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On Singular Integral Operators

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ABSTRACT

We introduce the subject of harmonic analysis through the perspective of singular integral theory. The Hilbert transform is motivated through problems regarding convergence of Fourier series and the study of boundary values of harmonic functions. The boundedness of the Hilbert transform in $L^p, 1 < p < \infty$, is proved through real-variable methods using Plancherel's Theorem and the Calderón Zygmund decomposition, along with Marcinkiewicz Interpolation Theorem. The same methods can be used to prove the boundedness of certain singular integrals on $\mathbb{R}^n$, and another approach using the Riesz transforms in $\mathbb{R}^n$ gives similar results for homogeneous singular integrals of degree $-n$. Measure theory and some basic complex analysis is assumed, along with a few results on functions spaces listed in the appendix.

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Contents

Introduction

Throughout this text we will study various singular integral operators. The simplest example of such an operator is the Hilbert transform $\mathcal{H}$. For $f \in L^1((-\frac{1}{2}, \frac{1}{2}))$, it is given by the expression

$$\mathcal{H}f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(t)}{\tan \pi(x-t)}\, \mathrm{d}t. \tag{1}$$

$\frac{1}{\tan \pi t}$ is not integrable near the origin (since $\frac{1}{\tan \pi t}$ - $\frac{1}{\pi t}$ is bounded), so the existence of the integral in (1) is a problem. For example, if $|f(t)| \geqslant C$ for all $t \in (x-\delta, x+\delta)$ for some $C, \delta > 0$, then it does not exist in the Lebesgue sense, since the integrand would be bounded below by $\frac{C}{|\tan \pi(x-t)|}$ near x. However, this can be handled by considering the integral in the principal value sense. That is, we define the Hilbert transform by

$$\mathcal{H}f(x) = \text{p.v.} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(t)}{\tan \pi(x-t)}\, \mathrm{d}t = \lim_{\epsilon \to 0^+} \int_{\epsilon \leqslant |x-t| < \frac{1}{2}} \frac{f(t)}{\tan \pi(x-t)}\, \mathrm{d}t, \tag{2}$$

and the truncated Hilbert transform is defined by

$$\mathcal{H}_\epsilon f = \int_{\epsilon \leqslant |x-t| < \frac{1}{2}} \frac{f(t)}{\tan \pi(x-t)}\, \mathrm{d}t. \tag{3}$$

By a change of variables we can write

$$\mathcal{H}_\epsilon f = \int_{\epsilon \leqslant |t| < \frac{1}{2}} \frac{f(x-t)}{\tan \pi t}\, \mathrm{d}t = -\int_{\epsilon \leqslant |t| < \frac{1}{2}} \frac{f(x+t)}{\tan \pi t}\, \mathrm{d}t = \int_{\epsilon}^{\frac{1}{2}} \frac{f(x-t) - f(x+t)}{\tan \pi t}\, \mathrm{d}t. \tag{4}$$

Note that $H_\epsilon f$ exists when $f \in L^1((-\frac{1}{2}, \frac{1}{2}))$, since $\frac{1}{\tan \pi t}$ is bounded on $[\epsilon, \frac{1}{2})$. It is still not obvious that the limit in (2) exists. However, we see that it holds if f satisfies a Hölder condition of order $\alpha > 0$, since then the last integral in (4) is bounded near the origin by a constant multiple of $\frac{t^\alpha}{\tan \pi t}$, which is integrable on $(0, \frac{1}{2})$. By Dominated Convergence Theorem, we also have

$$\mathcal{H}f(x) = \int_{0}^{\frac{1}{2}} \frac{f(x-t) - f(x+t)}{\tan \pi t}\, \mathrm{d}t. \tag{5}$$

The limit in (2) also turns out to exist almost everywhere for every $f \in L^1((-\frac{1}{2}, \frac{1}{2}))$. We will prove this in the second chapter.

The Hilbert transform appears in the study of holomorphic functions in the unit disc. Let $F = u + iv$ be a function holomorphic on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\overline{D}$. Suppose $u = f$ on ∂D where f satisfies a Hölder condition. u is unique by the maximum principle, and it is well known that it is given by the Poisson integral

$$u(re^{2\pi i\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(e^{2\pi i t}) P_r(\theta - t)\, \mathrm{d}t, \tag{6}$$

where

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos 2\pi t + r^2} \tag{7}$$

is the Poisson kernel. To see this result, we write

$$F(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(w)}{w - z}\, \mathrm{d}w, \tag{8}$$

which holds for all $z \in D$ by the Cauchy Integral Formula. The inverse point of z with respect to ∂D is $z^* = (\overline{z})^{-1}$. By Cauchy's Theorem we can write

$$F(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(w)}{w - z}\, \mathrm{d}w - \frac{1}{2\pi i} \int_{\partial D} \frac{F(w)}{w - z^*}\, \mathrm{d}w \tag{9}$$

since $z^* \notin D$. We have

$$\frac{1}{w-z} - \frac{1}{w-z^*} = \frac{1}{w-z} - \frac{\overline{z}}{w(\overline{z} - \overline{w})} = \frac{1 - |z|^2}{w\,|w-z|^2}.$$

If we now set $z = re^{2\pi i\theta}$ and $w = e^{2\pi it}$, we get

$$F(re^{2\pi i\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(1-r^2)F(e^{2\pi it})}{|e^{2\pi it} - re^{2\pi i\theta}|^2}\,\mathrm{d}t = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(e^{2\pi it})\frac{1-r^2}{1-2r\cos 2\pi(\theta - t) + r^2}\,\mathrm{d}t.$$

Hence the real part of $F(re^{2\pi i\theta})$ is given by the expression in (6). Since u is unique, v will be unique up to a constant. To uniquely determine v, we require that $v(0) = 0$. To calculate v, do the same calculation on

$$F(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{F(w)}{w - z}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{\partial D} \frac{F(w)}{w - z^*}\,\mathrm{d}w,$$

which will result in

$$v(re^{2\pi i\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(e^{2\pi it})\,\mathrm{d}t + \int_{-\frac{1}{2}}^{\frac{1}{2}} f(e^{2\pi it})Q_r(\theta - t)\,\mathrm{d}t, \tag{10}$$

where

$$Q_r(t) = \frac{2r\sin 2\pi t}{1 - 2r\cos 2\pi t + r^2} \tag{11}$$

is called the conjugate Poisson kernel. By the mean value property of harmonic functions, the first integral in (10) equals $v(0) = 0$, so we have

$$v(re^{2\pi i\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(e^{2\pi it})Q_r(\theta - t)\,\mathrm{d}t = \int_0^{\frac{1}{2}} \left(f(e^{2\pi i(\theta - t)}) - f(e^{2\pi i(\theta + t)})\right) Q_r(t)\,\mathrm{d}t.$$

If we let $r \to 1^-$ we see that $Q_r(t) \to \frac{1}{\tan \pi t}$. Since F is continuous on $\overline{D}$ and f satisfies a Hölder condition we get $v(re^{2\pi i\theta}) \xrightarrow{r\to 1^-} \mathcal{H}f(e^{2\pi i\theta})$, which can be shown to hold almost everywhere for all $f \in L^1((-\frac{1}{2}, \frac{1}{2}))$. From this perspective the Hilbert transform maps the boundary function of an harmonic function in D to the boundary function of its harmonic conjugate. For this reason it is sometimes called the Hilbert transform on the circle.

One can do a similar calculation for a bounded holomorphic function f defined on upper half space $\{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. The boundary function of the harmonic conjugate will then be given by the Hilbert transform on the real line,

$$\mathcal{H}f(x) = \mathrm{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x - t}\,\mathrm{d}t. \tag{12}$$

This linear operator has many interesting properties. For example, it commutes with translations, complex conjugation, positive dilations and differentiation. Moreover, it is a Hilbert space isomorphism on $L^2(\mathbb{R})$ with $\mathcal{H}^{-1} = -\mathcal{H}$, and both forms of the Hilbert transform map L^p into L^p when $1 < p < \infty$. They are also continuous on $L^p, 1 < p < \infty$, which is a deep result proved by Marcel Riesz. In the second chapter we give a proof of this for the Hilbert transform on the circle. In the first chapter we introduce Fourier series and discuss their convergence, which is deeply connected to the Hilbert transform. The L^2-theory of Fourier series will prove the boundedness of the Hilbert transform in $L^2((-\frac{1}{2}, \frac{1}{2}))$, which together with an interpolation theorem can be used to prove the more general L^p-boundedness when $1 < p < \infty$. Moreover, the L^p-boundedness of the Hilbert transform will then imply that the Fourier series of $f \in L^p((-\frac{1}{2}, \frac{1}{2})), 1 < p < \infty$, converges to f in L^p-norm! In fact, assuming the boundedness of the Hilbert transform in L^p, we can prove that

$$\widehat{\mathcal{H}f}(j) = -i\,\mathrm{sgn}\,(j)\,\widehat{f}(j), \tag{13}$$

where $\widehat{f}(j)$ are the Fourier coefficients of f. In chapter 1 we will prove this fact, and that if $-i\,\mathrm{sgn}\,(j)\,\widehat{f}(j)$ is the Fourier series of some $\widetilde{f} \in L^p((-\frac{1}{2}, \frac{1}{2}))$ for every $f \in L^p((-\frac{1}{2}, \frac{1}{2}))$ then the L^p-convergence of Fourier series holds in general.

In the third chapter we generalize the operator in (12) to a more general class of singular integral operators. For example, we have the Riesz transforms $\mathcal{R}_j, j \in \{1, 2, ..., n\}$, defined by

$$\mathcal{R}_j f(x) = \mathrm{p.v.}C_n \int_{\mathbb{R}^n} f(y)\frac{x_j - y_j}{|x - y|^{n+1}}\,\mathrm{d}y \tag{14}$$

for some constant C_n. The Riesz transforms appear when studying conjugate harmonic functions on upper half space $\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$. Let $u(x,t)$ be harmonic on $\mathbb{R}^{n+1}_+$ and $u(x,0) = f(x)$ where u has similar properties to the u in (6). Then we can find a system of conjugate harmonic functions $v_j, j \in \{1,2,...,n\}$, such that $\{u, v_1, ..., v_n\}$ is a set of harmonic functions satisfying the generalized Cauchy-Riemann equations,

$$\begin{cases} \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \\ \frac{\partial u_1}{\partial x_1} + ... + \frac{\partial u_n}{\partial x_n} = 0 \end{cases} . \tag{15}$$

In chapter 3 we find $v_j(x,t)$ and show that they converge to $\mathcal{R}_j f(x)$ as $t \to 0^+$, so the Riesz transforms generalize the complex analysis approach to the Hilbert transform (note that when $n = 1$ and $C_n = \frac{1}{\pi}$ in (14), the one Riesz transform is the Hilbert transform on the real line). We can get an identity similar to (13) for the Riesz transforms if we apply the Fourier transform $f \mapsto \widehat{f}$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \tag{16}$$

which we will prove is a Hilbert space isomorphism on $L^2(\mathbb{R}^n)$. For the Hilbert transform we will show that this identity is, for L^2-functions f,

$$\widehat{\mathcal{H}f}(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi), \tag{17}$$

and more generally for the Riesz transforms, with the proper choice of C_n,

$$\widehat{\mathcal{R}_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi). \tag{18}$$

From this identity we get

$$\sum_{j=1}^n \widehat{\mathcal{R}_j^2 f}(\xi) = -\sum_{j=1}^n \frac{\xi_j^2}{|\xi|^2} \widehat{f}(\xi) = -\widehat{f}(\xi).$$

Hence we get the inversion formula

$$\sum_{j=1}^n \mathcal{R}_j^2 f = -f, \tag{19}$$

and in the case $n = 1$,

$$\mathcal{H}^{-1} = -\mathcal{H}. \tag{20}$$

We also note that (18) would prove the L^2-boundedness of the Riesz transforms, since using the fact that the Fourier transform is an isometry on $L^2(\mathbb{R}^n)$ (Plancherel's Theorem), we get

$$\|\mathcal{R}_j f\|_2 = \left\|\widehat{\mathcal{R}_j f}\right\|_2 \leqslant \left\|\widehat{f}\right\|_2 = \|f\|_2 .$$

The L^p-boundedness, $1 < p < \infty$, can also be proved for the Riesz transforms. We prove this in chapter 3 by a calculation which shows that the L^p-boundedness of the Hilbert transform implies the L^p-boundedness of the Riesz transforms. Note that a continuity argument would then extend (19) and (20) to $L^p(\mathbb{R}^n), 1 < p < \infty$.

The Riesz transforms appear in the study of partial differential equations. Consider the Laplacian Δu where $u \in C_c^2(\mathbb{R}^n)$. Then a simple calculation gives $\widehat{\Delta u}(\xi) = -(2\pi)^2 |\xi|^2 \widehat{f}(\xi)$ (recall that $\widehat{\partial f/\partial x_j}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$). Now, by (18),

$$\widehat{\mathcal{R}_j \mathcal{R}_k \Delta u}(\xi) = \left(-i\frac{\xi_j}{|\xi|}\right)\left(-i\frac{\xi_k}{|\xi|}\right) \widehat{\Delta u}(\xi) = -\left(2\pi i \xi_j\right)\left(2\pi i \xi_k\right) \widehat{u}(\xi) = -\widehat{\frac{\partial^2 u}{\partial x_j \partial x_k}}(\xi).$$

Thus we get

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = -\mathcal{R}_j \mathcal{R}_k \Delta u. \tag{21}$$

Hence if $u \in C_c^2(\mathbb{R}^n)$ satisfies the Poisson equation $\Delta u = f$ where $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then $\partial^2 u/\partial x_j \partial x_k = -\mathcal{R}_j \mathcal{R}_k f$, so we can derive properties of the second derivatives of u through the Riesz transforms. Of course the Riesz transforms are quite complicated, since $x_j/|x|^{n+1}$ is not locally integrable near 0. Other operators with similar complications also appear in the study of partial differential equations, where a more general theory of singular integral operators would have to be considered. We will consider general operators of the type

$$\mathcal{T}f = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(\frac{y}{|y|})}{|y|^n} \, \mathrm{d}y, \tag{22}$$

where Ω is a function on the unit sphere $\Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. We are particularly interested in the boundedness properties of these operators. For the expression in (22) to make any sense we will need to assume that $\Omega \in L^1(\Sigma^{n-1})$ and $\int_{\Sigma^{n-1}} \Omega(y')dy' = 0$. The Riesz transforms are singular integrals of this type where Ω is an odd function. When Ω is odd the L^p-boundedness of $\mathcal{T}$ follows similarly to how the boundedness of the Riesz transforms will follow. For general Ω we can then consider the decomposition $\Omega = \Omega_O + \Omega_E$, where

$$\Omega_O = \frac{\Omega(x) - \Omega(-x)}{2} \;, \; \Omega_E = \frac{\Omega(x) + \Omega(-x)}{2}.$$

Ω_O is an odd function so this part of $\mathcal{T}$ is L^p-bounded when $1 < p < \infty$. Ω_E is even, so it suffices to prove boundedness for even operators. For an attempt to prove L^p-boundedness when Ω is even we will use (19) to write

$$\mathcal{T} = -\sum_{j=1}^{n} \mathcal{R}_j(\mathcal{R}_j\mathcal{T}). \tag{23}$$

Since the Riesz transforms are bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$, it suffices to prove the boundedness of the operators $\mathcal{R}_j\mathcal{T}$. The strategy here will be to show that each $\mathcal{R}_j\mathcal{T}$ is an operator of the type (22) with an odd Ω. We give a proof of this in the case $\Omega \in L^r(\Sigma^{n-1})$ for some $r > 1$, which is the most general boundedness theorem we treat here. Most of these general results were proved by Alberto Calderón and Antoni Zygmund. Therefore the theory of singular integrals often goes by the name Calderón-Zygmund theory. The theory includes much more than what is covered in this text. For a more extensive treatment of singular integral theory, along with other related topics, a great choice for a textbook is *Real-Variable Methods in Harmonic Analysis* by Alberto Torchinsky.

Convergence of Fourier Series

We introduce Fourier series and discuss their convergence, both pointwise and in norm. In particular, we will show how the L^p-boundedness of the Hilbert transform solves the problem of L^p-convergence of Fourier series.

Properties of Fourier Coefficients

When studying Fourier series we work with complex functions on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A function on $\mathbb{T}$ is essentially a 1-periodic function on $\mathbb{R}$. Hence spaces like $C(\mathbb{T}), C^k(\mathbb{T})$ and $L^p(\mathbb{T})$ are defined as the spaces of functions $f : \mathbb{T} \to \mathbb{C}$ whose corresponding periodic function belongs to $C(\mathbb{R}), C^k(\mathbb{R})$ and $L^p([-\frac{1}{2}, \frac{1}{2}])$, respectively. $\mathbb{T}$ is naturally identified with the interval $[0, 1)$, which gives us definitions of Lebesgue measure and the Lebesgue integral on $\mathbb{T}$. We highlight the fact that the measure on $\mathbb{T}$ is translation-invariant. This property will be used frequently throughout the text.

Definition 1.1. *A trigonometric polynomial of degree n on $\mathbb{T}$ is given by*

$$\varphi(x) = \sum_{j=-n}^{n} c_j e^{2\pi i j x}, \tag{24}$$

for some complex coefficients c_j where at least one of c_n and c_{-n} are non-zero.

To calculate the coefficients c_j we use the identity

$$\int_{\mathbb{T}} e^{2\pi i j t}\,\mathrm{d}t = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}. \tag{25}$$

Multiplying φ by $e^{-2\pi i k t}$ and then integrating we obtain

$$c_k = \int_{\mathbb{T}} \varphi(t) e^{-2\pi i k t}\,\mathrm{d}t. \tag{26}$$

This motivates the definition of Fourier series of functions in $L^1(\mathbb{T})$. Note that this will also define Fourier series of functions in $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, $p > 1$, since $\mathbb{T}$ has finite measure.

Definition 1.2. *For $f \in L^1(\mathbb{T})$, its Fourier series is given by the formal expression*

$$f \sim \sum_{j=-\infty}^{\infty} c_j e^{2\pi i j x}, \tag{27}$$

along with its partial sums

$$s_n(f; x) = \sum_{j=-n}^{n} c_j e^{2\pi i j x}, \tag{28}$$

where the Fourier coefficients $c_j = \widehat{f}(j)$ are given by

$$\widehat{f}(j) = \int_{\mathbb{T}} f(t) e^{-2\pi i j t}\,\mathrm{d}t. \tag{29}$$

If, for example, f equals a series in (27) that converges uniformly we can perform the same calculation as for the trigonometric polynomials above, and conclude that $c_j = \widehat{f}(j)$. We also see that $s_n(f; x) \to f$ if f is a trigonometric polynomial, both pointwise and in L^p-norm. We expect $s_n(f; x)$ to approximate f in some sense at least when f has certain properties, but in general a Fourier series need not converge, and it may converge to something else than its function. In fact, we don't necessarily have pointwise convergence of Fourier series in $L^1(\mathbb{T})$, not even in $C(\mathbb{T})$!

We proceed to list a few basic properties of Fourier series.

Lemma 1.3. *Let $f, g \in L^1(\mathbb{T})$ and $\alpha \in \mathbb{C}$.*

(i) *$f \mapsto \widehat{f}$ is linear. That is, $\widehat{f+g} = \widehat{f} + \widehat{g}$ and $\widehat{\alpha f} = \alpha \widehat{f}$.*

(ii) $\widehat{\overline{f}}(j) = \overline{\widehat{f}(-j)}$, *where* $\overline{f}$ *is the complex conjugate to* f.

(iii) $\widehat{f_t}(j) = \widehat{f}(j)e^{-2\pi ijt}$, *where* f_t *is the translation map* $f_t(x) = f(x-t)$.

(iv) *If* $g(x) = e^{2\pi inx}f(x)$, *then* $\widehat{g}(j) = \widehat{f}(j-n)$.

(v) $\widehat{f'}(j) = 2\pi ij\widehat{f}(j)$, *for* $f \in C^1(\mathbb{T})$

(vi) *The Fourier coefficients are bounded and* $\left|\widehat{f}(j)\right| \leqslant \|f\|_1$.

(vii) *The coefficients go to zero:* $\widehat{f}(j) \xrightarrow{j\to\pm\infty} 0$.

(i)-(vi) are all trivial. (vii) is a direct consequence of the following.

Lemma 1.4. (Riemann-Lebesgue) *Let* $f \in L^1((a,b)), -\infty \leqslant a < b \leqslant \infty$. *Then*

$$\lim_{P\to\infty} \int_a^b f(x)e^{iPt}\,\mathrm{d}t = 0.$$

Proof. It clearly holds for constant functions and thus also for step functions (see Appendix II). We now use the fact that the step functions are dense in L^1. Given $\epsilon > 0$ there is a step function g such that $\|f-g\|_1 < \epsilon/2$, and a P_0 such that for $P > P_0$ we have $\left|\int_a^b g(t)e^{iPt}dt\right| < \epsilon/2$. When $P > P_0$ we get

$$\left|\int_a^b f(t)e^{iPt}\,\mathrm{d}t\right| \leqslant \left|\int_a^b (f(t)-g(t))e^{iPt}\,\mathrm{d}t\right| + \left|\int_a^b g(t)e^{iPt}\,\mathrm{d}t\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

The function e^{iPt} may be replaced by $\sin Pt$ or $\cos Pt$ in the lemma above and the same result will hold with the exact same proof. Also note that (a,b) may be generalized to any measurable set $X \subset \mathbb{R}$, since any integrable function f on X can be identified with an integrable function on $\mathbb{R}$, by letting $f(x) = 0$ for $x \notin X$.

A Criterion for Pointwise Convergence

We rewrite the partial sums of a Fourier series as follows.

$$s_n(f;x) = \sum_{j=-n}^{n}\left(\int_{\mathbb{T}} f(t)e^{-2\pi ijt}\,\mathrm{d}t\right)e^{2\pi ijx} = \int_{\mathbb{T}} f(t)\sum_{j=-n}^{n} e^{2\pi ij(x-t)}\,\mathrm{d}t = \int_{\mathbb{T}} f(t)D_n(x-t)\,\mathrm{d}t,$$

where D_n is given by

$$D_n(x) = \sum_{j=-n}^{n} e^{2\pi ijx}. \tag{30}$$

The family $\{D_n\}_{n=0}^{\infty}$ is called the Dirichlet kernel. If we sum the geometric series in (30) we get

$$D_n(x) = \frac{\sin\pi(2n+1)x}{\sin\pi x}. \tag{31}$$

Note that D_n is an even function for every n. Using this we can write

$$s_n(f;x) = \int_{\mathbb{T}} f(t)D_n(x-t)\,\mathrm{d}t = \int_{\mathbb{T}} f(x-t)D_n(t)\,\mathrm{d}t = \int_{\mathbb{T}} f(x+t)D_n(t)\,\mathrm{d}t \tag{32}$$

$$= \int_{\mathbb{T}} \frac{f(x+t)+f(x-t)}{2}D_n(t)\,\mathrm{d}t. \tag{33}$$

D_n has the following properties,

(i) $\int_{\mathbb{T}} D_n(t)\,\mathrm{d}t = 1$.

(ii) $\int_{|x|\geqslant\delta} |D_n(t)|\,\mathrm{d}t \xrightarrow{n\to\infty} 0$ for every $\delta > 0$.

(i) follows from (30) and (25), and (ii) follows from (31) and Riemann-Lebesgue Lemma, since $\left|\frac{1}{\sin\pi x}\right|$ is bounded by $\frac{1}{\sin\pi\delta}$ when $|x| \geqslant \delta$.

If we want to make an attempt to prove $s_n(f;x_0) \to \ell$, the following calculation using (i) is very useful.

$$s_n(f;x) - \ell = \int_{\mathbb{T}} (f(x-t) - \ell)\, D_n(t)\,\mathrm{d}t. \tag{34}$$

Using (33), we can write

$$\begin{aligned} s_n(f;x_0) - \ell &= \int_{\mathbb{T}} \left(\frac{f(x_0+t) + f(x_0-t)}{2} - \ell \right) D_n(t)\,\mathrm{d}t \\ &= \int_{\mathbb{T}} \left(\frac{f(x_0+t) + f(x_0-t)}{2} - \ell \right) \frac{1}{\sin\pi t} \sin\pi(2n+1)t\,\mathrm{d}t. \end{aligned}$$

Here we would like to apply Riemann-Lebesgue Lemma. The problem is that $\frac{1}{\sin\pi t}$ is not integrable on $\mathbb{T}$. It is, however, integrable on $|x| \geqslant \delta$ for any $\delta > 0$, so we only need to consider the integral over $|x| < \delta$. We can apply Riemann-Lebesgue Lemma again if we have the following condition,

$$\int_{|t|<\delta} \left| \left(\frac{f(x_0+t) + f(x_0-t)}{2} - \ell \right) \frac{1}{\sin\pi t} \right| \mathrm{d}t < \infty. \tag{35}$$

A more natural condition can be obtained by replacing $\frac{1}{\sin\pi t}$ by $\frac{1}{t}$. This is equivalent since $|2t| \leqslant |\sin\pi t| \leqslant |\pi t|$. We can also replace $|t| < \delta$ with $0 < |t| < \delta$ since the integrand is even. We have proved the following theorem.

Theorem 1.5. (Dini) *Let $f \in L^1(\mathbb{T})$ and $x_0 \in \mathbb{T}$. Suppose that for some $\delta > 0$ and for some constant ℓ,*

$$\int_0^\delta \left| \left(\frac{f(x_0+t) + f(x_0-t)}{2} - \ell \right) \frac{1}{t} \right| \mathrm{d}t < \infty.$$

Then $s_n(f;x_0) \xrightarrow{n\to\infty} \ell$.

We define the left and right limits, $f(x_0-)$ and $f(x_0+)$, by $f(x_0-) = \lim_{t\to0^-} f(x_0+t)$ and $f(x_0+) = \lim_{t\to0^+} f(x_0+t)$, respectively. We also introduce the left and right derivatives, $f'(x_0-)$ and $f'(x_0+)$. They are defined by

$$f'(x_0-) = \lim_{t\to0^-} \frac{f(x_0+t) - f(x_0-)}{t} \quad , \quad f'(x_0+) = \lim_{t\to0^+} \frac{f(x_0+t) - f(x_0+)}{t}.$$

We have two important corollaries to Theorem 1.5.

Corollary 1.6.

(i) *If f is differentiable at $x_0 \in \mathbb{T}$, then $s_n(f;x_0) \xrightarrow{n\to\infty} f(x_0)$. More generally, if $f'(x_0+)$ and $f'(x_0-)$ exist, then $s_n(f;x_0) \xrightarrow{n\to\infty} \frac{f(x_0+)+f(x_0-)}{2}$.*

(ii) *If f satisfies a Hölder condition of order $\alpha > 0$, then its Fourier series converges uniformly to f.*

Proof. (i) is immediate from Dini's criterion with $\ell = \frac{f(x_0+)+f(x_0-)}{2}$, and (ii) can be derived directly from (34) by using Riemann-Lebesgue's Lemma with $\ell = f(x)$.

