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Torsion Classes and Support Tilting Modules for Path Algebras

Signe Lundkvist

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Handledare: Martin Herschend
Examinator: Veronica Crispin Quinonez
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Department of Mathematics
Uppsala University

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SIGNE LUNDVIST

Abstract. In this thesis, torsion classes and support tilting modules will be defined, and a bijection between them will be constructed. This will be done following the paper "Noncrossing partitions and representations of quivers" by Colin Ingalls and Hugh Thomas [4].

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1. Introduction

In this thesis, I introduce part of the paper Noncrossing partitions and representations of quivers by Colin Ingalls and Hugh Thomas, [4], as well as the background needed to understand it. In the paper Ingalls and Thomas find bijections between several algebraic constructions associated to the category of representations of some finite acyclic quiver Q. The category of representations of Q turns out to be equivalent to modules over the path algebra of Q, and as the path algebra is associative and unital most of the theory will be constructed for modules over associative, unital algebras.

The object of this paper is to construct a bijection between basic support tilting modules and torsion classes. This bijection is constructed in the first part of [4].

Constructing this bijection requires some tilting theory which will be introduced here. Perhaps most importantly it requires the notion of torsion pairs and torsion classes, and the notion of tilting modules. A torsion pair $(\mathcal{T},\mathcal{F})$ is a pair of full subcategories, where $\mathcal{T}$ is called the torsion class and $\mathcal{F}$ is called the torsion free class. One definition of a torsion class is that $\mathcal{T}$ is a full subcategory which is closed under images, direct sums and extensions, and a basic example of a torsion class is torsion abelian groups, i.e the abelian groups in which each element has finite order.

It turns out that torsion classes can be induced by taking $\operatorname{Gen} T$ of some tilting module T. It turns out that taking $\operatorname{Gen} T$ for a support tilting module T also induces torsion classes, and in this case the function which sends T to $\operatorname{Gen} T$ is, in fact, a bijection.

Some familiarity with homological algebra and category theory is assumed. The necessary background not covered in this thesis, as well as further reading, can be found in [1], [2], [3] and [5].

2. Background

In this section, the necessary background to understand the paper will be introduced. This section introduces some fundamental constructions, most importantly the Ext-functors, and some important theorems, such as the existance of a long exact sequence of Ext-functors, which will be one of the main tools in the rest of the paper.

2.1. Conventions and properties.

To start with, some conventions need to be established. Throughout, K will denote a field, and A will denote an algebra over K. Furthermore, A will always be assumed to be associative, unital and finite dimensional. The category of right modules over A will be denoted $\operatorname{Mod} A$, and the subcategory of finitely generated right A-modules will be denoted $\operatorname{mod} A$. An A-module M will refer to a module $M \in \operatorname{mod} A$. We will need some properties of A-modules.

Proposition 2.1. *Let A be a K-algebra, and let $M \in \operatorname{mod} A$. Then M is Artinian, which means that every descending chain of submodules of M must stabilize.*

Proof. This follows as M is assumed to be finitely generated, and so is finitely dimensional. □

Lemma 2.2. *Let $M \in \operatorname{mod} A$ be indecomposable. Then*

a *The algebra $B = \operatorname{End}_A(M)$ is local, i.e B has a unique maximal left ideal J.*
b *The maximal right ideal J is a nilpotent two sided ideal.*

Proof. (a) By [1] Corollary I.4.8 b, the algebra $B = \operatorname{End}_A(M)$ is local, which by [1] Lemma I.4.6 means that there is a unique maximal left ideal of B.

(b) By [1] Corollary I.2.3 the ideal $\operatorname{rad}(B)$ (see [1] Definition I.1.2) is nilpotent, as B is a finite dimensional algebra. By definition of $\operatorname{rad}(B)$, the unique maximal ideal J must be equal to $\operatorname{rad}(B)$, so J is nilpotent. □

Theorem 2.3 (Krull-Schmidt)**.** *Take $M \in \operatorname{mod} A$. Then M can be written uniquely as a direct sum of indecomposable A-modules, up to isomorphism and order of factors.*

Proof. For a proof of this, see [1] Theorem I.4.10. □

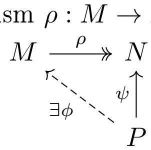
Given the Krull-Schmidt theorem the following definitions make sense.

Definition 2.4. Let $M \in \operatorname{mod} A$. Then, by the Krull-Schmidt theorem $M \cong \bigoplus M_i^{d_i}$, where each M_i is indecomposable. Then $\operatorname{bsc} M \cong \bigoplus M_i$, and M is basic if $M \cong \operatorname{bsc} M$. Furthermore, $\operatorname{add} M$ is defined to be the modules N such that $N \cong \bigoplus M_i^{d'_i}$. It follows directly from these definitions that $\operatorname{add} \operatorname{bsc} M = \operatorname{add} M$.

2.2. Projective and injective modules.

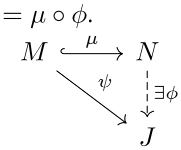
To construct the Ext-functors, which will be one of our main tools, we are going to use the notion of a projective resolution of a module M. So we first need the definition of a projective module. The material presented here can be found in [3], and [1] A.4

Definition 2.5. A module P is projective if for every morphism $\psi : P \to N$ and every surjective homomorphism $\rho : M \to N$ there exists some homomorphism $\phi : P \to M$ such that $\psi = \phi \circ \rho$.



This means that the function $\operatorname{Hom}_A(P, \rho) : \operatorname{Hom}_A(P, M) \to \operatorname{Hom}_A(P, N)$ is surjective if ρ is a surjective morphism $\rho : M \to N$. By definition, $\operatorname{Hom}_A(P, \rho)$ maps an element in $\operatorname{Hom}_A(P, M)$ to its composition with ρ, so if P is projective, for every $\psi \in \operatorname{Hom}_A(P, N)$ there is an element $\phi \in \operatorname{Hom}_A(P, M)$ which maps to it under $\operatorname{Hom}_A(P, \rho)$.

The dual of a projective module is an injective module. A module J is injective if for every morphism $\psi : M \to J$ and every injective morphism $\mu : M \to N$ there exists a morphism $\phi : N \to J$ such that $\psi = \mu \circ \phi$.



Definition 2.6. A projective resolution of an A-module M is an exact sequence $P_\bullet : ...P_n \longrightarrow P_{n-1} \longrightarrow ... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ such that all P_i are projective.

An injective resolution of M is an exact sequence $I^\bullet : 0 \longrightarrow M \longrightarrow I^0 \longrightarrow ... \longrightarrow I^n \longrightarrow I^{n+1} \longrightarrow ...$ such that all I_i are injective.

The following theorem will enable the construction of the Ext-functors.

Theorem 2.7. *Every module has a projective resolution, and every module has an injective resolution.*

Proof. For a proof of the existance of a projective resolution, see the proof of [1] Lemma I.5.3 c. The proof of the existance of an injective resolution is dual. □

Definition 2.8. The projective dimension of an A-module M, $\text{pd}(M)$, is the smallest integer m such that there exists a projective resolution
$P_\bullet : 0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow ... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$. Such an m does not necessarily exist, and if it does not, $\text{pd}\, M$ is said to be infinite.

In particular, if an A-module M has projective dimension $\text{pd}\, M \leq 1$, there is a short exact sequence $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, where P_1 and P_0 are projective.

2.3. **The Ext-functors.** The functor $\text{Ext}^m_A(-,-)$ is constructed as follows.

Let A be a K-algebra, M and N be two A-modules and let $P_\bullet$ be a projective resolution of M. Apply the functor $\text{Hom}_A(-, N)$ to get the cochain complex

$\text{Hom}_A(P_\bullet, N) : ... \longrightarrow \text{Hom}_A(0, N) \longrightarrow \text{Hom}_A(M, N) \longrightarrow \text{Hom}_A(P_0, N) \longrightarrow \text{Hom}_A(P_1, N) \longrightarrow ...$

Removing the entry $\text{Hom}_A(M, N)$ from the complex leaves the cochain complex

$C^\bullet : ... \longrightarrow \text{Hom}_A(0, N) \longrightarrow \text{Hom}_A(P_0, N) \longrightarrow \text{Hom}_A(P_1, N) \to \text{Hom}_A(P_2, N) \longrightarrow ...$

The m:th extension functor $\text{Ext}^m_A(M, N)$ is defined as the m:th cohomology of $C^\bullet$. Up to isomorphism $\text{Ext}^m_A(M, N)$ does not depend on the choice of projective resolution.

$\text{Ext}^m_A(-, N)$ is a contravariant functor $\text{Mod}\, A \to \text{Mod}\, K$, and the functor $\text{Ext}^m_A(M, -)$ is a covariant functor $\text{Mod}\, A \to \text{Mod}\, K$.

The functors can also be constructed using an injective resolution $I^\bullet$ of N, and considering the cohomologies of $\text{Hom}_A(M, I^\bullet)$.

The following theorem will be heavily used in the rest of the paper.

