gamma_\alpha \circ f_i & \gamma_\alpha \circ f'_i \end{bmatrix} = \begin{bmatrix} f_j & f'_j \end{bmatrix} \circ \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} = g_j \circ \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix}$$

□

Definition 2.9. $M \in \text{Rep}_K Q$ is indecomposable if and only if $M \neq 0$ and there are no non-zero representations L, N such that $M = L \oplus N$

Theorem 2.10 (Krull-Schmidt Theorem)**.** *Given a representation $M \in \text{Rep}_K Q$, M admits a unique (up to reordering of summands) direct sum decomposition into indecomposable summands, i.e $M \simeq M_1 \oplus \cdots \oplus M_n$ such that $M_1, \ldots, M_n$ are indecomposable.*

Proof. Theorem 1.2 in [7] □

Proposition 2.11. *$\mathrm{Rep}_K Q$ has pullbacks, i.e., given the following diagram in $\mathrm{Rep}_K Q$:*

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & D & \end{tikzcd}$$

there is a representation $X \in \mathrm{Rep}_K Q$ together with morphisms $\rho_A \in \mathrm{Hom}(X, A), \rho_B \in \mathrm{Hom}(X, B)$ constituting a pullback of the diagram.

Lemma 2.12. *The category R-Mod of left R-modules (for some ring R) has pullbacks.*

Proof. Given a diagram as above in R-Mod, let $X = \{(x, y) \in A \oplus B : f(x) = g(y)\} = \mathrm{Ker}(d)$, where $d = (f, -g) : A \oplus B \to D$, and let ρ_A, ρ_B be the restrictions of canonical projections π_A, π_B to X. Clearly $g \circ \rho_B = f \circ \rho_A$, so X completes the pullback square, thus it remains to show that the square is universal, i.e. given $Y \in R\text{-Mod}, q_A \in \mathrm{Hom}(Y, A), q_B \in \mathrm{Hom}(Y, B)$ such that $f \circ q_A = g \circ q_B$, $\exists! h : Y \to X$ such that $q_B = \rho_B \circ h, q_A = \rho_A \circ h$, which is illustrated by following diagram:

$$\begin{tikzcd} Y \arrow[dr, dashed, "h"] \arrow[drr, bend left, "q_B"] \arrow[ddr, bend right, "q_A"'] & & \\ & X \arrow[r, "\rho_B"] \arrow[d, "\rho_A"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & D \end{tikzcd}$$

We achieve that by letting $h = (q_A, q_B) = \left[\begin{smallmatrix} q_A \\ q_B \end{smallmatrix}\right]$ (which is well-defined as $f \circ q_A = g \circ q_B$). Clearly we have

$$\rho_A \circ h = \rho_A \circ (q_A, q_B) = q_A, \rho_B \circ h = \rho_B \circ (q_A, q_B) = q_B$$

so the diagram commutes. Uniqueness of h can be proven as following: given $h' \in \mathrm{Hom}(Y, X)$, $\rho_A \circ h' = \rho_A \circ h$, $\rho_B \circ h' = \rho_A \circ h$ and thus $h' = h$, but both those conditions are necessary for h' to satisfy the diagram above, and thus if h' does so, we get $h' = h$. □

This gives us a foundation to prove the same property for $\mathrm{Rep}_K Q$ just like we did with earlier statements: by borrowing the pointwise construction from Vect_K ($= K$-Mod) and endowing it with some canonical choice of arrow-wise structure so that all the morphisms are well defined.

Proof of proposition 2.11. Let $X = (X_i, \left[\begin{smallmatrix} \varphi_\alpha\restriction_{X_i} & 0 \\ 0 & \psi_\alpha\restriction_{X_i} \end{smallmatrix}\right])_{i \in Q_0, \alpha \in Q_1}$ (where $A = (A_i, \varphi_\alpha), B = (B_i, \psi_\alpha)$), where X_i is the pointwise pullback of

$$\begin{tikzcd} A_i \arrow[dr, "f_i"'] & & B_i \arrow[dl, "g_i"] \\ & D_i & \end{tikzcd}$$

Let $\rho_A : X \to A = (\rho_{A_i})_{i \in Q_0}$, $\rho_B : X \to B = (\rho_{B_i})_{i \in Q_0}$. Then $\rho_A \in \mathrm{Hom}(X, A)$ as $\varphi_\alpha \circ \rho_{A_i} = \varphi_\alpha \circ \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]_i = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]_i \circ \left[\begin{smallmatrix} \varphi_\alpha\restriction_{X_i} & 0 \\ 0 & \psi_\alpha\restriction_{X_i} \end{smallmatrix}\right]$. Analogous statement holds for ρ_B. Thus, X completes the commutative square. We still need to prove that X does it in a universal manner. Let $Y = (Y_i, \omega_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{Rep}_K Q, q_A \in \mathrm{Hom}(Y, A), q_B \in \mathrm{Hom}(Y, B)$ such that $f \circ q_A = g \circ q_B$ Let $h = (h_i)_{i \in Q_0}$ be the morphism defined pointwise by X_i being pullback pointwise (i.e. X_i's universal property of pullback of diagram above in $Vect_K$). It is clear that h is the only possible candidate, as for all $i \in Q_0$, h_i is the unique morphism making the diagram commute at i, which is necessary for h to make the corresponding diagram in $\mathrm{Rep}_K Q$ commute. Thus we only need to show that h is well-defined, i.e. $h \in \mathrm{Hom}(Y, X)$. Recall from Lemma 2.12 that $h_i = (q_{A_i}, q_{B_i}) = \left[\begin{smallmatrix} q_A \\ q_B \end{smallmatrix}\right]$ We need to show that given $\alpha : i \to j \in Q_1$, we have $\left[\begin{smallmatrix} \varphi_\alpha\restriction_{X_i} & 0 \\ 0 & \psi_\alpha\restriction_{X_i} \end{smallmatrix}\right] \circ h_i = h_j \circ \omega_\alpha$ In particular, the following holds:

$$\left[\begin{smallmatrix} \varphi_\alpha\restriction_{X_i} & 0 \\ 0 & \psi_\alpha\restriction_{X_i} \end{smallmatrix}\right] \circ h_i = \left[\begin{smallmatrix} \varphi_\alpha\restriction_{X_i} & 0 \\ 0 & \psi_\alpha\restriction_{X_i} \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} q_{A_i} \\ q_{B_i} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \varphi_\alpha\restriction_{X_i} \circ q_{A_i} & 0 \\ 0 & \psi_\alpha\restriction_{X_i} \circ q_{B_i} \end{smallmatrix}\right]$$

Now since $q_A \in \mathrm{Hom}(Y, A)$ we have that $\varphi_\alpha \circ q_{A_i} = q_{A_j} \circ \omega_\alpha$ and that equality implies $\varphi_\alpha \restriction_{X_i} \circ q_{A_i} = q_{A_j} \circ \omega_\alpha$ A similar equality holds for B and ψ. Thus

$$\begin{bmatrix} \varphi_\alpha \restriction_{X_i} & 0 \\ 0 & \psi_\alpha \restriction_{X_i} \end{bmatrix} \circ h_i = \begin{bmatrix} \varphi_\alpha \restriction_{X_i} \circ q_{A_i} & 0 \\ 0 & \psi_\alpha \restriction_{X_i} \circ q_{B_i} \end{bmatrix} = \begin{bmatrix} q_{A_j} \circ \omega_\alpha \\ q_{B_j} \circ \omega_\alpha \end{bmatrix} = \begin{bmatrix} q_{A_j} \\ q_{B_j} \end{bmatrix} \circ \omega_\alpha = h_j \circ \omega_\alpha$$

□

Proposition 2.13. $\mathrm{Rep}_K Q$ *has pushouts, i.e. given the following diagram in* $\mathrm{Rep}_K Q$ *:*

$$\begin{tikzcd} & D \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

There is a representation $X \in \mathrm{Rep}_K Q$ *together with morphisms* $\iota_A \in \mathrm{Hom}(A, X), \iota_B \in \mathrm{Hom}(B, X)$ *constituting a pushout of the diagram.*

Lemma 2.14. R-Mod *has pushouts.*

Proof. Consider a diagram as above in R-Mod. Let $X = (A \oplus B)/K$, where $K = \{(f(d), g(-d) \mid d \in D\} = (f, -g)(D)$. Let $\iota : A \to A \oplus B, \iota' : B \to A \oplus B$ be the usual inclusions, $\pi : A \oplus B \to X$ be the canonical projection. Let $\gamma_A = \pi \circ \iota$, $\gamma_B = \pi \circ \iota'$. Clearly $\gamma_A \circ f = \gamma_B \circ g$ as $\pi \circ \iota \circ f - \pi \circ \iota' \circ g = \pi \circ (\iota \circ f - \iota' \circ g) = 0$ as $\mathrm{Im}(\iota \circ f - \iota' \circ g) = \mathrm{Ker}(\pi)$. So the pushout square commutes. Now we need to show that it indeed is a pushout square, i.e. has the following universal property: given $Y \in R$-Mod, $q_A \in \mathrm{Hom}(A, Y), q_B \in \mathrm{Hom}(B, Y)$ such that $q_A \circ f = q_B \circ g$, there is a unique morphism $h \in \mathrm{Hom}(X, Y)$ such that $\beta = h \circ \gamma_A, \beta' = h \circ \gamma_B$, which is illustrated by the diagram below:

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "\gamma_B"] \arrow[ddr, bend left, "q_B"] & \\ A \arrow[r, "\gamma_A"] \arrow[drr, bend right, "q_A"'] & X \arrow[dr, dashed, "h"] & \\ & & Y \end{tikzcd}$$

Let $\omega = (q_A, q_B)$ i.e. the unique morphism such that $\omega \circ \iota = q_A, \omega \circ \iota' = q_B$. We have

$$\omega \circ (\iota \circ f - \iota' \circ g) = \omega \circ \iota f - \omega \circ \iota' \circ g = q_A \circ f - q_B \circ g = 0$$

so $K \subset \mathrm{Ker}(\omega)$, and thus by factorization theorem there is a unique morphism $\eta : X \to Y$ such that $\eta \circ \pi = \omega$ Thus $\eta \circ \gamma_A = \eta \circ \pi \circ \iota = \omega \circ \iota = q_A$ and similarly for B, so the diagram commutes. η is also unique such morphism: if $\eta' \in \mathrm{Hom}(X, Y)$ makes the diagram commute, it must have $\eta' \circ \pi \circ \iota = q_A = \eta \circ \pi \circ \iota$, and $\eta' \circ \pi \circ \iota' = q_B = \eta \circ \pi \circ \iota'$ so by universal property of coproducts we have $\eta' \circ \pi = \eta \circ \pi$, and so $\eta' = \eta$ as π is epi. □

Proof of proposition 2.13. Let $X = (X_i, \chi_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where X_i is the pullback of

$$\begin{tikzcd} & D_i \arrow[dl, "f_i"'] \arrow[dr, "g_i"] & \\ A_i & & B_i \end{tikzcd}$$

and $\chi_\alpha : X_i \to X_j$ is defined by $\chi_\alpha((a, b) + K_i) = (\varphi_\alpha(a), \psi_\alpha(b)) + K_j$. (where $A = (A_i, \varphi_\alpha), B = (B_i, \psi_\alpha)$). In analogy to the proof about pullbacks, we first show that $\gamma_A = (\gamma_{A_i})_{i \in Q_0}$ is a well-defined morphism:

$$\chi_\alpha \circ \gamma_{A_i}(a) = \chi_\alpha \circ \pi_i \circ \iota_i(a) = (\varphi_\alpha(\bar{a}), 0) + K_j = \pi_j \circ \iota_j \circ \varphi_\alpha = \gamma_{A_j} \circ \varphi_\alpha$$

so $\gamma_A \in \mathrm{Hom}(A, X)$, and similarly for B, and thus X completes the square. Let $Y = (Y_i, \kappa_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $q_A = (q_{A_i})_{i \in Q_0}$, $q_B = (q_{B_i})_{i \in Q_0}$ such that $q_A \circ f = q_B \circ g$, i.e. Y completes the square. Let $h : X \to Y = (h_i)_{i \in Q_0}$ consist of morphisms induced pointwise by pushout property. Clearly, h is the only possible candidate for the induced pushout morphism. It remains to show that h is a well-defined morphism from X to Y, which it is if and only if for all arrows $\alpha \in Q_1$, $\alpha : i \to j$, the following equation is satisfied: $\kappa_\alpha \circ h_i = h_j \circ \chi_\alpha$. We obtain that result as following: for $q_A \in \mathrm{Hom}(A, Y)$ we have $\kappa_\alpha \circ q_{A_i} = q_{A_j} \circ \varphi_\alpha$ Now by the pointwise pushout property of h we know that $q_{A_i} = h_i \circ \gamma_{A_i}$ and that $q_{A_j} = h_j \circ \gamma_{A_j}$. Now, by putting that to our former equation, we have the following: $\kappa_\alpha \circ h_i \gamma_{A_i} = h_j \circ \gamma_{A_j} \circ \varphi_\alpha$ and finally, as $\gamma_A \in \mathrm{Hom}(A, X)$, we have $\chi_\alpha \circ \gamma_{A_i} = \gamma_{A_j} \circ \varphi_\alpha$ we get

$$\kappa_\alpha \circ h_i \circ \gamma_{A_i} = h_j \circ \chi_\alpha \circ \gamma_{A_j} \iff \kappa_\alpha \circ h_i \circ \pi_i \circ \iota_i = h_j \circ \chi_\alpha \circ pi_j \circ \iota_j$$

and similarly for B we get $\kappa_\alpha \circ h_i \circ \pi_i \circ \iota'_i = h_j \circ \chi_\alpha \circ pi_j \circ \iota'_j$ and thus by universal property of coproducts we get $\kappa_\alpha \circ h_i \circ \pi_i = h_j \circ \chi_\alpha \circ \pi_i$ so $\kappa_\alpha \circ h_i = h_j \circ \chi_\alpha$ as π_i is epi. □

We can use our construction to directly obtain the kernel and cokernel representations.

Definition 2.15. Let $\mathcal{C}$ be a category with a zero object 0. Let $A, B \in \mathcal{C}$ and $f \in \mathrm{Hom}(A, B)$. A pair $(\mathrm{Ker}(f), \iota)$ such that $\mathrm{Ker}(f) \in \mathcal{C}, \iota \in \mathrm{Hom}(\mathrm{Ker}(f), A)$ is said to be the kernel of f it it satisfies the following universal property:
Given $D \in \mathcal{C}$ and $h \in \mathrm{Hom}(D, A)$ such that $f \circ h = 0$ (i.e. the unique morphism that factors through 0, that is $f \circ h = \bar{0} \circ \varphi, \varphi \in \mathrm{Hom}(D, 0)$ with $\bar{0}$ being the unique morphism in $\mathrm{Hom}(0, B)$), there is a unique morphism $\phi : D \to \mathrm{Ker}(f)$ such that $h = \iota \circ \phi$, i.e. the following diagram commutes:

$$\begin{tikzcd} & D \arrow[dl, "\exists! \phi"'] \arrow[dr, "h"] & & \\ \mathrm{Ker}(f) \arrow[rr, "\iota"] & & A \arrow[r, "f"] & B \end{tikzcd}$$

Proposition 2.16. *In a category $\mathcal{C}$ with a zero object 0 and pullbacks, given $f \in \mathrm{Hom}(A, B)$, f has a kernel (X, γ) and it is given by the following pullback:*

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, "\gamma"] & 0 \arrow[d, "0"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Proof. The universal property of kernels defined above is properly contained (as a diagram) in the following diagram induced by the universal property of pullbacks:

$$\begin{tikzcd} D \arrow[dr, dashed, "\exists! \phi"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "h"'] & & \\ & X \arrow[r, "0"] \arrow[d, "\gamma"] & 0 \arrow[d, "0"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

□

This yields the following definition:

Definition 2.17. Given $A = (A_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}, B \in \mathrm{Rep}_K Q, f \in \mathrm{Hom}(A, B)$ we define the kernel of f as the representation $\mathrm{Ker}(f) \in \mathrm{Rep}_K Q$ given by $\mathrm{Ker}(f) = (\mathrm{Ker}(f_i), \psi_\alpha \restriction_{\mathrm{Ker}(f_i)})_{i \in Q_0, \alpha \in Q_1}, \iota \in \mathrm{Hom}(\mathrm{Ker}(f), A) = (\iota_i)_{i \in Q_0}$, where ι_i is the canonical inclusion of $\mathrm{Ker}(f_i)$ into A_i in Vect_K

We treat cokernels similarly:

Definition 2.18. Let $\mathcal{C}$ be a category with a zero object 0. Let $A, B \in \mathcal{C}, f \in \mathrm{Hom}(A, B)$. A pair $(\mathrm{Coker}(f), \pi)$ such that $\mathrm{Coker}(f) \in \mathcal{C}, \pi \in \mathrm{Hom}(B, \mathrm{Coker}(f))$ is said to be the cokernel of f if it satisfies the following universal property:
Given $D \in \mathcal{C}$ and a morphism $h \in \mathrm{Hom}(B, D)$ such that $h \circ f = 0$, the following commutative diagram exists:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[rr, "\pi"] \arrow[dr, "h"] & & \mathrm{Coker}(f) \arrow[dl, "\exists! \phi"] \\ & & D & \end{tikzcd}$$

Proposition 2.19. *In a category $\mathcal{C}$ with a zero object 0 and pushouts, given $f \in \mathrm{Hom}(A, B)$, f has a cokernel (X, γ) and it is given by the following pushout:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "0"] & B \arrow[d, "\gamma"] \\ 0 \arrow[r, "0"] & X \end{tikzcd}$$

Proof. Once again, the universal property of cokernels is properly contained in the following diagram induced by the universal property of pushouts:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "0"] & B \arrow[d, "\gamma"] \arrow[ddr, bend left, "h"] & \\ 0 \arrow[r, "0"] \arrow[drr, bend right, "0"] & X \arrow[dr, dashed, "\exists!\phi"] & \\ & & D \end{tikzcd}$$

□

This motivates the following definition:

Definition 2.20. Given representations $A, B = (B_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$, and a morphism $f \in \mathrm{Hom}(A, B)$ we define the cokernel of f as the representation $\mathrm{Coker}(f) \in \mathrm{Rep}_K Q$ given by $\mathrm{Coker}(f) = (\mathrm{Coker}(f_i), \chi_\alpha)$, where $\chi_\alpha(b_i + f_i(B_i)) = \psi_\alpha(b_i) + f_j(B_j)$ and $\pi \in \mathrm{Hom}(B, \mathrm{Coker}(f)) = (\pi_i)_{i \in Q_0}$, where π_i is the canonical projection from B_i onto $\mathrm{Coker}(f_i)$

Definition 2.21. A morphism of quiver representations f is:
a) mono if $\mathrm{Ker}(f) = 0$
b) epi if $\mathrm{Coker}(f) = 0$
c) an isomorphism if it is both epi and mono.

Definition 2.22. A representation $L \in \mathrm{Rep}_K Q$ is said to be a subrepresentation of $M \in \mathrm{Rep}_K Q$ if there is a monomorphism $g : L \to M$.

Theorem 2.23 (First Isomorphism Theorem)**.** *Let $A = (A_i, \varphi_\alpha), B = (B_i, \psi_\alpha \in \mathrm{Rep}_K Q, f \in \mathrm{Hom}(A, B)$. Then $\mathrm{Coker}(\mathrm{Ker}(f) \simeq \mathrm{Ker}(\mathrm{Coker}(f))$*

Proof. Fortunately, the corresponding theorem for vector spaces equips us with an explicit isomorphism, i.e. $\mathrm{Coker}(\mathrm{Ker}(f_i)) = A_i/\mathrm{Ker}(f_i) \simeq \mathrm{Im}(f_i) = \mathrm{Ker}(\mathrm{Coker}(f_i))$ With $\gamma_i : A_i/\mathrm{Ker}(f_i) \xrightarrow{\sim} \mathrm{Im}(f_i)$ defined as follows: $\gamma_i(x_i + \mathrm{Ker}(f_i)) = f(x_i)$. By applying the definitions in said order we get that $\mathrm{Coker}(\mathrm{Ker}(f)) = (A_i/\mathrm{Ker}(f_i), \chi_\alpha)$ with $\chi_\alpha(x + \mathrm{Ker}(f_i)) = \varphi_\alpha(x) + \mathrm{Ker}(f_j)$, and $\mathrm{Ker}(\mathrm{Coker}(f)) = (\mathrm{Im}(f_i), \psi_\alpha \restriction_{\mathrm{Im}(f_i)})$
Now let $\gamma = (\gamma_i)_{i \in Q_0}$. Then γ is a morphism as for all $\alpha : i \to j \in Q_1$ and all $x + \mathrm{Ker}(f_i) \in A_i/\mathrm{Ker}(f_i)$ we have $(\psi_\alpha \restriction_{\mathrm{Im}(f_i)} \circ \gamma_i)(x + \mathrm{Ker}(f_i) = (\psi_\alpha \restriction_{\mathrm{Im}(f_i)})(f_i(x)) = (\psi_\alpha \circ f_i)(x) = (f_j \circ \varphi_\alpha)(x) = \gamma_j(\varphi_\alpha(x) + \mathrm{Ker}(f_j)) = (\gamma_j \circ \chi_\alpha)(x + \mathrm{Ker}(f_i))$ so $(\psi_\alpha \restriction_{\mathrm{Im}(f_i)}) \circ \gamma_i = \gamma_j \circ \chi_\alpha$. Moreover, since γ_i is an isomorphism for all $i \in Q_0$, its corresponding kernel and cokernel representations assign the 0 vector space to every i, and thus (as 0 is the zero object), they both are the zero representation, so γ is mono and epi, and thus an isomorphism. □

It's about time now to formally introduce the notion of image.

