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On the q-Lie group of q-Appell polynomial matrices and related factorizations

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Abstract: In the spirit of our earlier paper [10] and Zhang and Wang [16], we introduce the matrix of multiplicative q-Appell polynomials of order $M \in \mathbb{Z}$. This is the representation of the respective q-Appell polynomials in ke-ke basis. Based on the fact that the q-Appell polynomials form a commutative ring [11], we prove that this set constitutes a q-Lie group with two dual q-multiplications in the sense of [9]. A comparison with earlier results on q-Pascal matrices gives factorizations according to [7], which are specialized to q-Bernoulli and q-Euler polynomials. We also show that the corresponding q-Bernoulli and q-Euler matrices form q-Lie subgroups. In the limit $q \to 1$ we obtain corresponding formulas for Appell polynomial matrices. We conclude by presenting the commutative ring of generalized q-Pascal functional matrices, which operates on all functions $f \in C_q^\infty$.

Keywords: q-Lie group; multiplicative q-Appell polynomial matrix; commutative ring; q-Pascal functional matrix

MSC: Primary 17B99; Secondary 17B37, 33C80, 15A23

1 Introduction

In this paper we will introduce several new concepts, some of which were previously known only in the q-case from the articles of the author. By the logarithmic method for q-calculus, this transition will be almost automatic, with the q-addition being replaced by ordinary addition. Some of the matrix formulas in this paper were previously published for Bernoulli polynomials in [16] and for Pascal matrices in [17]. In the article [9] q-Lie matrix groups with two dual multiplications, and in [8] the concept multiplicative q-Appell polynomial were introduced. Now the interesting situation occurs, that the formula [16, p. 1623] for Bernoulli polynomial matrices, which are multiplicative Appell polynomial matrices, also holds for the latter ones. Thus we devote Section 2 to Lie groups of Appell matrices and to the new morphism formula (18). But first we repeat the summation matrix $\overline{\mathrm{G}_{n,k}}(x)$ and the difference matrix $\overline{\mathrm{F}_{n,k}}(x)$ and all the other matrices from [15] in Section 1.

To prepare for the matrix factorizations of the q-Lie matrices in Section 4, we present the relevant q-Pascal and q-unit matrices from [7] in Section 3. In Subsection 4.2 we first repeat the matrix forms of the q-Bernoulli and q-Euler polynomials from [10] to prepare for the computation of their inverses and factorizations. The main purpose of Section 4 is the introduction of the multiplicative q-Appell polynomial matrix and its functional equation, a general so-called q-morphism. In Section 4.1 generalizations of factorizations of Bernoulli matrices to q-Appell polynomial matrices are presented. Finally, in Section 5 the existence of a commutative ring of generalized q-Pascal polynomial functional matrices is proved.

We start our presentation with a brief repetition of some of our matrices.

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Definition 1. Matrix elements will always be denoted (i, j). Here i denotes the row and j denotes the column. The matrix elements range from 0 to $n-1$. The matrices I_n, S_n, A_n, D_n, $\mathrm{S}_n(x)$ and $\mathrm{D}_n(x)$ are defined by

$$\mathrm{I}_n \equiv \mathrm{diag}(1, 1, \ldots, 1) \tag{1}$$

$$\mathrm{S}_n(i,j) \equiv \begin{cases} 1, & \text{if } j \leq i, \\ 0, & \text{if } j > i, \end{cases} \tag{2}$$

$$\mathrm{A}_n(t)(i,j) \equiv \begin{cases} t^i, & \text{if } j = i, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

$$\mathrm{D}_n(i,i) \equiv 1 \text{ for all } i, \tag{4}$$

$$\mathrm{D}_n(i+1,i) \equiv -1, \text{ for } i = 0, \ldots, n-2 \tag{5}$$

$$\mathrm{D}_n(i,j) \equiv 0, \text{ if } j > i \text{ or } j < i-1 \tag{6}$$

$$\mathrm{S}_n(x)(i,j) \equiv \begin{cases} x^{i-j}, & \text{if } j \leq i, \\ 0, & \text{if } j > i, \end{cases} \tag{7}$$

$$\begin{aligned} &\mathrm{D}_n(x;i,i) \equiv 1, i = 0, \ldots, n-1, \; \mathrm{D}_n(x;i+1,i) \equiv -x, \text{ for } i = 0, \ldots, n-2, \\ &\mathrm{D}_n(x;i,j) \equiv 0, \text{ when } j > i \text{ or } j < i-1. \end{aligned} \tag{8}$$

We note that D_n is a special case of $\mathrm{D}_n(x)$, and S_n is a special case of $\mathrm{S}_n(x)$.

The summation matrix $\overline{\mathrm{G}_{n,k}}(x)$ and its inverse, the difference matrix $\overline{\mathrm{F}_{n,k}}(x)$, are defined by [15, p. 52,54]:

$$\begin{aligned} &\overline{\mathrm{G}_{n,k}}(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{S}_k(x) \end{bmatrix}, \; k = 3, \ldots, n, \; \overline{\mathrm{G}_{n,n}}(x) \equiv \mathrm{S}_n(x), n > 2, \\ &\overline{\mathrm{F}_{n,k}}(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{D}_k(x) \end{bmatrix}, \; k = 3, \ldots, n, \; \overline{\mathrm{F}_{n,n}}(x) \equiv \mathrm{D}_n(x), n > 2. \end{aligned} \tag{9}$$

2 The Lie group of Appell matrices

We first define Appell polynomials and multiplicative Appell polynomials.

Definition 2. Let $\mathcal{A}$ denote the set of real sequences $\{u_\nu\}_{\nu=0}^{\infty}$ such that

$$\sum_{\nu=0}^{\infty} |u_\nu| \frac{r^\nu}{\nu!} < \infty, \tag{10}$$

for some convergence radius $r > 0$.

Definition 3. For $f_n(t) \in \mathbb{R}[[t]]$, let $p_\nu \in \mathcal{A}$ and let $p_\nu^{(n)}$ denote the Appell numbers of degree ν and order $n \in \mathbb{Z}$ with the following generating function

$$f_n(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} p_\nu^{(n)}. \tag{11}$$

Definition 4. For every formal power series $f_n(t) = h(t)^n$, let $p_{\mathcal{M},\nu} \in \mathcal{A}$ and let $p_{\mathcal{M},\nu}^{(n)}$ denote the multiplicative Appell numbers of degree ν and order $n \in \mathbb{Z}$ with the following generating function

$$h(t)^n = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} p_{\mathcal{M},\nu}^{(n)}. \tag{12}$$

Definition 5. For every formal power series $f_n(t) = h(t)^n$ given by (12), the multiplicative Appell polynomials or $p_\nu^{(n)}(x)$ polynomials of degree ν and order $n \in \mathbb{Z}$ have the following generating function

$$f_n(t)e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} p_\nu^{(n)}(x). \tag{13}$$

The proof of the following formula is relegated to (43).

