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The Vladimirov Heat Kernel in the Program of Jorgenson-Lang

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Introduction

This thesis discusses possible p-adic analogues to the classical heat equation. The main motivation for this is the fact that the kernel to the heat equation

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

combined with the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \hat{f}\,(k)\, e^{2\pi i k x}$$

yields the theta inversion formula

$$\theta(\frac{1}{z}) = \sqrt{z}\theta(z),$$

which constitutes a main ingredient in many proofs in analytic number theory (for example the functional equation of the Riemann zeta function, by applying the Mellin transform, and quadratic reciprocity [11]). The most direct way to try to model this in a p-adic setting is to consider continuous functions

$$f : \mathbb{Q}_p \to \mathbb{C},$$

as one then can use the machinery of harmonic analysis on locally compact groups. Some work has been done in this direction, initiated by V.S. Vladimirov [19] with his introduction of a certain pseudo-differential operator on complex-valued functions of a p-adic variable, in some respects analogue to the classical Laplacian. This as well as subsequent research seem to mainly have been motivated by finding p-adic analogues to the mathematical models in modern physics, for example constructing a p-adic quantum theory, as well as p-adic versions of several equations of classical physics (see [20] and references therein). Specifically, an analogue to the heat equation has been proposed in this setting. We will give an account of the one-dimensional version of this, confirming results in [3, 14, 19], and connect it to a Poisson summation formula. We will also introduce a more "regularized" operator, and pass it through the same scheme.

On the other hand, p-adic numbers have been used extensively in number theory. Notably, John Tate used the adele ring, a construction considering all p-adic fields and the reals "at once", in combination with an adelic Poisson summation formula to explain all the components of the functional equation for the Riemann zeta function. As a part of a "global Gaussian", the indicator function on respective set of p-adic integers was used, mainly due to its properties with respect to the Fourier transform.

Nonetheless, attempts to p-adically mimic the intimate relation between the heat equation and the Gaussian apparent in the real case have been few, and on a whole, studying different analogues of the heat equation for number theory's sake seem to be a rather novel concept, initiated by Lang and Jorgenson [2, 10]. Essentially, the program consists of trying to find analogues to the heat equation, the heat kernel, the Poisson summation formula on different mathematical structures, with the goal of acquiring functional equations for "zeta functions" in each setting. We will try to follow this scheme, using the spectral zeta function one may associate with the Vladimirov operator [15]. Although the attained relation is trivial, we show that it could be viewed as a functional equation for the spectral zeta function.

The thesis is largely self-contained, apart from the spectral theory for the Vladimirov operator. [15] and the references within is a good introduction to this theory. Also, the construction of the p-adic numbers as well as their most basic properties will be assumed and very briefly stated. Some important features are summarized and proved in Proposition 1.1.1. For a thorough exposition, see [13].

Contents

Chapter 1

Prerequisites

1.1 The p-adic numbers $\mathbb{Q}_p$

The p-adic numbers $\mathbb{Q}_p$ are for each prime number p the field one gets after carrying out completion on $\mathbb{Q}$ with respect to the multiplicative norm

$$|x|_p = \begin{cases} p^{-n} & \text{if } x = p^n a/b, \\ 0 & \text{if } x = 0, \end{cases}$$

where $a, b, n \in \mathbb{Z}$, and a, b are co-prime numbers not divisible by p, using the fundamental theorem of arithmetic. Essentially, the norm of an element $x \in \mathbb{Q}$ is small if x contains many factors of p, and large if x contains many factors of p^{-1}. Further, this norm on $\mathbb{Q}$ carries over to $\mathbb{Q}_p$ and thus induces a non-archimedean metric with an associated topology on $\mathbb{Q}_p$ in the usual way. A central feature of the p-adic norm is the *strong triangle inequality*,

$$|x - y|_p \leq \max\{|x|_p, |y|_p\},$$

also known as the non-archimedean property. Due to this, some fundamental peculiarities of $\mathbb{Q}_p$ arises, see Proposition 1.1.1 below.

Each element has a unique representation

$$\sum_{i=-n}^{\infty} a_i p^i$$

with $0 \leq a_i < p$ [13]. The p-adic integers $\mathbb{Z}_p$, i.e. may be characterized in several ways, as the elements of the form

$$\sum_{i=0}^{\infty} a_i p^i$$

or equivalently $B_{\leq 1}(0)$, i.e. the closed ball centered in 0 with radius 1. This follows from the fact that the rational integers lie in the unit ball, so the elements in the ring of integers, satisfying

$$x^n + c_{n-1}nx^{n-1} + ...c_1x + c_0 = 0$$

where each c_k is a rational integer, also satisfies

$$|x^n|_p \leq \max\{|c_{n-1}x^{n-1}|_p, ..., |c_0x^0|_p\} = |c_jx^j|_p,$$

which implies $|x|_p \leq 1$. We will now prove some interesting facts about $\mathbb{Q}_p$ [12].

Proposition 1.1.1. *The following holds in* $\mathbb{Q}_p$*:*

(i) If $y \in B_{<r}(x)$*, then* $B_{<r}(x) = B_{<r}(y)$*. The same is true for closed balls.*

(ii) All open balls are clopen, i.e. both open and closed.

(iii) A sequence $\{a_i\}$ *is Cauchy (and thus converges) iff* $a_i - a_{i-1} \to 0$*.*

(iv) $\mathbb{Q}_p$ *is totally disconnected.*

(v) A set is compact iff it is closed and bounded.

(vi) $\mathbb{Q}_p$ *is locally compact, and* $\mathbb{Z}_p$ *is compact.*

Proof. (i) If $z \in B_{<r}(x)$, then $|y - z|_p = |y - x + x - z|_p \leq \max(|y - x|_p, |x - z|_p) < r$ so $z \in B_{<r}(y)$. Similarly if $z \in B_{<r}(y)$, then $z \in B_{<r}(x)$. Similar reasoning for closed balls.

(ii) Denote a general open ball $B_{<r}(a)$, and write its complement $\{x \in \mathbb{Q}_p : |x - a|_p \geq r\}$ as the union $\{x \in \mathbb{Q}_p : |x - a|_p = r\} \cup \{x \in \mathbb{Q}_p : |x - a|_p > r\}$. The latter is open in all metric spaces, so it is enough to show that the sphere $S_r(a)$ is open. Take $x \in S_r(a), \epsilon < r, y \in B_{<\epsilon}(x)$. Then

$$|y - a|_p = |y - x + x - a|_p \leq \max\{|y - x|_p, |x - a|_p\} = |x - a|_p$$

but $|x - a|_p = |x - y + y - a|_p \leq \max\{|x - y|_p, |y - a|_p\} = |y - a|_p$ so $|y - a|_p = |x - a|_p = r$ and $y \in S_r(a)$.

(iii) "⇒" holds for all metric spaces. To prove "⇐", assume

$$\lim_{i\to\infty} |a_i - a_{i-1}|_p = 0,$$

i.e. for any $\epsilon > 0$ there exists a natural number N such that if $n > N$ then

$$|a_n - a_{n-1}|_p < \epsilon.$$

If $m > n$, then

$$|a_m - a_n|_p = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + ... - a_n|_p < \max(|a_m - a_{m-1}|_p, ..., |a_{n+1} - a_n|_p) < \epsilon$$

due to the strong triangle inequality, so Cauchy.

(iv) We show that the only connected sets (i.e. cannot be written as the union of two disjoint non-empty open sets in the induced topology) are the singletons $\{a\}, a \in \mathbb{Q}_p$. Note that for each $n \in \mathbb{N}$, the ball $B_{<p^{-n}}(a)$ is a clopen neighborhood of a as noted in (i). Now take a set $A \neq \{a\}$ but $a \in A$. Then there exists an n such that $B_{<p^{-n}}(a) \cap A \neq A$, and thus

$$A = (B_{<p^{-n}}(a) \cap A) \cup (\mathbb{Q}_p \setminus B_{<p^{-n}}(a) \cap A).$$

Since $(\mathbb{Q}_p \setminus B_{<p^{-n}}(a), B_{<p^{-n}}(a)$ are open, this is a decomposition of A into disjoint open sets in the induced topology, and thus A is disconnected.

(v) "$\Leftarrow$": We will establish sequential compactness, as this is equivalent to compactness in metric spaces due to the Heine-Borel theorem. Denote the sequence $\{a_k\}$. Since the sequence is bounded, all elements are contained in some ball $B_{\leq p^n}(0)$, $n \in \mathbb{Z}$, which may be written $p^n\mathbb{Z}_p$. Using the digit expansion of a_k, we may write $a_k = p^n \sum_{n=0}^{\infty} a_{k,n}p^n = p^n \cdot ...a_{k,1}a_{k,0}$. We will call $a_{k,0}$ the first digit, $a_{k,1}$ the second digit, and so on. Since there is only finitely many possibilities for the first digit, we may find an integer $0 \leq b_0 < p$ such that there exists sub-sequence $\{a_k^0\}$ where each element has b_0 as its first digit. In the same way, we may from $\{a_k^0\}$ find a sub sequence $\{a_k^1\}$ where the first two digits in each element are b_0, b_1. Continuing iteratively, this creates

$$\{a_k^0\} = a_1^0, a_2^0, a_3^0, ...$$
$$\{a_k^1\} = a_1^1, a_2^1, a_3^1, ...$$
$$\{a_k^2\} = a_1^2, a_2^2, a_3^2, ...$$
$$\{a_k^3\} = a_1^3, a_2^3, a_3^3, ...,$$

taking the diagonal yields a sub-sequence to the original series $\{a_k\}$, converging to $p^n \cdot ...b_2b_1b_0$. Since the set is closed, this limit is contained in the set, and thus the set is sequentially compact.

"$\Rightarrow$": Let A be a sequentially compact set and take therein an arbitrary sequence $\{a_k\}$. We claim $\sup_k |a_k|_p < M$: otherwise we may find a sub-sequence $\{b_k\}$ with $|b_{k+1}|_p > |b_k|_p$ with no convergent sub-sequence. Thus A is bounded. A is closed since if $\{c_k\}$ converges to a point c in $\bar{A}$, each sub-sequence converge to the point.

(vi) Since each element $a \in \mathbb{Q}_p$ has a bounded closed neighborhood, $\mathbb{Q}_p$ is locally compact by (v). Z_p is bounded and closed by definition, thus also compact. □

1.2 Integration on $\mathbb{Q}_p$

$(\mathbb{Q}_p, +)$ is a locally compact group [8] and thus admits itself to techniques of harmonic analysis. Indeed, it has an additive Haar measure μ, unique up to a constant, which is

(i) *inner regular:* The measure of any Borel set B is the supremum of $\mu(K)$ over all compact subsets K of B.

(ii) *outer regular:* The measure of any Borel set B is the infimum of $\mu(U)$ over all open sets U containing B.

(iii) *locally finite:* Every point has a neighborhood U for which the measure is finite, and the measure is finite on compact sets.

Furthermore, this measure is unique if we normalize such that $\mu(\mathbb{Z}_p) = 1$. With this measure, one can in the usual way define an integral concept, with its usual properties. Let us get acquainted with p-adic integration through a few standard examples.