□

Corollary 1.7.

(i) *Suppose $f \in L^1(\mathbb{T})$ vanishes on an open interval I. Then $s_n(f)$ converges to 0 on I.*

(ii) *Let $f, g \in L^1(\mathbb{T})$ and suppose $f = g$ on an open interval I. Then $s_n(f)$ and $s_n(g)$ converge to the same values simultaneously on I.*

Example 1.8. The function $f(x) = x^2$ on $[-\frac{1}{2}, \frac{1}{2})$ satisfies the hypothesis of Corollary 1.6 (i), so we know that its Fourier series will converge to $f(x)$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$. The Fourier coefficients can be calculated to $\widehat{f}(n) = \frac{(-1)^n}{2\pi^2 n^2}$ for $n \neq 0$ and $\widehat{f}(0) = \frac{1}{12}$. This results in

$$x^2 = \frac{1}{12} + \sum_{n=1}^{\infty} (-1)^n \frac{e^{2\pi i n x} + e^{-2\pi i n x}}{2\pi^2 n^2} \quad , \quad x \in [-\frac{1}{2}, \frac{1}{2}].$$

If we set $x = 0$ we obtain the following identity.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

If we instead set $x = \frac{1}{2}$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

which is the solution to the famous Basel problem.

Convolutions and Kernels

The convolution product $*$ is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(t) g(x-t) \, \mathrm{d}t. \tag{36}$$

Unlike pointwise multiplication, this operation is closed in $L^p(\mathbb{T})$, by the following.

Theorem 1.9. (Young) *Let $1 \leqslant p \leqslant \infty$. Then*

$$\|f * g\|_p \leqslant \|f\|_p \|g\|_1 .$$

Hence, if $f, g \in L^p(\mathbb{T})$, then $f * g \in L^p(\mathbb{T})$ and exists almost everywhere on $\mathbb{T}$.

Proof. The cases $p = 1$ and $p = \infty$ are obvious. For general p we use Hölder's inequality in the following way.

$$\begin{aligned} |(f * g)(x)| &\leqslant \int_{\mathbb{T}} |f(t)| \, |g(x-t)| = \int_{\mathbb{T}} |f(t)| \, |g(x-t)|^{\frac{1}{p}} \, |g(x-t)|^{\frac{1}{q}} \\ &\leqslant \|g\|_1^{\frac{1}{q}} \left(\int_{\mathbb{T}} |f(t)|^p \, |g(x-t)| \, \mathrm{d}t \right)^{\frac{1}{p}} . \end{aligned}$$

Now integrate to get

$$\begin{aligned} \int_{\mathbb{T}} |(f * g)(x)|^p \, \mathrm{d}x &\leqslant \|g\|_1^{\frac{p}{q}} \int_{\mathbb{T}} \int_{\mathbb{T}} |f(t)|^p \, |g(x-t)| \, \mathrm{d}t \, \mathrm{d}x \\ &= \|g\|_1^{\frac{p}{q}} \left(\int_{\mathbb{T}} |g(x-t)| \, \mathrm{d}x \right) \left(\int_{\mathbb{T}} |f(t)|^p \, \mathrm{d}t \right) \\ &= \|g\|_1^{\frac{p}{q}} \|g\|_1 \|f\|_p^p = \|f\|_p^p \|g\|_1^p . \end{aligned}$$

Hence $\|f * g\|_p \leqslant \|f\|_p \|g\|_1$, and since $\|g\|_1 \leqslant \|g\|_p$ we have closure in $L^p(\mathbb{T})$. □

The following identity shows how convolutions behave with Fourier coefficients.

Lemma 1.10. $\widehat{f * g} = \widehat{f}\widehat{g}$.

The proof is easily verified using Fubini's theorem. One may also verify that convolution is commutative, associative and distributive (with pointwise addition). Together with Young's theorem, another way of phrasing this is to say that $L^p(\mathbb{T})$ is a commutative Banach algebra under pointwise addition and the convolution product. This algebra is not unital, in other words, there is no element $e \in L^p(\mathbb{T})$ such that $f * e = e * f = f$. To see this, assume the contrary. Then $f_n * e = f_n$ would be true for the functions $f_n(x) = e^{inx}$. Letting $n \to \infty$ pointwise, we get a contradiction by Riemann-Lebesgue Lemma, since $f_n * e$ converges to 0 but f_n does not.

With the notion of convolutions introduced, we can write the partial sums of a Fourier series as

$$s_n(f) = f * D_n. \tag{37}$$

We are interested in determining when $f * D_n \xrightarrow{n\to\infty} f$, pointwise and in norm. Seen from another perspective, we want to determine in what sense D_n is an "approximate identity". So far we have not made an approach to norm convergence and we have only been able to determine pointwise convergence for functions satisfying some smoothness condition. The next definition provides a more general approach to determining when $f * K_n \xrightarrow{n\to\infty} f$ for a family of functions $\{K_n\}_{n=0}^{\infty}$.

Definition 1.11. *A summability kernel is a family $\{K_n\}_{n=0}^{\infty}$ of integrable functions on $\mathbb{T}$ satisfying the following three conditions.*

(i) $\int_{\mathbb{T}} |K_n(t)|\,\mathrm{d}t \leqslant C$, *for some constant C (for all n).*

(ii) $\int_{\mathbb{T}} K_n(t)\,\mathrm{d}t = 1$.

(iii) $\int_{|x|\geqslant\delta} |K_n(t)|\,\mathrm{d}t \xrightarrow{n\to\infty} 0$ *for every $\delta > 0$.*

We will also encounter families $\{K_\epsilon\}_{\epsilon>0}$ and $\{K_r\}_{0\leqslant r<1}$. In these cases $n\to\infty$ will be replaced by $\epsilon\to 0^+$ and $r\to 1^-$, respectively. This will not change any of the results that will be developed below. The results will also apply if we integrate over $\mathbb{R}^n$ instead of $\mathbb{T}$ (and Young's inequality still applies as well).

Theorem 1.12. *Let $f \in L^p(\mathbb{T})$, $1 \leqslant p < \infty$, and $\{K_n\}_{n=0}^{\infty}$ be a summability kernel. Then $f * K_n \to f$ in L^p-norm. That is,*

$$\lim_{n\to\infty} \|f * K_n - f\|_p = 0.$$

To prove this we need the following lemma.

Lemma 1.13. *The translation operation $t \mapsto f_t$ is continuous on $L^p(\mathbb{T})$, $1 \leqslant p < \infty$. That is,*

$$\lim_{t\to t_0} \|f_t - f_{t_0}\|_p = 0.$$

Proof. The result is clear for continuous functions. Since the continuous functions are dense in $L^p(\mathbb{T})$ we can, given $\epsilon > 0$, choose such a function g such that $\|f - g\|_p < \epsilon/3$. Now choose $\delta > 0$ such that $\|g_t - g_{t_0}\| < \epsilon/3$ when $|t - t_0| < \delta$. For $|t - t_0| < \delta$ we have

$$\|f_t - f_{t_0}\|_p \leqslant \|f_t - g_t\|_p + \|g_t - g_{t_0}\|_p + \|g_{t_0} - f_{t_0}\|_p < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Proof of Theorem 1.12. We begin as in (34) and use Hölder's inequality to write

$$\begin{aligned}
|(f * K_n)(x) - f(x)| &\leqslant \int_{\mathbb{T}} |f(x-t) - f(x)|\,|K_n(t)|\,\mathrm{d}t \\
&\leqslant \left(\int_{\mathbb{T}} |f(x-t) - f(x)|^p\,|K_n(t)|\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_{\mathbb{T}} |K_n(t)|\,\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leqslant C^{\frac{1}{q}} \left(\int_{\mathbb{T}} |f(x-t) - f(x)|^p\,|K_n(t)|\,\mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus we get,

$$\begin{aligned}
\int_{\mathbb{T}} |(f * K_n)(x) - f(x)|^p\,\mathrm{d}x &\leqslant C^{\frac{p}{q}} \int_{\mathbb{T}}\int_{\mathbb{T}} |f(x-t) - f(x)|^p\,|K_n(t)|\,\mathrm{d}t\,\mathrm{d}x \\
&= C^{\frac{p}{q}} \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f(x-t) - f(x)|^p\,\mathrm{d}x\right) |K_n(t)|\,\mathrm{d}t \\
&= C^{\frac{p}{q}} \int_{\mathbb{T}} \|f_t - f\|_p^p\,|K_n(t)|\,\mathrm{d}t.
\end{aligned}$$

It suffices to prove that the last integral converges to zero. By Lemma 1.13, given $\epsilon > 0$ there is a $\delta > 0$ such that $\|f_t - f\|_p^p < \epsilon/2C$ when $|t| < \delta$. We split into the following two integrals.

$$I_1 = \int_{|t|<\delta} \|f_t - f\|_p^p \, |K_n(t)| \, \mathrm{d}t \ , \ I_2 = \int_{|t|\geqslant\delta} \|f_t - f\|_p^p \, |K_n(t)| \, \mathrm{d}t.$$

Clearly $I_1 < \epsilon/2$. Now, by condition (iii) of summability kernels, we can choose n_0 such that $\int_{|t|\geqslant\delta} |K_n(t)| \, \mathrm{d}t < \epsilon/(2^{p+1} \|f\|_p^p)$ for $n > n_0$. Hence, for $n > n_0$, we have $I_1 + I_2 < \epsilon/2 + \epsilon/2 = \epsilon$, which proves that $\|f * K_n - f\|_p^p \xrightarrow{n\to\infty} 0$, and so we are done. □

The following useful conditions provide a more restrictive class of summability kernels.

(i*) $K_n(x) \geqslant 0$.

(iii*) $\sup_{|x|\geqslant\delta} |K_n(x)| \xrightarrow{n\to\infty} 0$ for every $\delta > 0$.

It is clear that (i*) together with (ii) imply (i), and that (iii*) implies (iii). With the conditions (i*), (ii) and (iii*) we will be able to prove slightly more powerful results. We now turn to pointwise convergence of $f * K_n$.

Theorem 1.14. *Let $f \in L^1$ and $\{K_n\}_{n=0}^{\infty}$ be a summability kernel satisfying* (iii*).

(i) *If f is continuous at $x_0 \in \mathbb{T}$ then $(f * K_n)(x_0) \xrightarrow{n\to\infty} f(x_0)$ and the convergence is uniform on every compact set of continuity.*

(ii) *If K_n is even and $f(x_0\pm)$ exists then $(f * K_n)(x_0) \xrightarrow{n\to\infty} \frac{f(x_0+)+f(x_0-)}{2}$.*

Proof. To prove the first part of the theorem, we write

$$|(f * K_n)(x_0) - f(x_0)| \leqslant \int_{\mathbb{T}} |f(x_0 - t) - f(x_0)| \, |K_n(t)| \, \mathrm{d}t.$$

Now split the problem into the two integrals,

$$I_1 = \int_{|t|<\delta} |f(x_0 - t) - f(x_0)| \, |K_n(t)| \, \mathrm{d}t \ , \ I_2 = \int_{|t|\geqslant\delta} |f(x_0 - t) - f(x_0)| \, |K_n(t)| \, \mathrm{d}t,$$

where δ is chosen so that $I_1 < \epsilon/2$ for a given $\epsilon > 0$, which is possible since f is continuous at x_0. For I_2, we see that $I_2 \leqslant 2\|f\|_1 \sup_{|t|\geqslant\delta} |K_n(t)|$ and by (iii*) we can choose n_0 such that $I_2 < \epsilon/2$ for $n > n_0$, so the proof is complete. The uniform convergence follows from the fact that every continuous function on a compact set is uniformly continuous, so we can choose δ independently of x_0 on any given compact set of continuity of f. Now, for the second part, since K_n is even, we can write

$$\begin{aligned} |(f * K_n)(x_0) - \ell| &\leqslant \int_{\mathbb{T}} \left| \frac{f(x_0 + t) + f(x_0 - t)}{2} - \ell \right| |K_n(t)| \, \mathrm{d}t \\ &= 2\int_0^{\frac{1}{2}} \left| \frac{f(x_0 + t) + f(x_0 - t)}{2} - \ell \right| |K_n(t)| \, \mathrm{d}t, \end{aligned}$$

and the rest follows by the same proof as for the first part with $\ell = \frac{f(x_0+)+f(x_0-)}{2}$. □

We remark that if we assume that f is bounded in Theorem 1.14 we can remove (iii*), since a closer look at the proof will reveal that (iii) would be sufficient to bound I_2 in this case. In particular we have uniform convergence of $f * K_n$ for arbitrary summability kernels $\{K_n\}_{n=0}^{\infty}$ when f is continuous on $\mathbb{T}$. Also note that the existence of $f(x_0\pm)$ can be generalized to the existence of $\check{f}(x_0) = \lim_{t\to 0^+} \frac{f(x_0+t)+f(x_0-t)}{2}$, and we would have $(f * K_n)(x_0) \to \check{f}(x_0)$ in Theorem 1.14 (ii).

Examples of Summability

In the previous section we proved some very good convergence properties of $f * K_n$ for summability kernels K_n. We have already seen that D_n satisfies properties (ii) and (iii) of Definition 1.11. Unfortunately the Dirichlet kernel is not a summability kernel because it does not satisfy (i), by the following lemma.

Lemma 1.15. $\|D_n\|_1 \xrightarrow{n\to\infty} \infty$.

Proof. By (31) and $\sin \pi x \leqslant |\pi x|$ we get

$$\begin{aligned}\int_{\mathbb{T}} |D_n(x)| \, \mathrm{d}x > 2\int_0^{\frac{1}{2}} \left|\frac{\sin \pi(2n+1)x}{\pi x}\right| \mathrm{d}x = \frac{2}{\pi}\int_0^{n+\frac{1}{2}} \left|\frac{\sin \pi x}{x}\right| \mathrm{d}x \\ > \frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{j}\int_{j-1}^{j} |\sin \pi x| \, \mathrm{d}x = \frac{4}{\pi^2}\sum_{j=1}^{n}\frac{1}{j} > \frac{4}{\pi^2}\log(n+1) \xrightarrow{n\to\infty} \infty.\end{aligned}$$

□

If we redo the proof with $|\sin \pi x| \geqslant |2x|$ we get

$$\|D_n\|_1 < \frac{2}{\pi}\sum_{j=1}^{n+1}\frac{1}{j} < \frac{2}{\pi}(1+\log(n+1)), \tag{38}$$

which shows that D_n grows to infinity like a constant multiple of $\log n$. By Lemma 1.15 we can not deduce convergence of $f * D_n$ from Theorem 1.12 and Theorem 1.14, but there are other modes of convergence where these theorems will be useful. One of them comes from a method discovered by Fejér which shows that the problem of convergence in arithmetic means can be reduced to studying a certain summability kernel. A sequence $\{x_n\}_{n=1}^{\infty}$ in some normed linear space is said to convergence in mean if the sequence $\frac{x_1+\ldots+x_n}{n}$ converges. A series that converges in mean is sometimes called Cesàro summable, and its means are called Cesàro means. The following proposition is readily verified by the reader.

Proposition 1.16. *If $\{x_n\}_{n=1}^{\infty}$ converges, then it converges in mean to the same value.*

Note that the converse is not true. For example, the sequence $\{(-1)^n\}$ converges to 0 in mean. Hence convergence in mean extends convergence to a larger collection of sequences.

We denote by $\sigma_n(f;x)$ the Cesàro means of $s_n(f;x)$ and proceed to calculate $\sigma_n(f;x)$.

$$\sigma_n(f) = \frac{1}{n+1}\sum_{j=0}^{n} s_j(f) = \frac{1}{n+1}\sum_{j=0}^{n} f * D_j = f * \frac{1}{n+1}\sum_{j=0}^{n} D_j = f * F_n,$$

where F_n is given by

$$F_n = \frac{1}{n+1}\sum_{j=0}^{n} D_j. \tag{39}$$

Lemma 1.17.

$$F_n(x) = \frac{1}{n+1}\left(\frac{\sin \pi(n+1)x}{\sin \pi x}\right)^2.$$

Proof. F_n is the imaginary part of a geometric series, since

$$\sum_{j=0}^{n} D_j(x) = \sum_{j=0}^{n}\frac{\sin \pi(2j+1)x}{\sin \pi x} = \frac{1}{\sin \pi x}\sum_{j=0}^{n}\operatorname{Im}\left(e^{\pi i(2j+1)x}\right).$$

From here we just sum the geometric series to verify the result. □

The kernel $\{F_n\}_{n=0}^{\infty}$ is called the Fejér kernel. We see that $\|F_n\|_1 = 1$, by (30) and (39). By Lemma 1.17, F_n is even and positive, and also $|F_n(x)| < \frac{1}{(n+1)\sin^2 \pi\delta}$ when $|x| \geqslant \delta$, so we have $\sup_{|x|\geqslant\delta} |F_n(x)| \xrightarrow{n\to\infty} 0$ for all $\delta > 0$. Hence the Fejér kernel satisfies all the hypothesis of Theorem 1.12 and Theorem 1.14. We restate these theorems for F_n, along with three important corollaries that are all readily verified.

Theorem 1.18. (Fejér) *Let $f \in L^1(\mathbb{T})$.*

(i) *$\sigma_n(f) \to f$ in L^p-norm, $1 \leqslant p < \infty$.*

(ii) *If f is continuous at $x_0 \in \mathbb{T}$ then $\sigma_n(f;x_0) \xrightarrow{n\to\infty} f(x_0)$ and the convergence is uniform on every compact set of continuity.*

(iii) *If $\check{f}(x_0) = \lim_{t\to 0^+} \frac{f(x_0+t)+f(x_0-t)}{2}$ exists then $\sigma_n(f;x_0) \xrightarrow{n\to\infty} \check{f}(x_0)$.*

Corollary 1.19. *Let $f \in L^1(\mathbb{T})$.*

(i) *If the Fourier series of $f \in L^p(\mathbb{T})$ converges in L^p-norm, $1 \leqslant p < \infty$, then it must converge to f. A similar result holds if we replace $L^p(\mathbb{T})$ by $C(\mathbb{T})$.*

(ii) *If the Fourier series of f converges at $x_0 \in \mathbb{T}$ and x_0 is a point of continuity of f, then it must converge to $f(x_0)$. More generally, if $\check{f}(x_0)$ exists, then the Fourier series must converge to $\check{f}(x_0)$.*

Corollary 1.20. *Let $f, g \in L^1(\mathbb{T})$.*

(i) *If $\widehat{f} = 0$ then $f = 0$ a.e.*

(ii) *If $\widehat{f} = \widehat{g}$ then $f = g$ a.e.*

Corollary 1.21. *The trigonometric polynomials on $\mathbb{T}$ are dense in $C(\mathbb{T})$ and $L^p(\mathbb{T}), 1 \leqslant p < \infty$.*

For all $f \in L^1(\mathbb{T})$, $\sigma_{n_k}(f) \to f$ almost everywhere for some subsequence n_k, by Theorem 1.18 (i). It even turns out that the almost everywhere convergence of $\sigma_n(f)$ holds for every $f \in L^1(\mathbb{T})$, as stated by the theorem below.

Theorem 1.22. *$\sigma_n(f;x_0) \xrightarrow{n\to\infty} f(x_0)$ if almost everywhere on $\mathbb{T}$.*

Proof. It suffices to prove it for every Lebesgue point $x_0 \in \mathbb{T}$, by Lebesgue Differentiation Theorem (see e.g. [5, Chapter 7]). As always, we write

$$|\sigma_n(f;x_0) - f(x_0)| \leqslant 2\int_0^{\frac{1}{2}} \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right| |F_n(t)| \, \mathrm{d}t = 2I.$$

we split I into

$$I_1 = \int_0^{\delta} \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right| |F_n(t)| \, \mathrm{d}t \, ,$$
$$I_2 = \int_{\delta}^{\frac{1}{2}} \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right| |F_n(t)| \, \mathrm{d}t.$$

For I_2, we use $|F_n(t)| < \frac{1}{(n+1)\sin^2 \pi\delta} < \frac{1}{(n+1)(2\delta)^2}$ on $[\delta, \frac{1}{2})$ to get

$$I_2 < \frac{2\,\|f\|_1}{(n+1)\sin^2 \pi\delta} < \frac{\|f\|_1}{2(n+1)\delta^2} \xrightarrow{n\to\infty} 0$$

if $(n+1)\delta^2 \xrightarrow{n\to\infty} \infty$, hence if we choose $\delta = n^{-\frac{1}{4}}$. We split I_1 into the integral over $[0, n^{-1}]$, say I_3, and the integral over $(n^{-1}, n^{-\frac{1}{4}})$, say I_4. Define $\Phi(h) = \int_0^h \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right| dt$. Since $|D_j(t)| \leqslant 2j+1$, by (30), we have $|F_n(t)| \leqslant \frac{1}{n+1}\sum_{j=0}^{n}(2j+1) = n+1$. Hence $I_3 \leqslant (n+1)\Phi(\frac{1}{n})$, which converges to 0 since x_0 is a Lebesgue point. Lastly, for I_4 we write

$$
\begin{aligned}
I_4 &\leqslant \frac{1}{4(n+1)} \int_{n^{-1}}^{n^{-\frac{1}{4}}} \left| \frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0) \right| \frac{1}{t^2}\,\mathrm{d}t \\
&= \frac{1}{4(n+1)} \int_{n^{-1}}^{n^{-\frac{1}{4}}} \frac{\Phi'(t)}{t^2}\,\mathrm{d}t = \frac{1}{4(n+1)} \left.\frac{\Phi(t)}{t^2}\right|_{n^{-1}}^{n^{-\frac{1}{4}}} + \frac{1}{2(n+1)} \int_{n^{-1}}^{n^{-\frac{1}{4}}} \frac{\Phi(t)}{t^3}\,\mathrm{d}t.
\end{aligned}
$$

The first term converges to 0 since x_0 is a Lebesgue point. For the last term, pick $\epsilon > 0$. Then for sufficiently large n, we have $\frac{F(t)}{t} < \epsilon$, by Lebesgue Differentiation Theorem, and

$$\frac{1}{2(n+1)} \int_{n^{-1}}^{n^{-\frac{1}{4}}} \frac{\Phi(t)}{t^3}\,\mathrm{d}t < \frac{\epsilon}{2(n+1)} \int_{n^{-1}}^{n^{-\frac{1}{4}}} \frac{1}{t^2}\,\mathrm{d}t = \frac{\epsilon(n-n^{\frac{1}{4}})}{2(n+1)} < \frac{\epsilon}{2}.$$

Hence $\limsup_{n\to\infty} |\sigma_n(f;x_0) - f(x_0)| < \epsilon$. Since ϵ was arbitrary the result follows. □

Corollary 1.23.

(i) *If a Fourier series converges on a set $E \subset \mathbb{T}$, then it converges to f almost everywhere on E.*

(ii) *If a Fourier series converges to 0 almost everywhere on $\mathbb{T}$ then all its coefficients must be 0.*

Another useful summability method is the Abel summability of Fourier series. This is described by viewing the Fourier series of $f \in L^1(\mathbb{T})$ as the values on the unit circle $\{z : |z| = 1\}$ of the function

$$F(z) = \widehat{f}(0) + \sum_{j=1}^{\infty} \left(\widehat{f}(j) + \widehat{f}(-j)\right) z^j,$$

which is holomorphic on the unit disc $\{z : |z| < 1\}$ since $\widehat{f}$ is bounded. If we let $z = re^{2\pi i x}$ we get

$$F(re^{2\pi i x}) = \widehat{f}(0) + \sum_{j=1}^{\infty} \left(\widehat{f}(j) + \widehat{f}(-j)\right) r^j e^{2\pi i j x} = \int_{\mathbb{T}} f(t) P_r(x-t)\,\mathrm{d}t = (f * P_r)(x),$$

where

$$P_r(x) = 1 + \sum_{j=1}^{\infty} r^j e^{2\pi i j x} + \sum_{j=1}^{\infty} r^j e^{-2\pi i j x} = 1 + \frac{re^{2\pi i x}}{1-re^{2\pi i x}} + \frac{re^{-2\pi i x}}{1-re^{-2\pi i x}} = \frac{1-r^2}{1-2r\cos 2\pi x + r^2}$$

is the Poisson kernel. It is not hard to verify that the Poisson kernel satisfies (i*), (ii) and (iii*) of summability kernels, so Theorem 1.12 and Theorem 1.14 will apply here as well. First, to show (i*), we write

$$P_r(x) = \frac{1-r^2}{1-2r\cos 2\pi x + r^2} \geqslant \frac{1-r^2}{(1+r)^2} \geqslant 0.$$

To show (ii), we write

$$\int_{\mathbb{T}} P_r(t)\,\mathrm{d}t = \int_{\mathbb{T}} \left(1 + \sum_{j=1}^{\infty} r^j e^{2\pi i j t} + \sum_{j=1}^{\infty} r^j e^{-2\pi i j t}\right) \mathrm{d}t = \int_{\mathbb{T}} \mathrm{d}t = 1,$$

and for (iii*), when $|x| \geqslant \delta$, we have

$$P_r(x) \leqslant \frac{1-r^2}{1-2r\cos 2\pi\delta + r^2} \xrightarrow{r\to 1^-} 0.$$

Hence the statements in Theorem 1.18 also work for the Poisson kernel. The Fourier series of f is said to be Abel summable if $f * P_r$ converges pointwise as $r \to 1^-$. More generally, the series $\sum_{j=0}^{\infty} c_j$ is said to be Abel summable with limit ℓ if $\lim_{r\to 1^-} \sum_{j=0}^{\infty} c_j r^j = \ell$. A question of interest is if Abel summability extends convergence of series. This turns out to be true, and it even extends Cesàro summability. We leave the proof for the reader to verify (or see e.g. [2, Chapter VII]).

Theorem 1.24. *If $\sum_{j=0}^{\infty} c_j$ is Cesàro summable, then it is also Abel summable with the same limit.*

From this we conclude that Theorem 1.18 and Theorem 1.22 work for Abel summability as well. This summability method will turn out to be useful to us later.

Convergence in Norm

We now turn to a general discussion of convergence in norm for Fourier series. We begin by proving an equivalent statement using results from functional analysis.