Theorem 2.1. *Assume that M and K are the x-order and y-order, respectively.*

$$p_\nu^{(M+K)}(x+y) = \sum_{k=0}^{\nu} \binom{\nu}{k} p_k^{(M)}(x) p_{\nu-k}^{(K)}(y). \tag{14}$$

Definition 6. We will use the following vector forms for the Appell polynomials and numbers:

$$\Pi_n(x) \equiv (p_0(x), p_1(x), \ldots, p_{n-1}(x))^T, \tag{15}$$

$$\Pi_n \equiv \Pi_n(0). \tag{16}$$

Definition 7. The multiplicative Appell polynomial matrix of order $M \in \mathbb{Z}$ is defined by

$$\overline{p}_n^{(M)}(x)(i,j) \equiv \binom{i}{j} p_{i-j}^{(M)}(x), \; 0 \le i, j \le n-1. \tag{17}$$

We refer to (56) for the proof of the next theorem.

Theorem 2.2. *In the following formula we assume that M and K are the x-order and y-order, respectively.*

$$\overline{p}_n^{(M+K)}(x+y) = \overline{p}_n^{(M)}(x)\overline{p}_n^{(K)}(y). \tag{18}$$

Theorem 2.3. *The multiplicative Appell polynomial matrices $(\mathcal{M}, \odot)$ with elements $\overline{p}_n^{(M)}(x)$ is an Abelian matrix Lie group with multiplication given by (18) and inverse $\overline{p}_n^{(-M)}(-x)$.*

Proof. The set $\mathcal{M}$ is closed under the operation $\odot$ by (18). The group element $\overline{p}_n^{(-M)}(-x)$ is inverse to $\overline{p}_n^{(M)}(x)$ by the subtraction of real numbers. The unit element is the unit matrix I_n. The associativity and commutativity follow by (18). □

3 The q-Pascal matrix and the q-unit matrices

Definition 8. The q-Pascal matrix $\mathrm{P}_{n,q}(x)$ [7] is given by the familiar expression

$$\mathrm{P}_{n,q}(i,j)(x) \equiv \binom{i}{j}_q x^{i-j}, \; i \ge j. \tag{19}$$

The following special case is often used.

Definition 9.

$$\mathrm{P}_{n,q} \equiv \mathrm{P}_{n,q}(1). \tag{20}$$

We now recall some formulas from [7].

Definition 10. The matrices $\overline{\mathrm{P}_{n,k,q}}(x)$, $\mathrm{P}_{k,q}\star(x)$ and $\overline{\mathrm{P}_{n,k,q}}\star(x)$ are defined by

$$\overline{\mathrm{P}_{n,k,q}}(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{P}_{k,q}(x) \end{bmatrix}, \tag{21}$$

$$\mathrm{P}_{k,q}\star(x;i,j) = \binom{i}{j}_q (qx)^{i-j},\ i,j = 0,\ldots,k-1, \tag{22}$$

$$\overline{\mathrm{P}_{n,k,q}}\star(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{P}_{k,q}\star(x) \end{bmatrix},\ k = 3,\ldots,n,\ \overline{\mathrm{P}_{n,n,q}}\star(x) \equiv \mathrm{P}_{n,q}\star(x). \tag{23}$$

Let the two matrices $\mathrm{I}_{k,q}(x)$, and its inverse, $\mathrm{E}_{k,q}(x)$, be given by:

$$\begin{aligned} &\mathrm{I}_{k,q}(x;i,i) \equiv 1,\ i = 0,\ldots,k-1,\ \mathrm{I}_{k,q}(x;i+1,i) \equiv x(q^{i+1}-1),\ i = 0\ldots,k-1, \\ &\mathrm{I}_{k,q}(x;i,j) \equiv 0 \text{ for other } i,j. \\ &\mathrm{E}_{k,q}(x;i,j) \equiv \langle j+1;q\rangle_{i-j}x^{i-j},\ i \geq j,\ \mathrm{E}_{k,q}(x;i,j) \equiv 0 \text{ for other } i,j. \end{aligned} \tag{24}$$

Similarly, let the two matrices $\overline{\mathrm{I}_{n,k,q}}(x)$, and its inverse, $\overline{\mathrm{E}_{n,k,q}}(x)$, be given by:

$$\overline{\mathrm{I}_{n,k,q}}(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{I}_{k,q}(x) \end{bmatrix},\ \overline{\mathrm{I}_{n,n,q}}(x) \equiv \mathrm{I}_n. \tag{25}$$

$$\overline{\mathrm{E}_{n,k,q}}(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{E}_{k,q}(x) \end{bmatrix},\ \overline{\mathrm{E}_{n,n,q}}(x) \equiv \mathrm{I}_n. \tag{26}$$

We call $\overline{\mathrm{I}_{n,k,q}}(x)$ the q-unit matrix function. We will use a slightly q-deformed version of the D- and F-matrices:

$$\begin{aligned} &\mathrm{D}_{k,q}\star(x;i,i) \equiv 1,\ i = 0,\ldots,k-1,\ \mathrm{D}_{k,q}\star(x;i+1,i) \equiv -xq^i,\ i = 0,\ldots,k-1, \\ &\mathrm{D}_{k,q}\star(x;i,j) \equiv 0,\ \text{ if } j > i \text{ or } j < i-1. \end{aligned} \tag{27}$$

$$\overline{\mathrm{F}_{n,k,q}}\star(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{D}_{k,q}\star(x) \end{bmatrix}. \tag{28}$$

The $q-$summation matrices are defined by

$$\begin{aligned} &\mathrm{G}_k\star(x) \equiv \begin{cases} \mathrm{QE}\left(\binom{i-j+1}{2} + j(i-j)\right) x^{i-j}, & \text{if } j \leq i, \\ 0, & \text{if } j > i, \end{cases}, \\ &\overline{\mathrm{G}_{n,k,q}}\star(x) \equiv \begin{bmatrix} \mathrm{I}_{n-k} & 0^T \\ 0 & \mathrm{G}_{k,q}\star(x) \end{bmatrix}. \end{aligned} \tag{29}$$

We have the inverse relation:

$$\overline{\mathrm{F}_{n,k,q}}\star(x)^{-1} = \overline{\mathrm{G}_{n,k,q}}\star(x). \tag{30}$$

The inverse of $\mathrm{P}_{k,q}\star(x)$ is given by

$$(\mathrm{P}_{k,q}\star(x))^{-1}(i,j) = \binom{i}{j}_q (-x)^{i-j} q^{\binom{i-j+1}{2}},\ i,j = 0,\ldots,k-1. \tag{31}$$

The following matrix will be used in formula (52).

Definition 11. The q-Cauchy matrix is given by

$$W_{n,q}(x)(i,j) \equiv (x \oplus_q \overline{j_q})^i. \tag{32}$$

Theorem 3.1. *[7]. A q-analogue of [15, p.53 (1)]. If $n \geq 3$, the q-Pascal matrix $\mathrm{P}_{n,q}(x)$ can be factorized by the summation matrices and by the q-unit matrices as*

$$\mathrm{P}_{n,q}(x) = \prod_{k=n}^{3} \left(\overline{\mathrm{I}_{n,k,q}}(x)\overline{\mathrm{G}_{n,k}}(x)\right) \overline{\mathrm{G}_{n,2,q}} \star (x), \tag{33}$$

where the product is taken in decreasing order of k.