Theorem 2.9. *Let M be an A-module, and let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence of A-modules.*

Then there is a long exact sequence $0 \longrightarrow \text{Hom}_A(M, X) \longrightarrow \text{Hom}_A(M, Y) \longrightarrow \text{Hom}_A(M, Z) \longrightarrow \text{Ext}^1_A(M, X) \longrightarrow \text{Ext}^1_A(M, Y) \longrightarrow \text{Ext}^1_A(M, Z) \longrightarrow \text{Ext}^2_A(M, X) \longrightarrow \text{Ext}^2_A(M, Y) \longrightarrow ...$

There is also a long exact sequence $0 \longrightarrow \text{Hom}_A(Z, M) \longrightarrow \text{Hom}_A(Y, M) \longrightarrow \text{Hom}_A(X, M) \longrightarrow \text{Ext}^1_A(Z, M) \longrightarrow \text{Ext}^1_A(Y, M) \longrightarrow \text{Ext}^1_A(X, M) \longrightarrow \text{Ext}^2_A(Z, M) \longrightarrow \text{Ext}^2_A(Y, M) \longrightarrow ...$

Proof. For a proof of this see [3] Theorem XII.4.4. □

2.4. **The group of extensions.** Now we introduce the concept of extensions of two modules M and N. Defining an addition on extensions will give a different way of considering the first Ext-group. The material presented here can be found in [1] A.5.

Let A be a K-algebra, and let M and N be A-modules. An extension $\boldsymbol{E}$ of M by N is a short exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$.

Two extensions $\boldsymbol{E}$: $0 \longrightarrow M \xrightarrow{f} E \xrightarrow{f'} N \longrightarrow 0$ and $\boldsymbol{E}'$: $0 \longrightarrow M \xrightarrow{g} E' \xrightarrow{g'} N$ are said to be equivalent if there is a diagram of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & E & \xrightarrow{f'} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & M & \xrightarrow{g} & E' & \xrightarrow{g'} & N & \longrightarrow & 0 \end{array}$$

This is clearly an equivalence relation. Moreover, the sum $\boldsymbol{E} + \boldsymbol{E}'$ is defined to be the extension $\boldsymbol{E} + \boldsymbol{E}'$: $0 \longrightarrow M \xrightarrow{h} E^* \xrightarrow{h'} N$, where E^* is the quotient of $U = \{(m, m') \in E \oplus E' \mid f'(m) = g'(m')\}$

and $V = \{(f(m), -g(m')) \in E \bigoplus E'; m \in M\}$, and the functions h and h' are induced by $h : M \to U$ which maps m to $(f(m), 0)$ and $h' : U \to N$ which maps (m, m') to $g'(m')$. The set of extensions together with the addition defined above form an abelian group, where the split short exact sequences are the identity.

Let L and N be A-modules, and let $\boldsymbol{E}$ be an extension of L by N represented by the short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Consider a projective resolution $P^\bullet$ of N. There is a commutative diagram

$$\begin{array}{ccccccccc} P_2 & \xrightarrow{h_2} & P_1 & \xrightarrow{h_1} & P_0 & \xrightarrow{h_0} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle id_N} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Define $\chi(\boldsymbol{E}) = t_1 + \operatorname{im} \operatorname{Hom}_A(h_1, L) \in \operatorname{Ext}^1_A(N, L)$.

Theorem 2.10. *For a pair of A-modules M and N, χ defines a functorial isomorphism from the group of extensions of M by N to $\operatorname{Ext}^1_A(N, M)$. Furthermore, it can be shown that χ does not depend on the choice of t_1.*

Proof. See [1] Theorem A.5.9 □

2.5. The Grothendieck group.

Definition 2.11. Let A be a K-algebra. Let F be the free abelian group with basis consisting of isomorphism classes of A-modules, and define the subgroup F' of F to be the subgroup generated by the expressions $[L] - [M] - [N]$, whenever there is a short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$.

The Grothendieck group $K_0(A)$ is defined as $K_0(A) = F/F'$.

Theorem 2.12. *Let A be an algebra. Then $K_0(A)$ is the free abelian group with basis the simple A-modules, so for every A-module M, it is possible to write $[M] = \sum \mu_i [S_i]$ in $K_0(A)$, where S_i runs through the simple A-modules.*

Proof. See the proof of [2] Theorem I.1.7. □

2.6. The Jordan-Hölder theorem.

Definition 2.13. A composition series of an A-module M is a chain of inclusions of submodules $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = M$ such that the quotients M_i/M_{i-1} is simple for all i. The simple modules M_i/M_{i-1} are called the composition factors of M.

Every A-module M has such a composition series, which is unique up to permutation in the following sense.

Theorem 2.14 (The Jordan-Hölder theorem)**.**
If $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = M$ and $0 = N_0 \subset N_1 \subset ... \subset N_{m-1} \subset N_m = M$ are two composition series of M, then $m = n$ and there is a permutation $\sigma \in S_n$ such that for any $j \in \{1, ..., n\}$ there is an isomorphism $M_j/M_{j-1} \cong N_{\sigma(j)}/N_{\sigma(j-1)}$

Proof. Let $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = 0$ and $0 = N_0 \subset N_1 \subset ... \subset N_{m-1} \subset N_m = M$ be two composition series of M. Theorem 2.12 says that $[M] = \sum \mu_{M,S}[S]$ in $K_0(A)$. First note that, if there is a short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, then by definition of $K_0(A)$, $\mu_{M,S} = \mu_{L,S} + \mu_{N,S}$. Because there are short exact sequences $0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$, $\mu_{M_i,S} = \mu_{M_{i-1},S} + \mu_{M_i/M_{i-1},S}$, where $\mu_{M_i/M_{i-1},S} = 1$ if $M_i/M_{i-1} \cong S$, and 0 otherwise. So $\mu_{M,S}$ is the number of times S appears as a composition factor in the composition series $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = 0$. Clearly, the same thing can be done for the composition series $0 = N_0 \subset N_1 \subset ... \subset N_{m-1} \subseteq N_m = M$. As the coefficients $\mu_{M,S}$ are unique, this gives that each composition factor appears as many times in both of the composition factors, so the result follows. Note especially that the composition series must have as many composition factors, so $n = m$. □

3. Quiver representations and path algebras

In this section, the concept of quivers is introduced. Quivers are essentially directed graphs, which turn out to be a useful way of representing some associative algebras. The information presented in this section as well as further reading on quivers can be found in [1] chapters II and III.

Definition 3.1. A quiver Q consists of a set Q_0 of points, or vertices, a set Q_1 of arrows, and two maps $s, t : Q_1 \to Q_0$ which determine the start- and endpoints of the arrow, i.e. $s(a) \in Q_0$ is the source of the arrow a, and $t(a) \in Q_0$ is the endpoint, or target, of the arrow.

Definition 3.2. A K-linear representation of the quiver $Q = (Q_0, Q_1, s, t)$, or more briefly a representation of Q, assigns a K-vector space V_p to each point $p \in Q_0$, and assigns a linear map $\phi_a : V_p \to V_q$ for each $a \in Q_1$ with starting point p and endpoint q.

Denote a representation M of Q by (V_p, ϕ_a), where V_p and ϕ_p are as above.

A quiver Q is acyclic if it contains no cycles, finite if the sets Q_0 and Q_1 are finite,

Definition 3.3. A morphism between two quivers is defined as a family of maps $f = (f_p)_{p \in Q_p}$ between two representations $M_1 = (V_p, \phi_a)$ and $M_2 = (U_p, \psi_a)$ of some quiver Q, such that the maps in f make the following square commute for each arrow $a \in Q_1$:

$$\begin{array}{ccc} V_p & \xrightarrow{\phi_a} & V_q \\ \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_q} \\ U_q & \xrightarrow{\psi_a} & U_q \end{array}$$

Definition 3.4. Let Q be a quiver.

A path $p \to q$ in Q is a sequence $a_1, ..., a_n$ of arrows such that $s(a_1) = p$, $t(a_n) = q$, and $t(a_k) = s(a_{k+1})$. The length of the path is defined to be the number n of arrows in the path. For every vertex there is an associated path of length 0, called the trivial path.

The path algebra KQ of Q is a K-algebra with underlying vector space spanned by the paths in Q, where multiplication of two paths $a_1, ..., a_n$ from p_0 to p_1 and $b_1, ..., b_m$ from q_0 to q_1 is defined to be $a_1, ..., a_n, b_1, ..., b_m$ if $p_1 = q_0$, and 0 otherwise. The multiplication is extended to arbitrary elements by bilinearity.

Theorem 3.5. *Let Q be a quiver. The category $Rep_Q(K)$ of K-linear representations of Q is equivalent to the category of KQ-modules.*

Proof. See [1] Corollary III.1.7. □

Proposition 3.6. *If M is a KQ-module, then* $\operatorname{pd} M \leq 1$.

Proof. See note in [4] Definition 2.3. □

Example 3.7. Let K be a field, and let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$.