Theorem 1.25. *Let $1 \leqslant p < \infty$. Then $s_n(f)$ converges to f for every $f \in L^p(\mathbb{T})$ if and only if the partial sum operators s_n are uniformly bounded, that is, if $\|s_n(f)\|_p \leqslant C_p \|f\|_p$ for some constant C_p independent of n and f.*

Proof. If $s_n(f)$ converges in L^p-norm then $s_n(f)$ is bounded in $L^p(\mathbb{T})$ for each $f \in L^p(\mathbb{T})$. By Banach-Steinhaus Theorem (see Appendix II) we immediately get the result. For the converse, given $\epsilon > 0$ choose a trigonometric polynomial φ such that $\|f - \varphi\|_p < \epsilon/(C_p + 1)$ (this is possible by Corollary 1.21). For $n > \deg \varphi$,

$$\|s_n(f) - f\|_p \leqslant \|s_n(f) - \varphi\|_p + \|\varphi - f\|_p = \|s_n(f - \varphi)\|_p + \|\varphi - f\|_p \leqslant (C_p + 1) \|f - \varphi\|_p < \epsilon.$$

□

We are now interested in the boundedness of the linear operators s_n in various Banach spaces. By (37) and Theorem 1.9, we have

$$\|s_n(f)\|_p \leqslant \|f\|_p \|D_n\|_1 \tag{40}$$

when $1 \leqslant p \leqslant \infty$, so

$$\|s_n\|_{L^p(\mathbb{T})} \leqslant \|D_n\|_1 , \tag{41}$$

where $\|s_n\|_{L^p(\mathbb{T})}$ is the operator norm of the partial sum operators $s_n : L^p(\mathbb{T}) \to L^p(\mathbb{T})$. Hence we know that $\|s_n\|_{L^p(\mathbb{T})} = \mathcal{O}(\log n)$. We claim that (41) is an equality in the cases $p = 1$ and $p = \infty$.

Lemma 1.26. *The following equalities hold:*

(i) $\|s_n\|_{L^1(\mathbb{T})} = \|D_n\|_1$

(ii) $\|s_n\|_{L^\infty(\mathbb{T})} = \|D_n\|_1$

(iii) $\|s_n\|_{C(\mathbb{T})} = \|D_n\|_1$

Proof. Since $\|s_n\|_{C(\mathbb{T})} \leqslant \|s_n\|_{L^\infty(\mathbb{T})}$, (41) holds for $C(\mathbb{T})$ as well. Thus there is only one inequality left to prove. We begin with (i). First, note that $s_n(F_m) = \sigma_m(D_n) \xrightarrow{m\to\infty} D_n$ (by Theorem 1.18 (ii)), so for every $\epsilon > 0$ there is an m such that $\|s_n(F_m)\|_1 > \|D_n\|_1 - \epsilon$. Hence,

$$\|s_n\|_{L^1(\mathbb{T})} \geqslant \|s_n(F_m)\|_1 > \|D_n\|_1 - \epsilon,$$

so (i) is proved. Next, for $L^\infty(\mathbb{T})$, let $f_n(t) = \operatorname{sgn} D_n(t)$. Clearly $\|f_n\|_\infty = 1$, and $f_n D_n = |D_n|$. Hence,

$$\|s_n\|_{L^\infty(\mathbb{T})} \geqslant \|s_n(f_n)\|_\infty \geqslant |s_n(f_n; 0)| = \int_{\mathbb{T}} f_n(t) D_n(t)\, \mathrm{d}t = \|D_n\|_1 ,$$

so (ii) is proved. Since the functions f_n are not continuous we have not yet proved (iii), but we can easily fix this problem. D_n has exactly $2n$ zeros on $\mathbb{T}$ and we know that f_n is discontinuous only at these zeros. For each (sufficiently small) $\epsilon > 0$, we change the values of f_n in neighbourhoods of length $\epsilon/2n$ of each zero such that each f_n becomes continuous and $\|f_n\|_\infty$ remains equal to 1 (e.g. draw straight line segments). Now,

$$\|s_n\|_{C(\mathbb{T})} \geqslant \|s_n(f_n)\|_\infty \geqslant |s_n(f_n; 0)| = \int_{\mathbb{T}} f_n(t) D_n(t)\, \mathrm{d}t > \|D_n\|_1 - \epsilon,$$

so (iii) is also proved. □

By Lemma 1.15 we see that each of these norms are unbounded and so, by Theorem 1.25 and Lemma 1.26, $s_n(f)$ does not converge to f for every $f \in L^1(\mathbb{T})$. If we examine the proof of Theorem 1.25 we see that the first direction will hold in any Banach space of integrable functions on $\mathbb{T}$. Hence we can also conclude that norm convergence does not hold in $L^\infty(\mathbb{T})$ and $C(\mathbb{T})$ (however, the case $L^\infty(\mathbb{T})$ was already obvious). For the other direction of the proof we note that the only extra information we need is that the trigonometric polynomials are dense in the space. In particular, this means that Theorem 1.25 holds for $C(\mathbb{T})$ as well.

Part (iii) of Lemma 1.26 is particularly interesting. In its proof we saw that the bounded linear functionals $f \mapsto s_n(f;0)$ are not uniformly bounded on $C(\mathbb{T})$. By Banach-Steinhaus Theorem,

$$\sup_n |s_n(f;0)| = \infty$$

for some $f \in C(\mathbb{T})$. In other words, there is a continuous function whose Fourier series diverges at 0, even a whole dense G_δ of continuous functions (see Appendix II). We get the same result for any $x \in \mathbb{T}$ if we redefine f_n in the proof of Lemma 1.26 by $f_n(t) = \operatorname{sgn} D_n(x-t)$. We have proved the following.

Theorem 1.27. *Given $x \in \mathbb{T}$ there is a dense G_δ-set E_x of continuous functions on $\mathbb{T}$ whose Fourier series diverge at x.*

The complement of a dense G_δ is meagre (a countable union of nowhere dense sets). Hence divergence at a given point is in some sense typical. We may take this even further. Pick out countably many points $x_j \in \mathbb{T}$. The set $E = \bigcap_{j=1}^{\infty} E_{x_j}$ is a dense G_δ, by Baire Category Theorem. If we look at the set of points where the Fourier series of $f \in E$ diverges unboundedly, we see that

$$\left\{x : \sup_n |s_n(f;x)| = \infty\right\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{x : |s_n(f;x)| > m\}.$$

Since $s_n(f;x)$ is continuous, this is a G_δ-set, and we may choose the points x_j such that the set is dense. In this case it turns out that the Fourier series must diverge at more points.

Theorem 1.28. *In a complete metric space with no isolated points every dense G_δ is uncountable.*

Proof. Suppose $\{x_j\}_{j=1}^{\infty}$ is a countable dense G_δ. Then it is equal to $\bigcap_{i=1}^{\infty} U_i$ for some open dense sets U_i. Let $V_i = U_i - \bigcup_{j=1}^{i} \{x_j\}$. Then each V_i is also open and dense, and $\bigcap_{i=1}^{\infty} V_i = \varnothing$, which contradicts Baire Category Theorem. □

The Hilbert Transform on $\mathbb{T}$

The Hilbert transform on $\mathbb{T}$ is given by the principal value convolution

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{T}} \frac{f(t)}{\tan \pi(x-t)}\, \mathrm{d}t = \lim_{\epsilon \to 0^+} \mathcal{H}_\epsilon f(x), \tag{42}$$

where the truncated Hilbert transform $\mathcal{H}_\epsilon f$ is defined by

$$\mathcal{H}_\epsilon f(x) = \int_{|x-t| \geqslant \epsilon} \frac{f(t)}{\tan \pi(x-t)}\, \mathrm{d}t. \tag{43}$$

We can write $\mathcal{H}_\epsilon f = f * K_\epsilon$, where $K_\epsilon(x) = \frac{1}{\tan \pi x} \mathbb{1}_{\{|t| \geqslant \epsilon\}}(x)$. Since K_ϵ is bounded for all $\epsilon > 0$, Theorem 1.9 implies that $\mathcal{H}_\epsilon$ is a bounded linear operator on $L^p(\mathbb{T}), 1 \leqslant p \leqslant \infty$. The most important result regarding the Hilbert transform is that $\mathcal{H}$ is a bounded linear operator, stated in the following theorem.

Theorem 1.29. (M. Riesz) *There is a bounded linear operator $\mathcal{H} : L^p(\mathbb{T}) \to L^p(\mathbb{T}), 1 < p < \infty$, such that $\|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \to 0$ as $\epsilon \to 0^+$ for all $f \in L^p(\mathbb{T})$.*

In fact, if $\|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \xrightarrow{\epsilon \to 0^+} 0$, then $\mathcal{H}$ is bounded. To see this, first note that it would imply that $\sup_\epsilon \|\mathcal{H}_\epsilon f\|_p < \infty$ for every $f \in L^p(\mathbb{T})$. By Banach-Steinhaus Theorem, we have $\sup_\epsilon \|\mathcal{H}_\epsilon\|_{L^p(\mathbb{T})} \leqslant M$ for some M. Hence, $\|\mathcal{H}f\|_p \leqslant \|\mathcal{H}f - \mathcal{H}_\epsilon f\|_p + \|\mathcal{H}_\epsilon f\|_p$, and taking the limit we see that $\|\mathcal{H}f\|_p \leqslant M \|f\|_p$. The goal of chapter 2 will be to prove Theorem 1.29. We will do this by first proving that $\mathcal{H}$ exists as the pointwise almost everywhere limit of $\mathcal{H}_\epsilon f$, for all $f \in L^1(\mathbb{T})$, then prove that this linear operator is bounded on $L^p(\mathbb{T}), 1 < p < \infty$. For the rest of this section we discuss the implications of Theorem 1.29 to the study of Fourier series.

Theorem 1.30. *Suppose there is a linear operator $\mathcal{H}$ on $L^p(\mathbb{T})$ such that $\|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \xrightarrow{\epsilon\to 0^+} 0$ for all $f \in L^p(\mathbb{T}), 1 \leqslant p \leqslant \infty$. Then $\mathcal{H}$ is a multiplier operator on $L^p(\mathbb{T})$ with multiplier $-i\,\mathrm{sgn}\,(j)$. That is,*

$$\widehat{\mathcal{H}f}(j) = -i\,\mathrm{sgn}\,(j)\,\widehat{f}(j).$$

Proof. By Lemma 1.10, $\widehat{\mathcal{H}_\epsilon f} = \widehat{K_\epsilon}\widehat{f}$. We now compute $\widehat{K_\epsilon}$.

$$\widehat{K_\epsilon}(j) = \int_{|t|\geqslant\epsilon} \frac{1}{\tan \pi t} e^{-2\pi ijt}\,\mathrm{d}t = -i\int_{|t|\geqslant\epsilon} \frac{\sin 2\pi jt \cos \pi t}{\sin \pi t}\,\mathrm{d}t.$$

Since we see that $\widehat{K_\epsilon}(-j) = -\widehat{K_\epsilon}(j)$ and $\widehat{K_\epsilon}(0) = 0$ we only need to consider positive j. By (31), we can rewrite

$$\int_{|t|\geqslant\epsilon} \frac{\sin 2\pi jt \cos \pi t}{\sin \pi t}\,\mathrm{d}t = \int_{|t|\geqslant\epsilon} (D_j(t) - \cos 2\pi jt)\,\mathrm{d}t.$$

By Dominated Convergence Theorem,

$$\lim_{\epsilon\to 0^+}\int_{|t|\geqslant\epsilon} (D_j(t) - \cos 2\pi jt)\,\mathrm{d}t = \int_{\mathbb{T}} (D_j(t) - \cos 2\pi jt)\,\mathrm{d}t = 1.$$

Hence $\widehat{K_\epsilon}(j) \xrightarrow{\epsilon\to 0^+} -i\,\mathrm{sgn}\,(j)$. We also have

$$\left|\widehat{\mathcal{H}_\epsilon f}(j) - \widehat{\mathcal{H}f}(j)\right| \leqslant \|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \xrightarrow{\epsilon\to 0^+} 0,$$

so $\widehat{\mathcal{H}_\epsilon f}(j) \xrightarrow{\epsilon\to 0^+} \widehat{\mathcal{H}f}(j)$. Thus $\widehat{\mathcal{H}f}(j) = -i\,\mathrm{sgn}\,(j)\,\widehat{f}(j)$ and we are done. □

Definition 1.31. *Let B be a Banach space of integrable functions on $\mathbb{T}$. B is said to admit conjugation if for every $f \in B$, there is a function $\widetilde{f} \in B$, such that*

$$\widetilde{f} \sim \sum_{j=-\infty}^{\infty} -i\,\mathrm{sgn}\,(j)\,\widehat{f}(j)e^{2\pi ijx}.$$

The series is called the conjugate Fourier series of $f \in L^1(\mathbb{T})$ and $\widetilde{f}$ is called the conjugate function of f.

Lemma 1.32. *If B admits conjugation and $\|\cdot\|_1 \leqslant \|\cdot\|_B$ then $f \mapsto \widetilde{f}$ is a bounded linear operator.*

Proof. Suppose $f_n \to f$ and $\widetilde{f_n} \to g$ in B. By Closed Graph Theorem, it is sufficient to prove $g = \widetilde{f}$. This follows by Corollary 1.20 (ii), since

$$\widehat{g}(j) = \lim_{n\to\infty}\widehat{\widetilde{f_n}}(j) = -i\,\mathrm{sgn}\,(j)\lim_{n\to\infty}\widehat{f_n}(j) = -i\,\mathrm{sgn}\,(j)\,\widehat{f}(j) = \widehat{\widetilde{f}}(j).$$

□

The property of conjugation gains interest when we find out that it is related to norm convergence of Fourier series.

Theorem 1.33. *Norm convergence of Fourier series in $L^p(\mathbb{T}), 1 \leqslant p < \infty$ holds if and only if $L^p(\mathbb{T})$ admits conjugation.*

Proof. If $L^p(\mathbb{T})$ admits conjugation then there is a function $g \in L^p(\mathbb{T})$ such that $g \sim \sum_{j=0}^{\infty}\widehat{f}(j)e^{2\pi ijx}$, and $g = \frac{1}{2}\widehat{f}(0) + \frac{1}{2}f + \frac{i}{2}\widetilde{f}$. Conversely, if such a g exists then L^p admits conjugation since $\widetilde{f} = -2ig + if + i\widehat{f}(0)$. Also note that $f \mapsto g$ is a bounded linear operator, by Lemma 1.32. If such a g exists we say that $L^p(\mathbb{T})$ admits projections and we denote its partial sums by $P_n(f)$. The strategy will be to show that $L^p(\mathbb{T})$ admits projections if and only if the operators s_n are uniformly bounded. From there the result follows by Theorem 1.25 and the previous remarks. First, suppose $\|s_n(f)\|_p \leqslant C_p\|f\|_p$ independent of n and f. By Lemma 1.3 (iv) we see that

$$e^{2\pi inx}s_n(e^{-2\pi inx}f;x) = \sum_{j=0}^{2n}\widehat{f}(j)e^{2\pi ijx} = P_{2n}(f;x).$$

By assumption we get $\|P_{2n}(f)\|_p \leqslant C_p \|f\|_p$. Given $f \in L^p(\mathbb{T})$, pick a polynomial φ such that $\|f-\varphi\|_p \leqslant \epsilon/2C_p$. Then

$$\|P_{2n}(f) - P_{2n}(\varphi)\|_p \leqslant C_p \|f-\varphi\|_p \leqslant \epsilon/2.$$

For $n, m \geqslant \frac{1}{2}\deg\varphi$ we now get

$$\|P_{2n}(f) - P_{2m}(f)\|_p \leqslant \|P_{2n}(f) - P_{2n}(\varphi)\|_p + \|P_{2m}(\varphi) - P_{2m}(f)\|_p \leqslant \epsilon.$$

Hence $P_{2n}(f)$ is a Cauchy sequence in $L^p(\mathbb{T})$ and thus it converges to some $g \in L^p(\mathbb{T})$. Hence $\widehat{g}(j) = \lim_{n\to\infty} \widehat{P_{2n}(f)}(j) = \widehat{f}(j)\mathbb{1}_{\{j\geqslant 0\}}(j)$. In other words, $g \sim \sum_{j=0}^{\infty} \widehat{f}(j)e^{2\pi ijx}$. Now for the other direction, assume $g \in L^p(\mathbb{T})$ satisfies $\widehat{g}(j) = \widehat{f}(j)\mathbb{1}_{\{j\geqslant 0\}}(j)$. Then, as previously discussed, there is constant C_p such that $\|g\|_p \leqslant C_p \|f\|_p$. Similarly, there is an $h \in L^p(\mathbb{T})$ such that $\widehat{h}(j) = \widehat{f}(j+2n+1)\mathbb{1}_{\{j\geqslant 0\}}(j)$ and $\|h\|_p \leqslant C_p \|f\|_p$, since by Lemma 1.3 (iv), $\widehat{f}(j+2n+1)$ are the Fourier coefficients of $e^{-2\pi i(2n+1)x}f(x)$ (which is an L^p-function). Now,

$$P_{2n}(f;x) = g(x) - e^{2\pi i(2n+1)x}h(x),$$

which can be seen by calculating the Fourier coefficients of both sides. This shows that $\|P_{2n}(f)\|_p \leqslant 2C_p \|f\|_p$. We also have

$$s_n(f;x) = e^{-2\pi inx}P_{2n}(e^{2\pi inx}f;x).$$

Hence $\|s_n(f)\|_p \leqslant 2C_p \|f\|_p$, so the theorem is proved.

□

Once Theorem 1.29 is proved, Theorem 1.30 and Theorem 1.33 prove norm convergence of Fourier series in $L^p(\mathbb{T}), 1 < p < \infty$. Lemma 1.26 has already ruled out $L^1(\mathbb{T})$ and $C(\mathbb{T})$. Hence $L^1(\mathbb{T})$ does not admit conjugation, and the proof of Theorem 1.33 applies to $C(\mathbb{T})$ just as well, so $C(\mathbb{T})$ also does not admit conjugation. By Theorem 1.30, $\mathcal{H}_\epsilon f$ cannot converge in norm in $L^1(\mathbb{T})$ and $C(\mathbb{T})$.

Fourier series in L^2

We will now solve the problem of norm convergence in $L^2(\mathbb{T})$. This space turns out to be the best suited for studying Fourier series. Recall that $L^2(\mathbb{T})$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} f(t)\overline{g(t)}\,\mathrm{d}t.$$

Let $\varphi_j(x) = e^{2\pi ijx}$. The set $\{\varphi_j\}_{j\in\mathbb{Z}}$ is orthonormal, by (25), and its linear span is the set of trigonometric polynomials on $\mathbb{T}$. Let $\varphi = \sum_{j=-n}^{n} c_j\varphi_j$ be a trigonometric polynomial. Then $c_j = \langle\varphi, \varphi_j\rangle$ by (3), and by Pythagoras Theorem,

$$\|\varphi\|_2^2 = \sum_{j=-n}^{n} |\langle\varphi,\varphi_j\rangle|^2.$$

With the inner product notation we may rewrite the partial sums $s_n(f)$ as

$$s_n(f) = \sum_{j=-n}^{n} \langle f,\varphi_j\rangle\varphi_j. \tag{44}$$

Hence $s_n(f)$ is the orthogonal projection of f onto the subspace spanned by $\{\varphi_j\}_{j=-n}^{n}$. That is, $s_n(f)$ is the best approximation of f by trigonometric polynomials of degree n. Moreover, by Pythagoras Theorem,

$$\|s_n(f) - f\|_2^2 = \|f\|_2^2 - \sum_{j=-n}^{n} |\langle f,\varphi_j\rangle|^2, \tag{45}$$

so we get the following inequality of Bessel,

$$\sum_{j=-\infty}^{\infty} |\langle f,\varphi_j\rangle|^2 \leqslant \|f\|_2^2. \tag{46}$$

Lemma 1.34. *Suppose $f_n \to f$ in $L^2(\mathbb{T})$. Then for every $g \in L^2(\mathbb{T})$,*

$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle$$

Proof. By Cauchy-Schwartz inequality,

$$|\langle f_n, g\rangle - \langle f, g\rangle| \leqslant \|f_n - f\|_2 \|g\|_2 \to 0.$$

□

Lemma 1.35. *Let $f \in L^2(\mathbb{T})$. Suppose $f = \sum_{j=-\infty}^{\infty} c_j\varphi_j$ in L^2-norm. Then $c_j = \widehat{f}(j)$.*

Proof. By Lemma 1.34,

$$\widehat{f}(j) = \langle f, \varphi_j\rangle = \lim_{n\to\infty} \left\langle \sum_{k=-n}^{n} c_k\varphi_k, \varphi_j \right\rangle = c_j.$$

□

Theorem 1.36. (Riesz-Fischer) *Suppose $\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty$. Then there is an $f \in L^2(\mathbb{T})$ such that $f = \sum_{j=-\infty}^{\infty} c_j\varphi_j$ in L^2-norm, and $c_j = \widehat{f}(j)$.*

Proof. Given $\epsilon > 0$, choose N such that $\sum_{|j|\geqslant N} |c_j|^2 < \epsilon$. Let $s_n = \sum_{j=-n}^{n} c_j\varphi_j$. For $n > m > N$,

$$\|s_n - s_m\|_2^2 = \sum_{m<|j|\leqslant n} |c_j|^2 < \epsilon.$$

Hence s_n converges to some $f \in L^2(\mathbb{T})$, and by Lemma 1.35, $c_j = \widehat{f}(j)$. □

By Bessel's inequality, $\sum_{j=-\infty}^{\infty} |\langle f, \varphi_j\rangle|^2 < \infty$ for all $f \in L^2(\mathbb{T})$, so by Riesz-Fischer Theorem there is a $g \in L^2(\mathbb{T})$ such that $g = \sum_{j=-\infty}^{\infty} \langle f, \varphi_j\rangle\varphi_j$ and $\langle g, \varphi_j\rangle = \langle f, \varphi_j\rangle$. Hence $g = f$ by Corollary 1.20 (ii), so we have proved norm convergence of Fourier series in L^2, and by (45) we get the following identity of Parseval,

$$\sum_{j=-\infty}^{\infty} |\langle f, \varphi_j\rangle|^2 = \|f\|_2^2.$$

More generally we have the following result.

Theorem 1.37. (Parseval) *Let $f, g \in L^2(\mathbb{T})$. Then*

$$\langle f, g\rangle = \sum_{j=-\infty}^{\infty} \langle f, \varphi_j\rangle\overline{\langle g, \varphi_j\rangle}$$

Proof. By Lemma 1.34,

$$\langle f, g\rangle = \lim_{n\to\infty} \left\langle \sum_{j=-n}^{n} \langle f, \varphi_j\rangle\varphi_j, g \right\rangle = \sum_{j=-\infty}^{\infty} \langle f, \varphi_j\rangle\overline{\langle g, \varphi_j\rangle}.$$

□

These results on Fourier series in L^2 are summarized by the following Theorem.

Theorem 1.38. *The map $f \mapsto \widehat{f}$ is a Hilbert space isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.*

Proof. It is injective by Corollary 1.20 (ii), surjective by Riesz-Fischer Theorem, and preserves inner products by Parseval's Theorem. □

Boundedness of the Hilbert Transform

The goal of this chapter is to prove the boundedness of the Hilbert transform in $L^p(\mathbb{T}), 1 < p < \infty$. There are a few different proofs of this, and while the one presented in this chapter is not necessarily the most elementary one, the same methods will also be useful for proving boundedness properties of more general singular integral operators.

The Calderón-Zygmund Decomposition

We begin the chapter with a decomposition method which will be useful to us later. We fix a number $\lambda > \langle |f| \rangle$, where $\langle f \rangle_E = \frac{1}{m(E)} \int_E f$ is the average value of f on $E \subset \mathbb{T}$. If $E = \mathbb{T}$ we may write $\langle f \rangle$ instead of $\langle f \rangle_{\mathbb{T}}$. Subdivide $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2})$ into $I = (-\frac{1}{2}, 0)$ and $J = (0, \frac{1}{2})$. Note that

$$\langle |f| \rangle_I + \langle |f| \rangle_J = 2 \int_I |f| + 2 \int_J |f| = 2 \int_{\mathbb{T}} |f| = 2 \langle |f| \rangle < 2\lambda,$$

so at least one of $\langle |f| \rangle_I$ and $\langle |f| \rangle_J$ are still greater than λ. We also have $\langle |f| \rangle_I < 2\lambda$ as well as $\langle |f| \rangle_J < 2\lambda$. If $\langle |f| \rangle_I > \lambda$ we subdivide I into two new open intervals by removing the middle point of I, and if $\lambda \leqslant \langle |f| \rangle_I < 2\lambda$ we separate I. Do the same for J. We now treat the new intervals similarly, so we get an infinite process going, which separates a countable collection of intervals $\{Q_j\}$, with $\lambda \leqslant \langle |f| \rangle_{Q_j} < 2\lambda$, and

$$\sum_j m(Q_j) \leqslant \sum_j \frac{1}{\lambda} \int_{Q_j} |f| \leqslant \frac{1}{\lambda} \int_{\mathbb{T}} |f| .$$

We now define a function g by

$$g(x) = f(x) \mathbb{1}_{\left(\bigcup_j Q_j\right)^c}(x) + \sum_j \langle f \rangle_{Q_j} \mathbb{1}_{Q_j}(x).$$

Clearly $g(x) < 2\lambda$ when $x \in \bigcup_j Q_j$. We are interested in a similar bound for other x. If $x \notin \bigcup_j Q_j$ and x is not an endpoint of any of the intervals then there is a sequence of intervals $I_j \ni x$ with $\lim_{j \to \infty} \operatorname{diam} I_j = 0$ and $\bigcap_j I_j = \{x\}$, such that for all j,

$$\frac{1}{m(I_j)} \int_{I_j} |f| < \lambda.$$

Letting $j \to \infty$ we get $|f(x)| \leqslant \lambda$ almost everywhere on $\left(\bigcup_j Q_j\right)^c$, by Lebesgue Differentiation Theorem. Hence $\|g\|_\infty \leqslant 2\lambda$. From this we conclude $\|g\|_p^p \leqslant (2\lambda)^{p-1} \|f\|_1$, since

$$\int_{\mathbb{T}} |g|^p \leqslant (2\lambda)^{p-1} \int_{\mathbb{T}} |g| \leqslant (2\lambda)^{p-1} \int_{\mathbb{T}} |f| = (2\lambda)^{p-1} \|f\|_1 .$$

Now we define a function b by

$$b(x) = f(x) - g(x) = \sum_j \left(f(x) - \langle f \rangle_{Q_j} \right) \mathbb{1}_{Q_j}(x).$$

Clearly the support of b lies in $\bigcup_j Q_j$ and $\langle b \rangle_{Q_j} = 0$ for all j. We also have

$$\langle |b| \rangle_{Q_j} = \frac{1}{m(Q_j)} \int_{Q_j} |b| \leqslant \frac{2}{m(Q_j)} \int_{Q_j} |f| < 4\lambda,$$

and $\|b\|_1 \leqslant 2 \|f\|_1$.

We have now constructed the Calderón-Zygmund decomposition $f = g + b$ for $f \in L^1(\mathbb{T})$ at height $\lambda > \langle |f| \rangle$. The function g is referred to as the good part of f, while b is the bad part. We saw that g has good boundedness properties, while b contains the part of f where the function behaves worse. However, the construction ensures that b has average value 0 over each of our intervals Q_j, which is a property that will be useful when proving certain theorems, as we shall see very soon. We state everything we have proved in the theorem below.