Definition 2.24. Given $A \xrightarrow{f} B$ in $\mathrm{Rep}_K Q$, we let $\mathrm{Im}(f) \in \mathrm{Rep}_K Q$ be the representation given by $\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{Coker}(f))$.

Definition 2.25. A sequence $\cdots \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots$ in $\mathrm{Rep}_K Q$ is exact at M_i if $\mathrm{Im}(f_{i-1}) = \mathrm{Ker}(f_i)$, and is an exact sequence if it is exact for all i. In particular, exact sequences of the form $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ are said to be short exact.

Definition 2.26.
a) $f : L \to M$ in $\mathrm{Rep}_K Q$ is called a section if there is $h \in \mathrm{Hom}(M, L)$ such that $h \circ f = 1_L$
b) $g : M \to N$ in $\mathrm{Rep}_K Q$ is called a retraction if there is $h \in \mathrm{Hom}(N, M)$ such that $g \circ h = 1_N$
c) A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is said to split (equivalently, is said to be split exact) if f is a section.

Proposition 2.27. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\mathrm{Rep}_K Q$. Then:*
a) f is a section if and only if g is a retraction
b)If f is a section, then $M \simeq L \oplus N$. Moreover, let η be the morphism induced by g being a retraction and let h be the morphism induced by f being a section. Let ι_L, ι_N be the canonical injections from L respectively M to $L \oplus N$ and let π_L, π_N be the canonical projections from $L \oplus N$ to L respectively N. Then the following diagrams are isomorphic:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, shift left, "f"] & M \arrow[l, shift left, "h"] \arrow[r, shift left, "g"] & N \arrow[l, shift left, "\eta"] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, shift left, "\iota_L"] & L \oplus N \arrow[l, shift left, "\pi_L"] \arrow[r, shift left, "\pi_N"] & N \arrow[l, shift left, "\iota_N"] \arrow[r] & 0 \end{tikzcd}$$

To prove that proposition we employ the strategy used earlier: we first prove the same result in category R-Mod to obtain candidate morphisms that satisfy our desired property pointwise, and then prove that those candidate morphisms indeed are morphisms.

Lemma 2.28. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in R-Mod. Then:*
a) f is a section if and only if g is a retraction
b) If f is a section, then $M \simeq L \oplus N$. Moreover, let η be the morphism induced by g being a retraction and let h be the morphism induced by f being a section. Let ι_L, ι_N be the canonical injections from L respectively M to $L \oplus N$ and let π_L, π_N be the canonical projections from $L \oplus N$ to L respectively N. Then the following diagrams are isomorphic:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, shift left, "f"] & M \arrow[l, shift left, "h"] \arrow[r, shift left, "g"] & N \arrow[l, shift left, "\eta"] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, shift left, "\iota_L"] & L \oplus N \arrow[l, shift left, "\pi_L"] \arrow[r, shift left, "\pi_N"] & N \arrow[l, shift left, "\iota_N"] \arrow[r] & 0 \end{tikzcd}$$

Proof. a) " $\Longrightarrow$ ": If g was mono (it is epi due to exactness of the sequence) we would just set $\eta = g^{-1}$, but that need not be the case, and although no elements in N have an empty preimage, some preimages need not be singletons, making the inverse mapping not well-defined. We thus "fix" that problem with diagram chasing. Let $n \in N$, $m, m' \in g^{-1}(n)$. We have $m - m' \in \mathrm{Ker}(g)$, so $m - m' \in \mathrm{Im}(f)$ and hence there is $f^{-1}(m - m')$, which is unique as f is mono by exactness of the sequence. Now as f is a section, we have $h \in \mathrm{Hom}(M, L)$ such that $h \circ f = id_L$, and $(h \circ f)(f^{-1}(m - m')) = f^{-1}(m - m') = h(m - m')$, so $(f \circ h)(m - m') = f(f^{-1}(m - m')) = m - m'$ and thus $m - (f \circ h)(m) = m' - (f \circ h)(m')$ and $g(m - (f \circ h)(m)) = g(m) - (g \circ f)(h(m) = g(m)$ as $g \circ f = 0$. So, given $n \in N$, we let $\eta(n) = m - (f \circ h)(m)$ for any $m \in g^{-1}(n)$ which by above is well-defined and satisfies $g \circ \eta = id_N$ And so g is a retraction.
" $\Longleftarrow$ ": in analogy to the construction above, given $m \in M$, and letting $\eta \in \mathrm{Hom}(N, M)$ be the morphism induced by g being a retraction (i.e. $g \circ \eta = id_N$), we have $g(m - (\eta \circ g)(m)) = g(m) - (g \circ \eta \circ g)(m) = g(m) - (g \circ \eta)(g(m) = g(m) - g(m) = 0$ so we have $m - (\eta \circ g)(m) \in \mathrm{Ker}(g)$ and thus there is a unique element $f^{-1}(m - (\eta \circ g)(m))$ and we let $h(m) = f^{-1}(m - (\eta \circ g)(m))$ and given $l \in L$ we have $(h \circ f)(l) = f^{-1}(f(l) - (\eta \circ g \circ f)(l)) = f^{-1}(f(l)) = l$. So $h \circ f = id_L$ and we're done.

Remark 2.29. *The constructions given above commute, i.e. given a section f with $h \in \mathrm{Hom}(M, L)$ such that $h \circ f = id_L$, if we first construct a "retraction morphism" η from h as we did above, and then construct a "section morphism" h' from η as above, we get $h' = h$, as*

$$h'(m) = f^{-1}(m - \eta(g(m))) = f^{-1}(m - (g^{-1}(g(m)) - (f \circ h)(g^{-1}(g(m)))) = f^{-1}(m - m + (f \circ h)(m)) = h(m)$$

b) We prove this statement in several steps:
i) $0 \leftarrow L \xleftarrow{h} M \xleftarrow{\eta} N \leftarrow 0$ is a short exact sequence
Proof: $id_N = g \circ \eta$ is mono, so η is mono, $id_L = h \circ f$ is epi, hence h is epi. Now all that remains is to show $\mathrm{Ker}(h) = \mathrm{Im}(\eta)$. Let $m \in \mathrm{Im}(\eta)$. Thus there is a unique element n such that $m = \eta(n)$, so $g(m) = (g \circ \eta)(n) = n$, and $m \in g^{-1}(n)$, so by earlier definition of η we have $m = \eta(n) = m - (f \circ h)(m)$, hence $m = m-(f \circ h)(m)$, so $(f \circ h)(m) = 0$. Moreover, $h(m) = 0$ as f is mono, so $\mathrm{Im}(\eta) \subseteq \mathrm{Ker}(h)$. Now let $m \in \mathrm{Ker}(h)$. Then $h(m) = 0$, so by the definition of h we get $f^{-1}(m-(\eta \circ g)(m)) = 0$, so $m-(\eta \circ g)(m) = 0$ as f is mono, and thus $m = \eta(g(m))$. Hence $m \in \mathrm{Im}(\eta)$, so $\mathrm{Ker}(h) \subseteq \mathrm{Im}(\eta)$, and thus $\mathrm{Ker}(h) = \mathrm{Im}(\eta)$. Thus the sequence is exact.
ii) $f \circ h + \eta \circ g = id_M$
Proof: Given $m \in M$ we have

$$\begin{aligned} m - (f \circ h + \eta \circ g)(m) &= m - (f \circ h)(m) - (\eta \circ g)(m) \\ &= m - f(h(m) - (g^{-1}(g(m)) - (f \circ h)(g^{-1}(g(m)))) \\ &= m - f(h(m)) - (m - f(h(m))) = 0 \end{aligned}$$

iii) $M \simeq L \coprod N$ with injections f, η
Claim: Given $X \in R$-Mod, $\varphi \in \mathrm{Hom}(L, X), \psi \in \mathrm{Hom}(N, X)$, there is a unique $\gamma \in \mathrm{Hom}(M, X)$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[dr, "\varphi"'] & M \arrow[d, "\gamma"] & N \arrow[l, "\eta"'] \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

Proof of claim: Let $\gamma = \varphi \circ h + \psi \circ g$, so

$$\gamma \circ \eta = \varphi \circ h \circ \eta + \psi \circ g \circ \eta = \psi \circ g \circ \eta$$

as $h \circ \eta = 0$, and $g \circ \eta = id_N$ means that $\psi \circ g \circ \eta = \psi$, so $\gamma \circ \eta = \psi$. Similarly,

$$\gamma \circ f = \varphi \circ h \circ f + \psi \circ g \circ f = \varphi \circ h \circ f = \varphi$$

And so the diagram commutes. For uniqueness, let γ' such that $\gamma' \circ \eta = \psi$, $\gamma' \circ f = \varphi$. But $f \circ h + \eta \circ g = id_M$, so

$$\gamma' = \gamma' \circ (f \circ h + \eta \circ g) = (\gamma' \circ f) \circ h + (\gamma' \circ \eta) \circ g = \varphi \circ h + \psi \circ g = \gamma$$

iv) $M \simeq L \prod N$ with projections h, g
Claim: given a module $X \in R$-Mod, and morphisms of modules $\varphi \in \mathrm{Hom}(X, L), \psi \in \mathrm{Hom}(X, N)$, there is a unique $\gamma \in \mathrm{Hom}(X, M)$ such that the following diagram commutes:

$$\begin{tikzcd} L & M \arrow[l, "h"'] \arrow[r, "g"] & N \\ & X \arrow[ul, "\varphi"] \arrow[u, "\gamma"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Proof of claim: let $\gamma = \eta \circ \psi + f \circ \varphi$. Then

$$g \circ \gamma = g \circ \eta \circ \psi + g \circ f \circ \varphi = g \circ \eta \circ \psi = \psi$$

and

$$h \circ \gamma = h \circ \eta \circ \psi + h \circ f \circ \varphi = h \circ f \circ \varphi = \varphi$$

For uniqueness, let γ' such that $h \circ \gamma' = \varphi$, $g \circ \gamma = \psi$. But

$$\gamma' = id_M \circ \gamma' = (f \circ h + \eta \circ g) \circ \gamma' = f \circ (h \circ \gamma') + \eta \circ (g \circ \gamma') = f \circ \varphi + \eta \circ \psi = \gamma$$

$\square$

Now we're ready to prove the statement for $\mathrm{Rep}_K Q$:

Proof of proposition 2.27. Note that we assume that f is a section, and thus there is a morphism $h = (h_i)_{i\in Q_0} \in \mathrm{Hom}(M, L)$ such that $h \circ f = id_L$, and thus in particular, for all $i \in Q_0$ we have $h_i \circ f_i = id_{L_i}$. Now we let $\eta = (\eta_i)_{i\in Q_0}$, where η_i is constructed from h_i as above. Now, as mentioned earlier, all the desired properties stated in a) and b) will follow as long as η is a well-defined morphism. To show this we let $L = (L_i, \rho_\alpha)_{i\in Q_0,\alpha\in Q_1}$, $M = (M_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1}$, and $N = (N_i, \psi_\alpha)_{i\in Q_0,\alpha\in Q_1}$. We need to show that for all $\alpha : i \to j \in Q_1$ we have $\eta_j \circ \psi_\alpha = \varphi_\alpha \circ \eta_i$. But since the properties proven earlier still hold pointwise, i.e. for all $i \in Q_0$, $M_i \simeq L_i \oplus N_i$, we can represent our morphisms with matrices in following manner:

$$\begin{tikzcd}
L_i \arrow[r, shift left, "f_i"] \arrow[d, "\rho_\alpha"] & M_i \arrow[l, shift left, "h_i"] \arrow[r, shift left, "g_i"] \arrow[d, "\varphi_\alpha"] & N_i \arrow[l, shift left, "\eta_i"] \arrow[d, "\psi_\alpha"] & \simeq & L_i \arrow[r, shift left, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_i}"] \arrow[d, "\rho_\alpha"] & L_i \oplus N_i \arrow[l, shift left, "{[1\ 0]_i}"] \arrow[r, shift left, "{[0\ 1]_i}"] \arrow[d, "\varphi_\alpha"] & N_i \arrow[l, shift left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]_i}"] \arrow[d, "\psi_\alpha"] \\
L_j \arrow[r, shift left, "f_j"] & M_j \arrow[l, shift left, "h_j"] \arrow[r, shift left, "g_j"] & N_i \arrow[l, shift left, "\eta_j"] & & L_j \arrow[r, shift left, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_j}"] & L_j \oplus N_j \arrow[l, shift left, "{[1\ 0]_j}"] \arrow[r, shift left, "{[0\ 1]_j}"] & N_i \arrow[l, shift left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]_j}"]
\end{tikzcd}$$

And in particular, $\varphi_\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, but since f is a morphism we have $\varphi_\alpha \circ f_i = f_j \circ \rho_\alpha$, so $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_i = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_j \rho_\alpha$, so $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] = \left[\begin{smallmatrix} \rho_\alpha \\ 0 \end{smallmatrix}\right]$, so $a = \rho_\alpha, c = 0$, similarly for g we obtain $g_j \circ \varphi_\alpha = \psi_\alpha \circ g_i$ and thus $\left[\begin{smallmatrix} 0 & \psi_\alpha \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & d \end{smallmatrix}\right]$. Therefore we have $d = \psi_\alpha$, and finally with help of h we get $h_j \circ \varphi_\alpha = \rho_\alpha \circ h_i$, which means that $\left[\begin{smallmatrix} \rho_\alpha & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \end{smallmatrix}\right]$, so $b = 0$, so that $\varphi_\alpha = \begin{bmatrix} \rho_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix}$ Now we have $\eta_j \circ \psi_\alpha = \left[\begin{smallmatrix} 0 \\ \psi_\alpha \end{smallmatrix}\right] = \begin{bmatrix} \rho_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \varphi_\alpha \circ \eta_i$ and so our candidate morphism is well-defined, and the results follow. □

Definition 2.30. Given a representation X, we define the covariant Hom-functor of X, $\mathrm{Hom}(X, -) : \mathrm{Rep}_K Q \to \mathrm{Vect}_K$ by:

- Sending a representation $M \in \mathrm{Rep}_K Q$ to the vector space of morphisms $\mathrm{Hom}(X, M)$
- Sending a morphism of representations $f \in \mathrm{Hom}(L, M)$ to the K-linear map $\mathrm{Hom}(X, f) : \mathrm{Hom}(X, L) \to \mathrm{Hom}(X, M)$ which, given a morphism of representations $\varphi \in \mathrm{Hom}(X, L)$, sends it to $f \circ \varphi \in \mathrm{Hom}(X, M)$

Proposition 2.31. *A sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N$$

in $\mathrm{Rep}_K Q$ *is exact if and only if for all representations* $X \in \mathrm{Rep}_K Q$ *the following sequence is exact in* Vect_K*:*

$$0 \to \mathrm{Hom}(X, L) \xrightarrow{\mathrm{Hom}(X,f)} \mathrm{Hom}(X, M) \xrightarrow{\mathrm{Hom}(X,g)} \mathrm{Hom}(X, N)$$

In particular, the covariant Hom*-functor* $\mathrm{Hom}(X, -)$ *is left exact.*

Proof. " $\Longrightarrow$ " First, if $\varphi \in \mathrm{Hom}(X, L)$ such that $\mathrm{Hom}(X, f)(g) = f \circ g = 0$ then by universal property of kernels, $\varphi = \iota \circ h$ for a unique $h \in \mathrm{Hom}(X, \mathrm{Ker}(f))$. But f is mono, so $\mathrm{Ker}(f) = 0$, so $h = 0$, and thus $\varphi = 0$ and so $\mathrm{Hom}(X, f)$ is mono. Now we need to show $\mathrm{Im}(\mathrm{Hom}(X, f)) = \mathrm{Ker}(\mathrm{Hom}(X, g))$. Clearly, $\mathrm{Im}(\mathrm{Hom}(X, f)) \subset \mathrm{Ker}(\mathrm{Hom}(X, g))$ as $g \circ f = 0$ means that $\mathrm{Hom}(X, g)(\mathrm{Hom}(X, f)(\varphi)) = g \circ f \circ \varphi = 0 \circ \varphi = 0$. We also have $\mathrm{Ker}(\mathrm{Hom}(X, g)) \subset \mathrm{Im}(\mathrm{Hom}(X, f))$, since $\psi \in \mathrm{Ker}(\mathrm{Hom}(X, g)) \iff g \circ \psi = 0$ and by universal property of kernels we thus get $\exists! h \in \mathrm{Hom}(X, \mathrm{Ker}(g))$ such that $\psi = \iota \circ h$, with ι being the kernel morphism. But $\mathrm{Ker}(g) = \mathrm{Im}(f)$ and f is epi onto $\mathrm{Im}(f)$, so there is a unique isomorphism $\bar{f} : L \xrightarrow{\sim} \mathrm{Im}(f)$ and $\iota \circ \bar{f} = f$, and finally $\psi = f \circ \bar{f}^{-1} \circ h$, so ψ is in $\mathrm{Im}(\mathrm{Hom}(X, f))$.
" $\Longleftarrow$ " First we show that $\mathrm{Hom}(X, f)$ is mono implies that f is mono, arguing by contraposition: if f is not mono, then $\mathrm{Ker}(f) \neq 0$, $\iota \neq 0$ (ι being the kernel morphism), but $f \circ \iota = 0 = f \circ 0$, so $\mathrm{Hom}(X, f)$ is not mono. Next we have $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$ as $\mathrm{Hom}(X, g)(\mathrm{Hom}(X, f)(id_L)) = 0 = g \circ f \circ id_L = g \circ f$, and finally $Ker(\mathrm{Hom}(X, g)) \subset \mathrm{Im}(\mathrm{Hom}(X, f))$ as the kernel morphism of $\mathrm{Ker}(g), \iota_g : \mathrm{Ker}(g) \to M$ is in $\mathrm{Ker}(\mathrm{Hom}(X, g))$, and thus factors through f. □

Definition 2.32. Given a representation X, we define the contravariant Hom-functor of X, $\mathrm{Hom}(-, X) : \mathrm{Rep}_K Q \to \mathrm{Vect}_K$ by:

- Sending a representation $M \in \mathrm{Rep}_K Q$ to the vector space of morphisms $\mathrm{Hom}(M, X)$
- Sending a morphism of representations $g \in \mathrm{Hom}(M, N)$ to the K-linear map $\mathrm{Hom}(f, X) : \mathrm{Hom}(N, X) \to \mathrm{Hom}(M, X)$ which, given a morphism of representations $\psi \in \mathrm{Hom}(N, X)$, sends it to $\psi \circ g \in \mathrm{Hom}(M, X)$

Proposition 2.33. *A sequence*

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in $\mathrm{Rep}_K Q$ *is exact if and only if for all representations* $X \in \mathrm{Rep}_K Q$ *the following sequence is exact in* Vect_K*:*

$$0 \to \mathrm{Hom}(N,X) \xrightarrow{\mathrm{Hom}(g,X)} \mathrm{Hom}(M,X) \xrightarrow{\mathrm{Hom}(f,X)} \mathrm{Hom}(L,X)$$

In particular, the contravariant Hom*-functor* $\mathrm{Hom}(-,X)$ *is left exact.*