Then, up to isomorphism there are six indecomposable representations of Q, namely:

P_1: $K \xrightarrow{1} K \xrightarrow{1} K$

P_2: $0 \longrightarrow K \xrightarrow{1} K$

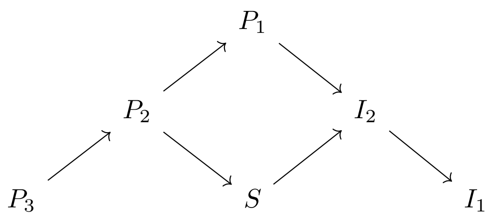
P_3: $0 \longrightarrow 0 \longrightarrow K$

I_2: $K \xrightarrow{1} K \longrightarrow 0$

I_1: $K \longrightarrow 0 \longrightarrow 0$

S: $0 \longrightarrow K \longrightarrow 0$

The indecomposable modules can be arranged in the following diagram, where the downwards arrows represent surjections and the upwards arrows represent injections.


P_1
P_2
I_2
P_3
S
I_1

4. Tilting theory

The following section introduces the necessary background in tilting theory. First and foremost, the notion of torsion pairs and torsion classes, tilting modules and support tilting modules. The necessary properties and lemmas to define the bijection between torsion classes and support tilting modules will also be introduced. The theory introduced in this section can be found in [1] chapter VI and [4].

4.1. Torsion pairs.

Here the notions of torsion pairs, torsion classes and torsion free classes will be defined.

Definition 4.1. Let $\mathcal{T}$ and $\mathcal{F}$ be full subcategories of $\operatorname{mod} A$. The pair $(\mathcal{F}, \mathcal{T})$ is a torsion pair if:

(1) $\operatorname{Hom}_A(M, N) = 0 \ \forall M \in \mathcal{T}$ and $N \in \mathcal{F}$
(2) $\operatorname{Hom}_A(M, N) = 0 \ \forall N \in \mathcal{F} \implies M \in \mathcal{T}$
(3) $\operatorname{Hom}_A(M, N) = 0 \ \forall M \in \mathcal{T} \implies N \in \mathcal{F}$

Then $\mathcal{T}$ is called a torsion class, and $\mathcal{F}$ is called a torsion-free class.

Definition 4.2. A subfunctor of the identity functor on $\operatorname{mod} A$ is a functor $t : \operatorname{mod} A \to \operatorname{mod} A$ such that t assigns a submodule $tM \subset M$ to every A module M, and every homomorphism $M \to N$ restricts to a homomorphism $tM \to tN$.
Such a functor is called an idempotent radical if for every A-module M, $t(tM) = tM$ and $t(M/tM) = 0$.

The following theorem gives some conditions which are equivalent to being a torsion class and a torsion free class, respectively. Crucially, the theorem says that closure under certain properties is enough to be a torsion class, or a torsion free class.

Theorem 4.3. *Let A be an algebra.*

a. *Let $\mathcal{T}$ a full subcategory of* $\operatorname{mod} A$*. Then the following are equivalent:*
   (1) *$\mathcal{T}$ is a torsion class with some corresponding torsion-free class $\mathcal{F}$.*
   (2) *$\mathcal{T}$ is closed under images, direct sums and extensions.*
   (3) *There is an idempotent radical t such that $\mathcal{T} = \{M \mid tM = M\}$*
b. *Let $\mathcal{F}$ be a full subcategory of* $\operatorname{mod} A$*. Then the following are equivalent:*
   (1) *$\mathcal{F}$ is a torsion-free class, with some corresponding torsion class $\mathcal{T}$.*
   (2) *$\mathcal{F}$ is closed under submodules, direct sums and extensions.*
   (3) *There is an idempotent radical t such that $\mathcal{F} = \{N \mid tN = 0\}$*

Proof. The proof follows the proof of [1] Proposition VI.1.4.
We only show the first statement. The proof of the second statement is dual.

(1) implies (2). Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence. The short exact sequence induces a left exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(M'', -)|_{\mathcal{F}} \longrightarrow \operatorname{Hom}_A(M, -)|_{\mathcal{F}} \longrightarrow \operatorname{Hom}_A(M', -)|_{\mathcal{F}}$$

of functors, where $\mathcal{F}$ is the corresponding torsion-free class of $(\mathcal{T}, \mathcal{F})$.
If $M \in \mathcal{T}$, $\operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0$, so $\operatorname{Hom}_A(M', -)|_{\mathcal{F}} = 0$ and $M' \in \mathcal{T}$. This shows that $\mathcal{T}$ is closed under images. Similarly, if $M', M'' \in \mathcal{T}$, then $M \in \mathcal{T}$, so $\mathcal{T}$ is closed under extensions. Considering the short exact sequence $0 \longrightarrow M' \longrightarrow M' \oplus M'' \longrightarrow M'' \longrightarrow 0$ with $M', M'' \in \mathcal{T}$ gives that $\mathcal{T}$ is closed under direct sums.

(2) implies (3). Let the trace of M, denoted tM, be defined as the sum of images of A-module morphisms from $\mathcal{T}$ to M. Because $\mathcal{T}$ is closed under images and sums, tM lies in $\mathcal{T}$, and is actually the largest submodule of M which is in $\mathcal{T}$. Next, we show that the trace is a subfunctor of the indentity functor.

Take a homomorphism $f : M \longrightarrow N$. Then $f(tM) \subseteq tN$, so f restricts to a morphism $tM \longrightarrow tN$. If $X \in \mathcal{T}$ and there is a morphism $g : X \longrightarrow M$, the image of the morphism $f \circ g : X \longrightarrow N$ lies in tN by definition.

It is clear that $t(t(M)) \subseteq t(M)$. It is possible to write $t(M) = \sum N_i$, where each N_i is the image of some morphism $f_i : X_i \to M$. Because M is finite dimensional, we may consider only finitely many distinct N_i. Now consider the morphism $f : \bigoplus X_i \to M$, which in each coordinate is f_i. Clearly the image of this function is tM, so $tM \subseteq t(t(M))$, and $tM = t(t(M))$

It remains to show that $t(M/t(M)) = 0$. Assume that $t(M/tM) = M'/t(M)$, with $t(M) \subseteq M' \subseteq M$. Because $\mathcal{T}$ is closed under extensions, and since $t(M) \in \mathcal{T}$, and $M'/t(M) \in \mathcal{T}$, the extension $0 \longrightarrow tM \longrightarrow$

$M' \longrightarrow M'/tM \longrightarrow 0$ gives that $M' \in \mathcal{T}$. As $M' \subset M$, and $M' \in \mathcal{T}$, it follows that $M' \subseteq tM$, so $t(t(M)) = M'/t(M) = 0$. The conclusion is that the trace is an idempotent radical satisfying the condition that $\mathcal{T} = \{M \mid t(M) = M\}$.

If $tM = M$, M is the sum of images of A-homomorphism, which implies that $M \in \mathcal{T}$ as $\mathcal{T}$ is closed under images and direct sums. If $M \in \mathcal{T}$, $M = t(M)$ as M is the image of the identity morphism from M to itself, so $M \in \mathcal{T}$ if and only if $M = tM$, and $\mathcal{T} = \{M \mid t(M) = M\}$

(3) implies (1). Let $\mathcal{F} = \{N | tN = 0\}$. Then for all $M \in \mathcal{T}$, $\mathrm{Hom}_A(M, -)|_{\mathcal{F}} = 0$, as every morphism $f : M \longrightarrow N$ restricts to a morphism $f|_{tM} : tM \longrightarrow tN$, and as $tM = M$ and $tN = 0$ the morphism must be 0.

Next we show that, $\mathrm{Hom}(M, -)|_{\mathcal{F}} = 0$ implies that $M \in \mathcal{T}$. As $t(M/tM) = 0$, $M/tM \in \mathcal{F}$. If $\mathrm{Hom}(M, -)|_F = 0$, the projection $M \longrightarrow M/tM$ must be zero. As the projection is surjective, $M/tM = 0$, which implies that $M = tM$, so $M \in \mathcal{T}$.

Next we show that $\mathrm{Hom}(-, N)|_{\mathcal{T}} = 0$ implies that $N \in \mathcal{F}$. Note that $t(tN) = tN$ implies that $tN \in \mathcal{T}$, so the inclusion $tN \longrightarrow N$ is zero, so $tN = 0$, and $N \in \mathcal{F}$. So the pair of full subcategories $(\mathcal{T}, \mathcal{F})$ is a torsion pair. □

Theorem 4.3 makes it possible to induce a torsion pair $(\mathcal{T}, \mathcal{F})$ from an arbitrary class $\mathcal{C}$ of A-modules by setting $\mathcal{F} = \{M | \mathrm{Hom}(-, M)|_{\mathcal{C}} = 0\}$, and $\mathcal{T} = \{\mathrm{Hom}(M, -)|_{\mathcal{F}} = 0\}$. This is possible because $\mathcal{F}$ satisfies the condition (2) of Theorem 4.3 b. To show that $\mathcal{F}$ is closed under submodules, take $M \in \mathcal{F}$, $N \subseteq M$, and let $M \in \mathcal{C}$. Applying $\mathrm{Hom}(C, -)$, for an arbitrary $C \in \mathcal{C}$ to a short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ with $L, N \in \mathcal{F}$ gives $\mathrm{Hom}(C, M) = 0$, so $M \in \mathcal{F}$. Because there is an extension $0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0$, the direct product of M and N is in $\mathcal{F}$, and applying $\mathrm{Hom}(C, -)$ to the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ shows that if $\mathrm{Hom}(C, M) = 0$ it follows that $\mathrm{Hom}(C, N) = 0$. So by Theorem 4.3, $\mathcal{F}$ is the torsion free class of some torsion pair $(\mathcal{T}, \mathcal{F})$, and given $\mathcal{F}$ the only possible choice for the torsion class $\mathcal{T}$ is $\mathcal{T} = \{\mathrm{Hom}(M, -)|_{\mathcal{F}} = 0\}$.

