Research Article

Thomas Ernst*

On the $q$-Lie group of $q$-Appell polynomial matrices and related factorizations

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Abstract: In the spirit of our earlier paper [10] and Zhang and Wang [16], we introduce the matrix of multiplicative $q$-Appell polynomials of order $M \in \mathbb{Z}$. This is the representation of the respective $q$-Appell polynomials in $ke$-$ke$ basis. Based on the fact that the $q$-Appell polynomials form a commutative ring [11], we prove that this set constitutes a $q$-Lie group with two dual $q$-multiplications in the sense of [9]. A comparison with earlier results on $q$-Pascal matrices gives factorizations according to [7], which are specialized to $q$-Bernoulli and $q$-Euler polynomials. We also show that the corresponding $q$-Bernoulli and $q$-Euler matrices form $q$-Lie subgroups. In the limit $q \to 1$ we obtain corresponding formulas for Appell polynomial matrices. We conclude by presenting the commutative ring of generalized $q$-Pascal functional matrices, which operates on all functions $f \in C^\infty_q$.

Keywords: $q$-Lie group; multiplicative $q$-Appell polynomial matrix; commutative ring; $q$-Pascal functional matrix

MSC: Primary 17B99; Secondary 17B37, 33C80, 15A23

1 Introduction

In this paper we will introduce several new concepts, some of which were previously known only in the $q$-case from the articles of the author. By the logarithmic method for $q$-calculus, this transition will be almost automatic, with the $q$-addition being replaced by ordinary addition. Some of the matrix formulas in this paper were previously published for Bernoulli polynomials in [16] and for Pascal matrices in [17]. In the article [9] $q$-Lie matrix groups with two dual multiplications, and in [8] the concept multiplicative $q$-Appell polynomial were introduced. Now the interesting situation occurs, that the formula [16, p. 1623] for Bernoulli polynomial matrices, which are multiplicative Appell polynomial matrices, also holds for the latter ones. Thus we devote Section 2 to Lie groups of Appell matrices and to the new morphism formula (18). But first we repeat the summation matrix $G_{n,k}(x)$ and the difference matrix $F_{n,k}(x)$ and all the other matrices from [15] in Section 1.

To prepare for the matrix factorizations of the $q$-Lie matrices in Section 4, we present the relevant $q$-Pascal and $q$-unit matrices from [7] in Section 3. In Subsection 4.2 we first repeat the matrix forms of the $q$-Bernoulli and $q$-Euler polynomials from [10] to prepare for the computation of their inverses and factorizations. The main purpose of Section 4 is the introduction of the multiplicative $q$-Appell polynomial matrix and its functional equation, a general so-called $q$-morphism. In Section 4.1 generalizations of factorizations of Bernoulli matrices to $q$-Appell polynomial matrices are presented. Finally, in Section 5 the existence of a commutative ring of generalized $q$-Pascal polynomial functional matrices is proved.

We start our presentation with a brief repetition of some of our matrices.

*Corresponding Author: Thomas Ernst: Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden, E-mail: thomas@math.uu.se
Definition 1. Matrix elements will always be denoted \((i, j)\). Here \(i\) denotes the row and \(j\) denotes the column. The matrix elements range from 0 to \(n - 1\). The matrices \(I_n, S_n, A_n, D_n, S_n(x)\) and \(D_n(x)\) are defined by

\[I_n \equiv \text{diag}(1, 1, \ldots, 1)\] (1)

\[S_n(i, j) \equiv \begin{cases} 1, & \text{if } j \leq i, \\ 0, & \text{if } j > i, \end{cases}\] (2)

\[A_n(t)(i, j) \equiv \begin{cases} t^i, & \text{if } j = i, \\ 0, & \text{otherwise} \end{cases}\] (3)

\[D_n(i, i) \equiv 1 \text{ for all } i,\] (4)

\[D_n(i + 1, i) \equiv -1, \text{ for } i = 0, \ldots, n - 2\] (5)

\[D_n(i, j) \equiv 0, \text{ if } j > i \text{ or } j < i - 1\] (6)

\[S_n(x)(i, j) \equiv \begin{cases} x^{i-j}, & \text{if } j \leq i, \\ 0, & \text{if } j > i, \end{cases}\] (7)

\[D_n(x; i, i) \equiv 1, \text{ if } i = 0, \ldots, n - 1, \text{ for } i = 0, \ldots, n - 2,\]

\[D_n(x; i, j) \equiv 0, \text{ when } j > i \text{ or } j < i - 1.\] (8)

We note that \(D_n\) is a special case of \(D_n(x)\), and \(S_n\) is a special case of \(S_n(x)\).

