Uppsala University

Bachelor Thesis in Physics 15 HP

Department of Physics and Astronomy
Division of Theoretical Physics

Higgs inflation

Author:
Erik Schildt

Supervisor:
Lorenzo Ruggeri
Subject reader:
Giuseppe Dibitetto

12 June, 2018

Abstract

In this project a recent model of inflation in which the Standard Model Higgs field with a nonminimal coupling to gravity takes on the role of the inflaton field is investigated. The tensor to scalar ratio, spectral index and the running of the spectral index is calculated for a tree level analysis and compared with the Planck experiment. The value of the nonminimal coupling constant ξ is estimated by obtaining a relation between the amplitude of scalar perturbations and the Higgs mass, it is found that $\xi \sim 10^4$. The basic aspects of how the results are modified through quantum corrections and what the consequences of the nonminimal coupling are for the effective field theory description is discussed. It is found that a tree level analysis yields predictions which are inside the allowed regions of the cosmological parameters given by the Planck experiment. The large value of the nonminimal coupling leads to unitarity problems for this model of inflation. However quantum effects will have a significant effect and how they modify the results of the tree level analysis is what decides if Higgs inflation is a viable theory.

Sammanfattning

I detta projekt undersöker vi en modell av kosmisk inflation där Higgsfältet med en ickeminimal koppling till tyngdkraften är mekanismen bakom inflation. Vi utför en klassisk analys och beräknar modellens föresägelser för ett antal kosmologiska parametrar som jämförs med Planck experimentet. Vi uppskattar värdet på den ickeminimala kopplingen ξ och finner att $\xi \sim 10^4$. De grundläggande aspekterna bakom kvantanalysen samt vad effekten av den ickeminimala kopplingen har på beskrivningen i termer av en effektiv fältteori diskuteras. Vi finner att en klassisk analys ger förutsägelser som passar väl med Planckexperimentet men att den ickeminimala kopplingen leder till unitaritetsproblem för denna modell av inflation. Kvanteffekter kan dock ha en avsevärd effekt på resultat och en utförlig analys som tar dem till hänsyn krävs för att avgöra om Higgsinflation är en möjlig modell för inflation.

Contents

Chapter 1

Introduction

1.1 Introduction

In 1927 George Lemaitre first noted that if one traced back the current expansion of the Universe it would emanate from an initial singularity and the idea of the Big Bang was born. The Big Bang model has since been successful in explaining the abundance of light elements, the Cosmic Microwave Background radiation and Hubble's law. But the Big Bang model is not without it's flaws, certain properties such as the flatness of the Universe and the homogeneity of the Cosmic Microwave Background temperature require extremely fine tuned initial conditions. We know today that the Universe is nearly flat, it could also have a positive or negative curvature depending on the energy content of it's constituent parts. A more careful analysis based on the Big Bang model shows that any deviations from flatness increase with time in the Big Bang model and would quickly evolve into a Universe with a larger curvature. A similar problem exists regarding the uniformity of the Cosmic Microwave Background(CMB) radiation, it consists of many regions of space which would have no time to come into contact and reach thermal equilibrium in the Big Bang model. What was expected was a background radiation with a much wider and less homogeneous distribution of temperatures. It is possible that the Universe was incredibly close to absolute flatness and with a distribution of temperature that was extremely homogeneous in the early universe. However this seems unlikely and presents a fine tuning problem in the Big Bang model.

These two problems are known as the flatness and the horizon problem respectively. In 1981 Alan Guth proposed a new idea which would solve the flatness and horizon problems. He postulated that the universe went through a period of accelerated expansion early in it's history called cosmic inflation. This period of accelerated expansion drove the universe towards flatness and allowed the Universe to reach thermal equilibrium in it's early stages, solving both the horizon and flatness problems.There are still some unsolved problems in inflation, the most predominant is that at this point it's not known what the mechanism which drove inflation is. However a number of inflationary models exist. Each inflationary model presents a set of predictions on certain cosmological parameters which can be used to evaluate these models. Experiments have significantly reduced the parameter space for an inflationary model through experiments such as COBE, WMAP and most recently Planck. Many models are built on an scalar field called the inflaton that drives inflation, this would be a new field that we haven't observed yet. Perhaps it's natural to first look at scalar fields in the Standard Model to see if they could be the mechanism behind inflation. The only scalar field in the Standard Model is the Higgs field, an analysis shows that the Higgs field can drive inflation but gives predictions which are incompatible with the observed fluctuations in the CMB.

However there are still degrees of freedom in the Standard Model one can use to construct a plausible theory. The Higgs field can have a nonminimal coupling to the Ricci scalar which causes gravity to act in a different way at higher energies, then a model which supports inflation and gives predictions that are more compatible with the observed fluctuations in the CMB is found. The goal of this thesis is to study this model and calculate quantities which are measured by the Planck experiment such as the tensor to scalar ratio, the spectral index and the running of the spectral index. The initial analysis is performed classically put the results are modified when quantum effects are accounted for, an overview of where these quantum effects arise from is also presented.

1.2 Background

References [1-5] consists of resources which provide the necessary tools to analyze the Higgs sector of the Standard Model. This includes classical and quantum field theory, basics of particle physics and group theory. References [7-13] consists of resources used to gain an insight into cosmology and inflation.
In [17] Shaposhinkov and Bezrukov consider a nonminimal coupling between the Higgs field and the Ricci scalar. They show that a classical analysis provides cosmological observables that fit with current cosmological data and discuss quantum effects. In [19] Shaposhnikov and Bezrukov consider quantum effects in more detail by obtaining upper bound and lower bounds on the Higgs mass needed for the Higgs field to be the inflaton based on 1-loop calucluation. A similar analysis performed to 2-loop is performed in [16] by Wilczek, De Simone and Hertzberg. They obtain a lower bound on the Higgs mass for Higgs inflation to be a viable model subject to data from WMAP.

1.3 Method

This thesis is based on a literature study. Through studying the Higgs sector of the Standard Model necessary background material for understanding Higgs inflation is obtained. Inflation is studied to understand how it solves the horizon and flatness problems and how adding quantum mechanics produces inhomogenities in the universe which can be observed in the CMB. By merging these two areas of physics Higgs inflation and it's results can be analyzed classically. How quantum effects modify the results in the classical case is studied through the use of renormalization.

Chapter 2

Classical Field Theory and Renormalization

2.1 Aspects of Classical Field Theory

2.1.1 Basics

Units We work in units where $c = \hbar = 1$ and from this one gets $[length] = [time] = [energy]^{-1} = [mass]^{-1}$
Hence we only need one unit for specifying all these quantities, the unit of energy eV is most commonly chosen.
Metric and fourvectors
We use the metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{2.1}$$

A fourvector $A^\mu = (A_0, A_1, A_2, A_3)$ is a vector with four components that transform in a certain way under Lorentz transformations . If the index is up we say that the vector is a contravariant vector, if it's down we say that it's a covariant vector. We can raise and lower the index of any fourvector as

$$\begin{aligned} A^\mu &= \eta^{\mu\nu} A_\nu \\ A_\mu &= \eta_{\mu\nu} A^\nu \end{aligned} \tag{2.2}$$

A sum over repeated indices is implicit. An invariant dot product between fourvectors is given by

$$A_\mu B^\mu = \eta^{\mu\nu} A_\mu B_\nu \tag{2.3}$$

The fourgradient is defined as

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\frac{\partial}{\partial t}, \nabla) \tag{2.4}$$

And the contravariant fourgradient is.

$$\partial^\mu = \eta^{\mu\nu} \partial_\nu = (\frac{\partial}{\partial t}, -\nabla) \tag{2.5}$$

2.1.2 Equations of motion

In this section we will give an introduction to classical field theory, it's primarily based on [1, 2, 3]. In classical mechanics the central object is the action S, which is the time integral of the Lagrangian L

$$S = \int L dt \tag{2.6}$$

The Lagrangian is a function of a set of generalized coordinates $q_a(t)$ and their derivatives $\dot{q}_a(t)$. In classical field theory the dynamical quantity we consider is instead fields $\phi_a(x)$ which have a value at every point in spacetime $x = (\mathbf{x}, t)$. The dynamics of fields is governed by the Lagrangian density $\mathcal{L}$ which is a function of the fields $\phi_a(x)$ and their derivatives $\partial_\mu \phi_a(x)$. The Lagrangian is the spatial integral of the Lagrangian density

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{2.7}$$

Hence the action can be written as

$$S = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{2.8}$$

We will simply refer to $\mathcal{L}$ as the Lagrangian from now on. In natural units $c = \hbar = 1$ from which it follows that $[d^4x] = L^4 = M^{-4}$, since the action is dimensionless we require that $[\mathcal{L}] = M^4$.

As in particle mechanics the dynamics are determined by an action principle which states that the system traverses a path in configuration space such that the action is an extremum $\delta S = 0$. We consider displacing the fields by an amount $\delta\phi_a$. The variation of the Lagrangian is

$$\delta\mathcal{L} = \mathcal{L}(\phi_a + \delta\phi_a, \partial_\mu(\phi_a + \delta\phi_a)) - \mathcal{L}(\phi_a, \partial_\mu\phi_a) = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a) \tag{2.9}$$

This gives

$$\begin{aligned} \delta S &= \int d^4x \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a) \\ &= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right) \right) \delta\phi_a + \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a \right) \end{aligned} \tag{2.10}$$

Where we have integrated the second term by parts. The last term is a total derivative which is zero for any variation that vanishes at spatial infinity. For any such $\delta\phi_a$ we conclude that for an extremum the Lagrangian satisfies the *Euler-Lagrange equation*

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0 \tag{2.11}$$

In particle mechanics we define the Hamiltonian as the Legendre transform of the Lagrangian.

$$H(q,p) = p^i \dot{q}_i - L(q_i, \dot{q}_i) \tag{2.12}$$

With the conjugate momenta defined as

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.13}$$

The definition is analagous in field theory, we first define a conjugate momentum density

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \tag{2.14}$$

The Hamiltonian density is then given by

$$\mathcal{H} = \pi^a \dot{\phi}_a - \mathcal{L} \tag{2.15}$$

and hence the Hamiltonian is given by

$$H = \int d^3x \mathcal{H} \tag{2.16}$$

Example Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{2.17}$$

The first term in the Euler-Lagrange equation is given by

$$\begin{aligned} &\frac{1}{2}\partial_\mu \left(\frac{\partial}{\partial(\partial_\mu \phi)} (\partial_\nu \phi \partial^\nu \phi) \right) = \frac{1}{2}\partial_\mu \left(\frac{\partial}{\partial(\partial_\mu \phi)} (\eta^{\nu\sigma} \partial_\nu \phi \partial_\sigma \phi) \right) \\ &= \frac{1}{2}\partial_\mu \left(\eta^{\nu\sigma} \delta_{\mu\nu} \partial_\sigma \phi + \eta^{\nu\sigma} \partial_\nu \delta_{\sigma\mu} \right) = \partial_\mu \partial^\mu \phi \end{aligned} \tag{2.18}$$

The equation of motion is thus

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0 \tag{2.19}$$

This is the *Klein-Gordon equation.* It can also be easily shown that $\pi = \dot{\phi}$, and hence the Hamiltonian is given by

$$H = \int d^3x \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{2.20}$$

2.1.3 Spontaneous symmetry breaking

We now consider a Lagrangian which has a discrete $\mathbb{Z}_2$ symmetry, $\phi \to -\phi$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{2.21}$$

where the potential is given by.

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2.22}$$

λ is a dimensionless coupling constant and the potential is clearly invariant under $\mathbb{Z}_2$ transformation. We once again have that $\pi = \dot{\phi}$ and the Hamiltonian is given by

$$H = \int d^3x(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4) \tag{2.23}$$

From this we see that for the energy to be bounded below we must pick $\lambda > 0$, otherwise the energy would go to negative infinity for large field values, m^2 is not constrained and can be positive or negative. The equations of motion are given by

$$\partial_\mu\partial^\mu\phi + \phi(m^2 + \frac{\lambda}{6}\phi^2) = 0 \tag{2.24}$$

An easily obtained solution is one in which the field is constant, such a set of solutions is called a *vacua* of the theory if it minimizes the potential.

- If $m^2 > 0$ then $\phi = 0$ is the only solution, it's also easily seen that it's the minimum of the potential and invariant under $\mathbb{Z}_2$ transformation.
- If $m^2 < 0$ then $\phi = 0$ is still a possible solution, but there are two additional solutions.
  $$\phi = \pm v = \pm\sqrt{\frac{-6m^2}{\lambda}} \tag{2.25}$$
  Now $\phi = 0$ is now a maximum while $\phi = \pm v$ minimize the potential and are the vacua of the theory and v is the *vacuum expectation value.* Interestingly the vacua are not invariant under $\mathbb{Z}_2$ transformation but instead map to eachother, whenever a symmetry of the Lagrangian is not respected by the vacua of the theory we say that the symmetry is *spontaneously broken.*

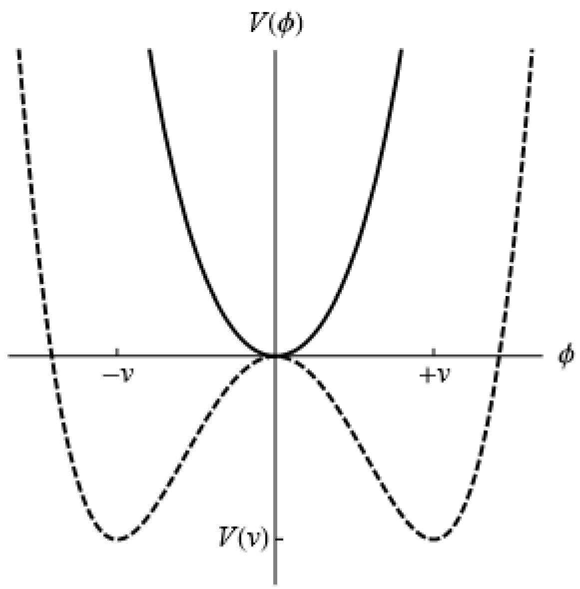


Figure 2.1: The Higgs potential for $m^2 > 0$ (solid line) and $m^2 < 0$ (dashed line).[1]

For the case $m^2 < 0$ we can consider excitations of the vacua, $\eta(x)$, such that

$$\phi = v + \eta(x) \tag{2.26}$$

In terms of the excitations the Lagrangian, with a constant dropped, is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - m^2\eta^2 - \sqrt{\frac{\lambda}{6}}\eta^3 - \frac{\lambda}{4}\eta^4 \tag{2.27}$$

We see that the field η has a mass $m_\eta = \sqrt{2}m$ and one more interaction term has appeared. Glancing at this form of the Lagrangian it appears that the original $\mathbb{Z}_2$ has completely disappeared since the above Lagrangian isn't invariant under $\eta \to -\eta$. This isn't actually true, the $\mathbb{Z}_2$ transformation acts differently on η

$$\eta = \phi - v \to -\phi - v = -\eta - 2v \tag{2.28}$$

So the original symmetry is still present in the Lagrangian but hidden in a subtle way. [3]
We now consider spontaneous symmetry breaking with a continuous symmmetry. Let $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ be a complex scalar field and consider a Lagrangian of the form.