Proof. " $\implies$ ": First, g epi means that $\mathrm{Hom}(g,X)$ is mono, since by universal property of cokernels, if $\phi \in \mathrm{Hom}(N,X)$ such that $\mathrm{Hom}(g,X)(\phi) = \phi \circ g = 0$, $\phi = h \circ \pi$ for a unique $h \in \mathrm{Hom}(\mathrm{Coker}(g), X)$ (π being the cokernel projection), but $\mathrm{Coker}(g) = 0$) means that $h = 0$, and hence $\phi = 0$)
Secondly, $\mathrm{Im}(\mathrm{Hom}(g,X)) \subset \mathrm{Ker}(\mathrm{Hom}(f,X)$ as $\mathrm{Hom}(f,X)(\mathrm{Hom}(g,X)(\phi)) = \phi \circ g \circ f = \phi \circ 0 = 0$.
And finally $\mathrm{Ker}(\mathrm{Hom}(f,X)) \subset \mathrm{Im}(\mathrm{Hom}(g,X))$ as $\phi \in \mathrm{Ker}(\mathrm{Hom}(f,X))$ is equivalent to $\phi \circ f = 0$, which implies that there is a unique morphism $h \in \mathrm{Hom}(\mathrm{Coker}(f), X)$ such that $\phi = h \circ \pi$, π being the cokernel morphism of $\mathrm{Coker}(f)$ (statement following from universal property of cokernels), and $\mathrm{Coker}(f) \simeq \mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f))) =: A$, since given the kernel morphism ι of $\mathrm{Ker}(\mathrm{Coker}(f))$, we have $\pi \circ \iota = 0$, and thus there is a unique morphism $\psi \in \mathrm{Hom}(A, \mathrm{Coker}(f))$ such that $\psi \circ \rho = \pi$, ρ denoting the cokernel morphism of A, so $\phi = h \circ \psi \circ \rho$, and $h \circ \psi$ acts as cokernel morphism, and is unique such: $\kappa \in \mathrm{Hom}(A,X)$ such that $\kappa \circ \rho = \phi$, so $\kappa \circ \rho = h \circ \psi \circ \rho$, and thus $\kappa = h \circ \psi$ as ρ is epi, and the equality follows by earlier part of this proof. Now $\mathrm{Coker}(f) = \mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f))) = \mathrm{Coker}(\mathrm{Ker}(g)) = \mathrm{Ker}(\mathrm{Coker}(g)) = \mathrm{Ker}(0) = N$, and thus ϕ factors through g.
" $\impliedby$ ": First $\mathrm{Hom}(g,X)$ being mono implies that g is epi. We argue by contraposition: if g is not epi, then $\mathrm{Coker}(g) \neq 0 \implies \pi \neq 0$, but $\mathrm{Hom}(g,X)(\pi) = \pi \circ g = 0 = 0 \circ g = \mathrm{Hom}(g,X)(0)$ so $\mathrm{Hom}(g,X)$ is not mono.
Secondly $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$ as $\mathrm{Hom}(f,X)(\mathrm{Hom}(g,X)(id_N)) = id_N \circ g \circ f = 0 \implies g \circ f = 0$
Thirdly $\mathrm{Ker}(g) \subset \mathrm{Im}(f)$: Let π denote the cokernel morphism of $\mathrm{Coker}(f)$. By definition we have $\pi \circ f = 0$, so $\pi \in \mathrm{Ker}(\mathrm{Hom}(f,X))$. Thus $\pi \in \mathrm{Im}(\mathrm{Hom}(g,X))$ and therefore there is a morphism $\varphi \in \mathrm{Hom}(N, \mathrm{Coker}(f))$ such that $\pi = \varphi \circ g$. Now let ι denote the kernel morphism of $\mathrm{Ker}(g)$. We have $\pi \circ \iota = \varphi \circ g \circ \iota = \varphi \circ 0 = 0$ so $\mathrm{Ker}(g) \subset \mathrm{Ker}(\mathrm{Coker}(f)) = \mathrm{Im}(f)$. □

Proposition 2.34. *A sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *in* $\mathrm{Rep}_K Q$ *is split exact if and only if for all* $X \in \mathrm{Rep}_K Q : 0 \to \mathrm{Hom}(X,L) \xrightarrow{\mathrm{Hom}(X,f)} \mathrm{Hom}(X,M) \xrightarrow{\mathrm{Hom}(X,g)} \mathrm{Hom}(X,N) \to 0$ *is exact.*

Proof. " $\implies$ ": Since $\mathrm{Hom}(X,-)$ is left exact, to show exactness of the sequence, we only need to show that $\mathrm{Hom}(X,g)$ is epi. But the corresponding sequence in $\mathrm{Rep}_K Q$ is split exact, so g is a retraction, and thus there is a morphism $\eta \in \mathrm{Hom}(N,M)$ such that $g \circ \eta = id_N$. Now let $\varphi \in \mathrm{Hom}(X,N)$. We have $\varphi = g \circ \eta \circ \varphi$ which means that $\mathrm{Hom}(X,g)$ is epi.
" $\impliedby$ ": Here we need to both show that g is epi and also that the sequence splits.
We let $X = N$, and since $\mathrm{Hom}(X,g)$ is epi, we get that $id_N = g \circ \eta$ for some $\eta \in \mathrm{Hom}(N,M)$. But id_N being epi means that g is epi, and also g is a retraction, so the sequence splits. □

Proposition 2.35. *A sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *in* $\mathrm{Rep}_K Q$ *is split exact if and only if for all* $X \in \mathrm{Rep}_K Q : 0 \to \mathrm{Hom}(N,X) \xrightarrow{\mathrm{Hom}(g,X)} \mathrm{Hom}(M,X) \xrightarrow{\mathrm{Hom}(f,X)} \mathrm{Hom}(L,X) \to 0$ *is exact.*

Proof. " $\implies$ ": all we need to show is that $\mathrm{Hom}(f,X)$ is epi. But the corresponding sequence in $\mathrm{Rep}_K Q$ splits, so f is a section, i.e. there is a morphism $\eta \in \mathrm{Hom}(M,L)$ such that $\eta \circ f = id_L$. Given $\varphi \in \mathrm{Hom}(L,X)$, we have $\varphi = \varphi \circ \eta \circ f$, thus $\varphi \in \mathrm{Im}(\mathrm{Hom}(f,X))$ and hence $\mathrm{Hom}(f,X)$ is epi.
" $\impliedby$ ": In analogy to the proof of the preceding proposition, $id_L = h \circ f$ for some $h \in \mathrm{Hom}(M,L)$. Thus f is mono and is a section, and the result follows. □

3. Projective and Injective Representations

Definition 3.1. Given $Q = (Q_0, Q_1, s, t)$, and vertices $i, j \in Q_0$, a path from i to j is a sequence of arrows $\{\alpha_k\}_{k=1}^{n}$ such that for all k, $t(\alpha_k) = s(\alpha_{k+1})$, and $s(\alpha_1) = i$, $t(\alpha_n) = j$

Definition 3.2. We say that a quiver is acyclic if and only if it contains no cycles, i.e. non-empty paths from i to i, for some $i \in Q_0$

Definition 3.3. Let $Q = (Q_0, Q_1, s, t)$. $i \in Q_0$ is a source if and only if there is no $\alpha \in Q_1$ such that $i = t(\alpha)$. Similarly, i is a sink if and only if there is no $\alpha \in Q_1$ such that $i = s(\alpha)$.

Proposition 3.4. *If Q is acyclic then:*
a) There is a source in Q
b) There is a sink in Q

Proof. a) Assume Q has no source. Then for all $i \in Q_0$ there is an arrow $\alpha \in Q_1$ such that $i = t(\alpha)$. Thus if we start at some fixed vertex, we can go back to another vertex by that arrow and continue so indefinitely, inducing an infinite path in the quiver. But we assume (throughout) that our quiver is finite, so some vertices must have been visited multiple times, thus we can separate a subpath of our path that starts and ends at one of those vertices, and thus is a directed loop, contradicting our hypothesis.
b) Similarly, assume Q has no sink. Then, after choosing some $i \in Q_0$, we can produce an infinite path by starting with arrow that has its source at i, and continuing with an arrow that has its source at the target of our arrow etc. And thus, just like in a), we obtain directed loops, leading us to contradiction. $\square$

From now on we assume that all quivers are acyclic, unless otherwise noted.

Definition 3.5. Let $i \in Q_0$ Define $S(i)$, the simple representation at vertex i as $(S(i)_j, \varphi_\alpha)_{j\in Q_0,\alpha\in Q_1}$, where $S(i)_j = K$ if $j = i$ and $S(i)_j = 0$ else, and for all $\alpha \in Q_1$ we have $\varphi_\alpha = 0$

Proposition 3.6. *Given a representation $M = (M_i, \gamma_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}_K Q$, M is simple (i.e. is not zero and has no non-zero subrepresentations) if and only if it is isomorphic to $S(i)$ for some $i \in Q_0$*

Proof. " $\Longrightarrow$ ":
Choose a sink j in Q. We have two cases:
1) $M_j \neq 0$
2) $M_j = 0$

For 1) We have a monomorphism $\iota : S(j) \to M$, $\iota = (\iota_i)_{i\in Q_0}$ defined by $\iota_j = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\iota_i = 0$ for $i \neq j$, which is well-defined, as the following square clearly commutes:

$$\begin{tikzcd} S(i)_m \arrow[r, "0"] \arrow[d, "0"] & S(i)_j \arrow[d, "\iota_j"] \\ M_m \arrow[r, "\gamma_\alpha"] & M_j \end{tikzcd}$$

For 2) We remove j and all the arrows into it (as there are no arrows from it to remove) from Q to obtain Q' and a representation M' of it, and consider the same problem for Q'. If we happen to be in case 1) this time and thus have a subrepresentation $S(j')$ of M', it will also be a subrepresentation of M after we put j and its arrows back to our quiver, since we extend by 0 morphisms for both $S(j')$ and M'.
This procedure terminates after at most $|Q_0|$ steps, and the only way it could avoid case 1) in all steps is by all its vector spaces being 0, and thus $M = 0$, which is not allowed by the definition of a simple object.
" $\Longleftarrow$ ":
For $N \neq 0$ to be a subrepresentation of $M \simeq S(i)$, i.e. have a monomorphism $N \to M$, the morphism must be pointwise mono, so $N_j = 0$ for $j \neq i$ and $N_j \simeq K \simeq S(i)_i$, and all the arrows being 0 now follows from 0 being a zero object in $Vect_K$ $\square$

Definition 3.7. Let $i \in Q_0$. Define the projective representation at vertex i, $P(i) = (P(i)_j, \varphi_\alpha)_{j\in Q_0,\alpha\in Q_1}$, by $P(i)_j \simeq K^n$, where n is the number of paths from i to j (from now on referred to as i-j paths), with a canonical basis formally consisting of those paths. Given $\alpha : j \to k$, the map φ_α is defined on that basis by $\varphi_\alpha : p \to p\alpha$, where p is any of the i-j paths in the basis.

Proposition 3.8. *For all $i \in Q_0$, $P(i)$ is a projective object of* $\mathrm{Rep}_K Q$ *, i.e. given $M,N \in \mathrm{Rep}_K Q$, $g \in \mathrm{Hom}(M, N)$ epi, and $h \in \mathrm{Hom}(P(i), N)$, there is a (not necessarily unique) morphism $f \in \mathrm{Hom}(P(i), M)$*

such that $g \circ f = h$. This is illustrated by the following diagram:

$$\begin{tikzcd} & P(i) \arrow[dl, "\exists f"'] \arrow[dr, "h"] & \\ M \arrow[rr, "g"'] & & N \end{tikzcd}$$

Proof. Let $M = (M_i, \gamma_\alpha)_{i \in Q_0, \alpha \in Q_1}, N = (N_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Set $Q_i = \{j \in Q_0$ such that there is an i-j path$\}$ and set $\overline{Q_i} := \{j \in Q_0$ such that there is no i-j path$\}$. On $\overline{Q_i}$ the problem is trivial: $j \in \overline{Q_i}$, so $P(i)_j = 0$, so we must have $f_j = 0 = h_j$, and thus clearly $g_j \circ f_j = h_j$. Our choice is also well-defined: given $\alpha : j \to m$ we have $f_m \circ \varphi_\alpha = f_m \circ 0 = 0 = \gamma_\alpha \circ 0 = \gamma_\alpha \circ f_j$ We thus focus on Q_i as we define f.
First choose $f_i \in \mathrm{Hom}(P(i)_i, M_i)$ so that $g_i \circ f_i = h_i$. This is possible as vector spaces are themselves projective (more explicitly, let $f_i(\epsilon_i) \in g^{-1}(h(\epsilon_i))$) (note that our canonical basis for $P(i)_i$ consists only of the empty path ϵ_i, and thus $P(i)_i \simeq K$, and $\mathrm{Hom}(P(i)_i, M_i) \simeq K^{dim\, M_i} \simeq M_i$) Later it will be clear that this choice uniquely determines all of f.
Let $m \in Q_i$ and let R_m be the canonical basis for $P(i)_m$. Choose $r \in R_m$ and write $r = \alpha_1\alpha_2 ... \alpha_n$. Now we introduce some convenient notation: given a set of K-linear maps $\{\gamma_\alpha\}_{\alpha \in Q_1}$, let γ_r denote $\gamma_{\alpha_n} \circ \gamma_{\alpha_{n-1}} \circ \cdots \circ \gamma_{\alpha_1}$, so that γ_r composes γ along the path r. Now let $f_m(r) = \gamma_r \circ f_i(\epsilon_i)$. It is now clear that this definition extends to all of Q_0, so the distinction between Q_i and $\overline{Q_i}$ is superfluous. We now need to show two things: 1) f is well defined, and 2) The diagram above is satisfied pointwise.
1) Let $j, m \in Q_0$. We need to show that this commutes:

$$\begin{tikzcd} P(i)_j \arrow[r, "\varphi_\alpha"] \arrow[d, "f_j"] & P(i)_m \arrow[d, "f_m"] \\ M_j \arrow[r, "\gamma_\alpha"] & M_m \end{tikzcd}$$

Let R_j be the canonical basis for $P(i)_j$, $r \in R_j$, $r = \alpha_1\alpha_2 ... \alpha_n$. Now we have $f_m \circ \varphi_\alpha(r) = f_m(\alpha_1\alpha_2 ... \alpha_n\alpha) = \gamma_\alpha \circ (\gamma_r \circ f_i(\epsilon_i)) = \gamma_\alpha \circ f_j(r)$ and we're done.
2) Let $m \in Q_0$ We need to show $g_m \circ f_m = h_m$. We have $(g_m \circ f_m)(r) = (g_m \circ \gamma_r \circ f_i)(\epsilon_i)$, which, since g is a morphism, is equal to $(\psi_r \circ g_i \circ f_i)(\epsilon_i) = (\psi_r \circ h_i)(\epsilon_i) = (h_m \circ \varphi_r)(\epsilon_i) = h_m(r)$ □

Proposition 3.9. *Given a representation $M = (M_i, \gamma_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{Rep}_K Q$, the following vector spaces are isomorphic:* $\mathrm{Hom}(P(i), M) \simeq \mathrm{Hom}(P(i)_i, M_i) \simeq M_i$

Proof. Choose a K-linear map $\psi \in \mathrm{Hom}(P(i)_i, M_i)$, and let $f \in \mathrm{Hom}(P(i), M)$ be a morphism of representations such that $f_i = \psi$, which is then constructed as the morphism f in the preceding proposition, i.e., given a vertex $j \in Q_0$ and an i-j path r in the canonical basis R_j for $P(i)_j$, we let $f_j(r) = \gamma_r \circ f_i(\epsilon_i)$.
Now let $f' \in \mathrm{Hom}(P(i), M)$ be a morphism of representations such that $f'_i = f_i$. In order for f' to be well defined, we in particular need that the following diagram commutes for all vertices $j \in Q_0$ and all i-j paths $r \in R_j$:

$$\begin{tikzcd} P(i)_i \arrow[r, "\varphi_r"] \arrow[d, "f'_i"] & P(i)_j \arrow[d, "f'_j"] \\ M_i \arrow[r, "\gamma_r"] & M_j \end{tikzcd}$$

Hence, under the canonical choice of basis, we need $f'_j \circ \varphi_r(\epsilon_i) = \gamma_r \circ f'_i(\epsilon_i)$, but we know that $\varphi_r(\epsilon_i) = r$ and that $f'_i = f_i$ and hence we need $f'_j(r) = \gamma_r \circ f_i(\epsilon_i)$. But if that is the case, then for all vertices j and all basis elements $r \in R_j$ we have $f'_j(r) = f_j(r)$ and so we conclude that $f' = f$. Thus there is a bijection between K-linear maps in $\mathrm{Hom}(P(i)_i, M_i)$ and morphisms of representations in $\mathrm{Hom}(P(i), M)$. That bijection is clearly linear, so we conclude that $\mathrm{Hom}(P(i)_i, M_i) \simeq \mathrm{Hom}(P(i), M)$. The isomorphism $\mathrm{Hom}(P(i)_i, M_i) \simeq M_i$ follows by earlier discussion of dimensions. □

Proposition 3.10. *$P, P' \in \mathrm{Rep}_K Q$ are projective if and only if $P \oplus P'$ is projective.*

Proof. First, we let $\iota : P \to P \oplus P'$, $\pi : P \oplus P' \to P$, $\iota' : P' \to P \oplus P'$, $\pi' : P \oplus P' \to P'$ be the canonical projections and injections of P and P'.

" $\Longleftarrow$ ": We show that given representations $M, N \in \mathrm{Rep}_K Q$, a morphism $h \in \mathrm{Hom}(P, N)$ and an epimorphism $g \in \mathrm{Hom}(M, N)$, there is a morphism $f \in \mathrm{Hom}(P, M)$ such that $h = g \circ f$.
Let $\gamma : P \oplus P' \to M$ be a morphism such that $g \circ \gamma = h \circ \pi$, existing due to $P \oplus P'$ being projective. We let $f = \gamma \circ \iota$, so that $g \circ f = (g \circ \gamma) \circ \iota = (h \circ \pi) \circ \iota = h \circ (\pi \circ \iota) = h \circ id_P = h$
Diagrammatically this gives:

$$\begin{tikzcd}
P \oplus P' \arrow[r, "\exists\gamma"] \arrow[d, shift left, "\pi"] & M \arrow[d, "g"] \\
P \arrow[u, shift left, "\iota"] \arrow[r, "f"'] \arrow[ur, "\gamma\circ\iota"'] & N
\end{tikzcd}$$

And P' being projective is proven by analogy.

" $\Longrightarrow$ ": Given $M, N \in \mathrm{Rep}_K Q$, $h \in \mathrm{Hom}(P \oplus P', N)$, $g \in \mathrm{Hom}(M, N)$ epi, we define $\gamma : P \oplus P' \to M$ so that $h = g \circ \gamma$. Since both P and P' are projective, there are morphisms $f : P \to M$, $f' : P' \to M$ such that $g \circ f = h \circ \iota$, $g \circ f' = h \circ \iota'$. And by the coproduct property of $P \oplus P'$, we have a unique $\gamma = (f, f')$ such that $\gamma \circ \iota = f$, $\gamma \circ \iota' = f'$, and since $id_{P \oplus P'} = \iota \circ \pi + \iota' \circ \pi'$, we have $g \circ \gamma = g \circ \gamma \circ (\iota \circ \pi + \iota' \circ \pi') = g \circ (\gamma \circ \iota) \circ \pi + g \circ (\gamma \circ \iota') \circ \pi' = (g \circ f) \circ \pi + (g \circ f') \circ \pi' = h \circ \iota \circ \pi + h \circ \iota' \circ \pi' = h \circ id_{P \oplus P'} = h$ Diagrammatically we have:

$$\begin{tikzcd}
P \arrow[dr, "\iota"] \arrow[drr, bend left, "\exists h"] & & & \\
& P \oplus P' \arrow[r, "\exists\gamma"] \arrow[rr, bend left, "f"] & M \arrow[r, "g"] & N \\
P' \arrow[ur, "\iota'"] \arrow[urr, bend right, "\exists h'"] & & &
\end{tikzcd}$$

$\square$

Proposition 3.11. *For all $i \in Q_0$, $P(i)$ is indecomposable.*

Proof. Assume $P(i)$ is not indecomposable, i.e. there are non-zero $M = (M_i, \gamma_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $N = (N_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{Rep}_K Q$ such that $P(i) \simeq M \oplus N$. Then in particular we have $M_i \oplus N_i \simeq P(i)_i \simeq K$, so either $M_i = 0, N_i \simeq K$ or $M_i \simeq K, N_i = 0$. Without loss of generality assume $M_i = 0$. But then by proposition 3.9 we know that $\mathrm{Hom}(P(i), M) \simeq M_i = 0$, so in particular the canonical projection $\pi_M : P(i) \to M$ induced by the product property of $P(i) \simeq L \oplus N$ must be zero. Hence $M = 0$, which contradicts our assumption. $\square$

Now we introduce some injective representations and present results about them that are dual to those concerning the projective representations discussed above.