Definition 4.4. Let T be an A-module.
Then $\mathrm{Gen}\, T$ is the class of modules M such that there is an integer $d \leq 1$ and an epimorphism $T^d \to M$.
$\mathrm{Cogen}\, T$ is the class of all modules M such that there exists an integer $d \leq 1$ and a monomorphism $M \to T^d$.

Lemma 4.5. *Let T and M be A-modules, and let $B = \mathrm{End}_A(T)$*

(1) $M \in \mathrm{Gen}\, T$ *if and only if the canonical homomorphism* $\epsilon_M : \mathrm{Hom}(T, M) \otimes_B T \to M$ *defined by* $\epsilon(f \otimes t) = f(t)$ *is surjective.*
(2) $M \in \mathrm{Cogen}\, T$ *if and only if the homomorphism* $\eta : M \to Hom_B(\mathrm{Hom}_A(M, T), T)$ *defined by* $\eta(x) = (g \mapsto g(x))$ *is injective.*

Proof. This proof follows [1] Lemma VI.1.8.
We only prove the first statement, as the proof of the second statement is dual. Take $M \in \mathrm{Gen}\, T$. Let $\{f_1, f_2, ..., f_d\}$ be a basis of the vector space $\mathrm{Hom}_A(T, M)$. Define $f : T^d \to M$ by $f(t_1, t_2, ..., t_d) = f_1(t_1) + f_2(t_2) + ... + f_d(t_d)$. We claim that f is an epimorphism. Because $M \in \mathrm{Gen}\, T$ by assumption, there is some $k \geq 1$ and some epimorphism $g : T^k \to M$. In the i:th coordinate, g can be written as a linear combination $g_i = \Sigma_{j=1}^{d} a_{ij} f_j$, as $\{f_1, f_2, ..., f_d\}$ was chosen to be a basis of $\mathrm{Hom}_A(T, M)$. Letting h be multiplication by the coefficient matrix $(a_{ij})_{ij}$ gives $h : T^k \to T^d$ such that $g = f \circ h$, so f is surjective since g is.

Now apply the functor $\mathrm{Hom}_A(T, -)$ to the short exact sequence $0 \longrightarrow \ker f \longrightarrow T^d \xrightarrow{f} M \longrightarrow 0$.
As $\mathrm{Hom}_A(T, f)$ is an epimorphism the sequence:

$$0 \longrightarrow \mathrm{Hom}_A(T, \ker f) \longrightarrow \mathrm{Hom}_A(T, T^d) \xrightarrow{\mathrm{Hom}_A(T,f)} \mathrm{Hom}_A(T, M) \longrightarrow 0$$

is exact.
Applying $- \otimes_B T$ to that sequence yields the following diagram, which is commutative as ϵ is a natural transformation:

$$\begin{array}{ccccccccc}
\cdots \operatorname{Tor}^1_B(T,M) & \longrightarrow & \operatorname{Hom}_A(T,\ker f)\otimes_B T & \longrightarrow & \operatorname{Hom}_A(T,T^d)\otimes_B T & \longrightarrow & \operatorname{Hom}_A(T,M)\otimes_B T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_{kerf}} & & \downarrow{\scriptstyle \epsilon_{T^d}} & & \downarrow{\scriptstyle \epsilon_M} & & \\
0 & \longrightarrow & \ker f & \longrightarrow & T^d & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The homomorphism $\epsilon_{T^d} : \operatorname{Hom}_A(T,T^d) \cong B^d \otimes T \cong T^d$ is an isomorphism, which by commutativity of the diagram means that ϵ_M is surjective.

For the converse, assume that ϵ_M is surjective. Because $\operatorname{Hom}_A(T,M)$ is a finitely generated B-module, there exists an $m \geq 1$ and an epimorphism $g : B^m \to \operatorname{Hom}_A(T,M)$, so there is an epimorphism

$T^m \cong B^m \otimes_B T \xrightarrow{g\otimes_B T} \operatorname{Hom}_A(T,M)\otimes_B T \xrightarrow{\epsilon_M} M.$

So, by definition, $M \in \operatorname{Gen} T$. □

The following Lemma gives a condition for $\operatorname{Gen} T$ to be a torsion class.

Lemma 4.6. *Let $T \in \operatorname{mod} A$.*

(1) *If $\operatorname{Ext}^1_A(T,-)|_{\operatorname{Gen} T} = 0$, $\operatorname{Gen} T$ is a torsion class with corresponding torsion-free class $\{M | \operatorname{Hom}_A(T,M) = 0\}$*

(2) *If $\operatorname{Ext}^1_A(-,T)|_{\operatorname{Cogen} T} = 0$, $\operatorname{Cogen} T$ is a torsion-free class with corresponding torsion class $\{M | \operatorname{Hom}_A(M,T) = 0\}$*

Proof. The proof follows [1] Lemma VI.1.9.
We only prove the first statement, as the proof of the second statement is dual.

Note that $\operatorname{Gen} T$ is closed under images and direct sums, so $\operatorname{Gen} T$ is a torsion class if $\operatorname{Gen} T$ is closed under extensions. Now, let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence, with $M', M'' \in \operatorname{Gen} T$. Since $M' \in \operatorname{Gen} T$, $\operatorname{Ext}^1_A(T,M') = 0$ by assumption. By Theorem 2.9 there is a short exact sequence $0 \longrightarrow \operatorname{Hom}_A(T,M') \longrightarrow \operatorname{Hom}_A(T,M) \longrightarrow \operatorname{Hom}_A(T,M'') \longrightarrow 0$. Apply the functor $- \otimes_B T$ to get the upper row in the commutative diagram

$$\begin{array}{ccccccccc}
\operatorname{Tor}_B(T,M'') & \longrightarrow & \operatorname{Hom}_A(T,M')\otimes_B T & \longrightarrow & \operatorname{Hom}_A(T,M)\otimes_B T & \longrightarrow & \operatorname{Hom}_A(T,M'')\otimes_B T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_{M'}} & & \downarrow{\scriptstyle \epsilon_M} & & \downarrow{\scriptstyle \epsilon_{M''}} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

By Lemma 4.5, $\epsilon_{M'}$ and $\epsilon_{M''}$ are isomorphisms, so by the five lemma, ϵ_M is an isomorphism. Again by Lemma 4.5, this means that $M \in \operatorname{Gen} T$, so $\operatorname{Gen} T$ is closed under extensions, and therefore a torsion class. □

4.2. **Tilting modules.** In order to define support tilting modules, and construct the bijection between support tilting modules and torsion classes, the definition of partial tilting and tilting modules, as well as some properties of tilting and partial tilting modules need to be introduced. The material presented here can be found in [1] chapter IV.

Definition 4.7. An A-module T is called partial tilting if:

(1) $\operatorname{pd}(T) \leq 1$

(2) $\operatorname{Ext}^1_A(T,T) = 0$

Definition 4.8. An A-module T is tilting if it is partial tilting and there exists a short exact sequence
$0 \longrightarrow A_A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ where $T', T'' \in \operatorname{add}(T)$.

The following definitions will be crucial in constructing the bijection between support tilting modules and torsion classes.

Definition 4.9. Let $\mathcal{T}$ be a subcategory of $\operatorname{mod} A$. Then a module $P \in \mathcal{T}$ is split projective in $\mathcal{T}$ if every surjective morphism $f : M \longrightarrow P$ splits, i.e there is some morphism $g : P \longrightarrow M$ such that $f \circ g = id_P$
$P \in \mathcal{T}$ is Ext projective in $\mathcal{T}$ if $\operatorname{Ext}^1_A(P,M) = 0$ for all $M \in \mathcal{T}$.

The following Lemma gives the relationship between Ext-projective and split projective.

Lemma 4.10. *Let $\mathcal{T}$ be a subcategory of $\operatorname{mod} A$. If $\mathcal{T}$ is closed under extensions and P is split projective in $\mathcal{T}$, P is Ext-projective in $\mathcal{T}$.*

Proof. Take $M \in \mathcal{T}$, and take an extension $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ of M by P. As $\mathcal{T}$ is closed under extensions, $N \in \mathcal{T}$, and as P is split projective in $\mathcal{T}$, the surjection $f : N \twoheadrightarrow P$ splits. This implies that the extension splits, and as any extension in $\operatorname{Ext}^1_A(P, M)$ splits, $\operatorname{Ext}^1_A(P, M) = 0$, so P is Ext-projective in $\mathcal{T}$. □

The following lemmas show some of the ways the notion of Ext-projective relates to partial tilting modules and torsion classes.

