Equivariant Localization in Supersymmetric Quantum Mechanics

Bachelor Thesis in Physics, 15c

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Emil Hössjer

Department of Physics and Astronomy

Division of Theoretical Physics

Supervisor: Konstantina Polydorou

Subject reader: Marco Chiodaroli

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Abstract

We review equivariant localization and through the Feynman formalism of quantum mechanics motivate its role as a tool for calculating partition functions. We also consider a specific supersymmetric theory of one boson and two fermions and conclude that by applying localization to its partition function we may arrive at a known result that has previously been derived using different approaches. This paper follows a similar article by Levent Akant [1].

Sammanfattning

Vi studerar ekvivariant lokalisering och motiverar medelst Feynmans vägintegralsformalism av kvantmekanik lokaliseringens användning till att beräkna partitionsfunktioner. Vi betraktar därefter ett särskilt supersymmetriskt system beståendes av en boson och två fermioner och tillämpar lokalisering för att beräkna partitionsfunktionen. Detta leder till ett resultat som tidigare har visats med hjälp av andra metoder. Uppsatsen är inspirerad av en artikel av Levent Akant [1].

Contents

1 Introduction

We start this paper with a math problem:

$$\int_{-1}^{1} x \cos x dx.$$

Although it is not that much of a dilemma to find the primitive function, evaluating it at the boundary to find that the contributions cancel would annoy any mathematician, as one had then missed an 'obvious' symmetry in the problem. There is an unofficial ruling in math that if a problem possesses symmetry, one should exploit this to simplify the problem. This paper is all about that, in the context of equivariant cohomology.

The symmetries we are considering are manifolds that permit Lie group actions, and the associated problems are integrals over the manifolds. We want to distinguish those differential forms for which the problem of integration may be simplified - this will be the cohomology - and see what this simplification means - this will be the localization. Denoting the manifold and the Lie group M and G respectively, it is a known result that if G acts freely on M then the quotient M/G is a smooth manifold. In this case the new, equivariant, cohomology that we want to define is

$$H^*_{eq}(M) := H^*(M/G),$$

i.e. the ordinary de Rham cohomology of the quotient space. If the action is not free however this definition fails, and a more refined one is needed. But the above equation is still the motivational one that in the case of free action should be valid.

The first equivariant localization formulae came in the early 1980's and were due to Duistermaat and Heckman on one side, Berline, Vergne; Atiyah, Bott on another. These powerful theorems show how certain integrals, largely motivated by physics, may be reduced to discrete sums. Equivariant cohomology has since been an active field of research attended by mathematicians and physicists alike.

According to the Feynman formalism of quantum mechanics, transition amplitudes may be calculated through an infinite dimensional integral, called a path integral. Not surprisingly, this path integral is very difficult to solve - at the time it was introduced, the only known examples were the harmonic oscillator and the free particle! What one wants to calculate in quantum physics is generally the partition function. Expressed in the Feynman formalism the partition function too becomes an infinite dimensional integral. Using a localization theorem to, if possible, simplify such an integral is therefore very appealing.

We start this paper by reviewing some of the mathematics required for equivariant cohomology. We assume the reader knows ordinary de Rham theory and Lie group basics. We do however cover some examples of Lie groups that we will carry with us to later chapters and we cover Lie group actions on manifolds. The chapter on symplectic geometry is what links physics to geometry and enables the application of cohomological results to physics. The chapter on Grassmannians is important both to the mathematics and the physics of this paper.

We use the Cartan model for equivariant cohomology and discuss some of its features, especially integration. We also consider equivariant cohomology for symplectic manifolds. Regarding the localization we prove the BVAB theorem already mentioned (*Berline, Vergne; Atiyah, Bott*).

We then get into some physics. We motivate why equivariant cohomology in general might be relevant to calculating partition functions, and we see how a specific supersymmetry by itself seems to generate a setting of equivariant cohomology. In the final chapter we take a common example of a supersymmetric system and combine the two ideas above to put the partition function into a setting of equivariant cohomology. Finally, we apply the BVAB theorem to show that the partition function localizes.

2 Mathematical Preliminaries

2.1 Lie groups

Most Lie groups of interest are matrix Lie groups, where the operations are simply matrix multiplication and matrix inverse, and the topological structure as a manifold comes from the identification of a matrix with an element of $\mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$. Examples include the group of invertible $n \times n$- matrices $GL(n)$ and some of its subsets the orthogonal group $O(n)$, which is a real matrix Lie group, and the unitary group $U(n)$, which is a complex one. We also have the more intricate matrix Lie group $O(1,3)$ of transformations on $\mathbb{R}^4$ preserving the Minkowski metric and $Sp(V)$, the set of symplectomorphisms from a vector space V to itself, which is a matrix lie group under a fixed basis.

Let us review some examples more thoroughly, together with some of the key concepts.
Example 2.1. The circle group S^1 is the circle in the complex plane

$$S^1 := \{z \in \mathbb{C} : |z|^2 = 1\}$$

with group multiplication inherited as the usual multiplication from the complex numbers, well defined as

$$|z_1 \cdot z_2|^2 = |z_1|^2 |z_2|^2 = 1.$$

Note that we may also write

$$S^1 = \{e^{it} \in \mathbb{C} : t \in [0, 2\pi)\}.$$

Example 2.2. The group $SO(2)$ is the matrix Lie group of orthogonal matrices with positive determinant, that is, linear maps on $\mathbb{R}^2$ preserving both norm and orientation; known to be the set of rotations in the plane!
Example 2.3. Consider $\mathbb{R}$ with the operation of regular addition, $+$. This is a familiar group whose operations

$$a \cdot b := a + b$$

and

$$a^{-1} := -a$$

are commonly known smooth functions of calculus. Thus $G = \mathbb{R}$ with this operation is a Lie group. The identity for G is $e = 0$ and the Lie algebra is $\mathfrak{g} = T_0 G \equiv \mathbb{R}$. Let us find the exponential map for some $X \in \mathfrak{g}$. In local coordinate we have $X = s\frac{d}{dx}|_0$ for some scalar s. Let c be the curve $c(t) = ts$ in G so $c(0) = 0$ and $c'(0) = X$ and calculate the left invariant vector field X at a point $a \in G$ by

$$X|_a = L_{a*}X = \frac{d}{dt}(L_a \circ c)|_{t=0} = \frac{d}{dt}(a + ts)|_{t=0} = s\frac{d}{dx}|_a.$$

Thus X is constant wrt the standard coordinate on $\mathbb{R}$,

$$X = s\frac{d}{dx},$$

and the exponential map satisfies

$$\frac{d}{dt}exp(tX)|_{t_0} = X|_{exp(t_0X)} = s\frac{d}{dx}|_{exp(t_0X)}.$$

Together with the condition $exp(0) = 0$ we see that we must have $exp(tX) = ts$.

2.1.1 Actions on manifolds

Definition 2.4. A Lie group G can, beyond act on itself, act on another manifold M. We call this the *action* of G on M and write

$$G \times M \to M$$

$$g \times p \to g \cdot p,$$

if this map is smooth and satisfies the two criteria

$$e \cdot p = p \tag{1}$$

$$(gh) \cdot p = g \cdot (h \cdot p), \tag{2}$$

for $g, h \in G$.

Definition 2.5. We say that the action of G on M is free if for all $p \in M$

$$g \cdot p \implies g = e,$$

i.e. no point is non-trivially fixed under action of G.

Example 2.6. As hinted about in example (2.2), the Lie group $SO(2)$ of rotations may act on the manifold $\mathbb{R}^2$ by matrix multiplication. Clearly this satisifies criteria (1) and (2):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$(AB)\begin{pmatrix} a \\ b \end{pmatrix} = A\left(B\begin{pmatrix} a \\ b \end{pmatrix}\right).$$

We know from intuition that this action has one fixed point, at origo. Thus this action is not free.

Example 2.7. The circle group S^1 from example (2.1) may act on the complex plane $\mathbb{C}$ by multiplication:

$$e^{it} \cdot z = e^{it}z = e^{it}r(\cos\theta + i\sin\theta) = r(\cos(\theta + t) + i\sin(\theta + t)).$$

Note that both S^1 and $SO(2)$ act by rotation. If we identify $\mathbb{C} \equiv \mathbb{R}^2$ in the standard way as real manifolds, we see that these Lie groups act in exactly the same way. We also see, since

$$S^1 = \{e^{it} \in \mathbb{C} : t \in [0, 2\pi)\},$$

that we may identify

$$S^1 \equiv \mathbb{R} \mod (2\pi)$$

where the group action on ($\mathbb{R} \mod (2\pi)$) is additive and modulo 2π:

$$t_1 \cdot t_2 = t_1 + t_2 \mod (2\pi).$$

This is an important identification that we will make much use of.

Example 2.8. Using the above identification $S^1 \equiv \mathbb{R} \mod (2\pi)$, we may use example (2.3) to obtain the exponential map on S^1. For any $X \in T_eS^1 \equiv \mathbb{R}$ with $X = s\frac{d}{dx}|_e$:

$$exp(tX) = ts \mod (2\pi).$$

The Lie algebra induces a flow and a vector field on the manifold in much the same way as it does on the Lie group.

Definition 2.9. Let G act on M and $X \in \mathfrak{g}$. Then the *exponential map* at $p \in M$ induced by $\mathfrak{g}$ is

$$p(t) = \exp(Xt) \cdot p$$

and the corresponding induced vector field is

$$X^{\#}(t) = \frac{d}{dt}\big(\exp(Xt) \cdot p\big)|_{t=0}.$$

Since the exponential map on G is a flow this makes the corresponding exponential map on M a flow as well. This in turn ensures that the induced vector field is well defined.

Example 2.10. Let us consider the action of S^1 on the sphere S^2 by rotation through the z-axis. The action is free except at the north and south pole of the sphere. We use

$$S^1 \equiv \mathbb{R} \mod (2\pi).$$

In spherical coordinates, a point $p \in S^2$ has

$$p = (\phi, \theta)$$

and $t \in S^1$ acts by

$$t \cdot p = (\phi, \theta + t) \mod (2\pi).$$

Take as before $X = s\frac{d}{dx}|_e \in T_eS^1$. Using the exponential map on S^1 that we obtained in example (2.8) we get the induced flow

$$\sigma(t, p) = exp(tX) \cdot p = ts \cdot p = (\phi, \theta + ts) \mod (2\pi)$$

and the induced vector field

$$X^{\#}|_p = \frac{d}{dt}\sigma(t, p)|_{t=0} = (0, s) = s\frac{\partial}{\partial\theta}|_p.$$

The action of a group on a manifold as in definition (2.4) induces an action of the group on certain related structures, specifically the Lie algebra, the Lie coalgebra and the exterior algebra. We will study these here.