The summation matrix \(\overline{G}_{n,k}(x)\) and its inverse, the difference matrix \(\overline{F}_{n,k}(x)\), are defined by [15, p. 52,54]:

\[
\overline{G}_{n,k}(x) \equiv \begin{bmatrix} I_{n-k} & 0^T \\ 0 & S_k(x) \end{bmatrix}, \quad k = 3, \ldots, n, \quad \overline{G}_{n,n}(x) \equiv S_n(x), \quad n > 2,
\]

\[
\overline{F}_{n,k}(x) \equiv \begin{bmatrix} I_{n-k} & 0^T \\ 0 & D_k(x) \end{bmatrix}, \quad k = 3, \ldots, n, \quad \overline{F}_{n,n}(x) \equiv D_n(x), \quad n > 2.
\] (9)

2 The Lie group of Appell matrices

We first define Appell polynomials and multiplicative Appell polynomials.

Definition 2. Let \(\mathcal{A}\) denote the set of real sequences \(\{u_v\}_v \in \mathbb{R}\) such that

\[
\sum_{v=0}^{\infty} |u_v| \frac{r^v}{v!} < \infty,
\] (10)

for some convergence radius \(r > 0\).

Definition 3. For \(f_n(t) \in \mathbb{R}[[t]]\), let \(p_v \in \mathcal{A}\) and let \(p_v^{(n)}\) denote the Appell numbers of degree \(v\) and order \(n \in \mathbb{Z}\) with the following generating function

\[
f_n(t) = \sum_{v=0}^{\infty} \frac{t^v}{v!} p_v^{(n)}.
\] (11)
**Definition 4.** For every formal power series \( f_n(t) = h(t)^n \), let \( p_{M,ν} \in A \) and let \( p^{[n]}_{M,ν} \) denote the multiplicative Appell numbers of degree \( ν \) and order \( n \in \mathbb{Z} \) with the following generating function

\[
h(t)^n = \sum_{ν=0}^{∞} \frac{t^n}{ν!} p^{[n]}_{M,ν}.
\]  

(12)

**Definition 5.** For every formal power series \( f_n(t) = h(t)^n \) given by (12), the multiplicative Appell polynomials or \( p^{(n)}_ν(x) \) polynomials of degree \( ν \) and order \( n \in \mathbb{Z} \) have the following generating function

\[
f_n(t)e^{xt} = \sum_{ν=0}^{∞} \frac{t^n}{ν!} p^{(n)}_ν(x).
\]  

(13)

The proof of the following formula is relegated to (43).

**Theorem 2.1.** Assume that \( M \) and \( K \) are the \( x \)-order and \( y \)-order, respectively.

\[
p^{(M+K)}_ν(x + y) = \sum_{k=0}^{ν} \frac{ν!}{k!(ν-k)!} p^{(M)}_k(x)p^{(K)}_{ν-k}(y).
\]  

(14)

**Definition 6.** We will use the following vector forms for the Appell polynomials and numbers:

\[
Π_n(x) \equiv (p_0(x), p_1(x), \ldots, p_{n-1}(x))^T,
\]  

(15)

\[
Π_n \equiv Π_n(0).
\]  

(16)

**Definition 7.** The multiplicative Appell polynomial matrix of order \( M \in \mathbb{Z} \) is defined by

\[
p^{(M)}_n(x, y) = \left(\begin{array}{c} i \\ j \end{array}\right) p^{(M)}_n(i-j, 0 ≤ i, j ≤ n-1).
\]  

(17)

We refer to (56) for the proof of the next theorem.

**Theorem 2.2.** In the following formula we assume that \( M \) and \( K \) are the \( x \)-order and \( y \)-order, respectively.

\[
p^{(M+K)}_n(x + y) = p^{(M)}_n(x)p^{(K)}_n(y).
\]  

(18)

**Theorem 2.3.** The multiplicative Appell polynomial matrices \((M, ⋄)\) with elements \( p^{(M)}_n(x) \) is an Abelian matrix Lie group with multiplication given by (18) and inverse \( p^{(-M)}_n(-x) \).

**Proof.** The set \( M \) is closed under the operation \( ⋄ \) by (18). The group element \( p^{(-M)}_n(-x) \) is inverse to \( p^{(M)}_n(x) \) by the subtraction of real numbers. The unit element is the unit matrix \( I_n \). The associativity and commutativity follow by (18).

\[\square\]

### 3 The \( q \)-Pascal matrix and the \( q \)-unit matrices

**Definition 8.** The \( q \)-Pascal matrix \( P_{n,q}(x) \) [7] is given by the familiar expression

\[
P_{n,q}(i,j) = \binom{i}{j}_q x^{i-j}, i ≥ j.
\]  

(19)

The following special case is often used.
Definition 9.

\[ P_{n,q} = P_{n,q}(1). \]  

We now recall some formulas from [7].

\begin{equation}
\mathbf{P}_{n,k,q}(x), \mathbf{P}_{k,q}^* (x) \text{ and } \mathbf{P}_{n,k,q}^* (x) \text{ are defined by}
\end{equation}

\begin{equation}
\mathbf{P}_{n,k,q}(x) \equiv \begin{bmatrix}
I_{n-k} & 0^T \\
0 & P_{k,q}(x)
\end{bmatrix}, 
\end{equation}

\begin{equation}
P_{k,q}^* (x; i, j) = \binom{i}{j}_q (qx)^{j-i}, i, j = 0, \ldots, k - 1,
\end{equation}

\begin{equation}
\mathbf{P}_{n,k,q}^* (x) \equiv \begin{bmatrix}
I_{n-k} & 0^T \\
0 & \mathbf{P}_{k,q}^* (x)
\end{bmatrix}, k = 3, \ldots, n, \mathbf{P}_{n,n,q}^* (x) \equiv \mathbf{P}_{n,q}^* (x).
\end{equation}