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi^* - m^2\varphi\varphi^* - \frac{\lambda}{4}(\varphi\varphi^*)^2 \tag{2.29}$$

This Lagrangian is clearly invariant under the $U(1)$ transformation,

$$\varphi \to e^{i\alpha}\varphi \tag{2.30}$$

where α is a constant. The equations of motion can be obtained by considering φ and φ^* as independent fields.

$$\begin{aligned} \partial_\mu\partial^\mu\varphi^* + \varphi^*(m^2 + \frac{\lambda}{2}(\varphi\varphi^*)) = 0 \\ \partial_\mu\partial^\mu\varphi + \varphi(m^2 + \frac{\lambda}{2}(\varphi\varphi^*)) = 0 \end{aligned} \tag{2.31}$$

We can now do a similar analysis as in the case of a discrete symmetry and consider the solutions to the equation of motion when φ is a constant. When $m^2 > 0$ the symmetry is unbroken and the minimum of the potential is $\varphi = 0$. When $m^2 < 0$ $\varphi = 0$ is a maximum and the set of vacua is

$$|\varphi| = \sqrt{\frac{-2m^2}{\lambda}} = \frac{v}{\sqrt{2}}, \qquad v = \sqrt{\frac{-4m^2}{\lambda}} \tag{2.32}$$

which is a circle. Clearly this is due to the symmetry of the $U(1)$ transformation. By choosing a particular direction in this set of vacua the symmetry is spontaneously broken. Let's pick $\varphi = \frac{1}{\sqrt{2}}v$, we can then parametrise the excitations around this minimum as

$$\varphi = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\theta(x)} \tag{2.33}$$

where $\eta(x)$ describes displacement in the radial direction and $\theta(x)$ in the circular direction. Inserting this into the Lagrangian gives

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta\partial^\mu\eta + (v+\eta)^2\partial_\mu\theta\partial^\mu\theta) - \frac{m^2}{2}(v+\eta)^2 - \frac{\lambda}{16}(v+\eta)^4 \tag{2.34}$$

Through expanding this expression one can obtain the coefficients of the mass terms, η^2 and θ^2. One finds that the masses of the fields are

$$m_\eta = \sqrt{-2m^2} \tag{2.35}$$
$$m_\theta = 0 \tag{2.36}$$

The excitations of the field η have gained a mass while the excitations of θ are massless. In hindsight this isn't surprising, the η field describes oscillations in the radial direction in which the potential has a curvature while there is no curvature in the circular direction and hence the excitations of the θ field are massless. Finding massless particles when a symmetry is spontaneously broken is a general property which we will study in more detail.

Goldstones Theorem
Suppose the Lagrangian is invariant under a particular symmetry group. Every broken generator of the symmetry group gives rise to a massless field and the corresponding massless particles are called *Goldstone bosons*
Proof
Suppose the Lagrangian is a function of a set of scalar fields $\phi = (\phi_1, \phi_2, ..., \phi_n)$ and the the potential is given by $V(\phi)$ The minimum of the potential is obtained at $\mathbf{v} = (v_1, v_2, ...v_n)$ such that

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi=\mathbf{v}} = 0 \tag{2.37}$$

Taylor expanding around the minimum one finds

$$V(\phi) = V(\mathbf{v}) + \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\phi=\mathbf{v}} (\phi_i - v_i)(\phi_j - v_j) + ... \tag{2.38}$$

We define the mass matrix M as

$$M_{ij} = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\phi=\mathbf{v}} \tag{2.39}$$

and by choosing a suitable basis in which the mass matrix is diagonal the eigenvalues of this matrix is the masses of the fields. Suppose the fields transform under the symmetry group as

$$\phi_i \rightarrow \phi_i + i\epsilon^a T^a_{ij} \phi_j \tag{2.40}$$

where T^a_{ij} are the generators of the symmetry group. The potential is invariant under this transformation which yields.

$$V(\phi_i) = V(\phi_i + i\epsilon^a T^a_{ij} \phi_j) \tag{2.41}$$

Through a Taylor expansion one obtains

$$V(\phi_i) = V(\phi_i) + \frac{\partial V}{\partial \phi_i} i\epsilon^a T^a_{ij} \phi_j + ... \tag{2.42}$$

Differentiating once again and evaluating at $\phi = \mathbf{v}$ one finds using Equations (2.37) and (2.39)

$$\left.\frac{\partial V}{\partial \phi_k}\right|_{\phi_k=v_k} = M_{ki} i\epsilon^a T^a_{ij} v_j = 0, \quad k = 1, 2, 3, ...n \tag{2.43}$$

This can be written in matrix form as

$$MT^a\mathbf{v} = 0 \tag{2.44}$$

A generator is said to be broken if

$$T^a\mathbf{v} \neq 0 \tag{2.45}$$

Using (2.44) we see that whenever a generator is broken the mass matrix must have an eigenvector with the corresponding eigenvalue zero. The eigenvalues of the mass matrix are the masses of the fields as seen earlier and hence every broken generator leads to a massless field.

2.1.4 Vector Field Theories

Maxwell's equation in terms of the magnetic field $\mathbf{B}$,the electric field $\mathbf{E}$, the charge density ρ and the current density $\mathbf{j}$ are given by

$$\nabla \cdot \mathbf{E} = \rho \tag{2.46}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2.47}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.48}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{2.49}$$

The electric and magnetic field can be written in terms of the scalar potential ϕ and the vector potential $\mathbf{A}$

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{2.50}$$

It can be shown that these equations are invariant under the *gauge transformations* of the scalar and vector potential.

$$\phi \to \phi - \frac{\partial \xi}{\partial t} \tag{2.51}$$

$$\mathbf{A} \to \mathbf{A} + \nabla\xi \tag{2.52}$$

Where ξ is an arbitrary function, this shows that the choice of the potentials is not unique and the gauge transformation relates physically identical choices of the potential.
A consequence of Maxwell's equations is current conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{2.53}$$

The above results can be written more elegantly in a relativistic framework by introducing the fourcurrent $j^\mu = (\rho, \mathbf{j})$ and the electromagnetic field tensor $F_{\mu\nu}$.
The electromagnetic field tensor $F_{\mu\nu}$ is an antisymmetric rank 2 tensor and hence

has 6 independent parameters which will be the all components of the electric and magnetic fields. We start by finding an equation which obeys current conservation

$$\partial_\mu j^\mu = 0 \tag{2.54}$$

One choice is

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{2.55}$$

Taking the partial derivative on both sides we obtain

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu \tag{2.56}$$

Using the antisymmetry of $F^{\mu\nu}$ and the equality of mixed partial derivatives the left hand side is given by.

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} \to \partial_\mu \partial_\nu F^{\mu\nu} = 0 \tag{2.57}$$

Current conservation follows immediately from this. The components of $F^{\mu\nu}$ are defined as:

$$F_{0i} = E_i, \quad i = 1, 2, 3 \tag{2.58}$$

and

$$F_{ij} = -\epsilon_{ijk} B_k, \quad i, j = 1, 2, 3 \tag{2.59}$$

In the matrix form get

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & B_1 \\ -E_3 & B_2 & B_1 & 0 \end{pmatrix} \tag{2.60}$$

Two of Maxwells equations, Amperes law and Gauss's law, are then obtained by

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{2.61}$$

The last two, Gauss's law for the magnetic field and Faraday's law, are obtained by the Bianchi identity

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0 \tag{2.62}$$

We now introduce the gauge potential A^μ, in terms of the scalar and vector potential it can be written as $A^\mu = (\phi, \mathbf{A})$. It can be used to write the electromagnetic field tensor as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.63}$$

When considering Maxwell's equations in their standard form we showed that the electric and magnetic fields where invariant under certain gauge transformations of the potential. A similar result also holds here. Consider the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu f(x) \tag{2.64}$$

Under this gauge transformation the field tensor is invariant

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\mu \partial_\nu f - \partial_\nu A_\mu - \partial_\nu \partial_\mu f \tag{2.65}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{2.66}$$

We would now like to find a Lagrangian that describes electromagnetism, starting with the case of no sources. We know it should be Lorentz invariant and gauge invariant, with this in mind we look for a Lagrangian with all indices contracted and which depends on the gauge potential through the gauge invariant electromagnetic field tensor. From earlier we also demanded that the mass dimension was four, $F_{\mu\nu}F^{\mu\nu}$ is the essentially the only term we have which satisfies all these requirements. Adding a constant in front for cosmetic purposes we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.67}$$

The equations of motion are given by,

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.68}$$

which is just what we wanted. Turning to the case with sources we see that adding the term $A_\mu j^\mu$ in the Lagrangian gives the equations of motion but leads to a Lagrangian which isn't manifestly gauge invariant. The added term transforms as

$$A_\mu j^\mu \to A_\mu j^\mu + (\partial_\mu f)j^\mu = A_\mu j^\mu + \partial_\mu(fj^\mu) - f\partial_\mu j^\mu \tag{2.69}$$

If the current conservation equation $\partial_\mu j^\mu = 0$ is satisfied we see that the only difference in the Lagrangian is a total derivative. Hence there is no influence on the equations of motion, this is the manifestation of gauge invariance in the Lagrangian. The Lagrangian for electromagnetism is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \tag{2.70}$$

with equations of motion given by

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{2.71}$$

2.1.5 Higgs mechanism

We saw earlier that the Lagrangian (2.29) had a global $U(1)$ symmetry. We will now consider what happens when we *gauge* the symmetry, making it a local symmetry, $\alpha = \alpha(x)$.

$$\varphi \to e^{i\alpha(x)}\varphi \tag{2.72}$$

The derivatives in the kinetic term will destroy the invariance, instead transforming as

$$\partial_\mu(e^{i\alpha(x)}\varphi) \to e^{i\alpha(x)}(\partial_\mu + i\partial_\mu\alpha(x))\varphi \tag{2.73}$$

By adding a *gauge field* A_μ which transforms as (2.64) we can create a Lagrangian which is invariant under a local $U(1)$ symmetry. Consider the term $(\partial_\mu - iA_\mu)\varphi$. It transforms as

$$\begin{aligned}(\partial_\mu - iA_\mu)\varphi &\to e^{i\alpha(x)}[\partial_\mu + i\partial_\mu\alpha(x) - i(A_\mu + \partial_\mu f)]\varphi \\ &= e^{i\alpha(x)}[\partial_\mu - iA_\mu + i(\partial_\mu(\alpha - f))]\varphi\end{aligned} \tag{2.74}$$

where $f(x)$ is an arbitrary function, but if we use the gauge freedom to choose $f(x) = \alpha(x)$ we obtain a quantity that is invariant under the local transformation, the *covariant derivative* $D_\mu = \partial_\mu - iA_\mu$. To obtain gauge invariant kinetic term all we have to do is replace ∂_μ with D_μ, but now there is a coupling between the gauge field and the complex scalar field. The Lagrangian is given by.

$$\mathcal{L} = (D_\mu\varphi)(D^\mu\varphi)^* - m^2\varphi\varphi^* - \frac{\lambda}{4}(\varphi\varphi^*)^2 \tag{2.75}$$

The next step is to add the kinetic term from (2.70), obtaining the Lagrangian for scalar electrodynamics.

$$\mathcal{L} = (D_\mu\varphi)(D^\mu\varphi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m^2\varphi\varphi^* - \frac{\lambda}{4}(\varphi\varphi^*)^2 \tag{2.76}$$

The vacua we obtained for the the case of a global symmetry (2.32) still holds here and we can parametrize around the vacua in the same way.

$$\varphi = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\theta(x)} \tag{2.77}$$

Now that the symmetry is local we can use our gauge freedom to choose $\alpha(x) = -\theta(x)$ yielding

$$\varphi' = e^{-i\theta(x)}\frac{1}{\sqrt{2}}(v + \eta(x))e^{i\theta(x)} = \frac{1}{\sqrt{2}}(v + \eta(x)) \tag{2.78}$$

This gauge is called the *unitary gauge*. The vector field hence transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\theta(x) \tag{2.79}$$

The Lagrangian (2.76) is gauge invariant and can therefore be evaluated in any gauge. We will evaluate it in the unitary gauge, the covariant derivatives are

$$(D_\mu\varphi)'(D^\mu\varphi)'^* = \frac{1}{2}(\partial_\mu + i(v+\eta)A'_\mu)(\partial^\mu - i(v+\eta)A'^\mu) = \frac{1}{2}(\partial_\mu\eta\partial^\mu\eta + (v+\eta)^2A'_\mu A'^\mu) \tag{2.80}$$

The full Lagrangian in the unitary gauge is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta\partial^\mu\eta + (v+\eta)^2A'_\mu A'^\mu) - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - V(\varphi'\varphi'^*) \tag{2.81}$$

where

$$V(\varphi'\varphi'^*) = \frac{1}{4}m^2v^2 - m^2\eta^2 + \frac{1}{4}\lambda v\eta^3 + \frac{\lambda}{16}\eta^4 \tag{2.82}$$

As before, the field $\eta(x)$ describing the radial excitation has gained a mass $m_\eta = \sqrt{-2m^2}$, however the Goldstone mode has dropped out of the Lagrangian and the vector field has gained a mass $m_{A'} = v$. The number of degrees of freedom in the system are the same since a massive vector fields has an additional longitudinal polarization mode compared to the two transversal polarization modes in a massless vector field. This is one example of the famous *Higgs mechanism*, the degree of freedom from the Goldstone mode has been turned into the longitudinal polarization of the associated vector field which hence has gained a mass. The particle which arises from the excitations in the field η is called a Higgs scalar.This is the same mechanism with which the $W^{\pm}$ and Z bosons get their mass in the Standard Model. However the symmetry group in the Glashow-Weinberg-Salam theory which describes this is $SU(2)_L \times U(1)_Y$ where L indicates that the symmetry acts on lefthanded doublets and Y is the hypercharge. This symmetry is spontaneously broken to $U(1)_{EM}$.

Following [4] we will show how the Higgs mechanism is applied in the Standard model. Consider the complex doublet

$$\Phi(x) = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{2.83}$$

A Lagrangian which is invariant under $SU(2)_L \times U(1)_Y$ is

$$\mathcal{L} = (D_\mu \Phi)(D^\mu \Phi)^\dagger - V(\Phi^\dagger \Phi) \tag{2.84}$$

where D_μ is the covariant derivative corresponding to $SU(2)_L \times U(1)_Y$ and the potential is given by

$$V(\Phi^\dagger \Phi) = m^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \tag{2.85}$$

The $SU(2)_L \times U(1)_Y$ symmetry acts on the doublet $\Phi(x)$ as

$$\Phi(x) \to \Phi'(x) = \exp\left(i\alpha_a(x) \frac{\sigma_a}{2} + i\chi(x) \frac{Y}{2} \right) \Phi(x) \tag{2.86}$$

where σ_a are the Pauli spin matrices which are the generators of $SU(2)_L$ and $Y = 1$ which is the generator of $U(1)_Y$. We then consider spontaneous symmetry breaking in this theory. As before we assume $m^2 < 0$, the vacua of the theory are obtained by minimizing the potential

$$V'(\Phi^\dagger \Phi) = m^2 + 2\lambda \Phi^\dagger \Phi = 0 \to \Phi^\dagger \Phi = -\frac{m^2}{2\lambda} \tag{2.87}$$

By choosing a doublet which satisfies this the symmetry is spontaneously broken, one choice is

$$\Phi = \nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v = \sqrt{\frac{-m^2}{\lambda}} \tag{2.88}$$

It is easily verified that all generators of $SU(2)_L$ are broken

$$\sigma_1 \nu = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0 \tag{2.89}$$

$$\sigma_2 \nu = -\frac{i}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0 \tag{2.90}$$

$$\sigma_3 \nu = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0 \tag{2.91}$$

Similarly with the generator of $U(1)_Y$

$$Y\nu = \nu \neq 0 \tag{2.92}$$

The generator that isn't broken is a linear combination of σ_2 and Y and corresponds to the electric charge.

$$Q = \frac{1}{2}(\sigma_3 + Y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.93}$$

It's easily verified that

$$Q\nu = 0 \tag{2.94}$$

which corresponds to the fact that the vacuum is electrically neutral and hence the breaking pattern is $SU(2)_L \times U(1)_Y \to U(1)_{EM}$

We can expand the excitations of the vacua as

$$\Phi(x) = \exp(i\theta_a(x)\sigma_a)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \tag{2.95}$$

Using the unitary gauge we get

$$\Phi'(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \tag{2.96}$$

Inserting this into the potential one obtains

$$V(\Phi'^{\dagger}\Phi') = \frac{1}{4}m^2v^2 - m^2H^2 - \lambda v H^3 + \frac{1}{4}\lambda H^4 \tag{2.97}$$

The mass of the Higgs boson is then found to be

$$m_{Higgs} = \sqrt{-2m^2} = \sqrt{2\lambda}v \tag{2.98}$$

which depends on both the coupling constant and the vaccum expectation value v. The vacuum expectation value can be obtained from muonic decays and measurements yield

$$v \approx 246 \quad \text{GeV} \tag{2.99}$$

The Lagrangian (2.84) describes a wide range of phenomena but we will mainly be interested in the dynamics of the field $H(x)$ which has the Lagrangian

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu H \partial^\mu H - V(\Phi'^{\dagger}\Phi') \tag{2.100}$$

This Lagrangian together with a coupling between gravity and the Higgs field will be the starting point for analyzing Higgs inflation.