The Lie algebra. Consider the conjugation map κ_{g_0} which for fixed $g_0 \in G$ maps $g \in G$ to G by

$$\kappa_{g_0} g \to g_0 g g_0^{-1}.$$

Observe that $\kappa_{g_0} e = e$ so the pushforward $\kappa_{g_0 *}$ at e is a map

$$\kappa_{g_0 *}|_e : T_e G \to T_e G$$

and identifying $\mathfrak{g} = T_e G$ we have, for every $g \in G$, obtained an induced map κ_{g*} on $\mathfrak{g}$. This will be called the *adjoint action* of G on $\mathfrak{g}$, denoted $Ad_g := \kappa_{g*}$, and for $X \in \mathfrak{g}$

$$g \cdot X = Ad_g X.$$

It is straight forward to check the criteria in definition (2.4) for this group action. We have

$$e \cdot X = Ad_e X = \frac{d}{dt}(\kappa_e exp(tX))|_{t=0} = \frac{d}{dt}(e\ exp(tX)e^{-1})|_{t=0} = X$$

which shows (1) and

$$\kappa_{gh} p = ghp(gh)^{-1} = g(hph^{-1})g^{-1} = k_g k_h p$$

together with the chain rule gives

$$Ad_{gh} = \kappa_{gh*} = \kappa_{g*}\kappa_{h*} = Ad_g Ad_h$$

which shows (2).

The Lie coalgebra. The adjoint action directly gives a corresponding *coadjoint action* of G on the dual space of the lie algebra $\mathfrak{g}^*$ by, for $X \in \mathfrak{g}, \phi \in \mathfrak{g}^*$, requiring

$$(g \cdot \phi)(X) = \phi(g^{-1} \cdot X),$$

or denoting the coadjoint action $Ad_g^{\#}$,

$$(Ad_g^{\#}\phi)(X) = \phi(Ad_{g^{-1}}X). \tag{3}$$

This is easily seen to be well defined by picking a basis $(X^i)_{i=1}^{dim(G)}$ in $\mathfrak{g}$ and a basis dual to it $(\phi^i)_{i=1}^{dim(G)} \in \mathfrak{g}^*$ and checking that $Ad_g^{\#}\phi$ exists and is uniquely defined. Since the adjoint action is a pushforward it is linear and it is enough to consider the case $X = X^a, \phi = \phi^b$ of two basis elements. We have

$$Ad_{g^{-1}}X^a = c_i^a X^i \tag{4}$$

for some constants c_i^a and so the righthandside of (3) is

$$\phi^b(Ad_{g^{-1}}X^a) = c_b^a.$$

This equation now reads

$$(Ad_g^{\#}\phi^b)(X^a) = c_b^a$$

so we must have

$$Ad_g^{\#}\phi^b = c_b^i\phi^i$$

with the same constants c_i^j as in (4) which uniquely determines the coadjoint action and so finishes the claim. We also see that the adjoint and the coadjoint actions correspond to the same linear transformation in the respective dual basis.

The first criterion (1) in definition (2.4) follows immediately from (3) as

$$(Ad_e^{\#}\phi)(X) = \phi(Ad_{e^{-1}}X) = \phi(X) \implies Ad_e^{\#}\phi = \phi$$

and the second criterion (2) follows from

$$Ad_{gh}^{\#}\phi(X) = \phi(Ad_{(gh)^{-1}}X),$$

which from the corresponding property of the adjoint map just showed equals

$$\phi(Ad_{(gh)^{-1}}X) = \phi(Ad_{h^{-1}}Ad_{g^{-1}}X) = Ad_h^{\#}\phi(Ad_{g^{-1}}X) = Ad_g^{\#}Ad_h^{\#}\phi(X)$$

so

$$Ad_{gh}^{\#} = Ad_g^{\#}Ad_h^{\#}.$$

The exterior algebra. An element $g \in G$ acts on a form $\omega \in \Omega^*(M)$ by pullback of the inverse element:

$$g \cdot \omega = g^{-1*}\omega.$$

Criterion (1) is obviously satisfied and (2) follows from

$$gh \cdot \omega = (gh)^{-1*}\omega = (h^{-1}g^{-1})^*\omega = g^{-1*}h^{-1*}\omega = g \cdot (h \cdot \omega).$$

Remark: If the group action is commutative then clearly the conjugation map κ_g is just the identity, and the same follows for the adjoint and the coadjoint actions.

Example 2.11. For a matrix lie group G acting by matrix multiplication we have for $g \in G$, $X \in \mathfrak{g}$

$$Ad_gX = \frac{d}{dt}(\kappa_g exp(tX))|_{t=0} = \frac{d}{dt}(g\ exp(tX)g^{-1})|_{t=0} = \frac{d}{dt}(exp(tgXg^{-1}))|_{t=0} = gXg^{-1}.$$

where in the third equality we expand the exponential in a sum and use $gX^ng^{-1} = (gXg^{-1})^n$.

Example 2.12. Let G act on M and take $f \in \Omega^0(M)$ a function. For $g \in G$ and $p \in M$ we have one action $f(g \cdot p)$ and another $g \cdot f(p) = g^{-1*}f(p) = f(g^{-1} \cdot p)$, related by

$$f(g \cdot p) = \big(g^{-1} \cdot f\big)(p).$$

2.1.2 Infinitesimal actions

Assume G acts on a manifold M and let $g = exp(X)$ for some $X \in \mathfrak{g}$. Consider the action of $g(t) := exp(tX)$ on a point $p \in M$:

$$g(t) \cdot p = exp(tX) \cdot p. \tag{5}$$

Taking the derivative at $t = 0$ we obtain

$$\frac{d}{dt} g(t) \cdot p|_{t=0} = X^{\#}|_p, \tag{6}$$

with $X^{\#}$ the corresponding fundamental vector field to X at p. So a first order approximation of the action of g to p is

$$g(t) \cdot p \approx p + tX^{\#},$$

which is accurate for an infinitesimal value on t. We say that X is the infinitesimal generator of the action g, referring to (5), and the infinitesimal action of g on the manifold M is $X^{\#}$, referring to (6). Motivated by this, we generalize as follows.

Definition 2.13. Let $g \in G$ act on $x \in N$. If g is in the image of the exponential map, $g = exp(V)$, we define the *infinitesimal action* of g on N as

$$\frac{d}{dt}(exp(tV) \cdot x)|_{t=0}.$$

Example 2.14. Let us consider the infinitesimal action of $g = exp(tV)$ on a form ω in the exterior algebra. We have

$$\frac{d}{dt}(exp(tV) \cdot \omega)|_{t=0} = \lim_{t \to 0} \frac{exp(-tV)^* \omega - \omega}{t},$$

which is the definition of the lie derivative of ω along the induced vector field $V^{\#}$, $\pounds_{V^{\#}}\omega$.

Definition 2.15. We say that a point $x \in N$ is *invariant* under group action $g \in G$ if $g \cdot x = x$. We also say that x is *infinitesimally invariant* under $V \in \mathfrak{g}$ if the infinitesimal action on x is 0.

It follows immediately from the definition that finite group action invariance implies infinitesimal invariance. We will now show an important converse.

Theorem 2.16. *Consider the image of the exponential map $H := Im(exp) \subseteq G$. If points in N are invariant under infinitesimal action, then they are invariant under group actions in H.*

Proof. For any $x \in N$ and any $V \in \mathfrak{g}$ we have

$$\frac{d}{dt}(exp(tV) \cdot x)|_{t=0} = 0. \tag{7}$$

We want to show that $g \cdot x = x$ for $g \in H$. By definition of H, $g = exp(V)$ for some V in the Lie algebra. Consider the map $g(t) = exp(tV)$. If

$$\frac{d}{dt}(g(t) \cdot x) = 0$$

for all $0 \leqslant t \leqslant 1$ then we are done as we must have $g(1) \cdot x = x$. Let us therefore show that

$$\frac{d}{dt}(g(t) \cdot x)|_{t=t_0} = 0$$

for an arbitrary t_0. We have

$$\begin{aligned}
\frac{d}{dt}(g(t) \cdot x)|_{t=t_0} &= \frac{d}{dt}(g(t+t_0) \cdot x)|_{t=0} = \frac{d}{dt}(exp((t+t_0)V) \cdot x)|_{t=0} \\
&= \frac{d}{dt}(exp(tV) \cdot (exp(t_0 V) \cdot x))|_{t=0} \\
&= \frac{d}{dt}(exp(tV) \cdot x')|_{t=0} = 0,
\end{aligned}$$

where in the last line we put $x' = exp(t_0 V) \cdot x$, and the whole thing vanishes due to (7). □

Corollary 2.16.1. *For any compact and connected Lie group G, invariance and infinitesimal invariance of the action are equivalent.*

Proof. Follows since the exponential map on any both compact and connected manifold is surjective and theorem (2.16) just proved. In the context of this theorem, $H = G$. □

2.2 Symplectic geometry

Consider the following 2-form on $\mathbb{R}^{2n}$ with coordinates $(q^i, p_i)_{i=1}^n$:

$$\omega = dq^i \wedge dp_i \tag{8}$$

whose matrix elements are constant

$$(\omega_{ij}) = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

This form is called the *standard symplectic form* on Euclidean space and enjoys several useful properties

$$\begin{aligned} &1)\ \omega_{ij} = -\omega_{ji}, && \text{(antisymmetry)} \\ &2)\ \det \omega_{ij} \neq 0, && \text{(nondegeneracy)} \\ &3)\ d\omega = 0, && \text{(closedness)} \end{aligned}$$

The usefulness claim is legitimized by classical mechanics where the problem of finding the equations of motion of a specific system is made into a geometrical one, in a setting called *symplectic geometry.* Namely, the dynamics of an n-coordinate system $(q_1, \ldots, q_n)$ may be solved from the Hamiltonian H in phase space $M = (q_1, \ldots, q_n, p_1, \ldots p_n)$, by solving the differential equations

$$\frac{\partial H}{\partial p_i} = \dot{q}^i \tag{9}$$

$$-\frac{\partial H}{\partial q^i} = \dot{p}_i \tag{10}$$

This may be expressed using the symplectic form ω above and Poisson brackets $\{\cdot, \cdot\}_\omega$ defined by

$$\{f, g\}_\omega = (\omega_{ij})^{-1} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^j}$$

as for any function on phase space $f \in \Omega^0(M)$ we then have

$$\dot{f} = \{f, H\}_\omega.$$

Specifically the cases $f = q^i$ and $f = p^i$ yield the Hamilton equations (9)-(10). Before discussing this any further, Let us define a symplectic form generally.