Lemma 4.11. *Let T be an A-module. If T is a partial tilting A-module, then $\operatorname{Gen} T$ is a torsion class in which T is Ext-projective, and the corresponding torsion-free class is $\mathcal{F}(T) = \{M \mid \operatorname{Hom}_A(T, M) = 0\}$*

Proof. The proof follows the proof of [1] Lemma VI.2.3.
Take $M \in \operatorname{Gen} T$. Then there is some $m \geq 1$ such that there is a surjection $f : T^m \to M$. It follows from the construction of $\operatorname{Ext}^n_A(T, -)$ that $\operatorname{pd} T \leq 1$ implies that $\operatorname{Ext}^2_A(T, -) = 0$. By Theorem 2.9 the short exact sequence

$$0 \longrightarrow ker f \xrightarrow{\iota} T^m \xrightarrow{f} M \longrightarrow 0$$

induces an exact sequence $\dots \longrightarrow \operatorname{Ext}^1_A(T, ker f) \longrightarrow \operatorname{Ext}^1_A(T, T^m) \longrightarrow \operatorname{Ext}^1_A(T, M) \longrightarrow 0$, so there is a surjection $f' : \operatorname{Ext}^1_A(T, T^m) \longrightarrow \operatorname{Ext}^1_A(T, M)$, and since $\operatorname{Ext}^1_A(T, T^m) = 0$, $\operatorname{Ext}^1_A(T, M) = 0$. This means that T is Ext-projective in $\operatorname{Gen} T$, and by Lemma 4.6 $\operatorname{Gen} T$ is a torsion class with corresponding torsion-free class $\mathcal{F}(T) = \{M | \operatorname{Hom}_A(T, M)\} = 0$. □

Lemma 4.12. *If T is a partial tilting module, there exists an A-module E such that $T \oplus E$ is a tilting module.*

Proof. The proof is in [1], see Lemma V.I.2.4. □

The module E is known as Bongartz's complement of T. Furthermore, in the proof of Lemma 4.12 a short exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow T^d \longrightarrow 0$ is constructed, where E is the A-module such that $E \oplus T$ is tilting. This sequence is known as Bongartz's exact sequence.

The following theorem gives some conditions on partial tilting modules which are equivalent to being tilting.

Theorem 4.13. *Define $\mathcal{T}(T) = \{M | \operatorname{Ext}^1_A(T, M) = 0\}$. If T is a partial tilting module, the following are equivalent:*

(1) *T is a tilting module*
(2) $\operatorname{Gen} T = \mathcal{T}(T)$
(3) *For every $M \in \mathcal{T}(T)$ there is a short exact sequence $0 \longrightarrow L \longrightarrow T_0 \longrightarrow M \longrightarrow 0$, with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$.*
(4) *Let X be an A-module. Then $X \in \operatorname{add} T$ if and only if X is Ext-projective in $\mathcal{T}(T)$*

Proof. The proof follows [1] Theorem VI.2.5.
(1) implies (2). Let $M \in \mathcal{T}(T)$. Showing that $M \in \operatorname{Gen} T$ is equivalent to showing that $M \cong tM$, where t is the idempotent radical associated to the torsion pair $(\operatorname{Gen} T, \mathcal{F}(T))$. Applying $\operatorname{Hom}_A(T, -)$ to the canonical sequence $0 \to tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$ yields an epimorphism $\operatorname{Ext}^1_A(T, M) \longrightarrow \operatorname{Ext}^1_A(T, M/tM)$. Since $M \in \mathcal{T}(T)$, $\operatorname{Ext}^1_A(T, M) = 0$, which implies that $\operatorname{Ext}^1_A(T, M/tM) = 0$.
Because T is a tilting module, there is a short exact sequence $0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$, with $T', T'' \in \operatorname{add} T$. Applying $\operatorname{Hom}_A(-, M/tM)$ yields that the sequence $\operatorname{Hom}_A(T', M/tM) \longrightarrow \operatorname{Hom}_A(A, M/tM) \longrightarrow \operatorname{Ext}^1_A(T'', M/tM)$ is exact. Because $M/tM \in \mathcal{F}(T)$ and $T' \in \operatorname{add} T$, $\operatorname{Hom}_A(T', M/tM) = 0$, and since $T'' \in \operatorname{add} T$, $\operatorname{Ext}^1_A(T'', M/tM) = 0$. This gives $M/tM \cong \operatorname{Hom}_A(A, M/tM) = 0$, so $M = tM \in \operatorname{Gen} T$.
(2) implies (3).Take $M \in \mathcal{T}(T)$. Because $M \in \operatorname{Gen} T$ by assumption, the function $f = [f_1, ..., f_d] : T^d \to M$, where $f_1, ..., f_d$ is a basis of $\operatorname{Hom}^1_A(T, M)$, is surjective. Applying the functor $\operatorname{Hom}_A(T, -)$ to the short exact sequence

(1) $$0 \longrightarrow \ker f \longrightarrow T^d \longrightarrow M \longrightarrow 0$$

gives, by Theorem 2.9, an exact sequence ... $\longrightarrow \operatorname{Hom}_A(T, T^d) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow \operatorname{Ext}^1_A(T, \ker f) \longrightarrow 0$. $\operatorname{Hom}_A(T, f) : \operatorname{Hom}_A(T, T^d) \to \operatorname{Hom}_A(T, M)$ is an epimorphism by construction. By exactness, this means that $\operatorname{Ext}^1_A(T, \ker f) = 0$, and $\ker f \in \mathcal{T}(T)$.
Since $T^d \in \operatorname{add} T$, (1) is a sequence as in (3).
(3) implies (4). Take $X \in \operatorname{add} T$. Then $\operatorname{Ext}^1_A(X, M) = 0$ for any M such that $\operatorname{Ext}^1_A(T, M) = 0$, so X is Ext-projective in $\mathcal{T}(T)$.
For the converse, assume M is Ext-projective in $\mathcal{T}(T)$. Consider the exact sequence

$$0 \longrightarrow L \longrightarrow T_0 \xrightarrow{f} X \longrightarrow 0$$

with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$. Apply $\operatorname{Hom}_A(X, -)$ to this sequence, to get an exact sequence $0 \longrightarrow \operatorname{Hom}_A(X, L) \longrightarrow \operatorname{Hom}_A(X, T_0) \longrightarrow \operatorname{Hom}_A(X, X) \longrightarrow \operatorname{Ext}^1_A(X, L) \longrightarrow \cdots$. Because X is Ext-projective in $\mathcal{T}$, $\operatorname{Ext}^1_A(X, L) = 0$. Exactness then implies that $\operatorname{Hom}_A(X, f)$ is surjective, which implies that f splits. Because f splits, so does the short exact sequence, and $X \in \operatorname{add} T$.
(4) implies (1). Let T be partial tilting and $0 \longrightarrow A \longrightarrow E \longrightarrow T^d \longrightarrow 0$ be Bongartz's exact sequence corresponding to T. To show that T is tilting it is enough to show that $E \in \operatorname{add} T$, because $T \oplus E$ is a tilting module. By assumption it is equivalent to showing that E is Ext-projective in $\mathcal{T}(T)$. Because $T \oplus E$ is tilting, $\operatorname{Ext}^1_A(T, E) = 0$, so $E \in \mathcal{T}(T)$. Applying $\operatorname{Hom}_A(-, M)$, for some $M \in \mathcal{T}(T)$ to the given Bongartz sequence gives an exact sequence:

$$\ldots \longrightarrow \operatorname{Ext}^1_A(T^d, M) \longrightarrow \operatorname{Ext}^1_A(E, M) \longrightarrow \operatorname{Ext}^1_A(A, M) \longrightarrow \ldots$$

Because $\operatorname{Ext}^1_A(T^d, M) = \operatorname{Ext}^1_A(A, M) = 0$, exactness gives $\operatorname{Ext}^1_A(E, M) = 0$, so E is Ext-projective in $\mathcal{T}(T)$. □

Definition 4.14. The support of a module M, $\operatorname{supp} M$, is the set of simple modules that occur in the Jordan-Hölder series for M.

The following lemma gives an equivalent definition of the support of a module M.

Lemma 4.15. *The support* $\operatorname{supp} M$ *is the set of simple modules that occur as quotients of finite sums of* M*.*

Proof. If $S \in \operatorname{supp} M$, S is a subquotient of M by definition.