Theorem 3.2. *[7]. A q-analogue of [15, p. 54]. The inverse of the q-Pascal matrix is given by*

$$\mathrm{P}_{n,q}(x)^{-1} = \overline{\mathrm{F}_{n,2,q}} \star (x) \prod_{k=3}^{n} (\overline{\mathrm{F}_{n,k}}(x)\overline{\mathrm{E}_{n,k,q}}(x)). \tag{34}$$

4 The q-Lie group of q-Appell polynomial matrices

We first repeat and extend some definitions from [9].

Definition 12. A q-Lie group $(G_{n,q,\cdot,\cdot_q}, \mathrm{I}_g) \supseteq \mathrm{E}_q(\mathfrak{g}_q)$, is a possibly infinite set of matrices $\in \mathrm{GL}_q(n, \mathbb{R})$, and a manifold, with two multiplications: $\cdot$, the usual matrix multiplication, and the twisted $\cdot_q$, which is defined separately. Each q-Lie group has a unit, denoted by I_g, which is the same for both multiplications. Each element $\Phi \in G_{n,q}$ has an inverse Φ^{-1} with the property $\Phi \cdot_q \Phi^{-1} = \mathrm{I}_g$.

Definition 13. If $(G_1, \cdot_1, \cdot_{1:q})$ and $(G_2, \cdot_2, \cdot_{2:q})$ are two q-Lie groups, then $(G_1 \times G_2, \cdot, \cdot_q)$ is a q-Lie group called the product q-Lie group. This has group operations defined by

$$(g_{11}, g_{21}) \cdot (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}), \tag{35}$$

and

$$(g_{11}, g_{21}) \cdot_q (g_{12}, g_{22}) = (g_{11} \cdot_{1:q} g_{12}, g_{21} \cdot_{2:q} g_{22}). \tag{36}$$

Definition 14. If $(G_{n,q}, \cdot, \cdot_q)$ is a q-Lie group and $H_{n,q}$ is a nonempty subset of $G_{n,q}$, then $(H_{n,q}, \cdot, \cdot_q)$ is called a q-Lie subgroup of $(G_{n,q}, \cdot, \cdot_q)$ if

1.
$$\Phi \cdot \Psi \in H_{n,q} \text{ and } \Phi \cdot_q \Psi \in H_{n,q} \text{ for all } \Phi, \Psi \in H_{n,q}. \tag{37}$$

2.
$$\Phi^{-1} \in H_{n,q} \text{ for all } \Phi \in H_{n,q}. \tag{38}$$

3. $H_{n,q}$ is a submanifold of $G_{n,q}$.

Definition 15. An invertible mapping $f : (G_{n,q}, \cdot_1, \cdot_{1:q}) \to (H_{n,q}, \cdot_2, \cdot_{2:q})$ is called a q-Lie group morphism between $(G_{n,q}, \cdot_1, \cdot_{1:q})$ and $(H_{n,q}, \cdot_2, \cdot_{2:q})$ if

$$f(\phi \cdot_1 \psi) = f(\phi) \cdot_2 f(\psi), \text{ and } f(\phi \cdot_{1:q} \psi) = f(\phi) \cdot_{2:q} f(\psi). \tag{39}$$

It is obvious that $(\mathbb{Z}, +)$ is a q-Lie group with only one operation. We will use this fact in formula (61).

The most general form of polynomial in this article is the q-Appell polynomial, which we will now define.

Definition 16. Let $\mathcal{A}_q$ denote the set of real sequences $\{u_{\nu,q}\}_{\nu=0}^{\infty}$ such that

$$\sum_{\nu=0}^{\infty} |u_{\nu,q}| \frac{r^{\nu}}{\{\nu\}_q!} < \infty, \tag{40}$$

for some q-dependent convergence radius $r = r(q) > 0$, where $0 < q < 1$.

Definition 17. Assume that $h(t,q),\ h(t,q)^{-1} \in \mathbb{R}[[t]]$. For $f_n(t,q) = h(t,q)^n$, let $\Phi_{\nu,q} \in \mathcal{A}_q$ and let $\Phi_{\nu,q}^{(n)}$ denote the multiplicative q–Appell number of degree ν and order n given by the generating function

$$f_n(t,q) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)},\ \Phi_{0,q} = 1,\ n \in \mathbb{Z}. \tag{41}$$

Definition 18. For every formal power series $f_n(t,q)$ given by (41), the multiplicative q–Appell polynomials or Φ_q polynomials of degree ν and order n have the following generating function:

$$f_n(t,q)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu,q}^{(n)}(x),\ n \in \mathbb{Z}. \tag{42}$$

Theorem 4.1. *In the following formula we assume that M and K are the x-order and y-order, respectively.*

$$\Phi_{\nu,q}^{(M+K)}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k,q}^{(M)}(x) \Phi_{\nu-k,q}^{(K)}(y). \tag{43}$$

Proof. This is proved in the same way as in [4, 4.242, p. 136]. □

The following vector forms for q-Appell polynomials and numbers will be used in formulas (80), (81), (101) and (102).

Definition 19.

$$\phi_{n,q}(x) \equiv (\Phi_{0,q}(x), \Phi_{1,q}(x), \ldots, \Phi_{n-1,q}(x))^T, \tag{44}$$

$$\phi_{n,q} \equiv \phi_{n,q}(0). \tag{45}$$

Definition 20. Define the q-Appell polynomial matrix by

$$\overline{\Phi}_{n,q}(x)(i,j) \equiv \binom{i}{j}_q \Phi_{i-j,q}(x),\ 0 \le i,j \le n-1. \tag{46}$$

Definition 21. The multiplicative q-Appell polynomial matrices $(\mathcal{M}_{x,q})$ with elements $\overline{\Phi}_{n,q}^{(M)}(x)$ of order $M \in \mathbb{Z}$ are defined by

$$\overline{\Phi}_{n,q}^{(M)}(x)(i,j) \equiv \binom{i}{j}_q \Phi_{i-j,q}^{(M)}(x),\ 0 \le i,j \le n-1. \tag{47}$$

Definition 22. The multiplicative q-Appell number matrices or the q-transfer matrices $(\mathcal{M}_q)$ with elements $\overline{\Phi}_{n,q}^{(M)}$ of order $M \in \mathbb{Z}$ are defined by

$$\overline{\Phi}_{n,q}^{(M)}(i,j) \equiv \overline{\Phi}_{n,q}^{(M)}(0)(i,j),\ 0 \le i,j \le n-1. \tag{48}$$

Theorem 4.2. *A q-analogue of [2, (3.9), p. 432]*

$$\overline{\Phi}_{n,q}^{(M)}(x) = \overline{\Phi}_{n,q}^{(M)} \xi_n(x),\ \text{where} \tag{49}$$

$$\xi_n(x) \equiv (1, x, x^2, \ldots, x^{n-1})^T. \tag{50}$$

We define a generalization of formulas (82) and (103).

Definition 23. The shifted q-Appell polynomial matrix $\widetilde{\Phi_{n,q}(x)}$ is defined by

$$\widetilde{\Phi_{n,q}(x)}(i,j) \equiv \Phi_{i,q}(x \oplus_q \bar{j}_q),\ 0 \le i,j \le n-1. \tag{51}$$

Corollary 4.3. *A generalization of [10]. The shifted q-Appell polynomial matrix can be written as the product of the q-Appell number matrix and the q-Cauchy matrix.*

$$\widetilde{\Phi_{n,q}(x)} = \overline{\Phi}_{n,q} W_{n,q}(x). \tag{52}$$

Proof. We show that the matrix indices are equal.

$$\sum_{k=0}^{i} \binom{i}{k}_q \Phi_{i-k,q}(x \oplus_q \bar{j}_q)^k = \Phi_{i,q}(x \oplus_q \bar{j}_q). \tag{53}$$

□

We remark that a special case of this equation can be found in [16, p. 1631].