Definition 2.17. A *symplectic form* on a $2n$-dimensional manifold M is a nondegenerate 2-form ω that is closed

$$d\omega = 0.$$

A manifold M that admits a symplectic form is called a *symplectic manifold.*

It turns out that a symplectic manifold always describes a mechanical system, as by the following theorem it may be put in the standard form (8) and be interpreted as a phase space [6].

Theorem 2.18 (Darboux Theorem)**.** *For a symplectic manifold* (M, ω) *we may locally find coordinates* $(q^i, p_i)_{i=1}^n$ *so that*

$$\omega = dq^i \wedge dp_i.$$

This geometric approach to mechanics will now grant the following two advantages. We may independently of coordinates state the dynamics of a mechanical system (M, ω) with Hamiltonian H as

$$\dot{f} = \{f, H\}_\omega$$

for an observable f of the system. And the shift from the classical formalism to a quantum formalism may now be made. This process is called geometric quantization and although we will not go into details results in the following simple transition. Functions become operators and the Poisson bracket becomes a Lie bracket:

$$f \to \hat{f}$$

$$\dot{\hat{f}} = i\hbar[\hat{f}, \hat{H}]_\omega.$$

This result will be used in chapter 5 when we go into quantum dynamics. We end this chapter with a couple more useful concepts stemming from symplectic manifolds and an example from geometry.

One can easily show that an even dimensional nondegenerate form is not nilpotent unless the incurred degree is above that of the space, in which case it is definitely zero. Specifically one can show for a symplectic manifold (M, ω) of dimension $2n$ that

$$\omega^n = c(x) dx^1 \wedge \ldots \wedge dx^{2n} \neq 0$$

is a top form with coefficient function c [6]. Passing to Darboux coordinates we obtain

$$\omega^n = (dq^i \wedge dp_i)^n = n!\; dq^1 \ldots dq^n dp_1 \ldots dp_n$$

so a symplectic manifold has a natural volume form called the *Liouville volume form* μ_L defined by

$$\mu_L = \frac{1}{n!}\omega^n. \tag{11}$$

In arbitrary coordinates this becomes

$$\mu_L = \sqrt{\det \omega_{ij}(x)} dx^1 \ldots dx^{2n}. \tag{12}$$

Example 2.19. Let us show that the sphere S^2 is symplectic. The volume form on $\mathbb{R}^3$ induces an area form on S^2 through

$$\omega := \iota_N dx \wedge dy \wedge dz$$

where N is the normal unit vector to the sphere in the outer direction. Thus $N = (x, y, z)$ at (x, y, z) and

$$\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

Passing to local spherical coordinates ϕ, θ with for example

$$\begin{aligned} x &= \sin\phi\cos\theta \\ dx &= \cos\phi\cos\theta d\phi - \sin\phi\sin\theta d\theta \end{aligned}$$

and substituting into ω we obtain

$$\omega = \sin\phi d\phi \wedge d\theta. \tag{13}$$

Clearly this satisfies being a symplectic form and the Liouville form $\mu_L = \omega^1$ is the same as the euclidean area form as

$$\int_{S^2} \omega = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin\phi d\phi \wedge d\theta = 4\pi.$$

Observe that the Darboux coordinates should here satisfy

$$\begin{aligned} dq &= \sin\phi d\phi \\ dp &= d\theta \end{aligned}$$

and thus

$$\begin{aligned} q &= \cos\phi \\ p &= \theta \end{aligned}$$

with the standard symplectic form

$$\omega = dq \wedge dp = d\cos\phi \wedge d\theta.$$

Definition 2.20. Let (M, ω) be symplectic with a group action G that preserves ω, i.e.

$$g \cdot \omega = \omega \iff g^{-1*}\omega = \omega.$$

Also let $\mu : M \to \mathfrak{g}^*$ and define for $X \in \mathfrak{g}$ the function $\mu^X \in \Omega^0(M)$ by

$$\mu^X|_p := \iota_X \mu|_p.$$

For $X^\#$ the induced vector field on M, μ is called a *moment map* if

$$1)\ d\mu^X = \iota_{X^\#}\omega \tag{14}$$
$$2)\ \mu|_{g\cdot p} = g \cdot \mu|_p. \tag{15}$$

Criterion (2) is that μ should be an equivariant map and is equivalent to

$$\mu|_{gp} = Ad_g^\# \mu|_p.$$

Example 2.21. Let us include a group action of S^1 in example (2.19) just studied to obtain a moment map on the symplectic manifold S^2. This same group action was studied in example (2.8) where we found that the induced vector field by $X = s\frac{d}{dx}|_e \in \mathfrak{s}^1 \equiv \mathbb{R}$ in S^2 is

$$X^{\#} = s\frac{\partial}{\partial\theta}.$$

Thus taking the symplectic form in eq. (13) gives for the moment map

$$d\mu^X = \iota_{X^{\#}}\omega = \sin\phi d\phi \wedge d\theta(s\frac{\partial}{\partial\theta}) = -s\sin\phi d\phi.$$

Integrating yields

$$\mu^X = s\cos\phi + s\lambda,$$

where we include the s in the constant since the moment map at any point $p \in S^2$ should be linear as a map from the lie algebra $\mathfrak{s}^1$. We may then conclude

$$\mu = \cos\phi + \lambda \in \mathbb{R}^*$$

where we identify $\mathfrak{s}^{1*} = \mathbb{R}^*$.

Remark. If the Lie algebra is finitely generated by some X^a then it is enough to consider the moment map evaluated at these, as for $X = c_a X^a$ we have

$$\mu^X = c_a \mu^{X^a}$$

following the linearity of the dual Lie algebra.

2.3 Grassmann algebra

In our further discussion we are going to need a new type of numbers, anticommuting ones generally called *Grassmann numbers*. In physical contexts these numbers can be interpreted as fermions, whereas usual commuting numbers in $\mathbb{C}$ are interpreted as bosons. This is due to the Pauli principle. We will define the basics of these numbers and state the important theorems that will be of use later. For proofs we refer to Nakahara [3].

Definition 2.22. A *Grassmann algebra* Λ is an algebra over some field generated by finitely many numbers $(\eta_1, \ldots, \eta_n)$ satisfying

$$\eta_i\eta_j = -\eta_j\eta_i.$$

The generators η_i are called *Grassmannians.*

The prominent and intuitive example of a Grassmann algebra is the exterior algebra over a manifold whose generators can be taken as linearly independent 1-forms, e.g. the dx^i. Derivation and integration in a Grassmann algebra are funnily enough defined to be equivalent. We have

$$\int d\eta_i\eta_j = \frac{\partial\eta_j}{\partial\eta_i} := \delta_{ij} \tag{16}$$

$$\int d\eta_i 1 = \frac{1}{\partial\eta_i} := 0. \tag{17}$$

So derivation and integration of scalars vanish and the volume of a Grassmann algebra is normalized:

$$\int_\Lambda d\eta_n \dots d\eta_1 \eta_1 \dots \eta_n = 1.$$

From this we have the following important properties of Grassmann integration.

Proposition 2.23. *A linear change of coordinates* $\tilde{\eta}^i = a_{ij}\eta^j$ *satisfies*

$$d^n\tilde{\eta} = \frac{1}{\det a_{ij}} d^n\eta. \tag{18}$$

A Gaussian integral in Grassmannians satisfies

$$\int_\Lambda d^{2n}\eta e^{\frac{1}{2}a_{ij}\eta^i\eta^j} = \sqrt{\det a_{ij}}. \tag{19}$$

These should be compared with their commutating number analogues

$$d^n\tilde{x} = \det a_{ij} d^n x \tag{20}$$

and

$$\int d^n x e^{-a_{ij}x^i x^j} = \frac{\pi^{n/2}}{\sqrt{\det a_{ij}}}. \tag{21}$$

We will now show a useful way of rewriting an ordinary integral. A k-form $\alpha \in \Omega^k(M)$ may be written

$$\alpha = f_{i_1 \dots i_k}(x) dx^{i_1} \dots dx^{i_k}$$

where as noted before the dx^i are part of a Grassmann algebra. We may therefore let $\eta^i = dx^i$ and consider α as an element of $\Omega^0(M) \otimes \Lambda$,

$$\alpha(x, \eta) = f_{i_1 \dots i_k}(x)\eta^{i_1} \dots \eta^{i_k}. \tag{22}$$

We may then use $Vol(\Lambda) = 1$ to rewrite an integral over M as an integral over $M \times \Lambda$, according to the following proposition.

Proposition 2.24. *For an integrable form* α *on* M *and the identification in eq.* (22) *we have*

$$\int_M \alpha = \int_{M\times\Lambda} d^n x d^n \eta \alpha(x, \eta).$$

Proof. Assume first $\deg(\alpha) < n = dim(M)$. Then the left side of the claimed equality vanishes, and after integrating out the existing grassmann numbers in $\alpha(x, \eta)$ we are left with a scalar that vanishes according to eq. (17).

Assume then $\deg(\alpha) = n$. Thus $\alpha = f(x)dx^1 \dots dx^n$ and the right side of the proposition reads

$$\int_{M\times\Lambda} d^n x d^n \eta f(x)\eta^1 \dots \eta^n = \int_M d^n x f(x) = \int_M \alpha$$

as the Grassmannians can just be integrated out to yield unity. □

The space $M \times \Lambda$ with both commuting and anticommuting coordinates is a supermanifold and will be called the odd tangent bundle over M, denoted ΠTM.

3 Equivariant Cohomology

Let, as when we were discussing Lie groups, G act on M. We want to introduce a cohomology on M that in the case of free action is

$$H^*_{eq}(M) = H^*(M/G). \tag{23}$$

When the action is not free there is an intuitive way of patching the above definition. Take a contractible space E on which G does act freely, and consider the space

$$M \times E.$$

The Lie group G acts on each factor independently in this space, called a diagonal action, and is thus free, and since $M \times E$ is homotopy equivalent to M we have

$$H^*(M) \equiv H^*(M \times E).$$

Hence it becomes natural to define

$$H^*_{eq}(M) = H^*((M \times E)/G), \tag{24}$$

which clearly is the same as (23) in the case of free action. This definition can both be shown to be independent of the choice of contractible space E, and to be always applicable, i.e. such a space E can always be found [2]. This definition although elegant does not open itself to easy calculations. Instead we will go on to look at a model called the Cartan model, which uses a differential complex and a new differential operator that in the same way as the de Rham complex calculates the cohomology. According to a theorem of Cartan these cohomologies are the same.