In the proof of Theorem 2.14, it is established that $\mu_{M,S}$ is the number of times the simple module S occurs as a composition factor in some composition series of M. So it follows that $\operatorname{supp} M = \{S \mid S$ is simple, and $\mu_{M,S} \neq 0\}$. If S is a subquotient of M^i for some i, there is a short exact sequence $0 \longrightarrow L \longrightarrow M' \longrightarrow S \longrightarrow 0$, where $M' \subseteq M^i$, so $\mu_{M',S} \neq 0$. Since $M' \subseteq M^i$, there is also a short exact sequence $0 \longrightarrow M' \longrightarrow M^i \longrightarrow M^i/M' \longrightarrow 0$, so $\mu_{M^i,S} \neq 0$. Because there is a short exact sequence $0 \longrightarrow M^{i-1} \longrightarrow M^i \longrightarrow M \longrightarrow 0$, it follows that $\mu_{M,S} \neq 0$ if and only if $\mu_{M^i,S} \neq 0$, so $S \in \operatorname{supp} M$. So $S \in \operatorname{supp} M$ if and only if S is a subquotient of M^i for some i. □

Lemma 4.16. *Let Q be a finite acyclic quiver. Let C be a partial tilting KQ-module, and let M be a KQ-module. Then* $\operatorname{supp} M \subseteq \operatorname{supp} C$ *if and only if M is a subquotient of C^i for some i.*

Proof. This proof follows the proof of [4] Lemma 2.4.
Assume that $\operatorname{supp} M \subseteq \operatorname{supp} C$. This means that the Jordan-Hölder series of M is made up of subquotients of C^i for some i, by Lemma 4.15. This means that the statement follows if the set of quotients of C^i for some i is closed under extensions.
Let x, y be submodules of X and Y respectively, which are quotients of C^i for some i. Applying the functor $\operatorname{Hom}_A(x, -)$ to the short exact sequence $0 \longrightarrow y \longrightarrow Y \longrightarrow Y/y \longrightarrow 0$ gives a map $\operatorname{Ext}^1_{KQ}(x, y) \to \operatorname{Ext}^1_{KQ}(x, Y)$. Similarly applying the functor $\operatorname{Hom}_{KQ}(-, Y)$ to the short exact sequence $0 \longrightarrow x \longrightarrow X \longrightarrow X/x \longrightarrow 0$ gives a map $\operatorname{Ext}^1_{KQ}(X, Y) \to \operatorname{Ext}^1_{KQ}(x, Y)$, which will be surjective as $\operatorname{Ext}^2_{KQ}$ vanishes, as the $\operatorname{pd} M \leq 1$ for all KQ-modules M. Mapping an extension $e \in \operatorname{Ext}^1_{KQ}(x, y) \to \operatorname{Ext}^1_{KQ}(x, Y)$ and lifting via the surjective map yields an extension $E \in \operatorname{Ext}^1_{KQ}(X, Y)$. As C is a partial tilting module, $\operatorname{Gen} C$ is a torsion

class closed under extensions, so $E \in \operatorname{Gen} C$, so E is a subquotient of C^i for some i.
Now let $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = M$ be a composition series of M. As $M_0 = 0$, $M_1/M_0 = M_1$, so M_1 is in $\operatorname{supp} M$, and thus a subqotient of C^i for some i, by definition. Now assume that M_{i-1} is a subquotient of C^i for some i. There is an extension $0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$, where $M_i/M_{i-1} \in \operatorname{supp} M$ by definition, and M_{i-1} is a subquotient of C^i for some i by assumption, so as the set of modules which are subquotients of C^i for some i is closed under extensions, M_i is a subquotient of C^i for some i. It follows by induction that $M_n = M$ is a subquotient of C^i for some i.
Now assume M is a subquotient of C^i for some i. If $N \in \operatorname{supp} M$, N is a subquotient of M^k for some k. As M is a subquotient of C^i for some i, so is M^k, and therefore also N. So $\operatorname{supp} M \subseteq \operatorname{supp} C$. $\square$

4.3. **Support tilting modules.** Now, support tilting modules can be defined, as well as the bijection between support tilting modules and torsion classes. The material presented here is mostly from the paper by Ingalls and Thomas, [4], together with some lemmas from [1] chapter VI.

Throughout, fix a finite acyclic quiver Q.

Proposition 4.17. *If C is a partial tilting module, the following are equivalent:*

(1) *C is tilting as a $KQ/\operatorname{Ann} C$ module, where $\operatorname{Ann} C$ is the annihilator ideal, defined by: $\operatorname{Ann} C = \{a \in KQ \mid Ca = 0\}$*
(2) *If M is a subquotient of C^i for some i, and $\operatorname{Ext}^1_{KQ}(C, M) = 0$, then $M \in \operatorname{Gen} C$.*
(3) *If $\operatorname{supp} M \subseteq \operatorname{supp} C$ and $\operatorname{Ext}^1_{KQ}(C, M) = 0$, then $M \in \operatorname{Gen} C$.*

Proof. See Proposition 2.5 in [4] for an outline of a proof that (1) is equivalent to (2). That (2) is equivalent to (3) follows directly from Lemma 4.16. □

Definition 4.18. A KQ-module C is called support tilting if it is partial tilting and satisfies any of the equivalent conditions in proposition 4.17.

Definition 4.19. A subcategory $\mathcal{T}$ of $\operatorname{mod} KQ$ is generated by $P \subseteq \mathcal{T}$ if $\mathcal{T} \subseteq \operatorname{Gen} P$. Furthermore, $\mathcal{T}$ is finitely generated if there is a finite set P of indecomposable modules such that P generates $\mathcal{T}$.
U is a minimal generator of $\mathcal{T}$ if U is a generator of $\mathcal{T}$ and for every direct sum decomposition $U \cong U' \oplus U''$ U' is not generated by U''.

The next few lemmas will be used to show that the bijection between basic support tilting modules and torsion classes is well-defined, but also that it actually is a bijection.

Lemma 4.20. *A finitely generated torsion class $\mathcal{T}$ has a minimal generator, which is unique up to isomorphism and given by the direct sum of the indecomposable split projectives in $\mathcal{T}$.*

Proof. This is proven following the proof of Lemma 2.8 in [4] If $\mathcal{T}$ is finitely generated, it has a generator U. Because generators of $\mathcal{T}$ are partially ordered by the relation of being a direct summand, and descending chains of A-modules stabilize, $\mathcal{T}$ has a minimal generator. Now what remains to show is that this minimal generator is the direct sum of all indecomposable split projectives in $\mathcal{T}$. Let the generator U be the direct sum $U \cong \bigoplus_{i=1}^n U_i$, where the U_i are distinct indecomposables. Take an indecomposable split projective U' in $\mathcal{T}$. Since U generates $\mathcal{T}$, there is a surjection $U^i \to U'$, which must split as U' is split projective. This implies that U' is a direct summand of U, and as the decomposition of U into indecomposables is unique, this implies that $U' = U_i$ for some i. So all indecomposable split projectives are direct summands of the minimal generator.

Now assume that U_1 is an indecomposable direct summand of the minimal generator U. Let $B = \operatorname{End}_A(U_1)$, and consider some epimorphism $v : E \twoheadrightarrow U_1$, where $E \in \mathcal{T}$. Because $E \in \operatorname{Gen}(U) = \mathcal{T}$, there is some epimorphism $h : U^m \twoheadrightarrow E$. Let $R = \bigoplus_{i=2}^n U_i$, meaning that $U = R \bigoplus U_1$, and $U^m = R^m \oplus U_1^m$. So we can write $v \circ h = [g, f_0, ..., f_m]$, with $f_i \in B$ for each i, and $g \in \operatorname{Hom}_A(R^m, U_1)$. Because U_1 is indecomposable, B has a unique maximal ideal J, and there is an integer N such that $J^N = 0$, by Lemma 2.2. Furthermore, J must consist of exactly the elements of B which are not invertible.

Assume that all $f_i \in J$. Then $U_1 = g(R^m) + \sum_{i=1}^n f_i(U_1) \subseteq g(R^m) + JU_1$. This implies that $J^k U_1 \subseteq J^k g(R^m) + J^{k+1} U_1 \subseteq g(R^m) + J^{k+1} U_1$, for any k, which in turn implies that $U_1 \subseteq g(R^m) + J^k U_1$. As there is an N such that $J^N = 0$, it follows that $U_1 \subseteq g(R^m)$. But by definition this means that $U_1 \in \operatorname{Gen} R$, which contradicts the choice of U as a minimal generator.

So there must be at least one f_i such that $f_i \notin J$. Then f_i must be invertible. Then the composition $v \circ h \circ [0, ..., f_i^{-1}, ..., 0]^T = id_{U_1}$, so v splits. Since V was arbitrary, it follows that U_1 is split projective in $\operatorname{Gen} U = \mathcal{T}$. □

Lemma 4.21. *Let U be a Gen-minimal module, i.e there is no direct sum decomposition so that $U \cong U' \oplus U''$ with $U' \in \operatorname{Gen}(U'')$, such that $\operatorname{Gen}(U)$ is a torsion class. Then U is Ext-projective in $\operatorname{Gen} U$.*

Proof. See the proof of [1] Lemma VI.6.1. □

Lemma 4.22. *Let U be a Gen-minimal faithful, i.e. $\operatorname{Ann} U$ vanishes, A-module such that $\operatorname{Gen}(U)$ is a torsion class. Then U is partial tilting.*

Proof. For a proof of this, see the proof of [1] Corollary VI.6.3. □

Lemma 4.23. *An A-module M is faithful, if and only if, for any basis $\{f_1, f_2, ..., f_n\}$ of $\mathrm{Hom}_A(A, M)$ the function $f = [f_1, f_2, ..., f_n]^T : A \longrightarrow M^n$ is injective.*