In [7] we proved the formula

$$\mathrm{P}_{n,q}(s \oplus_q t) = \mathrm{P}_{n,q}(s)\mathrm{P}_{n,q}(t),\ s,t \in \mathbb{R}. \tag{54}$$

This can be generalized to

Theorem 4.4. *We assume that M and K are the x-order and y-order, respectively. The formula (43) can be rewritten in the following matrix form, where $\cdot$ on the* RHS *denotes matrix multiplication.*

$$\overline{\Phi}_{n,q}^{(M+K)}(x \oplus_q y) = \overline{\Phi}_{n,q}^{(M)}(x) \cdot \overline{\Phi}_{n,q}^{(K)}(y). \tag{55}$$

For the following proof, compare with [16, p. 1624].

Proof. We compute the (i,j) matrix element of the matrix multiplication on the RHS.

$$\begin{aligned} &\sum_{k=j}^{i} \binom{i}{k}_q \Phi_{i-k,q}^{(M)}(x) \binom{k}{j}_q \Phi_{k-j,q}^{(K)}(y) = \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q \Phi_{i-k,q}^{(M)}(x)\Phi_{k-j,q}^{(K)}(y) \\ &= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q \Phi_{i-j-k,q}^{(M)}(x)\Phi_{k,q}^{(K)}(y) = \binom{i}{j}_q \Phi_{i-j,q}^{(M+K)}(x \oplus_q y) = \mathrm{LHS}. \end{aligned} \tag{56}$$

□

By formula (47), the $\overline{\Phi}_{n,q}^{(M)}(x)$ are matrices with matrix elements q-Appell polynomials multiplied by q-binomial coefficients, and we arrive at the next crucial definition.

Definition 24. We define the second matrix multiplication $\cdot_q$ by

$$\overline{\Phi}_{n,q}^{(M)}(x) \cdot_q \overline{\Phi}_{n,q}^{(K)}(y) \equiv \overline{\Phi}_{n,q}^{(M+K)}(x \boxplus_q y). \tag{57}$$

Theorem 4.5. *The set $(\mathcal{M}_{x,q}, \cdot, \cdot_q, \mathrm{I}_n)$ is a q-Lie group with multiplications given by (55) and (57), and inverse $\overline{\Phi}_{n,q}^{(-M)}(-x)$. The unit element is the unit matrix I_n.*

Proof. The set $\mathcal{M}_{x,q}$ is closed under the two operations by (55) and (57). By (57) we have

$$\overline{\Phi}_{n,q}^{(M)}(x)\cdot_q \overline{\Phi}_{n,q}^{(-M)}(-x)=\overline{\Phi}_{n,q}^{(0)}(\theta)=\mathrm{I}_n, \tag{58}$$

which shows the existence of an inverse element and a unit.

The associative law reads:

$$\left(\overline{\Phi}_{n,q}^{(M)}(x)\cdot\overline{\Phi}_{n,q}^{(K)}(y)\right)\cdot_q\overline{\Phi}_{n,q}^{(J)}(z)=\overline{\Phi}_{n,q}^{(M)}(x)\cdot\left(\overline{\Phi}_{n,q}^{(K)}(y)\cdot_q\overline{\Phi}_{n,q}^{(J)}(z)\right), \tag{59}$$

which is equivalent to

$$\overline{\Phi}_{n,q}^{(M+K+J)}((x\oplus_q y)\boxplus_q z)=\overline{\Phi}_{n,q}^{(M+K+J)}(x\oplus_q(y\boxplus_q z)). \tag{60}$$

However, formula (60) follows from the associativity of the two q-additions. □

Definition 25. In the definition of product q-Lie group, put

$$(\mathcal{R}_q,\cdot,\cdot_q)\equiv(\mathbb{R}_q,\oplus_q,\boxplus_q)\times(\mathbb{Z},+). \tag{61}$$

It is clear that formulas (55) and (57) defines a q-Lie group morphism from $\mathcal{R}_q$ to $\mathcal{M}_{x,q}$.

Let

$$\left(\overline{\Phi}_{n,q}^{(M)}(x)\right)^k\equiv\overline{\Phi}_{n,q}^{(M)}(x)\cdot\overline{\Phi}_{n,q}^{(M)}(x)\cdot\dots\cdot\overline{\Phi}_{n,q}^{(M)}(x), \tag{62}$$

where the right hand side denotes the product of k equal matrices $\overline{\Phi}_{n,q}^{(M)}(x)$.

In [7] we proved the formula

$$\mathrm{P}_{n,q}^k=\mathrm{P}_{n,q}(\overline{k}_q). \tag{63}$$

This can be generalized to

$$\left(\overline{\Phi}_{n,q}^{(M)}(x)\right)^k=\overline{\Phi}_{n,q}^{(kM)}(\overline{k}_q x). \tag{64}$$

Furthermore, the formulas in [16, p. 1624] can be generalized to the special cases

$$\left(\overline{\Phi}_{n,q}^{(M)}\right)^k=\overline{\Phi}_{n,q}^{(kM)},\ \left(\overline{\Phi}_{n,q}^{(1)}\right)^k=\overline{\Phi}_{n,q}^{(k)}. \tag{65}$$

4.1 Two factorizations

We show that our q-Appell polynomials allow simple extensions to factorizations by Fibonacci number matrices. The first Fibonacci numbers F_k have the following values:

$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0	1	1	2	3	5	8

Definition 26. [12] We use the following notation for the matrix form of the Fibonacci numbers: $\mathcal{F}_n(i,j)\equiv F_{i-j}$.

It follows that [12, p. 205]

$$\begin{aligned}&\mathcal{F}_n^{-1}(i,i)\equiv 1,\ i=0,\dots,n-1,\ \mathcal{F}_n^{-1}(i+1,i)\equiv -1\ i=0,\dots,n-2,\\&\mathcal{F}_n^{-1}(i+2,i)\equiv -1\ i=0,\dots,n-3,\ \mathcal{F}_n^{-1}(i,j)=0\text{ otherwise.}\end{aligned} \tag{66}$$

Definition 27. The matrix $\mathcal{M}_{n,q}(x)$ has matrix elements

$$m_{i,j}=\binom{i}{j}_q\Phi_{i-j,q}(x)-\binom{i-1}{j}_q\Phi_{i-j-1,q}(x)-\binom{i-2}{j}_q\Phi_{i-j-2,q}(x). \tag{67}$$

For the following formula, compare with [16, p. 1627], where the corresponding formula for the Bernoulli matrix was given. Note that we do not need the order of the polynomials.