Example 1.2.1. *For $n \geq 0$, we have $\mu(p^n\mathbb{Z}_p) = \frac{1}{p^n}$.*

Proof. We will show this by induction. The base case is clear. Assume $\mu(p^{n-1}\mathbb{Z}_p) = \frac{1}{p^{n-1}}$. As $p^{n-1}\mathbb{Z}_p/p^n\mathbb{Z}_p \simeq \mathbb{F}_p$, with representatives $0, p^{n-1}, 2p^{n-1}, ..., (p-1)p^{n-1}$, we may write

$$p^{n-1}\mathbb{Z}_p = p^n\mathbb{Z}_p \cup (p^{n-1} + p^n\mathbb{Z}_p) \cup (2p^{n-1} + p^n\mathbb{Z}_p) \cup ... \cup ((p-1)p^{n-1} + p^n\mathbb{Z}_p).$$

All of these sets are disjoint and have the same measure since the measure is additive, i.e. $\mu(A+a) = \mu(A)$, and since there are p of them, $\mu(p^n\mathbb{Z}_p) = \frac{1}{p^n}$. □

Example 1.2.2. *Let d be an integer and s be a real number, both greater or equal to 0. Then*

$$\int_{\mathbb{Z}_p} |x^d|_p^s dx = \frac{p-1}{p-p^{-sd}}.$$

Proof. Note that $|x^d|_p^s$ are constant on each set $p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$, since if $x \in p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$ then $|x^d|_p^s = |x|_p^{ds} = \frac{1}{p^{nds}}$. From the previous example, $\mu(p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p) = \frac{1}{p^n} - \frac{1}{p^{n+1}} = \frac{p-1}{p^{n+1}}$, so

$$\int_{\mathbb{Z}_p} |x^d|_p^s dx = \sum_{n=0}^{\infty} \frac{p-1}{p^{n+1}} \frac{1}{p^{nds}} = \frac{(p-1)}{p} \sum_{n=0}^{\infty} (\frac{1}{p^{1+ds}})^n = \frac{p-1}{p-p^{-sd}}.$$

□

In fact, from the first example, we obtain [1, 4]

Lemma 1.2.3. *For any Borel set A, $x \in \mathbb{Q}_p$, we have $\mu(xA) = |x|_p \mu(A)$.*

Proof. This is clearly true when $x = 0$, so assume $x \neq 0$. Then $M_x(y) = x^{-1}y$ is an isomorphism from $(\mathbb{Q}_p, +)$ to itself. Since $\mathbb{Q}_p$ is locally compact, this map is continuous and thus we may define the push-forward measure

$$\mu_x(A) = \mu(x^{-1}A).$$

Due to the homomorphism properties, this is also a Haar measure, and since $\mu_x(\mathbb{Q}_p) = \mu(\mathbb{Q}_p) = \infty$, it is not trivial and thus $\mu_x(A) = c_x\mu(A)$ for some $c_x > 0$. Furthermore, if $|x|_p = \frac{1}{p^n}$, we may write $x = p^n y$ where $y \in \mathbb{Z}_p^X$ due to the representation in Section 1.1. From Example 1.2.1 we then have

$$c_x\mu(\mathbb{Z}_p) = \mu(x\mathbb{Z}_p) = \mu(p^n y\mathbb{Z}_p) = \mu(p^n\mathbb{Z}_p) = \frac{1}{p^n}\mu(\mathbb{Z}_p) = |x|_p\mu(\mathbb{Z}_p).$$

so $c_x = |x|_p$ and $\mu_x(A) = |x|_p\mu(A)$.

□

We will often confide ourselves to the function space $L^1(\mathbb{Q}_p)$, hereafter denoted L^1, as all measurable functions $f : \mathbb{Q}_p \to \mathbb{C}$ satisfying

$$||f||_1 = \int_{\mathbb{Q}_p} |f(x)| dx < \infty.$$

Identifying $f, g \in L^1$ if $||f - g||_1 = 0$, this becomes a proper norm, and bestows L^1 which the structure of a Banach space (this holds for general measure spaces, see [5]). Introducing the convolution

$$f * g(x) = \int_{\mathbb{Q}_p} f(x-y)g(y)dy,$$

it also become a Banach algebra [8].

1.3 The Fourier transform

Our first task in order to define the Fourier transform, is to answer the following question: what are the characters $\mathbb{Q}_p \to \mathbb{C}$?

Theorem 1.3.1. *The additive continuous homomorphisms $\mathbb{Q}_p \to \mathbb{C}$ are all unitary, and of the form*

$$x \mapsto \psi_y(x) := e^{2\pi i \{xy\}_p},$$

where $\{\bullet\}_p$ is the fractional part of a p-adic number, i.e.

$$\{\sum_{i=-n}^{\infty} a_i p^i\}_p = \sum_{i=-n}^{-1} a_i p^i,$$

and $y \in \mathbb{Q}_p$.

Proof. Let χ be character [7]. We have $\chi(0) = 1$, and since χ is continuous, for x sufficiently close to 0 we have $|\chi(x) - 1|< 1$. In particular, the subgroup $p^N \mathbb{Z}_p$ is sent into $\{z \in C^X : |z-1|< 1\}$ for large N. Since the image $\chi(p^N \mathbb{Z}_p)$ is a group, we conclude that $\chi(p^N \mathbb{Z}_p) = \{1\}$. Now define the character $\chi_0 = \chi(p^N x)$, constant on $\mathbb{Z}_p$ and in particular $\chi_0(1) = 1$ and thus $\chi_0(1/p^n)$ is a p^n-th root of unity, say

$$\chi_0(1/p^n) = e^{2\pi i c_n / p^n}$$

for some integer $0 \leq c_n \leq p^n - 1$. On the other hand,

$$\chi_0(1/p^n) = \chi_0(p/p^{n+1}) = \chi_0(1/p^{n+1})^p = e^{2\pi i c_{n+1}/p^n},$$

so $c_{n+1} \equiv c_n \mod p^n$. This means that $|c_{n+1} - c_n|< \frac{1}{p^n}$, and thus the sequence $\{c_1, c_2, ...\}$ is Cauchy, converging p-adically to some p-adic integer c, and $c \equiv c_n \mod p^n$ for all n. In particular, this may be reformulated as

$$\{\frac{c}{p^n}\}_p = \frac{c_n}{p^n},$$

since $0 \leq c_n \leq p^n - 1$. Now take any $r \in \mathbb{Q}$ and write $r = s/p^m t$ with $(p,t) = 1$. Then

$$\chi_0(r)^t = \chi_0(tr) = \chi_0(\frac{s}{p^m}) = \chi_0(\frac{1}{p^m})^s = e^{2\pi s i c_m/p^m} = e^{2\pi s i \{\frac{c}{p^m}\}_p} = e^{2\pi i \{\frac{cs}{p^m}\}_p} = e^{2\pi i \{crt\}_p} = e^{2\pi i \{cr\}_p t},$$

and so the ratio $\frac{\chi_0(r)}{e^{2\pi i \{cr\}_p}}$ is a t-th root of unity. Since $(p,t) = 1$, this ratio is equal to 1 and $\chi_0(r) = e^{2\pi i \{cr\}_p}$. This formula then extends by continuity to $\mathbb{Q}_p$ since $\mathbb{Q}$ is a dense subset (which follows from the fact that $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$), and thus

$$\chi(x) = e^{2\pi i \{xy\}_p}$$

with $y = c/p^N$. □

The group of characters, where the group operation is defined by point-wise multiplication, is called the *Pontryagin dual group of* $\mathbb{Q}_p$, denoted $\widehat{\mathbb{Q}}_p$. Since

$$\psi_y(x)\cdot\psi_z(x) = e^{2\pi i\{xy\}_p}e^{2\pi i\{xz\}_p} = e^{2\pi i\{x(y+z)\}_p} = \psi_{y+z}(x)$$

we see that $\psi : y \mapsto \psi_y$ is a surjective group homomorphism, injective since

$$\forall x, \psi_y(x) = 1 \implies y = 0,$$

so $\mathbb{Q}_p \simeq \widehat{\mathbb{Q}}_p$ as groups. In accordance with the general theory [8], we endow $\hat{\mathbb{Q}}_p$ with the compact-open topology, i.e. declaring that the sets

$$W(K,V) = \{\chi \in \widehat{\mathbb{Q}}_p : \chi(K) \subset V\},$$

where $K \subset G$ is compact, V is an open neighborhood of the identity in S^1, constitute a neighborhood base of the trivial character $\psi_0 : x \mapsto 1$. With regards to this, one has [8, 18]:

Theorem 1.3.2. *The map $\psi : y \mapsto \psi_y$ is a topological isomorphism.*

Proof. We will show that y close to 0 maps to ψ_y close to ψ_0 and vice versa, as this extends to the entire group through the homomorphic properties of ψ.

Consider the compact sets $C_m := \{x \in \mathbb{Q}_p : |x|_p \leq p^m\}$. Due to prop 1. (iv), all compact sets are contained in some C_m, and thus $W(C_m,V) \subset W(C,V)$. For any fixed V, the larger the m, the "smaller" the neighborhood of the trivial character ψ_0.

If y is close to zero, say $|y|_p = p^{-M}$, then $\psi_y|_{C_m} = \psi_0|_{C_m}$, for all $m \leq M$, so $\psi_y \in W(C_m,V)$ for all V, and close to the trivial character. On the other hand, if $\psi_y \in W(C_m,V)$ for all V, then $\psi_y(C_m) \equiv 1$, and thus $|y|_p \leq p^{-m}$. This establishes bi-continuity. □

We are now in position to introduce the Fourier transform.

Definition 1.3.3. *The Fourier transform of a function $f \in L^1$ is a function $\hat{f} : \widehat{\mathbb{Q}}_p \to \mathbb{C}$ defined by*

$$\hat{f}(\psi_y) = \int_{\mathbb{Q}_p} f(x)\overline{\psi_y}(x)dx.$$

The reason we demand that $f \in L^1$ is the usual one, namely that it guarantees that the above formula converges, since

$$|\hat{f}(\psi_y)| = |\int_{\mathbb{Q}_p} f(x)\overline{\psi_y}(x)dx| \leq \int_{\mathbb{Q}_p} |f(x)||\overline{\psi_y}(x)|dx = \int_{\mathbb{Q}_p} |f(x)|dx < \infty.$$

As we have identified $\mathbb{Q}_p \simeq \widehat{\mathbb{Q}}_p$ through $y \mapsto \psi_y$ both topologically and as groups, we may instead view $\hat{f}$ as acting on $\mathbb{Q}_p$, i.e.

$$\hat{f}(w) = \int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{xw\}_p}dx.$$

The usual properties of the Fourier transform holds [8].

Lemma 1.3.4. *Let $f,g \in L^1$. For all $a, w \in \mathbb{Q}_p$ we then have* *(i)* $\widehat{(af+bg)}(w) = a\hat{f}(w) + b\hat{g}(w)$.

(ii) $\widehat{(f*g)}(w) = \hat{f}(w)\hat{g}(w)$.

(iii) $\widehat{f(x-a)}(w) = e^{-2\pi i\{aw\}_p}\hat{f}(w)$.

(iv) $\widehat{e^{-2\pi i\{ax\}_p}f(x)}(w) = \hat{f}(w+a)$.

Proof. Linearity is a property of the integral. Since $f*g \in L^1$, we have by Fubini's theorem

$$\begin{aligned}\widehat{f*g}(w) &= \int_{\mathbb{Q}_p} f*g(x)e^{-2\pi i\{xw\}_p}dx = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} f(x-y)g(y)e^{-2\pi i\{xw\}_p}dydx \\ &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} f(x-y)g(y)e^{-2\pi i\{xw\}_p}dxdy = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} f(x)g(y)e^{-2\pi i\{(x+y)w\}_p}dxdy \\ &= \int_{\mathbb{Q}_p} g(y)e^{-2\pi i\{yw\}_p}\int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{xw\}_p}dxdy = \hat{f}(w)\hat{g}(w).\end{aligned}$$

To show (iii), note that

$$\begin{aligned}f\widehat{(x-a)}(w) &= \int_{\mathbb{Q}_p} f(x-a)e^{-2\pi i\{xw\}_p}dx = \int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{(x+a)w\}_p}dx \\ &= e^{-2\pi i\{aw\}_p}\int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{xw\}_p}dx = e^{-2\pi i\{aw\}_p}\hat{f}(w),\end{aligned}$$

due to translation invariance. The proof of (iv) is similar.

□

Furthermore, since $\hat{\mathbb{Q}}_p \simeq \mathbb{Q}_p$ using the isomorphism ψ, $\hat{\mathbb{Q}}_p$ is a locally compact group, and thus admits a Haar measure as well. Denote this $\hat{\mu}$ at let it be normalized such that $\hat{\mu}(\psi(\mathbb{Z}_p)) = 1$. From the general theory of harmonic analysis on locally compact groups, we have then have the following theorem.

Theorem 1.3.5. *Let $f \in L^1$ such that f is continuous and $\hat{f} \in L^1(\widehat{Q_p})$. With respect to the measure $\hat{\mu}$, we then have*

$$f(x) = \int_{\mathbb{Q}_p} \hat{f}(w)e^{2\pi i\{xw\}_p}d\hat{\mu}.$$

Proof. See [8], Theorem 4.32. □

1.4 Do complex-valued functions on $\mathbb{Q}_p$ have a derivative?

In order to address the problem of formulating a p-adic analogue to the classical heat equation

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0,$$

we would want to formulate an analogue to the differential operator, or at least the Laplacian. As it turns out, it is not straight-forward to define a derivative of functions $\mathbb{Q}_p \to \mathbb{C}$. The main obstacle is the absence of continuous linear maps. Even worse so, not even non-trivial continuous additive functions exist, since if J is defined for some neighborhood D_j of 0, we may find a subgroup $p^N\mathbb{Z}_p$ of $\mathbb{Q}_p$ within D_j. Then $J(p^N\mathbb{Z}_p)$ is arbitrary close to 0 in $\mathbb{C}$, yet all additive subgroups of $\mathbb{C}$ are unbounded, so $J(p^N\mathbb{Z}_p) = \{0\}$. Taking an arbitrary $a \in D_j$, we may find M such that $p^M a \in p^N\mathbb{Z}_p$ and then $J(p^M a) = p^M J(a) = 0$ so $J(a) = 0$. This makes any of the common constructs virtually impossible to generalize. Some p-adic derivatives have however been defined, essentially constructed so that the characters are eigenfunctions to the operator [17]. We will however not employ this approach.