2.2 Renormalization

In this section we give a very brief introduction to renormalization based on [6, 14] Renormalization is one of the cornerstones of quantum field theory. It's used to cure the infinites often encountered in perturbation theory in different kinds of quantum field theories. In it's earliest applications it was used to calculate a finite value for the Lamb shift and to find finite perturbative corrections in quantum electrodynamics. A quantum field theory is a merger between quantum mechanics and special relativity, fundamental concepts in both of these areas in physics are at the heart of renormalization. A new length scale can be constructed from the deBroglie wavelength in quantum mechanics and the energy mass equivalence in special relativity, $\frac{\hbar}{mc}$. Due to Heisenberg uncertainty relations probing shorter wavelengths than this gives rise to creation of virtual particles. Virtual particles are not real particles in the sense that they don't obey the energy momentum relationship $P_\mu P^\mu = m^2$, but they can still interact with real particles. It's this interaction which causes the infinities in perturbation theory.

Feynman diagrams are used when calculating corrections in quantum field theory. These diagrams are used to represent physical processes, each theory has a set of Feynman rules which interactions must obey. As long as these rules, which enforce momentum and energy conservation, are applied any process is allowed. Most importantly, diagrams which show creation and annihilation of virtual particles are acceptable. To perform an exact calulation one must sum over all allowed Feynman diagrams, as these diagrams become more complicated this becomes impossible. However, diagrams which are more complicated give a smaller contributions, hence one can use a perturbative approach to perform calculations. The perturbation series is in terms of some coupling constant g which specifies the strength of the interaction, if $F(x)$ is some physical quantity then one often finds the form

$$F(x) = g + g^2 F_1(x) + g^3 F_2(x) + ... \tag{2.101}$$

and that the integrals needed to calculate $F_1(x)$ and $F_2(x)$ are divergent when integrating over all allowed momenta. As these higher order interaction depends on the coupling constant it's as if the coupling constant itself is changed through the interaction with virtual particles. One example is from quantum electrodynamics in which the coupling constant involves the electric charge, through the creation and annihilation of virtual particles the electron effectively behaves as if it had a different charge. Going back to our more general example the physical quantity $F(x)$ is parametrized in terms of g, but due to the interaction with virtual particles g isn't a physical quantity. One should instead reparametrize the theory in terms of some new coupling constant g_R defined at some renormalization point $x = \mu$ such that.

$$F(\mu) = g_R \tag{2.102}$$

The infinities that arise will then be absorbed into the difference between the two coupling constants g and g_R and the corrections will yield a finite result.

But this choice of scale is arbitrary, the theory can equally well be described by some other coupling constant g'_R defined at $x = \mu'$ or by the coupling constant g''_R at $x = \mu''$. We are now in the realm of the renormalization group. As the scale of the system changes it is parametrized by a new set of variables at each step, the change in parametrization between two scales (g_R, μ) and (g'_R, μ') can be seen as a group action which produces a self similar copy. This group law can be obtained thorugh the remormalization group equation.

$$\beta(g) = \frac{\partial g}{\partial \log \mu} \tag{2.103}$$

$\beta(g)$ is the beta function which is specific for each theory, through integrating the beta function the variation of the coupling constant between each scale can be obtained, one says that the coupling constant is "running". The beta function is often specified at one-loop or two-loop, this means that the analysis only considered Feynman diagrams with one loop or two loops and et cetera. As the number of loops increases the results get more accurate but the beta functions get more complicated. As an example lets consider the Higgs boson, the main contributions is from it's interaction with itself, the top quark and the electroweak bosons. The coupling constant in the theory λ runs with the energy as

$$\frac{d\lambda}{dt} = \beta_\lambda \tag{2.104}$$

where $t = \log Q^2$ where Q is the energy. The 1-loop beta function is[14]

$$\beta_\lambda = \frac{3}{4\pi^2}(\lambda^2 + \frac{1}{2}\lambda y_t^2 - y_t^4 + \mathcal{B}(g, g')) \tag{2.105}$$

where $\mathcal{B}(g, g')$ is the contribution from the gauge couplings in electroweak theory and y_t is the Yukawa coupling. Lets study the beta function in the regime in which $\lambda \gg y_t, g, g'$ where the renormalization group equation takes the simple form

$$\frac{d\lambda}{dt} = \frac{3}{4\pi^2}\lambda^2 \tag{2.106}$$

Lets consider two scales, the first one is the vacuum expectation value of the Higgs field v. As the standard model is thought to be an effective field theory, valid up to some energy Λ we take the second scale to be this cutoff. Through integration of the renormalization group equation one obtains.

$$\lambda(\Lambda) = \frac{\lambda(v)}{1 - \frac{3\lambda}{2\pi^2}\log\frac{\Lambda}{v}} \tag{2.107}$$

Hence we have related the coupling constant at two different scales. One can also notice that the coupling constant is singular at some cutoff Λ, which is called a *Landau Pole*

$$\Lambda = v \exp\left(\frac{2\pi^2}{3\lambda(v)}\right) \tag{2.108}$$

If the Standard Model is to be valid all the way up to some cutoff scale this puts an upper bound on the Higgs mass.There is a maximal value of the coupling constant

$$\lambda_{max}(v) = \frac{2\pi^2}{3\log\frac{\Lambda}{v}} \tag{2.109}$$

As the Higgs mass is $m = \sqrt{2\lambda}v$ one finds

$$m_{h,max} = \frac{2\pi v}{\sqrt{3\log\frac{\Lambda}{v}}} \tag{2.110}$$

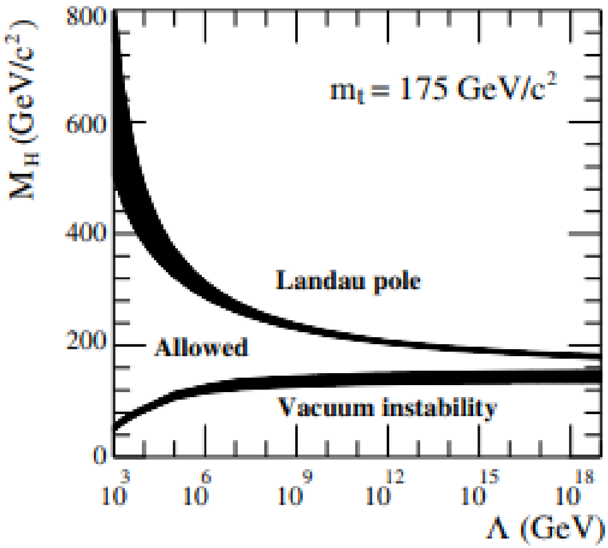


Figure 2.2: The bounds on the Higgs mass due to vacuum to instability and triviality as a function of the cutoff.[14]

In fact, from considering the regime in which the gauge coupling and the top quark coupling dominate the beta function one can find a lower bound on the Higgs mass from demanding vacuum stability, which requires that the coupling constant is positive. The details of this computation are a bit more involved and hence we will skip it.

Chapter 3

Cosmology

3.1 Cosmology

In this section we review some aspects of cosmology leading up to inflation and it's formulation in terms of scalar fields and some basic aspects of perturbations during inflation. This section is primarily based on [7, 8, 9, 10, 12, 13]

3.1.1 Basics of cosmology

The cosmological principle postulates that on large scale the universe is homogeneous and isotropic. A homogeneous space is one which is the same at every point and a isotropic space is the same in every direction. The most general metric that is consistent with the conditions of the cosmological principle is the Friedmann-Robertson-Walker(FRW) metric, producing the line element.

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.1}$$

The curvature of the universe is described with the curvature parameter k which specifies the geometry of the universe. The spherical coordinates (r, θ, ϕ) is a set of *comoving coordinates*, meaning that the distance between two objects in these coordinates is unchanged as the universe expands. The physical distance between two objects as the universe expands is governed by the *scale factor* $a(t)$ and is given by $R(t) = a(t)r$. The scale factor will depend on the constituents of the universe as we will see below.
Photons follow null geodesics $ds^2 = 0$, if we restrict to photons which are propagating radially and using the FRW metric one obtains

$$\int_0^t \frac{dt}{a(t)} = \int_0^r \frac{1}{\sqrt{1-kr^2}} \tag{3.2}$$

At this point it's useful to introduce *conformal time* τ, which is defined as

$$d\tau = \frac{dt}{a(t)} \tag{3.3}$$

The maximum comoving distance an object can travel after a time t is given by the particle horizon x_h, in units that have $c = 1$ this is also the elapsed conformal time

$$x_h = \tau - \tau_i = \int_0^t \frac{dt}{a(t)} \tag{3.4}$$

where t=0 corresponds to the initial singularity, this may not correspond to $\tau = 0$ as we will see later. The physical distance is obtained by multiplying the particle with the scale factor.

$$R(t) = a(t) \int_0^t \frac{dt}{a(t)} \tag{3.5}$$

Through defining the Hubble parameter, $H = \frac{\dot{a}}{a}$, it will be useful to rewrite the expression for the particle horizon as.

$$x_h = \int_0^t \frac{dt}{a(t)} = \int_0^a d\log a \left(\frac{1}{aH} \right) \tag{3.6}$$

The quantity $(aH)^{-1}$ is called the *comoving* Hubble radius. As the Hubble parameter corresponds to an inverse time the inverse Hubble parameter multiplied with the scale factor corresponds to the distance of casually connected points at a given moment of time.

Another useful aspect of conformal time is that lightrays follow straight lines in conformal time. Let the comoving coordinates be specified by $\mathbf{x}$, hence the metric takes the form

$$ds^2 = a^2(\tau)\left[-d\tau^2 + |d\mathbf{x}|^2 \right] \tag{3.7}$$

As photons follow null geodesics one obtains

$$\frac{|d\mathbf{x}|}{d\tau} = \pm 1 \tag{3.8}$$

and lightrays travel at 45 degree angles as shown in Figure 3.1.

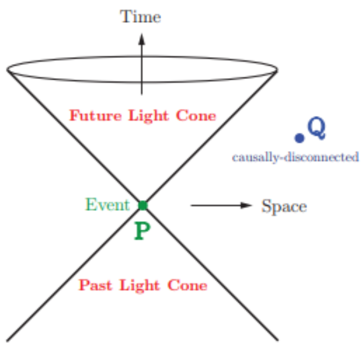


Figure 3.1: In conformal time light follows straight lines leading to a lightcone separating casually connected and disconnected regions.[7]

Following the cosmological principle one assumes that the universe can be described with a stress energy tensor of a perfect fluid, which has the form

$$T^{\mu}_{\nu} = \text{diag}(\rho, -p, -p, -p) \tag{3.9}$$

where ρ is the energy density and p is the pressure of the fluid. When combining this choice of the stress energy tensor with the FRW metric Einsteins equations gives the *Friedmann equations*

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{3.10}$$

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3p) \tag{3.11}$$

Individually the first one is often called the Friedmann equation while the second one is called the acceleration equation. Either from the Friedmann equations or from demanding $\nabla_\mu T^{\mu\nu} = 0$ the *continuity equation* is found.

$$\frac{\partial \rho}{\partial t} + 3H(\rho + p) = 0 \tag{3.12}$$

At this point it will prove useful to introduce the equation of state w as

$$w = \frac{p}{\rho} \tag{3.13}$$

Using the equation of state the pressure can be eliminated in the continuity equation to give

$$\rho \propto a^{-3(1+w)} \tag{3.14}$$

For a flat universe the evolution of the scale factor can be obtained using the Friedmann equation combined with Equation (3.14)

$$a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & \text{for} \quad w \neq -1 \\ e^{Ht} & \text{for} \quad w = -1 \end{cases} \tag{3.15}$$

We will consider three equations of state in particular.

- Non-relativistic matter: $w = 0, \quad a(t) \propto t^{2/3}, \quad \rho \propto a^{-3}$
- Relativistic matter or radiation: $w = \frac{1}{3}, \quad a(t) \propto t^{1/2}, \quad \rho \propto a^{-4}$
- Cosmological constant: $w = -1, \quad a(t) \propto e^{Ht}, \quad \rho \propto a^0$

The density of a matter dominated universe simply scales with the volume, while a radiation dominated universe scales with the volume combined with the redshift of the radiation which scales with the length. At the initial time the density will also be infinite for these types of universes, indicating a Big Bang scenario. The density of the cosmological constant doesn't scale at all and the scale factor is exponentially increasing, such a universe is called a *DeSitter* universe.
The curvature of the universe depends on the density, as can be seem by rearranging the first Friedmann equation

$$k = a^2(t)\big(\frac{\rho}{3} - H^2\big) \tag{3.16}$$

We can define a critical density $\rho_c = 3H^2$ and specify the density in terms of the critical density.

$$\Omega = \frac{\rho}{\rho_c} \tag{3.17}$$

The value of Ω specifies the curvature of the Universe as,

$$k = a^2(t)H^2\big(\Omega - 1\big) \tag{3.18}$$

hence the universe if closed if $\Omega > 1$, open if $\Omega < 1$ and flat if $\Omega = 1$. One can also split Ω into it's constituent pieces which contribute to the energy density

$$\Omega = \Omega_\gamma + \Omega_\Lambda + \Omega_m \tag{3.19}$$

where Ω_γ, Ω_Λ and Ω_m is the contribution from radiation, the cosmological constant and matter respectively. In our universe these take on the values [8],

$$\begin{aligned} \Omega_\Lambda &\approx 0.7 \\ \Omega_m &\approx 0.3 \\ \Omega_\gamma &\approx 10^{-4} \end{aligned} \tag{3.20}$$

indicating that the Universe is flat.