Theorem 4.6. *The q-Appell polynomial matrix can be factorized as*

$$\overline{\Phi}_{n,q}(x) = \mathcal{F}_n \mathcal{M}_{n,q}(x). \tag{68}$$

Proof. It would suffice to prove that

$$\mathcal{F}_n^{-1} \overline{\Phi}_{n,q}(x) = \mathcal{M}_{n,q}(x). \tag{69}$$

The matrix index of the left hand side is given by

$$\begin{aligned} \mathcal{F}_n^{-1}\overline{\Phi}_{n,q}(x)(i,j) &= \sum_{k=0}^{n-1} \mathcal{F}_n^{-1}(i,k) \binom{k}{j}_q \Phi_{k-j,q}(x) \\ &= \binom{i}{j}_q \Phi_{i-j,q}(x) - \binom{i-1}{j}_q \Phi_{i-j-1,q}(x) - \binom{i-2}{j}_q \Phi_{i-j-2,q}(x) = \text{RHS}. \end{aligned} \tag{70}$$

□

We shall now prove a similar formula.

Definition 28. The matrix $\mathcal{R}_{n,q}(x)$ has matrix elements

$$r_{i,j} = \binom{i}{j}_q \Phi_{i-j,q}(x) - \binom{i}{j+1}_q \Phi_{i-j-1,q}(x) - \binom{i}{j+2}_q \Phi_{i-j-2,q}(x). \tag{71}$$

For the following formula, compare with [17, p. 2372], where the corresponding formula for Pascal matrices was given.

Theorem 4.7. *The q-Appell polynomial matrix can be factorized as*

$$\overline{\Phi}_{n,q}(x) = \mathcal{R}_{n,q}(x)\mathcal{F}_n. \tag{72}$$

Proof. It suffices to prove that

$$\overline{\Phi}_{n,q}(x)\mathcal{F}_n^{-1} = \mathcal{R}_{n,q}(x). \tag{73}$$

The matrix index of the left hand side is given by

$$\begin{aligned} \overline{\Phi}_{n,q}(x)\mathcal{F}_n^{-1}(i,j) &= \sum_{k=0}^{n-1} \binom{i}{k}_q \Phi_{i-k,q}(x)\mathcal{F}_n^{-1}(k,j) \\ &= \binom{i}{j}_q \Phi_{i-j,q}(x) - \binom{i}{j+1}_q \Phi_{i-j-1,q}(x) - \binom{i}{j+2}_q \Phi_{i-j-2,q}(x) = \text{RHS}. \end{aligned} \tag{74}$$

□

4.2 q-Bernoulli and q-Euler polynomials

We will also consider the special cases q-Bernoulli and q-Euler polynomials.

Definition 29. There are two types of q-Bernoulli polynomials, called $\mathrm{B}_{\mathrm{NWA},\nu,q}(x)$, NWA q-Bernoulli polynomials, and $\mathrm{B}_{\mathrm{JHC},\nu,q}(x)$, JHC q-Bernoulli polynomials. They are defined by the two generating functions

$$\frac{t}{(\mathrm{E}_q(t)-1)}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathrm{B}_{\mathrm{NWA},\nu,q}(x)}{\{\nu\}_q!}, \; |t| < 2\pi. \tag{75}$$

and

$$\frac{t}{(\mathrm{E}_{\frac{1}{q}}(t)-1)}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}\mathrm{B}_{\mathrm{JHC},\nu,q}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{76}$$

Definition 30. The Ward q-Bernoulli numbers are given by

$$\mathrm{B}_{\mathrm{NWA},n,q}\equiv\mathrm{B}_{\mathrm{NWA},n,q}(0). \tag{77}$$

The Jackson q-Bernoulli numbers are given by

$$\mathrm{B}_{\mathrm{JHC},n,q}\equiv\mathrm{B}_{\mathrm{JHC},n,q}(0). \tag{78}$$

The following table lists some of the first Ward q-Bernoulli numbers.

$n=0$	$n=1$	$n=2$	$n=3$
1	$-(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(1-q)q^3(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}_q^2\{3\}_q\{5\}_q)^{-1}$

To save space, we will use the following abbreviation in equations (80) - (84), (87), (88), (91), (95), (97), (98), (101)-(105), (108)-(109), (112)-(114).

$$\mathrm{NWA}=\mathrm{NWA}\vee\mathrm{JHC}. \tag{79}$$

We will use the following vector forms for the q-Bernoulli polynomials corresponding to q-analogues of [1, p. 239].

$$b_{\mathrm{NWA},n,q}(x)\equiv(\mathrm{B}_{\mathrm{NWA},0,q}(x),\mathrm{B}_{\mathrm{NWA},1,q}(x),\ldots,\mathrm{B}_{\mathrm{NWA},n-1,q}(x))^T. \tag{80}$$

The corresponding vector forms for numbers are

$$b_{\mathrm{NWA},n,q}\equiv(\mathrm{B}_{\mathrm{NWA},0,q},\mathrm{B}_{\mathrm{NWA},1,q},\ldots,\mathrm{B}_{\mathrm{NWA},n-1,q})^T. \tag{81}$$

Let us introduce the NWA and JHC shifted q-Bernoulli matrices.

Definition 31.

$$\mathcal{B}_{\mathrm{NWA},n,q}(x)\equiv(b_{\mathrm{NWA},q}(x)\ \ \mathrm{E}(\oplus_q)b_{\mathrm{NWA},q}(x)\ \cdots\ \mathrm{E}(\oplus_q)^{\overline{n-1}_q}b_{\mathrm{NWA},q}(x)), \tag{82}$$

where $\mathrm{E}(\oplus_q)^{\overline{n-1}_q}(x^n)\equiv(x\oplus_q\overline{n-1}_q)^n$.

We will need two similar matrices based on the $\mathrm{B}_{\mathrm{NWA}}$ and $\mathrm{B}_{\mathrm{JHC}}$ polynomials and numbers.

Definition 32. Two q-analogues of [3, p. 193]. The NWA and JHC q-Bernoulli polynomial matrices are defined by

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x)(i,j)\equiv\binom{i}{j}_q\mathrm{B}_{\mathrm{NWA},i-j,q}(x),\ 0\le i,j\le n-1. \tag{83}$$

Definition 33. The NWA and JHC q-Bernoulli number matrices are defined by

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(i,j)\equiv\binom{i}{j}_q\mathrm{B}_{\mathrm{NWA},i-j,q},\ 0\le i,j\le n-1. \tag{84}$$

Definition 34. The matrix $\mathcal{D}_{\mathrm{NWA},n,q}$ has matrix elements

$$d_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q}\binom{i}{j}_q \text{ if } i \geq j, \\ 0 \text{ otherwise.} \end{cases} \tag{85}$$

Definition 35. The matrix $\mathcal{D}_{\mathrm{JHC},n,q}$ has matrix elements

$$d_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{q^{\binom{i-j+1}{2}}}{\{i-j+1\}_q}\binom{i}{j}_q \text{ if } i \geq j, \\ 0 \text{ otherwise.} \end{cases} \tag{86}$$

Theorem 4.8. *The inverses of the q-Bernoulli number matrices are given by*

$$(\overline{\mathrm{B}}_{\mathrm{NWA},n,q})^{-1} = \mathcal{D}_{\mathrm{NWA},n,q}. \tag{87}$$

This implies that

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}^{-k} = \mathcal{D}_{\mathrm{NWA},n,q}^{k}. \tag{88}$$

The following proof is very similar to [16, p. 1624].