Theorem 2.1. **(Calderón-Zygmund Decomposition)** *Suppose $f \in L^1(\mathbb{T})$ and $\lambda > \langle |f| \rangle$. Then there is a decomposition $f = g + b$ and a sequence of open and disjoint dyadic intervals $\{Q_j\}$ such that*

(i) $\lambda \leqslant \langle |f| \rangle_{Q_j} < 2\lambda$ *for all j.*

(ii) $m(\bigcup_j Q_j) \leqslant \frac{1}{\lambda} \int_{\bigcup_j Q_j} |f|$.

(iii) $|f(x)| \leqslant \lambda$ *for almost every* $x \in \left(\bigcup_j Q_j\right)^c$.

(iv) $g(x) = f(x) \mathbb{1}_{(\bigcup_j Q_j)^c}(x) + \sum_j \langle f \rangle_{Q_j} \mathbb{1}_{Q_j}(x)$.

(v) $\|g\|_\infty \leqslant 2\lambda$.

(vi) $\|g\|_p^p \leqslant (2\lambda)^{p-1} \|f\|_1, 1 \leqslant p < \infty$.

(vii) $b(x) = \sum_j \left(f(x) - \langle f \rangle_{Q_j}\right) \mathbb{1}_{Q_j}(x)$.

(viii) $\int_{Q_j} b = 0$ *for all j.*

(ix) $\langle |b| \rangle_{Q_j} < 4\lambda$ *for all j.*

(x) $\|b\|_1 \leqslant 2 \|f\|_1$.

The Hilbert Transform in L^2

We can apply the results about Fourier series in L^2 to derive properties of the Hilbert transform. These results will play an important part in the proof of Theorem 1.29.

Theorem 2.2. *There is a constant C such that $\|\mathcal{H}_\epsilon f\|_2 \leqslant C \|f\|_2$ for all $f \in L^2(\mathbb{T})$ and $\epsilon > 0$.*

Proof. By Parseval,

$$\|\mathcal{H}_\epsilon f\|_2 = \left\|\widehat{\mathcal{H}_\epsilon f}\right\|_2 = \sup_j \left|\widehat{K_\epsilon}(j)\right| \left\|\hat{f}\right\|_2 = \sup_j \left|\widehat{K_\epsilon}(j)\right| \|f\|_2 .$$

Using $|\tan \pi x| \geqslant |\pi x|$, we get

$$\left|\widehat{K_\epsilon}(j)\right| = \left|\int_{|t| \geqslant \epsilon} \frac{\sin 2\pi j t}{\tan \pi t} \,\mathrm{d}t\right| \leqslant \left|\int_{|t| \geqslant \epsilon} \frac{\sin 2\pi j t}{\pi t} \,\mathrm{d}t\right| \leqslant \frac{2}{\pi} \left|\int_{2\pi\epsilon|j|}^{\pi|j|} \frac{\sin s}{s} \,\mathrm{d}s\right| .$$

Now, a simple argument using integration by parts shows that

$$\left|\int_a^b \frac{\sin s}{s} \,\mathrm{d}s\right| \leqslant 4$$

for all $a, b \geqslant 0$. Hence $\sup_\epsilon \|\mathcal{H}_\epsilon f\|_2 \leqslant \frac{8}{\pi} \|f\|_2$. □

Theorem 2.3. $\left\|\mathcal{H}_\epsilon f - \tilde{f}\right\|_2 \xrightarrow{\epsilon \to 0^+} 0$ *for all $f \in L^2(\mathbb{T})$.*

Proof. We begin by proving it for trigonometric polynomials $\varphi(x) = \sum_{j=-n}^n c_j e^{2\pi i j x}$. First, it follows pointwise, since

$$\mathcal{H}_\epsilon \varphi(x) = \int_{|t| \geqslant \epsilon} \frac{\varphi(x-t)}{\tan \pi t} \,\mathrm{d}t = \sum_{j=-n}^n c_j e^{2\pi i j x} \int_{|t| \geqslant \epsilon} \frac{e^{-2\pi i j t}}{\tan \pi t} \,\mathrm{d}t \xrightarrow{\epsilon \to 0^+} \sum_{j=-n}^n -i \operatorname{sgn}(j) c_j e^{2\pi i j x} = \tilde{\varphi}(x),$$

where the limit is taken of the same integral as in the calculation in the proof of Theorem 1.30. $\mathcal{H}_\epsilon \varphi$ is bounded independent of ϵ, since

$$|\mathcal{H}_\epsilon \varphi(x)| = \left|\sum_{j=-n}^n c_j e^{2\pi i j x} \int_{|t| \geqslant \epsilon} \frac{e^{-2\pi i j t}}{\tan \pi t} \,\mathrm{d}t\right| \leqslant \sum_{j=-n}^n |c_j| \left|\widehat{K_\epsilon}(j)\right| \leqslant \frac{8}{\pi} \sum_{j=-n}^n |c_j| .$$

By Dominated Convergence Theorem we have $\|\mathcal{H}_\epsilon\varphi - \widetilde{\varphi}\|_2 \xrightarrow{\epsilon\to 0^+} 0$. From here, it follows by a simple density argument. For each trigonometric polynomial φ, we write

$$\left\|\mathcal{H}_\epsilon f - \widetilde{f}\right\|_2 \leqslant \|\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon\varphi\|_2 + \|\mathcal{H}_\epsilon\varphi - \widetilde{\varphi}\|_2 + \left\|\widetilde{\varphi} - \widetilde{f}\right\|_2.$$

The first term is bounded by $C\|f-\varphi\|_2$, where C is as in Theorem 2.2, and by Parseval the last term equals $\|\varphi - f\|_2 - \left|\widehat{(\varphi - f)}(0)\right|^2 \leqslant \|\varphi - f\|$. Now, given $\eta > 0$, choose ϵ small enough so that the second term is less than $\eta/2$, and choose φ such that $\|f-\varphi\|_2 < \eta/2(C+1)$. Then $\left\|\mathcal{H}_\epsilon f - \widetilde{f}\right\|_2 < \eta$ and the proof is complete. □

In the proof above we derived a result worth stating as a separately.

Lemma 2.4. *When $1 \leqslant p < \infty$ we have $\|\mathcal{H}_\epsilon\varphi - \widetilde{\varphi}\|_p \xrightarrow{\epsilon\to 0^+} 0$ for all trigonometric polynomials φ.*

We define $\mathcal{H}f = \widetilde{f}$ for all $f \in L^2(\mathbb{T})$. By Theorem 2.3 and Theorem 1.30 we now know that $\widehat{\mathcal{H}f}(j) = -i\,\mathrm{sgn}\,(j)\,\widehat{f}(j)$. From this we can easily derive many properties of $\mathcal{H}f$ for $f \in L^2(\mathbb{T})$ (and also $f \in L^p(\mathbb{T}), 1 < p < \infty$, once we have proved the general boundedness result). For example, we can obtain an inversion formula for the Hilbert transform in $L^p(\mathbb{T}), 1 < p < \infty$, by the following calculation.

$$\widehat{\mathcal{H}\mathcal{H}f}(j) = -i\,\mathrm{sgn}\,(j)\,\widehat{\mathcal{H}f}(j) = (-i\,\mathrm{sgn}\,(j))^2\widehat{f}(j) = \begin{cases} 0 & j = 0 \\ -\widehat{f}(j) & j \neq 0 \end{cases}. \tag{47}$$

By Corollary 1.20 (ii) we get the inversion formula for $\mathcal{H}$,

$$\mathcal{H}\mathcal{H}f = \langle f\rangle - f \text{ a.e.} \tag{48}$$

We can define a new linear operator $\widetilde{\mathcal{H}}$ by

$$\widetilde{\mathcal{H}}f = i\langle f\rangle + \mathcal{H}f. \tag{49}$$

Clearly $\widetilde{\mathcal{H}}$ is bounded if and only if $\mathcal{H}$ is bounded. By the calculation in (47),

$$\widehat{\widetilde{\mathcal{H}}\widetilde{\mathcal{H}}f} = -\widehat{f}, \tag{50}$$

and the we thus get the nicer inversion formula,

$$\widetilde{\mathcal{H}}\widetilde{\mathcal{H}}f = -f \qquad \text{a.e.} \tag{51}$$

By similar arguments we may derive the following selected properties of $\mathcal{H}$ (under suitable conditions on f and g).

$$\mathcal{H}\overline{f} = \overline{\mathcal{H}f}, \tag{52}$$

$$\mathcal{H}(f * g) = \mathcal{H}f * g, \tag{53}$$

$$\mathcal{H}f' = (\mathcal{H}f)'. \tag{54}$$

Lemma 2.5. *$\widetilde{\mathcal{H}}$ is a unitary operator on $L^2(\mathbb{T})$.*

Proof. By Parseval,

$$\left\langle\widetilde{\mathcal{H}}f, \widetilde{\mathcal{H}}g\right\rangle = \sum_{j=-\infty}^{\infty}\widehat{\widetilde{\mathcal{H}}f}(j)\overline{\widehat{\widetilde{\mathcal{H}}g}(j)} = \sum_{j=-\infty}^{\infty}\widehat{f}(j)\overline{\widehat{g}(j)} = \langle f, g\rangle.$$

□

Using the inversion formula, we may write

$$\left\langle\widetilde{\mathcal{H}}f, g\right\rangle = \left\langle f, -\widetilde{\mathcal{H}}g\right\rangle. \tag{55}$$

In other words, $\widetilde{\mathcal{H}}^* = \widetilde{\mathcal{H}}^{-1} = -\widetilde{\mathcal{H}}$. By (49) we get

$$\langle\mathcal{H}f, g\rangle = -\langle f, \mathcal{H}g\rangle, \tag{56}$$

so $\mathcal{H}^* = -\mathcal{H}$, and by (52) we get the following lemma.

Lemma 2.6. *For $f, g \in L^2(\mathbb{T})$, we have*

$$\int_{\mathbb{T}} \mathcal{H}f(t)g(t)\,\mathrm{d}t = -\int_{\mathbb{T}} f(t)\mathcal{H}g(t)\,\mathrm{d}t.$$

We can also prove the same identity if we replace $\mathcal{H}$ with $\mathcal{H}_\epsilon$, since by Parseval's identity and Lemma 1.3 (ii) we get

$$\begin{aligned}\int_{\mathbb{T}} \mathcal{H}_\epsilon f(t)g(t)\,\mathrm{d}t &= \sum_{j=-\infty}^{\infty} \widehat{\mathcal{H}_\epsilon}(j)\overline{\widehat{\overline{g}}(j)} = \sum_{j=-\infty}^{\infty} \widehat{K_\epsilon}(j)\widehat{f}(j)\widehat{g}(-j) \\ &= -\sum_{j=-\infty}^{\infty} \widehat{f}(j)\widehat{K_\epsilon}(-j)\widehat{g}(-j) = -\sum_{j=-\infty}^{\infty} \widehat{f}(j)\widehat{\mathcal{H}_\epsilon g}(-j) = -\int_{\mathbb{T}} f(t)\mathcal{H}_\epsilon g(t)\,\mathrm{d}t.\end{aligned}$$

We finish the section by proving that our definition of the Hilbert transform in $L^2(\mathbb{T})$ agrees with the pointwise one.

Theorem 2.7. $\mathcal{H}_\epsilon f \xrightarrow{\epsilon \to 0^+} \widetilde{f}$ *almost everywhere for all* $f \in L^2(\mathbb{T})$.

Proof. Since $L^2(\mathbb{T})$ admits conjugation we know that the conjugate Fourier series of f is Abel summable to $\widetilde{f}$ almost everywhere, and we have

$$-i\sum_{j=1}^{\infty}\left(\widehat{f}(j) - \widehat{f}(-j)\right) r^j e^{2\pi i j x} = \int_{\mathbb{T}} f(t)Q_r(x-t)\,\mathrm{d}t = (f * Q_r)(x),$$

where

$$Q_r(x) = -i\left(\sum_{j=1}^{\infty} r^j e^{2\pi i j x} - \sum_{j=1}^{\infty} r^j e^{-2\pi i x}\right) = \frac{-ire^{2\pi i j x}}{1 - re^{2\pi i x}} - \frac{-ire^{-2\pi i x}}{1 - re^{-2\pi i x}} = \frac{2r\sin 2\pi x}{1 - 2r\cos 2\pi x + r^2}$$

is the conjugate Poisson kernel. From here, we are done if we can prove that

$$(f * Q_{1-\epsilon})(x) - \mathcal{H}_\epsilon f(x) \xrightarrow{\epsilon \to 0^+} 0 \qquad \text{a.e.}$$

We prove this for all $f \in L^1(\mathbb{T})$. First, since $Q_1(x) = 1/\tan \pi x$, we can write

$$(f * Q_{1-\epsilon})(x) - \mathcal{H}_\epsilon f(x) = \int_{|t|<\epsilon} f(x-t)Q_{1-\epsilon}(t)\,\mathrm{d}t + \int_{|t|\geqslant\epsilon} f(x-t)\left(Q_{1-\epsilon}(t) - Q_1(t)\right)\,\mathrm{d}t.$$

We begin by proving convergence of the first integral. Note that we have the estimate

$$|Q_r(x)| = \left|\frac{2r\sin 2\pi x}{1 - 2r\cos 2\pi x + r^2}\right| \leqslant \frac{4\pi r\,|x|}{(1-r)^2}.$$

Hence $Q_{1-\epsilon}(x) \leqslant 8\pi(1-\epsilon)\,|x|\,/\epsilon^2 < 4\pi/\epsilon$ when $|x| < \epsilon$. Now, since $Q_{1-\epsilon}$ is odd, we can write

$$\left|\int_{|t|<\epsilon} f(x-t)Q_{1-\epsilon}(t)\,\mathrm{d}t\right| = \left|\int_{|t|<\epsilon} \left(f(x-t) - f(x)\right)Q_{1-\epsilon}(t)\,\mathrm{d}t\right| \leqslant \frac{4\pi}{\epsilon}\int_{|t|<\epsilon} |f(x-t) - f(x)|\,\mathrm{d}t,$$

so the first integral converges to 0 at every Lebesgue point. To prove convergence of the second integral, an easy calculaton shows that

$$Q_{1-\epsilon}(x) - Q_1(x) = \frac{2(1-\epsilon)\sin 2\pi x}{1 - 2(1-\epsilon)\cos 2\pi x + (1-\epsilon)^2} - \frac{2\sin 2\pi x}{2 - 2\cos 2\pi x} = \frac{\epsilon}{\epsilon - 2}Q_1(x)P_{1-\epsilon}(x),$$

and when $|x| \geqslant \epsilon$ we have

$$\frac{\epsilon}{2-\epsilon}\,|Q_1(x)| = \frac{\epsilon}{2-\epsilon}\left|\frac{1}{\tan \pi x}\right| \leqslant \frac{\epsilon}{|\sin \pi x|} \leqslant \frac{\epsilon}{|2x|} \leqslant \frac{1}{2}.$$

Hence

$$\begin{aligned}\left|\int_{|t|\geqslant\epsilon} f(x-t)\left(Q_{1-\epsilon}(t) - Q_1(t)\right)\,\mathrm{d}t\right| &= \left|\int_{|t|\geqslant\epsilon} \left(f(x-t) - f(x)\right)\left(Q_{1-\epsilon}(t) - Q_1(t)\right)\,\mathrm{d}t\right| \\ &\leqslant \frac{1}{2}\int_{|t|\geqslant\epsilon} |f(x-t) - f(x)|\,P_{1-\epsilon}(t)\,\mathrm{d}t.\end{aligned}$$

To estimate $P_{1-\epsilon}(x)$, we use that $1-\cos 2\pi x = 2\sin^2 \pi x \geqslant 8x^2$ and write

$$P_r(x) = \frac{1-r^2}{1-2r\cos 2\pi x + r^2} \leqslant \frac{1-r^2}{2r(1-\cos 2\pi x)} \leqslant \frac{1-r}{8x^2}.$$

Hence $P_{1-\epsilon}(x) \leqslant \epsilon/(8(1-\epsilon)x^2)$, so it suffices to prove the convergence of

$$\epsilon \int_{|t|\geqslant \epsilon} |f(x-t)-f(x)| \frac{1}{t^2}\,\mathrm{d}t,$$

which we showed converges to 0 at every Lebesgue point in the proof of Theorem 1.22, so we are done. $\square$

Existence of the Hilbert Transform

We now extend our definition of the Hilbert transform to all $f \in L^1(\mathbb{T})$, by the following theorem.

Theorem 2.8. *The Hilbert transform, $\mathcal{H}$, exists as the pointwise almost everywhere limit of $\mathcal{H}_\epsilon f$ for all $f \in L^1(\mathbb{T})$. That is,*

$$\lim_{\epsilon\to 0^+} \mathcal{H}_\epsilon f = \mathcal{H} f \qquad \text{a.e.}$$

Proof. We apply the Calderón-Zygmund decomposition with $\lambda > \langle |f| \rangle$. Let g, b, and Q_j be as in Theorem 2.1, and let $\Omega = \bigcup_j 2Q_j$, where $2Q_j$ is the interval concentric with Q_j of measure $2m(Q_j)$. As $g \in L^2(\mathbb{T})$, Theorem 2.7 shows that $\mathcal{H}_\epsilon g$ converges almost everywhere in $\mathbb{T}$, so it suffices to prove the almost everywhere convergence of $\mathcal{H}_\epsilon b$. It even suffices to prove almost everywhere convergence of $\mathcal{H}_\epsilon b$ on Ω^c, since then the same holds on $\bigcup_k \Omega_k^c = \left(\bigcap_k \Omega_k\right)^c$, where Ω_k are the sets where the Q_j's correspond to λ_k. We can let $\lambda_k \to \infty$, and then $\bigcap_k \Omega_k$ has measure zero, by (ii) of Theorem 2.1. The problem is now reduced to showing that $\mathcal{H}_\epsilon b(x)$ is Cauchy for almost every $x \in \Omega^c$. So we pick $\eta > 0$ and assume $\epsilon > \delta > 0$. First we write

$$|\mathcal{H}_\epsilon b(x) - \mathcal{H}_\delta b(x)| \leqslant \int_{(x-\epsilon,x-\delta)} \left|\frac{b(t)}{\tan \pi(x-t)}\right| \mathrm{d}t + \int_{(x+\delta,x+\epsilon)} \left|\frac{b(t)}{\tan \pi(x-t)}\right| \mathrm{d}t.$$

We show that the second term is less than $\eta/2$ for sufficiently small ϵ and δ, and the exact same methods will apply to show the same for the first term. Since the support of b lies in $\bigcup_j Q_j$, we only need to consider the integral over the part some Q_j intersects. There may be an interval intersecting $x+\delta$, and an interval intersecting $x+\epsilon$, but only one interval for each of the points since the Q_j's are disjoint. We first single out the integral over $Q \cap (x+\delta, x+\epsilon)$, where Q is the interval containing $x+\delta$. Note that $m(Q) < 2\delta$, since if $m(Q) \geqslant 2\delta$ then $x \in 2Q$, which contradicts that $x \in \Omega^c$. Hence $Q \subset (x, x+\delta+m(Q)) \subset (x, x+3\delta)$, so $Q \cap (x+\delta, x+\epsilon) \subset (x+\delta, x+3\delta)$. From here we can write

$$\begin{aligned}\int_{(x+\delta,x+\epsilon)\cap Q} \left|\frac{b(t)}{\tan \pi(x-t)}\right| \mathrm{d}t &\leqslant \int_{(x+\delta,x+3\delta)} \left|\frac{b(t)}{\pi(x-t)}\right| \mathrm{d}t \leqslant \frac{1}{\pi\delta}\int_{(x+\delta,x+3\delta)} |b(t)|\,\mathrm{d}t \\ &= \frac{1}{\pi\delta}\int_{(\delta,3\delta)} |b(x+t)|\,\mathrm{d}t = \frac{1}{\pi\delta}\int_{(\delta,3\delta)} |b(x+t)-b(x)|\,\mathrm{d}t,\end{aligned}$$

where we used $\left|\frac{1}{\tan \pi t}\right| \leqslant \left|\frac{1}{\pi t}\right|$ in the third inequality and $b(x) = 0$ in the last equality. Hence we see that this integral is less than $\eta/6$ for sufficiently small $\delta > 0$ when $x \in \Omega^c$ is a Lebesgue point, and in the case when $x+\epsilon \in Q$ we get similar bounds with the same methods. Hence, for small enough ϵ and δ we have

$$\left|\int_{(x+\delta,x+\epsilon)} \frac{b(t)}{\tan \pi(x-t)}\,\mathrm{d}t\right| < \frac{\eta}{6} + \frac{\eta}{6} + \sum_{Q_j \subset (x+\delta,x+\epsilon)} \left|\int_{Q_j} \frac{b(t)}{\tan \pi(x-t)}\,\mathrm{d}t\right|.$$

Note that the collection of all $Q_j \subset (x+\delta, x+\epsilon)$ depends on ϵ and δ. We prove that the sum converges almost everywhere independent of the collection. Let x_j be the center point of $Q_j \subset (x+\delta, x+\epsilon)$. We first write

$$\int_{Q_j} \frac{b(t)}{\tan \pi(x-t)}\,\mathrm{d}t = \int_{Q_j} \frac{b(t)}{\tan \pi(x-t)} - \frac{b(t)}{\tan \pi(x-x_j)}\,\mathrm{d}t = \int_{Q_j} \frac{b(t)\sin \pi(t-x_j)}{\sin \pi(x-t)\sin \pi(x-x_j)}\,\mathrm{d}t.$$

If $|x - x_j| \leqslant m(Q_j)/2$ then $x \in 2Q_j$, so $|x - x_j| > m(Q_j)/2$. Hence $|x - t| \leqslant |x - x_j| + |x_j - t| \leqslant |x - x_j| + m(Q_j) < 3\,|x - x_j|$, and similarly $|x - x_j| \leqslant 3\,|x - t|$. Using this and $2\,|x| \leqslant \sin \pi x \leqslant \pi\,|x|$ we get

$$\left|\int_{Q_j} \frac{b(t)}{\tan \pi(x-t)}\,\mathrm{d}t\right| \leqslant \int_{Q_j} \frac{b(t)\pi\,|t - x_j|}{2\,|x-t|\,2\,|x - x_j|}\,\mathrm{d}t \leqslant \frac{3\pi m(Q_j)}{8\,|x - x_j|^2}\int_{Q_j} |b(t)|\,\mathrm{d}t$$
$$\leqslant \frac{3\pi m(Q_j)}{8\,|x - x_j|^2}\int_{Q_j} |f(t)|\,\mathrm{d}t.$$

It suffices to conclude the convergence of

$$\Delta(f;x) = \sum_j \frac{m(Q_j)}{|x - x_j|^2}\int_{Q_j} |f(t)|\,\mathrm{d}t$$

for almost every $x \in \Omega^c$. We do this by proving that $\Delta(f;x)$ is integrable on Ω^c. Interchanging the order of integration we have

$$\int_{\Omega^c} \Delta(f;x)\,\mathrm{d}x = \sum_j \int_{\Omega^c} \frac{m(Q_j)}{|x - x_j|^2}\,\mathrm{d}x \int_{Q_j} |f(t)|\,\mathrm{d}t,$$

and to bound the first integral we write

$$\int_{\Omega^c} \frac{m(Q_j)}{|x - x_j|^2}\,\mathrm{d}x \leqslant \int_{(2Q_j)^c} \frac{m(Q_j)}{|x - x_j|^2}\,\mathrm{d}x \leqslant \int_{y \geqslant m(Q_j)/2} \frac{m(Q_j)}{y^2}\,\mathrm{d}y \leqslant 2.$$

Thus we have

$$\int_{\Omega^c} \Delta(f;x)\,\mathrm{d}x \leqslant 2\sum_j \int_{Q_j} |f(t)|\,\mathrm{d}t \leqslant 2\,\|f\|_1,$$

and the sum converges almost everywhere. Now, for some n we have

$$\sum_{j=n+1} \frac{3\pi m(Q_j)}{8\,|x - x_j|^2}\int_{Q_j} |f(t)|\,\mathrm{d}t < \frac{\eta}{6}.$$

When ϵ is small enough the first n intervals always lie outside of $(x+\delta, x+\epsilon)$. Then

$$\sum_{Q_j \subset (x+\delta, x+\epsilon)} \left|\int_{Q_j} \frac{b(t)}{\tan \pi(x-t)}\,\mathrm{d}t\right| \leqslant \sum_{j=n+1}^{\infty} \frac{12\pi m(Q_j)}{|x - x_j|^2}\int_{Q_j} |f(t)|\,\mathrm{d}t < \frac{\eta}{6},$$

and the proof is complete. □

The Marcinkiewicz Interpolation Theorem

We have already seen that H_ϵ is not Cauchy in $L^1(\mathbb{T})$. In general, $\mathcal{H}f$ is not integrable (as can be seen for example by calculating the Hilbert transform of the indicator function of an interval). However, we can prove that $\mathcal{H}$ does map $L^1(\mathbb{T})$ into a certain larger space, introduced through the definition below.

Definition 2.9. *Let X be a measure space with measure μ. The space weak $L^p(X,\mu), 1 \leqslant p < \infty$, denoted $L^{p,\infty}(X,\mu)$, is the space of all measurable functions $f : X \to \mathbb{C}$, for which there is a constant C with*

$$\lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) \leqslant C^p, \qquad \lambda > 0.$$

The weak L^p-norm of f, denoted $\|f\|_{1,\infty}$, (which is not a norm) is defined as the infimum over all the C's. Hence we have

$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{\frac{1}{p}}.$$

In the case $p = \infty$ we define weak $L^\infty(X,\mu)$ to be the same as $L^\infty(X,\mu)$

A weak L^p-function, $p < \infty$, need not even be locally in L^p, which can be verified for the functions $1/x^{1/p}$. On the other hand, we have $L^p(X,\mu) \subset L^{p,\infty}(X,\mu)$, by the well-known Chebyshev's inequality,

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leqslant \frac{1}{\lambda^p} \int_X |f(x)|^p \,\mathrm{d}\mu(x). \tag{57}$$

More generally, suppose $T : L^{p_1}(X,\mu) \to L^{p_2}(Y,\nu)$, $p_2 < \infty$, is a bounded linear operator, that is, there is a constant C such that $\|Tf\|_{p_2} \leqslant C\|f\|_{p_1}$ for all $f \in L^{p_1}(X,\mu)$. Then we also have the weak bound $\|Tf\|_{p_2,\infty} \leqslant C\|f\|_{p_1}$, with the same C in both bounds. That is,

$$\lambda\nu(\{x \in X : |Tf(x)| > \lambda\})^{\frac{1}{p_2}} \leqslant C\|f\|_{p_1}$$

for every $\lambda > 0$ and every $f \in L^{p_1}(X,\mu)$. To see this, we write

$$\begin{aligned} \nu(\{x \in X : |Tf(x)| > \lambda\}) = \int_{\{|Tf(x)|>\lambda\}} \mathrm{d}\nu(x) &\leqslant \int_{\{|Tf(x)|>\lambda\}} \frac{|Tf(x)|^{p_2}}{\lambda^{p_2}} \,\mathrm{d}\nu(x) \\ &\leqslant \frac{1}{\lambda^{p_2}} \|Tf\|_{p_2}^{p_2} \leqslant \frac{C^{p_2}}{\lambda^{p_2}} \|f\|_{p_1}^{p_2}. \end{aligned} \tag{58}$$

Note that if T is the identity operator on $L^p(X,\mu)$, $p < \infty$, then this is just Chebyshev's inequality. The weak L^p-spaces are mainly of interest because of the theorem below, which takes us a great step further towards proving the L^p-boundedness of the Hilbert transform.