Proof. This proof is part of the proof of [1] Lemma VI.2.2.
Note that $f(a) = 0$ if and only if $g(a) = 0$ for any $g \in \mathrm{Hom}_A(A, M)$. Given the canonical homomorphism $\mathrm{Hom}_A(A, M)$, this is equivalent to saying that $Ma = 0$. This implies the equivalence, as if M is faithful, the only $a \in A$ such that $Ma = 0$ is 0, which implies that f is injective, and if f is injective, the only $a \in A$ satisfying $Ma = 0$ is 0. □

Lemma 4.24. *Let Q be a finite acyclic quiver. Let $\mathcal{T}$ be a finitely generated torsion class in $\mathrm{mod}\, KQ$, and let C be the direct sum of the indecomposable Ext-projectives in $\mathcal{T}$. Then C is support tilting.*

Proof. This is proven following [4] Lemma 2.9.
Let U be the sum of all split projectives in $\mathcal{T}$. By Lemma 4.20, $\mathcal{T} = \mathrm{Gen}\, U$. Also let $A' = KQ/\,\mathrm{Ann}\, U$. First note that there is an injection $\mathrm{mod}\, A' \hookrightarrow \mathrm{mod}\, A$. Next, note that, as $U \in \mathrm{mod}\, A'$ and U generates $\mathcal{T}$, we have $\mathcal{T} \subseteq \mathrm{mod}\, A'$. Moreover, if $\mathcal{T}$ is a torsion class in $\mathrm{mod}\, A$ it is also a torsion class in $\mathrm{mod}\, A'$, as Theorem 4.3 a. (2) holds for $\mathcal{T}$ in $\mathrm{mod}\, A'$.

Let $\{f_1, f_2, ..., f_n\}$ be a basis of $\mathrm{Hom}_{A'}(A', U)$. Let $f = [f_1, f_2, ..., f_n] : A' \to U^n$. U is faithful, so f is injective, by Lemma 4.23. This gives a short exact sequence

$$0 \longrightarrow A' \longrightarrow U^n \longrightarrow \mathrm{coker}(f) = U' \longrightarrow 0.$$

Now $U' \in \mathrm{Gen}\, U = \mathcal{T}$. We want to show that U' is Ext-projective in $\mathcal{T}$. So take $M \in \mathcal{T}$, and apply $\mathrm{Hom}_{A'}(-, M)$ to the previous short exact sequence to get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{A'}(U', M) \longrightarrow \mathrm{Hom}_{A'}(U^n, M) \longrightarrow \mathrm{Hom}_{A'}(A', M) \longrightarrow \mathrm{Ext}^1_{A'}(U', M) \longrightarrow \mathrm{Ext}^1_{A'}(U^n, M) = 0.$$

That $\mathrm{Ext}^1_{A'}(U^n, M) = 0$ follows because U is Ext-projective in $\mathcal{T}$ by Lemma 4.21. If $\mathrm{Hom}_{A'}(f, M) : \mathrm{Hom}_{A'}(U^n, M) \longrightarrow \mathrm{Hom}_{A'}(A', M)$ is surjective, it will follow from exactness that $\mathrm{Ext}^1_{A'}(U', M) = 0$.

The morphism $\mathrm{Hom}_{A'}(f, M)$ is surjective if for every $h \in \mathrm{Hom}_{A'}(A', M)$ their is some $u \in \mathrm{Hom}_{A'}(U^n, M)$ such that $h = u \circ f$.

Because $M \in \mathcal{T} = \mathrm{Gen}\, U$, there is a surjection $g : U^m \twoheadrightarrow M$, for some m. Because A' is projective as a module over itself, there is some morphism p making the following diagram commute:

$$\begin{array}{ccc} A' & & \\ \downarrow p & \searrow^{h} & \\ U^m & \xrightarrow{g} & M \end{array}$$

Furthermore, by definition of f there is some $q \in \mathrm{Hom}_{A'}(U^n, U^m)$, making the following diagram commute:

$$\begin{array}{ccc} U^n & \xleftarrow[f]{} & A' \\ \downarrow q & \swarrow_{p} & \\ U^m & & \end{array}$$

This gives that $g \circ q \circ f = g \circ p = h$, so letting $u = g \circ q$, u will satisfy $u \circ f = h$, so $\mathrm{Hom}_{A'}(f, M)$ is surjective, and it follows by exactness that $\mathrm{Ext}^1_{A'}(U', M) = 0$, and U' is Ext-projective in $\mathcal{T}$.

Note also that $\mathrm{pd}\, U \leq 1$ as U is a partial tilting module by Lemma 4.22, which implies $\mathrm{pd}\, U' \leq 1$. So U' is partial tilting.

Now let $T = U \oplus U'$. Then $\mathrm{pd}\, T \leq 1$, as both U and U' have projective dimension smaller than or equal to 1, and $\mathrm{Ext}^1_{A'}(T, T) = 0$, as both U and U' are Ext-projective in $\mathcal{T}$. Furthermore, the short exact sequence $0 \longrightarrow A' \longrightarrow U^n \longrightarrow U' \longrightarrow 0$ shows that T is tilting as a module over A', as $U', U \in \mathrm{add}\, T$. This gives that T is support tilting.

It follows from Theorem 4.13 that all Ext-projectives in $\mathcal{T}$ are summands of T, so the sum of all Ext-projectives in $\mathcal{T}$ is a support tilting module. □

Definition 4.25. Let $\mathcal{T}$ be a subcategory of $\mathrm{mod}\, A$, and $X \in \mathrm{mod}\, A$.
A $\mathcal{T}$ right approximation of X is a map $f : Y \longrightarrow X$, where $Y \in \mathcal{T}$ and for every morphism $g : Z \longrightarrow X$ factors through f, i.e there is a morphism $f' : Z \longrightarrow Y$ such that $f \circ f' = g$.

Theorem 4.26. *Let Q be an acyclic quiver, and let C be a support tilting module in* $\operatorname{mod} KQ$*. Then* $\operatorname{Gen} C$ *is a torsion class, and the indecomposable Ext-projectives of* $\operatorname{Gen} C$ *are all the indecomposable summands of* C*. So* $\operatorname{bsc} C$ *is the sum of all indecomposable Ext-projectives of* $\operatorname{Gen} C$*.*

Proof. This proof follows [4] Theorem 2.10.

Let V be an indecomposable Ext-projective of $\operatorname{Gen} C$. First we claim that an $\operatorname{add} C$ right approximation of V exists. Take $V \in \operatorname{Gen} C$. Let the indecomposable direct summands of C be $\{U_1, ..., U_n\}$, and let $f_{i,1}, ..., f_{i,m_i}$ be a basis for $\operatorname{Hom}_{KQ}(U_i, V)$. Then $f : B = \bigoplus_{i=1}^{n} U_i^{m_i} \to V$ given by $(f_{i,j})$ in each coordinate is an $\operatorname{add} C$ right approximation of V (see proof of Lemma 4.5 for more details).

Because V is in $\operatorname{Gen} C$, there is a surjection $g : C^i \to V$ for some i, a which must factor through f since $C^i \in \operatorname{add} C$. This gives that f is surjective. Applying the functor $\operatorname{Hom}_{KQ}(C, -)$ to the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow B \longrightarrow V \longrightarrow 0 \tag{2}$$

gives an exact sequence

$$\operatorname{Hom}_{KQ}(C, B) \longrightarrow \operatorname{Hom}_{KQ}(C, Q) \longrightarrow \operatorname{Ext}^1_{KQ}(C, \ker f) \longrightarrow \operatorname{Ext}^1_{KQ}(C, B) = 0.$$

Since $B \in \operatorname{add} C$, and C is partial tilting, we have that $\operatorname{Ext}^1_{KQ}(C, B) = 0$. Furthermore, the map $\operatorname{Hom}_{KQ}(C, B) \to \operatorname{Hom}(C, V)$ is surjective, because f is a right approximation of V, which gives that $\operatorname{Ext}^1_{KQ}(C, \ker f) = 0$. Furthermore, $\operatorname{supp} \ker f \subseteq \operatorname{supp} C$, which means, by Theorem 4.17, that $\ker f \in \operatorname{Gen} C$, as C is assumed to be support tilting. Applying $\operatorname{Hom}_{KQ}(V, -)$ to (2) gives an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{KQ}(V, \ker f) \longrightarrow \operatorname{Hom}_{KQ}(V, B) \longrightarrow \operatorname{Hom}_{KQ}(V, V) \longrightarrow \operatorname{Ext}^1_{KQ}(V, \ker f) = 0.$$

We know $\operatorname{Ext}^1_{KQ}(V, \ker f) = 0$ as V is Ext-projective in $\operatorname{Gen} C$, and $\ker f \in \operatorname{Gen} C$. Exactness gives that $\operatorname{Hom}_{KQ}(V, f)$ is surjective, so f splits, and V is a direct summand of B, and therefore also of C, as $B \in \operatorname{add} C$. This gives that any Ext-projective of $\operatorname{Gen} C$ is a direct summand of C.
As $\operatorname{Ext}^1_{KQ}(C, C) = 0$, it follows that C is Ext-projective in $\operatorname{Gen} C$. It also follows from Lemma 4.6 that $\operatorname{Gen} C$ is a torsion class. □

We can now construct the bijection between basic support tilting modules and torsion classes.