3.1.2 Flatness and Horizon problems

The flatness of the universe presents a problem, rewriting equation (3.18) the deviations from flatness is obtained.

$$\Omega - 1 = \frac{k}{(aH)^2} \tag{3.21}$$

For a matter dominated universe one finds $|\Omega - 1| \propto t^{\frac{2}{3}}$ and for a radiation dominated universe $|\Omega - 1| \propto t$. In any of these types of universes any deviation from absolute flatness, $\Omega = 1$, grows with time. For example to match the flatness seen today the value needs to be extremely close to one in the past, in the Big Bang Nucleosynthesis(BBN) period and the era of Grand Unified Theories(GUT) the deviations from flatness must be[7]

$$\begin{aligned} |\Omega_{BBN} - 1| &\leq 10^{-16} \\ |\Omega_{GUT} - 1| &\leq 10^{-55} \end{aligned} \tag{3.22}$$

These incredibly small deviations from flatness required earlier in the history of the universe seem very unlikely, and hence this is known as the *flatness problem*.
The Cosmic Microwave Background(CMB) is the electromagnetic radiation which decoupled from plasma approximately $380,000$ years after the Big Bang. The wavelength of the radiation has been shifted to the microwave range through the expansion of the Universe. The spectrum we see today is that of a black body that is incredibly uniform with fluctuations in the temperature of the order [8]

$$\frac{\Delta T}{T} \sim 10^{-5} \tag{3.23}$$

From standard cosmology there is no reason that the temperature should be as uniform as we will see.
Using (3.6) one finds that the scale factor for a matter or radiation dominated universe grows with the conformal time as,

$$a(\tau) = \begin{cases} \tau & \text{Matter} \\ \tau^2 & \text{Radiation} \end{cases} \tag{3.24}$$

this means that the fraction of the universe that is casually connected increases with time and the biggest contribution occurs at later times. So areas that haven't been casually connected up to this point have never been casually connected before, then why does the CMB show such a uniform distribution of temperature? It is of course possible that the Universe was extremely homogeneous and had nearly the same temperature throughout, but this seems unlikely. Instead the uniform temperature indicates that somehow areas which shouldn't have been casually connected have been in contact and reached thermal equilibrium. This is the *horizon problem.*

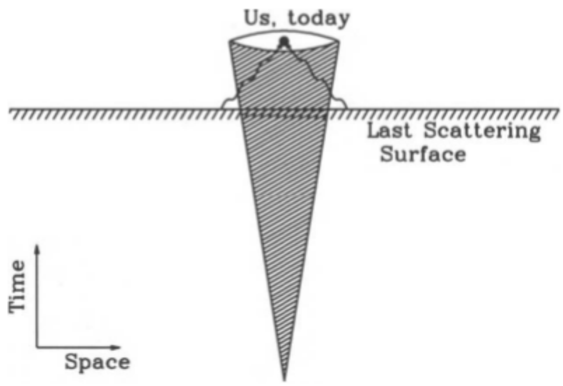


Figure 3.2: The Horizon problem: The two regions of the CMB are outside the region inside the cone which is casually connected[9].

The horizon and flatness problem where unsolved problem until Alan Guth came up which the concept of cosmological inflation which says that the universe went through a period of accelerated expansion in the early universe. This solves the horizon and flatness problems as we will see below and also gives a mechanism for structure formation. We will first understand inflation through the concept of the comoving Hubble radius as it is fundamental in the horizon and flatness problems, we will then show how this is related to an exponential expansion of the universe. At the heart of the horizon and flatness problems was the increase of the comoving Hubble radius, what would happen if it instead decreased a sufficient amount during a period in the early universe? The flatness of the universe wouldn't be a mystery, from Equation (3.21) we see that this behaviour of the comoving Hubble radius drives the universe towards flatness. The solution of the horizon problem through the decrease of the comoving Hubble radius is more subtle. As the comoving Hubble radius if the measure of casual connected areas at a given timestep it is possible that it was much larger in the past than it is now. As the comving Hubble radius decreases it "zooms" in on a smaller subpatch that is in thermal equilibrium, eventually it start to expand as the universe enters a radiation or matter dominated era.

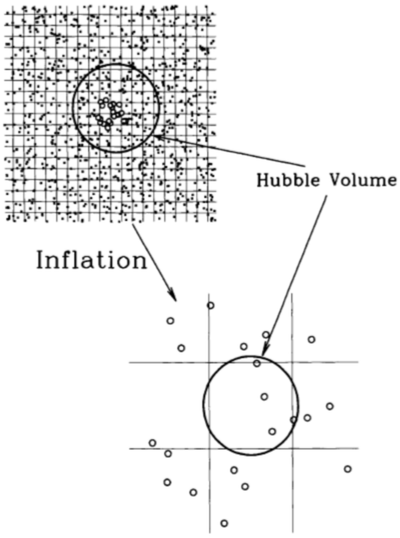


Figure 3.3: The decreasing comoving horizon zooms in on a region in thermal equilibrium [9].

But what does it actually mean that the comoving Hubble radius decreases? It would necessarily mean that aH is increasing, $\frac{d}{dt}(aH) > 0$, this can be written as.

$$\frac{d}{dt}(a\frac{\dot{a}}{a}) = \frac{d^2 a}{dt^2} > 0 \tag{3.25}$$

But this means that the scale factor of the universe is going through a period of accelerated expansion, this is exactly the idea that inflation is based on! From the acceleration equation we can also see that this requires $p < -\frac{\rho}{3}$, which is a negative pressure.Hence we have three equivalent conditions for inflation

- Decreasing comoving Hubble radius $\frac{d}{dt}(aH)^{-1} < 0$
- Accelerated expansion: $\frac{d^2a}{dt^2} > 0$
- Negative pressure: $p < -\frac{\rho}{3}$

It is constructive to consider inflation in conformal time, we assume for simplicity sake that the equation of state is $w = -1$ and hence the scale factor grows as

$$a(t) \propto e^{Ht} \tag{3.26}$$

and the Hubble parameter is constant. In conformal time one obtains.

$$a(\tau) = -\frac{1}{H\tau} \tag{3.27}$$

The initial singularity happens at τ_s when $a(\tau_s) = 0$ and hence $\tau_s = -\infty$. Remarkably the initial singularity is pushed back arbitrarily far into conformal time and this provides perhaps the most elegant illustration of how inflation solves the horizon problem. When there is no inflation and the universe is dominated by an equation of state in which the comoving Hubble radius decreases, for example matter or radiation, one can see from Equation (3.24) that the initial singularity happens at $\tau = 0$. This doesn't leave enough time for distant regions of the CMB to become casually connected before decoupling, but with inflation there is an infinite amount of conformal time before decoupling during which interaction is possible.

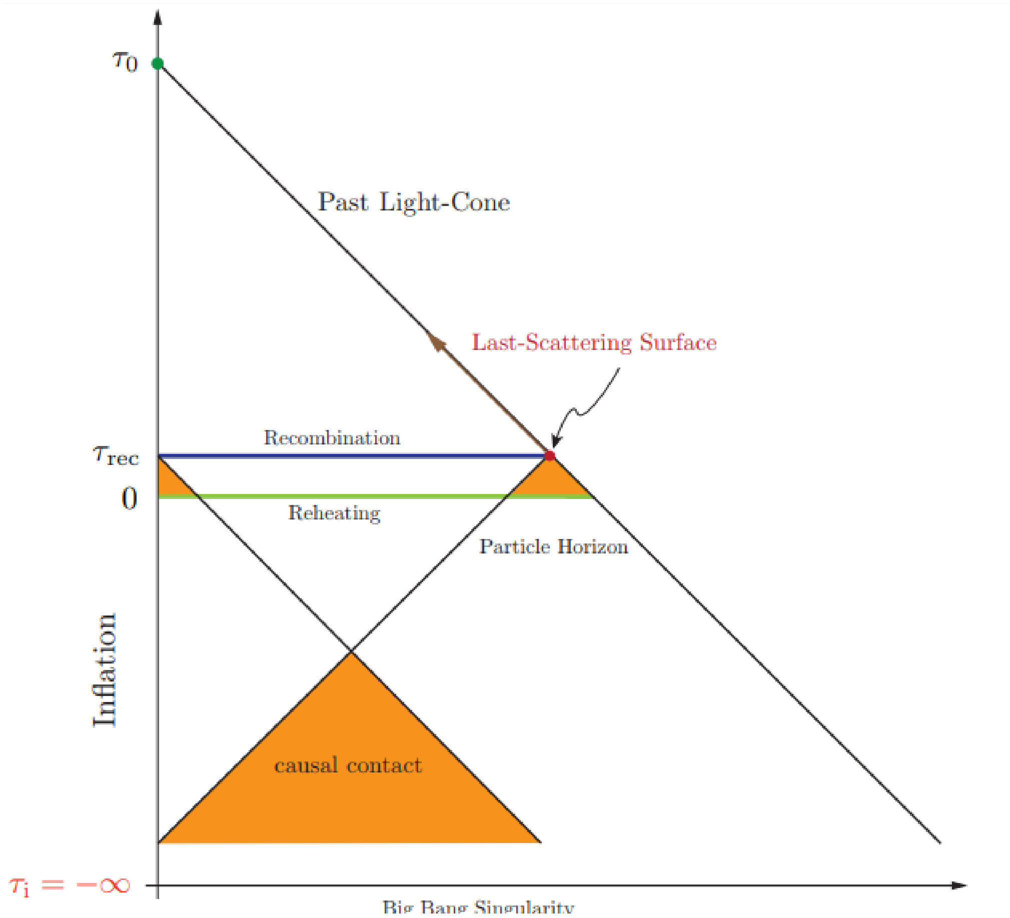


Figure 3.4: The solution of the horizon problem through the extension of conformal time produced by inflation[7]

3.1.3 Inflation from scalar fields

Inflation can be modelled from a single scalar field ϕ with a specified potential $V(\phi)$. The Lagrangian is given by

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \tag{3.28}$$

The metric, $g^{\mu\nu}$, is not necessarily the Minkowski metric so the action we constructed earlier needs to be modified. The invariant volume element is obtained by introducing a factor of $\sqrt{-g}$, yielding the following action for the scalar field

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi \tag{3.29}$$

Combining this with the Einstein-Hilbert action one obtains

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = S_{EH} + S_\phi \tag{3.30}$$

The energy momentum tensor corresponding to the scalar field is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \tag{3.31}$$

Varying (3.29) with respect to the metric one obtains

$$\delta S_\phi = \int d^4x[\delta(\sqrt{-g})\Big(- \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big) + \sqrt{-g}\Big(- \frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big)] \tag{3.32}$$

The variation with respect to the determinant of the metric is

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu} \tag{3.33}$$

and hence

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{3.34}$$

Using this and relabeling the indices in Equation (3.32) one finds

$$\delta S_\phi = \int d^4x\sqrt{-g}\Big[- \frac{1}{2}g_{\alpha\beta}\big(- \frac{1}{2}(\partial\phi)^2 - V(\phi)\big) - \frac{1}{2}\partial_\alpha\phi\partial_\beta\phi\Big]\delta g^{\alpha\beta} \tag{3.35}$$

Using Equation (3.31), the stress energy tensor for the field is found.

$$T_{\mu\nu} = \partial_\mu\phi\partial\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}(\partial\phi)^2 + V(\phi)\Big) \tag{3.36}$$

We will assume a homogeneous field such that $\phi(\mathbf{x}, t) = \phi(t)$ and a flat space time, hence the FRW metric takes the form.

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \tag{3.37}$$

With these assumptions the equations of motion for the scalar field are found from the Euler-Lagrange equations to be,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{3.38}$$

which looks like the equations of motion of a free field combined with a friction term $3H\dot{\phi}$ from the expansion of the universe. Furthermore,the energy momentum tensor takes the form of a perfect fluid with

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.39}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.40}$$

This gives the equation of state

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{3.41}$$

Recall that one of the conditions for inflation was that $w < -\frac{1}{3}$, from the above equation of state one concludes that this is satisfies whenever the potential energy

dominates. The cosmological equation of state $w = -1$, which leads to exponential growth of the scale factor corresponds to $\dot{\phi} = 0$, but is nearly satisfied when $\dot{\phi}^2 \ll V(\phi)$
The two Friedmann equations can be written as

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{1}{6}(\rho + 3p) \\ H^2 &= \frac{1}{3}\rho = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\end{aligned} \tag{3.42}$$

It's useful to introduce the *slow roll* parameter ε which is defined as

$$\varepsilon = \frac{3}{2}(w + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{3.43}$$

where the equation of state and the first Friedmann equation has been used. Using the slow roll parameter the second Friedmann equation can be written as

$$\frac{\ddot{a}}{a} = H^2(1 - \varepsilon) \tag{3.44}$$

The value of the slow roll parameter is useful in determining if the condition for inflation are satisfied. In (3.43) $\epsilon > 1$ when the equation of state for inflation $w < -\frac{1}{3}$ is violated and in (3.44) the condition for the accelerated expansion is violated whenever $\varepsilon > 1$. The slow roll parameter vanishes when the cosmological constant equation of state $w = -1$ is satisfied. The slow roll parameter can also be directly related to the Hubble parameter

$$\epsilon = -\frac{d \log H}{dN} \tag{3.45}$$

where we have defined N as $dN = Hdt$, which also can be written as

$$a(t) = e^N \tag{3.46}$$

and hence N is the number of e-folds of inflation. It will be useful to introduce a second slow roll parameter η which expresses the relation between the friction term in the equations of motion and the second time derivative of the field.

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{dN} \tag{3.47}$$

The condition $\eta < 1$ ensures that inflation is sustained. Using the equations of motion for the scalar field (3.38) the slow roll parameters can be approximated with the potential slow roll parameters which are found to be

$$\epsilon_v = \frac{M_{pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \tag{3.48}$$

$$\eta_v = M_{pl}^2 \frac{V''(\phi)}{V(\phi)} \tag{3.49}$$

In practice this is what is used to perform calculations in different inflationary models. As mentioned before the inflationary conditions are violated when $\epsilon = 1$, to obtain the field value at this time ϕ_e when inflation ends one requires

$$\epsilon_v(\phi_e) = 1 \tag{3.50}$$

The number of e-folds of inflation necessary between an initial field value ϕ_0 and the final field value ϕ_e is

$$N(\phi_0) = \frac{1}{\sqrt{2}M_{pl}} \int_{\phi_e}^{\phi_0} \frac{1}{\sqrt{\varepsilon_v}} d\phi \tag{3.51}$$

To solve the flatness and horizontal problems we require that $N \approx 60$, this can be obtained by requiring that the Universe fits into the comoving Hubble radius at the start of inflation.[12]

3.2 Perturbations during inflation

Up to this point we have considered a Universe which is isotropic and homogeneous, in accordance with the cosmological principle. In order to describe the universe we must account and explain the inhomogeneities which arise. If one combines inflation with quantum mechanical effects in the early universe these inhomogeneities can be accounted for.