Theorem 2.10. (Marcinkiewicz) *Suppose (X,μ) and (Y,ν) are measure spaces, X is σ-finite, $1 \leqslant p_1 < p_2 \leqslant \infty$, and T is a sublinear operator $L^{p_1}(X) \to L^{p_1,\infty}(Y)$ and $L^{p_2}(X) \to L^{p_2,\infty}(Y)$ satisfying*

$$\|Tf\|_{p_1,\infty} \leqslant C_1\|f\|_{p_1} \,(\text{all } f \in L^{p_1}(X)), \quad \|Tf\|_{p_2,\infty} \leqslant C_2\|f\|_{p_2} \,(\text{all } f \in L^{p_2}(X))$$

for some constants C_1, C_2. Then T is also a sublinear operator $L^p(X) \to L^p(Y)$ for all p between p_1 and p_2, satisfying $\|Tf\|_p \leqslant C\|f\|_p$, where

$$C = 2\left(\frac{p}{p-p_1} + \frac{p}{p_2-p}\right)^{\frac{1}{p}} C_1^{\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}} C_2^{\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}}.$$

The proof of Theorem 2.10 requires the following lemma.

Lemma 2.11. *Let (X,μ) be a σ-finite measure space. Then we have the identity*

$$\|f\|_p^p = p\int_0^\infty \lambda^{p-1} m(\{x \in X : |f(x)| > \lambda\}) \,\mathrm{d}\lambda$$

for every $f \in L^p(X), 1 \leqslant p < \infty$.

Proof.

$$\begin{aligned} p\int_0^\infty \lambda^{p-1} m(\{x \in X : |f(x)| > \lambda\}) \,\mathrm{d}\lambda &= \int_0^\infty p\lambda^{p-1} \int_X \mathbb{1}_{\{|f(x)|>\lambda\}}(x) \,\mathrm{d}\mu(x) \,\mathrm{d}\lambda \\ &= \int_X \int_0^\infty p\lambda^{p-1} \mathbb{1}_{\{|f(x)|>\lambda\}}(x) \,\mathrm{d}\lambda \,\mathrm{d}\mu(x) \\ &= \int_X \int_0^{|f(x)|} p\lambda^{p-1} \,\mathrm{d}\lambda \,\mathrm{d}\mu(x) \\ &= \int_X |f(x)|^p \,\mathrm{d}\mu(x) = \|f\|_p^p. \end{aligned}$$

□

Note that the change of order of integration in the calculation above is why we need to assume X is σ-finite.

Proof of Theorem 2.10. Fix $\lambda > 0$ and $f \in L^p(X)$, $p_1 < p < p_2$. Consider the decomposition $f = f_1 + f_2$ where $f_1 = f\mathbb{1}_{\{|f(x)|>\alpha\lambda\}}$ and $f_2 = f\mathbb{1}_{\{|f(x)|\leqslant\alpha\lambda\}}$, where $\alpha > 0$ is a constant we will choose properly. Note that $f_1 \in L^{p_1}(X)$ and $f_2 \in L^{p_2}(X)$. By subadditivity $|Tf| \leqslant |Tf_1| + |Tf_2|$, so we have

$$\{x \in X : |Tf(x)| > \lambda\} \subset \left\{x \in X : |Tf_1(x)| > \frac{\lambda}{2}\right\} \cup \left\{x \in X : |Tf_2(x)| > \frac{\lambda}{2}\right\},$$

which implies

$$m(\{x \in X : |Tf(x)| > \lambda\}) \leqslant m(\left\{x \in X : |Tf_1(x)| > \frac{\lambda}{2}\right\}) + m(\left\{x \in X : |Tf_2(x)| > \frac{\lambda}{2}\right\}).$$

We have to separate the cases $p_2 < \infty$ and $p_2 = \infty$. We begin by assuming $p_2 < \infty$. By (58), we have

$$m(\left\{x \in X : |Tf_1(x)| > \frac{\lambda}{2}\right\}) \leqslant \frac{2^{p_1} C_1^{p_1}}{\lambda^{p_1}} \|f_1\|_{p_1}^{p_1} \quad , \quad m(\left\{x \in X : |Tf_2(x)| > \frac{\lambda}{2}\right\}) \leqslant \frac{2^{p_2} C_2^{p_2}}{\lambda^{p_2}} \|f_2\|_{p_2}^{p_2}.$$

Now combine this with Lemma 2.11, and write

$$\begin{aligned}
\|Tf\|_p^p &\leqslant p \int_0^\infty \lambda^{p-1} m(\left\{x \in X : |Tf_1(x)| > \frac{\lambda}{2}\right\}) \,\mathrm{d}\lambda + p \int_0^\infty \lambda^{p-1} m(\left\{x \in X : |Tf_2(x)| > \frac{\lambda}{2}\right\}) \,\mathrm{d}\lambda \\
&\leqslant p 2^{p_1} C_1^{p_1} \int_0^\infty \lambda^{p-1-p_1} \|f_1\|_{p_1}^{p_1} \,\mathrm{d}\lambda + p 2^{p_2} C_2^{p_2} \int_0^\infty \lambda^{p-1-p_2} \|f_2\|_{p_2}^{p_2} \,\mathrm{d}\lambda \\
&= p 2^{p_1} C_1^{p_1} \int_0^\infty \lambda^{p-1-p_1} \int_{\{|f(x)| > \alpha\lambda\}} |f(x)|^{p_1} \,\mathrm{d}\mu(x) \,\mathrm{d}\lambda \\
&\qquad + p 2^{p_2} C_2^{p_2} \int_0^\infty \lambda^{p-1-p_2} \int_{\{|f(x)| \leqslant \alpha\lambda\}} |f(x)|^{p_2} \,\mathrm{d}\mu(x) \,\mathrm{d}\lambda \\
&= p 2^{p_1} C_1^{p_1} \int_X |f(x)|^{p_1} \int_0^{\frac{|f(x)|}{\alpha}} \lambda^{p-1-p_1} \,\mathrm{d}\lambda \,\mathrm{d}\mu(x) \\
&\qquad + p 2^{p_2} C_2^{p_2} \int_X |f(x)|^{p_2} \int_{\frac{|f(x)|}{\alpha}}^\infty \lambda^{p-1-p_2} \,\mathrm{d}\lambda \,\mathrm{d}\mu(x) \\
&= p 2^{p_1} C_1^{p_1} \frac{1}{p - p_1} \frac{1}{\alpha^{p-p_1}} \int_X |f(x)|^{p_1} |f(x)|^{p-p_1} \,\mathrm{d}\mu(x) \\
&\qquad + p 2^{p_2} C_2^{p_2} \frac{1}{p_2 - p} \frac{1}{\alpha^{p-p_2}} \int_X |f(x)|^{p_2} |f(x)|^{p-p_2} \,\mathrm{d}\mu(x) \\
&= \left(p 2^{p_1} C_1^{p_1} \frac{1}{p - p_1} \frac{1}{\alpha^{p-p_1}} + p 2^{p_2} C_2^{p_2} \frac{1}{p_2 - p} \frac{1}{\alpha^{p-p_2}} \right) \|f\|_p^p.
\end{aligned}$$

We have thus proved L^p-boundedness when $p_2 < \infty$ and we may choose α to get the desired constant C in the statement of the theorem. Now consider the case $p_2 = \infty$. Then we have

$$\|Tf_2\|_\infty \leqslant C_2 \|f_2\|_\infty \leqslant C_2 \alpha \lambda,$$

so if we choose $\alpha = 1/(2C_2)$ we have

$$\|Tf_2\|_\infty \leqslant \frac{\lambda}{2}.$$

Hence,

$$\begin{aligned}
m(\{x \in X : |Tf(x)| > \lambda\}) &\leqslant m(\left\{x \in X : |Tf_1(x)| > \frac{\lambda}{2}\right\}) + m(\left\{x \in X : |Tf_2(x)| > \frac{\lambda}{2}\right\}) \\
&= m(\left\{x \in X : |Tf_1(x)| > \frac{\lambda}{2}\right\}).
\end{aligned}$$

From here we have just half of the previous case, and we have already done the calculation, so

$$\|Tf\|_p^p \leqslant p 2^{p_1} C_1^{p_1} \frac{1}{p - p_1} \frac{1}{\alpha^{p-p_1}} \|f\|_p^p = p 2^p C_1^{p_1} C_2^{p-p_1} \frac{1}{\alpha^{p-p_1}} \|f\|_p^p,$$

and the reader may check that this does in fact coincide with the C in the statement of the theorem. $\square$

An example where Marcinkiewicz Interpolation Theorem can be applied is the Hardy-Littlewood maximal function $\mathcal{M}$. On $\mathbb{T}$ it is defined by

$$\mathcal{M}f(x) = \sup_{0<\delta\leqslant\frac{1}{2}} \frac{1}{2\delta}\int_{(x-\delta,x+\delta)} |f(t)|\ \mathrm{d}t. \tag{59}$$

This operator appears for example when studying differentiation theorems for measures (and is used to prove Lebesgue Differentiation Theorem). Note that it maps $L^\infty(\mathbb{T})$ boundedly into $L^\infty(\mathbb{T})$, since

$$\frac{1}{m(Q)}\int_Q |f(t)|\ \mathrm{d}t \leqslant \|f\|_\infty$$

for every measurable set $Q \subset \mathbb{T}$ (and the operator norm is equal to 1, since there is no better bound for constant functions). Also note that $\mathcal{M}$ is sublinear, and we have weak L^1-boundedness, by the famous Hardy-Littlewood Maximal Theorem:

Theorem 2.12. (Hardy-Littlewood Maximal Theorem) *There is a constant C such that for every $\lambda > 0$ and $f \in L^1(\mathbb{T})$, we have*

$$\lambda m(\{x \in \mathbb{T} : |\mathcal{M}f(x)| > \lambda\}) \leqslant C\int_{\mathbb{T}} |f(t)|\ \mathrm{d}t.$$

Proof. See e.g. [5, Chapter 7] or, for another proof involving the Calderón-Zygmund decomposition, see [2, Chapter IV]. □

Hence $\mathcal{M}$ is bounded on $L^p(\mathbb{T}), 1 < p < \infty$, by Marcinkiewicz Interpolation Theorem. We would like to make a similar argument for the Hilbert transform. We already have boundedness of $\mathcal{H}$ in the case $p = 2$. Now we would like to show weak L^1-boundedness. For this, Theorem 2.12 will turn out to be useful.

Boundedness of the Hilbert Transform

Theorem 2.13. (Kolmogorov) *There is a constant C such that for all $f \in L^1(\mathbb{T})$ and all $\lambda > 0$,*

$$\lambda m(\{x \in \mathbb{T} : |\mathcal{H}f(x)| > \lambda\}) \leqslant C\|f\|_1,$$

and for $\mathcal{H}_\epsilon$ we have (independent of ϵ)

$$\lambda m(\{x \in \mathbb{T} : |\mathcal{H}_\epsilon f(x)| > \lambda\}) \leqslant C\|f\|_1.$$

Proof. We begin by proving the result for $\mathcal{H}_\epsilon$. It is not hard to see that it suffices to prove the result for all sufficiently large λ. We attempt to prove it for all $\lambda > \langle|f|\rangle$, where we can apply the Calderón-Zygmund decomposition. Let Ω and Δ be as in the proof the Theorem 2.8. First, we write

$$m(\{x \in \mathbb{T} : |\mathcal{H}_\epsilon f(x)| > \lambda\}) \leqslant m(\Omega) + m(\{x \in \Omega^c : |\mathcal{H}_\epsilon f(x)| > \lambda\}).$$

By (ii) of Theorem 2.1 we have $m(\Omega) \leqslant 2\sum_j m(Q_j) \leqslant \frac{2}{\lambda}\|f\|_1$, so it remains to prove a similar bound for the last term. We now decompose f into g and b, and write

$$m(\{x \in \Omega^c : |\mathcal{H}_\epsilon f(x)| > \lambda\}) \leqslant m(\left\{x \in \Omega^c : |\mathcal{H}_\epsilon g(x)| > \frac{\lambda}{2}\right\}) + m(\left\{x \in \Omega^c : |\mathcal{H}_\epsilon b(x)| > \frac{\lambda}{2}\right\}).$$

For the "good term", we use (57), Theorem 2.2, and (vi) of Theorem 2.1, to get

$$m(\left\{x \in \Omega^c : |\mathcal{H}_\epsilon g(x)| > \frac{\lambda}{2}\right\}) \leqslant \frac{1}{\left(\frac{\lambda}{2}\right)^2}\int_{\mathbb{T}} |\mathcal{H}_\epsilon g(t)|^2\ \mathrm{d}t \leqslant \frac{4C}{\lambda^2}\|g\|_2^2 \leqslant \frac{8C\|f\|_1}{\lambda}.$$

We are now left with the "bad term". We make use of the estimate of $|\mathcal{H}_\epsilon - \mathcal{H}_\delta|$ we got in the proof of Theorem 2.8. In the case $\epsilon = 1/2$, this results in (for almost every $x \in \Omega^c$)

$$\begin{aligned} |\mathcal{H}_\epsilon b(x) - \mathcal{H}_\delta b(x)| = |\mathcal{H}_\delta| &\leqslant \frac{3\pi}{8}\Delta(f;x) + \frac{1}{\pi\delta}\int_{(x+\delta,x+3\delta)} |b(t)|\ \mathrm{d}t + \frac{1}{\pi\delta}\int_{(x-\delta,x-3\delta)} |b(t)|\ \mathrm{d}t \\ &\leqslant \frac{3\pi}{8}\Delta(f;x) + \frac{6}{\pi}\frac{1}{6\delta}\int_{(x-3\delta,x+3\delta)} |b(t)|\ \mathrm{d}t \leqslant \frac{3\pi}{8}\Delta(f;x) + \frac{6}{\pi}\mathcal{M}b(x). \end{aligned}$$

Now we get

$$m(\left\{x\in\Omega^c : |\mathcal{H}_\epsilon b(x)| > \frac{\lambda}{2}\right\}) \leqslant m(\left\{x\in\Omega^c : |C\Delta(f;x)| > \frac{\lambda}{4}\right\}) + m(\left\{x\in\Omega^c : |C\mathcal{M}b(x)| > \frac{\lambda}{4}\right\})$$

for some constant C. Recall from the proof of Theorem 2.8 that

$$\int_{\Omega^c}\Delta(f;x)\,\mathrm{d}x \leqslant 2\sum_j\int_{Q_j}|f(t)|\;\mathrm{d}t \leqslant 2\left\|f\right\|_1.$$

By this and Chebyshev's inequality we get

$$m(\left\{x\in\Omega^c : |C\Delta(f;x)| > \frac{\lambda}{4}\right\}) \leqslant \frac{2C\left\|f\right\|_1}{\lambda}.$$

For the last term we apply Hardy-Littlewood Maximal Theorem and (x) of Theorem 2.1 to get

$$m(\left\{x\in\Omega^c : |C\mathcal{M}b(x)| > \frac{\lambda}{4}\right\}) \leqslant \frac{C'}{\lambda}\left\|b\right\|_1 \leqslant \frac{2C'}{\lambda}\left\|f\right\|_1,$$

so we are done with $\mathcal{H}_\epsilon$, and the same proof will hold for $\mathcal{H}$ except that have to replace the estimate

$$|\mathcal{H}_\delta| \leqslant \frac{3\pi}{8}\Delta(f;x) + \frac{1}{\pi\delta}\int_{(x+\delta,x+3\delta)}|b(t)|\;\mathrm{d}t + \frac{1}{\pi\delta}\int_{(x-\delta,x-3\delta)}|b(t)|\;\mathrm{d}t$$

We do this by letting $\delta\to 0^+$ above and we arrive at

$$|\mathcal{H}b(x)| \leqslant \frac{3\pi}{8}\Delta(f;x)$$

almost everywhere on Ω^c. Hence the proof for $\mathcal{H}$ is even easier. $\square$

We are finally ready to prove the boundedness of the Hilbert transform. Marcinkiewicz now gives us the result for $1<p<2$, and the rest will follow by a duality argument. We restate Theorem 1.29 below.

Theorem 2.14. (M.Riesz) *$\mathcal{H}$ is a bounded linear operator on $L^p(\mathbb{T}), 1<p<\infty$, and $\mathcal{H}_\epsilon f$ converges in L^p-norm to $\mathcal{H}f$ for all $f\in L^p(\mathbb{T})$.*

Proof. By Marcinkiewicz, the boundedness for $1<p<2$ follows from the Theorem 2.3 and Theorem 2.13. Now, we want to prove the same in the case $2<q<\infty, q=\frac{p}{p-1}$. First we prove it on the trigonometric polynomials in $L^q(\mathbb{T})$. By Lemma 2.6, we have for every $g\in L^2(\mathbb{T})$,

$$\int_{\mathbb{T}}\mathcal{H}f(t)g(t)\,\mathrm{d}t = -\int_{\mathbb{T}}f(t)\mathcal{H}g(t)\,\mathrm{d}t.$$

We can extend this definition to $g\in L^p(\mathbb{T})$, since we now know that $\mathcal{H}$ is continuous on $L^p(\mathbb{T})$, and since $f,\mathcal{H}f\in L^q(\mathbb{T})$ when f is a trigonometric polynomial. That is, we pick a sequence g_n in $L^2(\mathbb{T})$ with $g_n\to g$ in L^p-norm. Then

$$\int_{\mathbb{T}}\mathcal{H}f(t)g(t)\,\mathrm{d}t = \lim_{n\to\infty}\int_{\mathbb{T}}\mathcal{H}f(t)g_n(t)\,\mathrm{d}t = -\lim_{n\to\infty}\int_{\mathbb{T}}f(t)\mathcal{H}g_n(t)\,\mathrm{d}t = -\int_{\mathbb{T}}f(t)\mathcal{H}g(t)\,\mathrm{d}t.$$

Now, by the converse Hölder inequality, and Hölder's inequality,

$$\left\|\mathcal{H}f\right\|_q = \sup_{\left\|g\right\|_p=1}\left|\int_{\mathbb{T}}f(t)\mathcal{H}g(t)\,\mathrm{d}t\right| \leqslant \left\|f\right\|_q\sup_{\left\|g\right\|_p=1}\left\|\mathcal{H}g\right\|_p = C_p\left\|f\right\|_q,$$

where C_p is the operator norm of $\mathcal{H}: L^p(\mathbb{T})\to L^p(\mathbb{T})$. From here we can define an operator $\mathcal{H}'$ through extension by continuity so that it is a bounded linear operator on $L^q(\mathbb{T})$. We want to show that $\mathcal{H}'f=\mathcal{H}f$ for all $f\in L^q(\mathbb{T})$. To see this, we show that they have the same Fourier coefficients. We know that $\mathcal{H}'\varphi=\mathcal{H}\varphi$ for trigonometric polynomials. Let φ_n be a sequence of polynomials converging to f in L^q-norm. Then

$$\widehat{\mathcal{H}'f}(j) = \lim_{n\to\infty}\widehat{\mathcal{H}'\varphi_n}(j) = \lim_{n\to\infty}\widehat{\mathcal{H}\varphi_n}(j) = \widehat{\mathcal{H}f}(j),$$

where the first equality follows since

$$\left|\widehat{\mathcal{H}'\varphi_n}(j) - \widehat{\mathcal{H}'f}(j)\right| \leqslant \left\|\mathcal{H}'\varphi_n - \mathcal{H}'f\right\|_q \to 0,$$

and the third equality follows since

$$\left|\widehat{\mathcal{H}\varphi_n}(j) - \widehat{\mathcal{H}f}(j)\right| \leqslant \|\mathcal{H}\varphi_n - \mathcal{H}f\|_2 \leqslant C_2 \|\varphi_n - f\|_2 \leqslant C_2 \|\varphi_n - f\|_q \to 0.$$

Hence $\mathcal{H}f$ is bounded on $L^q(\mathbb{T})$, and we have the bound $\|\mathcal{H}f\|_p \leqslant C_p \|f\|_p$ for all $f \in L^p(\mathbb{T})$, $1 < p < \infty$. The same arguments also prove that $\|\mathcal{H}_\epsilon f\|_p \leqslant C_p \|f\|_p$, $1 < p < \infty$, independent of ϵ and f (but we already know that each $\mathcal{H}_\epsilon$ is continuous on $L^p(\mathbb{T})$, by Theorem 1.9, so we do not need to extend by continuity here). Now we can prove that $\|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \xrightarrow{\epsilon \to 0^+} 0, f \in L^p(\mathbb{T}), 1 < p < \infty$. Given $\eta > 0$, pick a polynomial φ such that $\|f - \varphi\|_p < \eta/(4C_p)$. Then

$$\|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \leqslant \|\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon \varphi\|_p + \|\mathcal{H}_\epsilon \varphi - \mathcal{H}\varphi\|_p + \|\mathcal{H}\varphi - \mathcal{H}f\|_p \leqslant 2C_p \|f - \varphi\|_p + \|\mathcal{H}_\epsilon \varphi - \mathcal{H}\varphi\|_p.$$

In the last expression the first term is bounded by $\eta/2$, and for ϵ small enough the last term is also bounded by $\eta/2$, by Lemma 2.4. Hence $\|\mathcal{H}_\epsilon f - \mathcal{H}f\|_p \xrightarrow{\epsilon \to 0^+} 0$. Finally, $L^p(\mathbb{T})$ admits conjugation and $\mathcal{H}f = \widetilde{f}$ by Theorem 1.30, which means the Fourier series of every $f \in L^p(\mathbb{T})$ converges to f in L^p-norm. $\square$

Singular Integrals on $\mathbb{R}^n$

In this chapter we discuss the Riesz Transforms, the analogue to the Hilbert transform in $\mathbb{R}^n$. We also generalize these to a larger class of singular integral operators, the ones whose kernel is homogeneous of degree $-n$. Once again we are mainly concerned with the L^p-continuity of these operators. For odd kernels we will show that this is equivalent to the continuity of the Hilbert transform on the real line. To prove general boundedness results, it is natural to begin with the L^2-theory of singular integrals, where the Fourier transform will be an important tool.

The Hilbert Transform on $\mathbb{R}$

The Hilbert transform appears when studying conjugate harmonic functions. Historically, it first appeared through the Riemann-Hilbert problem. For a continuous function f on $\mathbb{R}$, the problem is to find a function F, holomorphic on $\{z \in \mathbb{C} : \mathrm{Im} z \neq 0\}$, such that

$$\lim_{y \to 0^+} (F(x+iy) - F(x-iy)) = f(x). \tag{60}$$

This is achieved through the Cauchy-type integral

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} \,\mathrm{d}t. \tag{61}$$

The function is easily verified to be holomorphic and a simple calculation shows that

$$\begin{aligned} F(x+iy) - F(x-iy) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-x-iy} \,\mathrm{d}t - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-x+iy} \,\mathrm{d}t \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{y f(t)}{(t-x)^2 + y^2} \,\mathrm{d}t = (f * P_y)(x), \end{aligned}$$

where $\{P_y\}_{y>0}$ is the Poisson kernel on $\mathbb{R}$ given by

$$P_y(x) = \frac{1}{\pi} \frac{y}{y^2 + x^2}. \tag{62}$$

The Poisson kernel satisfies the conditions (i*), (ii) and (iii*) for a summability kernel, except that the kernel is defined on $\mathbb{R}$ instead of $\mathbb{T}$. The proofs of Theorem 1.12-1.14 will remain the same for such kernels (mainly because the invariance properties of the measure hold for $\mathbb{R}^n$ as well). By Theorem 1.14 (ii), $(f * P_y)(x) \xrightarrow{y \to 0^+} f(x)$, so (60) is established.

Now we expect the Hilbert transform to appear as the boundary values of the imaginary part of F. By a calculation similar to the one we just did we see that

$$F(x+iy) + F(x-iy) = (if * Q_y)(x),$$

where $\{Q_y\}_{y>0}$ is the conjugate Poisson kernel on $\mathbb{R}$ given by

$$Q_y(x) = \frac{1}{\pi} \frac{x}{y^2 + x^2}.$$

Hence $F(x+iy) = \frac{1}{2}((f * P_y)(x) + i(f * Q_y)(x)$ and we are interested in determining when $(f * Q_y)(x) \xrightarrow{y \to 0^+} \mathcal{H}f(x)$, where $\mathcal{H}$ is the Hilbert transform on $\mathbb{R}$ given by

$$\mathcal{H}f(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x-t} \,\mathrm{d}t = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-t)}{t} \,\mathrm{d}t = \lim_{\epsilon \to 0^+} \mathcal{H}_\epsilon(x),$$

where $\mathcal{H}_\epsilon$ is defined in the obvious way. $\mathcal{H}_\epsilon$ exists for all $f \in L^p(\mathbb{R}), 1 \leqslant p < \infty$ by Hölder's inequality. When f satisfies a Hölder condition $\mathcal{H}_\epsilon(x)$ also converges to the integral of $f(x-t)/t$, since the integrand is bounded by a constant multiple of $t^{\alpha-1}, \alpha > 0$, which is locally integrable near the origin. In this case $(f * Q_y)(x) \xrightarrow{y \to 0^+} \mathcal{H}f(x)$, and in the next section we prove the pointwise convergence of $f * Q_y$ in $L^p(\mathbb{R}), 1 \leqslant p < \infty$. For now we prove that $\mathcal{H}_\epsilon$ converges pointwise almost everywhere for all $f \in L^p(\mathbb{R}), 1 \leqslant p < \infty$. To see this, we rewrite $\mathcal{H}_\epsilon f(x)$ as

$$\int_{\frac{1}{2} > |t| \geqslant \epsilon} \frac{f(x-t)}{\tan \pi t} \,\mathrm{d}t + \int_{\frac{1}{2} > |t| \geqslant \epsilon} f(x-t) \left(\frac{1}{\pi t} - \frac{1}{\tan \pi t} \right) \mathrm{d}t + \frac{1}{\pi} \int_{|t| \geqslant \frac{1}{2}} \frac{f(x-t)}{t} \,\mathrm{d}t.$$

The first integral is identified with the Hilbert transform on $\mathbb{T}$, so it converges almost everywhere. The second integral converges everywhere since $\frac{1}{\pi t} - \frac{1}{\tan \pi t}$ is bounded, and the last integral is finite by Hölder's inequality. The boundedness in $L^p(\mathbb{R}), 1 < p < \infty$ is also true, but we can not prove it here. It can be proved similarly to how we proved it for the Hilbert transform on $\mathbb{T}$ with Marcinkiewicz Interpolation Theorem and duality, but this means that we first need prove it in $L^2(\mathbb{R})$. We state the result as a theorem below but prove it later after the Fourier transform has been introduced. For now we discuss the consequences of the result to generalizations of the Hilbert transform.