3.1 Cartan model

Definition 3.1. Let M and N be two manifolds on which the Lie group G has an action. A map between these manifolds, $f : M \to N$ is said to be *equivariant* with respect to G if it commutes with the group action, i.e.

$$f(g \cdot p) = g \cdot f(p). \tag{25}$$

Definition 3.2. An *equivariant differential form* is a map $\alpha : \mathfrak{g} \to \Omega^*(M)$ that is equivariant, i.e. for $X \in \mathfrak{g}$

$$\alpha(g \cdot X) = g \cdot \alpha(X) \iff \alpha(Ad_g X) = g^{-1*}\alpha(X). \tag{26}$$

The set of such forms is called the *equivariant differential complex* and is denoted $\Omega^\infty_G(M)$.

We will show that the equivariant differential complex, like the ordinary de Rham complex, constitutes an algebra under the wedge product and that we may introduce a differential operator on it, justifying the name of a *differential* complex. We will then see how the very explicit subset - in fact subalgebra - of polynomial maps of $\Omega^\infty_G(M)$ has the correct cohomology in accordance with (23) and (24).

Definition 3.3. We extend the wedge product to equivariant forms by, for $\alpha, \beta \in \Omega_G^\infty(M)$

$$\big(\alpha \wedge \beta\big)(X) := \alpha(X) \wedge \beta(X).$$

This is well defined as a direct computation shows that the product is too an equivariant form:

$$\begin{aligned}\big(\alpha \wedge \beta\big)(g \cdot X) &= \alpha(g \cdot X) \wedge \beta(g \cdot X) \\ &= g \cdot \alpha(X) \wedge g \cdot \beta(X) = g \cdot \big((\alpha \wedge \beta)(X)\big).\end{aligned}$$

Definition 3.4. We define the *equivariant differential operator* d_G on $\Omega_G^\infty(M)$ by, for α an equivariant form,

$$(d_G\alpha)(X) = (d + \iota_{X^\#})(\alpha(X)).$$

We must similarly show that this is well defined by showing that the equivariant complex is closed under d_G. Direct computation gives

$$(d_G\alpha)(g \cdot X) = (d + \iota_{(Ad_g X)^\#})(\alpha(g \cdot X)) = (d + \iota_{(Ad_g X)^\#})(g \cdot \alpha(X)),$$

which we would like to be equal to

$$g \cdot \big[(d + \iota_{X^\#})(\alpha(X))\big].$$

Now, the ordinary differential commutes with pullpacks which solves the d-term. For the interior product-term we have the operator identity

$$\iota_{(Ad_g X)^\#} = g \cdot \iota_{X^\#} \cdot g^{-1}$$

whose validity can be seen by expressing both sides of the equation in local coordinates. Thus

$$\iota_{(Ad_g X)^\#}(g \cdot \alpha(X)) = (g \cdot \iota_{X^\#})\alpha(X)$$

and we are done.

To define an equivariant cohomology on $\Omega_G^\infty(M)$ analogous to de Rham cohomology we would like the product of two closed forms to be a closed form and we would like the differential operator to square to zero. This is the case (otherwise our choice of the equivariant differential complex would not have been good).

Lemma 3.5. *For $\alpha, \beta \in \Omega_G^\infty(M)$*

$$\begin{aligned}&1) d_G(\alpha \wedge \beta) = d_G\alpha \wedge \beta + (-1)^{\deg \beta}\alpha \wedge d_G\beta \\ &2) d_G^2 = 0\end{aligned}$$

The first equation says that d_G is an anti-derivation.

Proof. 1) Follows as both d and ι are anti-derivations.

2) We will again solve this by direct computation. For $\alpha \in \Omega_G^\infty(M)$ we have

$$(d_G^2\alpha)(X) = (d + \iota_{X^\#})^2(\alpha(X)) = (d\iota_{X^\#} + \iota_{X^\#}d)(\alpha(X))$$

since both d and $\iota_{X^\#}$ squares to zero. By Cartan's magic formula this last expression is equal to the Lie derivative along $X^\#$. So

$$(d_G^2\alpha)(X) = \pounds_{X^\#}(\alpha(X)). \tag{27}$$

Now the equivariance of α comes into play. For any $g \in G$ we have

$$\alpha(g \cdot X) = g \cdot \alpha(X),$$

so specifically this is true for $g = exp(sX)$,

$$\alpha(\exp(sX) \cdot X) = \exp(sX) \cdot \alpha(X).$$

Taking the derivative at $s = 0$ in this equation we obtain the corresponding infinitesimal actions of g. Following example (2.14) the right hand side of this equation is the Lie derivative of $\alpha(X)$ along $X^\#$, which is what we have in (27) and want to show is zero. So if we can show that the left hand side vanishes we are done. We have

$$\frac{d}{ds}\alpha(\exp(sX) \cdot X)|_{s=0} = \frac{d}{ds}\alpha(Ad_{\exp(sX)}X)|_{s=0}$$

and using the definition of the adjoint action

$$\begin{aligned}\frac{d}{ds}\alpha(Ad_{\exp(sX)}X)|_{s=0} &= \frac{d}{ds}\left(\frac{d}{dt}\alpha\big(\exp(sX)\exp(tX)\exp(-sX)\big)|_{t=0}\right)|_{s=0} \\ &= \frac{d}{ds}\left(\frac{d}{dt}\alpha\big(\exp(tX)\big)|_{t=0}\right)|_{s=0} = 0,\end{aligned}$$

since the exponentials may be added to eliminate the s-dependency. □

Now we have the basis for cohomology. Consider the subset of polynomial maps $\mathfrak{g} \to \Omega^*(M)$ in $\Omega_G^\infty(M)$. Clearly this set is closed under the wedge product so it forms a subalgebra. We will denote this set $\Omega_G^*(M)$ (with a grading instead of the infinity superscript), whose elements can be made very explicit using the standard identification

$$\Omega_G^*(M) = \big(\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)\big)^G,$$

the set $\mathbb{C}[\mathfrak{g}]$ denoting the polynomial ring over $\mathfrak{g}$ and the G-superscript denoting the subset that satisfies G-equivariance as maps $\mathfrak{g} \to \Omega^*(M)$. We will call $\Omega_G^*(M)$ the *Cartan complex* of equivariant cohomology. An element $\alpha \in \Omega_G^*(M)$ may then be written

$$\alpha = \phi \otimes \omega$$

with

$$\alpha(X) = \phi(X)\omega \in \Omega^*(M),$$

and furthermore

$$d_G\alpha(X) = (d + \iota_{X^\#})\phi(X)\omega = \phi(X)\big(d + \iota_{X^\#})\omega.$$

We see that we may then make the equivariant differential explicit. Let $(X^a)_{a=1}^{\dim(G)}$ be generators of the Lie algebra and $(\phi^a)_{a=1}^{\dim(G)}$ their dual elements in $\mathfrak{g}^*$,

$$\phi^a(X^b) = \delta_{ab}.$$

We have $X = c_a X^a$ and likewise for the induced vector field $X^\# = c_a X^{a\#}$, so

$$d_G\alpha(X) = \phi(X)\big(d + c_a \iota_{X^{a\#}}\big)\omega.$$

Noting $\phi^a(X) = c_a$ we have

$$d_G\alpha(X) = \phi(X)\big(d + \phi^a(X)\iota_{X^{a\#}}\big)\omega = \bigg((1 \otimes d + \phi^a \otimes \iota_{X^{a\#}})\phi \otimes \omega\bigg)(X)$$

so in fact

$$d_G = 1 \otimes d + \phi^a \otimes \iota_{X^{a\#}}$$

whose relaxed version is

$$d_G \equiv d + \phi^a \iota_{X^a}.$$

This fits very well into the Cartan complex. We may consequently introduce a grading on $\Omega^*_G(M)$ by

$$\deg_G(\alpha) = 2\deg_P(\phi) + \deg_{dR}(\omega),$$

i.e. we count twice the polynomial degree in $\mathbb{C}[\mathfrak{g}]$ plus the ordinary de Rham degree in $\Omega^*(M)$. With this the equivariant differential becomes graded with degree 1, but only on the Cartan complex. At last we have arrived at the following.

Definition 3.6. The equivariant cohomology is

$$H^*_G(M) = \bigoplus H^i_G(M) := \bigoplus \ker(d_G|_{\Omega^i_G(M)})/im(d_G|_{\Omega^{i-1}_G(M)}).$$

Theorem 3.7 (Equivariant de Rham Theorem). *If a compact Lie group acts on a compact manifold, then the equivariant cohomology defined by the Cartan complex is isomorphic to the topologically one defined in* (24)*. [2]*

Let us finally go into integration and see a few examples of the properties this cohomology has.

Definition 3.8. Let $\alpha \in \Omega^\infty_G(M)$ and assume $\alpha(X) \in \Omega^*(M)$ is integrable over M. Then the integration operator $\int_M : \Omega^\infty_G(M) \to C(\mathfrak{g})$ is defined by

$$\Big(\int_M \alpha\Big)(X) = \int_M \big(\alpha(X)\big).$$

If $\alpha(X)$ does not contain a top form then this vanishes. For $\alpha = \phi \otimes \omega$ in the Cartan complex this becomes

$$\int_M \alpha = \phi \int_M \omega \in \mathbb{C}[\mathfrak{g}].$$

It will also be convenient to adopt the following notation. For an ordinary de Rham form $\alpha \in \Omega^*(M)$ we split it up into its different degrees by

$$\alpha = \alpha_{[0]} + \alpha_{[1]} + \ldots + \alpha_{[n]} \tag{28}$$

so that for example $\alpha_{[0]}$ is a function and $\alpha_{[n]}$ is a top-/volume form on M.

Example 3.9. Ordinary de Rham theory has the typical property that the integral of a boundaryless domain descends to cohomology. This is due to Stokes' theorem and is easily proved:

$$\int_M d\omega = \int_{\partial M} \omega = 0.$$

But the same is true for equivariant cohomology. The top form in an equivariantly exact form evaluated at any X, $d_G\alpha(X)$, must necessarily be of the form $d\alpha(X)$ as the interior product lowers the degree. Thus the same argument applies:

$$\int_M d_G\alpha(X)_{[n]} = \int_M d\alpha(X)_{[n]} = \int_{\partial M} \alpha(X)_{[n]} = 0.$$

Example 3.10. Consider similarly the bottom form, i.e. the function part in $\Omega^0(M)$ of an equivariantly exact form $d_G\alpha(X)$. This part must arise as the interior product of the 1-form in $\alpha(X)$. Thus

$$d_G\alpha(X)_{[0]} = \iota_{X^\#}\alpha(X)_{[1]}.$$

At a fixed point of the action then, where the induced vector field is zero, this vanishes. Thus evaluating the bottom form at a fixed point is another map that descends to equivariant cohomology.