Proof. For the NWA case, take away the factor $q^{\binom{k+1}{2}}$ and corresponding q-powers in the following equations. We show that $\overline{\mathrm{B}}_{\mathrm{JHC},n,q}\mathcal{D}_{\mathrm{JHC},n,q}$ is equal to the unit matrix. We know that

$$\sum_{k=0}^{n} \frac{q^{\binom{k+1}{2}}}{\{k+1\}_q}\binom{n}{k}_q \mathrm{B}_{\mathrm{JHC},n-k,q} = \delta_{n,0}. \tag{89}$$

Then we have

$$\begin{aligned} &\sum_{k=j}^{i} \frac{q^{\binom{k+1-j}{2}}}{\{k+1-j\}_q}\binom{i}{k}_q \mathrm{B}_{\mathrm{JHC},i-k,q}\binom{k}{j}_q \\ &= \binom{i}{j}_q \sum_{k=j}^{i} \frac{q^{\binom{k+1-j}{2}}}{\{k+1-j\}_q}\binom{i-j}{k-j}_q \mathrm{B}_{\mathrm{JHC},i-k,q} \\ &= \binom{i}{j}_q \sum_{k=0}^{i-j} \frac{q^{\binom{k+1}{2}}}{\{k+1\}_q}\binom{i-j}{k}_q \mathrm{B}_{\mathrm{JHC},i-j-k,q} \overset{\text{by}(89)}{=} \binom{i}{j}_q \delta_{i-j,0}. \end{aligned} \tag{90}$$

□

In [10] we considered the following q-analogues of [16, p. 1625]

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x \oplus_q y) = \mathrm{P}_{n,q}(x)\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(y). \tag{91}$$

These can be generalized to

Theorem 4.9.

$$\overline{\Phi}_{n,q}(x \oplus_q y) = \mathrm{P}_{n,q}(x)\overline{\Phi}_{n,q}(y). \tag{92}$$

In particular,

$$\overline{\Phi}_{n,q}(x) = \mathrm{P}_{n,q}(x)\overline{\Phi}_{n,q}. \tag{93}$$

For the following proof one should compare with [16, p. 1625].

Proof.

$$\begin{aligned}&\sum_{k=j}^{i}\binom{i}{k}_q x^{i-k}\binom{k}{j}_q \Phi_{k-j,q}(y)=\binom{i}{j}_q\sum_{k=j}^{i}\binom{i-j}{k-j}_q x^{i-k}\Phi_{k-j,q}(y)\\&=\binom{i}{j}_q\sum_{k=0}^{i-j}\binom{i-j}{k}_q x^{i-j-k}(x)\Phi_{k,q}(y)=\binom{i}{j}_q\Phi_{i-j,q}(x\oplus_q y)=\text{LHS}.\end{aligned}\tag{94}$$

□

Theorem 4.10. *Two q-analogues of [16, p. 1626] The inverses of the q-Bernoulli polynomial matrices are given by*

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x)\right)^{-1}=\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q}\right)^{-1}\mathrm{P}_{n,q}(x)^{-1}=\mathcal{D}_{\mathrm{NWA},n,q}\mathrm{P}_{n,q}(x)^{-1}.\tag{95}$$

When the order is increased, for $y=0$ in (55), we multiply the q-transfer matrix by $\overline{\Phi}_{n,q}^{(M)}(x)$. When the order is constant, in (93), we multiply the q-transfer matrix by the q-Pascal matrix.

We can now find a factorization of the q-Appell polynomial matrix and its inverse.

Theorem 4.11. *A q-analogue and generalization of [16, p. 1626].*

$$\overline{\Phi}_{n,q}(x)=\prod_{k=n}^{3}\left(\overline{\mathrm{I}_{n,k,q}}(x)\overline{\mathrm{G}_{n,k}}(x)\right)\overline{\mathrm{G}_{n,2,q}}\star(x)\overline{\Phi}_{n,q},\tag{96}$$

where the product is taken in decreasing order of k.

A factorization of the two q-Bernoulli matrices.

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x)\right)^{-1}=\mathcal{D}_{\mathrm{NWA},n,q}\overline{\mathrm{F}_{n,2,q}}\star(x)\prod_{k=3}^{n}(\overline{\mathrm{F}_{n,k}}(x)\overline{\mathrm{E}_{n,k,q}}(x)).\tag{97}$$

Proof. Use formulas (33), (34) and (93). □

We now return to the q-Lie groups. We find that

Theorem 4.12. *The functions of q-Bernoulli polynomial matrices* $(\mathcal{B}_{\mathrm{NWA},q},\cdot,\cdot_q,\mathrm{I}_n)$ *and* $(\mathcal{B}_{\mathrm{JHC},q},\cdot,\cdot_q,\mathrm{I}_n)$ *with elements*

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x)\tag{98}$$

are q-Lie subgroups of $\mathcal{M}_{x,q}$.

Proof. The sets $\mathcal{B}$ are closed under the two operations by (55) and (57). The existence of inverses follows as for $\mathcal{M}_{x,q}$.

□

Definition 36. There are two types of q-Euler polynomials, called $\mathrm{F}_{\mathrm{NWA},\nu,q}(x)$, NWA q-Euler polynomials, and $\mathrm{F}_{\mathrm{JHC},\nu,q}(x)$, JHC q-Euler polynomials. They are defined by the following two generating functions:

$$\frac{2\mathrm{E}_q(xt)}{\mathrm{E}_q(t)+1}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\mathrm{F}_{\mathrm{NWA},\nu,q}(x),\ |t|<\pi,\tag{99}$$

and

$$\frac{2\mathrm{E}_q(xt)}{\mathrm{E}_{\frac{1}{q}}(t)+1}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\mathrm{F}_{\mathrm{JHC},\nu,q}(x),\ |t|<\pi.\tag{100}$$

Definition 37. We will use the following vector forms for these polynomials.

$$f_{\mathrm{NWA},n,q}(x) \equiv (\mathrm{F}_{\mathrm{NWA},0,q}(x), \mathrm{F}_{\mathrm{NWA},1,q}(x), \ldots, \mathrm{F}_{\mathrm{NWA},n-1,q}(x))^T. \tag{101}$$

The corresponding q-Euler number vectors are

$$f_{\mathrm{NWA},n,q} \equiv (\mathrm{F}_{\mathrm{NWA},0,q}, \mathrm{F}_{\mathrm{NWA},1,q}, \ldots, \mathrm{F}_{\mathrm{NWA},n-1,q})^T. \tag{102}$$

Let us introduce the two shifted q-Euler matrices.

Definition 38.

$$\mathcal{F}_{\mathrm{NWA},n,q}(x) \equiv (f_{\mathrm{NWA},q}(x) \;\; \mathrm{E}(\oplus_q) f_{\mathrm{NWA},q}(x) \;\cdots\; \mathrm{E}(\oplus_q)^{\overline{n-1}_q} f_{\mathrm{NWA},q}(x)). \tag{103}$$

We will need two similar matrices, based on the $\mathrm{F}_{\mathrm{NWA}}$ polynomials.