Theorem 3.1. *$\mathcal{H}$ exists almost everywhere for all $f \in L^p(\mathbb{R}), 1 \leqslant p < \infty$, and when $p > 1$, there is some constant C_p such that $\|\mathcal{H}f\|_p \leqslant C_p \|f\|_p$ for all $f \in L^p(\mathbb{R})$. Also, $\|\mathcal{H}_\epsilon f\|_p \leqslant C_p \|f\|_p$ for every $\epsilon > 0$ and $\mathcal{H}_\epsilon f$ converges to $\mathcal{H}f$ in L^p-norm.*

The Riesz Transforms

We now want to generalize the Hilbert transform to $\mathbb{R}^n$, in such a way that it has similar properties to the ones just derived on $\mathbb{R}$. We begin by introducing the Poisson kernel on $\mathbb{R}^n$.

Definition 3.2. *The Poisson kernel on $\mathbb{R}^n$, $\{P_\epsilon\}_{\epsilon>0}$, is defined by*

$$P_\epsilon(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{\epsilon}{(\epsilon^2 + |x|^2)^{\frac{n+1}{2}}}.$$

The constant is chosen so that $\int_{\mathbb{R}^n} P_\epsilon = 1$. Thus the Poisson kernel satisfies (i*), (ii) and (iii*) for a summability kernel, so Theorem 1.12 and Theorem 1.14 can be applied with this kernel. To see that its integral is 1, we write

$$\int_{\mathbb{R}^n} \frac{\epsilon}{(\epsilon^2 + |x|^2)^{\frac{n+1}{2}}} \, \mathrm{d}x = \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{\frac{n+1}{2}}} \, \mathrm{d}x = \omega_{n-1} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{n+1}{2}}} \, \mathrm{d}r,$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$ (see Appendix I for more about the Gamma function and how to calculate the surface area of the sphere). Substitute $r = \tan\theta$ to get

$$\omega_{n-1} \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \, \mathrm{d}\theta.$$

Now subtitute $u = \sin^2\theta$, $\mathrm{d}\theta = \frac{\mathrm{d}u}{2\sin\theta\cos\theta} = \frac{\mathrm{d}u}{2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}}$ and we get

$$\frac{1}{2}\omega_{n-1} \int_0^1 u^{\frac{n}{2}-1}(1-u)^{-\frac{1}{2}} \, \mathrm{d}u.$$

Next, make the substitution $u = \frac{v}{1+v}$, $\mathrm{d}u = \frac{1}{1+v^2} \, \mathrm{d}v$, and we arrive at the integral

$$\frac{1}{2}\omega_{n-1} \int_0^\infty v^{\frac{n}{2}-1} \left(\frac{1}{1+v}\right)^{\frac{n+1}{2}} \mathrm{d}v$$

We claim that this is equal to $\frac{1}{2}\omega_{n-1}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} = \frac{1}{2}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$. The following calculation proves the result.

$$\begin{aligned}
&\Gamma\left(\frac{n+1}{2}\right) \int_0^\infty v^{\frac{n}{2}-1} \left(\frac{1}{1+v}\right)^{\frac{n+1}{2}} \mathrm{d}v = \int_0^\infty \int_0^\infty v^{\frac{n}{2}-1} \left(\frac{1}{1+v}\right)^{\frac{n+1}{2}} t^{\frac{n-1}{2}} e^{-t} \, \mathrm{d}t \, \mathrm{d}v \\
&= \begin{bmatrix} s = \frac{t}{1+v} \\ \mathrm{d}t = (1+v) \, \mathrm{d}t \end{bmatrix} = \int_0^\infty \int_0^\infty s^{\frac{n}{2}-1} e^{-(1+v)s} \, \mathrm{d}s \, \mathrm{d}v = \int_0^\infty s^{\frac{1}{2}} e^{-s} \int_0^\infty (sv)^{\frac{n}{2}-1} e^{-sv} \, \mathrm{d}v \, \mathrm{d}s \\
&= \left(\int_0^\infty s^{-\frac{1}{2}} e^{-s} \, \mathrm{d}s\right) \left(\int_0^\infty w^{-\frac{1}{2}} e^{-w} \, \mathrm{d}w\right) = \Gamma(\frac{n}{2})\Gamma(\frac{1}{2}).
\end{aligned}$$

Lemma 3.3. *Suppose K is bounded and $K(x) = \mathcal{O}(\frac{1}{|x|^{n+a}})$ as $n \to \infty$ for some $a > 0$. Let $K_\epsilon(x) = \frac{1}{\epsilon^n}K(\frac{x}{\epsilon})$. Then $(f * K_\epsilon)(x) \xrightarrow{\epsilon \to 0^+} f(x)\int_{\mathbb{R}^n} K(y) \, \mathrm{d}y$ at every Lebesgue point of f. In particular, if $\int_{\mathbb{R}^n} K(y) \, \mathrm{d}y = 1$, then K is a summability kernel and $f * K_\epsilon \xrightarrow{\epsilon \to 0^+} f$ almost everywhere*

Proof. It is easy to see that $K(x) = \mathcal{O}(\frac{1}{|x|^{n+a}})$ along with the boundedness of K combines to the estimate

$$K(x) \leqslant \frac{C}{1 + |x|^{n+a}}.$$

Define $F(r) = \int_{|y| \leqslant r} |f(x-y) - f(x)| \, \mathrm{d}y$. For a Lebesgue point x, given $\eta > 0$ there is some $\delta > 0$ such that $|F(r)| \leqslant \eta r$ when $r < \delta$. We now write

$$\begin{aligned}\left|(f * K_\epsilon)(x) - f(x)\int_{\mathbb{R}^n} K(y)\,\mathrm{d}y\right| &\leqslant \int_{\mathbb{R}^n} |f(x-y) - f(x)| \, |K(y)| \, \mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^n} |f(x-y) - f(x)| \frac{C\epsilon^a}{\epsilon^{n+a} + |y|^{n+a}} \, \mathrm{d}y = I.\end{aligned}$$

We split I into

$$I_1 = \int_{|t| \leqslant \epsilon} |f(x-y) - f(x)| \frac{C\epsilon^a}{\epsilon^{n+a} + |y|^{n+a}} \, \mathrm{d}y \ , \ I_2 = \int_{\epsilon < |x| < \delta} |f(x-y) - f(x)| \frac{C\epsilon^a}{\epsilon^{n+a} + |y|^{n+a}} \, \mathrm{d}y \ ,$$

$$I_3 = \int_{|x| \geqslant \delta} |f(x-y) - f(x)| \frac{C\epsilon^a}{\epsilon^{n+a} + |y|^{n+a}} \, \mathrm{d}y.$$

First, we have

$$I_1 \leqslant \frac{C\epsilon^a}{\epsilon^{n+a}} \int_{|t| \leqslant \epsilon} |f(x-y) - f(x)| \, \mathrm{d}y \xrightarrow{\epsilon \to 0^+} 0$$

when x is a Lebesgue point. For I_2, we have

$$\begin{aligned} I_2 = C\epsilon^a \int_{\epsilon < |x| < \delta} \frac{|f(x-y) - f(x)|}{|y|^a} \, \mathrm{d}y = C\epsilon^a \int_\epsilon^\delta \frac{\mathrm{d}F(r)}{r^{n+a}} &= C\epsilon^a \left(\left.\frac{F(r)}{r^{n+a}}\right|_\epsilon^\delta + (n+a) \int_\epsilon^\delta \frac{F(r)}{r^{n+a+1}} \, \mathrm{d}r \right) \\ &\leqslant C\epsilon^a \frac{F(\delta)}{r^{n+a}} + C\epsilon^a (n+a) \int_\epsilon^\infty \frac{\eta r^n}{r^{n+a+1}} \, \mathrm{d}r \xrightarrow{\epsilon \to 0^+} \frac{C\eta(n+a)}{a}. \end{aligned}$$

Lastly, for any $\delta > 0$,

$$C\epsilon^a \int_{|x| \geqslant \delta} \frac{|f(x-y) - f(x)|}{|y|^{n+a}} \, \mathrm{d}y \leqslant \frac{2C \, \|f\|_1 \, \epsilon^a}{\delta^{n+a}} \xrightarrow{\epsilon \to 0^+} 0.$$

Hence $\limsup_{\epsilon \to 0^+} I \leqslant \frac{C\eta(n+a)}{a}$. Since η was arbitrary the result follows. ☐

For $f \in L^p(\mathbb{R}^n), 1 \leqslant p < \infty$, define $u(x,t) = f * P_t(x)$. By Lemma 3.3, we have $f * P_t(x) \xrightarrow{t \to 0^+} f(x)$ almost everywhere. The reader may verify that $P_t(x)$ and $u(x,t)$ are harmonic on upper half space $\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$.

Recall that for $u \in C^2(\mathbb{R}^n)$, we have $\nabla \times \nabla u = 0$, and if u is harmonic we also have $\nabla \cdot \nabla u = 0$. Hence $\nabla u = (u_1, ..., u_n)$ satisfies the generalized Cauchy-Riemann equations,

$$\begin{cases} \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \\ \frac{\partial u_1}{\partial x_1} + ... + \frac{\partial u_n}{\partial x_n} = 0 \end{cases}. \tag{63}$$

If $(u_1, ..., u_n)$ satisfies these equations it is called a system of conjugate functions. In the case $n = 2$ we have seen that $(u(x,t), v(x,t)) = ((f * P_t)(x), (f * Q_t)(x))$ is a system of conjugate functions. We are interested in a set of functions $v_1, ..., v_n$ such that $(u, v_1, ..., v_n)$ is a system of conjugate functions, where $u(x,t) = f * P_t(x)$. The reader may verify this (under suitable conditions on f) for the functions

$$v_j(x,t) = f * Q_{t,j}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}} \, \mathrm{d}y, j = 1, ..., n, \tag{64}$$

where $Q_{t,j}, j = 1, ..., n$ are the conjugate Poisson kernels on $\mathbb{R}^n$. We are now interested in the boundary values of each $f * Q_{t,j}$. Informally taking the limit will lead us to the following definition.

Definition 3.4. *The Riesz transforms $\mathcal{R}_j, j \in \{1, 2, ..., n\}$, are defined by*

$$\mathcal{R}_j f(x) = \text{p.v.} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}} \, \mathrm{d}y.$$

Note that when $n = 1$ this is just the Hilbert transform on $\mathbb{R}$. In the next section we prove that $\mathcal{R}_j$ exists as a norm limit in $L^p(\mathbb{R}^n), 1 < p < \infty$, and that this operator is continuous. For now we prove the pointwise equiconvergence with the conjugate Poisson integral. Note that, by Theorem 3.1, this shows that $f * Q_\epsilon \xrightarrow{\epsilon \to 0^+} \mathcal{H}f$ almost everywhere in $\mathbb{R}$.

Theorem 3.5. *Let $f \in L^p(\mathbb{R}^n), 1 \leqslant p < \infty$. Then for almost every x,*

$$\lim_{\epsilon \to 0^+} \left(\int_{|y| \geqslant \epsilon} f(x-y) \frac{y_j}{|y|^{n+1}} \, \mathrm{d}y - \int_{\mathbb{R}^n} f(x-y) \frac{y_j}{(\epsilon^2 + |y|^2)^{\frac{n+1}{2}}} \, \mathrm{d}y \right) = 0.$$

Proof.

$$\frac{y_j}{|y|^{n+1}} \mathbb{1}_{\{|x| \geqslant \epsilon\}}(y) - \frac{y_j}{(\epsilon^2 + |y|^2)^{\frac{n+1}{2}}} = \frac{1}{\epsilon^n} K\left(\frac{y}{\epsilon}\right),$$

where

$$K(y) = -\frac{y_j}{(1+|y|^2)^{\frac{n+1}{2}}} \mathbb{1}_{\{|x|<1\}}(y) + \left(\frac{y_j}{|y|^{n+1}} - \frac{y_j}{(1+|y|^2)^{\frac{n+1}{2}}} \right) \mathbb{1}_{\{|x| \geqslant 1\}}(y).$$

For $|y| \geqslant 1$ we have

$$|K(y)| = \frac{|y_j| \left(\left(1 + |y|^2\right)^{\frac{n+1}{2}} - |y|^{n+1} \right)}{|y|^{n+1} (1+|y|^2)^{\frac{n+1}{2}}}.$$

By applying Binomial Theorem with $\sqrt{a+b} - \sqrt{b} \leqslant \sqrt{a}$ on the numerator we get

$$|K(y)| \leqslant \frac{|y| \sqrt{1 + |y|^2 + ... + |y|^{2n}}}{|y|^{n+1} (1+|y|^2)^{\frac{n+1}{2}}} \leqslant \frac{\sqrt{n+1}}{(1+|y|^2)^{\frac{n+1}{2}}}.$$

Thus K satisfies the hypothesis of Lemma 3.3, and since K is odd we have $\int_{\mathbb{R}^n} K(y) \, \mathrm{d}y = 0$, which gives us the result. □

Boundedness for Odd Kernels

We generalize the Riesz transforms to a larger class of singular integral operators of the type $\mathcal{T}f = \text{p.v.} f * K$, where f are functions on $\mathbb{R}^n$ and the function K, defined on $\mathbb{R}^n - \{0\}$, is called the kernel of $\mathcal{T}$. We will assume that K is positively homogeneous of degree $-n$, that is, $K(\lambda x) = \lambda^{-n} K(x)$ for all $\lambda > 0$. K is uniquely determined by its values on the unit sphere $\Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, since $K(x) = \frac{1}{|x|^n} K(\frac{x}{|x|})$. Set $\Omega = K|_{\Sigma^{n-1}}$. Then our singular integrals are of the following form,

$$\mathcal{T}f = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(\frac{y}{|y|})}{|y|^n} \, \mathrm{d}y, \tag{65}$$

along with the truncated integrals,

$$\mathcal{T}_\epsilon f = \int_{|y| \geqslant \epsilon} f(x-y) \frac{\Omega(\frac{y}{|y|})}{|y|^n} \, \mathrm{d}y. \tag{66}$$

We will require that $\Omega \in L^1(\Sigma^{n-1})$ and has mean value zero, i.e. $\int_{\Sigma^{n-1}} \Omega(y') dy' = 0$. The mean value zero condition turns out to be necessary for the limit of $\mathcal{T}_\epsilon f$ to exist even in $C_c^\infty(\mathbb{R}^n)$. To see this, let $f(x) = 1$ for $|x| \leqslant 2$. For $|x| < 1$ we have

$$\int_{\epsilon \leqslant |y| < 1} f(x-y) \frac{\Omega(\frac{y}{|y|})}{|y|^n} \, \mathrm{d}y = \int_{\epsilon \leqslant |y| < 1} \frac{\Omega(\frac{y}{|y|})}{|y|^n} \, \mathrm{d}y = \int_\epsilon^1 \frac{1}{r} \int_{\Sigma^{n-1}} \Omega(y') \, \mathrm{d}y' \, \mathrm{d}r.$$

Clearly the last integral converges if and only if Ω has mean value 0.

We see that The Riesz transforms are of the form (65), and by the mean value zero condition the only such singular integrals in $\mathbb{R}$ are constant multiples of the Hilbert transform. Note that for the Riesz transforms the kernel is an odd function. Using methods of Calderón and Zygmund, the existence and boundedness of singular integrals with odd kernel can be reduced to the case $n = 1$, so that Theorem 3.1 would give us the result.

Theorem 3.6. (Method of Rotations) *Suppose Ω is odd. Then $\|\mathcal{T}_\epsilon\|_{L^p(\mathbb{R}^n)} \leqslant \frac{\pi}{2}\|\Omega\|_1 \|\mathcal{H}\|_{L^p(\mathbb{R})}$ for all $\epsilon > 0$ and $1 < p < \infty$.*

Proof. First, since Ω is odd, we can write

$$\begin{aligned}\mathcal{T}_\epsilon f(x) = \int_{\Sigma^{n-1}} \Omega(y') \int_\epsilon^\infty \frac{f(x-ry')}{r}\,\mathrm{d}r\,\mathrm{d}y' &= \int_{\Sigma^{n-1}} \Omega(y') \int_\epsilon^\infty \frac{f(x-ry') - f(x+ry')}{2r}\,\mathrm{d}r\,\mathrm{d}y' \\ &= \int_{\Sigma^{n-1}} \Omega(y') \int_{|r|\geqslant\epsilon} \frac{f(x-ry')}{2r}\,\mathrm{d}r\,\mathrm{d}y'.\end{aligned}$$

Let $L(y') = \{ty' : t \in \mathbb{R}\}$. Each $x \in \mathbb{R}^n$ is written uniquely as $x = z + ty'$ where $z \in L(y')^\perp$. Thus $x - ry' = z + (t-r)y'$. Let $g(r, z, y') = f(z + ry')$. Then we see that the last integral is the Hilbert transform of πg in the r-variable. Hence,

$$\mathcal{T}_\epsilon f(x) = \frac{\pi}{2}\int_{\Sigma^{n-1}} \Omega(y')\mathcal{H}_\epsilon g(t, z, y')\,\mathrm{d}y'.$$

By Minkowski's inequality (see Appendix II),

$$\begin{aligned}\|\mathcal{T}_\epsilon f\|_p &\leqslant \frac{\pi}{2}\int_{\Sigma^{n-1}} |\Omega(y')| \left(\int_{\mathbb{R}^n} \left|\mathcal{H}_\epsilon g(t,z,y')\right|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \mathrm{d}y' \\ &= \frac{\pi}{2}\int_{\Sigma^{n-1}} |\Omega(y')| \left(\int_{L(y')^\perp}\int_{\mathbb{R}} \left|\mathcal{H}_\epsilon g(t,z,y')\right|^p \,\mathrm{d}t\,\mathrm{d}z\right)^{\frac{1}{p}} \mathrm{d}y' \\ &\leqslant \frac{\pi}{2}\int_{\Sigma^{n-1}} |\Omega(y')| \left(\int_{L(y')^\perp} C_p^p \int_{\mathbb{R}} \left|f(z+ty')\right|^p \,\mathrm{d}t\,\mathrm{d}z\right)^{\frac{1}{p}} \mathrm{d}y' = \frac{\pi}{2}\|\Omega\|_1 C_p \|f\|_p.\end{aligned}$$

□

Theorem 3.7. *Suppose Ω is odd. Then there is a bounded linear operator $\mathcal{T} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), 1 < p < \infty$, such that $\|\mathcal{T}_\epsilon f - \mathcal{T} f\|_p \xrightarrow{\epsilon\to 0^+} 0$ and $\|\mathcal{T}\|_p \leqslant \frac{\pi}{2}\|\Omega\|_1 \|\mathcal{H}\|_p$.*

Proof. We begin by proving it for functions g in the dense subset $C_c^1(\mathbb{R}^n)$. Let R be such that the support of g lies in the ball of of radius R centered at the origin. For $|y| \leqslant R$ and $|x| > 2R$ we have $|x-y|^n \geqslant |x|^n - |y|^n \geqslant \frac{1}{2}|x|^n > R$, so

$$|\mathcal{T}_\epsilon g(x)| \leqslant \int_{|x-y|\geqslant\epsilon} |g(y)| \frac{\left|\Omega(\frac{x-y}{|x-y|})\right|}{|x-y|^n}\,\mathrm{d}y \leqslant \int_{|y|\leqslant R} 2|g(y)| \frac{\left|\Omega(\frac{x-y}{|x-y|})\right|}{|x|^n}\,\mathrm{d}y \leqslant \frac{C_1}{|x|^n}$$

for some constant C_1. Now pick $h \in C_c(\mathbb{R}^n)$ such that $h(x) = 1$ in a neighbourhood around 0 and $h(x) = h(y)$ when $|x| = |y|$. Then

$$\int_{|y|\geqslant\epsilon} h(y)\frac{\Omega(\frac{y}{|y|})}{|y|^n}\,\mathrm{d}y = 0.$$

Hence,

$$\mathcal{T}_\epsilon g(x) = \int_{|y|\geqslant\epsilon} (g(x-y) - g(x)h(y))\frac{\Omega(\frac{y}{|y|})}{|y|^n}\,\mathrm{d}y.$$

For $|x| \leqslant 2R$ we have $|g(x-y) - g(x)h(y)| \leqslant C_2|y|$ for some constant C_2. To see this, first note that it holds outside any given neighbourhood around 0, since everything is bounded. Near the origin (where $h(y) = 1$) it follows by applying Mean-Value Theorem to $\varphi(t) = g(x - ty)$. Since g

and h have compact support we now have $|\mathcal{T}_\epsilon g(x)| \leqslant C_3$ for some constant C_3. If we combine this with the first estimate we get

$$|\mathcal{T}_\epsilon g(x)| \leqslant \frac{C}{1+|x|^n}$$

for some constant C. Clearly $\mathcal{T}_\epsilon g(x)$ converges to the finite integral

$$\mathcal{T}g(x) = \int_{\mathbb{R}^n} (g(x-y) - g(x)h(y)) \frac{\Omega(\frac{y}{|y|})}{|y|^n} \, \mathrm{d}y,$$

and from the estimate we get $\|\mathcal{T}_\epsilon g - \mathcal{T}g\|_p \xrightarrow{\epsilon \to 0^+} 0, 1 < p < \infty$, by Dominated Convergence Theorem, since $(1+|x|^n)^{-1}$ lies in $L^p(\mathbb{R}^n)$ when $p > 1$. From here all we need to do is prove that $\mathcal{T}_\epsilon$ is a Cauchy sequence, which follows by a familiar density argument using Theorem 3.6. $\square$

We are now familiar with the boundedness results for odd kernels. Similar results for arbitrary kernels will be proved in later sections. It suffices to study even kernels, since any function on Σ^{n-1} can be decomposed into its even and odd part $\Omega = \Omega_O + \Omega_E$, where

$$\Omega_O = \frac{\Omega(x) - \Omega(-x)}{2} \ , \ \Omega_E = \frac{\Omega(x) + \Omega(-x)}{2}.$$

The Fourier Transform

We now turn to the Fourier transform. It is the analogue to the Fourier coefficients for functions in $L^1(\mathbb{R}^n)$. Just like the Fourier coefficients are deeply connected to the Hilbert transform on $\mathbb{T}$ we expect the Fourier transform to have connections with the singular integrals we have just discussed.

Definition 3.8. *For $f \in L^1(\mathbb{R}^n)$ we define*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx. \tag{67}$$

Then the Fourier transform of f is the linear operator $\mathcal{F}$ defined by $\mathcal{F}f = \widehat{f}$.

Note that, unlike for the Fourier coefficients, this does define the Fourier transform for functions in $L^p(\mathbb{R}^n)$, since $\mathbb{R}^n$ has infinite measure. We begin our study of Fourier transforms by listing some useful calculations.

Lemma 3.9.

(i) $\widehat{\mathbb{1}_{|x|<h}}(\xi) = \frac{\sin 2\pi h\xi}{\pi\xi}, x \in \mathbb{R}$.

(ii) *If $f(x) = e^{-2\pi|x|}, x \in \mathbb{R}$, then $\widehat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+x^2}$.*

(iii) *If $g(x) = f(\lambda x), \lambda \in \mathbb{R}$. Then $\widehat{g}(\xi) = \frac{1}{|\lambda|^n}\widehat{f}(\frac{\xi}{\lambda})$.*

(iv) *If $f(x) = f_1(x_1)...f_n(x_n)$, then $\widehat{f}(\xi) = \widehat{f_1}(\xi_1)...\widehat{f_n}(\xi_n)$.*

(v) $\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}$.

(vi) *If $f_t(x) = f(x-t)$, then $\widehat{f_t}(\xi) = e^{-2\pi i\xi\cdot t}\widehat{f}(\xi)$.*

(vii) *If $g(x) = f(x)e^{2\pi i x\cdot t}$ then $\widehat{g}(\xi) = \widehat{f}(\xi - t)$.*

(i) shows that $\widehat{f}$ is not necessarily integrable, by an argument similar to the proof of lemma 1.15. $\mathcal{F}$ does however map $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. More precisely, the following theorem shows that it maps $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$.

Theorem 3.10. *Let $f \in L^1(\mathbb{R}^n)$.*

(i) *$\widehat{f}$ is bounded and $\left|\widehat{f}\right| \leqslant \|f\|_1$.*

(ii) *$\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

(iii) $\widehat{f}$ *is uniformly continuous.*

Proof. (i) is obvious. (ii) is just the Riemann-Lebesgue Lemma when $n = 1$. For arbitrary n the proof is similar; first verify for step functions, then use a density argument. The result for step functions is reduced to the one-dimensional case by Lemma 3.9 (iv), since $\mathbb{1}_{\{a_i \leqslant |x_i| \leqslant b_i\}} = \prod_i \mathbb{1}_{[a_i,b_i]}$. From there the proof is the exact same as for Lemma 1.4. To prove (iii), pick $\epsilon > 0$ and R such that

$$\int_{|x| \geqslant R} |f(x)| \, \mathrm{d}x < \epsilon/4.$$

We now have

$$\left|\widehat{f}(\xi + h) - \widehat{f}(\xi)\right| = \left|\int_{\mathbb{R}^n} f(x) \left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot\xi}\right) \mathrm{d}x\right| \leqslant \int_{\mathbb{R}^n} |f(x)| \left|e^{-2\pi i x\cdot h} - 1\right| \mathrm{d}x.$$

Split the last integral into

$$I_1 = \int_{|x|<R} |f(x)| \left|e^{-2\pi i x\cdot h} - 1\right| \mathrm{d}x, \; I_2 = \int_{|x| \geqslant R} |f(x)| \left|e^{-2\pi i x\cdot h} - 1\right| \mathrm{d}x.$$

Clearly $I_2 < \epsilon/2$. Now choose $\delta = \epsilon/(4\pi R \|f\|_1)$. By Mean-Value Theorem and Cauchy-Schwarz inequality we get

$$\left|e^{-2\pi i x\cdot h} - 1\right| \leqslant 2\pi |x| |h|.$$

Hence, for $|h| < \delta$, we get $I_1 < 2\pi R\delta \|f\|_1 < \epsilon/2$ so we are done. ☐

Lemma 3.11. *Let $f, g \in L^1(\mathbb{R}^n)$. Then $\widehat{f}g, f\widehat{g} \in L^1(\mathbb{R}^n)$ and*

(i) $\int_{\mathbb{R}^n} \widehat{f}g = \int_{\mathbb{R}^n} f\widehat{g}$.