Remark. In none of the examples above did we assume α was in the Cartan complex. Thus the above results actually apply to the larger and coarser cohomology group of the whole equivariant complex,

$$H_G^\infty(M) := \ker(d_G|_{\Omega_G^\infty(M)})/im(d_G|_{\Omega_G^\infty(M)}).$$

We will later encounter forms that are not included in the Cartan complex, so this cohomology group will too be important.

Example 3.11. Consider the integral of an equivariant form where we integrate both over the manifold and the Lie algebra:

$$\int_{\mathfrak{g}} dX \int_M \alpha(X).$$

Letting $(X^a)_{a=1}^{\dim(G)}$ again be generators of the Lie algebra we may write this as

$$\int_{\mathbb{R}} dc_a \int_M \alpha(c_a X^a).$$

3.2 Equivariant symplectic forms

Consider an equivariant differential form in the Cartan complex of degree 2, $\tilde{\omega} \in \Omega_G^2(M)$. Clearly this degree can only arise in two ways,

$$\Omega_G^2(M) = (\mathbb{C}^0[\mathfrak{g}] \otimes \Omega^2(M))^G, \oplus (\mathbb{C}^1[\mathfrak{g}] \otimes \Omega^0(M))^G$$

where the superscript in $\mathbb{C}^*[\mathfrak{g}]$ denotes the polynomial degree. Now, $\mathbb{C}^0[\mathfrak{g}] \equiv \mathbb{C}$ so the first part of the sum is just

$$(\mathbb{C}^0[\mathfrak{g}] \otimes \Omega^2(M))^G \equiv (\Omega^2(M))^G,$$

so we may write an element of $\mathbb{C}^0[\mathfrak{g}] \otimes \Omega^2(M))^G$ as $1 \otimes \omega \equiv \omega$ with ω a two-form invariant under G,

$$g \cdot \omega = g^{-1*}\omega = \omega.$$

An element in the second part of the sum is $\phi f \in (\mathbb{C}^1[\mathfrak{g}] \otimes \Omega^0(M))^G$, with ϕ a polynom of degree one and f a function on M.

Let us see what conditions closedness poses on $\tilde{\omega}$.

$$\begin{aligned} d_G\tilde{\omega} = 0 &\iff (1 \otimes d + \phi^a \otimes \iota_{V^a})(1 \otimes \omega + \phi f) = 0 \\ &\iff 1 \otimes d\omega + \phi^a \otimes \iota_{V^a}\omega + \phi \otimes df = 0 \\ &\iff 1 \otimes d\omega = 0 \text{ and } \phi^a \otimes \iota_{V^a}\omega + \phi \otimes df = 0 \end{aligned}$$

where in the last line we use that $d\omega \in \Omega^3(M)$ and $\iota_{V^a}\omega, df \in \Omega^2(M)$ are of different degree and must vanish independently. The first equality says that ω is closed in the de Rham cohomology and the second equality is equivalent to, for all $X \in \mathfrak{g}$

$$\iota_{X^\#}\omega + \phi(X)df = 0.$$

But this last expression may be written

$$d(\phi(X)f) = -\iota_{X^\#}\omega$$

which is the criterion 1 in definition (2.20) of the moment map if we identify $\mu = -\phi f$,

$$\begin{aligned} \mu : M &\to \mathfrak{g}^* \\ \mu : p \in M &\to -\phi f(p) \in \mathfrak{g}^*. \end{aligned}$$

But $-\phi f$ is also equivariant as in criterion 2 in the definition of the moment map and furthermore we have shown that ω is G-invariant and de Rham-closed. The only thing different from the moment map definition of symplectic geometry is that we have said nothing about the degeneracy of ω. We have shown the following.

Theorem 3.12. *For ω a symplectic form and μ its moment map,*

$$\tilde{\omega} := 1 \otimes \omega - \mu \equiv \omega - \mu$$

is a closed equivariant differential form.

Conversely, any $\bar{\omega} \in H^2_G(M)$ with $\bar{\omega}_{[2]}$ nondegenerate may be written as the difference of a sympletic form and its moment map.

We will call the $\bar{\omega}$ corresponding to a symplectic form ω the equivariant symplectic extension. This theorem gives us a mean of producing equivariant forms in the cohomology of a symplectic manifold, as also

$$\bar{\omega}^2, \bar{\omega}^3, \ldots$$

are equivariantly closed. Observe that unlike the case of the ordinary symplectic form ω the above sequence does not terminate,

$$(\omega^1, \omega^2, \ldots) = (\omega^1, \omega^2, \ldots, \omega^n, 0, 0, \ldots).$$

Consider also

$$e^{\bar{\omega}} := 1 + \bar{\omega} + \frac{\bar{\omega}^2}{2} + \ldots \tag{29}$$

which is a closed equivariant form as applying d_G kills every term. Since μ commutes with ω we have

$$e^{\bar{\omega}} = e^{\omega}e^{-\mu} = e^{-\mu}(1 + \omega + \frac{\omega^2}{2} + \ldots + \frac{\omega^n}{n!}). \tag{30}$$

Note in (29)-(30) that every finite partial sum is an element of the equivariant cohomology $H^*_G(M)$ of the Cartan complex but the limital series is not, as the coefficient map on $\mathfrak{g}$ is not a polynomial but a power series. However, $e^{\bar{\omega}}$ is an element of the bigger cohomology group $H^\infty_G(M)$ from remark (3.1). This form is of great importance as it arises in the classical partition function and is the focus of one of the big localization formulas in equivariant cohomology, the Duistermaat-Heckman formula. It will also appear later in our study of a supersymmetric quantum theory, in chapter 6. Thus considering the bigger cohomology group corresponding to the whole equivariant differential complex $\Omega^\infty_G(M)$ can be of importance.

We end this chapter with our returning example of a circle action on the sphere.

Example 3.13. Let S^1 act on S^2. We know from example (2.21) that the symplectic extension of the area form is

$$\tilde{\omega} = \omega - \mu = \sin\phi d\phi \wedge d\theta - (\cos\phi + \lambda),$$

and it is equivariantly closed. We can compute

$$\int_{S^2} \tilde{\omega} = \int_{S^2} \omega = 4\pi$$

as only the top form survives and thus this integral is constant as a function on $\mathfrak{g}$.

4 Equivariant Localization

There are a number of theorems of localization in equivariant cohomology, adjusted for different problems and assuming different properties of the Lie group action and the manifold. We will here prove one of the first and most fundamental localization theorems, catered for the action of a circle group.

Lemma 4.1. *For the action of compact G on M, there is a Riemannian metric r on M that is group invariant:*

$$g \cdot r = r.$$

Proof. The details are not important for us, but taking any metric r_0 on M and averaging it over the group invariant Haar measure dg over G proves the lemma by construction. Explicitly

$$r = \int_G (g \cdot r_0) dg$$

as can be done since G is compact. □

Theorem 4.2 (Berline, Vergne 1982; Atiyah, Bott 1984)**.** *Assume the following: M is compact, orientable, boundaryless and of even dimension n; G is compact, connected and 1-dimensional; the fixed points of the action are isolated; α is a closed equivariant differential form. Then*

$$\int_M \alpha = (-2\pi)^{n/2} \sum_k \alpha_{[0]}(y_k) \frac{1}{\sqrt{\det \partial_i V^j|_{y_k}}},$$

where V is the vector field induced by a nonzero $V \in \mathfrak{g}$ and the sum is over the fixed points y_k.

Remark. By examples (3.9)-(3.10) both the left- and right side of the above equation descend to cohomology (assuming the theorem to be true, any of them would have had to imply the other) so the theorem too descends to equivariant cohomology.

Proof. We divide the proof into several steps.
Claim (Localization principle)**.** Assume β satisfies $d_G^2\beta = 0$. Then the expression

$$Z(s) := \int_M \alpha e^{-sd_G\beta}$$

is independent of s.

Since M is boundaryless the integral of an exact form is zero - see example (3.9). Thus if we can show $\alpha(1 - e^{-sd_G\beta})$ is an exact form we are done as this will imply

$$\int_M \alpha = \int_M \alpha e^{-sd_G\beta},$$

and the right hand side is independent of s. But

$$\alpha(1 - e^{-sd_G\beta}) = \alpha \sum_{i=1} (-sd_G\beta)^i)/i!$$

$$= \alpha d_G \sum_{i=0} -s\beta(-sd_G\beta)^i/(i+1)! = d_G\left(\alpha \sum_{i=0} -s\beta(-sd_G\beta)^i/(i+1)!\right)$$

using that $d_G\beta$ is a closed form by assumption. △

Observe that the integral we want to calculate is $Z(0)$ for the function Z defined above, but we may choose any β and any s that we deem $Z(s)$ to be calculable for, as Z should render the same value independently. So let us abuse this.
Claim (Choice of β)**.** Take V as in the theorem and r as in the preceding lemma (4.1). We may then let

$$\beta := r(V, \cdot) = r_{\mu\nu}V^\mu dx^\nu.$$

We need to show $d_G^2\beta = 0$. We saw in the previous chapter that the equivariant operator squares to the Lie derivative. Thus the condition $d_G^2\beta = 0$ may be stated as: for any vector field W induced by the Lie algebra we have

$$\pounds_W\beta = 0.$$

But the Lie group is one dimensional so any two elements in the Lie algebra must differ by only a constant scalar and the same applies to the corresponding vector fields. Thus the above statement is equivalent to $\pounds_W\beta = 0$ for *some* nonzero $W \in \mathfrak{g}$, so specifically for $W = V$. Now r is G-invariant so by corollary (2.16.1) it is infinitesimally invariant,

$$\pounds_V r = 0.$$

But the Lie derivative of V^μ along V is also zero so we must have

$$\pounds_V\beta = 0$$

(we could also have expressed the above locally in coordinates to see $\mathfrak{L}_V\beta$ cancels). △

Claim (Localization at stable points)**.** With this choice of β, the integral

$$Z(s) = \int_M \alpha e^{-sd_G\beta}$$

only depends on finitely many, arbitrarily small neighborhoods of the stable points of the action.