Definition 39. The two q-Euler polynomial matrices are defined by

$$\overline{F}_{\mathrm{NWA},n,q}(x)(i,j) \equiv \binom{i}{j}_q f_{\mathrm{NWA},i-j,q}(x). \tag{104}$$

Definition 40. The NWA and JHC q-Euler matrices are defined by

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(i,j) \equiv \binom{i}{j}_q \mathrm{F}_{\mathrm{NWA},i-j,q}, 0 \le i,j \le n-1. \tag{105}$$

Definition 41. The matrix $\mathcal{C}_{\mathrm{NWA},n,q}$ has matrix elements

$$c_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{1}{2}\left[1+\delta_{i-j,0}\right]\binom{i}{j}_q \text{ if } i \ge j, \\ 0 \text{ otherwise.} \end{cases} \tag{106}$$

Definition 42. The matrix $\mathcal{C}_{\mathrm{JHC},n,q}$ has matrix elements

$$c_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{1}{2}\left[q^{\binom{i-j}{2}}+\delta_{i-j,0}\right]\binom{i}{j}_q \text{ if } i \ge j, \\ 0 \text{ otherwise.} \end{cases} \tag{107}$$

Theorem 4.13. *The inverses of the q-Euler number matrices are given by*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}\right)^{-1} = \mathcal{C}_{\mathrm{NWA},n,q}. \tag{108}$$

This implies that

$$\overline{\mathrm{F}}^{-k}_{\mathrm{NWA},n,q} = \mathcal{C}^{k}_{\mathrm{NWA},n,q}. \tag{109}$$

Proof. For the NWA case, replace the factor $q^{\binom{k}{2}}$ by 1. We show that $\overline{\mathrm{F}}_{\mathrm{JHC},n,q}\mathcal{C}_{\mathrm{JHC},n,q}$ is equal to the unit matrix. We know that

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q \mathrm{F}_{\mathrm{JHC},n-k,q} + \mathrm{F}_{\mathrm{JHC},n,q} = 2\delta_{n,0}. \tag{110}$$

Introduce a function $G(k)$. Then we have

$$\begin{aligned}&\sum_{k=j}^{i}\binom{i}{k}_q \mathrm{F}_{\mathrm{JHC},i-k,q}G(k-j)\binom{k}{j}_q\\ &=\binom{i}{j}_q\sum_{k=j}^{i}\binom{i-j}{k-j}_q \mathrm{F}_{\mathrm{JHC},i-k,q}G(k-j)\\ &=\binom{i}{j}_q\sum_{k=0}^{i-j}\binom{i-j}{k}_q \mathrm{F}_{\mathrm{JHC},i-j-k,q}G(k)\overset{\text{by(110)}}{=}\binom{i}{j}_q\delta_{i-j,0}.\end{aligned} \tag{111}$$

It is now obvious that $G(k)=\frac{1}{2}\left[q^{\binom{k}{2}}+\delta_{k,0}\right]$ solves this equation for JHC and similar for NWA. □

Theorem 4.14. *Compare with [16, p. 1626] The inverses of the q-Euler polynomial matrices are given by*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x)\right)^{-1}=\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}\right)^{-1}\mathrm{P}_{n,q}(x)^{-1}=\mathcal{C}_{\mathrm{NWA},n,q}\mathrm{P}_{n,q}(x)^{-1}. \tag{112}$$

Theorem 4.15. *A factorization of the two q-Euler matrices.*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x)\right)^{-1}=\mathcal{C}_{\mathrm{NWA},n,q}\overline{\mathrm{F}_{n,2,q}}\star(x)\prod_{k=3}^{n}(\overline{\mathrm{F}_{n,k}}(x)\overline{\mathrm{E}_{n,k,q}}(x)). \tag{113}$$

We now return to the q-Lie groups.

Theorem 4.16. *The sets of q-Euler polynomial matrices* $(\mathcal{F}_{\mathrm{NWA},q},\cdot,\cdot_q,\mathrm{I}_n)$ *and* $(\mathcal{F}_{\mathrm{JHC},q},\cdot,\cdot_q,\mathrm{I}_n)$ *with elements*

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x) \tag{114}$$

are q-Lie subgroups of $\mathcal{M}_{x,q}$.

5 A related ring of matrices

We first recall the commutative ring of q–Appell polynomials.

Definition 43. [11] We denote the set of all q–Appell polynomials (in the variable x) by $\mathcal{A}_{x;q}$.

Let $\Phi_{n,q}(x)$ and $\Psi_{n,q}(x)$ be two elements in $\mathcal{A}_{x;q}$. Then the operations $\bigoplus$ and $\bigodot$ are defined as follows:

$$(\Phi_q(x)\bigoplus\Psi_q(x))_n\equiv(\Phi_q(x)+\Psi_q(x))_n, \tag{115}$$

$$(\Phi_q(x)\bigodot\Psi_q(x))_n\equiv(\Phi_q(x)\oplus_q\Psi_q(x))^n=\sum_{k=0}^{n}\binom{n}{k}_q\Phi_{n-k,q}(x)\Psi_{k,q}(x). \tag{116}$$

We keep the usual priority between $\bigoplus$ and $\bigodot$.

Theorem 5.1. *[8], [11]* $(\mathcal{A}_{x;q},\bigoplus,\bigodot)$ *is a commutative ring.*

In the following, for clarity, we sometimes write $f(t)$ for $f(t,q)$.

Definition 44. Let $f(t)\in\mathbb{R}[[x]]$. The q-deformed Leibniz functional matrix is given by

$$(\mathcal{L}_{n,q})[f(t,q)](i,j)\equiv\begin{cases}\frac{\mathrm{D}_{q,t}^{i-j}f(t,q)}{\{i-j\}_q!} & \text{if } i\geq j;\\ 0, & \text{otherwise}\end{cases}\quad i,j=0,1,2\ldots,n-1. \tag{117}$$

Let the operator $\epsilon : \mathbb{R}[[x]] \to \mathbb{R}[[x]]$ be defined by

$$\epsilon f(x) \equiv f(qx). \tag{118}$$

We infer that by the q-Leibniz formula [5]

$$(\mathcal{L}_{n,q})[f(t,q)g(t,q)] = (\mathcal{L}_{n,q})[f(t,q)] \cdot_{\epsilon} (\mathcal{L}_{n,q})[g(t,q)], \tag{119}$$

where in the matrix multiplication for every term which includes $\mathrm{D}_q^k f$, we operate with ϵ^k on g. We denote this by $\cdot_\epsilon$. This operator can also be iterated, compare with [5].

For the following considerations, compare with [6], [13, p. 232] and [14, p. 67].

Definition 45. Assume that $f(t) \in \mathbb{R}[[x]]$. The i, j entries of the generalized q-Pascal functional matrix $\mathcal{PF}_{n,q}[f(t,q)]$ of size $n \times n$ are

$$\mathcal{P}_{n,q}[f(t,q)](i,j) \equiv \begin{cases} \binom{i}{j}_q \mathrm{D}_{q,t}^{i-j} f(t,q) & \text{if } i \geq j; \\ 0, & \text{otherwise} \end{cases} \quad i,j = 0,1,2\ldots,n-1. \tag{120}$$

The function $f(t)$ is called invertible if $f(0) \neq 0$.