3.2.1 Cosmological perturbation theory

To treat perturbations we will resort to cosmological perturbation theory. The perturbed quantities we consider are the inflaton field, the stress energy tensor and the metric, let Q denote any one of these. Then it can be written as

$$Q(x,t) = \overline{Q}(t) + \delta Q(x,t) \tag{3.52}$$

where $\overline{Q}$ represents the homogeneous background and $\delta Q(x,t)$ is the perturbation. Earlier we have used the unperturbed FRW metric, we will now consider perturbations of this metric. The metric is symmetric and hence has 10 independent entries which we can decompose in any way we want. One convenient way to do this is the SVT decomposition which decomposes these 10 degrees of freedom into scalars, vectors and tensors. We can write the perturbed metric as

$$\begin{aligned} g_{00} &= -(1+2\Phi) \\ g_{0i} &= 2aBi \\ g_{ij} &= a^2((1-2\Psi)\delta_{ij} + E_{ij}) \end{aligned} \tag{3.53}$$

where we have used the common convention that latin indices specify spatial components. Φ and Ψ are scalars while B_i is a vector and E_{ij} is a tensor. We will now use the SVT decomposition, the vector B_i is composed into a scalar and a divergenceless vector

$$B_i = \partial_i B + S_i, \quad \partial_i S_i = 0 \tag{3.54}$$

The tensor E_{ij} is decomposed into a scalar, vector and tensor part.

$$E_{ij} = 2\partial_i\partial_j E + \partial_{(i}F_{j)} + h_{ij} \tag{3.55}$$

where $\partial_j F_j = 0$ and $h_i^i = \partial_i h_{ij} = 0$. We have decomposed the 10 degrees of freedom into

- 4 scalars: Ψ,Φ, E and B

- 2 vectors F_j and S_i
- 1 tensor E_{ij}

A scalar has one degree of freedom, the vectors have two as we have removed one degree of freedom with the divergenceless requirement and the tensor also has two as we have removed one from the traceless condition and three from the requiring that it's divergenceless for each choice of j. Hence we have accounted for all degrees of freedom. The reason we write the metric perturbations in this way is that the scalar, vector and tensor parts will evolve independently which significantly simplies the treatment of perturbations[7]. However, vector perturbations are not created during inflation and will therefore be ignored in this thesis.

Before we continue there is an important point to consider. When defining the metric perturbations we have implicitly adopted a coordinate system x^μ, which corresponds to an unique threading of spacetime into lines of constant x_i and a slicing into hypersurfaces of constant x^0.

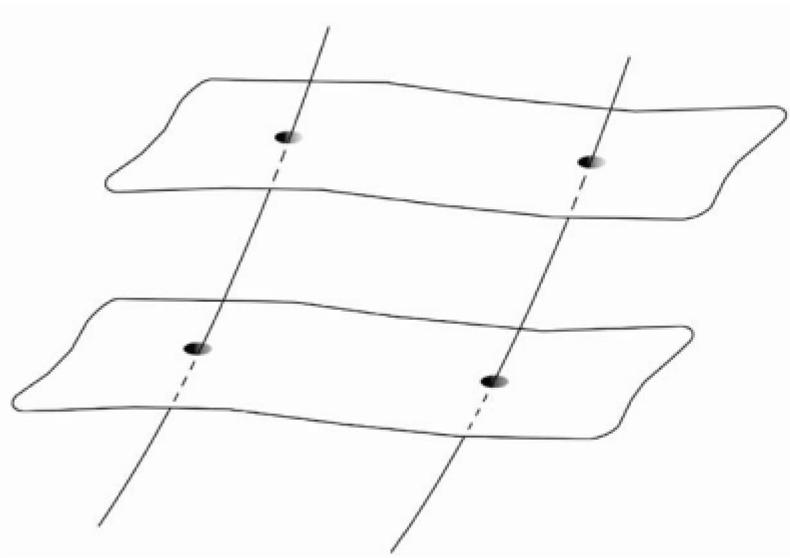


Figure 3.5: A slicing and threading of spacetime into lines and hypersurfaces.[13]

For the unperturbed FRW metric there was a preferred choice of coordinates which resulted in homogeneity and isotropy. The situation is more complicated for the perturbed metric, there is no coordinate system in which this is true. However a coordinate system must still be chosen, any set of coordinates which reduce to the unperturbed FRW metric in the absence of perturbations is allowed, such a choice is called a *gauge*[13]. Tensor perturbations are gauge invariant[7] but scalar perturbations are not and are therefore defined by the choice of gauge. One can consider a unperturbed universe with a homogeneous energy density in which a new coordinate choice will yield what appears to be perturbations, but these are entirely due to the choice of gauge. One can also consider a perturbed universe with a energy density which is inhomogeneous, but by choosing a hypersurface of constant energy which also is the hypersurface of constant time the perturbations have been removed by the choice of gauge.

To make the dependence on the gauge more explicit we consider the gauge transformation.

$$\begin{aligned} t &\rightarrow \tilde{t} = t + \xi_0(t, x_i) \\ x_i &\rightarrow \tilde{x}_i = x_i + \xi_i(t, x_i) \end{aligned} \tag{3.56}$$

The differentials of which are

$$\begin{aligned} d\tilde{t} &= (1 + \dot{\xi}_0)dt + \partial_i dx_i \\ d\tilde{x}_i &= dx_i + \dot{\xi}_i dt + \partial_j \xi_i dx_j \end{aligned} \tag{3.57}$$

Through using the invariance of the spacetime interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu$ and keeping terms to linear order one can identify the gauge transformations for the scalar perturbations as[7]

$$\begin{aligned} \tilde{\Phi} &= \Phi - \dot{\xi}_0 \\ \tilde{B} &= B + \frac{\xi_0}{a} - a\dot{\xi} \\ \tilde{E} &= E - \xi \\ \tilde{\Psi} &= \Psi + H\xi_0 \end{aligned} \tag{3.58}$$

We now want to consider how perturbations affect the density, pressure and the momentum density which are important quantities in the stress-energy tensor. It can be shown that they transform as[7]

$$\begin{aligned} \delta\tilde{\rho} &= \delta\rho - \dot{\overline{\rho}}\xi_0 \\ \delta\tilde{p} &= \delta p - \dot{\overline{p}}\xi_0 \\ \delta\tilde{q} &= \delta q + (\overline{\rho} + \overline{p})\xi_0 \end{aligned} \tag{3.59}$$

In order to perform calculations one must choose a gauge, as scalar perturbations aren't gauge invariant it is most convenient to pick a gauge and stick with it. However, to ensure that the perturbations aren't an artifact of our choice of gauge we want to express the perturbations in terms of a gauge invariant quantity. One such object is the comoving curvature perturbation which measures the curvature on comoving hypersurfaces, it is defined as

$$\mathcal{R} = \Psi - \frac{H}{\overline{\rho} + \overline{p}} \tag{3.60}$$

It is easily shown that it's gauge invariant using the gauge transformations in Eqns (3.58) and (3.59).

$$\tilde{\mathcal{R}} = \Psi + H\xi_0 - \frac{H}{\overline{\rho} + \overline{p}}(\delta q + (\overline{\rho} + \overline{p})\xi_0) = \mathcal{R} \tag{3.61}$$

One very useful property of $\mathcal{R}$ is that it's conserved on superhorizon scales. This allows us to compute it when it leaves the horizon and know that it has the same value when reentering the horizon.

When performing calculations there are several gauges one can use. The gauge freedom can be used to set two of the scalars to an arbitrary value, usually zero.Alternatively one can use one degree of freedom to make the inflaton, energy density or momentum density vanish. In this thesis we will perform calculations in the spatially flat gauge, it consists of picking $\Psi = E = 0$. The advantage with the spatially flat gauge is that the scalar perturbations are expressed entirely in terms of the perturbed inflation field, $\delta\phi$. In the spatially flat gauge $\mathcal{R}$ is related to $\delta\phi$ as

$$\mathcal{R} = \frac{H}{\dot{\phi}}\delta\phi \tag{3.62}$$

3.2.2 Quantum mechanics of the harmonic oscillator

As perturbations in the early universe are due to quantum effects we will quickly review the basics of the quantum mechanical harmonic oscillator as a basis from which we can perform analogous computations in quantum field theory. In quantum mechanics the position and momentum gets promoted to operators which satisfy the canonical commutation relation

$$[x, p] = i\hbar \tag{3.63}$$

We will work in the Heisenberg picture in which operators are time-dependent and states are time-independent. An operator A_H in the Heisenberg picture, with no explicit time dependence, satisfies the Heisenberg equation of motion

$$\frac{dA_H}{dt} = \frac{1}{i\hbar}[A_H, H] \tag{3.64}$$

Hence the time evolution of the operators is governed by the commutator between the operator and the Hamiltonian. We will mainly be concerned with the harmonic oscillator which has the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \tag{3.65}$$

It will prove useful to introduce the ladder operators a and $a^\dagger$ to find the eigenstates of the harmonic oscillator.

$$a = \sqrt{\frac{\hbar}{2m\omega}}(x + \frac{i}{m\omega}p), \quad a^\dagger = \sqrt{\frac{\hbar}{2m\omega}}(x - \frac{i}{m\omega}p) \tag{3.66}$$

They satisfy the commutation relation

$$[a, a^\dagger] = 1 \tag{3.67}$$

The Hamiltonian can be rewritten in terms of the ladder operators as

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{3.68}$$

From considering the commutation relations between the Hamiltonian and the two ladder operators it can be shown that they act as a lowering and raising operator respectively, moving between states,$|n\rangle$, with lower and higher energies.

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{3.69}$$

The vacuum state $|0\rangle$ is annihilated by the lowering operator

$$a|0\rangle = 0 \tag{3.70}$$

The excited states can be constructed by multiple applications of the raising operator.
The position operator can be written in terms of the ladder operators,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{3.71}$$

and hence by finding the time evolution of the ladder operators one finds the time evolution of the position operator.Using Eqs (3.64) and (3.68) one finds

$$\begin{aligned} \frac{da_H}{dt} &= ae^{-i\omega t} \\ \frac{da_H^\dagger}{dt} &= a^\dagger e^{i\omega t} \end{aligned} \tag{3.72}$$

where a and $a^\dagger$ are the ladder operators at $t = 0$, or equivalently in the Schrödinger picture. The time evolution of the position operator can be written in terms of a mode function $u(t) = \frac{\hbar}{2m\omega}e^{-i\omega t}$ as

$$x(t) = u(t)a + u^*(t)a^\dagger \tag{3.73}$$

3.2.3 Scalar perturbations

We will perform the calculations in the spatially flat gauge following [12] in which the metric is unperturbed and the perturbations only affect the inflation field.[12] We will start from the inflaton action, but in terms of the conformal time.

$$S = \int d\tau d^3x a(\tau)\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)] \tag{3.74}$$

In conformal time the inverse of the unperturbed FRW metric takes the form,

$$g^{\mu\nu} = \frac{1}{a(\tau)^2}\eta^{\mu\nu} \tag{3.75}$$

and combining with $\sqrt{-g} = a^3$ the inflaton action takes the form

$$S = \int d\tau d^3x[\frac{1}{2}a^2(\phi'^2 - (\nabla\phi)^2) - a^4V(\phi)] \tag{3.76}$$

where $\phi' = \partial_\tau\phi$. The perturbed inflation field can be written as

$$\phi(\tau, \mathbf{x}) = \overline{\phi}(\tau) + \delta\phi(\mathbf{x}, \tau) = \overline{\phi}(\tau) + \frac{f(\tau, \mathbf{x})}{a(\tau)} \tag{3.77}$$

where $\overline{\phi}$ is the unperturbed inflaton field. We have earlier obtained the equation of motion for the background field, in terms of conformal time it is

$$\overline{\phi}''(\tau) + 2\mathcal{H}\overline{\phi}'(\tau) + a^2\frac{dV(\overline{\phi})}{d\overline{\phi}} = 0 \tag{3.78}$$

We have defined the Hubble parameter in terms of the conformal time as $\mathcal{H} = \frac{a'}{a}$
We will now expand the action in terms of the fluctuations to second order $S \approx \overline{S} + S_1 + S_2$.

$$\begin{aligned} S \approx \int d\tau d^3x [\underbrace{\frac{1}{2}a^2(\overline{\phi}'^2 - V(\overline{\phi}))}_{\overline{S}} + \underbrace{\overline{\phi}'(f'a - a'f) - a^3 f\frac{dV}{d\overline{\phi}}}_{S_1} \\ + \underbrace{\frac{1}{2}(f'^2 - (\nabla f)^2) - \mathcal{H}ff' + \frac{1}{2}(\mathcal{H}^2 - a^2\frac{d^2V}{d\overline{\phi}^2})f^2]}_{S_2} \end{aligned} \tag{3.79}$$

We have identified the background of the action and it's first and second order expansion. The action to first order is

$$S_1 = \int d\tau d^3x[\overline{\phi}'(f'a - a'f) - a^3 f\frac{dV}{d\overline{\phi}}] = -\int d\tau d^3x[a[\overline{\phi}'' + 2\mathcal{H}\overline{\phi}' + a^2\frac{dV}{d\overline{\phi}}]f] \tag{3.80}$$

where we have integrated by parts and assumed that the fluctuations decay at spatial infinity. To first order the action is proportional to the background equations of motion and hence it vanishes.
To second order the action is

$$S_2 = \int d\tau d^3x[\frac{1}{2}(f'^2 - (\nabla f)^2) - \mathcal{H}ff' + \frac{1}{2}(\mathcal{H}^2 - a^2\frac{d^2V}{d\overline{\phi}^2})f^2] \tag{3.81}$$

From integrating the term that's proportional to f' by parts one gets

$$\begin{aligned} S_2 &= \int d\tau d^3x[\frac{1}{2}(f'^2 - (\nabla f)^2) + \frac{1}{2}(\mathcal{H}' + \mathcal{H}^2 - a^2\frac{d^2V}{d\overline{\phi}^2})f^2] \\ &= \frac{1}{2}\int d\tau d^3x[(f'^2 - (\nabla f)^2) + (\frac{a''}{a} - a^2\frac{d^2V}{d\overline{\phi}^2})f^2] \end{aligned} \tag{3.82}$$

We will now restrict to the case of a DeSitter spacetime, in which the scale factor is

$$a(\tau) \sim -\frac{1}{H\tau} \tag{3.83}$$

Using the slow roll approximation it can be shown that

$$\frac{V''}{\frac{a''}{a}} \sim \frac{\eta_v}{a^2} \tag{3.84}$$

During inflation the slow roll parameter satisfies $\eta_v \ll 1$ and hence we can simplify the action to the second order by neglecting the potential term.

$$S_2 = \frac{1}{2}\int d\tau d^3x[(f'^2 - (\nabla f)^2) + \frac{a''}{a}f^2] \tag{3.85}$$

Hence the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(f''^2 - \nabla^2 f) + \frac{1}{2}\frac{a''}{a}f^2 \tag{3.86}$$

Using the Euler-Lagrange equation one obtains the equation of motion

$$f'' - \nabla^2 f + \frac{a''}{a} f = 0 \tag{3.87}$$

This is the *Mukhanov-Sasaki equation*, by taking the Fourier transform one finds that the Fourier modes satisfies the equation

$$f_k'' + (k^2 + \frac{a''}{a}) f_k = 0 \tag{3.88}$$

We will consider this equation in two limits

- Far inside the horizon the wavelength is small and $k^2 \gg \frac{a''}{a}$, hence we get

  $$f_k'' + k^2 f_k = 0 \tag{3.89}$$

  which is a harmonic oscillator with a time-independent frequency.

- Far outside the horizon the wavelength is very long and hence $k^2 \ll \frac{a''}{a}$, which gives.

  $$f_k'' + \frac{a''}{a} f_k = 0 \tag{3.90}$$

  One solution is $f_k \propto a$, hence $\delta\phi_k = const.$ and corresponds to a frozen Fourier mode.