The problem lingers as well if one tries to imitate the fact that in the real case,

$$D(f(x)) = \mathcal{F}^{-1}(iw\mathcal{F}(f(x)))$$

where $\mathcal{F}$ is the Fourier transform. Despite having access to a Fourier transform for the p-adics, as it is not evident what "iw", being a continuous additive map, should be replaced by in the p-adic to complex case. Nonetheless, several such pseudo-differential operators occur in the literature [19, 9]. One of these in particular has been studied extensively, as well as its associated Laplacian and heat equation. We will give an account of this. Inspired by [9], we will also define another "p-adic Laplacian", and consider the heat equation it gives rise to.

Chapter 2

Two p-adic heat equations

2.1 The function space $C_c^\infty(\mathbb{Q}_p)$

When introducing a operator of the form

$$D(u(x)) = \mathcal{F}^{-1}(\varphi(w)\mathcal{F}(u(x))),$$

where "$\varphi(w)$" is some sensible analogous function $\mathbb{Q}_p \mapsto \mathbb{C}$ to the classical "iw", we must make sure that everything is well-defined. Clearly, for the Fourier transforms involved to make sense, we must a priori demand that $u(x) \in L^1(\mathbb{Q}_p)$, $\varphi(w)\hat{u}(w) \in L^1(\hat{\mathbb{Q}}_p)$. This space is however quite large, and our arsenal of techniques for carrying out manipulations is not that rich. Thus, in order to meet these criteria on u and facilitate explicit calculations, we will whenever possible limit ourselves to the function space $C_c^\infty(\mathbb{Q}_p)$ of compactly supported, locally constant functions, i.e. equal to zero outside some ball $B_{\leq p^m}$, and constant on the cosets of some ball $B_{\leq p^n}$, $m, n \in \mathbb{Z}$. The following proposition summarizes the most straight-forward features of this space [18].

Proposition 2.1.1. *The following holds:*

(i) If $f \in C_c^\infty(\mathbb{Q}_p)$, then f is continuous.

(ii) If $f \in C_c^\infty(\mathbb{Q}_p)$, then f only takes on finitely many values.

(iii) $C_c^\infty(\mathbb{Q}_p)$ is dense in $L^\alpha(\mathbb{Q}_p), \alpha \in [1, \infty)$.

Proof. (i) Since f is locally constant, there exists an neighborhood around each point x such that $f(y) = f(x)$ for all y close to x, say $|x-y|_p < \delta$, so $|f(x) - f(y)| = 0 < \epsilon$.

(ii) Let f be supported on $B_{\leq p^m}$, and constant on the cosets of $B_{\leq p^n}$, with $n \leq m$ (note that f is identically zero if $m < n$). Since

$$|B_{\leq p^m}/B_{\leq p^n}| = |\{x : x = a_{-m}p^{-m} + a_{-m+1}p^{-m+1} + ... + a_{-n-1}p^{-n-1}, 0 \leq a_i < p\}| = p^{m-n-1},$$

we have $|f(\mathbb{Q}_p)| < \infty$.

(iii) It follows from the construction of the integral that the vector space of simple functions is dense in L^α. Thus it is enough to construct a sequence converging to $\mathbf{1}_A$, where A has finite measure. Since the Haar measure is outer regular, there are open sets U_i containing A such that $\mu(U_i) - \mu(A) \leq 1/i$. Also, since the Haar measure is inner regular, there exist compact sets K_i such that $\mu(A) - \mu(K_i) \leq 1/i$. Let W_i be a open cover of K_i, consisting of finitely many open balls. Since $\mathbf{1}_{W_i \cap U_i} \in C_c^\infty$, and

$$
\begin{aligned}
||\mathbf{1}_A - \mathbf{1}_{W_i \cap U_i}||_\alpha &= (\int_{(W_i \cap U_i) \cup A \setminus (W_i \cap U_i) \cap A} 1^\alpha dx)^{1/\alpha} \\
&= (\mu((W_i \cap U_i) \cup A) - \mu((W_i \cap U_i) \cap A))^{1/\alpha} \\
&\leq (\mu(U_i) - \mu(K_i))^{1/\alpha} \leq 1/(2i)^{1/\alpha},
\end{aligned}
$$

the claim follows. □

We will now introduce another substitution of variables, which will be especially useful when manipulating integrals involving functions in C_c^∞ [4].

Lemma 2.1.2. *If $f \in L^1, x \neq 0$, then*

$$\int_{\mathbb{Q}_p} f(x^{-1}y)dy = |x|_p \int_{\mathbb{Q}_p} f(y)dy.$$

Proof. Using Lemma 1.2.3 and the push-forward measure, we have

$$\int_{\mathbb{Q}_p} f(x^{-1}y)dy = \int_{\mathbb{Q}_p} f \circ M_x(y)dy = |x|_p \int_{\mathbb{Q}_p} f(y)dy.$$

□

A simple but important example of locally constant, compactly supported functions are characteristic functions on balls in $\mathbb{Q}_p$. We will now introduce the notation ξ_{p^n} to denote the characteristic function of the set $B_{\leq p^n}$. They behave particularly well with respect to the Fourier transform [18]:

Proposition 2.1.3. *The Fourier transform of the characteristic function ξ_{p^n} is*

$$\widehat{\xi_{p^n}}(w) = p^n \xi_{p^{-n}}(w).$$

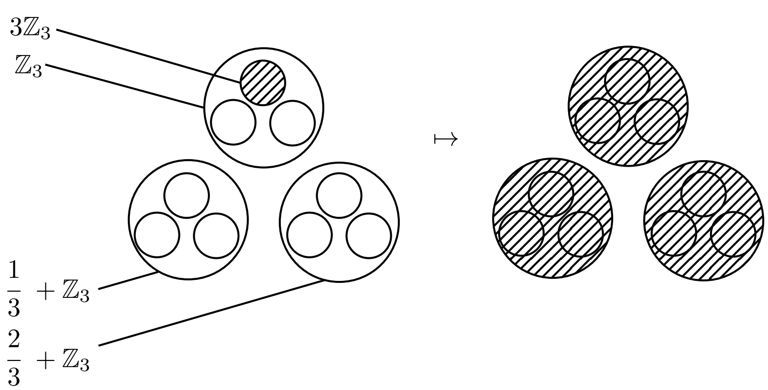

Figure 2.1: A depiction of the transform of the characteristic function on the set $3\mathbb{Z}_3$.

Proof. Note that

$$\widehat{\xi_{p^n}}(w) = \int_{\mathbb{Q}_p} e^{-2\pi i\{xw\}_p}\xi_{p^n}(x)dx = |w|_p^{-1}\int_{\mathbb{Q}_p} e^{-2\pi i\{x\}_p}\xi_{p^n}(w^{-1}x)dx$$
$$= |w|_p^{-1}\int_{\mathbb{Q}_p} e^{-2\pi i\{x\}_p}\xi_{p^n|w|_p}(x)dx = |w|_p^{-1}\int_{|x|_p\leq p^n|w|_p} e^{-2\pi i\{x\}_p}dx.$$

To calculate the integral, we divide into two cases. Let m be the integer such that $p^m = p^n|w|_p$. If $|w|_p\leq p^{-n}$, then $m \leq 0$, and

$$\int_{|x|_p\leq p^m} e^{-2\pi i\{x\}_p}dx = \int_{|x|_p\leq p^m} dx = p^m,$$

and if $m > 0$, then there exists an y with $|y|_p\leq p^m$ such that $e^{2\pi i\{y\}_p} \neq 1$, so translational invariance implies that

$$\int_{|x|_p\leq p^m} e^{-2\pi i\{x\}_p}dx = \int_{|x|_p\leq p^m} e^{-2\pi i\{y+x\}_p}dx = e^{2\pi i\{y\}_p}\int_{|x|_p\leq p^m} e^{-2\pi i\{w\}_p} = 0.$$

Thus

$$\widehat{\xi_{p^n}}(w) = |w|_p^{-1}\int_{|x|_p\leq p^n|w|_p} e^{-2\pi i\{x\}_p}dx = |w|_p^{-1}p^m\xi_{p^{-n}}(w) = p^n\xi_{p^{-n}}(w).$$

□

Notice that the characteristic function of $\mathbb{Z}_p$, ξ_{p^0}, is its own Fourier transform, and thus a good candidate for a p-adic Gaussian.

In fact, since any function in C_c^∞ may be written as the sum of characteristic functions of balls, the above proposition effectively permits us to calculate the transform of any function in C_c^∞ [18]. Indeed, let f be supported on $B_{\leq p^m}$, and constant on the cosets of $B_{\leq p^{-n}}$. Then we may choose a finite set of points $\{a_k\} \subset B_{\leq p^m}$ such that

$$B_{\leq p^m} = \bigsqcup_{k=0}^{l}(a_k + B_{\leq p^{-n}})$$

where f is constant on each set $B_{\leq p^{-n}}(a_k)$. Thus f may be written as a finite linear combination of indicator functions $\mathbf{1}_{B_{\leq p^{-n}}(a_k)}(x) = \xi_{p^{-n}}(x - a_k)$. Using Lemma 1.3.4, (iii) to handle the shift by a_k, one acquires:

Theorem 2.1.4. *Let f be supported on $B_{\leq p^m}$, and constant on the cosets of $B_{\leq p^{-n}}$, i.e. on the form*

$$\sum_{k=0}^{l} c_k\mathbf{1}_{B_{\leq p^{-n}}(a_k)}(x).$$

Then

$$\hat{f}(x) = \begin{cases} \sum_{k=0}^{l} c_k e^{-2\pi i\{a_k x\}_p}p^{-n} & if \ \ |x|_p\leq p^n. \\ 0 \quad if \ \ |x|_p> p^n. \end{cases}$$

We end this section by noting that $C_c^\infty(\mathbb{Q}_p)$ is often regarded as the p-adic Schwartz space, mainly due to the following theorem [18].

Theorem 2.1.5. *The map $f \mapsto \hat{f}$ is a bijection of $C_c^\infty(\mathbb{Q}_p)$ onto itself.*

Proof. Suppose f is supported on $B_{\leq p^m}$ and constant on cosets of the ball $B_{\leq p^n}$. We will show that this happens iff $\hat{f}$ is supported on $B_{\leq p^{-n}}$ and constant on cosets of the ball $B_{\leq p^{-m}}$.

Take $y \in B_{\leq p^{-m}}$. We then have

$$\begin{aligned}\hat{f}(u+y) &= \int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{x(u+y)\}_p}dx = \int_{B_{\leq p^m}} f(x)e^{-2\pi i\{x(u+y)\}_p}dx \\ &= \int_{B_{\leq p^m}} f(x)e^{-2\pi i\{xu\}_p}e^{-2\pi i\{xy\}_p}dx = \int_{B_{\leq p^m}} f(x)e^{-2\pi i\{xu\}_p}dx = \hat{f}(u),\end{aligned}$$

so $\hat{f}$ is constant on cosets of $B_{\leq p^{-m}}$. Now take $h \in B_{\leq p^n}$. Since f is constant on cosets of $B_{\leq p^n}$, we have

$$\hat{f}(u) = \int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{xu\}_p}dx = \int_{\mathbb{Q}_p} f(x+h)e^{-2\pi i\{xu\}_p}dx.$$

Making the change of variable $x \mapsto x - h$,

$$\hat{f}(u) = \int_{\mathbb{Q}_p} f(x)e^{-2\pi i\{(x-h)u\}_p}dx = e^{2\pi i\{hu\}_p}\hat{f}(u),$$

so indeed $\hat{f}(u) = 0$ whenever $|hu|_p > 1$, i.e. when $u \notin B_{\leq p^{-n}}$. This shows that $\hat{f} \in C_c^\infty$. The reverse implication may be shown with the same reasoning using Theorem 1.3.5. □

2.2 The Vladimirov operator D^α and its associated heat equation

The following operator has appeared several times in the literature, and has been used to formulate p-adic versions of some of the most classic equations of modern physics - notably a p-adic heat equation [19] which we shall soon describe.

Definition 2.2.1. *Whenever $u \in L^1(\mathbb{Q}_p), \hat{u} \in L^1(\hat{\mathbb{Q}}_p)$, the map*

$$u(x) \mapsto \mathcal{F}^{-1}_{w\to x}[|w|_p^\alpha \mathcal{F}_{x\to w}u]$$

is called Vladimirov's p-adic fractional differentiation operator, denoted D^α.