Our choice of β gives

$$d_G\beta = (d + \iota_V)r_{\mu\nu}V^\mu dx^\nu = r_{\mu\nu}V^\mu V^\nu + \partial_\gamma(r_{\mu\nu}V^\mu)dx^\gamma dx^\nu,$$

so

$$Z(s) = \int_M \alpha e^{-s(r_{\mu\nu}V^\mu V^\nu + \partial_\gamma(r_{\mu\nu}V^\mu)dx^\gamma dx^\nu)}.$$

But the first term

$$r_{\mu\nu}V^\mu V^\nu \in \Omega^0(M)$$

is just a function which commutes with forms of higher degree so

$$Z(s) = \int_M \alpha e^{-sr_{\mu\nu}V^\mu V^\nu} e^{-s\partial_\gamma(r_{\mu\nu}V^\mu)dx^\gamma dx^\nu}.$$

Rewriting this integral using proposition (2.24) as an integral over the odd tangent bundle, we obtain

$$Z(s) = \int_{\Pi TM} d^n x d^n \eta \alpha(x,\eta) e^{-sr_{\mu\nu}V^\mu V^\nu} e^{-s\partial_\gamma(r_{\mu\nu}V^\mu)\eta^\gamma\eta^\nu}.$$

Since r is a Riemannian metric it is positively definite and

$$r_{\mu\nu}V^\mu V^\nu \geqslant 0$$

with equality iff $V = 0$, i.e. where the action is infinitesimally invariant. Since the Lie group is assumed to be compact and connected though, infinitesimal invariance is the same as group invariance by corollary (2.16.1). Therefore, in the limit $s \to \infty$ we have

$$e^{-sr_{\mu\nu}V^\mu V^\nu} \to 0$$

except at the stable points of the action. But

$$\alpha(x) = \sum f_{i_1,\ldots,i_k}(x) dx^{i_1} \ldots dx^{i_k}.$$

is bounded (it is a form on a compact manifold), so at any point not stable

$$\alpha(x,\eta) = \sum f_{i_1,\ldots,i_k}(x)\eta^{i_1} \ldots \eta i_k.$$

gets killed by the exponential function. Also the other factor

$$e^{-s\partial_\gamma(r_{\mu\nu}V^\mu)\eta^\gamma\eta^\nu}$$

is just a polynomial in the η which cannot compete with the exponential factor. Hence in the limit $s \to \infty$ the whole integrand goes to zero except at the stable points of the action, where $V = 0$.

We conclude that only neighborhoods arbitrarily small around the stable points will contribute to the integral. By assumption these are isolated and since M is compact they are finitely many. △

Claim (Contribution of a single point)**.** Say that a stable point of the action is y. The contribution to the integral of this point is

$$(-2\pi)^{n/2}\alpha_(y,0))\frac{1}{\sqrt{\det \partial_i V^j}}.$$

Let us denote the finite number of stable points y_k. Then by the above claim we may find neighborhoods M_{y_k} of these points that are disjoint and the restriction of the integrand to these domains satisfies

$$\begin{aligned} Z(s) &= \int_{\Pi TM} d^n x d^n \eta \alpha(x,\eta) e^{-s r_{\mu\nu} V^\mu V^\nu} e^{-s\partial_\gamma (r_{\mu\nu} V^\mu)\eta^\gamma \eta^\nu} \\ &= \sum_k \int_{\Pi TM_{y_k}} d^n x d^n \eta \alpha(x,\eta) e^{-s r_{\mu\nu} V^\mu V^\nu} e^{-s\partial_\gamma (r_{\mu\nu} V^\mu)\eta^\gamma \eta^\nu}. \end{aligned}$$

Let us consider one of these integrals over an arbitrarily small M_y corresponding to the stable point y. Expanding $r_{\mu\nu}V^\mu V^\nu$ at $y = (y^1, \ldots, y^n)$ and using $V(y) = 0$ we have

$$\begin{aligned} r_{\mu\nu} V^\mu V^\nu|_x &= r_{\mu\nu} V^\mu V^\nu|_y + \partial_i r_{\mu\nu} V^\mu V^\nu|_y (x^i - y^i) \\ &\quad + \partial_{ij} r_{\mu\nu} V^\mu V^\nu|_y (x^i - y^i)(x^j - y^j) + O(x^3) \\ &= r_{\mu\nu} \partial_i V^\mu \partial_j V^\nu|_y (x^i - y^i)(x^j - y^j) + O((x-y)^3), \end{aligned}$$

as only terms containing differentiations of both the V-components *do not* vanish at y.

Let us now employ the change of coordinates

$$x^i \to \tilde{x}^i,$$

$$\eta^i \to \tilde{\eta}^i,$$

where

$$x^i = y^i + \frac{\tilde{x^i}}{\sqrt{s}}$$

$$\eta^i = \frac{\tilde{\eta^i}}{\sqrt{s}}.$$

This gives

$$r_{\mu\nu} V^\mu V^\nu|_{(y+\frac{\tilde{x}}{\sqrt{s}})} = r_{\mu\nu}\partial_i V^\mu \partial_j V^\nu|_y \frac{\tilde{x^i}\tilde{x^j}}{s} + O(\frac{\tilde{x^3}}{\sqrt{s}^3})$$

and

$$e^{-s r_{\mu\nu} V^\mu V^\nu|_x} = e^{-r_{\mu\nu}\partial_i V^\mu \partial_j V^\nu|_y \tilde{x^i}\tilde{x^j} + O(\frac{\tilde{x}^3}{\sqrt{s}})}$$

and in the limit $s \to \infty$ we have

$$\lim_{s\to\infty} e^{-s r_{\mu\nu} V^\mu V^\nu|_x} = e^{-r_{\mu\nu}\partial_i V^\mu \partial_j V^\nu|_y \tilde{x^i}\tilde{x^j}}$$

The other exponential factor in the new coordinates becomes

$$-s\partial_\gamma (r_{\mu\nu} V^\mu)|_x \eta^\gamma \eta^\nu = -\partial_\gamma (r_{\mu\nu} V^\mu)|_{(y+\frac{\tilde{x}}{\sqrt{s}})} \tilde{\eta^\gamma}\tilde{\eta^\nu}$$

and in the limit $s \to \infty$ we have

$$-\partial_\gamma(r_{\mu\nu}V^\mu)|_{(y+\frac{\tilde{x}}{\sqrt{s}})}\tilde{\eta^\gamma}\tilde{\eta^\nu} \to -\partial_\gamma(r_{\mu\nu}V^\mu)|_y\tilde{\eta^\gamma}\tilde{\eta^\nu}$$

independent of the x- and $\tilde{x}$-coordinates. Also, $V^\mu|_y = 0$ so

$$-\partial_\gamma(r_{\mu\nu}V^\mu)|_y = -r_{\mu\nu}\partial_\gamma V^\mu|_y.$$

The last factor in the integrand $\alpha(x,\eta)$ has

$$\alpha(x,\eta) = \alpha(y + \frac{\tilde{x}}{\sqrt{s}}, \frac{\tilde{\eta}}{\sqrt{s}})$$

which in the limit $s \to \infty$ becomes

$$\alpha(x,\eta) \to \alpha(y,0).$$

Using the formulas in equation (20) and proposition (18) for the changes in measure in the transformations we have

$$d^n\tilde{x} = (\sqrt{s})^n d^n x$$

and

$$d^n\tilde{\eta} = 1/(\sqrt{s})^n d^n\eta,$$

so these effects of the coordinate changes cancel. Finally, the integrand in

$$\int_{\Pi TM_y} d^n x d^n \eta \alpha(x,\eta) e^{-sr_{\mu\nu}V^\mu V^\nu} e^{-s\partial_\gamma(r_{\mu\nu}V^\mu)\eta^\gamma\eta^\nu}$$

is bounded and we may use the dominated convergence theorem and all our assertions on the limits of the integrand as $s \to \infty$ to conclude

$$\begin{aligned}
&\lim_{s\to\infty}\int_{\Pi TM_y} d^n x d^n \eta \alpha(x,\eta) e^{-sr_{\mu\nu}V^\mu V^\nu} e^{-s\partial_\gamma(r_{\mu\nu}V^\mu)\eta^\gamma\eta^\nu} \\
=&\int_{\Pi TM_y} \lim_{s\to\infty} d^n x d^n \eta \alpha(x,\eta) e^{-sr_{\mu\nu}V^\mu V^\nu} e^{-s\partial_\gamma(r_{\mu\nu}V^\mu)\eta^\gamma\eta^\nu} \\
=&\int_{\Pi TM_y} d^n\tilde{x} d^n\tilde{\eta}\alpha(y,0) e^{-r_{\mu\nu}\partial_i V^\mu\partial_j V^\nu|_y\tilde{x^i}\tilde{x^j}} e^{-r_{\mu\nu}\partial_\gamma V^\mu|_y\tilde{\eta^\gamma}\tilde{\eta^\nu}}.
\end{aligned}$$

Now this integral may be separated into two Gaussian integrals over M_y and the Grassmann algebra Λ respectively as

$$\alpha(y,0)\int_{M_y} d^n\tilde{x} e^{-r_{\mu\nu}\partial_i V^\mu\partial_j V^\nu|_y\tilde{x^i}\tilde{x^j}} \int_\Lambda d^n\tilde{\eta} e^{-r_{\mu\nu}\partial_\gamma V^\mu|_y\tilde{\eta^\gamma}\tilde{\eta^\nu}}.$$