Definition 3.12. Let $i \in Q_0$. Define the injective representation at vertex i, $I(i) = (I(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$, by $I(i)_j \simeq K^n$, where n is the number of j-i paths, with a canonical basis formally consisting of those paths. Let w be a j-i path in the canonical basis for $I(i)_j$. Given $\alpha : j \to k$, we define $\varphi_\alpha(w) = p$ if $w = \alpha p$ for some k-i path p, and as $\varphi_\alpha(w) = 0$ if that is not the case.

Proposition 3.13. *For all $i \in Q_0$, $I(i)$ is an injective object of $\mathrm{Rep}_K Q$, i.e. given $L, M \in \mathrm{Rep}_K Q$, $M = (M_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $N = (N_i, \gamma_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $g \in \mathrm{Hom}(L, M)$ mono, and $h \in \mathrm{Hom}(L, I(i))$, there is a (not necessarily unique) morphism $f \in \mathrm{Hom}(M, I(i))$ such that $f \circ g = h$. This is illustrated by the following diagram:*

$$\begin{tikzcd}
L \arrow[rr, "g"] \arrow[dr, "h"'] & & M \arrow[dl, "\exists f"] \\
& I(i) &
\end{tikzcd}$$

Proof. First choose f_i such that $f_i \circ g_i = h_i$ (let f_i be zero on cokernel of g and equal to h_i on its image). Let W_j be the canonical basis for $I(i)_j$ consisting of j-i paths. Given $m \in M_j$, define

$$f_j(m) = \sum_{w \in W_j} (f_i \circ \psi_w)(m) w$$

so

$$f_j = \begin{bmatrix} f_i \circ \psi_{w_1} & & \\ & \ddots & \\ & & f_i \circ \psi_{w_n} \end{bmatrix}$$

There are two things to be shown here: 1) f is well-defined, and 2) for all $j \in Q_0$ we have $f_j \circ g_j = h_j$
1) for $\alpha : m \to j$ we need the following diagram to commute:

$$\begin{tikzcd} M_m \arrow[r, "\psi_\alpha"] \arrow[d, "f_m"] & M_j \arrow[d, "f_j"] \\ I(i)_m \arrow[r, "\varphi_\alpha"] & I(i)_j \end{tikzcd}$$

and indeed we have

$$(f_j \circ \psi_\alpha)(x) = \sum\nolimits_{w \in W_j} (f_i \circ \psi_w)(\psi_\alpha(x))w = \sum\nolimits_{w \in W_j} (f_i \circ \psi_{\alpha w})(x)w$$

and

$$\begin{aligned}(\varphi_\alpha \circ f_m)(x) &= \varphi_\alpha(\sum\nolimits_{w' \in W_m} (f_i \circ \psi_{w'})(x)w') = \sum\nolimits_{w' \in W_m} (f_i \circ \psi_{w'})(x)\varphi_\alpha(w') \\ &= \sum\nolimits_{\alpha w, w \in W_j} (f_i \circ \psi_{\alpha w})(x)\varphi_\alpha(\alpha w) = \sum\nolimits_{w \in W_j} (f_i \circ \psi_{\alpha w})(x)w = (f_j \circ \psi_\alpha)(x)\end{aligned}$$

2) We have

$$\begin{aligned}(f_j \circ g_j)(l) &= \sum\nolimits_{w \in W_j} (f_i \circ \psi_w \circ g_j)(l)w = \sum\nolimits_{w \in W_j} (f_i \circ g_i \circ \gamma_w)(l)w \\ &= \sum\nolimits_{w \in W_j} (h_i \circ \gamma_w)(l)w = \sum\nolimits_{w \in W_j} (f_i \circ g_i \circ \gamma_w)(l)w \\ &= \sum\nolimits_{w \in W_j} (f_i \circ g_i \circ \gamma_w)(l)w = \sum\nolimits_{w \in W_j} (\varphi_w \circ h_j)(l)w\end{aligned}$$

Now if we denote $h_j(l)$ as $\sum_{v \in W_j} \rho_v v$ we get

$$\sum\nolimits_{w \in W_j} (\varphi_w)(h_j(l))w = \sum\nolimits_{w \in W_j} \sum\nolimits_{v \in W_j} (\rho_v \varphi_w(v))w = \sum\nolimits_{w \in W_j} \rho_w w = h_j(l)$$

$\square$

Proposition 3.14. *Given a representation* $M = (M_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{Rep}_K Q$, *the following vector spaces are isomorphic:* $\mathrm{Hom}(M, I(i)) \simeq \mathrm{Hom}(M_i, I(i)_i) \simeq M_i$

Proof. The argument is very similar to that given in proof of proposition 3.9: Pick a K-linear map $\eta \in \mathrm{Hom}(M_i, I(i)_i)$, and let $f \in \mathrm{Hom}(M, I(i)$ be a morphism of representations such that $f_i = \eta$, which is then constructed as in the preceding proposition, i.e, given a vertex $j \in Q_0$ and a basis element $m \in M_j$, we let $f_j(m) = \sum_{w \in W_j} (f_i \circ \psi_w)(m)w$. Let $f' \in \mathrm{Hom}(M, I(i))$ be a morphism of representations such that $f'_i = f_i$. In order to be well-defined, f' must satisfy the following commutative diagram:

$$\begin{tikzcd} M_j \arrow[r, "\psi_w"] \arrow[d, "f'_j"] & M_i \arrow[d, "f'_i"] \\ I(i)_j \arrow[r, "\varphi_w"] & I(i)_i \end{tikzcd}$$

Which in turn implies that $f'_j = f_j$, and hence $f = f'$. So there is a natural isomorphism $\mathrm{Hom}(M_i, I(i)_i) \simeq \mathrm{Hom}(M, I(i))$. Moreover, $\mathrm{Hom}(M_i, I(i)_i) \simeq M_i$ by equality of dimensions. $\square$

Proposition 3.15. $I, I' \in \mathrm{Rep}_K Q$ *are injective if and only if* $I \oplus I'$ *is injective*

Proof. We denote the projections and injections as we did in the proof of the dual statement (Proposition 3.10) " $\Longleftarrow$ ": Given $M, N \in \mathrm{Rep}_K Q$, $f \in \mathrm{Hom}(M, N)$ mono, $g \in \mathrm{Hom}(M, I)$. Since $I \oplus I'$ is projective, we let $\gamma : N \to I \oplus I'$ be such that $\gamma \circ f = \iota \circ g$, so now we have $(\pi \circ \gamma) \circ f = \pi \circ \iota \circ g = id_I \circ g = g$ so $\pi \circ \gamma$ completes the following diagram, proving the injective property of I:

$$\begin{tikzcd} M \arrow[d, "f"] \arrow[dr, "g"] & \\ N \arrow[r, "\pi\circ\gamma"] & I \end{tikzcd}$$

" $\implies$ ": Given

$$\begin{tikzcd} M \arrow[d, "f"] \arrow[dr, "g"] & \\ N & I \oplus I' \end{tikzcd}$$

in $\mathrm{Rep}_K Q$, we use the injectivity of I and I' to obtain $h : N \to I$, $h' N \to I'$ such that $h \circ f = \pi \circ g$, $h' \circ f = \pi' \circ g$, and by product property of $I \oplus I'$ we get $\gamma : N \to I \oplus I'$ such that $\pi \circ \gamma = h$, $\pi' \circ \gamma = h'$. We claim that γ completes the diagram above, thus proving the injectivity of $I \oplus I'$. That is because

$$\gamma \circ f = id_{I\oplus I'} \circ \gamma \circ f = \iota \circ \pi \circ \gamma \circ f + \iota' \circ \pi' \circ \gamma \circ f = \iota \circ h \circ f + \iota' \circ h' \circ f = \iota \circ \pi \circ g + \iota' \circ \pi' \circ g = g$$

$\square$

Proposition 3.16. *$I(i)$ is indecomposable.*

Proof. In analogy to the proof of indecomposability of $P(i)$, if we assume $I(i) \simeq M \oplus N$, with neither of summands zero, we can assume $M_i = 0$, and hence have $\mathrm{Hom}(M, I(i)) \simeq M_i = 0$, hence the canonical injection of M into $I(i)$ is zero, and thus M is zero, contradicting the hypothesis. $\square$

We now extend our definition of exactness (and that of homology) to a more general setting:

Definition 3.17. In an abelian category, given a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$, we define the homology of the sequence at B as the cokernel of the induced morphism $\varphi : \mathrm{Coker}(j) \to \mathrm{Ker}(g)$, illustrated by the following diagram:

$$\begin{tikzcd} & \mathrm{Coker}(j) \arrow[r, "\varphi"] & \mathrm{Ker}(g) \arrow[d, "\iota"] & \\ \mathrm{Ker}(f) \arrow[r, "j"] & A \arrow[u, "\rho"] \arrow[ur, "\psi"] \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

Where ψ is the morphism induced by f due to kernel property of $\mathrm{Ker}(g)$, and φ is the morphism induced by ψ due to cokernel property of $\mathrm{Coker}(j)$. We say that the sequence is exact at B whenever the homology at B is zero.

Definition 3.18. Given $M \in \mathrm{Rep}_K Q$, a projective resolution of M is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where P_l is projective for all l. We define the length of a resolution as greatest n such that $P_n \neq 0$.

Theorem 3.19. *$M = (M_i, \psi_\alpha)_{i\in Q_0, \alpha\in Q_1} \in \mathrm{Rep}_K Q$ admits a projective resolution of length 1,*

$$\cdots \to 0 \to P_1 \xrightarrow{\gamma} P_0 \xrightarrow{f} M \to 0$$

Proof. We see that our resolution is a short exact sequence, so if we manage to find a suitable $P_0 = (P_{0i}, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ together with an epimorphism $f : P_0 \to M$, we will have $P_1 \simeq \mathrm{Ker}(f)$, and all that will remain to be done will be to show that $\mathrm{Ker}(f)$ is projective. We let

$$P_0 = \bigoplus_{i\in Q_0} \bigoplus_{k=1}^{\dim(M_i)} P(i)^k$$

(superscript meaning "kth copy"; superscripts will be used to index copies throughout this proof). The K-linear maps of P_0 will be denoted as φ (being - due to the linear maps of its direct summands - a large diagonal matrix of morphisms composing arrows). By definition, given $i \in Q_0$, the canonical basis for $(P_0)_i$

consists of $\dim(M_j)$ copies of the set W_{ji} of j-i paths, for all $j \in Q_0$. We define $f : P_0 \to M$ on that basis. So fix $j \in Q_0$, $w \in W_{ji}$, and let (as mentioned earlier) w^l denote the lth copy of w. Now fix a basis E_j for M_j, $E_j = \{e_1, e_2, ..., e_{\dim(M_j)}\}$. We define f by letting $f_i(w^l) = \psi_w(e_l)$ Clearly then, given an arrow $\kappa \in Q_1$, $\kappa : i \to s$ we have

$$(\psi_\kappa \circ f_i)(w^l) = (\psi_\kappa \circ \psi_w)(e_l) = \psi_{w\alpha}(e_l) = (f_s \circ \varphi_\alpha)(w^l)$$

Hence the following diagram commutes:

$$\begin{tikzcd}
(P_0)_i \arrow[r, "\varphi_\kappa"] \arrow[d, "f_i"] & (P_0)_s \arrow[d, "f_s"] \\
M_i \arrow[r, "\psi_\kappa"] & M_j
\end{tikzcd}$$

and so f is well-defined. Clearly, f is an epimorphism, as the $\dim(M_j)$ copies of empty paths ϵ_j at j are mapped to respective basis elements of M_j by the definition of f. This also gives information about the dimension vector of $\mathrm{Ker}(f)$: each non-trivial path in our canonical basis contributes with one extra dimension to the kernel of f. We can already intuitively describe the projective structure of the kernel: we're left with non-trivial paths in our basis, and by removing the starting arrow out of each path we shift from projective representation (as a direct summand of P_0) at some vertex to projective subrepresentations at each vertex that is a target of an arrow starting from our initially chosen vertex. This is done for each arrow, and number of copies of the representation depends of numbers of copies of paths whose starting arrow we forget, thus we have the following hypothesis:

$$\mathrm{Ker}(f) \simeq \bigoplus_{\alpha \in Q_1} \dim(M_{s(\alpha)})\, P(t(\alpha))$$

And hence we let $P_1 = \bigoplus_{\alpha \in Q_1} . \dim(M_{s(\alpha)})\, P(t(\alpha))$ as above. We we denote the K-linear maps of P_1 by $\bar{\varphi}$.

By definition, given $i \in Q_0$ the canonical basis for P_{1i} is the following set:

$$\{(\textstyle\sum_{\alpha \in Q_1, t(\alpha)=j} \dim(M_{s(\alpha)})) \text{ copies of } W_{ji} \mid j \in Q_0\}$$

There is a natural bijection between that set and the set of non-trivial paths ending at i, which, as is described above, is in bijection with a basis for $\mathrm{Ker}(f)_i$. Given $j \in Q_0$, $\alpha \in Q_1$, $\alpha : k \to j$, let w_α^l be the lth copy of the j-i path w_α lying in the summand corresponding to α. We send w_α^l to $(\alpha w_\alpha)^l$ (as the ordering comes from numbering the copies of α in P_0, the order is naturally preserved). This clearly is a bijection. We denote αw_α by w, and use subscripts to denote paths that have 'forgotten' their starting arrows (represented by the subscript).

Now we start defining $\gamma : P_1 \to P_0$. We will show three properties of γ:

(1) γ is well-defined.
(2) $\mathrm{Im}(\gamma) \subset \mathrm{Ker}(f)$.
(3) γ is a monomorphism.

Once we have established those three properties we will be able to conclude that γ is an isomorphism between P_1 and $\mathrm{Ker}(f)$, since the pointwise equality of dimensions combined with γ being pointwise mono makes γ pointwise iso, and as we've seen before, a well-defined morphism of representations that is a pointwise isomorphism of vector spaces is an isomorphism of representations.

We define γ by defining it on a general basis element of P_1. Hence let $i \in Q_0$, $\alpha \in Q_1$ such that $\alpha : k \to j$ and let $w \in W_{ki}$ be a k-i path such that α is the starting arrow of w. Let w_α^l be the lth copy of w which has 'forgotten' its starting arrow, as described above. This is our basis element.

It would be natural (and most certainly well-defined) to set $\gamma(w_\alpha^l) = w^l$, but we need $\gamma(w_\alpha^l)$ to lie in the kernel of f_i, and that is not necessarily the case for w^l. By definition, if e_l is the lth element in our chosen basis E_k for M_k, we have $f(w^l) = \psi_w(e_l)$ Now we use that $w = \alpha w_\alpha$ to obtain

$$\psi_w(e_l) = \psi_{\alpha w_\alpha}(e_l) = \psi_{w_\alpha}(\psi_\alpha(e_l)) = \psi_{w_\alpha}(\textstyle\sum_{e_c \in E_j} \lambda_c e_c).$$

Note that the linear combination $\sum_{e_c \in E_j} \lambda_c e_c)$ is determined uniquely.

By definition of f we have that $\psi_{w_\alpha}(\sum_{e_c \in E_j} \lambda_c e_c) = f_i(\sum_c \lambda_c w_\alpha^c)$ and finally we conclude that $f_i(w^l) = f_i(\sum_c \lambda_c w_\alpha^c)$, so $f_i(w^l - \sum_c \lambda_c w_\alpha^c) = 0$. We thus set

$$\gamma_i(w_\alpha^l) = w^l - \sum_c \lambda_c w_\alpha^c$$

The quest now is to show that γ is well-defined. Thus we let $\kappa \in Q_1$, $\kappa : i \to s$, and aim to show that the following diagram commutes:

$$\begin{tikzcd} P_{1i} \arrow[r, "\bar{\varphi_\kappa}"] \arrow[d, "\gamma_i"] & P_{1s} \arrow[d, "\gamma_s"] \\ P_{0i} \arrow[r, "\varphi_\kappa"] & P_{0s} \end{tikzcd}$$

We have

$$(\varphi_\kappa \circ \gamma_i)(w_\alpha^l) = \varphi_\kappa(w^l - \sum_c \lambda_c w_\alpha^c) = (w^l)\kappa - (\sum_c \lambda_c w_\alpha^c)\kappa = (w\kappa)^l - \sum_c \lambda_c (w\kappa)_\alpha^c$$

(note that composing with a new arrow added at the end of the path does not change neither the 'forgotten' starting arrow nor the index of the path, so our subscript and superscript operations commute with φ and $\bar{\varphi}$)
and:

$$(\gamma_s \circ \bar{\varphi}_\kappa)(w_\alpha^l) = \gamma_s(w_\alpha^l \kappa) = \gamma_s((w\kappa)_\alpha^l) = (w\kappa)^l - \sum_v \lambda_v (w\kappa)_\alpha^v$$

For the unique set of coefficients $\{\lambda_v\}$ such that

$$f_s(w\kappa)^l - \sum_v \lambda_v (w\kappa)_\alpha^v = 0$$

But we also have

$$f_s((w\kappa)^l - \sum_c \lambda_c (w\kappa)_\alpha^c) = \psi_{w\kappa}(e_l) - \psi_{(w\kappa)_\alpha}(\sum_c \lambda_c e_c) =$$
$$\psi_\kappa(\psi_w(e_l) - \psi_{w_\alpha}(\sum_c \lambda_c e_c)) = \psi_\kappa(f_i(w^l) - f_i(\sum_c \lambda_c w_\alpha^c = \psi_\kappa(0) = 0$$

Hence, by uniqueness mentioned above we must have $\sum_v \lambda_v (w\kappa)_\alpha^v = \sum_c \lambda_c (w\kappa)_\alpha^c$ and hence γ is well-defined. Finally, γ is pointwise mono as the first term in $\gamma_i(w_\alpha^l)$ is a path longer than the others. Hence if the first terms of γ of two basis elements b_1, b_2 are non-equal, then $\gamma(b_1) \neq \gamma(b_2)$. And the first term is just the image of the basis element under the "remembering" bijection, hence if $b_1 \neq b_2$ then their first terms are non-equal, and finally, if $b_1 \neq b_2$, then $\gamma(b_1) \neq \gamma(b_2)$. □

Definition 3.20. Given $M \in \mathrm{Rep}_K Q$, a projective cover of M is a projective $P \in \mathrm{Rep}_K Q$ together with an epimorphism $g : P \to M$ such that given another pair of projective $P' \in \mathrm{Rep}_K Q$ together with an epimorphism $g' : P' \to M$, there is an epimorphism $h : P' \to P$ such that $g \circ h = g'$. Diagrammatically we have

$$\begin{tikzcd} & P' \arrow[dl, "h"'] \arrow[d, "g'"] \\ P \arrow[r, "g"] & M \end{tikzcd}$$

Definition 3.21. Dually, an injective envelope of M is an injective $I \in \mathrm{Rep}_K Q$ together with a monomorphism $f : M \to I$ such that given another injective $I' \in \mathrm{Rep}_K Q$ together with a monomorphism $f' : M \to I'$, there is a monomorphism $h : I \to I'$.

Definition 3.22. A projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is minimal if $P_0 \to M$ is a projective cover and so is $P_i \to \mathrm{Ker}_{i-1}$ for all $i > 0$.