Clearly, it is well-defined when $u \in C_c^\infty$, since "$|w|_p^\alpha \mathcal{F}_{x\to w}u$" also is an element in L_1, as one can quickly conclude from Example 1.2.2. For $\alpha > 0$, it is possible to define D^α in a more explicit version [19, 3], namely

$$D^\alpha(f) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x-y|_p^{\alpha+1}}dy,$$

which shows that for all $\alpha > 0$, D^α computes an weighted average of the difference between of f at x and all other points, and thus also could serve as a Laplacian. We will however restrict our attention to the case $\alpha = 2$, analogous to the real case

$$-\frac{d^2 f}{dx^2} = \mathcal{F}^{-1}(-x^2 \mathcal{F} f) = \mathcal{F}^{-1}(-|x|^2 \mathcal{F} f),$$

though the forthcoming material without effort extents to all $\alpha > 0$.

Let us now consider the equation

$$u'(x,t) + D^2(u(x,t)) = 0, \tag{2.1}$$

where $t > 0, x \in \mathbb{Q}_p, u(x,0) = \varphi(x) \in C_c^\infty(\mathbb{Q}_p)$, and $u'(x,t) := \frac{\partial u}{\partial t}(x,t)$. This equation and its solution, as well as generalizations, has been described in the literature [19, 3, 14], though to my knowledge without detailed proofs. One might object that this "mix" of real and p-adic analysis is not natural, and as this critique without doubt holds some merit, the approach has been successful in other contexts, notably in constructing sensible heat equations on graphs [11].

Theorem 2.2.2. *A solution to (2.1) is given by* $u(x,t) = K(x,t) * \varphi(x)$*, where*

$$K(x,t) = \sum_{k=-\infty}^{\infty} (e^{-tp^{2k}} - e^{-tp^{2k+2}}) p^k \xi_{p^{-k}}(x).$$

Proof. One would hope that taking Fourier transform on (2.1) would produce

$$\int_{\mathbb{Q}_p} e^{-2\pi i \{xw\}_p} u'(x,t) dx + |w|_p^2 \hat{u}(w,t) = \hat{u}'(w,t) + |w|_p^2 \hat{u}(w,t) = 0.$$

We will now justify this. We assume that

1. $u(x,t)$ is a Lebesgue-integrable function of x for each $t > 0$.
2. For all $x \in \mathbb{Q}_p, t > 0$, the derivative $u'(x,t)$ exists.
3. There is an integrable function $H : \mathbb{Q}_p \to \mathbb{R}$ such that $|u'(x,s)| \leq H(x)$ for all s in a neighborhood of t.

The first two of these are required to make the problem well-posed. The third condition, which we will later verify, is needed to apply the Dominated Convergence Theorem. Now write

$$\begin{aligned} \frac{\hat{u}(w,t+h) - \hat{u}(w,t)}{h} &= \int_{\mathbb{Q}_p} e^{-2\pi i \{xw\}_p} \frac{u(x,t+h) - u(x,t)}{h} dx \\ &= \int_{\mathbb{Q}_p} e^{-2\pi i \{xw\}_p} u'(x, c_x(h)) dx, \end{aligned}$$

where $c_w(h)$ is a point belonging to the interior of the segment joining $t, t+h$ by the Mean Value Theorem. Using H, the Dominated Convergence Theorem permits us to move the limit $h \to 0$ inside the integral, yielding $\hat{u}'(w,t) + |w|_p^2 \hat{u}(w,t) = 0$.

The transformed equation has the solution $\hat{\varphi}(w) e^{-|w|_p^2 t}$, a compactly supported function for each t. Write $e^{-|w|_p^2 t} = \hat{K}(w,t)$. Note that $\hat{K}(w,t)$ lies in L^1, since $e^{-|w|_p^2 t} < \frac{1}{|w|_p^2 t}$, and

$$t^{-1}\int_{\mathbb{Q}_p}|w|_p^{-2}dw = t^{-1}(\int_{\mathbb{Z}_p}|w|_p^{-2}dw+\sum_{k=1}^{\infty}\int_{|w|_p=p^k}|w|_p^{-2}dw) = t^{-1}(\int_{\mathbb{Z}_p}|w|_p^{-2}dw+\sum_{k=1}^{\infty}p^{-2k}p^k(1-p^{-1})) < \infty,$$

using similar calculations as in Example 1.2.2. Due to Lemma 1.3.4, it is enough to determine $K(x,t)$. The expression may be acquired by writing

$$\begin{aligned}
K(x,t) &= \int_{\mathbb{Q}_p} e^{2\pi i\{xw\}_p}e^{-t|w|_p^2}dw = \sum_{k=-\infty}^{\infty} e^{-tp^{2k}}\int_{|w|=p^k} e^{2\pi i\{xw\}_p}dw \\
&= \sum_{k=-\infty}^{\infty} e^{-tp^{2k}}(\int_{|w|\le p^k} e^{2\pi i\{xw\}_p}dw - \int_{|w|\le p^{k-1}} e^{2\pi i\{xw\}_p}dw) \\
&= \sum_{k=-\infty}^{\infty} (e^{-tp^{2k}} - e^{-tp^{2(k+1)}})\int_{|w|\le p^k} e^{2\pi i\{xw\}_p}dw \\
&= \sum_{k=-\infty}^{\infty} (e^{-tp^{2k}} - e^{-tp^{2(k+1)}})p^k\xi_{p^{-k}}(x).
\end{aligned}$$

using Proposition 2.1.3. We will now prove that H exists. Using Theorem 2.1.4, we have

$$\begin{aligned}
&|u'(x,t)| = |D^2(u(x,t))| = |\mathcal{F}^{-1}(|w|_p^2\hat{\varphi}(w)\hat{K}(w,t))| = |\int_{\mathbb{Q}_p} e^{2\pi i\{xw\}_p}|w|_p^2\hat{\varphi}(w)e^{-t|w|_p^2}dw| \\
&= |\int_{B_{\le p^n}} e^{2\pi i\{xw\}_p}|w|_p^2e^{-t|w|_p^2}\sum_{k=0}^{l} c_k e^{2\pi i\{a_k w\}_p}p^{-n}dw| \\
&= |\sum_{k=0}^{l} c_k p^{-n}\int_{B_{\le p^n}} e^{2\pi i\{(x+a_k)w\}_p}|w|_p^2 e^{-t|w|_p^2}dw| \\
&= |\sum_{k=0}^{l} c_k p^{-n}\sum_{i=-\infty}^{m} e^{-tp^{2i}}p^{2i}\int_{|w|=p^i} e^{2\pi i\{(x+a_k)w\}_p}dw| \\
&= |\sum_{k=0}^{l} c_k p^{-n}\sum_{i=-\infty}^{m} e^{-tp^{2i}}p^{2i}\int_{\mathbb{Q}_p} e^{2\pi i\{(x+a_k)w\}_p}(\xi_{p^i}(w)-\xi_{p^{i-1}}(w))dw| \\
&= |\sum_{k=0}^{l} c_k p^{-n}\sum_{i=-\infty}^{m} e^{-tp^{2i}}p^{2i}(p^i\xi_{p^{i-1}}(x+a_k) - p^{-i}\xi_{p^{1-i}}(x+a_k))| \\
&< C\sum_{k=0}^{l}\sum_{i=-\infty}^{m} p^{3i}\xi_{p^{1-i}}(x+a_k) = H(x).
\end{aligned}$$

This is essentially a finite sum of locally constant, compactly supported functions and functions of the form

$$\sum_{i=0}^{\infty} p^{-3i}\xi_{p^i}(x+a_k),$$

which are in L^1 since

$$\int_{\mathbb{Q}_p} \sum_{i=0}^{\infty} p^{-3i} \xi_{p^i}(x + a_k) dx = \int_{\mathbb{Z}_p} \sum_{i=0}^{\infty} (1/p^3)^i \xi_{p^i}(x) dx + \sum_{n=1}^{\infty} \int_{|x|_p = p^n} \sum_{i=0}^{\infty} (1/p^3)^i \xi_{p^i}(x) dx$$
$$= \frac{1}{1 - p^{-3}} + \sum_{n=1}^{\infty} \int_{|x|_p = p^n} \sum_{i=n}^{\infty} (1/p^3)^i \xi_{p^i}(x) dx$$
$$= \frac{1}{1 - p^{-3}} + \sum_{n=1}^{\infty} \int_{|x|_p = p^n} \frac{1}{1 - p^{-3}} - \frac{1 - p^{-3n}}{1 - p^{-3}} dx$$
$$= \frac{1}{1 - p^{-3}} + \sum_{n=1}^{\infty} (p^n - p^{n-1}) \frac{p^{-3n}}{1 - p^{-3}}$$
$$= \frac{1}{1 - p^{-3}} \left(\frac{1}{1 - p^{-2}} - \frac{1}{p - 1} + \frac{1}{p} \right) < \infty,$$

summing the geometric series. This shows $H \in L^1$.

□

We will prove another formula for $K(x, t)$, mentioned in [3, 14].[1]

Theorem 2.2.3. *If $x \neq 0$, then*

$$K(x, t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} \frac{1 - p^{2m}}{1 - p^{-2m-1}} |x|_p^{-2m-1}.$$

Proof. Using the exponential function's Taylor expansion, we may write

$$K(x, t) = \int_{\mathbb{Q}_p} e^{2\pi i \{xw\}_p} e^{-t|w|_p^2} dw = \sum_{n=-\infty}^{\infty} e^{-tp^{2n}} \int_{|w|_p = p^n} e^{2\pi i \{xw\}_p} dw$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} p^{2mn} \int_{|w|_p = p^n} e^{2\pi i \{xw\}_p} dw.$$

In order to continue, we will interchange the order of summation; this is justified by Fubini-Tonelli theorem if

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \left| \frac{(-1)^m t^m}{m!} p^{2mn} \int_{|w|_p = p^n} e^{2\pi i \{xw\}_p} dw \right| < \infty,$$

which we will establish later. Since $x \neq 0$, we may do the change of variable $w \to x^{-1} w$, yielding

$$\sum_{n=-\infty}^{\infty} p^{2mn} \int_{|w|_p = p^n} e^{2\pi i \{xw\}_p} dw = \frac{1}{|x|_p} \sum_{n=-\infty}^{\infty} p^{2mn} \int_{|w|_p = p^n |x|_p} e^{2\pi i \{w\}_p} dw$$
$$= \frac{1}{|x|_p} \sum_{n=-\infty}^{\infty} p^{2mn} |x|_p^{-2m} \int_{|w|_p = p^n} e^{2\pi i \{w\}_p} dw,$$

[1] In [14] an argument for the formula is given, but without substantial details. In correspondence with the author it was pointed out that the Fubini-Torelli theorem is indeed enough to justify the change of summation. In all other respects, the proof has been done independently.

since we are summing over all n. To calculate this sum, we note that

$$\int_{|w|_p=p^n} e^{2\pi i\{w\}_p} dw = \int_{|w|_p\le p^n} e^{2\pi i\{w\}_p} dw - \int_{|w|_p\le p^{n-1}} e^{2\pi i\{w\}_p} dw = \begin{cases} p^n(1-\frac{1}{p}), & n\le 0 \\ -1, & n=1 \\ 0, & n>1 \end{cases}$$

using Proposition 2.1.3. Thus

$$\begin{aligned}\sum_{n=-\infty}^{\infty} p^{2mn} \int_{|w|_p=p^n} e^{2\pi i\{w\}_p} dw &= -p^{2m} + \sum_{n=-\infty}^{0} p^{2mn} \int_{|w|_p=p^n} e^{2\pi i\{w\}_p} dw \\ = -p^{2m} + (1-\frac{1}{p}) \sum_{n=-\infty}^{0} p^{2mn} p^n &= -p^{2m} + (1-\frac{1}{p}) \sum_{n=0}^{\infty} p^{-(2m+1)n} \\ = -p^{2m} + (1-\frac{1}{p}) \frac{1}{1-p^{-2m-1}} &= \frac{1-p^{2m}}{1-p^{-2m-1}},\end{aligned}$$

and so

$$K(x,t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} \frac{1-p^{2m}}{1-p^{-2m-1}} |x|_p^{-2m-1}.$$

It remains to show that the Fubini-Tonelli theorem is applicable. According to previous calculations,

$$\begin{aligned}&\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \left|\frac{(-1)^m t^m}{m!} p^{2mn} \int_{|w|_p=p^n} e^{2\pi i\{xw\}_p} dw\right| = \sum_{m=0}^{\infty} \left|\frac{(-1)^m t^m}{m!}\right| \sum_{n=-\infty}^{\infty} \left|p^{2mn} \int_{|w|_p=p^n} e^{2\pi i\{xw\}_p} dw\right| \\ &< \sum_{m=0}^{\infty} \frac{t^m}{m!} |x|_p^{-2m-1} (p^{2m} + \frac{1+p^{-1}}{1-p^{-2m-1}}) < |x|_p^{-1} \sum_{m=0}^{\infty} \frac{(\frac{t}{|x|_p^2})^m}{m!} (p^{2m} + \frac{1+p^{-1}}{1-p^{-1}}) \\ &= |x|_p^{-1} (e^{tp^2/|x|_p^2} + \frac{(1+p^{-1}) e^{t/|x|_p^2}}{1-p^{-1}}) < \infty.\end{aligned}$$