These we can evaluate. Denoting the coefficient matrix for the quadratic form in the $\tilde{x}^i$

$$H_{ij} := r_{\mu\nu}\partial_i V^\mu\partial_j V^\nu|_y$$

and the suiting coefficient matrix for the quadratic form in the $\tilde{\eta}^i$

$$K_{ij} := 2r_{\mu j}\partial_i V^\mu|_y$$

we obtain, using the Gaussian integration formulas in equation (21) and proposition (19),

$$\int_{M_y} d^n\tilde{x} e^{-r_{\mu\nu}\partial_i V^\mu \partial_j V^\nu|_y \tilde{x^i}\tilde{x^j}} = \int_{M_y} d^n\tilde{x} e^{-H_{ij}\tilde{x^i}\tilde{x^j}} = \pi^{n/2}\frac{1}{\sqrt{\det H_{ij}}}$$

and

$$\int_\Lambda d^n\tilde{\eta} e^{-r_{\mu\nu}\partial_\gamma V^\mu|_y \tilde{\eta^\gamma}\tilde{\eta^\nu}} = \int_\Lambda d^n\tilde{\eta} e^{-\frac{1}{2}K_{ij}\tilde{\eta^i}\tilde{\eta^j}} = (-1)^{n/2}\sqrt{\det K_{ij}},$$

so putting these factors together the integral becomes

$$\alpha(y,0)(-\pi)^{n/2}\frac{\sqrt{\det K_{ij}}}{\sqrt{\det H_{ij}}}.$$

The last detail in the claim comes from noting that

$$H_{ij} = \frac{1}{2}(2r_{\mu\nu}\partial_i V^\mu)\partial_j V^\nu|_y = \frac{1}{2}K_{i\nu}\partial_j V^\nu|_y$$

is the product of two matrices so

$$\det H_{ij} = \frac{1}{2^n}\det K_{ij} \det \partial_i V^j|_y.$$

This gives the integral contribution

$$(-2\pi)^{n/2}\alpha(y,0)\frac{1}{\sqrt{\det \partial_i V^j|_y}}.$$

△

Discarding of the Grassmann variables as in $\alpha(y,0)$ just leaves the function part of α, so $\alpha(y,0) = \alpha_{[0]}(y)$. The complete proof now follows from the last claim by summing over all the stable points of the action,

$$\int_M \alpha = (-2\pi)^{n/2}\sum_k \alpha_{[0]}(y_k)\frac{1}{\sqrt{\det \partial_i V^j|_{y_k}}}.$$

□

Remark. An integral $\int_M \alpha$ is completely independent of anything but the topform in α - the rest vanishes upon taking the integral. But for α to be closed equivariantly, all its component forms of lower degree come into play, related to one another by the condition $(d + \iota_V)\alpha = 0$. In the end, the BVAB localization formula tells us that we can read the integral value of a closed form α from only its zero-form part, the function, at a certain set of points. Amazing!

Example 4.3. We may return to study our old example of the circle action on S^2. Say we wish to integrate the standard area form $\omega = \sin\phi d\phi \wedge d\theta$ (which we have done a few times already - it is 4π) using the theorem just proved. From chapter 3.2 we know that the equivariant extension $\tilde{\omega}$ is a closed equivariant form, and since the top form is the same in ω and $\tilde{\omega}$ they have the same value when integrated over M,

$$\int_{S^2} \omega = \int_{S^2} \tilde{\omega},$$

independently of where in $\mathfrak{g}$ we evaluate it. But this latter integral fulfils all requirements in the BVAB theorem just proved. Thus its integral should localize onto the two fixed points of the rotational circle action, which by inspection we see is the south- and north pole. The function part of $\tilde{\omega}$ is $-\mu = -z$ where we set the constant parameter to zero, and passing to euclidean coordinates which include the two poles we obtain

$$\int_{S^2} \bar{\omega} = (-2\pi)^{2/2} \sum_{np,sp} -z \frac{1}{\sqrt{\det \partial_i (x\partial y - y\partial x)}} = -2\pi(-1-1) = 4\pi.$$

5 Equivariant Cohomology in Quantum Mechanics

Equivariant cohomology is interesting in its own right, as the previous chapters have shown, but let us see how it may be relevant to quantum mechanics and even further to supersymmetric quantum mechanics. We will give two motivational ideas correspondingly, and in the final chapter we combine these to motivate a known result about a specific theory, namely that the partition function localizes on specific path configurations. To begin with we are going to need the Feynman formalism of quantum mechanics, that of path integrals.

5.1 Path integral formalism

Let us for clarity assume we have only one particle that is constrained to move within one degree of freedom. The Feynman formalism then states that given our system in a specific configuration at an initial time $|q_i, t_i\rangle$ the probability that the system will be in another configuration at a time later $|q_f, t_f\rangle$ is obtained by summing the probability of every possible path the system can take between these points. The probability of a single path is in turn determined by integrating the exponential of the action, where the action is the classical one

$$S = \int L dt.$$

Summing over every conceivable path results in an infinite dimensional integral - not well defined by any means - but which may still be calculable to give correct predictions. We showed in chapter 2.2 that a mechanical system may be identified with a symplectic manifold (M, ω) and that its dynamics follow

$$\dot{\hat{f}} = i\hbar[\hat{f}, \hat{H}]_\omega$$

for the Hamiltonian $H \equiv \hat{H}$ and an observable $f \equiv \hat{f}$. It is shown in for example [4] that the path integral described above from $|q_i, t_i\rangle$ to $|q_f, t_f\rangle$ is

$$\langle q_f, t_f | q_i, t_i \rangle = \int [dq\ dp]\ exp(\frac{i}{\hbar} S[q,p]) \delta(q(t_i) - q_i) \delta(q(t_f) - q_f)$$

where the *Feynman measure* $[dq\ dp]$ denotes summing over all paths and is defined by an infinite time slicing

$$[dq\ dp] \equiv \Pi_{t \in [0,T]} \frac{dp(t)}{2\pi\hbar} dq(t).$$

The action is

$$S = \int_{t_i}^{t_f} \dot{q}(t)p(t) - H(q,p)dt.$$

The last factor $\delta(q^i(t_i) - q^i_i)\delta(q^i(t_f) - q^i_f)$ ensures we have the correct start- and end points. It is conventional to set $\hbar = 1$ since it is irrelevant to the analysis and we will also assume $t_i = 0$, $t_f = T$. What one usually wants to calculate in quantum mechanics is the partition function, which expressed using the Feynman path integral formalism becomes

$$Z(T) = \int [dq\ dp]\ exp(iS[q,p])\delta(q(t_i) - q(t_f)),$$

i.e. the integral over all closed paths of period T, taking all starting points into account [4]. Finally generalizing this to $2n$ coordinates is straight forward. Recalling the Liouville volume form (12) we have in arbitrary coordinates

$$dq^1 \dots dq^n dp_1 \dots dp_n = \frac{1}{n!}\omega^n = \sqrt{\det \omega_{ij}(x)} dx^1 \dots dx^{2n}$$

so over a curve of period T we may write the general partition function as

$$Z(T) = \int [d^{2n}x]\Pi_{t\in[0,T]}\sqrt{\det \omega_{ij}(x(t))}\ e^{iS[x]}\Pi_i \delta\big(x^i(t_i) - x^i(t_f)\big).$$

5.2 Loop space

For M a manifold, we may introduce the space of loops, i.e. closed curves, on M of a specific period T. We denote this space LM, so that an element $x \in LM$ is a curve on M with $x(0) = x(T)$. Loop space is an infinite dimensional manifold, actually an infinite dimensional fiber bundle over M with fiber consisting of all loops starting at a point [4]. With this we may write the partition function as an integral over loop space of phase space M:

$$Z(T) = \int_{LM} [d^{2n}x]\Pi_{t\in[0,T]}\sqrt{\det \omega_{ij}(x(t))} e^{iS[x]}. \tag{31}$$

The connection of equivariant localization to quantum mechanics may now be made apparent. Loop space naturally holds a Lie group action by identifying loops with a circle group. Explicitly, if we rescale time so that $T = 2\pi$, we may let $\tau \in S^1$ act on a coordinate $x^a(t)$ by

$$\tau \cdot x^a(t) = x^a(t + \tau),$$

and we can define this action diagonally on as many of the coordinates as pleases. This allows for an equivariant cohomology on loop space, and one can hope to be able to calculate the partition function in (31) with the help of a localization theorem as the one proved previously, in chapter 4.

So loop space will be important to us; let us list some of its properties. A differential k-form is

$$\alpha = \frac{1}{k!}\int_0^T dt\ \alpha_{i_1 \dots i_k}(x(t))dx^{i_1}(t)\dots dx^{i_k}(t),$$

and the exterior derivative d_L and the contraction operator ι are respectively

$$d_L = \int_0^T dt\ dx^a(t)\frac{\delta}{\delta x^a(t)},$$
$$\iota_V = \int_0^T dt\ V^a[x;t](t)\frac{\delta}{\delta x^a(t)}.$$

So a symplectic form on loop space is a 2-form

$$\Omega = \frac{1}{2}\int_0^T dt\ \Omega_{\mu\nu}(x(t))dx^\mu(t)dx^\nu(t)$$

which is closed:

$$d_L\Omega = 0,$$

and its moment map with respect to some Lie group induced vector field V^a is defined by

$$d_L\mu_L^V = \iota_V\Omega,$$

where as a 0-form

$$\mu_L^V = \int_0^T dt\mu^V[x;t].$$

If (M,ω) is symplectic this canonically induces a symplectic form Ω on loop space by

$$\Omega_{\mu\nu} = \omega_{\mu\nu}.$$

One can then see that μ is a moment map for (M,ω) if and only if

$$\mu_L := \int_0^T dt\mu \tag{32}$$

is a moment map for (LM,Ω).

A final observation is the following. The factor $\sqrt{\det\omega_{ij}(x)}$ is what results from a Grassmann integration of the Gaussian $\exp\left(\frac{1}{2}\eta^i\omega_{ij}(x)\eta^j\right)$ as in proposition (2.23). Therefore

$$\begin{aligned}\Pi_{t\in[0,T]}\sqrt{\det\omega_{ij}(x(t))} &= \Pi_{t\in[0,T]}\int_\Lambda \exp\big(\frac{1}{2}\eta^i(t)\omega_{ij}(x(t))\eta^j(t)\big)\\ &= \int_\Lambda \exp\big(\int_0^T dt\frac{1}{2}\eta^i(t)\omega_{ij}(x(t))\eta^j(t)\big).\end{aligned}$$

The canonically induced symplectic form Ω with the identification $dx^i = \eta^i$ has

$$\Omega[x,\eta;t] = \frac{1}{2}\int_0^T dt\ \omega_{\mu\nu}(x(t))\eta^\mu(t)\eta^\nu(t)$$

and thus, again using proposition (2.24) to rewrite the partition function as an integral over the odd tangent bundle of LM,

$$Z = \int_{LM}[d^{2n}x]\Pi_{t\in[0,T]}\sqrt{\det(\omega(x(t)))}e^{iS[x]} = \int_{\Pi LM}[d^{2n}x][d^{2n}\eta]e^{i(S[x]+\Omega[x,\eta])}. \tag{33}$$

The odd tangent bundle ΠLM is called *super loop space* and the exponential

$$A = S + \Omega,$$

occurring in the above partition function is called the *augmented action* on ΠLM.