In fact for a DeSitter spacetime the Fourier modes of the Mukhanov-Sasaki equation have an exact solution, Eq (3.88) takes the form

$$f_k'' + (k^2 - \frac{2}{\tau^2}) f_k = 0 \tag{3.91}$$

The solution of which is

$$f_k = A \frac{e^{-ik\tau}}{\sqrt{2k}} (1 - \frac{1}{k\tau}) + B \frac{e^{ik\tau}}{\sqrt{2k}} (1 + \frac{1}{k\tau}) \tag{3.92}$$

3.2.4 Canonical quantisation

The inflaton perturbation $\delta\phi(\mathbf{x}, \tau) = \frac{f(\mathbf{x},\tau)}{a(\tau)}$ is at this point entirely classical. As we have shown, in quantum mechanics the position and momentum are promoted to operators which satisfy the canonical commutation relations. In our treatment of the perturbations so far we have used classical field theory, to quantize them we will hence need quantum field theory and the methods of canonical quantization. The field $f(\tau, \mathbf{x})$ and it's conjugate momenta $\pi(\mathbf{x}, \tau) = \frac{\partial \mathcal{L}}{\partial f'} = f'$ are required to satisfy the continuum analog of the canonical commutation relation

$$[f(\mathbf{x}, \tau), \pi(\mathbf{x}', \tau)] = \delta^{(\mathbf{3})}(\mathbf{x} - \mathbf{x}') \tag{3.93}$$

We can once again expand the field in terms of creation and annihilation operators

$$f(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} (f_k(\tau) a_{\mathbf{k}} e^{i\mathbf{kx}} + f_k^*(\tau) a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}}) \tag{3.94}$$

It's possible to show that if the creation and annihilation operators satisfy

$$\begin{aligned} [a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] &= \delta^{(3)}(\mathbf{k} - \mathbf{k}') \\ [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= [a^{\dagger}_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = 0 \end{aligned} \tag{3.95}$$

then the canonical commutation relations are satisfied aswell.
The mode function $f_k(\tau)$ in the expansion still satisfies (3.88). What we now need to consider is the boundary conditions of this equation. This consists of choosing a physical vacuum, we recall Eq (3.89) which describes the Fourier modes far into the horizon. For this case we have oscillating solutions

$$f_k = c_1 e^{ik\tau} + c_2 e^{-ik\tau} \tag{3.96}$$

We recall that the modefunction in quantum mechanics was $u(t) = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t}$, hence the most natural choice here is

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \tag{3.97}$$

This is the *Bunch-Davies vacuum.* This fixes the boundary conditions needed for specifying the solution in DeSitter space, by requiring that the solutions in Eq (3.92) reduce to the Bunch-Davies vacuum as $\tau \to -\infty$ we obtain the mode function

$$f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}\left(1 - \frac{1}{k\tau}\right) \tag{3.98}$$

Perturbations are described by random fields[13], let $g(\mathbf{x}, t)$ be such a random field. It can be Fourier expanded as

$$g(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.99}$$

We consider the ensamble average of the Fourier modes, from which we can define the *power spectrum* P_g[13]

$$\langle g(\mathbf{k}) g(\mathbf{k}') \rangle = \frac{2\pi^3}{k^3} \delta^3(\mathbf{k} + \mathbf{k}') P_g \tag{3.100}$$

In the case where g is a quantum operator the ensemble average can be identified with the expectation value of the vacuum state[13]. We will now find the power spectrum of the inflaton perturbations.

$$\langle \delta\phi_{\mathbf{k}} \delta\phi_{\mathbf{k}'} \rangle = \frac{2\pi^3}{k^3} \delta^3(\mathbf{k} + \mathbf{k}') P_{\delta\phi}(k) \tag{3.101}$$

From evaluating the expectation value one finds

$$P_{\delta\phi}(k) = \frac{k^3}{2\pi^2} |f_k(\tau)|^2 \tag{3.102}$$

Evaluating on superhorizon scales this gives

$$P_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \tag{3.103}$$

In the spatially flat gauge the perturbations in the inflaton field are related to $\mathcal{R}$ as

$$\mathcal{R} = -\frac{H}{\dot{\phi}}\delta\phi \tag{3.104}$$

Using the slow roll parameter ε we get

$$\mathcal{R} = -\frac{1}{\sqrt{2\varepsilon}M_{pl}}\delta\phi \tag{3.105}$$

Hence the power spectrum for $\mathcal{R}$ is

$$P_{\mathcal{R}}(k) = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M_{pl}^2}\bigg|_{k=aH} \tag{3.106}$$

The perturbations are quantum mechanical in origin but lose their quantum nature on superhorizon scales and can hence be described by a classical random field $\mathcal{R}$[13]. As $\mathcal{R}$ is constant on superhorizon the power spectrum is often evaluated at horizon crossing, even though the power spectrum for the inflaton perturbations was evaluated at superhorizon scales. As the quantities involved are slowly changing[9] this is a good approximation.

3.2.5 Tensor perturbations

We will now consider tensor perturbations, in the spatially flat gauge the tensor perturbations yield the line element

$$ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx_i dx_j \tag{3.107}$$

Through Einsteins equations one can show that this corresponds to two polarization modes of gravitational waves, h_+ and $h_\times$. By inserting the perturbed metric into the Einstein-Hilbert action one obtains two copies of a free scalar field with $\phi_{+,\times} = \frac{2}{M_{pl}}h_{+,\times}$, accounting for the two polarization modes. Hence the power spectrum for tensor perturbations is

$$P_t(k) = \frac{8}{M_{pl}^2}\left(\frac{H}{2\pi}\right)^2 \tag{3.108}$$

3.2.6 Cosmological parameters

Now that we have calculated the power spectrum for both the tensor and scalar perturbations we can define the tensor to scalar ratio as

$$r = \frac{P_t(k)}{P_{\mathcal{R}}(k)} = 16\varepsilon \tag{3.109}$$

The tensor to scalar ratio is a direct measure of the energy scale of inflation as[7]

$$V^{\frac{1}{4}} \sim \left(\frac{r}{0.01}\right)^{\frac{1}{4}} 10^{16}\text{GeV} \tag{3.110}$$

A deSitter equation of state during inflation produces a spectrum which is scale invariant, however the equation of state can be quasi de Sitter and still sustain inflation. Therefore we expect that the power spectrum is nearly scale independent, we can define the spectral index which measures deviation from measures deviations from scale invariance

$$n_s - 1 = \frac{dP_s}{d\log k} \tag{3.111}$$

In a similar fashion we define the running of the spectral index as

$$\alpha = \frac{dn_s}{d\log k} \tag{3.112}$$

We now want to express these deviations from scale invariance in terms of the slow roll parameters. It can be shown that the spectral index, at first order, is given by

$$n_s = 1 + 2\eta_v - 4\epsilon_v \tag{3.113}$$

As the potential slow roll parameters are more commonly used it will prove useful to express the above defined quantities in them. The two sets of slow roll parameters are related as

$$\begin{aligned} \varepsilon &\approx \epsilon_v \\ \eta &\approx \eta_v - \epsilon_v \end{aligned} \tag{3.114}$$

It will also be useful to introduce a third slow roll parameter in an analogous way[16].

$$\zeta_v^2 = M_{pl}\frac{V'''(\phi)V'(\phi)}{V(\phi)^2} \tag{3.115}$$

In terms of the potential slow roll parameters one gets [16] [7]

$$\begin{aligned} r &= 16\epsilon_v \\ n_s &= 1 - 6\epsilon_v + 2\eta_v \\ \alpha_s &= -24\epsilon_v^2 + 16\epsilon_v\eta_v - 2\zeta_v^2 \end{aligned} \tag{3.116}$$

Inflationary models predict different values of these quantities and can thus be compared with the observed values from measurements of the CMB. Note that since the slow roll parameters are small the deviations from scale invariance is small, the running of the spectral index is even smaller as it is second order in the slow roll parameters. We will give an example how these quantities are calculated below.

3.2.7 Observational constraints

The CMB is one of the main probes used in cosmology today. It has been used to constrain cosmological parameters by several experiments, most notably by COBE, WMAP and the Planck experiment. In 2015 the Planck experiments results considerably constrained the allowed values of several cosmological parameters, including the tensor to scalar ratio, spectral index and running of the spectral index[15]. Thorugh narrowing the allowed parameter space of the quantities inflationary models can be ruled out.

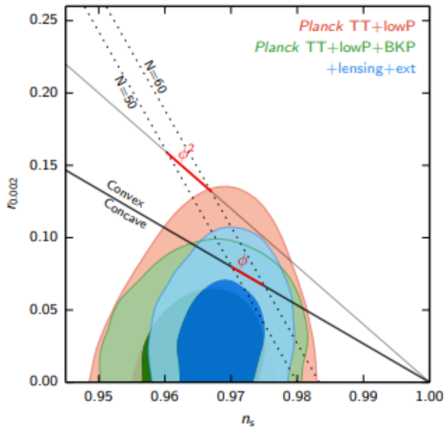


Figure 3.6: Constraints on the tensor to sclar ratio and spectral index from Planck [15]

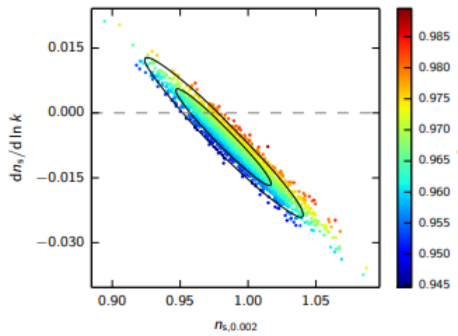


Figure 3.7: Constraints on the running of the spectral index and the running of the spectral index from Planck [15]

3.2.8 Inflation from a quartic potential

As an example we consider the potential $V(\phi) = \lambda\phi^4$, the tensor to scalar ratio and spectral index corresponding to this potential can be calculated to yield predictions to the measured values of these quantities. The slow roll parameters are

$$\begin{aligned} \epsilon_v &= \frac{M_{pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{8M_{pl}^2}{\phi^2} \\ \eta_v &= M_{pl}\frac{V''(\phi)}{V(\phi)} = M_{pl}^2\frac{12}{\phi^2} \end{aligned} \tag{3.117}$$

From which we can find the field value at the end of inflation

$$\epsilon_v(\phi_{end}) = 1 \rightarrow \phi_{end} = \sqrt{8}M_{pl} \tag{3.118}$$

The number of efolds of inflation is then found to be

$$N(\phi) = \frac{1}{\sqrt{2}M_{pl}}\int_{\phi_{end}}^{\phi}\frac{1}{\sqrt{\epsilon_v}}d\phi = \frac{1}{4M_{pl}^2}\int_{\phi_{end}}^{\phi}\phi d\phi \approx \frac{\phi^2}{8M_{pl}^2} \tag{3.119}$$

The tensor to scalar ratio and the spectral index can then be calculated

$$\begin{aligned} r &= 16\epsilon_v = \frac{128M_{pl}^2}{\phi^2} = \frac{16}{N} \\ n_s &= 1 + 2\eta_v - 6\epsilon_v = 1 - \frac{24M_{pl}^2}{\phi^2} = 1 - \frac{3}{N} \end{aligned} \tag{3.120}$$

3.3 Inflation through non-minimal coupling to gravity

In this we section review one type of modified theory which consists of adding non-minimal coupling to gravity and analyze inflation in such a theory. The basic idea is to reduce this nonminimally coupled theory of gravity to one which is minimally coupled so one can use the standard methods of analysis for an inflationary model. Consider an action of the form

$$S = \int d^4x\sqrt{-g}[\frac{1}{2}M_{pl}^2 f(\phi)R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)] \tag{3.121}$$

in which the nonminmal coupling is introduced through the inclusion of $f(\phi)$, and hence the action is not in the canonical Einstein form as in the minimally coupled case which was our basis for analyzing inflation from a scalar field. However the action can be brought into the Einstein form through the use of a conformal transformation and hence we can still use tools for analyzing inflation from a scalar field. How this conformal transformation is implemented will be explained below but the derivation will be left to the Appendix. The conformal transformation of the metric is

$$\hat{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \tag{3.122}$$

as the metric transforms the Ricci scalar will also transform as. [11, 18]

$$\hat{R} = \frac{1}{\Omega^2}(R - \frac{6}{\Omega}\Box\Omega) \tag{3.123}$$

Inserting these expression into the action and choosing $f(\phi) = \Omega^2$ will transform the action into the Einstein form, we will therefore call this frame the Einstein frame. However, performing the canonical transformation resulted in noncanonical kinetic terms and a modified potential. The potential was modified as

$$V(\phi) \to \frac{V(\phi)}{f^2(\phi)} \tag{3.124}$$

and hence we will define the potential in the Einstein frame as.

$$\hat{V} = \frac{V(\phi)}{f^2(\phi)} \tag{3.125}$$

The noncanonical kinetic term can be brought into a canonical form through the use of a field rescaling $\phi \to \hat{\phi}$ which is specified by the differential equation

$$\frac{d\hat{\phi}}{d\phi} = F(\phi) = \sqrt{\left(\frac{3}{2}M_{pl}^2\frac{f'(\phi)^2}{f(\phi)^2} + \frac{1}{f}\right)} \tag{3.126}$$

This will to an action in the Einstein form with canonical kinetic terms

$$S = \int d^4x\sqrt{-\hat{g}}[\frac{1}{2}M_{pl}^2\hat{R} - \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\phi - \hat{V}] \tag{3.127}$$

In this frame the standard method of analysis for scalar fields is applicable.

Chapter 4

Higgs inflation

4.1 Classical analysis

In this section we analyse the predictions of Higgs inflation, this is also done in [12, 17, 16, 21]. The Higgs doublet in the unitary gauge is given by

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \phi(x) \end{pmatrix} \approx \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \phi(x) \end{pmatrix} \tag{4.1}$$

where the vacuum expectation value has been neglected, this is justified as the field takes superplanckian values during inflation and hence $\phi \gg v$ as $v \approx 246$ GeV. The case with the minimal coupling between gravity and the Higgs field leads to unrealistic requirements of the Higgs mass[12].Instead we consider a nonminimal coupling between the Higgs doublet and gravity of the form

$$\mathcal{L}_{coupling} = \xi \Phi \Phi^{\dagger} R \tag{4.2}$$

where R is the Ricci scalar. The Lagrangian describing the Higgs sector together with the coupling is then found to be

$$\mathcal{L}_{Higgs} + \mathcal{L}_{coupling} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{4}\lambda\phi^4 + \frac{1}{2}\xi\phi^2 R \tag{4.3}$$

Again, as the vacuum expectation value is completely neglible the potential is well approximated by a quartic potential $V(\phi) = \frac{\lambda}{4}\phi^4$. We know want to consider the Higgs field as the inflaton, the first step is to set up the action of the Higgs sector with the coupling combined with the Einstein-Hilbert action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{pl}^2 f(\phi)R - \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right] \tag{4.4}$$

where $f(\phi) = 1 + \frac{\xi\phi^2}{M_{pl}^2}$, and hence the action is not in the canonical Einstein-Hilbert form. With the use of a conformal transformation to the Einstein frame and a field rescaling $\phi \to \hat{\phi}$ the action is in the canonical kinetic and canonical Einstein-Hilbert form with an action. In the Einstein frame the standard methods of analysis for scalar fields is applicable, the potential in the Einstein frame is found to be

$$\hat{V}(\phi) = \frac{1}{4}\frac{\lambda\phi^4}{(1 + \frac{\xi\phi^2}{M_{pl}^2})^2} \tag{4.5}$$

In the large field limit $\phi \gg \frac{M_{pl}}{\sqrt{\xi}}$ the potential flattens out and approaches a constant value $V_0 = \frac{\lambda M_{pl}^4}{4\xi^2}$ allowing slow roll inflation to take place.