□

We end this section by investigating whether it is possible to transform K with respect to t. One might try to calculate the Mellin transform, which we define as

$$\mathcal{M}[f](s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} f(t) dt,$$

in accordance with [16]. A quick calculation shows that

$$\int_0^\infty t^{s-1}K(x,t)dt = \int_0^\infty t^{s-1}\sum_{k=-\infty}^{\infty}(e^{-tp^{2k}} - e^{-tp^{2(k+1)}})p^k\xi_{p^{-k}}(x)dt$$
$$= \sum_{k=-\infty}^{\infty} p^k\xi_{p^{-k}}(x)(\int_0^\infty t^{s-1}e^{-tp^{2k}}dt - \int_0^\infty t^{s-1}e^{-tp^{2(k+1)}}dt)$$
$$= \sum_{k=-\infty}^{\infty} p^k\xi_{p^{-k}}(x)(p^{2k(1-s)}\int_0^\infty t^{s-1}e^{-t}dt - p^{2(k+1)(1-s)}\int_0^\infty t^{s-1}e^{-t}dt)$$
$$= \Gamma(s)\sum_{k=-\infty}^{\infty} p^k\xi_{p^{-k}}(x)(p^{2k(1-s)} - p^{2(k+1)(1-s)}) = \Gamma(s)(1-p^{2(1-s)})\sum_{k=-\infty}^{\infty} p^{k(2s-3)}\xi_{p^k}(x).$$

Remember that for fixed x, $|x|_p = p^N$, the two-sided sum is cut off at the negative end, so it is enough to require that $s < \frac{3}{2}$ for the sum to converge. Thus we may write

$$\sum_{k=-\infty}^{\infty} p^{k(2s-3)}\xi_{p^k}(x) = \sum_{k=N}^{\infty} p^{k(2s-3)} = \sum_{k=0}^{\infty} p^{(k+N)(2s-3)} = p^{N(2s-3)}\sum_{k=0}^{\infty} p^{k(2s-3)} = \frac{|x|_p^{2s-3}}{1-p^{2s-3}},$$

so

$$\mathcal{M}[K](x,s) = \frac{(1-p^{2-2s})|x|_p^{2s-3}}{1-p^{2s-3}}.$$

Note that we cannot apply the transform on K as written in Theorem 2.2.3, as the integral is then not absolutely summable, and we cannot switch places between summation and integration. This is however possible when applying the Laplace transform, yielding the following formula.

Theorem 2.2.4. *For $\Re(s) > (\frac{p}{|x|_p})^2$, we have*

$$\frac{1}{|x|_p s}\sum_{m=0}^{\infty}(\frac{-1}{|x|_p^2 s})^m\frac{1-p^{2m}}{1-p^{-2m-1}} = \sum_{k=-\infty}^{\infty}\frac{p^2-1}{(1+p^{2k}s)(p^2+p^{2k}s)}p^k\xi_{p^k}(x).$$

Proof. As estimated in the previous proof, for real s we have

$$\int_0^\infty e^{-st}K(x,t)dt = \int_0^\infty e^{-st}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1}dt$$
$$< \int_0^\infty e^{-st}|x|_p^{-1}e^{tp^2/|x|_p^2}dt + \int_0^\infty e^{-st}\frac{(1+p^{-1})e^{t/|x|_p^2}}{1-p^{-1}}dt$$
$$= |x|_p^{-1}\int_0^\infty e^{t(p^2/|x|_p^2-s)}dt + \frac{1+p^{-1}}{1-p^{-1}}\int_0^\infty e^{t(|x|_p^{-2}-s)}dt,$$

so the transform converges absolutely when $s > (\frac{p}{|x|_p})^2$, and more generally when $\Re(s) > (\frac{p}{|x|_p})^2$. This justifies the interchange of limits in the following calculations:

$$\begin{aligned}
\int_0^\infty e^{-st}K(x,t)dt &= \int_0^\infty e^{-st}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1}dt \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1}\int_0^\infty t^m e^{-st}dt \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1}\int_0^\infty t^m e^{-st}dt \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1}s^{-m-1} \\
&= \frac{1}{|x|_p s}\sum_{m=0}^{\infty}(\frac{-1}{|x|_p^2 s})^m\frac{1-p^{2m}}{1-p^{-2m-1}}.
\end{aligned}$$

where the integral may be calculated by repeatedly differentiating under the integral sign [6], or by applying the definition of the gamma function. Using the other representation of $K(x,t)$ consisting of positive terms, we obtain from the Fubini-Tonelli theorem

$$\begin{aligned}
\int_0^\infty e^{-st}K(x,t)dt &= \int_0^\infty e^{-st}\sum_{k=-\infty}^{\infty}(e^{-tp^{2k}}-e^{-tp^{2(k+1)}})p^k\xi_{p^{-k}}(x)dt \\
&= \sum_{k=-\infty}^{\infty}p^k\xi_{p^{-k}}(x)\int_0^\infty(e^{-t(p^{2k}+s)}-e^{-t(p^{2(k+1)}+s)})dt \\
&= \sum_{k=-\infty}^{\infty}p^k\xi_{p^{-k}}(x)(\frac{1}{p^{2k}+s}-\frac{1}{p^{2(k+1)}+s}) \\
&= \sum_{k=-\infty}^{\infty}\frac{p^2-1}{(1+\frac{s}{p^{2k}})(p^2+\frac{s}{p^{2k}})}p^{-k}\xi_{p^{-k}}(x) \\
&= \sum_{k=-\infty}^{\infty}\frac{p^2-1}{(1+p^{2k}s)(p^2+p^{2k}s)}p^k\xi_{p^k}(x).
\end{aligned}$$

□

2.3 The operator Δ_C and its associated heat equation

In previous discussion, we chose $\varphi=|\xi|_p$ when defining the operator in Section 2.1. Other choices have been considered. For example, [9] considers a slight variation of the Vladimirov operator: the map J^α defined by

$$u(x)\mapsto \mathcal{F}^{-1}_{w\to x}[|1,w|_p^{-\alpha}\mathcal{F}_{x\to w}u],$$

where $|1,x|_p=\max\{1,|x|_p\}$. Why either should have precedence over the other is not entirely clear. Due to the fact that in the analogous situation for the real numbers ($J^{-2N}=(1+\frac{\partial^2}{\partial x^2})^N$) [9], it could be worthwhile to consider the operator

$$\Delta_C := J^{-2}-C.$$

Once again, this is well-defined on C_c^∞ due to Theorem 2.1.5, and its heat equation yields to the same computational techniques as Theorem 2.2.2; in fact, calculations are considerably easier, as $|1,x|_p$ is locally constant even at 0.

Theorem 2.3.1. *The equation*

$$u'(x,t) + \Delta_C u(x,t) = 0, \quad u(x,0) = \varphi(x) \in C_c^\infty$$

has the solution $\varphi(x) * K_C(x,t)$, *where*

$$K_C(x,t) = e^{Ct}\sum_{k=0}^{\infty}(e^{-p^{2k}t} - e^{-p^{2k+2}t})p^k \xi_{p^{-k}}(x).$$

Furthermore, both K_C *and* $\hat{K}_C$ *are integrable.*

Proof. In contrast with the proof of Theorem 2.2.2, we will assume that for all $t > 0$, $u(x,t) \in C_c^\infty$, later verifying that this is indeed the case. This is sufficient to imply that there is an integrable function $H : \mathbb{Q}_p \to \mathbb{R}$ such that $|u'(x,s)|\leq H(x)$ for all s in a neighborhood of t, i.e. the existence of a dominating function. Again write

$$\begin{aligned}\frac{\hat{u}(w,t+h) - \hat{u}(w,t)}{h} &= \int_{\mathbb{Q}_p} e^{-2\pi i\{xw\}_p}\frac{u(x,t+h) - u(x,t)}{h}dx \\ &= \int_{\mathbb{Q}_p} e^{-2\pi i\{xw\}_p}u'(x,c_x(h))dx,\end{aligned}$$

and using the Dominated Convergence Theorem, we are permitted to move the limit $h \to 0$ inside the integral. Taking the transform, we arrive at

$$\hat{u}'(w,t) + (|1,w|_p^2 - C)\hat{u}(w,t) = 0,$$

with solution $\hat{u}(w,t) = \hat{\varphi}(w)e^{(C-|1,w|_p^2)t}$. This a locally constant, compactly supported function, so due to Theorem 2.1.5, we conclude that $u(x,t) \in C_c^\infty$ for all $t > 0$. Now denote $\hat{K}_C(w,t) := e^{(C-|1,w|_p^2)t}$. $\hat{K}(w,t)$ lies in L^1, since $e^{-|1,w|_p^2 t} \leq e^{-|w|_p^2 t} < \frac{1}{|w|_p^2 t}$, and

$$t^{-1}\int_{\mathbb{Q}_p}|w|_p^{-2}dw = t^{-1}(\int_{\mathbb{Z}_p}|w|_p^{-2}dw + \sum_{k=1}^{\infty}\int_{|w|_p=p^k}|w|_p^{-2}dw) = t^{-1}(\int_{\mathbb{Z}_p}|w|_p^{-2}dw + \sum_{k=1}^{\infty}p^{-2k}p^k(1-p^{-1})) < \infty,$$

using similar calculations as in Example 1.2.2. Finally, taking the inverse transform, we obtain

$$
\begin{aligned}
K_C(x,t) &= \int_{\mathbb{Q}_p} e^{2\pi i\{xw\}_p} e^{(C-|1,w|_p^2)t} dw = e^{(C-1)t}\int_{\mathbb{Z}_p} e^{2\pi i\{xw\}_p} dw + \sum_{k=1}^{\infty} e^{Ct-p^{2k}t}\int_{|w|=p^k} e^{2\pi i\{xw\}_p} dw \\
&= e^{(C-1)t}\xi_{\mathbb{Z}_p}(x) + \sum_{k=1}^{\infty} e^{Ct-p^{2k}t}(\int_{|w|\le p^k} e^{2\pi i\{xw\}_p} dw - \int_{|w|\le p^{k-1}} e^{2\pi i\{xw\}_p} dw) \\
&= e^{(C-1)t}\xi_{\mathbb{Z}_p}(x) + e^{Ct}\sum_{k=1}^{\infty} e^{-p^{2k}t}p^{k-1}(p\xi_{p^{-k}}(x) - \xi_{p^{1-k}}(x)) \\
&= e^{Ct}((e^{-t} - e^{-p^2 t})\xi_{\mathbb{Z}_p}(x) + \sum_{k=1}^{\infty}(e^{-p^{2k}t} - e^{-p^{2k+2}t})p^k\xi_{p^{-k}}(x)) \\
&= e^{Ct}\sum_{k=0}^{\infty}(e^{-p^{2k}t} - e^{-p^{2k+2}t})p^k\xi_{p^{-k}}(x),
\end{aligned}
$$

which is clearly in L^1 for all $t > 0$ since

$$
\sum_{k=0}^{\infty}(e^{-p^{2k}t} - e^{-p^{2k+2}t})p^k\xi_{p^{-k}}(x) < \sum_{k=0}^{\infty}(e^{-p^{2k}t} - p^{-2}e^{-p^{2k+2}t})p^k\xi_{p^0}(x) < \omega(t)\cdot\xi_{p^0}(x).
$$

□

Just as for K, there is analogously another formula for K_C.