5.3 Supersymmetry

Supersymmetry is for our purpose a transformation that for a given system relates bosonic, commutating fields x^a with fermionic, anticommutating fields ψ^a, where the latter are represented by Grassmannian variables. In this paper, to consider systems where equivariant lozalization may be applicable, we will study a system with the following assumed supersymmetric transformations: one fermionic δ_f- and one bosonic δ_b transformation acting on our fields as well as a series of auxiliary bosonic fields ϕ^a by

$$\delta_f x^a = \psi^a \tag{34}$$
$$\delta_f \psi^a = \delta_b x^a \tag{35}$$
$$\delta_f \phi^a = 0. \tag{36}$$

We will call a system of such supersymmetric transformations a BRST symmetry. To see the connection to equivariant cohomology, consider the equivariant operator $d_G = d + \iota_V$ on coordinates x^a:

$$\begin{aligned} d_G x^a &= dx^a \\ d_G dx^a &= £_V x^a = V^a \\ d_G V^a T_a &= 0. \end{aligned}$$

Recalling the identification $dx^a \equiv \psi^a$ which we have already used a number of times, we see that the fermionic transformation in a BRST-theory is truly similar to an equivariant differential. Pursuing this further, the auxiliary fields ϕ^a get identified with Lie algebra elements and the bosonic supersymmetry δ_b gets identified with the Lie derivative along V. We may then see the underlying Lie group action as defined by its induced vector fields via the Lie derivative in either of

$$£_V x^a = \delta_f^2 x^a = \delta_b x^a.$$

If the Lie group is compact and connected (e.g. a circle action) then this is well defined, as follows from the surjectivity of the exponential map.

The partition function for a system described above is of the form

$$Z = \int [dx^a][d\psi^a][d\phi^a] e^{iS[x^a, \psi^a, \phi^a]}$$

with the integration over super loop space. We will in the next chapter combine the two motivating arguments given in this chapter to calculate such a partition function using equivariant localization.

6 Equivariant Localization in Supersymmetric Quantum Mechanics

It is shown in [7] that for a system with one bosonic field x and two fermionic fields $\psi, \bar{\psi}$ the action and the partition function are respectively

$$A = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(X))^2 + \bar{\psi}[i\partial_t - W''(x)]\psi \right)$$

$$Z = \int [dx][d\psi][d\bar{\psi}]e^{iA[x,\psi,\bar{\psi}]},$$

and furthermore the action is invariant under the SUSY transformations

$$\begin{aligned} \delta x &= i\bar{\epsilon}\psi + i\epsilon\bar{\psi} \\ \delta\psi &= -\epsilon(\dot{x} - iW'(x)) \\ \delta\bar{\psi} &= -\bar{\epsilon}(\dot{x} + iW'(x)). \end{aligned}$$

Starting from this, we are interested in the specific case of BRST symmetry as in (34)-(36), suggesting $\epsilon = 0, \bar{\epsilon} = -i$:

$$\begin{aligned} \delta x &= \psi \\ \delta\psi &= 0 \\ \delta\bar{\psi} &= i\dot{x} - W'(x). \end{aligned}$$

The above does not completely resemble a theory with BRST symmetry as a commutating field with differential $\bar{\psi}$ is lacking. We will therefore add such a field π without changing the action. The system is then described by

$$Z = \int [dx][d\pi][d\psi][d\bar{\psi}]e^{iA[x,\psi,\bar{\psi}]} \tag{37}$$

$$A = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(X))^2 + \bar{\psi}[i\partial_t - W''(x)]\psi \right),$$

but letting

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(X))^2 \right)$$

and

$$\Omega = \int dt dt' d\pi(t)[i\partial_t - W''(x)]\delta(t - t')dx(t'),$$

we see that the partition function in (37) may actually be interpreted as a partition function on phase space $M = (x, \pi)$ written as an integral over super loop space with augmented action A. This will be key later in applying equivariant localization. Note also that Ω is symplectic on LM as it is obviously closed being a top form.

We now add an auxiliary field D which we want to identify with the Lie algebra, i.e. we want the Lie derivative to give us D as in (35) and hence we consider

$$\delta_f = \delta \tag{38}$$
$$\delta_f x = \psi \tag{39}$$
$$\delta_f \psi = 0 \tag{40}$$
$$\delta_f \pi = \bar{\psi} \tag{41}$$
$$\delta_f \bar{\psi} = D. \tag{42}$$

We can add such a field D if it can be integrated out of the action to return $i\dot{x} - W'(x)$. This leads us to consider the action

$$S' = \int dt\big((-i\dot{x} + W'(x))D + \frac{1}{2}D^2\big)$$

with then

$$\begin{aligned}\int [dD]S' &= \int dt\big((-i\dot{x} + W'(x))(i\dot{x} - W'(x)) + \frac{1}{2}(i\dot{x} - W'(x))^2\big) \\ &= \int dt\big(\frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(x))^2 - i\dot{x}W'(x)\big)\end{aligned}$$

which is just what we got in our action S except for the last term, but

$$i\dot{x}W'(x) = \frac{d}{dt}iW(x)$$

is an exact derivative, and since we are only considering closed paths the surface terms from integrating this cancel. Therefore we may indeed insert the auxiliary field D and consider the action $S \to S'$ to obtain

$$Z = \int [dD][dx][d\pi][d\psi][d\bar{\psi}]e^{iA[D,x,\psi,\bar{\psi}]}, \tag{43}$$

$$A = \int dt\bigg((-i\dot{x} + W'(x))D + \frac{1}{2}D^2 + \bar{\psi}[i\partial_t - W''(x)]\psi\bigg).$$

We will now show that the supersymmetry transformations in (38)-(42) do in fact define a circle action on LM as described in chapter (5.2). The group action should induce

$$\pounds_{V(D)}x = 0$$
$$\pounds_{V(D)}\pi = D$$

where D is Lie algebra valued so the induced vector field by S^1 is

$$V = D\frac{\partial}{\partial \pi}.$$

The action is then, for $\tau \in S^1$,

$$\tau \cdot (x(t), \pi(t)) = (x(t), \pi(t + \tau)).$$

Now that we have established the role of δ_f as an equivariant differential operator, let us see how equivariant localization can help calculate the partition function (43). It will turn out that A is the equivariant symplectic extension of

$$\Omega = \int dtdt'd\pi(t)[i\partial_t - W''(x)]\delta(t-t')dx(t')$$

on LM evaluated at D in the Lie algebra. The moment map μ_L satisfies

$$d_L\mu_L^\pi = \iota_{\partial_\pi}\Omega$$

or equivalently on phase space M, as noted in (32),

$$d\mu^\pi = \iota_{\partial_\pi}\big(d\pi[i\partial_t - W''(x)]dx\big)$$

so

$$\frac{\partial\mu^\pi}{\partial x} = i\partial_t - W''(x)$$
$$\frac{\partial\mu^\pi}{\partial\pi} = 0$$

and we have

$$\mu^\pi = i\dot{x} - W'(x)$$

ignoring the constant of integration. So the equivariant symplectic extension is

$$\bar{\Omega} = \Omega - \mu_L = \int dt\Big(- i\dot{x} + W'(x) + d\pi(t)[i\partial_t - W''(x)]dx(t)\Big)$$

but evaluating this at D gives, as in example (3.11),

$$\bar{\Omega}(D\partial_\pi) = \Omega - D\mu_L^\pi = \int dt\Big((-i\dot{x} + W'(x))D + d\pi(t)[i\partial_t - W''(x)]dx(t)\Big).$$

Now comparing this with A, we see that apart from the term $\frac{1}{2}D^2$ this is exactly the bosonic version of the integrand A over LM, again using proposition (2.24). Thus, rewriting using this proposition, we have

$$\begin{aligned} Z &= \int [dD][dx][d\pi][d\psi][d\bar{\psi}]e^{iA[D,x,\psi,\bar{\psi}]} \\ &= \int [dD]\int_{LM} exp[i\big(\Omega - D\mu_L^\pi + \int dt(\frac{1}{2}D^2)\big)] \\ &= \int [dD]\int dt\ exp(i\frac{1}{2}D^2)\Big(\int_{LM} e^{i\bar{\Omega}}\Big)(D\partial_\pi) \end{aligned}$$

where in the last line we divide the integral into one outer integral over the Lie algebra and one inner integral of $\bar{\Omega}$ over LM, as we did in example (3.11). Finally the equivariant symplectic extension $\bar{\Omega}$ is closed over LM, so we see that the partition function beautifully has wind up as the integral of an equivariantly closed form, just as we studied in chapter 4 only now infinite dimensional. We could hope to calculate this using some generalized version of the BVAB formula. Blindly disregarding the small detail of infinite dimensionality for a minute and using the BVAB formula as proved in this paper, this integral should localize on the fixed points of the action, where $D = 0$. Recalling $D = i\dot{x} - W'(x)$ we then obtain that the partition function localizes on those paths satisfying

$$i\dot{x} - W'(x) = 0.$$

We may strengthen this statement. Since $\dot{x}$ and $W'(x)$ are both real quantities they must in fact both vanish at the localized paths,

$$\dot{x} = 0$$
$$W'(x) = 0$$

so the localization is by the first equation onto the constant paths over phase space, which is just phase space! And the second equation further reduces the localization to those points with $W'(x)$=0. This result is very strong. We have reduced an infinite dimensional integral to a sum over those points in finite dimensional phase space that are critical points of the potential - a set that is typically finite!

This result copies that in [5], where a different approach is used to obtain the same localization result. Thus applying localization to this system does in fact yield the correct result.

7 Conclusion

The Cartan model makes equivariant cohomology, that intuitively was to be the de Rham cohomology modulo the group action, very explicit. Although this was the cohomology we set out to define we saw that there are cases when a bigger cohomology group, not restricted to polynomial maps on the Lie algebra as coefficients, should be considered. The properties we show in this paper descend to both cohomology groups. Furthermore, we see that symplectic geometry - whose role was completely necessary in creating the geometrical framework for dynamics in which cohomology could be applied - has a natural and useful role in equivariant cohomology via the moment map. At the end of our study of equivariant cohomology we prove the well known BVAB localization theorem.

Regarding the application to quantum mechanics we see that loop space readily permits a circle group action and that the specific case of BRST supersymmetry when interpreted as an equivariant differential operator can define a Lie group action. These two arguments enable equivariant cohomology in quantum mechanics. Lastly, taking a standard model of supersymmetry and applying the above ideas we rewrite the partition function as an integral of equivariant cohomology which, using the BVAB theorem, then localizes onto a small set of path configurations. This is a rederivation of a known result that shows how equivariant localization with success may be used in quantum mechanics.

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