Proposition 3.23. *Let (P, g), (P', g') be projective covers of M. Then $P \simeq P'$*

Proof. Let $h : P' \to P$ be an epimorphism induced by the projective cover property of P. Then we have the following exact sequence: $0 \to \mathrm{Ker}(h) \xrightarrow{\iota} P' \xrightarrow{h} P \to 0$ And since P is projective, this sequence splits. We thus have $P' \simeq \mathrm{Ker}(h) \oplus P$, and by using the projective cover property of P' in the same manner we obtain $P \simeq P' \oplus Ker(h') \simeq P \oplus \mathrm{Ker}(h) \oplus \mathrm{Ker}(h')$, so $\mathrm{Ker}(h) = 0 = \mathrm{Ker}(h')$, and both h and h' are isomorphisms. $\square$

Proof of the dual statement is dual:

Proposition 3.24. *Let (I, f), (I', f') be injective envelopes of M. Then $I \simeq I'$*

Proof. By injective envelope property of I, we have an exact sequence $0 \to I \xrightarrow{h} I' \xrightarrow{\pi} \mathrm{Coker}(h) \to 0$, and that sequence splits as I is injective. So $I' \simeq \mathrm{Coker}(h) \oplus I$, and similarly for I',

$$I \simeq \mathrm{Coker}(h') \oplus I' \simeq \mathrm{Coker}(h') \oplus \mathrm{Coker}(h) \oplus I.$$

Thus we have $\mathrm{Coker}(h) = 0 = \mathrm{Coker}(h')$, and so the morphisms are iso. $\square$

Definition 3.25. Let $A = \bigoplus_{i \in Q_0} P(i)$. $F \in \mathrm{Rep}_K Q$ is free if and only if $F \simeq A \oplus \cdots \oplus A$

Proposition 3.26. *i) $M \in \mathrm{Rep}_K Q$ is projective if and only if it is a direct summand of a free representation F.*
ii) M is projective if and only if it is a direct sum of some $P(i)$, $i \in Q_0$

Proof. i)
" $\Longleftarrow$ ": follows by repeated application of proposition 3.10. Using said proposition we know that A (being defined as a direct sum of projective indecomposable representations) is projective, hence so is F, and finally, using the proposition for the third time, we get that M, being a direct summand of F, is projective. " $\Longrightarrow$ " follows from the canonical projective resolution of M, $0 \to P_1 \to P_0 \to M \to 0$, since P_0 clearly is a direct summand of a large enough F (for example of rank equal to the maximal dimension of a space in M), and M is projective, so our sequence splits and M is a direct summand of P_0 and thus also a direct summand of that F.
ii) Follows directly from i), M is a direct summand of F, and by Krull-Schmidt theorem and indecomposability of $P(i)$, direct summands of F are (small enough) direct sums of $P(i)$, $i \in Q_0$ $\square$

Proposition 3.27. *Let $M = (M_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{Rep}_K Q$, and let P be a projective representation $P \in \mathrm{Rep}_K Q$. If M is a subrepresentation of P, then M is projective.*

Proof. Consider the following short exact sequence in $\mathrm{Rep}_K Q$:

$$0 \to M \xrightarrow{\iota} P \xrightarrow{\pi} P/M \to 0$$

where ι is the canonical injection and π is the canonical projection. By proposition 3.19 we know that there are projective representations P_0, P_1, P'_0, P'_1 such that the following commutative diagram exists:

$$\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & M \arrow[u] \arrow[r, "\iota"] & P \arrow[r, "\pi"] & P/M \arrow[u] \arrow[r] & 0 \\
 & P'_0 \arrow[u] & & P_0 \arrow[u] & \\
 & P'_1 \arrow[u] & & P_1 \arrow[u] & \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

Now by Horseshoe lemma (proof of which can be found in [6], p.349) we can complete the "missing" second and third rows of the diagram so that the rows split, and that the resulting middle column is a projective resolution of P. Using that P itself is projective, we conclude that the middle column also is split. Hence the morphisms $\bar{f}, \bar{g}$ in the diagram below are induced by f and g being sections.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M \arrow[r, "\iota"] \arrow[u] & P \arrow[r, "\pi"] \arrow[u] & P/M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "b"] \arrow[u] & Q_0 \arrow[r] \arrow[u] \arrow[d, shift left, "\bar{g}"] & P_0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_1' \arrow[r, shift left, "f"] \arrow[u, "a"] & Q_1 \arrow[r] \arrow[u, shift left, "g"] \arrow[l, shift left, "\bar{f}"] & P_1 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In particular we have

$$\bar{f} \circ \bar{g} \circ b \circ a = \bar{f} \circ \bar{g} \circ g \circ f = \bar{f} \circ id_{Q_1} \circ f = \bar{f} \circ f = id_{P_1'}.$$

Thus a is a section, and by proposition 2.27 we conclude that the first column is a split exact sequence, hence $P_0' = M \oplus P_1'$ and by proposition 3.10 we know that M is projective. $\square$

We thus have established that $\mathrm{Rep}_K Q$ is what is called a hereditary category. This has immediate consequences for some of the homological algebra we have managed to defined for quiver representations. Namely, recall that given a projective resolution of M,

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

and cochain complex given by $\mathrm{Hom}(-, X)$ of the sequence:

$$0 \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(P_0, X) \to \cdots$$

we can, independently of chosen resolution, obtain the $\mathrm{Ext}^i(M, X), i \geq 0$ by taking the cohomology of the so called deleted complex

$$0 \to \mathrm{Hom}(P_0, X) \to \mathrm{Hom}(P_1, X) \to \cdots$$

but since we know that every quiver representation has a projective resolution of length 1, we obtain the following result:

Proposition 3.28. Ext^2 *vanishes for quiver representations.*

Note that this can be established for any abelian category $\mathcal{C}$ that satisfies the following conditions:
a) It is hereditary
b) It has enough projectives, i.e. given $M \in \mathcal{C}$, there is a projective object P with an epimorphism $P \to M$
This is easily seen by exactness of $0 \to \mathrm{Ker}(f) \to P \xrightarrow{f} M \to 0$, yielding a projective resolution of length 1, putting us in the same situation as we had for quiver representations.

Lemma 3.29. *Additive functors preserve finite biproducts. (i.e. the coinciding notions of products and coproducts, see [5], corollary 8.2.4)*

Proof. Let $\mathcal{C}, \mathcal{D}$ be additive (i.e. *Ab*-Enriched, with finite coproducts) categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. (i.e. for morphisms f, g in $\mathcal{C}$, we have $F(f+g) = F(f) + F(g)$ whenever $f+g$ is well-defined.) Let $M, M' \in \mathcal{C}$, and let $M \oplus M'$ be their biproduct. We show that $F(M \oplus M') = F(M) \oplus F(M')$ by showing that $F(M \oplus M')$ satisfies the universal property of product of $F(M)$ and $F(M')$, and thus is the biproduct of those objects.
Let $N \in \mathcal{D}, f_1 \in \mathrm{Hom}(F(M), N), f_2 \in \mathrm{Hom}(F(M'), N)$. Let $f := f_1 \circ F(\pi) + f_2 \circ F(\pi')$.
First, we have:

$$f \circ F(\iota) = (f_1 \circ F(\pi) + f_2 \circ F(\pi')) \circ F(\iota) = f_1 \circ F(\pi \circ \iota) + f_2 \circ F(\pi' \circ \iota) = f_1 \circ F(id_M) + f_2 \circ F(0) = f_1$$

$f \circ F(\iota') = f_2$ is obtained in the same fashion. So f satisfies the necessary conditions for being the product morphism, and now we only need to show it's unique such. Thus let f' also satisfy those conditions. We get

$$f' = f' \circ id_{F(M \oplus M')} = f' \circ F(\iota \circ \pi + \iota' \circ \pi') = (f \circ F(\iota)) \circ F(\pi) + (f \circ F(\iota')) \circ F(\pi') = f_1 \circ \pi + f_2 \circ \pi' = f$$

□

Definition 3.30. Let Q^{op} denote the opposite quiver of Q i.e a quiver whose set of vertices is the same as that of Q, and its set of arrows consists of arrows of Q with their direction reversed. Let D be the contravariant functor: $\mathrm{Rep}_K Q \to \mathrm{Rep}_K Q^{op}$ sending representations $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ to $(M_i^*, \circ\varphi_\alpha)_{i \in Q_0^{op}, \alpha \in Q_1^{op}}$ where M^* denotes the dual space of M and $\circ\varphi$ denotes the pullback of φ, and sending the morphisms $\gamma = (\gamma_i)_{i \in Q_0} : M \to N$ to $D\gamma : DN \to DM$, where $D\gamma$ is defined as $(D\gamma_j)_{j \in Q_0^{op}} := (\circ\gamma_j)_{j \in Q_0^{op}}$.

Proposition 3.31. *Let* $\mathrm{Proj}\, Q$ *denote the subcategory of* $\mathrm{Rep}_K Q$ *of projective representations of* Q*, and* $\mathrm{Inj}\, Q$ *the subcategory of* $\mathrm{Rep}_K Q$ *of injective representations of* Q*.* D *is a contravariant equivalence of categories (a so called duality). In particular we have* $D(P_Q(i)) = I_{Q^{op}}(i)$*. Thus, by additivity of* D *and lemma 3.29, its restriction to* $\mathrm{Proj}\, Q$ *is a duality between* $\mathrm{Proj}\, Q$ *and* $\mathrm{Inj}\, Q^{op}$*.*

Proof. We show that i) $D_{Q^{op}} \circ D_Q = id_{\mathrm{Rep}_K Q}$ which is equivalent to $D_Q \circ D_{Q^{op}} = id_{\mathrm{Rep}_K Q^{op}}$, and thus the categories are equivalent.
Note that given a linear map f, its pullback is exactly its dual f^*. Now denote $D_{Q^{op}} \circ D_Q$ as D^2. Given $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{Rep}_K Q$, we have $D^2 M = (M_i^{**}, \varphi_\alpha^{**})_{i \in Q_0, \alpha \in Q_1} \simeq M$ (by isomorphisms, both point- and arrow-wise), and similarly for $f \in \mathrm{Hom}(M, N)$, $D^2 f = f^{**} \simeq f$, so we're done with that part.
Now we show $D(P_Q(i)) \simeq I_{Q^{op}}(i)$.
First note that for $m \in Q_0$, there is a canonical choice of basis for $DP_Q(i)_m$, namely the basis dual to the canonical basis for $P_Q(i)_m$. Such a basis consists of a set of 1-forms $\{w^* \mid w \in W_{im}$ (set of i-m paths) $\}$, where w^* is defined on the canonical basis by letting $w^*(w) = 1$ and letting $w^*(\eta) = 0$ for all $\eta \in W_{im} : \eta \neq w$.
Let $DP_Q(i) = (DP_Q(i)_j, \circ\varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$, $I_{Q^{op}} = (I_{Q^{op}}(i)_j, \psi_\alpha)_{j \in Q^op_0, \alpha \in Q^op_1}$. Given $\alpha : j \to m$ in Q_1 and an i-j path t in Q, we have $((\circ\varphi_\alpha) \circ \omega^*)(t) = \omega^* \circ \varphi_\alpha(t) = \omega^*(t\alpha) = \delta_\omega(t\alpha)$, giving the zero map if ω does not end with α and otherwise checking whether t is equal to ω if we forget the α at the end of it. Let this form be denoted by $\omega \ominus \alpha$. We define $f : DP_Q(i) \to I_{Q^{op}}(i)$ by $\omega^* \to \omega^{-1}$, where ω^{-1} denotes the path "opposite to ω". Clearly it is pointwise mono and epi (being a natural bijection between i-m paths in Q and m-i paths in Q^{op}), and in order to prove that it is an isomorphism we only need to show that it is well-defined, that is $f_j \circ (\circ\varphi_\alpha) = \psi_{\alpha^{-1}} \circ f_m$ (note that $\alpha \in Q_1$ corresponds to $\alpha^{-1} \in Q_1^{op}$). Choose an i-m path ω in canonical basis for $DP_Q(i)_m$. We have $(\psi_\alpha \circ f_m)(\omega^*) = \psi_\alpha(\omega^{-1}) = \omega^{-1} - \alpha^{-1}$ (which is 0 if ω^{-1} does not start at $\alpha^{-1} \iff \omega$ ends on α, else, by definition, one just cuts off the starting arrow α^{-1}). Clearly we also have $(f_j \circ (\circ\alpha))(\omega) = f_j(\omega \ominus \alpha) = \omega^{-1} - \alpha^{-1}$, so our isomorphism is well defined. □

We make one more observation about D, which will allow us to justify our earlier "omission" of injective resolutions.

Proposition 3.32. *An equivalence of Abelian Categories is exact.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence between them. By our definition of exactness, it is not hard to see that in order to establish the exactness of F, we only need to show that it preserves kernels and cokernels (as then the diagram defining the homology as in definition 3.17 is preserved under F). The claim is:

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\iota"] & L \arrow[r, "f"] & M \\ & \forall X \arrow[u, "\forall h"] \arrow[ul, "\exists! \varphi"] & \end{tikzcd} \quad \xrightarrow{F} \quad \begin{tikzcd} F(\mathrm{Ker}(f)) \arrow[r, "F(\iota)"] & F(L) \arrow[r, "F(f)"] & F(M) \\ & \forall X', X' \simeq F(X) \arrow[u, "\forall h' = F(h)"'] \arrow[ul, "\exists! F(\varphi)"] & \end{tikzcd}$$

Given $X' \in \mathcal{D}$, we have $X' \simeq F(X)$ for some $X \in \mathcal{C}$ as F is essentially surjective. Similarly, an arbitrary $h' \in \mathrm{Hom}(X', F(L))$ in $\mathcal{D}$ can be (under that isomorphism) expressed as $F(h)$ for some $h \in \mathrm{Hom}(X, L)$ in $\mathcal{C}$ as F is full. Finally, $F(\iota) \circ F(\varphi) = F(\iota \circ \varphi) = F(h)$, so $F(\iota)$ clearly satisfies the relation required to be a morphism induced by kernel property, and it is unique such since φ was unique such in $\mathcal{C}$. The proof for cokernels is analogous. □

Thus F is exact, and we can justify the following claim:

Lemma 3.33. *$M \in \mathrm{Rep}_K Q$ has an injective resolution of length 1.*

Proof. Take the standard projective resolution of DM in $\mathrm{Rep}_K Q^{op}$,

$$0 \to P_1 \to P_0 \to DM \to 0$$

and now apply D to that sequence to obtain

$$0 \to M \to DP_0 \to DP_1 \to 0$$

By results above we know that DP_0 and DP_1 are injective, and that the sequence is exact. $\square$

Now we introduce an auxiliary, "special hom-functor", which instead of sending quiver representations to vector spaces (the hom-sets), sends $\mathrm{Rep}_K Q$ to $\mathrm{Rep}_K Q^{op}$. Recall that we called a representation free if and only if it was isomorphic to $A \oplus \cdots \oplus A$. We define our functor using $\mathrm{Hom}(-, A)$.

Definition 3.34. $\mathrm{Hom}(-, A) : \mathrm{Rep}_K Q \to \mathrm{Rep}_K Q^{op}$ is defined as follows: Given $X \in \mathrm{Rep}_K Q$, let $\mathrm{Hom}(-, A)(X) = \mathrm{Hom}(X, A) = (M_i, \varphi_{\alpha^{-1}})_{i \in Q_0^{op}, \alpha^{-1} \in Q_1^{op}}$, where $M_i = \mathrm{Hom}(X, P(i))$ for all $i \in Q_0^{op}$, and now, using the fact that $\mathrm{Hom}(P(i), P(j))$ has a basis consisting of all j-i paths, given $\alpha^{-1} : j \to i \ \in Q_1^{op}$ we let $\varphi_{\alpha^{op}} : \mathrm{Hom}(X, P(j)) \to \mathrm{Hom}(X, P(i))$ take f to $\alpha \circ f$. Now for morphisms, given $g \in \mathrm{Hom}(X, X')$ we let $\mathrm{Hom}(-, A)(g) = \circ g$. Clearly for α^{op} as earlier we have $((\circ g_i) \circ \varphi'_{\alpha^{op}})(f) = (\alpha \circ f) \circ g = \alpha \circ (f \circ g) = (\varphi_{\alpha^{op}} \circ (\circ g_j))(f)$, so it is well defined, and just like for the usual hom-functor we have $(\circ(f \circ g)) = (\circ g) \circ (\circ f)$, so we have a contravariant functor.

Now we have the necessary tools to introduce the Nakayama functor, but there are still two remarks to be made.

Proposition 3.35. $\mathrm{Hom}(-, A)$ *is zero on non-projective indecomposable summands.*

Proof. Let $M \in \mathrm{Rep}_K Q$, and let $M = \bigoplus_k M_k$ be a decomposition of M into indecomposable summands. Since $\mathrm{Hom}(-, A)$ is additive (being a hom-functor), we have $\mathrm{Hom}(-, A)M = \bigoplus_k \mathrm{Hom}(M_k, A)$, and $\mathrm{Hom}(M_k, A) \neq 0$ if and only if M_k is projective, since for a non-zero morphism in $\mathrm{Hom}(M_k, P(j))$, the sequence $0 \to \mathrm{Ker}(f) \to M_k \to \mathrm{Im}(f) \to 0$ splits (as $\mathrm{Im}(f) = \mathrm{Ker}(\mathrm{Coker}(f))$ splits, since $\mathrm{Im}(f)$ is a subrepresentation of P_j and thus is projective), and thus $M_k = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$, so either $\mathrm{Ker}(f) = 0$ or $\mathrm{Im}(f) = 0$, but $f \neq 0$ implies that $\mathrm{Im}(f) \neq 0$, so $\mathrm{Ker}(f) = 0$, thus $M_k \simeq \mathrm{Im}(f)$, and f is injective, and so M_k is a subrepresentation of $P(j)$, and thus M_k is projective. $\square$

Remark 3.36. $\mathrm{Hom}(-, A)$ *restricted to* $\mathrm{Proj}\, Q$ *is a duality* $\mathrm{Proj}\, Q \to \mathrm{Proj}\, Q^{op}$

Proof. On objects we have $\mathrm{Hom}(-, A)(P_Q(i)) = P_{Q^{op}}(i)$: a canonical basis for $(\mathrm{Hom}(-, A)(P_Q(i)))_k = \mathrm{Hom}(P_Q(i), P_Q(k))$ consists of the set of all k-i paths. Map those to their "opposites" in the representation of the opposite quiver $P_{Q^{op}}(i)_k$. Call that map γ. γ is clearly a bijection of bases and thus a pointwise isomorphism. To show that it's well-defined we have, given a k-i path ω in Q and an arrow $\alpha : j \to k \in Q_1$, $(\gamma_j \circ (\circ\alpha))(\omega) = \gamma_j(\alpha\omega) = (\alpha\omega)^{-1} = \omega^{-1}\alpha^{-1} = (\varphi_{\alpha^{-1}} \circ \gamma_k)(\omega)$, where $\varphi_{\alpha^{-1}}$ denotes the linear map of $P_{Q^{op}}$ along α^{-1}, attaching α^{-1} to ω^{-1}. Thus we have $(\mathrm{Hom}_{Q^{op}}(-, A^{op}) \circ \mathrm{Hom}_Q(-, A))(P_Q(i))) = (\mathrm{Hom}_{Q^{op}}(-, A^{op})(P_{Q^{op}}(i)) = P_Q(i)$, using the fact that ${Q^{op}}^{op} = Q$.
On morphisms we only need to study what happens to morphisms between the indecomposable projectives, since Hom is additive. Take two indecomposable projectives in $\mathrm{Proj}\, Q$, $P(j), P(m)$, and let ω be the morphism corresponding to the m-j path ω. ω is sent to $\circ\omega$ under $\mathrm{Hom}(-, A)$, and then to ω^{op} under the isomorphism described above, and thus again, if we first apply $\mathrm{Hom}_Q(-, A)$ and then its (as is being proven) quasi-inverse $\mathrm{Hom}_{Q^{op}}(-, A^{op})$ on ω, we obtain $(\omega^{-1})^{-1} = \omega$. $\square$

Definition 3.37. The Nakayama functor is defined as $\nu : \mathrm{Rep}_K Q \to \mathrm{Rep}_K Q$, $\nu = D \circ Hom(-, A)$

Some properties of ν follow from the properties of D and $\mathrm{Hom}(-, A)$ discussed earlier. First of all, since $\mathrm{Hom}(-, A)$ is a duality $\mathrm{Proj}\, Q \to \mathrm{Proj}\, Q^{op}$, with $\mathrm{Hom}(-, A)(P_Q(i)) = P_{Q^{op}}(i)$, and D is a duality mapping projectives to injectives, we have:

Proposition 3.38. *The restriction of ν to projective representations is an equivalence* $\mathrm{Proj}\, Q \to \mathrm{Inj}\, Q$ *with* $\nu(P(i)) = I(i)$

And since Hom is left exact and D is exact and contravariant, we have:

Proposition 3.39. *The Nakayama functor ν is right exact.*

Definition 3.40. Given $M \in \mathrm{Rep}_K Q$ let

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

$$0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \longrightarrow 0$$

be minimal projective respectively minimal injective resolutions of M. (those exist by I.5.10 in [1])
We define:

(1) The Auslander-Reiten translate of M, τM, as $\mathrm{Ker}(\nu p_1)$ in the sequence
$$0 \to \mathrm{Ker}(\nu p_1) \xrightarrow{\iota} \nu P_1 \xrightarrow{p_1} \nu P_0 \xrightarrow{p_0} \nu M \xrightarrow{\pi} \mathrm{Coker}(\nu p_0)$$
obtained by applying nu to the projective resolution above
(2) Let $\mathcal{O}_{\mathrm{Rep}_K Q}$ denote the set of objects of $\mathrm{Rep}_K Q$ (the set of all representations of Q). We define The Auslander-Reiten translation as the map $\tau : \mathcal{O}_{\mathrm{Rep}_K Q} \to \mathcal{O}_{\mathrm{Rep}_K Q}$ by $M \to \tau M$.
(3) The inverse Auslander-Reiten translate of M, $\tau^{-1}M$, as $\mathrm{Coker}(\nu^{-1} i_1)$ in the sequence
$$0 \to \mathrm{Ker}(\nu^{-1} i_0) \xrightarrow{\iota} \nu^{-1} I_0 \xrightarrow{i_0} \nu^{-1} I_1 \xrightarrow{i_1} \nu^{-1} M \xrightarrow{\pi} \mathrm{Coker}(\nu^{-1} i_1)$$
obtained by applying ν^{-1} to the injective resolution above.
(4) The inverse Auslander-Reiten translation $\tau^{-1} : \mathcal{O}_{\mathrm{Rep}_K Q} \to \mathcal{O}_{\mathrm{Rep}_K Q}$ by $M \to \tau^{-1}M$

Note that this is well-defined by propositions 3.23 and 3.24, since as a consequence of those, minimal resolutions are unique up to isomorphism.