Theorem 2.3.2. *For $x \in \mathbb{Z}_p \setminus \{0\}$,*

$$
K_C(x,t) = e^{Ct}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}(\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1} + (\frac{1-p^{-2m}}{p-p^{-2m}})).
$$

Proof. We will proceed in the same manner as the proof of Theorem 2.2.3. We have

$$
\begin{aligned}
K_C(x,t) &= \int_{\mathbb{Q}_p} e^{2\pi i\{xw\}_p} e^{Ct-t|1,w|_p^2} dw = e^{Ct}\sum_{n=-\infty}^{\infty} e^{-t\max\{1,p^{2n}\}}\int_{|w|_p=p^n} e^{2\pi i\{xw\}_p} dw \\
&= e^{Ct}\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\max\{1,p^{2nm}\}\int_{|w|_p=p^n} e^{2\pi i\{xw\}_p} dw \\
&= e^{Ct}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\sum_{n=-\infty}^{\infty}\max\{1,p^{2nm}\}\int_{|w|_p=p^n} e^{2\pi i\{xw\}_p} dw \\
&= e^{Ct}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\sum_{n=-\infty}^{\infty}\max\{1,p^{2nm}\}\int_{|w|_p=p^n} e^{2\pi i\{xw\}_p} dw \\
&= e^{Ct}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\frac{1}{|x|_p}\sum_{n=-\infty}^{\infty}\max\{1,p^{2nm}\}\int_{|w|_p=p^n|x|_p} e^{2\pi i\{w\}_p} dw \\
&= e^{Ct}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}\frac{1}{|x|_p}\sum_{n=-\infty}^{\infty}\max\{1,p^{2nm}|x|_p^{-2m}\}\int_{|w|_p=p^n} e^{2\pi i\{w\}_p} dw.
\end{aligned}
$$

It follows immediately from the calculations for $K(x,t)$ that the interchange of summation above holds, and we may write

$$\sum_{n=-\infty}^{\infty} \max\{1, p^{2nm}|x|_p^{-2m}\} \int_{|w|_p=p^n} e^{2\pi i\{w\}_p} dw = -\max\{1, p^{2m}|x|_p^{-2m}\}$$
$$+ (1-\frac{1}{p}) \sum_{n=0}^{\infty} \max\{1, p^{-2nm}|x|_p^{-2m}\} p^{-n}.$$

Note that this sum is zero if $|x|_p > 1$, verifying that $K_C(x,t)$ indeed is supported on $\mathbb{Z}_p$. Now consider $|x|_p = p^{-N}$. Then the first term is $-p^{2m}|x|_p^{-2m}$, and sum within the second term may be written as

$$\sum_{n=0}^{\infty} \max\{1, p^{-2nm}|x|_p^{-2m}\} p^{-n} = \sum_{n=0}^{\infty} \max\{1, p^{-2m(n-N)}\} p^{-n} = p^{2mN} \sum_{n=0}^{N-1} p^{-(2m+1)n} + \sum_{n=N}^{\infty} p^{-n}$$
$$= \frac{p^{2mN} - p^{-N}}{1-p^{-(2m+1)}} + \frac{p^{-N}}{1-p^{-1}} = \frac{|x|_p^{-2m} - |x|_p}{1-p^{-(2m+1)}} + \frac{|x|_p}{1-p^{-1}}.$$

Thus the entire sum is equal to

$$(-p^{2m} + \frac{1-p^{-1}}{1-p^{-(2m+1)}})|x|_p^{-2m} + (1 - \frac{1-p^{-1}}{1-p^{-(2m+1)}})|x|_p = \frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m} + (\frac{1-p^{-2m}}{p-p^{-2m}})|x|_p,$$

so for $x \in \mathbb{Z}_p \setminus \{0\}$,

$$K_C(x,t) = e^{Ct} \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} (\frac{1-p^{2m}}{1-p^{-2m-1}}|x|_p^{-2m-1} + (\frac{1-p^{-2m}}{p-p^{-2m}})).$$

□

Writing

$$M(t) := \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} (\frac{1-p^{-2m}}{p-p^{-2m}}),$$

we notice that $K = K_0 - M$ for $x \in \mathbb{Z}_p \setminus \{0\}$. Thus this reasoning using the "regularized" heat kernel have brought some additional information about K, essentially giving another way of viewing the sides of the sum

$$K(x,t) = \sum_{k=-\infty}^{\infty} (e^{-tp^{2k}} - e^{-tp^{2k+2}}) p^k \xi_{p^{-k}}(x),$$

since K_0 corresponds to the positive side (compare Theorem 2.2.2, Theorem 2.3.1). It also provides us with a sanity check, as it implies that

$$-\sum_{m=1}^{\infty} \frac{(-1)^m t^m}{m!} (\frac{1-p^{-2m}}{p-p^{-2m}}) = \sum_{k=1}^{\infty} (e^{-tp^{-2k}} - e^{-tp^{2-2k}}) p^{-k},$$

which one verifies by writing the second factor on the left hand side as a geometric sum and contracting the Taylor series.

One quickly obtains the Mellin and Laplace transform of $\frac{K_C}{e^{Ct}} = K_0$ using the transforms for K. For $|x|_p = p^N, N \leq 0$, we have

$$\mathcal{M}(K_0) = \xi_{p^0}(x)(1-p^{2(1-s)})\sum_{k=N}^{0} p^{k(2s-3)} = \xi_{p^0}(x)(1-p^{2(1-s)})\sum_{k=0}^{N} p^{k(3-2s)}$$
$$= \frac{\xi_{p^0}(x)(1-p^{2-2s})(1-(p|x|_p)^{3-2s})}{1-p^{3-2s}} = \frac{\xi_{p^0}(x)(1-p^{2-2s})(p^{2s-3}-|x|_p^{3-2s})}{p^{2s-3}-1},$$

where the factor ξ_{p^0} is due to the fact that K_0 is supported on $\mathbb{Z}_p$, and

$$\mathcal{L}(K_0) = \sum_{k=0}^{\infty} \frac{p^2-1}{(1+p^{-2k}s)(p^2+p^{-2k}s)} p^{-k}\xi_{p^{-k}}(x).$$

To acquire the corresponding relation to Theorem 2.2.4, we will use that $K_0 = K\xi_{p^0} + M$, and calculate $\mathcal{L}(M)$. We have

$$\int_0^{\infty} e^{-st}M(t)dt = M(t) = \int_0^{\infty} e^{-st}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!}(\frac{1-p^{-2m}}{p-p^{-2m}})dt < \int_0^{\infty} e^{(1-s)t}dt,$$

so the integral converges absolutely if $s > 1$. Thus we have

$$\int_0^{\infty} e^{-st}M(t)dt = \sum_{m=0}^{\infty}(\frac{1-p^{-2m}}{p-p^{-2m}})\frac{(-1)^m}{m!}\int_0^{\infty} e^{-st}t^m dt$$
$$= \sum_{m=0}^{\infty}(\frac{1-p^{-2m}}{p-p^{-2m}})\frac{(-1)^m}{s^{m+1}},$$

and

$$\mathcal{L}(K_0) = \xi_{p^0}(x)\Big(\frac{1}{|x|_p s}\sum_{m=0}^{\infty}((\frac{-1}{|x|_p^2 s})^m \frac{1-p^{2m}}{1-p^{-2m-1}}) + \sum_{m=0}^{\infty}((\frac{1-p^{-2m}}{p-p^{-2m}})\frac{(-1)^m}{s^{m+1}})\Big)$$
$$= \sum_{m=0}^{\infty}\frac{(-\xi_{p^0}(x))^m}{s^{m+1}}(|x|_p^{-2m-1}+p^{-1})\frac{1-p^{2m}}{1-p^{-2m-1}}.$$

Chapter 3

A p-adic Poisson summation formula

3.1 Two proofs of the Poisson summation formula

We will now prove a p-adic version of the classic Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi i k x}.$$

In the real case, for sufficiently nice functions, this proved by periodization, i.e. involving the subgroup $\mathbb{Z}$, and the quotient $\mathbb{R}/\mathbb{Z}$. Using the p-adic analogues of these, i.e. the locally compact groups $\mathbb{Z}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$, and equipping them with suitable Haar measures, one acquires the p-adic Poisson summation formula below. In fact, this may be done for locally compact abelian groups in general whenever one has a closed subgroup [8].

Note however that the analogy is not perfect, as to get summation instead of integration, it is necessary to take the quotient with a discrete co-compact subgroup; such a subgroup does not exist in $\mathbb{Q}_p$. Indeed, let H be a discrete subgroup of $\mathbb{Q}_p$ and take a therein, with $|a| = p^N$ for some N. By definition, there is an open cover of $\mathbb{Q}_p$ in which every open subset contains exactly one element of H; denote the open subset that contains a by U_a. Since U_a is open, $B_{\leq p^{-n}}(a) \subset U_a$ for some $n \in \mathbb{N}$. But then $a - p^{N+n}a \in B_{<p^{-n}}(a)$, so U_a contains another element of H.

Thus, instead of becoming a relation connecting sums, we obtain on both sides integrals over $\mathbb{Z}_p$. We note that the sought relation is immediate from Theorem 2.1.4 when dealing with functions in C_c^{∞}.

Theorem 3.1.1 (Poisson summation formula)**.** *Whenever $f \in C_c^{\infty}$, we have*

$$\int_{\mathbb{Z}_p} f(x+y)dy = \int_{\mathbb{Z}_p} \hat{f}(y) e^{2\pi i \{xy\}_p} dy.$$

Proof. Using that each element in C_c^{∞} may be written as a finite sum of $c_k \mathbf{1}_{B_{\leq p^n}(a_k)}(x)$, it is enough to check the relation term by term, since the Fourier transform is linear. Thus assume $f(x) = c_k \mathbf{1}_{B_{\leq p^n}(a_k)}(x) = c_k \xi_{p^n}(x - a_k)$, for which we have

$$\int_{\mathbb{Z}_p} f(x+y)dy = \int_{\mathbb{Z}_p} c_k \mathbf{1}_{B_{\leq p^n}(a_k - x)}(y)dy = c_k \mu(\mathbb{Z}_p \cap B_{\leq p^n}(a_k - x)).$$

Due to the non-archimedean property of the p-adic metric, we have one of the following cases,

$$\begin{aligned}\mathbb{Z}_p &\subset B_{\le p^n}(a_k - x)\\ \mathbb{Z}_p &\supset B_{\le p^n}(a_k - x)\\ \mathbb{Z}_p &\cap B_{\le p^n}(a_k - x) = \emptyset,\end{aligned}$$

so

$$\int_{\mathbb{Z}_p} f(x+y)dy = \begin{cases} c_k & \text{if } \mathbb{Z}_p \subset B_{\le p^n}(a_k - x)\\ p^n c_k & \text{if } \mathbb{Z}_p \supset B_{\le p^n}(a_k - x),\\ 0 & \text{if } B_{\le p^n}(a_k - x) \cap \mathbb{Z}_p = \emptyset.\end{cases}$$

From Proposition 2.1.3, we furthermore have

$$\begin{aligned}\int_{\mathbb{Z}_p} \hat{f}(y)e^{2i\{xy\}_p}dy &= \int_{\mathbb{Z}_p} c_k e^{2\pi i\{((x-a_k)y\}_p} p^n \xi_{p^{-n}}(y)dy = c_k p^n \int_{\mathbb{Z}_p} e^{2\pi i\{(x-a_k)y\}_p} \xi_{p^{-n}}(y)dy\\ &= \begin{cases} c_k p^n \widehat{\xi_{p^{-n}}}(x-a_k) & \text{if } 0 \le n,\\ c_k p^n \widehat{\xi_{p^0}}(x-a_k) & \text{if } n \le 0\end{cases} = \begin{cases} c_k \xi_{p^n}(x-a_k) & \text{if } 0 \le n,\\ c_k p^n \xi_{p^0}(x-a_k) & \text{if } n \le 0\end{cases}\\ &= \begin{cases} c_k & \text{if } x-a_k \in B_{\le p^n}, \mathbb{Z}_p \subset B_{\le p^n},\\ p^n c_k & \text{if } x-a_k \in \mathbb{Z}_p, B_{\le p^n} \subset \mathbb{Z}_p,\\ 0 & \text{if } x-a_k \notin B_{\le p^n} \cap \mathbb{Z}_p\end{cases}\\ &= \begin{cases} c_k & \text{if } \mathbb{Z}_p \subset B_{\le p^n}(a_k - x),\\ p^n c_k & \text{if } \mathbb{Z}_p \supset B_{\le p^n}(a_k - x),\\ 0 & \text{if } B_{\le p^n}(a_k - x) \cap \mathbb{Z}_p = \emptyset.\end{cases}\end{aligned}$$

establishing the above formula.

□

It is straight-forward to obtain the p-adic analogue to a more "structure-showing" version of the Poisson summation formula from this. Assume $A \in \mathbb{Q}_p^*$. As

$$f(Ax) = c_k \xi_{B_{\le p^n}(a_k)}(Ax) = c_k \xi_{B_{\le p^n |A|_p^{-1}}(a_k)}(x)$$

has the transform

$$\widehat{f(|A|_p x)}(w) = c_k e^{2\pi i\{a_k w\}_p} p^n |A|_p^{-1} \xi_{B_{\le p^{-n}|A|_p}}(w) = c_k e^{2\pi i\{a_k w\}_p} p^n |A|_p^{-1} \xi_{B_{\le p^{-n}}}(\frac{w}{A}),$$

we get

$$\int_{\mathbb{Z}_p} f(Ax)dx = \frac{1}{|A|_p} \int_{\mathbb{Z}_p} \hat{f}(\frac{x}{A})dx. \tag{3.1}$$

Note that when $f = \xi_{p^0}$, this yields a candidate for a p-adic theta function,

$$\theta(x) = |1, x|_p.$$

We are now going to give another proof of the Poisson summation formula, which extends its validity. The proof is conceptually important, and is principally the same as in the general formula valid for all locally compact groups [8]. In particular, we will prove the relation for functions f with compact support that satisfy $\hat{f}|_{\mathbb{Z}_p} \in L^1(\mathbb{Z}_p)$; this will be enough to show that the Poisson summation formula holds for K_C. It is possible to weaken the assumptions on f even further, as noted in [8], and show that the relations holds almost everywhere for functions in L^1 with $\hat{f}|_{\mathbb{Z}_p} \in L^1(\mathbb{Z}_p)$. This would be enough to prove the relation for K as well (for a proof that $K \in L^1$, see [14]). We will however not prove this, mainly due to technical difficulties extending Lemma 3.1.3.