(ii) $\widehat{f * g} = \widehat{f}\widehat{g}$.

Proof. $\widehat{f}g$ and $f\widehat{g}$ lie in $L^1(\mathbb{R}^n)$ by Theorem 3.10 (i), and both identities are trivial consequences of Fubini's theorem. ☐

(ii) and (iii) of Lemma 3.9 shows that the Fourier transform of $e^{-2\pi\epsilon|x|}$ is the Poisson kernel in $\mathbb{R}$. One might guess that the same is true in $\mathbb{R}^n$. It turns out to be the case, but the computation is a bit complicated. However, the result is important, so we end this section by presenting a proof anyway.

Lemma 3.12.

$$\int_{\mathbb{R}^n} e^{-2\pi|x|} e^{-2\pi i\xi\cdot x} \, \mathrm{d}x = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1 + |\xi|^2)^{\frac{n+1}{2}}}.$$

Proof. Using calculus of residues on $\frac{e^{iaz}}{1+z^2}$ we can obtain

$$e^{-a} = \frac{2}{\pi} \int_0^\infty \frac{\cos ax}{1 + x^2} \, \mathrm{d}x, \qquad a > 0.$$

Using this and Fubini's Theorem we have

$$\begin{aligned}
e^{-a} &= \frac{2}{\pi} \int_0^\infty \frac{\cos ax}{1 + x^2} \, \mathrm{d}x = \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty e^{-(1+x^2)y} \, \mathrm{d}y\right) \cos ax \, \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^\infty e^{-y} \left(\int_0^\infty e^{-x^2 y} \cos ax \, \mathrm{d}x\right) \mathrm{d}y = \frac{1}{\pi} \int_0^\infty e^{-y} \left(\int_{-\infty}^\infty e^{-x^2 y} e^{iax} \, \mathrm{d}x\right) \mathrm{d}y \\
&= \frac{1}{\pi} \int_0^\infty e^{-y} \left(\int_{-\infty}^\infty e^{-(x\sqrt{y} - i\frac{a}{2\sqrt{y}})^2} e^{-\frac{a^2}{4y}} \, \mathrm{d}x\right) \mathrm{d}y = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} e^{-\frac{a^2}{4y}} \, \mathrm{d}y.
\end{aligned}$$

Now let $a = 2\pi |x|$ and substitute $e^{-2\pi|x|}$ with the last integral to get

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2\pi|x|} e^{-2\pi i\xi\cdot x} \, \mathrm{d}x &= \int_{\mathbb{R}^n} \left(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} e^{-\frac{4\pi^2|x|^2}{4y}} \, \mathrm{d}y\right) e^{-2\pi i\xi\cdot x} \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} \left(\int_{\mathbb{R}^n} e^{-\frac{\pi^2|x|^2}{y}} e^{-2\pi i\xi\cdot x} \, \mathrm{d}x\right) \mathrm{d}y.
\end{aligned}$$

The inner integral of the last integral is the Fourer transform of $e^{-\pi^2|x|^2/y}$. By Lemma 3.9 (iv) the calculation reduces to the one-dimensional case, and similar to a calculation we just did, the Fourier transform of $e^{-\pi^2 x_j^2/y}$ is calculated to $\sqrt{\frac{y}{\pi}}e^{-y\xi_j^2}$. Now we get

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2\pi|x|}e^{-2\pi i\xi\cdot x}\,\mathrm{d}x &= \frac{1}{\pi}\int_0^\infty \frac{e^{-y}}{\sqrt{y}}\left(\sqrt{\frac{y}{\pi}}\right)^n e^{-y|\xi|^2}\,\mathrm{d}y = \frac{1}{\pi^{\frac{n+1}{2}}}\int_0^\infty y^{\frac{n-1}{2}}e^{-y(1+|\xi|^2)}\,\mathrm{d}y \\ &= \frac{1}{\pi^{\frac{n+1}{2}}}\frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}\int_0^\infty t^{\frac{n-1}{2}}e^{-t}\,\mathrm{d}t = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.\end{aligned}$$

□

The Inversion Formula

We have seen that an integrable function on $\mathbb{T}$, under certain conditions, can be recovered from its Fourier coefficients by its Fourier series. In particular, we saw that the Fourier series of $f \in L^1(\mathbb{T})$ was summable to f in some sense. In similar fashion, we will prove that f can be recovered from the Fourier transform by

$$f(x) \sim \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi. \tag{68}$$

This integral expression is problematic, since we saw that $\widehat{f}$ is not integrable in general. To deal with this we introduce the following summability method.

Definition 3.13. *The integral $\int_{\mathbb{R}^n} f$ is said to be Abel-summable to the value I if*

$$\lim_{\epsilon\to 0+}\int_{\mathbb{R}^n} f(x)e^{-2\pi\epsilon|x|}\,\mathrm{d}x = I \tag{69}$$

and the integrals in (69) *are called the Abel means of $\int_{\mathbb{R}^n} f$*

By Lemma 3.10 (i) the Abel means of the integral in (68) is finite for all $f \in L^1(\mathbb{R}^n)$. Just like with the summability methods for Fourier series, the summability of the integral in (68) is reduced to the convergence of a convolution $f * K_\epsilon$. By Lemma 3.9 (vii), Lemma 3.11 (i) and Lemma 3.12,

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i\xi\cdot x}e^{-2\pi\epsilon|\xi|}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \widehat{f_{-x}}(\xi)e^{-2\pi\epsilon|\xi|}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} f(\xi+x)P_\epsilon(\xi)\,\mathrm{d}\xi = (f * P_\epsilon)(x), \tag{70}$$

and by Lemma 3.3, Theorem 1.12 and Theorem 1.14 this results in the following.

Theorem 3.14. *The integral in* (68) *is Abel-summable to f in L^1-norm, pointwise almost everywhere, and pointwise at every point of continuity. For $n = 1$, if the left and right limits exist then the integral is Abel-summable to the mean of the two limits.*

Corollary 3.15. *If $\widehat{f} = \widehat{g}$ then $f = g$ almost everywhere. If f and g are continuous at $x_0 \in \mathbb{R}^n$ then $f(x_0) = g(x_0)$.*

The integral in (68) is finite if and only if $\widehat{f}$ is integrable, and if it is finite then its Abel means converge to this integral, by Dominated Convergence Theorem. Together with the above theorem this gives us the inversion theorem.

Theorem 3.16. *If $\widehat{f}$ is integrable then*

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi = f(x)$$

almost everywhere. Equality holds at every point of continuity and, for $n = 1$, it equals $\check{f}(x_0) = \lim_{t\to 0^+}\frac{f(x_0+t)+f(x_0-t)}{2}$ for each $x_0 \in R$ where the limit exists.

Corollary 3.17. *If $\widehat{f} \in L^1(\mathbb{R}^n)$ and f is continuous at 0, then*

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)\,\mathrm{d}\xi = f(0).$$

We can generalize summability of integrals by replacing $e^{-2\pi\epsilon|x|}$ by $H(\epsilon|x|)$ where H is continuous at 0, $H(0) = 1$, and $\widehat{H}$ is integrable. The point of these conditions is motivated by the fact that we get a convolution with $\widehat{H(\epsilon|x|)}$ in (70), which we would like to be a summability kernel. By Lemma 3.3 and Lemma 3.9 (iii) this holds when its integral is 1, which is true when H is continuous at 0 and $H(0) = 1$, by Corollary 3.17. Examples of such H include the function $H(x) = e^{-\pi x^2}$. This function is actually a fixed point of the Fourier transform. The summability method corresponding to this function is called Gauss-Weierstrass summability. Another method in the one-dimensional case is Cesàro summability, that is, the convergence of

$$\sigma_R f(x) = \frac{1}{R}\int_0^R \int_{-t}^t \widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi\,\mathrm{d}t = (f * F_R)(x),$$

where the Fejér kernel F_R is given by

$$F_R(x) = \frac{1}{R}\int_0^R e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi = \frac{\sin^2 R\pi x}{R\pi^2 x^2}.$$

In this case H is the triangular function $H(x) = (1-|x|)\mathbb{1}_{|x|\leqslant 1}(x)$ and $\epsilon = \frac{1}{R}$. Since this H verifies all conditions for summability, the Inversion Theorem applies here as well.

Fourier Transforms in L^2

Theorem 3.18. *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widehat{f} \in L^2(\mathbb{R}^n)$ and*

$$\left\|\widehat{f}\right\|_2 = \|f\|_2\,.$$

More generally we have Plancherel's identity,

$$\left\langle \widehat{f}, \widehat{g}\right\rangle = \langle f, g\rangle\,.$$

Proof. Let $g(x) = \overline{f(-x)}$ and $h = f * g$. By Lemma 3.9 (iii) and (v), and Lemma 3.11 (ii), we have $\widehat{h} = \widehat{f}\widehat{g} = \widehat{f}\overline{\widehat{f}} = \left|\widehat{f}\right|^2$. We also have $h(0) = \int_{\mathbb{R}^n} f\overline{f} = \int_{\mathbb{R}^n} |f|^2$. Hence, if h satisfies the hypothesis of Corollary 3.17, then

$$\int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2\,\mathrm{d}\xi = h(0) = \int_{\mathbb{R}^n} |f(x)|^2\,\mathrm{d}x.$$

The continuity of h follows from Cauchy-Schwartz inequality and Lemma 1.13. To prove that $\widehat{h}$ is integrable, we use that

$$\int_{\mathbb{R}^n} \widehat{h}(x)e^{-2\pi\epsilon|x|}\,\mathrm{d}x = (h * P_\epsilon)(0) \xrightarrow{\epsilon\to 0^+} h(0).$$

Since h is non-negative, Fatou's Lemma applies to show that h is integrable. Hence the first identity is proved. The other one now follows by the polarization identity,

$$4\langle f, g\rangle = \|f+g\|_2^2 - \|f-g\|_2^2 + i\,\|f+ig\|_2^2 - i\,\|f-ig\|_2^2\,.$$

□

Although the Fourier transform $\mathcal{F}$ defined by (67) does not exist for all $f \in L^2(\mathbb{R}^n)$, we can extend it by continuity, since $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. That is, for $f \in L^2(\mathbb{R}^n)$ we choose a sequence of functions $f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $f_j \xrightarrow{j\to\infty} f$ in L^2-norm and define $\mathcal{F}f$ by the limit in norm of $\mathcal{F}f_j$. This limit exists since $\mathcal{F}f_j$ is Cauchy, by continuity, so it converges. To see that this is well-defined, pick another sequence $g_j \xrightarrow{j\to\infty} f$. Then $\mathcal{F}g_j$ is also Cauchy. The sequence $f_1, g_2, f_3, g_4, \ldots$ converges to f and so $\mathcal{F}f_1, \mathcal{F}g_2, \mathcal{F}f_3, \mathcal{F}g_4, \ldots$ is Cauchy and has limit $\mathcal{F}f$. Hence $\mathcal{F}g_j \xrightarrow{j\to\infty} \mathcal{F}f$.

Note that, by continuity of the norm, Plancherel's identity holds for all $f \in L^2(\mathbb{R}^n)$. We also have the following analogue to Lemma 3.11, whose proof follows by density arguments.

Lemma 3.19. *Let $f, g \in L^2(\mathbb{R}^n)$ and $h \in L^1(\mathbb{R}^n)$. Then $\widehat{f}g, f\widehat{g} \in L^1(\mathbb{R}^n)$, $f * h \in L^2(\mathbb{R}^n)$ and*

(i) $\int_{\mathbb{R}^n} \widehat{f} g = \int_{\mathbb{R}^n} f \widehat{g}$.

(ii) $\widehat{f * h} = \widehat{f}\widehat{h}$ *a.e.*

We would like to extend the inversion formula (68). We do this by defining the inverse Fourier transform for $f \in L^2(\mathbb{R}^n)$ by $\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$. We already know that $\mathcal{F}\mathcal{F}^{-1}f = f$ for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. For arbitrary $f \in L^2(\mathbb{R}^n)$, we have $f = \lim_{j\to\infty} f_j$ in L^2-norm for some sequence $f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\mathcal{F}\mathcal{F}^{-1}f = \mathcal{F}\mathcal{F}^{-1}\lim_{j\to\infty} f_j = \lim_{j\to\infty} \mathcal{F}\mathcal{F}^{-1}f_j = \lim_{j\to\infty} f_j = f$. We have proved the following.

Theorem 3.20. (Plancherel) *The Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$ and $\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$.*

For any sequence t_j such that $t_j \xrightarrow{j\to\infty} \infty$, let $f_j = f\mathbb{1}_{\{|t|\leqslant t_j\}}$ where $f \in L^2(\mathbb{R}^n)$. Then $f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $f_j \xrightarrow{j\to\infty} f$ in L^2-norm. Hence the Fourier transform and Fourier inversion are given by limits of the following truncated integrals,

$$\mathcal{F}_R f = \int_{|x|\leqslant R} f(x)e^{-2\pi i x\cdot\xi}dx \ , \ \mathcal{F}_R^{-1} f = \int_{|\xi|\leqslant R} f(\xi)e^{2\pi i \xi\cdot x}\,\mathrm{d}\xi.$$

Singular Integrals in L^2

We now return to our study of singular integrals. As always, the theory is simpler in L^2, since L^2 has the property of being a Hilbert space. The method will be to apply the Fourier transform, which we now know has very good properties in L^2. We begin by introducing the truncated kernels $K_{\epsilon,R}(x) = K(x)\mathbb{1}_{\epsilon\leqslant|x|\leqslant R}(x)$ and define $\mathcal{T}_{\epsilon,R}$ by $\mathcal{T}_{\epsilon,R}f = f * K_{\epsilon,R}$. Note that $K_{\epsilon,R} \in L^1(\mathbb{R}^n)$, so by Lemma 3.19 (ii) we have $\widehat{\mathcal{T}_{\epsilon,R}f} = \widehat{f}\widehat{K_{\epsilon,R}}$ for all $f \in L^2(\mathbb{R}^n)$, which is where the Fourier transform gets involved. From here we would like to conclude something similar to Theorem 2.2 using similar methods. To prove the desired boundedness properties the assumption $\Omega \in L^1(\Sigma^{n-1})$ will not be enough, but $\Omega \in L^r(\Sigma^{n-1})$ for some $r > 1$ will be. In fact, the minimal assumption needed for the proofs below is $\Omega \in L\log_+ L(\Sigma^{n-1}) = \left\{f : \int_{\Sigma^{n-1}} |f|\log_+|f| < \infty\right\}$, where $\log_+(x) = \log f(x)\mathbb{1}_{x\geqslant 1}(x)$ (of course we still assume that Ω has mean value zero).

Lemma 3.21. *Suppose $\Omega \in L\log_+ L(\Sigma^{n-1})$. Then $\left|\widehat{K_{\epsilon,R}}(\xi)\right| \leqslant C$ for some constant C independent of ϵ and R, and $\widehat{K_{\epsilon,R}} \to \widehat{K}$ as $\epsilon \to 0^+$ and $R \to \infty$, where*

$$\widehat{K}(\xi) = \int_{\Sigma^{n-1}} \Omega(y')\left(\log\frac{1}{\left|y'\cdot\frac{\xi}{|\xi|}\right|} - \frac{\pi i}{2}\operatorname{sgn}\left(y'\cdot\frac{\xi}{|\xi|}\right)\right)\mathrm{d}y', \qquad |\xi| > 0.$$

Proof.

$$\begin{aligned}\widehat{K_{\epsilon,R}}(\xi) = \int_{\epsilon\leqslant|x|\leqslant R} K(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x &= \int_{\Sigma^{n-1}} \Omega(y')\left(\int_\epsilon^R \frac{e^{-2\pi i(y'\cdot\frac{\xi}{|\xi|})|\xi|r}}{r}\,\mathrm{d}r\right)\mathrm{d}y' \\ &= \int_{\Sigma^{n-1}} \Omega(y')\left(\int_{\epsilon|\xi|}^{R|\xi|} \frac{e^{-2\pi i(y'\cdot\frac{\xi}{|\xi|})s}}{s}\,\mathrm{d}s\right)\mathrm{d}y' \\ &= \int_{\Sigma^{n-1}} \Omega(y')\left(\int_{\epsilon|\xi|}^{R|\xi|} \frac{e^{-2\pi i(y'\cdot\frac{\xi}{|\xi|})s} - \mathbb{1}_{(0,1)}(s)}{s}\,\mathrm{d}s\right)\mathrm{d}y'.\end{aligned}$$

From here we want to find a bound for the inner integral,

$$I_{\epsilon,R} = \int_{\epsilon|\xi|}^{R|\xi|} \frac{e^{-2\pi i(y'\cdot\frac{\xi}{|\xi|})s} - \mathbb{1}_{(0,1)}(s)}{s}\,\mathrm{d}s.$$

The claim is that

$$|I_{\epsilon,R}| \leqslant 2\log\frac{1}{\left|y'\cdot\frac{\xi}{|\xi|}\right|} + C$$

for some constant C whenever $y' \cdot \frac{\xi}{|\xi|}$ is non-zero. We assume $y' \cdot \frac{\xi}{|\xi|} > 0$. The other case then follows by looking at $\overline{I_{\epsilon,R}}$. We first consider the case $\epsilon |\xi| \leqslant 1 \leqslant R|\xi|$. Split $I_{\epsilon,R}$ into the two integrals

$$I_1 = \int_{\epsilon|\xi|}^{1} \frac{e^{-2\pi i (y' \cdot \frac{\xi}{|\xi|})s} - 1}{s} \,\mathrm{d}s, \; I_2 = \int_{1}^{R|\xi|} \frac{e^{-2\pi i (y' \cdot \frac{\xi}{|\xi|})s}}{s} \,\mathrm{d}s.$$

Using $\left|e^{ix} - 1\right| \leqslant |x|$ we immediately get $|I_1| \leqslant 2\pi$. For the second integral, we write

$$I_2 = \int_{y' \cdot \frac{\xi}{|\xi|}}^{R|\xi|\left(y' \cdot \frac{\xi}{|\xi|}\right)} \frac{e^{-2\pi i t}}{t} \,\mathrm{d}t.$$

If $R|\xi| \, y' \cdot \frac{\xi}{|\xi|} \leqslant 1$ then

$$|I_2| \leqslant \int_{y' \cdot \frac{\xi}{|\xi|}}^{1} \frac{1}{t} \,\mathrm{d}t = \log \frac{1}{\left|y' \cdot \frac{\xi}{|\xi|}\right|} + C,$$

and if $R|\xi| \left(y' \cdot \frac{\xi}{|\xi|}\right) > 1$ we have

$$|I_2| \leqslant \int_{y' \cdot \frac{\xi}{|\xi|}}^{1} \frac{1}{t} \,\mathrm{d}t = \log \frac{1}{\left|y' \cdot \frac{\xi}{|\xi|}\right|} + \left| \int_{1}^{R|\xi|\left(y' \cdot \frac{\xi}{|\xi|}\right)} \frac{e^{-2\pi i t}}{t} \,\mathrm{d}t \right|.$$

As we have previously seen, integration by parts shows that the last integral is bounded independent of $R|\xi| \, y' \cdot \frac{\xi}{|\xi|}$. Now consider the case $\epsilon |\xi| > 1$. Then we write

$$I_{\epsilon,R} = \left(\int_{1}^{\epsilon|\xi|} - \int_{1}^{R|\xi|} \right) \left(\frac{e^{-2\pi i (y' \cdot \frac{\xi}{|\xi|})s}}{s} \,\mathrm{d}s \right),$$

and the bounds here are the same as for I_2. The last case, $1 > R|\xi|$, is of similar nature, so the claim is proved. We now have

$$\left|\widehat{K_{\epsilon,R}}(\xi)\right| \leqslant \int_{\Sigma^{n-1}} \left|\Omega(y')\right| \left(2 \log \frac{1}{\left|y' \cdot \frac{\xi}{|\xi|}\right|} + C \right) \mathrm{d}y' \leqslant 2 \int_{\Sigma^{n-1}} \left|\Omega(y')\right| \frac{1}{\left|y' \cdot \frac{\xi}{|\xi|}\right|} \,\mathrm{d}y'.$$

To see that the last integral is finite we use the inequality $ab \leqslant \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,\mathrm{d}x$, where f is strictly increasing, $f(0) = 0$ and $a, b \geqslant 0$ (this is geometrically evident). Let $f(x) = \log(1 + x)$. Then we get $ab < a \log(1 + a) + e^b$. Let $a = |\Omega(y')|$ and $b = \frac{1}{2} \log \frac{1}{\left|y' \cdot \frac{\xi}{|\xi|}\right|}$. This gives us

$$\left|\Omega(y')\right| \log \frac{1}{\left|y' \cdot \frac{\xi}{|\xi|}\right|} \leqslant 2 \left|\Omega(y')\right| \log 1 + \left|\Omega(y')\right| + \frac{2}{\sqrt{\left|y' \cdot \frac{\xi}{|\xi|}\right|}}.$$

Since both terms are integrable the first part is proved. Now we want to calculate $\lim_{\epsilon \to 0^+} \lim_{R \to \infty} \widehat{K_{\epsilon,R}}(\xi)$. We have

$$\widehat{K_{\epsilon,R}}(\xi) = \int_{\Sigma^{n-1}} \Omega(y') \left(\int_{\epsilon|\xi|}^{R|\xi|} \frac{e^{-2\pi i (y' \cdot \frac{\xi}{|\xi|})s}}{s} \,\mathrm{d}s \right) \mathrm{d}y'.$$

Express the inner integral by its real and imaginary parts $A_{\epsilon,R} + iB_{\epsilon,R}$. $B_{\epsilon,R}$ converges to $-\frac{\pi}{2} \operatorname{sgn} y' \cdot \frac{\xi}{|\xi|}$, and we can put the limits inside the integral of $\Omega(y')B_{\epsilon,R}(y', \xi)$ by Dominated Convergence Theorem along with the bounds we have just established. For the real part, we first write

$$A_{\epsilon,R} = \int_{1}^{2\pi R|\xi|\left|y' \cdot \frac{\xi}{|\xi|}\right|} \frac{\cos t}{t} \,\mathrm{d}t + \int_{2\pi\epsilon|\xi|\left|y' \cdot \frac{\xi}{|\xi|}\right|}^{1} \frac{\cos t - 1}{t} \,\mathrm{d}t + \int_{2\pi\epsilon|\xi|\left|y' \cdot \frac{\xi}{|\xi|}\right|}^{1} \frac{1}{t} \,\mathrm{d}t.$$

The first and second integral converges independent of y', hence these terms will vanish in the final integral by the mean value zero condition. The last integral equals $\log \frac{1}{\left|y'\cdot\frac{\xi}{|\xi|}\right|} + \log\frac{1}{2\pi\epsilon|\xi|}$, and thus we have

$$\lim_{\epsilon\to 0^+}\lim_{R\to\infty}\widehat{K_{\epsilon,R}}(\xi) = \int_{\Sigma^{n-1}}\Omega(y')\left(\log\frac{1}{\left|y'\cdot\frac{\xi}{|\xi|}\right|} - \frac{\pi i}{2}\operatorname{sgn}\left(y'\cdot\frac{\xi}{|\xi|}\right)\right)\mathrm{d}y'.$$

□

After this computational nightmare of a lemma, we can now prove the boundedness of singular integrals in L^2.

Theorem 3.22. (Calderón-Zygmund) *Suppose $\Omega \in L\log_+ L(\Sigma^{n-1})$. Then there is a bounded linear operator $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, such that $\|\mathcal{T}_{\epsilon,R}f - \mathcal{T}f\|_2 \to 0$ as $\epsilon \to 0^+$ and $R \to \infty$, and $\|\mathcal{T}\| \leqslant \sup_{\xi\in\mathbb{R}^n}\left|\widehat{K}(\xi)\right|$.*

Proof. By Plancherel and Lemma 3.21,

$$\|\mathcal{T}_{\epsilon,R}f\|_2 = \left\|\widehat{\mathcal{T}_{\epsilon,R}f}\right\|_2 = \left\|\widehat{f}\widehat{K_{\epsilon,R}}\right\|_2 \leqslant C\left\|\widehat{f}\right\|_2 = C\|f\|_2.$$

Now define $\mathcal{T}f$ by $\widehat{\mathcal{T}f} = \widehat{f}\widehat{K}$. Since $\left|\widehat{K}\right| \leqslant C$, we have $\widehat{\mathcal{T}f} \in L^2(\mathbb{R}^n)$. Hence $\mathcal{T}f \in L^2(\mathbb{R}^n)$, by Plancherel's Theorem. The rest now follows since

$$\|\mathcal{T}_{\epsilon,R}f - \mathcal{T}f\|_2 = \left\|\widehat{\mathcal{T}_{\epsilon,R}f} - \widehat{\mathcal{T}f}\right\|_2 = \left\|\widehat{f}\left(\widehat{K_{\epsilon,R}} - \widehat{K}\right)\right\|_2 \xrightarrow{\epsilon\to 0^+, R\to\infty} 0,$$

by Dominated Convergence Theorem. □

Note that when Ω is odd, we have

$$\widehat{K}(\xi) = -\frac{\pi i}{2}\int_{\Sigma^{n-1}}\Omega(y')\operatorname{sgn}\left(y'\cdot\frac{\xi}{|\xi|}\right)\mathrm{d}y'.$$

In the case of the Riesz transforms, we have the following (compare with Theorem 1.30),

$$\widehat{\mathcal{R}_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\widehat{f}(\xi). \tag{71}$$

To see this, we need the following calculation.