Example 1. When $f(t) \equiv \mathrm{E}_q(at)$ we have

$$\mathcal{P}_{n,q}[f(t,q)] \equiv \mathcal{P}_{n,q}[\mathrm{E}_q(at)] = \mathrm{P}_{n,q}(a). \tag{121}$$

Theorem 5.2. *[6] Formulas for the generalized q-Pascal functional matrix.*

$$\begin{aligned} &\mathcal{P}_{n,q}[f(t)+g(t)] = \mathcal{P}_{n,q}[f(t)] + \mathcal{P}_{n,q}[g(t)], \\ &\mathcal{P}_{n,q}[f(t)] \cdot_\epsilon \mathcal{P}_{n,q}[g(t)] = \mathcal{P}_{n,q}[f(t)g(t)]. \end{aligned} \tag{122}$$

Definition 46. [6] Assume that $f(t)$ is invertible. If the inverse $(f(t)^{-1})^{(k)}$ exists for $k < n$, we can define the q-inverse of the generalized q-Pascal functional matrix as

$$\left[\mathcal{P}_{n,q}[f(t,q)]\right]^{-1} \equiv \mathcal{P}_{n,q}[f(t,q)]^{-1}]. \tag{123}$$

Inspired by this, we make the following definition:

Definition 47. The operations $\bigoplus$ and $\bigodot$ are defined as follows:

$$\mathcal{P}_{n,q}[f(t)] \bigoplus \mathcal{P}_{n,q}[g(t)] \equiv \mathcal{P}_{n,q}[f(t)+g(t)]. \tag{124}$$

$$\mathcal{P}_{n,q}[f(t)] \bigodot \mathcal{P}_{n,q}[g(t)] \equiv \mathcal{P}_{n,q}[f(t)] \cdot_\epsilon \mathcal{P}_{n,q}[g(t)] = \mathcal{P}_{n,q}[f(t)g(t)]. \tag{125}$$

We keep the usual priority between $\bigoplus$ and $\bigodot$.

The generalized q-Pascal polynomial functional matrix $(\mathcal{PF}, f(t), \bigoplus, \bigodot, \mathcal{O}_n, \mathrm{I}_n)$ is the set of all $\mathcal{P}_{n,q}[f]$ with the operations $\bigoplus$ and $\bigodot$ and units $\mathcal{O}_n$ and I_n.

Theorem 5.3. *Assume that $\mathcal{O}_n$ is the unit for $\bigoplus$, and that I_n is the unit for $\bigodot$. The generalized q-Pascal polynomial functional matrix*

$$(\mathcal{PF}, f(t), \bigoplus, \bigodot, \mathcal{O}_n, \mathrm{I}_n) \tag{126}$$

is a commutative ring.

Proof. We presume that $\mathcal{P}_{n,q}[f]$, $\mathcal{P}_{n,q}[g]$ and $\mathcal{P}_{n,q}[h]$ are three elements in $\mathcal{PF}$ corresponding to $f(t)$, $g(t)$ and $h(t)$ respectively.

We first show that $\bigoplus$ is well-defined. Assume that $f(t)+g(t)\neq 0$. Then

$$\mathcal{P}_{n,q}[f(t)]\bigoplus\mathcal{P}_{n,q}[g(t)]\in\mathcal{PF}, \tag{127}$$

The associative law for $\bigoplus$ reads:

$$\begin{aligned}&\mathcal{P}_{n,q}[f(t)]\bigoplus\left(\mathcal{P}_{n,q}[g(t)]\bigoplus\mathcal{P}_{n,q}[h(t)]\right)\\&=\left(\mathcal{P}_{n,q}[f(t)]\bigoplus\mathcal{P}_{n,q}[g(t)]\right)\bigoplus\mathcal{P}_{n,q}[h(t)])).\end{aligned} \tag{128}$$

This follows from the associativity of +.

The commutative law for $\bigoplus$ reads:

$$\mathcal{P}_{n,q}[f(t)]\bigoplus\mathcal{P}_{n,q}[g(t)]=\mathcal{P}_{n,q}[g(t)]\bigoplus\mathcal{P}_{n,q}[f(t)]. \tag{129}$$

This follows from the commutativity of +. The identity element with respect to $\bigoplus$ is the zero matrix $\mathcal{O}_n$. We have

$$\mathcal{P}_{n,q}[f(t)]\bigoplus\mathcal{O}_n=\mathcal{P}_{n,q}[f(t)]. \tag{130}$$

There exists $-\mathcal{P}_{n,q}[f(t)]$ such that

$$\mathcal{P}_{n,q}[f(t)]\bigoplus-\mathcal{P}_{n,q}[f(t)]=\mathcal{O}_n. \tag{131}$$

This follows from the corresponding property of real numbers.

Then we show that $\bigodot$ is well-defined.

Assume that $f(t)g(t)\neq 0$. Then we have

$$\mathcal{P}_{n,q}[f(t)]\bigodot\mathcal{P}_{n,q}[g(t)]\text{ by}(125)\in\mathcal{PF}. \tag{132}$$

The associative law for $\bigodot$ reads:

$$\begin{aligned}&\mathcal{P}_{n,q}[f(t)]\bigodot\left(\mathcal{P}_{n,q}[g(t)]\bigodot\mathcal{P}_{n,q}[h(t)]\right)\\&=\left(\mathcal{P}_{n,q}[f(t)]\bigodot\mathcal{P}_{n,q}[g(t)]\right)\bigodot\mathcal{P}_{n,q}[h(t)])).\end{aligned} \tag{133}$$

This follows from the associativity of the multiplication $\cdot$.

The commutative law for $\bigodot$ reads:

$$\mathcal{P}_{n,q}[f(t)]\bigodot\mathcal{P}_{n,q}[g(t)]=\mathcal{P}_{n,q}[g(t)]\bigodot\mathcal{P}_{n,q}[f(t)]. \tag{134}$$

This follows from the commutativity of the multiplication $\cdot$.

The identity element with respect to $\bigodot$ is the identity matrix I_n. We have

$$\mathcal{P}_{n,q}[f(t)]\bigodot\mathrm{I}_n=\mathcal{P}_{n,q}[f(t)]. \tag{135}$$

The distributive law reads:

$$\begin{aligned}&\mathcal{P}_{n,q}[f(t)]\bigodot\left(\mathcal{P}_{n,q}[g(t)]\bigoplus\mathcal{P}_{n,q}[h(t)]\right)\\&=\mathcal{P}_{n,q}[f(t)(g(t)+h(t))]=\mathcal{P}_{n,q}[f(t)g(t)+f(t)h(t)]\\&=\left(\mathcal{P}_{n,q}[f(t)]\bigodot\mathcal{P}_{n,q}[g(t)]\right)\bigoplus\left(\mathcal{P}_{n,q}[f(t)]\bigodot\mathcal{P}_{n,q}[h(t)]\right).\end{aligned} \tag{136}$$

This follows from the distributive law of real numbers.

□

6 Conclusion

We have united and q-deformed formulas from papers by Arponen, Aceto et al., Yang et al. and Zhang et al. to give a first synthesis of q-Appell polynomial matrices, which were previously only known in special cases. Some of the formulas for q-Pascal matrices and their factorizations are generalized as well as formulas for q-Bernoulli and q-Euler matrices. We have given the first concrete examples of q-Lie subgroups, and constructed a similar ring of matrices with many pleasant properties.

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