For the proof, we will require three lemmas.

Lemma 3.1.2. *The quotient $\mathbb{Q}_p/\mathbb{Z}_p$ is a locally compact, abelian group with respect to the quotient topology, and $\mathbb{Z}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$ are Pontryagin duals of each other.*

Proof. Each element in $\mathbb{Q}_p/\mathbb{Z}_p$ may be seen as the fractional part of a p-adic number x. Clearly, the quotient topology is discrete and thus $\mathbb{Q}_p/\mathbb{Z}_p$ is a locally compact abelian group. It was established in Theorem 1.3 that $\hat{\mathbb{Q}}_p \simeq \mathbb{Q}_p$, i.e. that the characters of $\mathbb{Q}_p$ could be indexed by a p-adic number y. Write $y = y_1 + y_2$, where $y_1 \in \mathbb{Z}_p$ and $y_2 = \{y\}_p$. Restricting the characters to $x \in \mathbb{Z}_p$, we see that

$$\xi_y(x) = e^{2\pi i\{xy\}_p} = e^{2\pi i\{x(y_1+y_2)\}_p} = e^{2\pi i\{xy_2\}_p},$$

and thus $\widehat{\mathbb{Z}_p} \simeq \mathbb{Q}_p/\mathbb{Z}_p$ as groups. We will now topologically investigate the isomorphism

$$\begin{aligned} \mathbb{Q}_p/\mathbb{Z}_p &\to \widehat{\mathbb{Z}_p} \\ a\mathbb{Z}_p &\mapsto \xi_a(x) = e^{2\pi\{ax\}_p}, \end{aligned}$$

with respect to the compact-open topology on the dual. Then the sets

$$W(K,V) = \{\xi \in \widehat{\mathbb{Z}_p} : \xi(K) \subset V\},$$

where $K \subset \mathbb{Z}_p$ is compact, V is an open neighborhood of the identity in S^1, constitute a neighborhood base of the trivial character $\xi_0 : x \mapsto 1$. For $a \in \mathbb{Q}_p/\mathbb{Z}_p$, denote $V_a = \{e^{2\pi ix} \in S^1 : 0 < x < a\}$. Then

$$W(\{1\}, V_a) \cap W(\{1\}, V_{a-|a|_p}) = \{\xi_a(x)\}$$

is an open set, so the topology is discrete, establishing that the isomorphism is bi-continuous.

It remains to show that $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \simeq \mathbb{Z}_p$. In fact, this follows in general from Pontryagin duality, but we will settle with a direct argument. Denote a character ψ, and let A_n denote the group generated by the element $\{p^{-n}\}_p$. Thus an element $x_n \in A_n$ must be mapped to a p^nth root, alas

$$\psi(x_n) = e^{2\pi i c_n x_n},$$

for some integer $0 \leq c_n \leq p^n - 1$. However, as we have a chain of subgroups $A_1 \leq A_2 \leq ... \leq \mathbb{Q}_p/\mathbb{Z}_p$, $x_n \in A_{n+1}$, and we may also write

$$\psi(x_n) = e^{2\pi i c_{n+1} x_n},$$

implying the relation $c_{n+1} \equiv c_n \mod p^n$. Similarly to the proof of Theorem 1.3, this means that $|c_{n+1} - c_n| < \frac{1}{p^n}$, and thus the sequence $\{c_1, c_2, ...\}$ is Cauchy, converging p-adically to some p-adic integer c. Thus for any x,

$$\psi(x) = e^{2\pi i\{cx\}_p}.$$

Clearly, the map

$$\begin{aligned} \mathbb{Z}_p &\to \widehat{\mathbb{Q}_p/\mathbb{Z}_p} \\ a &\mapsto \psi_a(x) = e^{2\pi\{ax\}_p} \end{aligned}$$

is also injective, which shows $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \simeq \mathbb{Z}_p$ as groups. To show that it is a homeomorphism, we essentially follow the proof of Theorem 1.3.2. Equipping the dual with the compact-open topology,

$$W(K,V) = \{\psi \in \widehat{\mathbb{Q}_p/\mathbb{Z}_p} : \psi(K) \subset V\},$$

where $K \subset \mathbb{Q}_p/\mathbb{Z}_p$ is compact, V is an open neighborhood of the identity in S^1, constitute a neighborhood base of the trivial character $\psi_0 : x \mapsto 1$. As the topology is discrete, $|K| < \infty$, and there exists an $m \in \mathbb{N}$ such that

$$K \subset K_m = \{x\mathbb{Z}_p \in \mathbb{Q}_p/\mathbb{Z}_p : x = \sum_{i=-m}^{-1} a_i p^i, 0 \leq a_i < p\}.$$

Thus $W(K_m, V) \subset W(K, V)$, and for fixed V, the larger the m, the smaller the neighborhood of ψ_0. there is a V_ϵ such that $W(\{0\}, V_\epsilon) \subset W(K, V)$.

If a is close to zero, say $|a|_p = p^{-N}$, then $\psi_a|_{C_m} = \psi_0|_{C_m}$, for all $m \leq N$, so $\psi_a \in W(C_m, V)$ for all V, and close to the trivial character. On the other hand, if $\psi_a \in W(C_m, V)$ for all V, then $\psi_a(C_m) \equiv 1$, and thus $|a|_p \leq p^{-m}$. This establishes bi-continuity. □

Lemma 3.1.3. *If f has compact support, then with regards to counting measure on $\mathbb{Q}_p/\mathbb{Z}_p$ and the Haar measure on $\mathbb{Z}_p$ with $\mu(\mathbb{Z}_p) = 1$, one has*

$$\int_{\mathbb{Q}_p} f(x)dx = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \int_{\mathbb{Z}_p} f(y+w)dyd(w\mathbb{Z}_p).$$

Proof. Say $f \equiv 0$ outside some ball B_m. Then

$$\int_{\mathbb{Q}_p} f(x)dx = \int_{B_n} f(x)dx = \sum_{a\in K_m} \int_{\mathbb{Z}_p + a} f(x)dx = \sum_{a\in K_m} \int_{\mathbb{Z}_p} f(x+a)dx = \int_{\mathbb{Z}_p} \sum_{a\in K_m} f(x+a)dx,$$

where $K_m := \{x \in \mathbb{Q}_p : x = \sum_{i=-m}^{-1} a_i p^i, 0 \leq a_i < p\}$ is a finite set. Identifying K_m with the corresponding set in $\mathbb{Q}_p/\mathbb{Z}_p$ then yields the formula above. □

As for $\mathbb{Q}_p$, we may analogously define the Fourier transform for suitable functions on $\mathbb{Z}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$ respectively, and using the correspondence established in Lemma 3.1.2, the transformed function is then defined on $\mathbb{Q}_p/\mathbb{Z}_p$, $\mathbb{Z}_p$ respectively. The last lemma states that we have an inverse Fourier transform for these cases as well.

Lemma 3.1.4. *Let $f \in L^1(\mathbb{Z}_p)$, $g \in L^1(\mathbb{Q}_p/\mathbb{Z}_p)$ such that f, g are continuous and such that $\hat{f} \in L^1(\mathbb{Q}_p/\mathbb{Z}_p)$, $\hat{g} \in L^1(\mathbb{Z}_p)$. Using the counting measure and normalizing such that $\mathbb{Z}_p$ always have the measure equal to 1, we then have*

$$f(x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \hat{f}(w) e^{2\pi i\{xw\}_p} d(w\mathbb{Z}_p),$$
$$g(w\mathbb{Z}_p) = \int_{\mathbb{Z}_p} \hat{g}(x) e^{2\pi i\{xw\}_p} dx.$$

Proof. See [8], Theorem 4.32. □

We are now ready to prove the more general Poisson summation formula.

Theorem 3.1.5. *Assume that $f \in C_c(\mathbb{Q}_p)$ and that $\hat{f}|_{\mathbb{Z}_p} \in L^1(\mathbb{Z}_p)$. Then*

$$\int_{\mathbb{Z}_p} f(x+y)dy = \int_{\mathbb{Z}_p} \hat{f}(y) e^{2\pi i\{xy\}_p} dy.$$

Proof. Define $F \in C_c(\mathbb{Q}_p/\mathbb{Z}_p)$ by $F(x+\mathbb{Z}_p) = \int_{\mathbb{Z}_p} f(x+y)dy$. Due to Lemma 3.1.2, the Fourier transform of F is a function on $\mathbb{Z}_p$. We may write

$$\begin{aligned}\hat{F}(w) &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} e^{-2\pi i\{xw\}_p} \int_{\mathbb{Z}_p} f(x+y)dyd(x\mathbb{Z}_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \int_{\mathbb{Z}_p} f(x+y)e^{-2\pi i\{xw\}_p} dyd(x\mathbb{Z}_p) \\ &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} \int_{\mathbb{Z}_p} f(x+y)e^{-2\pi i\{(x+y)w\}_p} dyd(x\mathbb{Z}_p) = \int_{\mathbb{Q}_p} f(z)e^{-2\pi i\{zw\}_p} dyd(x\mathbb{Z}_p) = \hat{f}(w).\end{aligned}$$

Applying the inverse Fourier transform then yields the formula. □

As noted, K_C satisfy the requirements for the Poisson summation formula to hold, and K satisfies the requirements for the formula to be valid almost everywhere. We will now complement this by showing that the formula (3.1) holds by direct calculation.

Theorem 3.1.6. *The heat kernels K, K_C satisfy*

$$\int_{\mathbb{Z}_p} K_C(Ax,t)dx = \frac{1}{|A|_p} \int_{\mathbb{Z}_p} \widehat{K}_C(\frac{x}{A},t)dx$$
$$\int_{\mathbb{Z}_p} K(Ax,t)dx = \frac{1}{|A|_p} \int_{\mathbb{Z}_p} \widehat{K}(\frac{x}{A},t)dx.$$

Proof. As $K_C \in L^1$, we may apply the Dominated Convergence Theorem, so

$$\begin{aligned}\int_{\mathbb{Z}_p} K_C(Ax,t)dx &= e^{Ct} \int_{\mathbb{Z}_p} \sum_{k=0}^{\infty} (e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \xi_{p^{-k}}(Ax)dx \\ &= e^{Ct} \sum_{k=0}^{\infty} (e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \int_{\mathbb{Z}_p} \xi_{p^{-k}|A|_p^{-1}}(x)dx.\end{aligned}$$

We will now consider two cases. If $|A|_p \geq 1$, this is equal to

$$\frac{1}{|A|_p} e^{Ct} \sum_{k=0}^{\infty} (e^{-p^{2k}t} - e^{-p^{2k+2}t}) = \frac{e^{(C-1)t}}{|A|_p} = \frac{1}{|A|_p} \int_{\mathbb{Z}_p} e^{(C - |1, \frac{x}{A}|_p^2)t} dx = \frac{1}{|A|_p} \int_{\mathbb{Z}_p} \widehat{K}_C(\frac{x}{A}, t) dx.$$

If $|A|_p = p^{-N} < 1$, then

$$\begin{aligned}
\int_{\mathbb{Z}_p} K_C(Ax, t) dx &= \sum_{k=0}^{N} e^{Ct} \Big((e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \int_{\mathbb{Z}_p} \xi_{p^{-k}|A|_p^{-1}}(x) dx \Big) \\
&+ \sum_{k=N+1}^{\infty} e^{Ct} \Big(e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \int_{\mathbb{Z}_p} \xi_{p^{-k}|A|_p^{-1}}(x) dx \Big) \\
&= \sum_{k=0}^{N} e^{Ct} \Big((e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \Big) + \sum_{k=N+1}^{\infty} e^{Ct} \Big((e^{-p^{2k}t} - e^{-p^{2k+2}t}) |A|_p^{-1} \Big) \\
&= \sum_{k=0}^{N} e^{Ct} \Big((e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \Big) + e^{Ct} e^{-p^{2N+2}t} |A|_p^{-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{|A|_p} \int_{\mathbb{Z}_p} \widehat{K}_C(\frac{x}{A}, t) dx &= \frac{1}{|A|_p} \Big(\int_{B_{-N}} \widehat{K}_C(\frac{x}{A}, t) dx + \sum_{i=0}^{N-1} \int_{|x|_p = p^{-i}} \widehat{K}_C(\frac{x}{A}, t) dx \Big) \\
&= e^{(C-1)t} + \frac{1}{|A|_p} \sum_{i=0}^{N-1} \int_{|x|_p = p^{-i}} \widehat{K}_C(\frac{x}{A}, t) dx = e^{(C-1)t} + \frac{e^{Ct}}{|A|_p} \sum_{i=0}^{N-1} \int_{|x|_p = p^{-i}} e^{-|\frac{x}{A}|_p^2 t} dx \\
&= e^{(C-1)t} + e^{Ct} \sum_{i=0}^{N-1} e^{-p^{2(N-i)}t} (p^{N-i} - p^{N-i-1}) = e^{(C-1)t} + e^{Ct} \sum_{k=1}^{N} e^{-p^{2k}t} (p^k - p^{k-1}) \\
&= e^{(C-1)t} + e^{Ct} (e^{-p^{2N+2}t} p^N - e^{-t} + \sum_{k=0}^{N} (e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k) = \int_{\mathbb{Z}_p} K_C(Ax, t) dx.
\end{aligned}$$