4. Irreducible morphisms and almost split sequences

Towards the end of the last section we have payed significant attention to a particular quiver representation, $A = \bigoplus_{i \in Q_0} P(i)$. We will now reveal why that representation plays a special role.

Definition 4.1. Given a quiver Q, we let KQ denote the algebra defined as follows:
as a K-vector space, we let KQ be the K-linearization of the set $\bar{Q}$ of paths in Q, and the product is defined (on the basis consisting of paths) by composition of paths; if the composition is impossible, the product is defined to be 0. More formally we have $(\sum_{\omega \in \bar{Q}} \lambda_\omega \omega) \cdot (\sum_{\kappa \in \bar{Q}} \alpha_\kappa) = \sum_{\omega,\kappa} \delta_{t(\omega),s(\kappa)} \lambda_\omega \alpha_\kappa \, \omega\kappa$

Lemma 4.2. *As composing paths is associative, the path algebra is an associative algebra. Moreover, under our assumptions about the quiver (being finite, acyclic), this algebra is unital (its unit being $\sum_{i \in Q_0} \epsilon_i$ and finite-dimensional (its dimension being equal to the number of paths in the quiver).*

Before the significance of A is revealed, we need one more result:

Proposition 4.3. *The category* mod-KQ *of right finite dimensional KQ-modules is equivalent to the category* $\mathrm{Rep}_K Q$ *of finite-dimensional representations of Q.*

Proof. Let $M \in$ mod-KQ. Note that $M = \bigoplus_{i \in Q_0} M\epsilon_i$ as the identity of the path algebra is given by $\sum_{i \in Q_0} \epsilon_i$, and $i \neq j$ implies that $M\epsilon_i \cap M\epsilon_j = 0$ as

$$m = m_1\epsilon_i = m_2\epsilon_j, \text{ so } m_1\epsilon_i^2 = m_1\epsilon_i = m = m_2(\epsilon_j\epsilon_i) = m_2 0 = 0$$

and that for $\alpha : i \to j \in Q_1$ the map $\cdot\alpha : M \to M$ is 0 on $M\epsilon_j$ for $j \neq i$ and $im(\cdot\alpha) \subset M\epsilon_j$. And by the definition of M we know that $\cdot\alpha$ is K-linear. Thus $\cdot\alpha : M\epsilon_i \to M\epsilon_j$ is a linear map. Now we will define the equivalence $F :$ mod-$KQ \to \mathrm{Rep}_K Q$. Let F send M to quiver representation $(M\epsilon_i, \cdot\alpha)_{i \in Q_0, \alpha \in Q_1}$ and for a module morphism $\varphi \in \mathrm{Hom}(M, N)$ (being a K-linear map commuting with the KQ-action) we have $\varphi(M\epsilon_i) \subset \varphi(M)\epsilon_i$, so φ can be viewed as a set of maps $\varphi_i : M\epsilon_i \to N\epsilon_i$, and all of them commute with mod-KQ action, so if we instead view that set as a morphism of quiver representations $F(M), F(N)$, our map commutes with arrows, and thus is well-defined. Clearly now F is a functor. Now we construct its quasi-inverse:

Let $M = (M_i, \psi_\alpha)_{i\in Q_0, \alpha\in Q_1} \in \mathrm{Rep}_K Q$. Now let $F^{-1}(M) = \bigoplus_{i\in Q_0} M_i$ and endow it with following action: for $m \in M$, let

$$m\epsilon_i = \begin{cases} m, & \text{if } m \in M_i \\ 0, & \text{otherwise} \end{cases}$$

, and for $m \in M_i, \alpha : i \to j \in Q_1$ let $m\alpha = \psi_\alpha(m)$ (for other m let $m\alpha = 0$). This clearly defines a KQ-module, and since F only "disconnects" a module, which is later "joined" back together in the same fashion by F^{-1}, these are clearly quasi-inverses. □

Proposition 4.4. *$A = \bigoplus_{i\in Q_0} P(i) = F(KQ_{KQ})$, where KQ_{KQ} denotes the path algebra as a module over itself.*

Proof. By definition of F we have a canonical basis for $F(KQ_{KQ})_i$ consisting of paths ending at i, which is exactly the basis for A_i (since basis for $P(j)_i$ consists of paths from j to i, but here we take the direct sum over all j). Thus $F(KQ_{KQ})_i = A_i$. And for an arrow $\alpha : i \to j \in Q_1$, the linear map corresponding to it acts by composing with α, by definition, thus we also obtain arrow-wise equality, and thus obtain our result. □

Proposition 4.5 (II.1.12 in [1])**.** *Let $|Q_0| = n$. Then*

$$KQ \simeq \begin{bmatrix} \epsilon_1(KQ)\epsilon_1 & 0 & \dots & 0 \\ \epsilon_2(KQ)\epsilon_1 & \epsilon_2(KQ)\epsilon_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \epsilon_n(KQ)\epsilon_1 & \epsilon_n(KQ)\epsilon_2 & \dots & \epsilon_n(KQ)\epsilon_n \end{bmatrix}$$

Proof. Since the quiver is acyclic, we can enumerate Q_0 so that $\alpha : i \to j \in Q_0$ implies $j \geq i$. Now let W denote the set of all paths in Q, and, just as before, let W_{ij} denote the set of all i-j paths in Q. Clearly then $W = \coprod_{j\geq i} W_{ij}$. Now for $x = \sum_{\omega\in W} \lambda_\omega \omega$ we can rewrite it as $x = \sum_{j\geq i} \sum_{\omega\in W_{ij}} \lambda_\omega \omega$, and that element we uniquely express as the lower triangular matrix $X = (\sum_{\omega\in W_{ij}} \lambda_\omega \omega)_{j\geq i}$. Now all we need to do is to show that our bijection $\chi : x \to X$ is a morphism of algebras. For $x, y \in KQ$ we have

$$x = \sum_{j\geq i, W_{ij}} \lambda_\omega \omega, y = \sum_{j\geq i, W_{ij}} \kappa_\eta \eta, \text{ so } xy = \sum_{j\geq k\geq i, \omega: j\to k, \eta: k\to i} \lambda_\omega \kappa_\eta \omega\eta$$

and fixing j, i in that expression yields $\chi(xy)_{ij}$ as a sum over k. On the other hand

$$(\chi(x)\chi(y))_{ij} = \sum_{k=1}^{n} \sum_{\omega: j\to k, \eta: k\to j} \lambda_\omega \kappa_\eta \omega\eta$$

and we get entry-wise matrix equality by noting that in order for an element in the sum to be non-zero we need $j \geq k \geq i$. Other criteria follow entry-wise. □

The remaining results of this chapter are proven within the framework of modules over some finite-dimensional algebra A, but by the equivalence shown above, all the results follow for the category of finite-dimensional representations of an acyclic quiver Q. We could thus even restrict the generality of our considerations and think of A as the path algebra of Q.

Definition 4.6 (IV.1.1 [1])**.** Let $L, M, N \in$ mod-A.

(1) $f \in \mathrm{Hom}(L, M)$ is left minimal if for every $h \in \mathrm{End}(M)$ we have: if $h \circ f = f$, then h is an automorphism.
(2) $g \in \mathrm{Hom}(M, N)$ is right minimal if for every $h \in \mathrm{End}(M)$ we have: if $g \circ h = g$ then h is an automorphism.
(3) $f \in \mathrm{Hom}(L, M)$ is left almost split if f is not a section and for all $h \in \mathrm{Hom}(L, X)$: if h is not a section, then there is a morphism $h' \in \mathrm{Hom}(M, X)$ such that $h' \circ f = h$.
(4) $g \in \mathrm{Hom}(M, N)$ is right almost split if g is not a retraction and for all $h \in \mathrm{Hom}(X, N)$: if h is not a retraction, then there is a morphism $h' \in \mathrm{Hom}(X, M)$ such that $g \circ h' = h$.
(5) $f \in \mathrm{Hom}(L, M)$ is left minimal almost split if it is left minimal and left almost split.
(6) $g \in \mathrm{Hom}(M, N)$ is right minimal almost split if it is right minimal and right almost split.

Proposition 4.7 (IV.1.2 in [1])**.**

(1) *If A-module homomorphisms $f : L \to M$, $f' : L \to M'$ are left minimal almost split, then there is an isomorphism $h : M \to M'$ such that $f' = h \circ f$*
(2) *If A-module homomorphisms $g : M \to N$, $g' : M' \to N$ are right minimal almost split, then there is an isomorphism $h : M \to M'$ such that $g' = g \circ h$*

Proof. (1) By "almost split" property of f we know that there is $\varphi : M \to M'$ such that $\varphi \circ f = f'$. By the same property of f' we know that there is $\psi : M' \to M$ such that $\psi \circ f' = f$, thus $\psi \circ \varphi \circ f = \psi \circ f' = f$, and $\varphi \circ \psi \circ f' = \varphi \circ f = f'$, and by "minimal" property of f and f' we know that $\psi \circ \varphi$ respectively $\varphi \circ \psi$ are isomorphisms. Now $\varphi \circ \psi$ is epi, hence so is φ, and $\psi \circ \varphi$ being mono means that so is φ, thus φ is mono and epi and thus an isomorphism, and by its definition we have $f' = \varphi \circ f$.
(2) Similarly to 1), we first use the "almost split" property to obtain $\varphi : M \to M', \psi : M' \to M$ such that $g = g' \circ \varphi$, $g' = g \circ \psi$, which yields $g \circ \psi \circ \varphi = g, g' \circ \varphi \circ \psi = g'$, and thus, by the "minimal" property we have $\varphi \circ \psi$ iso and $\psi \circ \varphi$ iso, and thus ψ satisfies said conditions.

□

Proposition 4.8 (IV.1.3 in [1])**.**

(1) *If $L \xrightarrow{f} M$ is left almost split, then L is indecomposable.*
(2) *If $M \xrightarrow{g} N$ is right almost split, then N is indecomposable.*

Proof.

(1) Assume L is not indecomposable, $L \simeq L_1 \oplus L_2$, $L_1, L_2 \neq 0$. Recall that then $id_L = \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2$, ι, π denoting the canonical inclusions and projections. By "almost split" property of f we know that there is φ_1 such that $\varphi_1 \circ f = \pi_1$, and φ_2 such that $\varphi_2 \circ f = \pi_2$, since clearly none of the projections is a section (as long as the other summand is non-zero). But then define $M \xrightarrow{\gamma} L$ as $\gamma = \iota_1 \circ \varphi_1 + \iota_2 \circ \varphi_2$. We have
$$\gamma \circ f = (\iota_1 \circ \varphi_1 + \iota_2 \circ \varphi_2) \circ f = \iota_1 \circ (\varphi_1 \circ f) + \iota_2 \circ (\varphi_2 \circ f) = \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2$$
and thus f is a section, contradicting the definition of f.
(2) This is analogous to 1): assume N is not indecomposable, $N \simeq N_1 \oplus N_2$. Then since g is almost split we have ψ_1, ψ_2 such that $g \circ \psi_1 = \iota_1$, $g \circ \psi_2 = \iota_2$ (as inclusions are not retractions). Then by letting $N \xrightarrow{\eta} M$, and letting $\eta = \psi_1 \circ \pi_1 + \psi_2 \circ \pi_2$, we get $g \circ \eta = id_N$, and thus g is a retraction, contradicting the definition of g.

□

Definition 4.9. A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in mod-A is an almost split sequence if f is left minimal almost split and g is right minimal almost split.

Remark 4.10. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ and $0 \to L' \xrightarrow{f'} M' \xrightarrow{g'} N' \to 0$ be two almost split exact sequences. Then the following statements are equivalent:*

(1) *The two sequences are isomorphic*
(2) *$L \simeq L'$*
(3) *$N \simeq N'$*

Proof. (1) implies (2) and (3) by definition. We show that (2) implies (1). (3) implies (1) in a very similar manner.
Let $L \underset{u}{\xrightarrow{\sim}} L'$. Then clearly $L \xrightarrow{f' \circ u} M'$ is left minimal almost split, and thus, by proposition 4.7, there is an isomorphism $t : M \xrightarrow{\sim} M'$ such that $t \circ f = f' \circ u$. Note that by exactness of sequences we have $N \simeq \mathrm{Coker}(f)$, $N' \simeq \mathrm{Coker}(f')$, and thus $g' \circ t$ induces a morphism $N \xrightarrow{p} N'$ such that $p \circ g = g' \circ t$, by the cokernel property of N. Similarly, $g \circ t^{-1}$ induces a morphism $N' \xrightarrow{p'} N$ such that $p' \circ g' = g \circ t^{-1}$, by the cokernel property of N'. Using those two expressions we conclude that
$$p \circ g \circ t^{-1} = (p \circ p') \circ g' = g' \circ (t \circ t^{-1}) = g'$$

and thus $p \circ p' = id_{N'}$ as g' is epi. Similarly we obtain $p' \circ p = id_N$, so p and p' are mutually inverse isomorphisms that also commute with g, g', t and t^{-1}, and thus we have:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "t"] & N \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are isomorphisms. □

We shift our focus from the almost split sequences and morphisms for a moment and instead introduce a closely related notion of irreducible morphisms:

Definition 4.11. A morphism $f : M \to N$ is irreducible if:

(1) f is neither a section nor a retraction
(2) if $f = f_1 \circ f_2$ for some $f_2 \in \mathrm{Hom}(M, X), f_1 \in \mathrm{Hom}(X, N)$, then either f_1 is a retraction or f_2 is a section.

Proposition 4.12. *Given a morphism $f : X \to Y$, if f is irreducible, then it is either a monomorphism or an epimorphism.*

Proof. Assume that f is not an epimorphism. Consider the canonical factorization of f through its image,

$$X \xrightarrow{f'} \mathrm{Im}(f) \xrightarrow{\iota} Y$$

Since f is not an epimorphism, neither is ι, and thus ι can't be a retraction. Thus, since f is irreducible, f' is a section, and thus f' is mono and so is f.
Now assume that f is not a monomorphism. We use the first isomorphism theorem to reverse the order of the universal properties we used in the first argument and obtain the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\
& \mathrm{Coker}(\iota) \arrow[ur, "\exists! g"] &
\end{tikzcd}$$

Now since f is not mono, neither is π, so π is not a section and thus g is a retraction, so g, and hence also f, is an epimorphism. □

In order to prove the first basic fact about irreducible morphisms, we first show some statements about pullbacks and pushouts.

Lemma 4.13.

(1) *Given a pullback square*

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "u"] & B \arrow[d, "v"] \\
C \arrow[r, "w"] & D
\end{tikzcd}$$

in mod-A, *if v is an monomorphism, then so is u. Similarly, if v is a epimorphism, then so is u.*
(2) *Dually, the same statements hold for pushout squares.*

Proof. First we show the statement for monomorphisms:
Let $X \in$ mod-A, $g, g' \in \mathrm{Hom}(X, A)$ such that $u \circ g = u \circ g'$. We will show that then $g = g'$, and thus u is mono. We have $u \circ g = u \circ g'$, and thus $w \circ u \circ g = w \circ u \circ g'$, and by commutativity of the square also $v \circ t \circ g = v \circ t \circ g'$, and thus, since v is a monomorphism, $t \circ g = t \circ g'$. But then both g and g' satisfy the properties of the pullback morphism for $t \circ g \in \mathrm{Hom}(X, B), u \circ g \in \mathrm{Hom}(X, C)$, but such morphism is unique, and thus $g = g'$.
Surprisingly, the analogous statement for epimorphisms is not as natural, and in the opposite to that for monomorphisms, it does not hold for general abelian categories. We prove the statement as follows:
Note that under our equivalence of categories established in proposition 4.3, mod-A has pullbacks, and these

pullbacks look just like those for K-modules (since all we do is "gathering" the pointwise K-module pullbacks of the quiver into one KQ-module), that is, given

$$\begin{tikzcd} B \arrow[dr, "v"'] & & C \arrow[dl, "w"] \\ & D & \end{tikzcd}$$

in mod-A, its pullback is defined as $\{(x, y) \in B \oplus C \mid v(x) = w(y)\}$. Now we have $\forall c \in C : \exists b \in B : w(c) = v(b)$ by choosing $b \in v^{-1}(w(c))$, which is non-empty since v is epi. Thus

$$u(P) = \{c \in C \mid \exists b \in B : w(c) = v(b)\} = C$$

and hence u is epi.
The proof for pushouts squares is dual, and dually to pullbacks, it is epimorphisms that are stable under pushouts in more general settings, whereas monomorphisms require the underlying category to have additional properties. □

Proposition 4.14. *Given the diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "r"] \arrow[d, "1_L"] & V' \arrow[r, "v'"] \arrow[d, "g'"] & V \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

V' is isomorphic to the pullback of $V \xrightarrow{v} N \xleftarrow{g} M$.
Dually, given the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "f'"] & N \arrow[r] \arrow[d, "1_N"] & 0 \\ 0 \arrow[r] & U \arrow[r, "u'"] & U' \arrow[r, "t"] & N \arrow[r] & 0 \end{tikzcd}$$

U' is isomorphic to the pushout of $U \xleftarrow{u} L \xrightarrow{f} M$.

In order to prove that result, we first state and show a lemma:

Lemma 4.15 (Theorem II.6.2 in [4])**.** *Let*

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

be a pullback square in an abelian category $\mathcal{C}$. Let J be the kernel of β and let μ be its canonical inclusion morphism into P. Then $\mathrm{Ker}(\varphi) = J$ and the canonical inclusion of $\mathrm{Ker}(\varphi)$ is equal to $\alpha \circ \mu$

Proof. First, $\alpha \circ \mu$ is mono, since the inclusion $\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] : P \to A \oplus B$ is mono and so is μ, hence $\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] \circ \mu = \left[\begin{smallmatrix}\alpha\circ\mu \\ 0\end{smallmatrix}\right]$ is mono and thus $\alpha \circ \mu$ is mono. Moreover, $\varphi \circ \alpha \circ \mu = \psi \circ \beta \circ \mu = 0$. Now we only need to show that $(J, \alpha \circ \mu)$ satisfies the universal property of kernels. So let $Z \in \mathcal{C}$, and $\tau \in \mathrm{Hom}(Z, A)$ be a morphism such that $\varphi \circ \tau = 0$. Then we have $\varphi \circ \tau = \psi \circ 0 = 0$ and that commutative square induces a unique morphism $\sigma : Z \to P$ by the pullback property of P such that $\alpha \circ \sigma = \tau$ and $\beta \circ \sigma = 0$. By the kernel property of μ, the second equality induces a unique morphism γ such that $\sigma = \mu \circ \gamma$. Thus we have $\tau = \alpha\mu\gamma$, and γ is the unique morphism satisfying that equality. □

Now we prove the preceding proposition.