Theorem 4.27. *Let Q be a finite acyclic quiver. Then there is a bijection between isomorphism classes of basic support tilting modules in* $\operatorname{mod} KQ$ *and torsion classes in* $\operatorname{mod} KQ$*.*

Proof. This is proven following [4] Theorem 2.11.

Let F be the function which takes a basic support tilting module M to $\operatorname{Gen}(M)$. By Theorem 4.26, $F(M)$ is a torsion class, so F is well-defined. Let G be the function which takes a torsion class $\mathcal{T}$ to the sum of all its indecomposable Ext-projectives. By Lemma 4.24 $G(\mathcal{T})$ is a basic support tilting module, so G is also well-defined.

Now we will show that G is the inverse of F. Let $\mathcal{T}$ be a torsion class. Let M be the sum of all Ext-projectives in $\mathcal{T}$. Let U be the sum of all indecomposable split projectives in $\mathcal{T}$. By Lemma 4.20, $\mathcal{T}$ is generated by U. Because split projective implies Ext-projective, U is a direct summand of M. It follows that $\mathcal{T} = \operatorname{Gen}(U) \subseteq \operatorname{Gen}(M) \subseteq \mathcal{T}$, so $\operatorname{Gen}(M) = \mathcal{T}$. The conclusion of this is that $G \circ F = id$.

Let M be support tilting and basic. Then $F \circ G(M)$ is the direct sum of all indecomposable Ext-projectives of $\operatorname{Gen} C$, which by Theorem 4.26 are exactly the indecomposable summands of M. So $F \circ G = id$, as M is assumed to be basic.

All of this gives that F and G are inverses, and therefore bijections. □

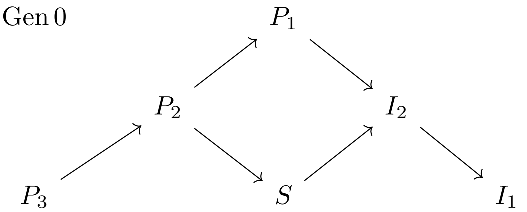
Example 4.28. We return to the example of the quiver $1 \longrightarrow 2 \longrightarrow 3$.

The support tilting representations of Q are

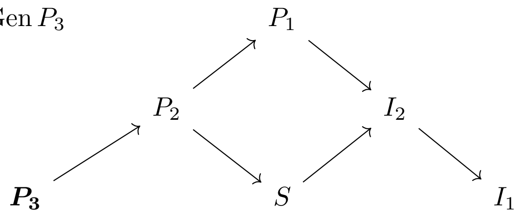
0	$P_2 \oplus P_3$	$P_1 \oplus P_2 \oplus P_3$
P_3	$P_2 \oplus S$	$P_1 \oplus P_2 \oplus S$
S	$S \oplus I_2$	$P_1 \oplus I_2 \oplus S$
I_1	$I_2 \oplus I_1$	$P_1 \oplus I_2 \oplus I_1$
	$P_3 \oplus I_1$	$P_1 \oplus P_3 \oplus I_1$

To find the indecomposables in the torsion classes in $Rep_K Q$, it is enough to find which indecomposables are generated by each of the support tilting modules. The diagram from Example 3.7 gives what is generated by each support tilting module, as the indecomposable modules in $\operatorname{Gen} T$ are the indecomposables which are direct summands of T as well as the indecomposables M such that $M \in \operatorname{Gen} T$.

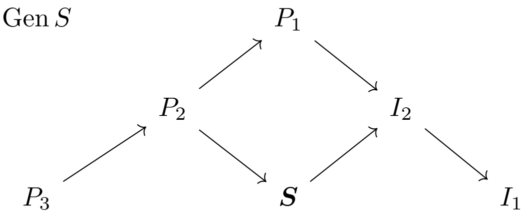
$\operatorname{Gen} 0$



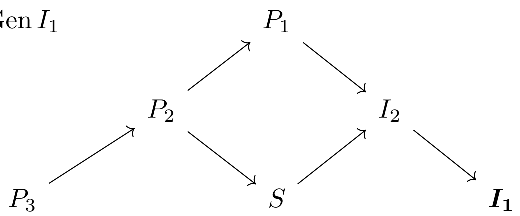
$\operatorname{Gen} P_3$



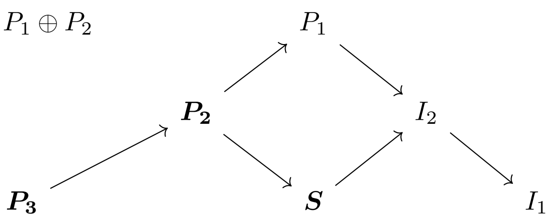
$\operatorname{Gen} S$



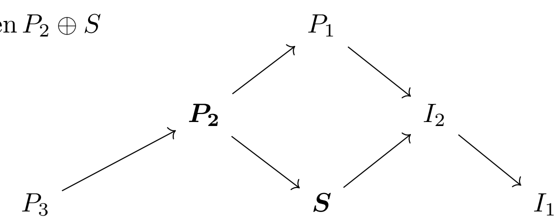
$\operatorname{Gen} I_1$



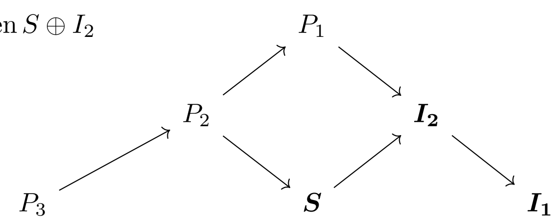
$\operatorname{Gen} P_1 \oplus P_2$



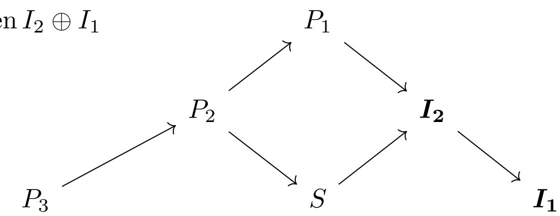
$\operatorname{Gen} P_2 \oplus S$



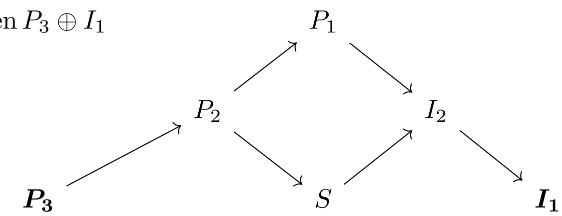
$\operatorname{Gen} S \oplus I_2$



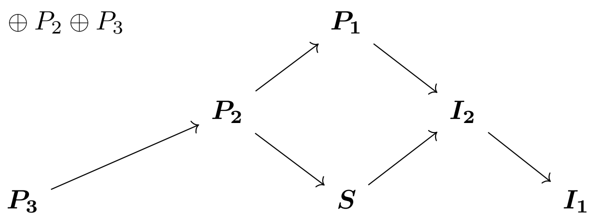
Gen $I_2 \oplus I_1$



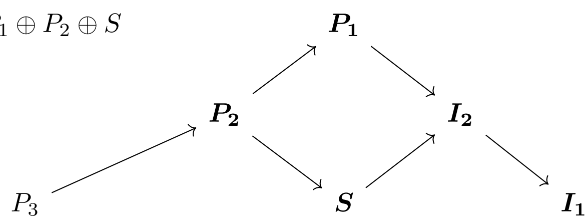
Gen $P_3 \oplus I_1$



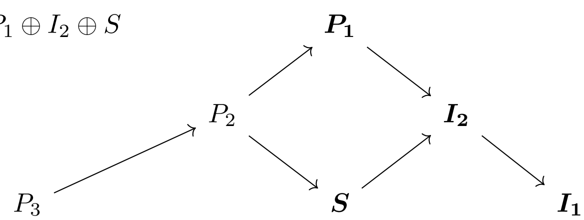
Gen $P_1 \oplus P_2 \oplus P_3$



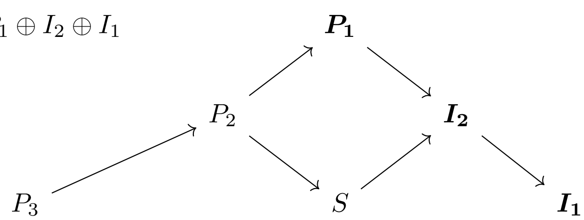
Gen $P_1 \oplus P_2 \oplus S$



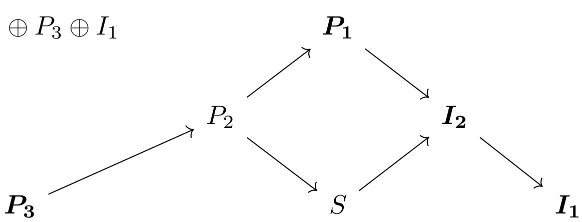
Gen $P_1 \oplus I_2 \oplus S$



Gen $P_1 \oplus I_2 \oplus I_1$



Gen $P_1 \oplus P_3 \oplus I_1$



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