The heat kernel K associated to the Vladimirov operator yields to similiar calculations. We will begin with the case when $|A|_p = p^{-N}, N \geq 0$. Since $K(x,t) = K_0(x,t) + \sum_{k=1}^{\infty} (e^{-p^{-2k}t} - e^{-p^{-2k-2}t}) p^{-k} \xi_{p^k}(x)$,

$$\begin{aligned}
\int_{\mathbb{Z}_p} K(Ax, t) dx &= \int_{\mathbb{Z}_p} K_0(Ax, t) + \sum_{k=1}^{\infty} (e^{-p^{-2k}t} - e^{-p^{-2k+2}t}) p^{-k} \xi_{p^{k+N}}(x) dx \\
&= \int_{\mathbb{Z}_p} K_0(Ax, t) dx + \sum_{k=1}^{\infty} (e^{-p^{-2k}t} - e^{-p^{-2k+2}t}) p^{-k} \\
&= \sum_{k=0}^{N} \Big((e^{-p^{2k}t} - e^{-p^{2k+2}t}) p^k \Big) + e^{-p^{2N+2}t} p^N + \sum_{k=1}^{\infty} (e^{-p^{-2k}t} - e^{-p^{-2k+2}t}) p^{-k} \\
&= e^{-p^{2N+2}t} p^N + \sum_{k=-N}^{\infty} (e^{-p^{-2k}t} - e^{-p^{-2k+2}t}) p^{-k}.
\end{aligned}$$

On the right hand side,

$$\frac{1}{|A|_p}\int_{\mathbb{Z}_p}\widehat{K}(A^{-1}x,t)dx = \frac{1}{|A|_p}\sum_{k=0}^{\infty}\int_{|x|_p=p^{-k}} e^{-p^{-2(k-N)}}dx = \sum_{k=0}^{\infty} e^{-p^{-2(k-N)}}(p^{N-k}-p^{N-k-1})$$
$$= \sum_{k=-N}^{\infty} e^{-p^{-2k}}(p^{-k}-p^{-k-1}) = e^{-p^{2N+2}t}p^{N} + \sum_{k=-N}^{\infty}(e^{-p^{-2k}} - e^{-p^{-2k+2}})p^{-k}.$$

Now assume that $|A|_p = p^N, N \geq 0$. From the previous calculations, we see that

$$\frac{1}{|A|_p}\int_{\mathbb{Z}_p}\widehat{K}(A^{-1}x,t)dx = e^{-p^{-2N+2}t}p^{-N} + \sum_{k=N}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}.$$

Writing

$$\int_{\mathbb{Z}_p} K(Ax,t)dx = \int_{\mathbb{Z}_p} K_0(Ax,t) + \sum_{k=1}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}\xi_{p^{k-N}}(x)dx$$
$$= \frac{e^{-t}}{|A|_p} + \int_{\mathbb{Z}_p}\sum_{k=1}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}\xi_{p^{k-N}}(x)dx$$
$$= \frac{e^{-t}}{|A|_p} + \sum_{k=1}^{\infty}\int_{\mathbb{Z}_p}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}\xi_{p^{k-N}}(x)dx,$$

using the dominating function

$$\sum_{k=1}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}\xi_{p^{k-N}}(x) < \sum_{k=1}^{\infty}2p^{-k}\xi_{\mathbb{Z}_p}(x),$$

the left side becomes

$$\int_{\mathbb{Z}_p} K(Ax,t)dx = \frac{e^{-t}}{|A|_p} + \sum_{k=1}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}\int_{\mathbb{Z}_p}\xi_{p^{k-N}}(x)dx$$
$$= \frac{e^{-t}}{|A|_p} + \sum_{k=1}^{N-1}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-N} + \sum_{k=N}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k}$$
$$= e^{-p^{-2(N-1)}t}p^{-N} + \sum_{k=N}^{\infty}(e^{-p^{-2k}t} - e^{-p^{-2k+2}t})p^{-k} = \frac{1}{|A|_p}\int_{\mathbb{Z}_p}\widehat{K}(A^{-1}x,t)dx.$$

□

3.2 Transforming the summation formula

As a special case of the summation formula, we have

$$\int_{\mathbb{Z}_p} K_0(x,t)dx = \int_{\mathbb{Z}_p} \widehat{K}_0(x,t)dx$$
$$\int_{\mathbb{Z}_p} K(x,t)dx = \int_{\mathbb{Z}_p} \widehat{K}(x,t)dx.$$

For K_0, we obtain $1 = 1$ when carrying out the Mellin transform:

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\int_{\mathbb{Z}_p} K_0(x,t)dxdt = \int_{\mathbb{Z}_p} \mathcal{M}[K_0(x,t)]dx &= \frac{(1-p^{2-2s})}{p^{2s-3}-1}\int_{Z_p}(p^{2s-3}-|x|_p^{3-2s})dx \\ &= \frac{(1-p^{2-2s})}{p^{2s-3}-1}(p^{2s-3}-\frac{p-1}{p-p^{3-2s}}) \\ &= \frac{(1-p^{2-2s})}{p^{2s-3}-1}\frac{p^{2s-2}-p}{p-p^{3-2s}} = 1,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\int_{\mathbb{Z}_p}\widehat{K_0}(x,t)dxdt &= \frac{1}{\Gamma(s)}\int_{\mathbb{Z}_p}\int_0^\infty t^{s-1}e^{-t|1,x|_p^2}dtdx \\ &= \frac{1}{\Gamma(s)}\int_{\mathbb{Z}_p}|1,x|_p^{2-2s}\int_0^\infty t^{s-1}e^{-t}dtdx = 1.\end{aligned}$$

For K, we obtain on the left hand side

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\int_{\mathbb{Z}_p}K(x,t)dxdt = \int_{\mathbb{Z}_p}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(x,t)dtdx &= \int_{\mathbb{Z}_p}\mathcal{M}[K](x,s)dx \\ &= \frac{(1-p^{2-2s})}{1-p^{2s-3}}\int_{Z_p}|x|_p^{2s-3}dx = \frac{(1-p^{2-2s})}{1-p^{2s-3}}\frac{p-1}{p-p^{3-2s}} \\ &= \frac{(p-1)}{p-p^{2s-2}},\end{aligned}$$

using Example 1.2.2. On the right hand side,

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\int_{\mathbb{Z}_p}\widehat{K}(x,t)dxdt = \frac{1}{\Gamma(s)}\int_{\mathbb{Z}_p}\int_0^\infty t^{s-1}e^{-t|x|_p^2}dtdx &= \frac{1}{\Gamma(s)}\int_{\mathbb{Z}_p}|x|_p^{2-2s}\int_0^\infty t^{s-1}e^{-t}dtdx \\ = \int_{Z_p}|x|_p^{2-2s}dx = \frac{(p-1)}{p-p^{2s-2}}.\end{aligned}$$

The "associated theta relation" for K is thus also trivially equal. It however striking when comparing with the spectral zeta function associated to the operator D^2 is calculated to be [15]

$$\zeta_{D^2}(s) = \frac{(p-1)}{(p^{2s}-p)}.$$

Compare this to the real case [16], where the spectral zeta function of the Laplacian on S^1 is

$$\zeta_{S^1} = 2\zeta(2s),$$

where ζ is the Riemann zeta function. Rewriting the trivial equality in terms of this function, we obtain

$$\begin{aligned} -\zeta_{D^2}(s-1) &= \frac{(p-1)}{p-p^{2s-2}} = \frac{(p-1)p^{3-2s}}{p^{4-2s}-p} \\ &= \frac{(p-1)p^{3-2s}}{p^{2(1-(s-1))}-p} = p^{1-2(s-1)}\zeta_{D^2}(1-(s-1)), \end{aligned}$$

which is equivalent to

$$\zeta_{D^2}(s) = -p^{1-2s}\zeta_{D^2}(1-s).$$

Again one might compare with the functional equation for the real spectral zeta function,

$$\zeta_{S^1}(s) = 4^s\pi^{2s-1}\sin(\pi s)\Gamma(1-2s)\zeta_{S^1}(\frac{1}{2}-s).$$

Bibliography

[1] S. Echterhoff A. Deitmar. *Principles of Harmonic Analysis.* Universitext. Springer Science+Business Media, 2014. ISBN: 978-3319057910.

[2] M. Neuhauser A. Karlsson. "Heat kernels, theta identities, and zeta functions on cyclic groups". In: *Contemporary Mathematics* 394 (2006), pp. 177–189.

[3] A.N. Kochubei A.Y. Khrennikov. "p-Adic Analogue of the Porous Medium Equation". In: *Journal of Fourier Analysis and Applications* (2017), pp. 1–24.

[4] J. Bell. *Harmonic analysis on the p-adic numbers.* `https://pdfs.semanticscholar.org/9d4f/170c.pdf`. Unpublished; accessed May 16 2018.

[5] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Universitext. Springer Science+Business Media, 2011. ISBN: 978-0-387-70913-0.

[6] K. Conrad. *Differentiating under the Integral Sign.* `http://www.math.uconn.edu/~kconrad/blurbs/analysis/diffunderint.pdf`. Unpublished; accessed May 28 2018.

[7] K. Conrad. *The Character Group of* $\mathbb{Q}$. `http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/characterQ.pdf`. Unpublished; accessed April 6 2018.

[8] G. B. Folland. *A Course in Abstract Harmonic Analysis.* Studies in Advanced Mathematics. CRC press, 1995. ISBN: 0-8493-8490-7.

[9] S. Haran. "Quantizations and Symbolic Calculus Over the p-adic Numbers". In: *Annales de l'institut Fourier* 43 (1993), pp. 997–1053.

[10] S. Lang J. Jorgenson. *The Heat Kernel and Theta Inversion on SL2(C).* Springer Monographs in Mathematics. Springer New York., 2008. ISBN: 978-0-387-38032-2.

[11] A. Karlsson. "Applications of heat kernels on abelian groups: $\zeta(2n)$, quadratic reciprocity, Bessel integrals". In: *Number theory, Analysis and Geometry* (2012), pp. 307–320.

[12] S. Katok. *p-adic Analysis Compared with Real.* Vol. 37. Student Mathematical Library. American Mathematical Society, 2007. ISBN: 978-0-8218-4220-1.

[13] N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions.* Graduate Texts in Mathematics. Springer-Verlag New York, Inc, 1984. ISBN: 0-387-96017-1.

[14] A. N. Kochubei. *Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields.* Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., 2001. ISBN: 0-8247-0655-2.

[15] W. A. Zúñiga-Galindo L. F. Chaćon-Cortés. *Heat Traces and Spectral Zeta functions for p-adic Laplacians.* `https://arxiv.org/pdf/1511.02146.pdf`. Unpublished; accessed June 8 2018.

[16] M. Vergne N. Berline E. Getzler. *Heat Hernels and Dirac Operators.* Springer-Verlag Berlin Heidelberg New York, 1996. ISBN: 3-540-53340-0.

[17] C. W. Onneweer. "On the Definition of Dyadic Differentiation". In: *Applicable Anal.* 9 (1979), pp. 267–278.

[18] P. J. Sally. “An Introduction to p-adic Fields, Harmonic Analysis and the Representation Theory of SL_2”. In: *Letters in Mathematical Physics* (1998), pp. 1–47.

[19] E. I. Zelenov V. S. Vladimirov I. V. Volovich. *p-adic Analysis and Mathematical Physics*. Vol. 1. Series on Soviet and East European Mathematics. World Scientific Publishing Co. Pte. Ltd., 1994. ISBN: 981-02-0880-4.

[20] W. A. Zúñiga-Galindo. *Pseudodifferential Equations Over Non-Archimedean Spaces*. Lecture Notes in Mathematics. Springer Nature, 2016. ISBN: 978-3-319-46737-5.