Lemma 3.23.

$$-\frac{\pi i}{2}\int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'\cdot\frac{\xi}{|\xi|}\right)y'_j\,\mathrm{d}y' = -i\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\frac{\xi_j}{|\xi|}.$$

Proof. Choose an orthogonal matrix $A = (a_{jk})_{1\leqslant j,k\leqslant n}$ such that $Ae_j = \frac{\xi}{|\xi|}$. Now,

$$\begin{aligned}\int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'\cdot\frac{\xi}{|\xi|}\right)y'_j\,\mathrm{d}y' &= \int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'\cdot Ae_j\right)y'_j\,\mathrm{d}y' = \int_{\Sigma^{n-1}}\operatorname{sgn}\left(A^Ty'\cdot e_j\right)AA^Ty'_j\,\mathrm{d}y' \\ &= \int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'\cdot e_j\right)Ay'_j\,\mathrm{d}y' = \int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'_j\right)\left(a_{j1}y'_1 + \ldots + \frac{\xi_j}{|\xi|}y'_j + \ldots + a_{jn}y'_n\right)\mathrm{d}y'.\end{aligned}$$

Since y'_k has integral 0 on the hemispheres $y'_j > 0$ and $y'_j < 0$, we know that all terms except the jth one vanishes. That is,

$$\int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'_j\right)y'_k\,\mathrm{d}y' = \begin{cases}0 & , \quad j\neq k \\ \int_{\Sigma^{n-1}}\left|y'_j\right|\mathrm{d}y' & , \quad j = k\end{cases}.$$

Thus we have

$$\int_{\Sigma^{n-1}}\operatorname{sgn}\left(y'\cdot\frac{\xi}{|\xi|}\right)y'_j\,\mathrm{d}y' = \frac{\xi_j}{|\xi|}\int_{\Sigma^{n-1}}\left|y'_j\right|\mathrm{d}y'.$$

By symmetry,

$$\int_{\Sigma^{n-1}}\left|y'_j\right|\mathrm{d}y' = \int_{\Sigma^{n-1}}\left|y'_k\right|\mathrm{d}y', \qquad 1\leqslant j,k\leqslant n,$$

so it suffices to prove the result when $j = 1$. Now we write (for the first change of variables, see [6, Appendix D]),

$$\int_{\Sigma^{n-1}} |y_1'| \, \mathrm{d}y' = \int_{-1}^{1} \frac{|t|}{1-t^2} \int_{\sqrt{1-t^2}\Sigma^{n-2}} \mathrm{d}\varphi \, \mathrm{d}t$$
$$= \omega_{n-2} \int_{-1}^{1} |t| \, (1-t^2)^{\frac{n-2}{3}} \, \mathrm{d}t = \omega_{n-2} \int_{0}^{1} s^{\frac{n-2}{3}} \, \mathrm{d}s = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{2}{n-1} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})},$$

and the result is proved. □

Theorem 3.24. *Let $f, g \in L^2(\mathbb{R}^n)$ and $h \in L^1(\mathbb{R}^n)$.*

(i) $\sum_{j=1}^{n} \mathcal{R}_j^2 f = -f$.

(ii) $\sum_{j=1}^{n} \langle \mathcal{R}_j f, \mathcal{R}_j g \rangle = \langle f, g \rangle$.

(iii) $\mathcal{R}_j(f * h) = \mathcal{R}_j f * h$.

Proof. By (71) and Plancherel's Theorem,

$$\sum_{j=1}^{n} \widehat{\mathcal{R}_j^2 f}(\xi) = -\sum_{j=1}^{n} \frac{\xi_j^2}{|\xi|^2} \widehat{f}(\xi) = -\widehat{f}(\xi).$$

Now take the inverse Fourier transform to get (i). Similarly,

$$\sum_{j=1}^{n} \langle \mathcal{R}_j f, \mathcal{R}_j g \rangle = \sum_{j=1}^{n} \left\langle \widehat{\mathcal{R}_j f}, \widehat{\mathcal{R}_j g} \right\rangle = \left\langle \widehat{f}, \widehat{g} \right\rangle = \langle f, g \rangle .$$

Lastly, (iii) follows by a similar argument using Lemma 3.19 (ii) □

Corollary 3.25. *Let $f, g \in L^2(\mathbb{R})$ and $h \in L^1(\mathbb{R})$.*

(i) $\mathcal{H}^{-1} = -\mathcal{H}$.

(ii) $\langle \mathcal{H} f, \mathcal{H} g \rangle = \langle f, g \rangle$

(iii) $\int_{\mathbb{R}} \mathcal{H} f \mathcal{H} g = \int_{\mathbb{R}} f g$.

(iv) $\int_{\mathbb{R}} f \mathcal{H} g = -\int_{\mathbb{R}} g \mathcal{H} f$.

(v) $\mathcal{H}(f * h) = \mathcal{H} f * h$.

Combining (i) and (ii) of Corollary 3.24 we see that $\mathcal{H}$ is a unitary operator on $L^2(\mathbb{R})$ and $\mathcal{H}^* = -\mathcal{H}$. As we now have the L^2-results for the Hilbert transform on $\mathbb{R}$, Theorem 3.1 follows through the same methods we used to prove it for the Hilbert transform on $\mathbb{T}$ (For the details, see e.g. [8]). Hence the L^p-continuity of all singular integral operators with odd kernel now follows by Theorem 3.6 and Theorem 3.7. Note that (i) and (iii) of Theorem 3.24 and (i) and (v) of Corollary 3.25 can now be extended to $f \in L^p(\mathbb{R}^n), 1 < p < \infty$ by a density argument.

Boundedness for Even Kernels

We continue our study of singular integrals of the type $\mathcal{T} f = \text{p.v.} f * K$, where $K(x) = \Omega(\frac{x}{|x|}) / |x|^n$. We will assume that $\Omega \in L^r(\Sigma^{n-1})$ for some $r > 0$, and as always, Ω is assumed to have mean value zero on Σ^{n-1}. As we noted in the study of odd kernels, the boundedness of $\mathcal{T}$ can be reduced to the assumption that Ω is even, so we assume this. By Theorem 3.24 (i) (and its extension to $L^p, 1 < p < \infty$), we can write

$$\mathcal{T} = -\sum_{j=1}^{n} \mathcal{R}_j(\mathcal{R}_j \mathcal{T}). \tag{72}$$

Since $\mathcal{R}_j f$ is odd and $\mathcal{T}f$ is even, $\mathcal{R}_j\mathcal{T}f$ is odd and homogeneous of degree $-n$. If there is an odd kernel $\widetilde{K}_j$ that represents $\mathcal{R}_j\mathcal{T}$ as a singular integral in (65), then we can conclude boundedness in $L^p(\mathbb{R}^n), 1 < p < \infty$, by (72) and Theorem 3.7. Note that $K_\epsilon \in L^r(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} |K_\epsilon(x)|^r \,\mathrm{d}x = \left(\int_{\Sigma^{n-1}} \left|\Omega(y')\right|^r \,\mathrm{d}y'\right)\left(\int_\epsilon^\infty \frac{1}{\rho^{1+n(r-1)}}\,\mathrm{d}\rho\right) < \infty, \tag{73}$$

For $f \in L^1(\mathbb{R}^n)$ we get

$$\mathcal{R}_j(f * K_\epsilon) = f * \mathcal{R}_j K_\epsilon, \tag{74}$$

by Theorem 3.24 (iii). The following result provides a successful attempt at finding $\widetilde{K}_j$.

Lemma 3.26. *$\widetilde{K}_j$ exists as the L^∞-limit of $\mathcal{R}_jK_\epsilon$ on every closed subset of $\mathbb{R}^n - \{0\}$. It is odd, homogeneous of degree $-n$, and integrable on Σ^{n-1}.*

Proof. We first prove that $\mathcal{R}_jK_\epsilon$ is Cauchy. Pick ϵ and η and assume $0 < \epsilon < \eta$. Let $\mathcal{R}_{j,\delta}$ be the truncated integrals of $\mathcal{R}_j, \delta > 0$. For sufficiently small δ, we can write

$$\begin{aligned}\mathcal{R}_{j,\delta}K_\epsilon(x) - \mathcal{R}_{j,\delta}K_\eta(x) &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\epsilon<|y|<\eta} \frac{x_j - y_j}{|x-y|^{n+1}}K(y)\,\mathrm{d}y\\ &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\epsilon<|y|<\eta}\left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right)K(y)\,\mathrm{d}y,\end{aligned}$$

where the last equality follows from the fact that $\widetilde{K}_j$ has mean value zero on the shell $\epsilon < |y| < \eta$. By Theorem 3.7 we can choose a sequence $\delta_k \to 0$ such that $\mathcal{R}_{j,\delta_k} \to \mathcal{R}_j$ almost everywhere, so replacing $\mathcal{R}_{j,\delta}$ with $\mathcal{R}_j$ above we see that equality still holds almost everywhere. Applying Mean-Value Theorem to the integrand, there is some $0 < \theta < 1$ such that

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\left|\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right| \leqslant \frac{M_n\,|y|}{|x-\theta y|^{n+1}},$$

where $M_n = (n+2)\Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$. We can assume $\eta < |x|/2$. Then $|x - \theta y| > |x| - |y| > \frac{1}{2}|x|$. Combining this with the above estimate we get

$$\begin{aligned}|\mathcal{R}_jK_\epsilon(x) - \mathcal{R}_jK_\eta(x)| \leqslant \frac{2M_n}{|x|^{n+1}}\int_{\epsilon<|y|<\eta}|y|\,|K(y)|\,\mathrm{d}y &= \frac{2M_n}{|x|^{n+1}}\int_\epsilon^\eta\int_{\Sigma^{n-1}}\left|\Omega(y')\right|\,\mathrm{d}y'\,\mathrm{d}\rho\\ &\leqslant \frac{2M_n\,\|\Omega\|_1\,\eta}{|x|^{n+1}} \leqslant \frac{2M_n\,\|\Omega\|_1\,\eta}{|a|^{n+1}},\end{aligned}$$

where a is chosen so that $|x| \geqslant a$ (which is possible since x lies in a closed subset of $\mathbb{R}^n - \{0\}$). Hence $\mathcal{R}_jK_\epsilon$ is Cauchy in L^∞ over any closed subset of $\mathbb{R}^n - \{0\}$, and so it converges. Let K'_j be the pointwise almost everywhere limit. Since $\mathcal{R}_jK_\epsilon$ is odd, K'_j is odd almost everywhere, and we can determine its values on the remaining zero set such that K'_j is odd everywhere. Now pick $\lambda > 0$. We have, for almost every x,

$$\begin{aligned}\mathcal{R}_{j,\delta}K_\epsilon(\lambda x) &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\lim_{k\to\infty}\int_{|\lambda x-y|\geqslant\delta_k}\frac{\lambda x_j - y_j}{|\lambda x - y|^{n+1}}K_\epsilon(y)\,\mathrm{d}y\\ &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\lim_{k\to\infty}\int_{|\lambda(x-y)|\geqslant\delta_k}\frac{\lambda(x_j - y_j)}{|\lambda(x-y)|^{n+1}}K_\epsilon(\lambda y)\lambda^n\,\mathrm{d}y = \frac{1}{\lambda^n}\mathcal{R}_jK_{\epsilon/\lambda}(x).\end{aligned}$$

So $K'_j(\lambda x) = \lambda^{-n}K'_j(x)$ everywhere except for a zero set, which unfortunately depends on λ, but the set $Z = \{(x,\lambda) : K'_j(\lambda x) \neq \lambda^{-n}K'_j(x)\}$ has measure zero. By Fubini's Theorem there is a sphere $\rho\Sigma^{n-1}$ for some $\rho > 0$ such that $Z \cap \rho\Sigma^{n-1}$ has measure zero in $\rho\Sigma^{n-1}$. Define $\widetilde{K}_j(x) = (\rho/|x|)^n K'_j(\rho x/|x|)$ when $\rho x/|x| \notin Z \cap \rho\Sigma^{n-1}$. Then $\widetilde{K}_j(x_0) = K'_j(x_0)$ when $x_0 = \rho x/|x|$. From this it follows that $\widetilde{K}_j = K'_j$ almost everywhere, since the set of these x has full measure and $\widetilde{K}_j(\lambda x_0) = \lambda^{-n}\widetilde{K}_j(x_0) = \lambda^{-n}K'_j(x_0) = K'_j(\lambda x_0)$. If we now define $\widetilde{K}_j(x) = 0$ for all other x, then

$\widetilde{K}_j$ is odd and homogeneous of degree $-n$. It remains to prove that $\widetilde{K}_j$ is integrable on Σ^{n-1}. To prove this we use polar coordinates to get

$$\begin{aligned}\int_{\Sigma^{n-1}} \left|\widetilde{K}_j(x')\right| \mathrm{d}x' &= \frac{1}{\int_1^2 \rho^{-1}\,\mathrm{d}\rho} \int_{1<|x|<2} \left|\widetilde{K}_j(x)\right| \mathrm{d}x \\ &\leqslant \frac{1}{\log 2} \int_{1<|x|<2} \left|\widetilde{K}_j(x) - \mathcal{R}_j K_{\frac{1}{2}}(x)\right| \mathrm{d}x + \frac{1}{\log 2} \int_{1<|x|<2} \left|\mathcal{R}_j K_{\frac{1}{2}}(x)\right| \mathrm{d}x.\end{aligned}$$

Earlier in the proof we concluded that $|\mathcal{R}_j K_\epsilon(x) - \mathcal{R}_j K_\eta(x)| \leqslant \frac{2M_n \|\Omega\|_1 \eta}{|x|^{n+1}} = \frac{M_n\|\Omega\|_1}{|x|^{n+1}}$ when $\eta = \frac{1}{2}$ and $|x| > 1$. If we let $\epsilon \to 0^+$ we get $\left|\widetilde{K}_j(x) - \mathcal{R}_j K_{\frac{1}{2}}(x)\right| \leqslant \frac{M_n\|\Omega\|_1}{|x|^{n+1}}$, so the first integral is bounded by a constant multiple of $\|\Omega\|_r$. Using Theorem 3.7 and the fact that $K_{\frac{1}{2}} \in L^r(\mathbb{R}^n)$ we can also bound the second integral. Hence $\widetilde{K}_j$ is integrable on Σ^{n-1} and there is a constant C such that $\int_{\Sigma^{n-1}} \left|\widetilde{K}_j(x')\right| \mathrm{d}x' \leqslant C \|\Omega\|_r$, so the proof is complete. □

We are now ready to prove the general boundedness theorem for our singular integrals.

Theorem 3.27. (Calderón-Zygmund) *Let* $\mathcal{T} = \text{p.v.} f * K$ *be one of our usual singular integral operators. Suppose* $\Omega_E \in L^r(\mathbb{R}^n)$ *for some* $r > 1$. *Then there is a bounded linear operator* $\mathcal{T} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), 1 < p < \infty$, *such that* $\|\mathcal{T}_\epsilon f - \mathcal{T} f\|_p \to 0$ *as* $\epsilon \to 0^+$.

Proof. We prove it when K is odd and the rest will follow by Theorem 3.7. Define $\Delta_{j,\epsilon} = \mathcal{R}_j K_\epsilon - \widetilde{K}_{j,\epsilon}$ (where $\widetilde{K}_{j,\epsilon}$ are the truncated kernels). We claim that $\Delta_{j,\epsilon} \in L^1(\mathbb{R}^n)$. Note that $\Delta_{j,\epsilon}(x) = \epsilon^{-n}\Delta_{j,1}(\epsilon^{-1}x)$, so $\|\Delta_{j,\epsilon}\|_1 = \|\Delta_{j,\eta}\|_1$ for all $\epsilon, \eta > 0$ and it suffices to prove integrability in the case $\epsilon = 1$. Now,

$$\begin{aligned}\|\Delta_{j,1}\|_1 &\leqslant \int_{|x|<2} |\mathcal{R}_j K_1(x)|\,\mathrm{d}x + \int_{|x|<2} \left|\widetilde{K}_{j,1}(x)\right| \mathrm{d}x + \int_{|x|\geqslant 2} \left|\mathcal{R}_j K_1(x) - \widetilde{K}_{j,1}(x)\right| \mathrm{d}x \\ &= \int_{|x|<2} |\mathcal{R}_j K_1(x)|\,\mathrm{d}x + \int_{1\leqslant|x|<2} \left|\widetilde{K}_j(x)\right| \mathrm{d}x + \int_{|x|\geqslant 2} \left|\mathcal{R}_j K_1(x) - \widetilde{K}_j(x)\right| \mathrm{d}x.\end{aligned}$$

The boundedness of each of these terms follows by similar arguments to ones used in the proof of Lemma 3.26. The first term is bounded by a constant multiple of $\|\Omega\|_r$ by (73) and Theorem 3.7. The same goes for the second term, as we saw when we proved that $\widetilde{K}_j$ is integrable on Σ^{n-1}. For the last term, we have $\left|\mathcal{R}_j K_1(x) - \widetilde{K}_j(x)\right| \leqslant \frac{2M_n\|\Omega\|_1}{|x|^{n+1}}$ (the case $\eta = 1$ in the proof that $\mathcal{R}_j K_\epsilon$ is Cauchy), so this is finite as well. Hence there is a constant C such that $\|\Delta_{j,\epsilon}\|_1 \leqslant C\|\Omega\|_r$ for every $\epsilon > 0$. We now claim that $\|\mathcal{T}_\epsilon f\|_p \leqslant C\|f\|_p$ for some constant C for all $f \in L^p(\mathbb{R}^n)$. It suffices to prove it for f in the dense subspace $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Here we can use (74) to write $\mathcal{R}_j(f * K_\epsilon) = f * \mathcal{R}_j K_\epsilon = f * \widetilde{K}_{j,\epsilon} + f * \Delta_{j,\epsilon}$. The first term is bounded in $L^p(\mathbb{R}^n)$ by Theorem 3.6, and the same is true for the last term by Theorem 1.9. Hence $f * K_\epsilon$ is bounded on $L^p(\mathbb{R}^n)$ by (72) and Theorem 3.6. The convergence in norm now follows with the same proof as for Theorem 3.7. □

Appendix I: Some Integrals over $\mathbb{R}^n$ and Σ^{n-1}

Consider the integrals

$$I_1 = \int_{|x|<R} \frac{1}{|x|^p}\, \mathrm{d}x \ , \ I_2 = \int_{|x|\geqslant R} \frac{1}{|x|^p}\, \mathrm{d}x.$$

where $x \in \mathbb{R}^n$. Then converting to polar coordinates we see that

$$I_1 = \omega_{n-1} \int_0^R r^{n-p-1}\, \mathrm{d}r \ , \ I_2 = \omega_{n-1} \int_R^\infty r^{n-p-1}\, \mathrm{d}r$$

where ω_{n-1} is the surface area of the sphere $\Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Hence I_1 converges if and only if $p < 1$ and I_2 converges if and only if $p > 1$. To calculate the surface area ω_{n-1}, we first note that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, \mathrm{d}x = \pi^{\frac{n}{2}},$$

which is immediate from the well known one-dimensional case. By converting to polar coordinates and then letting $t = r^2$ we have

$$\pi^{\frac{n}{2}} = \omega_{n-1} \int_0^\infty e^{-r^2} r^{n-1}\, \mathrm{d}r = \omega_{n-1} \int_0^\infty e^{-t} t^{\frac{n}{2}-1}\, \mathrm{d}r.$$

The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t.$$

Simple calculations show that $x\Gamma(x) = \Gamma(x+1)$ and $\Gamma(1) = 1$, so $\Gamma(n) = (n-1)!$ for positive integers n. Other interesting values include $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which can be seen by substituting $t = s^2$. With the gamma notation introduced we have

$$\omega_{n-1} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

To calculate the volume v_n of the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$ we write

$$\int_{B^n} \mathrm{d}x = \omega_{n-1} \int_0^1 r^{n-1}\, \mathrm{d}r = \frac{\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Appendix II: Some Results on Function Spaces

All proofs mentioned here can be found in e.g. [5] or [10]. Let X be a measure space. $L^p(X,\mu), 1 \leqslant p < \infty$, is the space of all measurable functions $f : X \to \mathbb{C}$ such that

$$\int_X |f(x)|^p \, \mathrm{d}\mu(x) < \infty.$$

We write $L^p(X)$ if it is clear what measure we consider. In particular, $L^p(\mathbb{R}^n)$ is assumed to have the Lebesgue measure $\mathrm{d}x$, unless otherwise stated. If μ is the counting measure we write $\ell^p(X)$ instead of $L^p(X,\mu)$. For example, $\ell^p(\mathbb{N})$ is the set of all complex sequences $\{x_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

We define a metric on $L^p(X,\mu)$ by

$$d(f,g) = \left(\int_X |f(x) - g(x)|^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}},$$

and identify two functions as equal if and only if they are equal almost everywhere, so that $f = g$ if and only if $d(f,g) = 0$. $\|f\|_p = d(f,0)$ is indeed a norm on $L^p(X,\mu)$. The triangle inequality $\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$ is called Minkowski's inequality. More generally, we have

$$\left(\int_X \left| \int_Y f(x,y) \, \mathrm{d}\mu_Y(y) \right|^p \mathrm{d}\mu_X(x) \right)^{\frac{1}{p}} \leqslant \int_Y \left(\int_X |f(x,y)|^p \, \mathrm{d}\mu_X(x) \right)^{\frac{1}{p}} \mathrm{d}\mu_Y(y).$$

In the case $Y = \{1,2\}$ with counting measure this is just the original triangle inequality.

When $p = 2$ the norm on $L^p(X,\mu)$ is generated by the inner product

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)} \, \mathrm{d}\mu(x),$$

and Cauchy-Schwartz inequality reads $|\langle f, g \rangle| \leqslant \|f\|_2 \|g\|_2$. More generally we have Hölder's inequality,

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. The number q is called the Hölder conjugate to p. Whenever we talk about p in the same context as here, q is understood to be the Hölder conjugate to p. $L^p(X,\mu)$ is complete under the metric d, and every Cauchy sequence has a subsequence that converges almost everywhere, and the pointwise limit agrees with its norm limit almost everywhere. A complete normed vector space is called a Banach space, and a complete inner product space is called a Hilbert space. An isomorphism $\Phi : X \to Y$ between Banach spaces X and Y is an invertible linear map that is also an isometry, that is, $\|\Phi f\|_Y = \|f\|_X$. If X and Y are Hilbert spaces then Φ is an isomorphism if it also preserves inner products, that is, if $\langle \Phi f, \Phi g \rangle_Y = \langle f, g \rangle_X$. If $Y = X$ such a map is called a unitary operator.

We also define the space $L^\infty(X,\mu)$. This is the space of all measurable functions $f : X \to \mathbb{C}$ that are essentially bounded, that is, there is a constant M such that $|f(x)| \leqslant M$ almost everywhere. The norm on this space is given by the infimum of these M. The triangle inequality and Hölder's inequality extend to this space as well (where the Hölder conjugate to ∞ is 1), and it is also a Banach space. An important subspace of $L^\infty(X,\mu)$ is $C_c(X)$, the space of continuous functions with compact support. In particular, when X is compact $C_c(X) = C(X)$ is a Banach space. $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ when $1 \leqslant p < \infty$, but in $L^\infty(\mathbb{R}^n)$ the completion of $C_c(\mathbb{R}^n)$ is $C_0(\mathbb{R}^n)$, where $C_0(X), X \subset \mathbb{R}^n$, is the space of all continuous functions such that for every $\epsilon > 0$ there is a compact set K such that $|f(x)| < \epsilon$ for all $x \notin K$. For example, in the case of $\mathbb{N}$ with counting measure and discrete topology, $C_0(\mathbb{N}) = c_0(\mathbb{N})$ is the space of all complex sequences converging to zero. Another dense subset of $L^p(\mathbb{R}^n), 1 < p < \infty$, is the set of step functions, i.e. finite linear combinations of (multi-)intervals. We can restrict ourselves to intervals where all endpoints lie in $\mathbb{Q}$ and the set will still be dense. Hence $L^p(\mathbb{R}^n)$ is a separable metric space.

When X has finite measure we have $L^s(X,\mu) \subset L^r(X,\mu)$ whenever $1 \leqslant r < s \leqslant \infty$. The integrals of $|x|^{-\alpha}$, $\alpha > 0$ provide counterexamples when $X = \mathbb{R}^n$, and we can find similar examples in any measure space of infinite measure. If we redefine the norms by

$$\|f\|_p = \left(\frac{1}{\mu(X)} \int_X |f(x)|^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}},$$

then we also have $\|f\|_r \leqslant \|f\|_s$.

Let X and Y be normed vector spaces and $T : X \to Y$ be a linear operator. T is said to be bounded if there is a constant C such that $\|Tf\|_Y \leqslant C \|f\|_X$ for all $f \in X$. It is easy to prove that T is bounded if and only if T is continuous if and only if it is continuous at a point. Let $\mathcal{B}(X,Y)$ be the space of all bounded linear operators. We define the operator norm of $T \in \mathcal{B}(X,Y)$ by

$$\|T\|_{X,Y} = \inf \{C : \|Tf\|_Y \leqslant C \|f\|_X\} = \sup \{\|Tf\|_Y : \|f\|_X \leqslant 1\} = \sup \{\|Tf\|_Y : \|f\|_X = 1\}.$$

If $Y = X$ we may write $\|\cdot\|_X$ instead of $\|\cdot\|_{X,X}$. If Y is complete this turns $\mathcal{B}(X,Y)$ into a Banach space. In particular, the dual space $X^* = \mathcal{B}(X,\mathbb{C})$ is complete. An interesting result is that $L^p(X,\mu)$ is isomorphic to $L^q(X,\mu)^*$. We introduce the dual pairing

$$\langle f, g \rangle = \int_X f(x) g(x) \, \mathrm{d}\mu(x),$$

which is finite by Hölder's inequality. Then $f \mapsto (g \mapsto \langle f, g \rangle)$ is an isomorphism. This means that we have the converse Hölder inequality,

$$\|f\|_p = \sup_{\|g\|_q = 1} \left| \int_X f(x) g(x) \, \mathrm{d}\mu(x) \right|.$$

We have these three fundamental theorems regarding bounded linear operators.

Theorem 5.1. (Banach-Steinhaus Theorem/Uniform Boundedness Principle) *Suppose X is a Banach space and $T_\alpha : X \to Y$ is a family of bounded linear operators. If $\sup_\alpha \|T_\alpha f\|_Y < \infty$ then $\sup_\alpha \|T\|_{X,Y} < \infty$.*

Theorem 5.2. (Open Mapping Theorem) *Suppose X and Y are Banach spaces and $T : X \to Y$ is a surjective bounded linear operator. Then T maps open sets to open sets.*

Corollary 5.3. (Closed Graph Theorem) *Suppose X and Y are Banach spaces and $T : X \to Y$ is a linear operator. Then T is bounded if and only if its graph, $G_T = \{(f,g) \in X \times Y : Tf = g\}$, is closed in $X \times Y$.*

The Closed Graph Theorem can be proved using Open Mapping Theorem. Banach-Steinhaus Theorem and Open Mapping Theorem are proved using Baire Category Theorem, which states that in complete metric spaces the intersection of a countable family of dense G_δ-sets is also a dense G_δ-set (a countable intersection of open sets). A set is said to be of first category if it is a countable union of nowhere dense sets, and of second category otherwise. Stated otherwise, Baire Category Theorem says that no complete metric space is of the first category. A metric space of first category is sometimes thought of as small. Hence Baire Category Theorem says something about the size of complete metric spaces. It also applies to show that if $\sup_\alpha \|T_\alpha f\|_Y = \infty$ for some $f \in X$ (where T_α is as in the statement of Banach Steinhaus Theorem), then $\sup_\alpha \|T_\alpha f\|_Y = \infty$ for all f expect possibly on a set of first category. That is, if the hypothesis of Banach Steinhaus Theorem is violated then divergence at a given point is in some sense typical.

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