Proof of proposition 4.14. Let

$$\begin{tikzcd} P \arrow[r, "\epsilon"] \arrow[d, "\varphi"] & V \arrow[d, "v"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

be the pullback square of $V \xrightarrow{v} N \xleftarrow{g} M$. By lemma 4.15 we know that φ induces an isomorphism $\mathrm{Ker}(\epsilon) \simeq \mathrm{Ker}(g) \simeq L$. We thus have the following short exact sequence: $0 \to L \xrightarrow{\iota} P \xrightarrow{\epsilon} V \to 0$, where ι is the kernel morphism of L as the kernel of ϵ. By pullback property of P there is a morphism $\omega \in \mathrm{Hom}(V', P)$ such that $\varphi \circ \omega = g'$ and $v' = \epsilon \circ \omega$. Moreover, since r is the kernel morphism of v', we have $\epsilon \circ \omega \circ r = v' \circ r = 0$, so $\mathrm{Im}(\omega \circ r) \subset \mathrm{Ker}(\epsilon)$. We also have $\varphi \circ \omega \circ r = g' \circ r = f = \varphi \circ \iota$. Since φ induces an isomorphism $\mathrm{Ker}(\epsilon) \to \mathrm{Ker}(g)$, we can treat φ like a monomorphism and thus conclude that $\omega \circ r = \iota$. We then have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "r"] \arrow[d, "id_L"] & V' \arrow[r, "v'"] \arrow[d, "\omega"] & V \arrow[r] \arrow[d, "id_V"] & 0 \\
0 \arrow[r] & L \arrow[r, "\iota"] & P \arrow[r, "\epsilon"] & V \arrow[r] & 0
\end{tikzcd}$$

and since both id_L and id_V are isomorphisms, then, by five lemma, so is ω.
Proof of the dual statement for pushouts is dual. □

Proposition 4.16 (IV.1.7, [1]). *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a nonsplit short exact sequence in* mod-A. *Then*

(1) *f is irreducible if and only if, for every module V, and every morphism $v \in \mathrm{Hom}(V, N)$, there exists $v_1 \in \mathrm{Hom}(V, M)$ such that $v = g \circ v_1$ or there exists $v_2 \in \mathrm{Hom}(M, V)$ such that $g = v \circ v_2$.*

(2) *g is irreducible if and only if, for every module U, and every morphism $u \in \mathrm{Hom}(L, U)$, there exists $u_1 \in \mathrm{Hom}(M, U)$ such that $u = g \circ u_1$ or there exists $u_2 \in \mathrm{Hom}(U, M)$ such that $g = u \circ u_2$.*

Proof. (1) " $\implies$ ": Diagrammatically we can represent our premises as

$$\begin{tikzcd}
 & & & V \arrow[d, "v"] & \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

Let P be the pullback of

$$\begin{tikzcd}
 & V \arrow[d, "v"] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

and let $g' \in \mathrm{Hom}(P, V), u \in \mathrm{Hom}(P, M)$ denote the morphisms completing the pullback square. Since g is epi, by lemma 4.13 we find that g' is epi. There is also a morphism $f' \in \mathrm{Hom}(L, P)$ induced by pullback property of P applied on square

$$\begin{tikzcd}
L \arrow[r, "0"] \arrow[d, "f"] & V \arrow[d, "v"] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

f' is mono as we have $u \circ f' = f$ and f is mono. Thus the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f'"] \arrow[d, "id_l"] & P \arrow[r, "g'"] \arrow[d, "u"] & V \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

Moreover, its rows are exact. By definition of f' we know $g' \circ f' = 0$, so $\mathrm{Im}(f') \subset \mathrm{Ker}(g')$. Now let $x \in \mathrm{Ker}(g')$. Then $(g \circ u)(x) = 0$, and by exactness of the lower row we have $u(x) \in \mathrm{Im}(f)$, and as f is mono, there is a unique $l \in L : f(l) = u(x)$, and so

$$(u \circ f')(l) = f(l) = u(x), \text{ hence } u(f'(l)) = u(x)$$

and as f' is mono, we know that $f'(l) = x$. So $\mathrm{Ker}(g') \subset \mathrm{Im}(f')$ and thus the lower row is exact.
Now since f is irreducible, either f' is a section or u is a retraction. In the first case we know by

proposition 2.27 that g' is a retraction and our desired morphism is $u \circ \bar{g'}$ ($\bar{g'}$ denoting the retraction morphism of g'), and in the second case we have $g' \circ \bar{u}$ ($\bar{u}$ denoting the retraction morphism of u.)

" $\Longleftarrow$ ": Since the given sequence is not split, f is not a retraction nor a section (thus satisfying one of the two defining properties of an irreducible morphism). Now suppose that there is a module X and morphisms $f_2 \in \mathrm{Hom}(L, X), f_1 \in \mathrm{Hom}(X, M)$ such that $f = f_1 \circ f_2$. Because f is mono, so is f_2 and we obtain the following commutative diagram with exact rows (note: rightmost vertical morphism is obtained by the universal property of cokernels):

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f_2"] \arrow[d, "id_L"] & X \arrow[r, "\pi"] \arrow[d, "f_1"] & \mathrm{Coker}(f_2) \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

Now we can show that f satisfies the second property defining an irreducible morphism, that is, either f_2 is a section, or f_1 is a retraction. For if there is $v_1 \in \mathrm{Hom}(\mathrm{Coker}(f_2), M)$ such that $v = g \circ v_1$, then $v \circ id_{\mathrm{Coker}(f_2} = g \circ v_1$ is a commutative square containing $M \to N \leftarrow \mathrm{Coker}(f_2)$, and thus by pullback property of X we have $\gamma \in \mathrm{Hom}(\mathrm{Coker}(f_2), X)$ such that $\pi \circ \gamma = id_{\mathrm{Coker}(f_2)}$. In other words π is a retraction, and thus f_2 is a section. Else if there is $v_2 \in \mathrm{Hom}(M, \mathrm{Coker}(f_2))$ such that $g = v \circ v_2$, then $v \circ v_2 = g \circ id_M$ is the commutative square that by pullback property of X induces a morphism $\omega \in \mathrm{Hom}(M, X)$ such that $id_M = f_1 \circ \omega$ and we obtain our result immediately as then f_1 is a retraction.

(2) Analogous to 1)

$\square$

This leads us to first connection between the irreducible morphisms and the almost split sequences:

Proposition 4.17.

(1) *If $f : L \to M$ is an irreducible monomorphism, then* $\mathrm{Coker}(f)$ *is indecomposable.*
(2) *If $g : M \to N$ is irreducible epimorphism, then* $\mathrm{Ker}(g)$ *is indecomposable.*

Proof. We aim to use the preceding lemma. Assume that $\mathrm{Coker}(f)$ is not indecomposable, $\mathrm{Coker}(f) = N_1 \oplus N_2$, $N_1, N_2 \neq 0$. We then have the following nonsplit (for if it were, f would be a section, contradicting its irreducibility) short exact sequence:

$$0 \to L \xrightarrow{f} M \xrightarrow{\rho} \mathrm{Coker}(f) \to 0$$

and if we add the decomposition of $\mathrm{Coker}(f)$ to the picture, we obtain the following diagram:

$$\begin{tikzcd}
 & & & N_1 \arrow[d, "\iota_1"] & \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "\rho"] & \mathrm{Coker}(f) \arrow[r] & 0 \\
 & & & N_2 \arrow[u, "\iota_2"'] &
\end{tikzcd}$$

Now we can view that diagram as union of two diagrams as the one in proposition 4.16, and thus for each $i \in \{1, 2\}$ we either have a morphism $h_i \in \mathrm{Hom}(M, N_i)$ such that $\iota_i \circ h_i = \rho$ or a morphism $g_i \in \mathrm{Hom}(N_i, M)$ such that $\iota_i = \rho \circ g_i$. But in both cases the first alternative yields a contradiction, as $\iota_i \circ h_i = \rho$ implies that ι_i is epi. But ι_i is also mono, and thus is an isomorphism, which contradicts the assumption that $N_i \neq 0$. Thus we have g_1, g_2 as described above. But then by coproduct property of $\mathrm{Coker}(f)$ we know that there is a morphism $[g_1, g_2] : \mathrm{Coker}(f) \to M$ and then $\rho \circ [g_1, g_2] = id_{\mathrm{Coker}(f)}$, so ρ is a retraction and thus f is a section, contradicting its irreducibility. Thus $\mathrm{Coker}(f)$ is indecomposable. 2) can be proven by dualizing the proof above. $\square$

The following proposition establishes a further connection between the two central notions of this chapter.

Proposition 4.18.

(1) *Let $f : L \to M$ be left minimal almost split. Then f is irreducible.*
(2) *Let $g : M \to N$ be right minimal almost split. Then g is irreducible.*

Proof. By definition f is not a section. By proposition 4.8, L is indecomposable. We also know that f is not an isomorphism, as then f would be both a section and a retraction. Assume f is a retraction. Then the following short exact sequence splits: $0 \to \mathrm{Ker}(f) \xrightarrow{\iota} L \xrightarrow{f} M \to 0$, and thus $L = M \oplus \mathrm{Ker}(f)$, with $M, \mathrm{Ker}(f) \neq 0$ (if $\mathrm{Ker}(f)$ was 0, f would be both epi and mono and thus iso). That contradicts L being indecomposable. So f is not a retraction. Now we move on to the second property: assume $f = f_1 \circ f_2$ for some $X \in \text{mod-}A, f_1 \in \mathrm{Hom}(X, M), f_2 \in \mathrm{Hom}(L, X)$. Assume f_2 is not a section (if it is, the proof ends here). Then as f is left almost split, there is $g \in \mathrm{Hom}(M, X)$ such that $g \circ f = f_2$, so $f = f_1 \circ f_2 = (f_1 \circ g) \circ f$, and now by minimality of f we know that $f_1 \circ g$ must be an automorphism. Call it h. Then $f_1 \circ (g \circ h^{-1}) = (f_1 \circ g) \circ h^{-1} = h \circ h^{-1} = id_M$, so f_1 is a retraction. And thus either f_2 is a section or f_1 is a retraction, and so f is irreducible.
Proof of (2) is similar. □

Moreover, the following weak converse of proposition 4.18 holds:

Proposition 4.19.
(1) *Let $f \in \mathrm{Hom}(L, M)$ be left minimal almost split. Then, given $f' : L \to M'$, f' is irreducible if and only if $M' \neq 0$, and there is a module M'' such that $M \simeq M' \oplus M''$ together with morphism $f'' : L \to M''$ such that $\begin{bmatrix} f' \\ f'' \end{bmatrix}$ is left minimal almost split.*
(2) *Let $g \in \mathrm{Hom}(M, N)$ be right minimal almost split. Then, given $g' : M' \to N$, g' is irreducible if and only if $M' \neq 0$, and there is a module M'' such that $M \simeq M' \oplus M''$ together with morphism $g'' : M'' \to N$ such that $\begin{bmatrix} g' & g'' \end{bmatrix}$ is right minimal almost split.*

Proof. See IV.1.10 in [1]. □

Proposition 4.20. *Let*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "v"] & N \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact non-split rows.

(1) *If L is indecomposable and w is an automorphism, then u and hence v are automorphisms.*
(2) *If N is indecomposable and u is an automorphism, then w and hence v are automorphisms.*

Theorem 4.21 (IV.1.13, [1])**.** *For a short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in mod-A, the following statements are equivalent:*

(1) *The given sequence is almost split.*
(2) *L is indecomposable, and g is right almost split.*
(3) *N is indecomposable, and f is left almost split.*
(4) *The homomorphism f is left minimal almost split.*
(5) *The homomorphism g is right minimal almost split.*
(6) *L and N are indecomposable, and f and g are irreducible.*

Proof. To outline the proof's structure, we start by listing the implications that will be proved in order to establish the equivalence (in the order in which they appear in the proof):
(1) implies (4); (1) implies (5); (1) implies (2); (1) implies (3); (1) implies (6); (5) implies (2); (4) implies (3); (2) implies (3); (3) implies (2); (2) and (3) imply (1); and finally (6) implies (2).
First two implications hold trivially by the definition of almost split sequences. Second two implications are an immediate consequence of proposition 4.8, and (1) implies (6) as an immediate consequence of propositions 4.8 and 4.18.
(5) implies (2):
If g is right minimal almost split, then g is irreducible by proposition 4.18. Now since the sequence is exact,

we have $L \simeq \mathrm{Ker}(g)$ and we also know that g is an epimorphism and thus by proposition 4.17 we know that L is indecomposable, and the implication follows.
(4) implies (3): Dual to the preceding implication.
(2) implies (3): We need to show that N is indecomposable, f is not a section, and given module U and $u \in \mathrm{Hom}(L, U)$ such that u is not a section, there is some $u' \in \mathrm{Hom}(M, U)$ such that $u = u' \circ f$. First two parts of the statement are obtained immediately: N is indecomposable by proposition 4.8, and f is not a section as g is right minimal almost split and thus is not a retraction. We prove the third part by contraposition. Thus, given $u \in Hom(L, U)$ such that there is no $u' \in \mathrm{Hom}(M, U)$ such that $u = u' \circ f$, we will show that u must be a section. Just as we did when proving proposition 4.17, we use the stability of epi- and monomorphisms under pullbacks and pushouts (lemma 4.13). Let V be the pushout of the triangle of U, L, and M. We then have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "v"] & N \arrow[r] \arrow[d, "id_N"] & 0 \\
0 \arrow[r] & U \arrow[r, "h"] & V \arrow[r, "k"] & N \arrow[r] & 0
\end{tikzcd}$$

Note that both the rows are non-split: first row is not split as g is right almost split so g is not a retraction, and second row is not split, for if h was a section with its section morphism $\overline{h}$, then $\overline{h} \circ v$ contradicts our assumption about u. So in particular, k is not a retraction. Now since g is right almost split, there is a morphism $\bar{v} \in \mathrm{Hom}(V, M)$ such that $k = g \circ \bar{v}$. We thus have $g \circ \bar{v} \circ h = k \circ h = 0$ and thus by the kernel property of L, there is a unique morphism $\bar{u}$ such that $f \circ \bar{u} = \bar{v} \circ h$. Therefore the following commutative diagram with exact rows exists:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\bar{u}\circ u"] & M \arrow[r, "g"] \arrow[d, "\bar{v}\circ v"] & N \arrow[r] \arrow[d, "id_N"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

and since id_N is an automorphism, by proposition 4.20 we know that so is $\bar{u} \circ u$, hence u is a section.
Dually, (3) implies (2).
(2) and (3) imply (1): All that needs to be shown is minimality of both f and g. Take $h \in \mathrm{End}(M)$ such that $h \circ f = f$. We want to show that h is an automorphism. We arrive at that conclusion immediately by applying proposition 4.20 on the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "id_L"] & M \arrow[r, "g"] \arrow[d, "h"] & N \arrow[r] \arrow[d, "id_N"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

From the same diagram, by analogous reasoning, we obtain the same result for g. Thus both f and g are minimal.
(6) implies (2): L is indecomposable by assumption, and thus we only need to prove the second part of (2), namely that g is right almost split. Since g is irreducible, it is not a retraction. All that remains to be proved is that given $v : V \to N$, where v is not a retraction, there exists $v' : V \to M$ such that $g \circ v' = v$. We thus let v be as above, and assume that V is indecomposable. We may do that, for if it is not such, we perform the procedure below on each of the direct summands of V and "glue together" the obtained morphisms by the product property of V to obtain the desired morphism. Since f is irreducible, by proposition 4.16, either there is a morphism $v' : V \to M$ such that $v = g \circ v'$ and we're done, or there is a morphism $h : M \to V$ such that $g = v \circ h$. But then since g is irreducible, either v is a retraction or h is a section, and by assumption v is not a retraction. Thus h is a section, and as such it is also a monomorphism. But then we have the following split short exact sequence: $0 \to M \xrightarrow{h} V \xrightarrow{\pi} \mathrm{Coker}(h) \to 0$ and so $V \simeq \mathrm{Coker}(h) \oplus M$. But V is indecomposable, and thus $\mathrm{Coker}(h) = 0$ and so h is epi. So h is both mono and epi, and thus is an isomorphism. Then we have $g \circ h^{-1} = v$ and thus h^{-1} is the desired morphism, and we're done. $\square$

5. Auslander-Reiten Translation Revisited

In this chapter we will continue establishing interesting properties of the Auslander-Reiten translation defined in the end of chapter 2. We will also continue using the language of finite dimensional algebras rather than that of quiver representations, but by now it should be clear that all results here have a direct correspondent in the realm of quiver representations.
Throughout this chapter we will assume that A is a path algebra of a finite acyclic quiver. The reason is that, as we have shown earlier, all the representations of such quiver, and thus all modules over such an algebra, have a projective resolution and an injective resolution both of length one. The corresponding theory for algebras and quivers not having that property is a bit more advanced, although similar.
First thing we will do is reformulating our definition of Auslander-Reiten translation by changing the order of the steps we took. Recall that our definition of τM for $M \in \mathrm{Rep}_K Q$ included four steps:

(1) Take a minimal projective resolution of M:

$$0 \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

(2) Apply the left exact contravariant functor $\mathrm{Hom}(-, A)$ (to simplify the notation we will from now on refer to that functor as $(-)^t$, so it sends a representation M to M^t and a morphism f to f^t) on said resolution to obtain:

$$0 \to M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \xrightarrow{\pi} \mathrm{Coker}(p_1^t) \to 0$$

(3) Apply the contravariant exact functor D on the sequence resulting from the preceding step and obtain

$$0 \to \mathrm{Ker}(\nu p_1) \xrightarrow{\iota} \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \to 0$$

(4) Obtain the Auslander-Reiten translate as $\tau M := \mathrm{Ker}(\nu p_1)$

But it should be clear that this process commutes with that obtained by switching steps (3) and (4), that is, we first "choose" $\mathrm{Coker}(p_1^t)$. We call it the transpose of M, denoting it by $\mathrm{Tr}\, M$. Now apply D to obtain $\tau M = D(\mathrm{Tr}\, M)$. In accordance with our earlier definition of the Nakayama functor $\nu = D\,\mathrm{Hom}(-, A)$, for which we've defined an inverse $\nu^{-1} = \mathrm{Hom}_{A^{op}}(-, A^{op})D = \mathrm{Hom}(DA, -)$, we define τ^{-1} as $\tau^{-1} = \mathrm{Tr}\, D$
The benefit of that change of perspective is that now it will be easier to describe properties of τ using properties of Tr and the fact that D is an exact functor (moreover, an equivalence).

Proposition 5.1 (IV.2.1, [1])**.** *Let M be an indecomposable A-module.*

(1) *The left A-module $\mathrm{Tr}\, M$ has no nonzero projective direct summands.*
(2) *If M is not projective, then the sequence $0 \to M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Tr}\, M \to 0$ obtained by applying $\mathrm{Hom}(-, A)$ on the minimal projective resolution of M is a minimal projective resolution of $\mathrm{Tr}\, M$.*
(3) *M is projective if and only if $\mathrm{Tr}\, M$ is zero. If M is not projective, then $\mathrm{Tr}\, M$ is indecomposable and $\mathrm{Tr}^2 M \simeq M$*
(4) *If M and N are indecomposable and not projective, then $M \simeq N$ if and only if $\mathrm{Tr}\, M \simeq \mathrm{Tr}\, N$*

Proof. First we show that $\mathrm{Tr}\, M = 0$ if and only if M is projective. If M is projective, then clearly $P_1 = 0$ and $P_1^t = 0$, and thus p_1^t is epi, and so $\mathrm{Tr}\, M = \mathrm{Coker}(p_1^t) = 0$. And if $\mathrm{Tr}\, M = 0$, then p_1^t is epi, and the sequence splits (as P_1 is projective), so p_1^t is a retraction. Now since $\mathrm{Hom}(-, A)$ is a contravariant equivalence when restricted to projective modules, we know that p_1 is a section, and thus the sequence splits and so M is isomorphic to a direct summand of P_0 and thus is projective.
Now we prove (2): since M is projective indecomposable, we know that $\mathrm{Hom}(M, A)$ is zero (by remark 3.35). Thus the first two objects of our sequence are 0, hence we can ignore the first one and obtain

$$0 \to P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Tr}\, M \to 0$$

This clearly is a projective resolution of $\mathrm{Tr}\, M$. Now we show that it is minimal. Assume it is not. Then (as a corollary of theorem I.5.8 in [1]) we have $P_1^t \simeq E_1' \oplus E_1''$ and $P_0^t \simeq E_0' \oplus E_0''$ with an isomorphism $v : E_0'' \xrightarrow{\sim} E_1''$ such that $p_1^t = \left[\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right]$. Note that as a consequence of (1) and proposition 3.35 we have $(\mathrm{Tr}\, M)^t = 0$. Now we apply $\operatorname{Hom}(-, A)$ to the sequence above, and use the fact that it is an equivalence when restricted to projective modules to obtain

$$0 \to (\operatorname{Tr} M)^t \to {E_1'}^t \oplus {E_1''}^t \xrightarrow{p_1^{tt}} {E_0'}^t \oplus {E_0''}^t \to \operatorname{Coker}(p_1)$$

which is clearly equal (ignoring the superfluous zero term in the beginning) to

$$0 \to {E_1'}^t \oplus {E_1''}^t \xrightarrow{p_1} {E_0'}^t \oplus {E_0''}^t \to \operatorname{Coker}(p_1)$$

and that in turn is isomorphic to

$$0 \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M$$

But then it is clear that this resolution is not minimal, as we can eliminate ${E_1''}^t$ and ${E_0''}^t$ from their respective sums, since they together with the isomorphism v^t between them have no contribution to the cokernel of the morphism, since

$$\operatorname{Coker}\left(\left[\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right]^t\right) = \operatorname{Coker}\left(\left[\begin{smallmatrix} u^t & 0 \\ 0 & v^t \end{smallmatrix}\right]\right) = {E_1'}^t \oplus {E_1''}^t / \operatorname{Im}\left(\left[\begin{smallmatrix} u^t & 0 \\ 0 & v^t \end{smallmatrix}\right]\right) = ({E_1'}^t / \operatorname{Im}(u^t)) \oplus ({E_1''}^t / \operatorname{Im}(v^t) \simeq {E_1'}^t / \operatorname{Im}(u^t)$$

as v^t is an isomorphism. Thus we conclude that our initial minimal resolution is not minimal, which is obviously a contradiction.
Now to prove (3) we use (2) by observing that there is a commutative diagram with exact rows and isomorphisms as vertical columns:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "p_1"] \arrow[d, "\varphi"] & P_0 \arrow[r, "p_0"] \arrow[d, "\psi"] & M \arrow[r] & 0 \\
0 \arrow[r] & P_1^{tt} \arrow[r, "p_1^{tt}"] & P_0^{tt} \arrow[r, "p_0^{tt}"] & \operatorname{Tr}^2 M \arrow[r] & 0
\end{tikzcd}$$