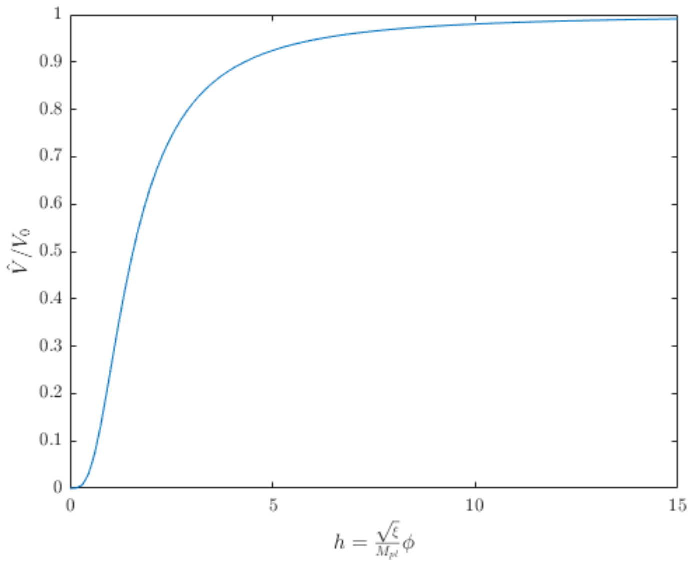


Figure 4.1: The potential in the Einstein frame in terms of $V_0 = \frac{\lambda M_{pl}^4}{4\xi^2}$ against a rescaled Higgs field h.

To do the analysis in the Einstein frame we first find the field in terms of the rescaled field in the large field limit. They are related through

$$\frac{d\hat{\phi}}{d\phi} = \sqrt{\frac{1}{f} + \frac{3}{2}M_{pl}^2\left(\frac{f'(\phi)}{f(\phi)}\right)^2} = \sqrt{\frac{f + {}^{6\xi^2\phi^2}/_{M_{pl}^2}}{f^2}} \tag{4.6}$$

We will consider this relation in two limits

- Small field limit $\phi \ll \frac{M_{pl}}{\sqrt{\xi}}$: Eq (4.6) simplifies significantly and we obtain.

$$\phi \sim \hat{\phi} \tag{4.7}$$

  The factor involved in the conformal transformation also simplifies $\Omega^2 \approx 1$. Hence at low energies the fields are the same the potential in the Einstein frame reduces to the regular Higgs potential. Thus the two frames are effectively the same.

- Large field limit: $\phi \gg \frac{M_{pl}}{\sqrt{\xi}}$ we get $f(\phi) \approx \frac{\xi\phi^2}{M_{pl}^2}$ and hence

$$\frac{d\hat{\phi}}{d\phi} \approx \frac{\sqrt{6}M_{pl}}{\phi} \tag{4.8}$$

  from which we find.

$$\phi = \frac{M_{pl}}{\sqrt{\xi}}\exp\left(\frac{\hat{\phi}}{\sqrt{6}M_{pl}}\right) \tag{4.9}$$

  However we should note that the constant in front is unspecified by any boundary condition. It has been set in such a way that the large field limit agrees in both frames.

The large field limit in the rescaled field is $\hat{\phi} \gg M_{pl}$, and the potential in the Einstein frame in terms of the rescaled field is

$$\hat{V}(\hat{\phi}) = \frac{\lambda M_{pl}^4}{4\xi^2}\left(1 + \exp\left(-\frac{2\hat{\phi}}{\sqrt{6}M_{pl}}\right)\right)^{-2} \approx \frac{\lambda M_{pl}^4}{4\xi^2}\left(1 - 2\exp\left(-\frac{2\hat{\phi}}{\sqrt{6}M_{pl}}\right)\right) \quad (4.10)$$

The conclusion regarding the flattening of the potential at large field values remains true after expressing the potential in terms of the rescaled field and it approaches the same value V_0. From this potential the slow roll parameters are readily calculated

$$\varepsilon_v = \frac{M_{pl}^2}{2}\left(\frac{\hat{V}'(\hat{\phi})}{\hat{V}(\hat{\phi})}\right)^2 \approx \frac{4}{3}\exp\left(-\frac{4\hat{\phi}}{\sqrt{6}M_{pl}}\right) = \frac{3}{4}\eta_v^2 \quad (4.11)$$

$$\eta_v = M_{pl}^2\frac{\hat{V}''(\hat{\phi})}{\hat{V}(\hat{\phi})} \approx -\frac{4}{3}\exp\left(-\frac{2\hat{\phi}}{\sqrt{6}M_{pl}}\right) \quad (4.12)$$

$$\zeta^2 = M_{pl}^4\frac{\hat{V}'''(\hat{\phi})\hat{V}'(\hat{\phi})}{\hat{V}(\hat{\phi})^2} \approx \frac{16}{9}\exp\left(-\frac{4\hat{\phi}}{\sqrt{6}M_{pl}}\right) = \eta_v^2 \quad (4.13)$$

Next we want to calculate the number of e-folds of inflation, inflation ends at $\hat{\phi}_{end}$, defined when $\varepsilon_v(\hat{\phi}_{end}) = 1$. One finds

$$\hat{\phi}_{end} = -\frac{\sqrt{6}}{4}M_{pl}\log\frac{3}{4} \quad (4.14)$$

Hence the number of efolds between the initial value of the field $\hat{\phi}$ to the final value is.

$$N = \frac{1}{\sqrt{2}M_{pl}}\int_{\hat{\phi}_{end}}^{\hat{\phi}}\frac{1}{\sqrt{\varepsilon_v}}d\hat{\phi} \approx -\frac{1}{\eta} \quad (4.15)$$

The slow roll parameters can be expressed through the number of e-folds of inflation and hence the spectral index, tensor to scalar ratio and the running of the spectral index is found depending on the number of e-folds of inflation. The spectral index is given by

$$\alpha = -24\varepsilon_v^2 + 16\varepsilon_v\eta_v - 2\zeta_v^2 = -\frac{27}{2}\frac{1}{N^4} - \frac{3}{4}\frac{1}{N^3} - \frac{2}{N^2} \quad (4.16)$$

and the running of the spectral index is found to be.

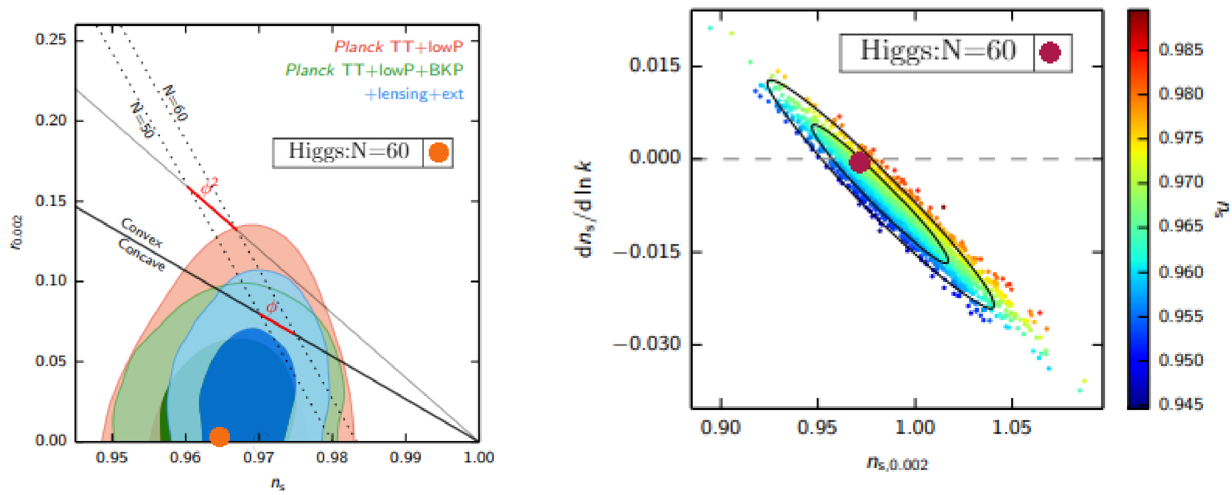
$$n_s = 1 - 6\varepsilon_v + 2\eta_v = 1 - \frac{9}{2}\frac{1}{N^2} - \frac{2}{N} \quad (4.17)$$

The tensor to scalar ratio is

$$r = 16\varepsilon_v = \frac{12}{N^2} \quad (4.18)$$

For $N = 60$ one predicts the following values for α, n_s and r

$$\begin{aligned} n_s &= 0.965 \\ \alpha &= -5.58 \times 10^{-4} \\ r &= 3.3 \times 10^{-3} \end{aligned} \quad (4.19)$$



(a) Tensor to scalar ratio and spectral index. (b) Spectral index and the running of the spectral index.

Figure 4.2: These figures shows the restriction on the tensor to scalar ratio, spectral index and the running of the spectral index from the Planck experiment combined with the predictions from Higgs inflation for $N = 60$ efolds. [15]

Using the observed amplitude of $P_\mathcal{R}$ a relation between the nonminimal coupling ξ and the mass of the Higgs boson can be found. The power spectrum is

$$P_\mathcal{R} = \frac{1}{8\pi^2}\frac{1}{\varepsilon_v}\frac{H^2}{M_{pl}^2} \tag{4.20}$$

Using $\varepsilon_v = \frac{3}{4N^2}$, $H^2 \approx \frac{V}{3M_{pl}^2}$ and approximating the potential as $V \approx \frac{\lambda M_{pl}^4}{4\xi^2}$ one finds

$$P_\mathcal{R} \approx \frac{\lambda N^2}{72\pi^2\xi^2} \tag{4.21}$$

Using the relation $m_{Higgs}^2 = 2\lambda v^2$ the coupling can be related to the Higgs boson mass as

$$\xi \approx \frac{N}{12\pi^2\sqrt{P_\mathcal{R}}}\frac{m_{Higgs}}{v} \tag{4.22}$$

From observation we know that $P_\mathcal{R} \approx 2\times10^{-9}$, $v = 246$ GeV, $N \approx 60$ and $m_{Higgs} =$ 125 GeV the coupling constant takes the value

$$\xi \approx 18 \times 10^3 \tag{4.23}$$

This value of the coupling constant is larger than what is expected which will have consequences when taking quantum corrections into account as we will do in the next section.

4.2 Quantum Analysis

In this section we'll discuss some basic aspects of the quantum analysis. In the classical analysis the coupling constants are fixed but when quantum effects are considered they change with the energy. The Higgs boson interacts with other particles in the Standard Model and this interaction effectively changes the coupling constants, they start to "run" with the energy. The exact behaviour is governed by the renormalization group equation

$$\frac{d\lambda}{dt} = \beta_\lambda \tag{4.24}$$

where β_λ is the beta function of Higgs self coupling and $t = \log \frac{\phi}{\mu}$ where μ is the renormalization point. At 1-loop it takes the form

$$\beta_\lambda = \frac{1}{(4\pi)^2}\Big(24\lambda^2 + 12\lambda y_t^2 - 9\lambda(g^2 + \frac{1}{3}g'^2) - 6y_t^4 + \frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g'^2 g^2\Big) \tag{4.25}$$

which involves the gauge couplings g, g' and the Yukawa coupling y_t apart from λ itself, hence it is necessary to run these couplings aswell to ensure accurate results. The Yukawa coupling has a 1-loop beta function

$$\beta_{y_t} = \frac{1}{(4\pi)^2}\Big(\frac{9}{2}y_t^3 - 8g_3^2 y_t - \frac{9}{4}g^2 y_t - \frac{17}{12}g'^2 y_t\Big) \tag{4.26}$$

which also involves the third gauge coupling g_3. The introduction of the nonminimal coupling constant also gives rise to a beta function for ξ

$$\beta_\xi = \frac{1}{(4\pi)^2}(\xi + \frac{1}{6})(12\lambda + 6y_t^2 - \frac{9}{2}g^2 - \frac{3}{2}g'^2) \tag{4.27}$$

The 6 coupled differential equations for the three gauge couplings, the Yukawa coupling, the Higgs self coupling and the nonminimal coupling can be solved numerically to obtain the running of the couplings.The result of this for the running of λ is seen in Figure 4.3.

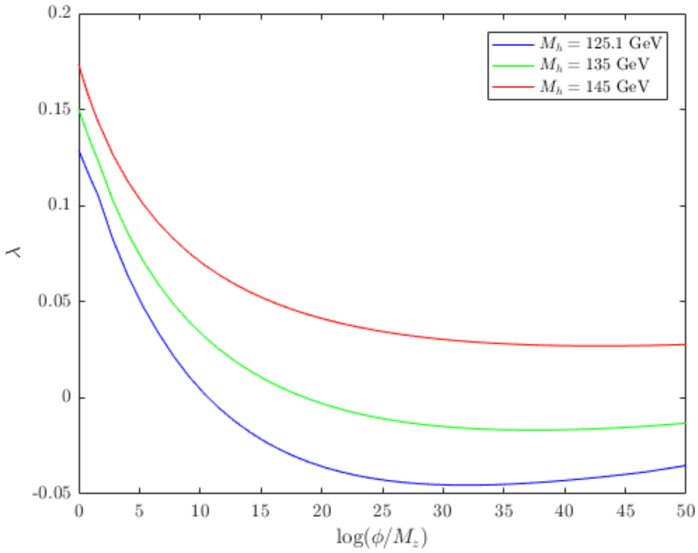


Figure 4.3: The running of λ for three different values of the Higgs mass from a 1-loop analysis with the renormalization point $\mu = M_Z$. The mass of the top quark has been set to $M_t = 172.44$ GeV

As can be seen in Figure 4.3 for certain values of the Higgs mass the coupling constant becomes negative for large field values. A negative value can signal metastability or instability depending on its magnitude, for more details see [20]. However the running is dependent on several initial conditions, not just the Higgs mass but also the mass of the top quark and the gauge coupling g_3. From the standpoint of Higgs inflation we require that λ is positive so that the inflationary potential doesn't acquire a second minimum[21]. Taking this requirement into account one can find that Higgs inflation sets requirements on the mass of the Higgs boson and the top quark. By comparing with the mass obtained in particle physics experiments Higgs inflation can be tested. However, just requiring that the potential doesn't require a second minimum isn't enough. The values of the cosmological parameters such as the tensor to scalar ratio, spectral index and the running of the spectral index depend on the shape of the potential which also relies on running of the coupling constant and loop corrections to the potential. In this way one can analyze the potential when quantum corrections are taken into account and obtain a relation between the cosmological parameters and the mass of the Higgs and top quark aswell as the value of g_3. As Higgs inflation needs to satisfy both the experiments on the cosmological parameters and on particle physics data this places more stringent demands than many other inflationary theories. A thorough analysis taking quantum corrections into account has been performed in [21, 16].