And we can now define an isomorphism γ using a standard diagram chasing argument: given $m \in M$, let $p, p' \in p_0^{-1}(m)$. We want to define $\gamma(m)$ as $p_0^{tt} \circ \psi(p)$ but in order to have that well-defined, we need to show $p_0^{tt} \circ \psi(p) = p_0^{tt} \circ \psi(p')$. Note that $p - p' \in \operatorname{Ker}(p_0)$ and thus $p - p' \in \operatorname{Im}(p_1)$, and since p_1 is mono, there is a unique $q \in p_1^{-1}(p - p')$. By exactness and commutativity of the diagram we have

$$p_0^{tt} \circ p_1^{tt} \circ \varphi(q) = 0 = p_0^{tt} \circ \psi \circ p_1(q) = p_0^{tt} \circ \psi(p - p')$$

and the result follows. Now since the diagram is symmetric, we can perform the same procedure to obtain the inverse of γ, and thus γ is an isomorphism.
Now we show that $\operatorname{Tr} M$ is indecomposable. We achieve that by showing that if M is decomposable, then so is $\operatorname{Tr} M$. Thus let $M = M' \oplus M''$. By horseshoe lemma we obtain the minimal projective resolution of M by taking the 'direct sum' (letting the modules in resolution being the direct sums of those in corresponding resolutions and letting the morphisms be diagonal) of the minimal projective resolutions of M' and M''. Applying $\operatorname{Hom}(-, A)$ on that resolution preserves the "direct sum" structure of the resolution described above, since Hom commutes with direct sums. But now since $p_1^t : (P_0' \oplus P_0'')^t \to (P_1' \oplus P_1'')^t$ is diagonal, we know that its cokernel can be decomposed into a direct sum of the cokernels of its "diagonal entries". Thus $\operatorname{Tr} M$ is indecomposable. Now we apply the statement we've proven to our situation. We know that M is indecomposable. But $M \simeq \operatorname{Tr}^2 M$, and so $\operatorname{Tr}^2 M = \operatorname{Tr}(\operatorname{Tr} M)$ is indecomposable. And thus $\operatorname{Tr} M$ too is indecomposable.
Finally we move on to (4). Clearly $M \simeq N$ implies $\operatorname{Tr} M \simeq \operatorname{Tr} N$. But then also $\operatorname{Tr} M \simeq \operatorname{Tr} N$ implies $\operatorname{Tr}(\operatorname{Tr} M) \simeq \operatorname{Tr}(\operatorname{Tr} N)$ and thus $M \simeq N$ □

As mentioned earlier, the statements about Tr readily yield properties of τ and τ^{-1}:

Corollary 5.2. *Let M, N be indecomposable A-modules.*

(1) *The module τM is zero if and only if M is projective.*
(2) *If M is a nonprojective module, then τM is indecomposable noninjective and $\tau^{-1}\tau M \simeq M$.*
(3) *If M and N are nonprojective, then $M \simeq N$ if and only if there is an isomorphism $\tau M \simeq \tau N$.*

Dual versions of each of those three statements hold for τ^{-1} (those are obtained by swapping "injective" with "projective" and τ with τ^{-1}), and are also proven by dualizing the proofs below.

Proof.

(1) Since D is an equivalence, $D(M) = 0$ implies $M = 0$. And since $\tau M = D \operatorname{Tr} M$, we know that τM is zero if and only if $\operatorname{Tr} M$ is zero, which it is if and only if M is projective by the preceding proposition.
(2) D is an equivalence, so τM is indecomposable if and only if $\operatorname{Tr} M$ is such. But we know that if M is indecomposable, then so is $\operatorname{Tr} M$. Moreover, $\tau M = D \operatorname{Tr} M$ is noninjective as by proposition 3.31 we know that $D \operatorname{Tr} M$ is injective if and only if $\operatorname{Tr} M$ is projective, which it can not be, as then $\operatorname{Tr}^2 M$ would be zero, and by proposition 5.1 we know that $\operatorname{Tr}^2 M \simeq M$. Finally we have $\tau^{-1}\tau M = (\operatorname{Tr} D)(D \operatorname{Tr})M = \sigma D^2 \operatorname{Tr} M = \operatorname{Tr}^2 M \simeq M$, as D is a duality.
(3) Clearly $M \simeq N$ implies $\tau M \simeq \tau N$. Conversely, $\tau M \simeq \tau N$ is by definition equivalent to $D \operatorname{Tr} M \simeq D \operatorname{Tr} N$, and since D is an equivalence, this implies that $\operatorname{Tr} M \simeq \operatorname{Tr} N$ and thus $M \simeq N$ by proposition 5.1.

□

Proposition 5.3. *Let M be a module over A.*
Then $\nu^{-1}A = \operatorname{Hom}(DA, \tau M) = 0$ and dually $\nu A = \operatorname{Hom}(\tau^{-1}M, A) = 0$.

Proof. We only prove the first part of the statement. Note that just as ν is right exact, so ν^{-1} is left exact. So given an exact sequence

$$0 \to \operatorname{Ker}(f) \xrightarrow{\iota} L \xrightarrow{f} M$$

in mod-A we have that

$$0 \to \nu^{-1} \operatorname{Ker}(f) \xrightarrow{\nu^{-1}\iota} \nu^{-1} L \xrightarrow{\nu^{-1} f} \nu^{-1} M$$

is exact, and thus $\nu^{-1} \operatorname{Ker}(f) \simeq \operatorname{Ker}(\nu^{-1} f)$. We thus say that left exact functors commute with kernels. In our case we take the minimal projective resolution

$$0 \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

of M, apply ν^{-1} to it and by definition have $\tau M = \operatorname{Ker}(\nu p_1)$ so

$$\nu^{-1}(\tau M) = \nu^{-1}(\tau M) = \nu^{-1} \operatorname{Ker}(\nu p_1) = \operatorname{Ker}(\nu^{-1}\nu p_1) = \operatorname{Ker}(p_1)$$

(as $\nu^{-1}\nu = id$ on the full subcategory of projective modules), which is equal to zero. □

We have now seen that τ can yield new indecomposable modules from old ones, and we have introduced the important notion of irreducible morphisms. We finish this chapter with a presentation of some results that will shed some light on the role of τ in the structure of the category mod-A, also involving the irreducible morphisms.
Regrettably, some of the results, most notably theorems 5.4 and 5.6 (we hope that their fundamental importance for the theory presented in the remaining part of the text still is conveyed in the text), are presented without proof. These are provided fully (and also for a more general case than that treated here) in [1].

Theorem 5.4 (Auslander-Reiten Formulas, IV.2.14 in [1])**.** *Let M and N be modules over A. There exists K-linear isomorphisms:*

$$\operatorname{Ext}^1_A(M, N) \simeq D \operatorname{Hom}_A(N, \tau M)$$
$$\operatorname{Ext}^1_A(M, N) \simeq D \operatorname{Hom}_A(\tau^{-1} N, M)$$

And both the isomorphisms are functorial in both variables.

Corollary 5.5. *There also exist following K-linear isomorphisms:*

(1) *For all modules M, N we have:*

$$\operatorname{Hom}_A(N, \tau M) \simeq \operatorname{Hom}_A(\tau^{-1} N, M)$$

(2) *If N is indecomposable nonprojective, we have:*

$$\operatorname{Hom}_A(\tau N, \tau M) \simeq \operatorname{Hom}_A(N, M)$$

(3) *If M is indecomposable noninjective, we have:*

$$\operatorname{Hom}_A(\tau^{-1} N, \tau^{-1} M) \simeq \operatorname{Hom}_A(N, M)$$

Proof. (1) is a direct consequence of the preceding theorem, since D is an equivalence.
To prove (2), recall that if N is indecomposable nonprojective, we have $\tau^{-1}\tau N \simeq N$. Thus $\mathrm{Hom}(N, M) \simeq \mathrm{Hom}(\tau^{-1}(\tau N), M)$, and this in turn, by (1) is isomorphic to $\mathrm{Hom}(\tau N, \tau M)$. Similarly, if M is indecomposable noninjective, then $\tau\tau^{-1}M \simeq M$ and $\mathrm{Hom}(N, M) \simeq \mathrm{Hom}(N, \tau(\tau^{-1}M))$ and again by (1) that is isomorphic to $\mathrm{Hom}(\tau^{-1}N, \tau^{-1}M)$ □

Theorem 5.6 (IV.3.1 in [1])**.**
(1) *For any indecomposable non-projective module M, there exists an almost split sequence*

$$0 \to \tau M \to E \to M \to 0$$

(2) *For any indecomposable non-injective module N, there exists an almost split sequence*

$$0 \to N \to F \to \tau^{-1}N \to 0$$

To present some of the results below, we need to introduce two more concepts:

Definition 5.7. Given a module M, we define its radical $\mathrm{Rad}\, M$ as the intersection of all the maximal submodules of M.

Definition 5.8. Given a module M, we define its socle $\mathrm{Soc}\, M$ as the submodule of M generated by all simple submodules of M.

A notion related to the radical of a module described above is that of radical of algebra, and we introduce it for the particular case of the algebra $\mathrm{Hom}(M, N)$ (where M and N are indecomposable A-modules, and the product operation is given by composition of morphisms.)

Definition 5.9. Given two indecomposable A-modules M and N, we define $\mathrm{Rad}(M, N)$ as the set of morphisms from M to N that are not isomorphisms.

Proposition 5.10. *If M and N are indecomposable, the set* $\mathrm{Rad}(M, N)$ *forms a vector space, which clearly then is a subspace of* $\mathrm{Hom}(M, N)$*. It is equal to* $\mathrm{Hom}(M, N)$ *if M is not isomorphic to N, and if M is isomorphic to N, then* $\mathrm{Rad}(M, N)$ *is the Jacobson radical of the endomorphism algebra* $\mathrm{End}(M)$ *(since we can, without loss of generality, assume equality $M = N$). This is the case as the endomorphism algebra of an indecomposable module is local and the Jacobson radical of a local algebra consists exactly of non-unit elements in the algebra. A more thorough exposition of the topic can be found in Chapter I of* [1].

There is an important characterization of irreducible morphisms. To formulate it, we need to give another definition:

Definition 5.11. Given two indecomposable modules M and N, we define $\mathrm{Rad}^2(M, N)$ as the subspace of $\mathrm{Rad}(M, N)$ formed by the set of morphisms

$$\{\sum_{i=1}^{n} g_i \circ f_i \mid f_i \in \mathrm{Rad}(M, X_i), g_i \in \mathrm{Rad}(X_i, N) \text{ for some indecomposable modules } X_i \text{ and some } n \in \mathbb{N}\}.$$

Now we state (unfortunately without proof) the aforementioned characterization of irreducible morphisms:

Proposition 5.12 (IV.1.6 in [1])**.** *Let M, N be indecomposable A-modules. A morphism $f : M \to N$ is irreducible if and only if it lies in* $\mathrm{Rad}(M, N) \setminus \mathrm{Rad}^2(M, N)$

This means that further study of irreducible morphisms will, in a sense, give us information about all non-isomorphisms (which will turn out to be of particular interest) that is complete up to the information about $\mathrm{Rad}^2(M, N)$. This motivates the following definition:

Definition 5.13. Let M and N be indecomposable A-modules. By $\mathrm{Irr}(M, N)$ we denote the quotient vector space $\mathrm{Rad}(M, N)/\mathrm{Rad}^2(M, N)$. We call it (in a bit misleading fashion) the space of irreducible morphisms from M to N.

We also state (also without proof) an important result about such spaces:

Proposition 5.14. *Let M be an indecomposable module, and let*

$$0 \to \tau M \to E \to M \to 0$$

be the almost-split sequence ending with M (which exists by 5.6, and moreover exists uniquely, due to 4.10). Then the dimensions of spaces $\operatorname{Irr}(\tau M, E)$ and $\operatorname{Irr}(E, M)$ are equal.
Dually, the same holds for the almost-split sequence beginning with M.

Proof. A direct consequence of proposition IV.4.1 in [1] □

Definition 5.15. The Auslander-Reiten quiver $\Gamma(\text{mod-}A)$ of the algebra A is defined as follows:

(1) The points of $\Gamma(\text{mod-}A)$ are the isomorphism classes of indecomposable A-modules.
(2) The arrows $M \to N$ represent basis vectors of space $\operatorname{Irr}(M, N)$ of irreducible morphisms from M to N. (which in particular can be identified with some set of irreducible morphisms such that the equivalence classes of those morphisms are a basis)

Now we present some basic properties of the Auslander-Reiten quiver of A.

Proposition 5.16. *The set of immediate predecessors of an indecomposable module M (or rather its iso-class) in the Auslander-Reiten quiver $\Gamma(\text{mod-}A)$ consists of*

(1) *The indecomposable direct summands of $\operatorname{Rad} M$ if M is projective*
(2) *The indecomposable direct summands of the middle term of the almost split exact sequence ending with M, if M is non-projective.*

Proposition 5.17. *The set of immediate successors of an indecomposable module M in the Auslander-Reiten quiver $\Gamma(\text{mod-}A)$ consists of*

(1) *The indecomposable direct summands of $M/\operatorname{Soc} M$ if M is injective*
(2) *The indecomposable direct summands of the middle term of the almost split exact sequence starting with M, if M is non-injective.*

The preceding two propositions are a consequence of combining multiple statements proven in chapters IV.2, IV.3 and IV.4 in [1]. Since $M, \operatorname{Rad} M$ and $\operatorname{Soc} M$ all are finite-dimensional, a consequence of those propositions is that $\Gamma(\text{mod-}A)$ is a locally finite quiver, that is, for any point in $\Gamma(\text{mod-}A)$, both the set of its direct predecessors and the set of its direct successors are finite, or, equivalently, the set of all its neighbors (the union of the two aforementioned sets) is finite.

Remark 5.18. *$\Gamma(\text{mod-}A)$ of A has no loops.*

Proof. A loop $X \to X$ in $\Gamma(\text{mod-}A)$ would correspond to an irreducible morphism $\varphi : X \to X$. By proposition 4.12 we know that φ must either be mono or be epi, but since φ is in particular an endomorphism of a finite dimensional vector space (the underlying space of the indecomposable module X), it is mono if and only if it is epi, hence it must be both, and hence also an isomorphism. But then φ is both a section and a retraction, both contradicting φ being irreducible.

□

Proposition 5.19. *If A is representation-finite (that is, there are only finitely many isoclasses of indecomposable A-modules), then $\Gamma(\text{mod-}A)$ has no multiple arrows.*

Theorem 5.20. *Let Γ' denote the full subquiver of $\Gamma(\text{mod-}A)$ corresponding to the full subcategory of projective modules, and let Γ'' denote the full subquiver of $\Gamma(\text{mod-}A)$ corresponding to the full subcategory of injective modules.*
The Auslander-Reiten translation τ is a translation of the Auslander-Reiten quiver $\Gamma(\text{mod-}A)$, i.e. the map $\tau : \Gamma(\text{mod-}A)_0 \setminus \Gamma'_0 \to \Gamma(\text{mod-}A)_0 \setminus \Gamma''_0$ is a bijection such that for any vertex $x \in \Gamma(\text{mod-}A)_0 \setminus \Gamma'_0$ there is also a bijection between the set of arrows having x as their target and the set of arrows having τx as their source.

Proof. By theorem 5.6, for any non-projective indecomposable module M there is an almost split sequence $0 \to \tau M \to E \to M \to 0$, and by remark 4.10 we know that that sequence is unique up to isomorphism. By corollary 5.2 we know that τM is indecomposable and non-injective. By existence of τ^{-1} together with

$\tau^{-1}\tau M \simeq M$ we know that τ is surjective onto $\Gamma(\text{mod-}A)_0 \setminus \Gamma_0''$ and since $\tau M \simeq \tau M'$ holds if and only if $M \simeq M$, we know that it is also injective. Thus τ is a bijection of said sets. The bijection of sets of arrows follows from proposition 5.14. □

Now we give an example of an Auslander-Reiten quiver:

Example 5.21. Let $Q =$

$$1 \to 2 \to 3$$

and let A be its path algebra. From earlier we know all the indecomposable projective, indecomposable injective and simple modules. We use the notation we've used earlier, so for example by $P(2)$ we mean the indecomposable projective representation at vertex 2, and we let it be represented by the following canonical representative:

$$0 \to K \xrightarrow{id_K} K$$

It is fairly clear that $\operatorname{Rad} P(1) = P(2), \operatorname{Rad} P(2) = P(3)$ and that $\operatorname{Soc} I(3) = P(3), \operatorname{Soc} I(2) = S(2)$, and hence also $I(3)/\operatorname{Soc} I(3) \simeq I(2), I(2)/\operatorname{Soc} I(2) \simeq I(1)$. Now, using propositions 5.4, 5.16, 5.17, we obtain a full subquiver of $\Gamma(\text{mod-}A)$:

$$\begin{tikzcd}
 & & P(1) \arrow[dr] & & \\
 & P(2) \arrow[ur] & & I(2) \arrow[dr] & \\
P(3) \arrow[ur] & & & & I(1)
\end{tikzcd}$$

And moreover, since the radicals and quotients described above are indecomposable, this quiver contains all predecessors of $P(3)$, $P(2)$, $P(1)$, and $I(1)$. Similarly, it also contains all successors of $P(3)$, $P(1)$ (which is equal to $I(3)$) and $I(2)$. As a consequence of proposition IV.3.11 in [1], the following sequence is almost split:

$$0 \to P(2) \to P(1) \oplus S(2) \to I(2)$$

Hence, by 5.17 $P(1)$ and $S(2)$ are the only successors of $P(2)$, and dually, by 5.16, these modules are the only predecessors of $I(2)$. Now we will show the following:

(1) $P(2)$ is the only predecessor of $S(2)$
(2) $I(2)$ is the only successor of $S(2)$

To prove (1), we note that the following sequence:

$$0 \to P(3) \xrightarrow{i} P(2) \to S(2) \to 0$$

(where i denotes the inclusion morphism) is the standard projective resolution of $S(2)$ given in 3.19, and moreover, since all the modules in the sequence are indecomposable, it is even the minimal projective resolution of $S(2)$, hence also our starting point in finding $\tau S(2)$. We start by applying $\operatorname{Hom}(-, A)$ and, using 3.35 and 3.36, we obtain

$$0 \to P_{Q^{op}}(2) \xrightarrow{\operatorname{Hom}(i,A)} P_{Q^{op}}(3) \to \operatorname{Coker}(\operatorname{Hom}(i, A))$$

where $\operatorname{Hom}(i, A)$ is mono, and hence $\operatorname{Coker}(\operatorname{Hom}(i, A)) \simeq I_{Q^{op}}(3)$. We now apply D on that cokernel to obtain (using 3.31) $\tau S(2) \simeq P(3)$. Now, by 5.6, there is an almost split sequence starting with $P(3)$ and ending with $S(2)$, which is also the unique almost split sequence ending with $S(2)$, by 4.10. But the left morphism in that sequence must be irreducible, and so by our earlier result about the successors of $P(3)$, we know that the middle term of the sequence is $P(2)$, hence being the only predecessor of $S(2)$.
(2) is proven in exact analogy by using the fact that

$$0 \to S(2) \to I(2) \to I(1)$$

is the minimal injective resolution of $S(2)$.
We need to make one more remark: since τ gives us a way to go "left" in our quiver, and the only way to terminate that procedure is to have $\tau M = 0$ for some module, we know that each connected component of

Γ(mod-A) must contain some projective indecomposable modules, since these are the only ones yielding 0 after τ was applied.
And since all of the indecomposable projective representations of the quiver appeared in the full subquiver we gave above, we know that Γ(mod-A) is connected. In fact, Γ(mod-A) is the following quiver:

$$\begin{tikzcd}
 & & P(1) \arrow[dr] & & \\
 & P(2) \arrow[ur] \arrow[dr] & & I(2) \arrow[dr] & \\
P(3) \arrow[ur] & & S(2) \arrow[ur] & & I(1)
\end{tikzcd}$$

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