The large value of the nonminimal coupling constant has caused some concern. The inflationary regime is believed to be an effective field theory which is only valid up to some energy cutoff Λ. However, the cutoff for a nonminimally coupled theory is[16]

$$\Lambda = \frac{M_{pl}}{\xi} \tag{4.28}$$

while inflation takes place at an energy scale $\phi \gg \frac{M_{pl}}{\sqrt{\xi}}$. For $\xi \gg 1$ inflation takes place at an energy scale which is far greater than the cutoff of the effective theory. The effective field theory contains higher dimensional operators which are suppressed by this cutoff, but if inflation takes place above the cutoff these operators will be relevant and may spoil the flatness of the potential required for inflation. One aspect worthy to consider is that it's the combination $\frac{\lambda}{\xi^2}$ which has to be small to match the amplitude of scalar perturbations. In the classical analysis λ was given by it's tree level value, but if it runs to small values in the inflationary regime a smaller value of ξ is needed. It remains possible that $\xi \sim 1$ in the inflationary regime in which case the cutoff approaches the inflationary regime in which the effective field description is valid once again. [21]

4.3 Conclusion

We have performed a tree level analysis of Higgs inflation with a nonminimal coupling to gravity, the value of the nonminimal coupling was found to be $\xi \sim 10^4$. The obtained value of the tensor to scalar ratio, spectral index and the running of the spectral index is within the allowed regions of the Planck experiment(2015). However, this isn't the ultimate test of Higgs inflation. One must perform an analysis taking quantum effects into account to ensure that Higgs inflation is a viable theory. The large value of the nonminimal coupling ξ which was obtained in the tree level analysis indicates unitarity problems for Higgs inflation as inflation takes place above the cutoff of the effective field theory description but an analysis taking into account the running of the coupling constants must be performed in order to assure this.

Acknowledgements

First of all I would like to thank my supervisor Lorenzo Ruggeri for patiently answering all my questions and providing excellent help throughout the entire project. Secondly I would like to thank all my friends in the Bachelor Programme for making these three years more enjoyable. Lastly I would like to thank my family for their love and support.

Chapter 5

Appendix

5.1 Conformal transformations of the metric

A conformal transformation is a transformation which locally scales lengths. As a means to analyze Higgs inflation a conformal transformation combined with a field rescaling is needed to bring the action into a form with canonical Einstein-Hilbert action and canonical kinetic terms for the scalar field. This section follows [18].
The action in the Jordan frame is given by

$$S = \int d^4x\sqrt{-g}(\frac{1}{2}M_{pl}^2 f(\phi)R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)) \tag{5.1}$$

where $f(\phi)$ is positive definite function of the scalar field, R is the Ricci scalar and M_{pl} is the Planck mass. We then consider conformal transformations of the metric which have the form,

$$\hat{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \tag{5.2}$$

hence the inverse metric transforms as

$$\hat{g}^{\mu\nu} = \frac{1}{\Omega^2}g^{\mu\nu} \tag{5.3}$$

and the determinant of the metric transforms as.

$$\sqrt{-\hat{g}} = \Omega^4(x)\sqrt{-g} \tag{5.4}$$

Under this transformation the Christoffel symbols will change as we change the metric, this implies that the Ricci scalar will transform aswell, one finds[11, 18]

$$\hat{R} = \frac{1}{\Omega^2}(R - \frac{6}{\Omega}\Box\Omega) \tag{5.5}$$

where $\Box\Omega = g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega]$. We then apply these transformations to the first term in (5.1), using (5.5) and (5.4) we get

$$\begin{aligned} \int d^4x\sqrt{-g}\frac{M_{pl}^2}{2}f(\phi)R &= \int d^4x\frac{\sqrt{-\hat{g}}}{\Omega^4}\frac{M_{pl}^2}{2}f(\phi)[\Omega^2\hat{R} + \frac{6}{\Omega}\Box\Omega] \\ = \int d^4x\sqrt{-\hat{g}}\frac{M_{pl}^2}{2}&[\frac{f(\phi)}{\Omega^2}\hat{R} + \frac{6f(\phi)}{\Omega^5}\Box\Omega] \end{aligned} \tag{5.6}$$

Hence if we identify $\Omega^2 = f(\phi)$ the Einstein-Hilbert canonical form is obtained, this frame is therefore often called the *Einstein frame*. The price we pay for this is an added term to the action

$$\int d^4x\sqrt{-\hat{g}}\frac{M_{pl}^2}{2}\frac{6}{\Omega^3}\Box\Omega \tag{5.7}$$

where we have written $f(\phi)$ in terms of Ω. We want to express $\Box$ in terms of the new metric, the derivative is the same in the new metric since the position itself is unaffected by the conformal transformation. Using (5.3) and (5.4) one gets

$$\begin{aligned}\Box\Omega &= \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega] = \frac{\Omega^4}{\sqrt{-\hat{g}}}\partial_\mu[\frac{\sqrt{-\hat{g}}}{\Omega^4}\Omega^2\hat{g}^{\mu\nu}\partial_\nu\Omega] \\ &= \frac{\Omega^4}{\sqrt{-\hat{g}}}\partial_\mu[\frac{\sqrt{-\hat{g}}}{\Omega^2}\hat{g}^{\mu\nu}\partial_\nu\Omega] = -2\Omega\hat{g}^{\mu\nu}\partial_\nu\Omega\partial_\mu\Omega + \Omega^2\hat{\Box}\Omega\end{aligned} \tag{5.8}$$

Using (5.8) the second term (5.7) can be rewritten as

$$\int d^4x\sqrt{-\hat{g}}3M_{pl}^2(\frac{-2}{\Omega^2}\hat{g}^{\mu\nu}\partial_\nu\Omega\partial_\mu\Omega + \frac{1}{\Omega}\hat{\Box}\Omega) \tag{5.9}$$

Integrating the first term by parts and using the metric compatibility of the covariant derivative to rewrite the boundary term as a covariant derivative which vanishes.

$$\begin{aligned}&\int d^4x\sqrt{-\hat{g}}3M_{pl}^2(\frac{1}{\Omega}\hat{g}^{\mu\nu}\nabla_\mu\nabla_\nu\Omega - 2\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega) \\ &= -\int d^4x\sqrt{-\hat{g}}3M_{pl}^2\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega\end{aligned} \tag{5.10}$$

We now want to reintroduce $f(\phi)$ again, this gives

$$-\int d^4x\sqrt{-\hat{g}}\frac{3M_{pl}^2}{4f^2}\hat{g}^{\mu\nu}\partial_\mu f\partial_\nu f \tag{5.11}$$

The part of the action that governs the scalar field transforms as

$$\int d^4x\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)] = \int d^4x\sqrt{-\hat{g}}[-\frac{1}{2}\frac{1}{f(\phi)}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi\hat{\nabla}\phi - \hat{V}] \tag{5.12}$$

where $\hat{V} = \frac{V}{f(\phi)^2}$ is the potential in the Einstein frame. The total action (5.1) in the Einstein frame is hence given by

$$S = \int d^4x\sqrt{-\hat{g}}[\frac{M_{pl}^2}{2}\hat{R} - \frac{3M_{pl}^2}{4f^2}\hat{g}^{\mu\nu}\partial_\mu f\partial_\nu f - \frac{1}{2}\frac{1}{f(\phi)}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \hat{V}] \tag{5.13}$$

The next step is to rescale the fields which gives canonical kinetic terms aswell. Note that $\partial_\mu f = f'(\phi)\partial_\mu\phi$ and hence we look for a field rescaling $\phi \to \hat{\phi}$ such that

$$-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} = -\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi(\frac{3}{2}M_{pl}^2\frac{f'(\phi)^2}{f(\phi)^2} + \frac{1}{f}) \tag{5.14}$$

We assume the field definition can be written in the form

$$\frac{d\hat{\phi}}{d\phi} = F(\phi) \tag{5.15}$$

from which one finds.

$$F(\phi) = \sqrt{\Big(\frac{3}{2}M_{pl}^2\frac{f'(\phi)^2}{f(\phi)^2} + \frac{1}{f}\Big)} \tag{5.16}$$

In terms of the rescaled field the action can be written with canonical kinetic terms giving

$$S = \int d^4x\sqrt{-\hat{g}}[\frac{M_{pl}^2}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \hat{V}(\hat{\phi})] \tag{5.17}$$

5.2 Group Theory

In this section we will develop some of the group theory that is needed in the section on Classical Field Theory, it is based on [1, 2, 5].
A group G is a set of elements $g \in G$ together with a multiplication law

$$\begin{aligned} &G \times G \to G \\ &(g_1, g_2) \mapsto g_1 * g_2 \end{aligned} \tag{5.18}$$

satisfying the axioms

1. **Associativity**:For any $g_1, g_2, g_3 \in G$, $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$
2. **Identity**: There exists an identity element $e \in G$ such that $e * g = g * e = g$ for any $g \in G$
3. **Inverse**: For every $g \in G$ there exists an inverse g^{-1} such that $g * g^{-1} = g^{-1} * g = e$

Some additional remarks on groups are

- A subgroup $H \subset G$ is a subset of the elements of the group G that form a group themselves with the same multiplication law.
- A group with the property that $g_1 * g_2 = g_2 * g_1$ for all $g_1, g_2 \in G$ is called *abelian*, if it doesn't satisfy this property it is *nonabelian*
- Groups can be divided into continuous and discrete groups, we will almost exclusively consider continuous groups

Some examples of groups include

- $\mathbf{U(1)}$: The group of complex numbers with $|z| = 1$ under multiplication. Clearly $|z_1 z_2| = |z_1||z_2| = 1$, the identity element is $z = 1$ and the inverse is z^{-1} which has the property $|z^{-1}| = \frac{1}{|z|} = 1$
- $\mathbf{O(n)}$: The group of orthogonal $n \times n$ matrices under matrix multiplication, $OO^T = O^T O = 1$. Consider the product of two orthogonal matrices O_1 and O_2, $(O_1 O_2)^T (O_1 O_2) = O_2^T O_1^T O_1 O_2 = 1$ and hence the product is orthogonal aswell. The determinant of an orthogonal matrix is $det(O) = \pm 1$ and therefore there exists an inverse.

- **SO(n)** We consider orthogonal matrices but limit ourselves to the case $det(O) = 1$. We then get the group $SO(n)$ which is a subgroup $SO(n) \subset O(n)$,the S stands for "special" and means that the determinant is one. Let $A, B \in SO(n)$, then $det(AB) = det(A)det(B) = 1$ and $AB \in SO(n)$.

- **U(n)**: The group of unitary $n \times n$ matrices under matrix multiplication, $U^\dagger U = UU^\dagger = 1$. We can then proceed in a similar way as in $O(n)$ to prove that it satisfies the group properties, we can also restrict to the case $detU = 1$ similarly.

We will review some properties of Lie groups. A Lie group is a continuous group that is specified by a set of parameters, $U(1)$ and $SU(n)$ are two examples of Lie groups. Suppose that every element of the Lie group is specified by a set of parameters θ_a, we start by considering elements close to the identity

$$g(d\theta_a) = 1 + iT_a d\theta_a + \mathcal{O}((d\theta)^2) \tag{5.19}$$

where T_a are the *generators* of the Lie group. By repeatedly applying this relation for infinitesimal elements any finite element can be obtained as.

$$g(\theta_a) = exp(i\theta_a T_a) \tag{5.20}$$

The generators of the Lie group satisfy a Lie algebra of the form

$$[T_a, T_b] = if_{abc}T_c \tag{5.21}$$

where f_{abc} are the structure constants of the Lie algebra.
Example
SU(n)is the group of special unitary $n \times n$ matrices.We start by considering group elements close to the identity, $U = 1 + i\theta_a T_a + \mathcal{O}(\theta^2)$, the unitary constraint gives.

$$UU^\dagger = (1 + i\theta_a T_a)(1 - i\theta_a T_a^\dagger) = 1 + i\theta_a(T_a - T_a^\dagger) + \mathcal{O}(\theta_a^2) \tag{5.22}$$

From this we conclude that the generators must be hermitian, $T_a = T_a^\dagger$. An element of $SU(N)$ is given by

$$U(\theta_a) = exp(i\theta_a T_a) \tag{5.23}$$

Using $det(e^A) = e^{tr(A)}$ and $detU = 1$ we find that

$$det(U) = exp(i\theta_a Tr(T_a)) = 1 \Rightarrow Tr(T_a) = 0 \tag{5.24}$$

There are $2N^2$ parameters in a unitary $N \times N$ matrix, the hermitian and traceless condition on the generators reduce the parameters to $N^2 - 1$, and these are also the number of generators we have. We now consider $SU(2)$ which is one of the most important Lie groups in physics. It has three generators that are 2×2, traceless and hermitian. One choice is

$$\tau_a = \frac{\sigma_a}{2} \tag{5.25}$$

where σ_a are the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.26}$$

The Pauli matrices satisfy the identity $\sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$ and hence the Lie algebra for $SU(2)$ is

$$[\tau_a, \tau_b] = \frac{1}{4}[\tau_a, \tau_b] = \epsilon_{abc}\tau_c \tag{5.27}$$

and the structure constants are $f_{abc} = \epsilon_{abc}$.

Bibliography

[1] Joseph Conlon, *Lecture notes on Classical Field Theory*, http://www-thphys.physics.ox.ac.uk/people/JosephConlon/C6Notes/classic.pdf

[2] Michael Peskin Daniel V Schroeder, *An Introduction to Quantum Field Theory*. Perseus Books. Reading, Massachusetts,1995

[3] Matthew D. Schwartz *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014

[4] Cristopher G Tully, *Elementary Particle Physics in a Nutshell*. Princeton University Press, 2011.

[5] Christoph Ludeling, *Group Theory(For Physicists* http://www.th.physik.uni-bonn.de/nilles/people/luedeling/grouptheory/data/grouptheorynotes.pdf

[6] Bertrand Delamotte, A hint of renormalization, arXiv:arXiv:hep-th/0212049v3

[7] Daniel Baumann, *TASI Lectures on Inflation*, arXiv:0907.5424 [hep-th]

[8] William H. Kinney, *TASI Lectures on Inflation*, arXiv:0902.1529v2 [astro-ph.CO]

[9] Scott Dodelson, *Modern Cosmology*, Academic Press, 2003.

[10] Lars Bergström Ariel Goobar, *Cosmology and Particle Astrophysics*, Springer, 2006

[11] Sean Carroll, *Spacetime and Geometry, An Introduction to General Relativity*, Pearson, 2014

[12] Daniel Baumann, *Lecture Notes on Cosmology*, http://www.damtp.cam.ac.uk/user/db275/Cosmology2015.pdf

[13] David H. Lyth Andrew R.Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure*, Cambridge University Press,2009

[14] Ivo van Vulpen, *The Standard Model Higgs boson*, https://www.nikhef.nl/ ivov/HiggsLectureNote.pdf

[15] The Planck Collaboration, Planck 2015 results. XIII.Cosmological Parameters, arXiv:1502.01589v3 [astro-ph.CO]

[16] Frank Wilczek, Andrea De Simone, Mark Hertzberg, *Running Inflation in the Standard Model*, arXiv:0812.4946 [hep-ph]

[17] Mikael Shaposhnikov, Fedor Bezrukov, *The Standard Model Higgs boson as the inflaton*, arXiv:0710.3755 [hep-th]

[18] David I. Kaiser, *Conformal Transformations with Multiple Scalar Fields*,arXiv:1003.1159 [gr-qc]

[19] Mikael Shaposhnikov, Fedor Bezrukov, Standard Model Higgs boson mass from inflation,arXiv:0812.4950v2[hep-ph]

[20] Elias-Miró, Joan; Espinosa, José R.; Giudice, Gian F.; Isidori, Gino; Riotto, Antonio; Strumia, Alessandro, *Higgs mass implications on the stability of the electroweak vacuum*, arXiv:1112.3022v1 [hep-th]

[21] Kyle Allison,*Higgs* ξ *inflation for the* $125-126$ *GeV Higgs: A two-loop analysis*, arXiv:1306.6931v3 [hep-ph]