Let the two matrices \( \mathbf{I}_{k,q}(x) \), and its inverse, \( \mathbf{E}_{k,q}(x) \), be given by:

\begin{equation}
\mathbf{I}_{k,q}(x; i, j) \equiv 1, i = 0, \ldots, k - 1, \mathbf{I}_{k,q}(x; i + 1, i) \equiv x(q^{i+1} - 1), i = 0, \ldots, k - 1, \\
\mathbf{I}_{k,q}(x; i, j) \equiv 0 \text{ for other } i, j.
\end{equation}

\begin{equation}
\mathbf{E}_{k,q}(x; i, j) \equiv (j + 1; q)_{i-j}x^{i-j}, i \geq j, \mathbf{E}_{k,q}(x; i, j) \equiv 0 \text{ for other } i, j.
\end{equation}

Similarly, let the two matrices \( \mathbf{I}_{n,k,q}(x) \), and its inverse, \( \mathbf{E}_{n,k,q}(x) \), be given by:

\begin{equation}
\mathbf{I}_{n,k,q}(x) \equiv \begin{bmatrix}
I_{n-k} & 0^T \\
0 & \mathbf{I}_{k,q}(x)
\end{bmatrix}, \mathbf{I}_{n,n,q}(x) \equiv \mathbf{I}_n.
\end{equation}

\begin{equation}
\mathbf{E}_{n,k,q}(x) \equiv \begin{bmatrix}
I_{n-k} & 0^T \\
0 & \mathbf{E}_{k,q}(x)
\end{bmatrix}, \mathbf{E}_{n,n,q}(x) \equiv \mathbf{I}_n.
\end{equation}

We call \( \mathbf{I}_{n,k,q}(x) \) the \( q \)-unit matrix function. We will use a slightly \( q \)-deformed version of the D- and F-matrices:

\begin{equation}
\mathbf{D}_{k,q}^* (x; i, i) \equiv 1, i = 0, \ldots, k - 1, \mathbf{D}_{k,q}^* (x; i + 1, i) \equiv -qx^i, i = 0, \ldots, k - 1, \\
\mathbf{D}_{k,q}^* (x; i, j) \equiv 0 \text{ if } j > i \text{ or } j < i - 1.
\end{equation}

\begin{equation}
\mathbf{F}_{n,k,q}^* (x) \equiv \begin{bmatrix}
I_{n-k} & 0^T \\
0 & \mathbf{D}_{k,q}^* (x)
\end{bmatrix}.
\end{equation}

The \( q \)-summation matrices are defined by

\begin{equation}
\mathbf{G}_k^* (x) \equiv \begin{cases}
Q E \left( \frac{(i-j-1)}{2} \right) + j(i-j) \right) x^{i-j}, & \text{if } j \leq i, \\
0, & \text{if } j > i,
\end{cases}
\end{equation}

\begin{equation}
\mathbf{G}_{n,k,q}^* (x) \equiv \begin{bmatrix}
I_{n-k} & 0^T \\
0 & \mathbf{G}_{k,q}^* (x)
\end{bmatrix}.
\end{equation}

We have the inverse relation:

\begin{equation}
\mathbf{F}_{n,k,q}^* (x)^{-1} = \mathbf{G}_{n,k,q}^* (x).
\end{equation}

The inverse of \( \mathbf{P}_{k,q}^* (x) \) is given by

\begin{equation}
(\mathbf{P}_{k,q}^* (x))^{-1}(i, j) = \binom{i}{j}_q (-x)^{j-i} q^{\binom{i-j}{2}}, i, j = 0, \ldots, k - 1.
\end{equation}
The following matrix will be used in formula (52).

**Definition 11.** The $q$-Cauchy matrix is given by

$$W_{n,q}(x)(i, j) \equiv (x \oplus_q T_q)^i.$$  

(32)

**Theorem 3.1.** [7]. A $q$-analogue of [15, p.53 (1)]. If $n \geq 3$, the $q$-Pascal matrix $P_{n,q}(x)$ can be factorized by the summation matrices and by the $q$-unit matrices as

$$P_{n,q}(x) = \prod_{k=3}^3 \left( \frac{1}{3} \left( \sum_{k=3}^n (\frac{1}{3} \sum_{k=3}^n g_{n,k}(x)) \ G_{n,2,q} \right)^* (x), \right.$$  

(33)

where the product is taken in decreasing order of $k$.

**Theorem 3.2.** [7]. A $q$-analogue of [15, p. 54]. The inverse of the $q$-Pascal matrix is given by

$$P_{n,q}(x)^{-1} = \prod_{k=3}^n \left( \frac{1}{3} \left( \sum_{k=3}^n (\frac{1}{3} \sum_{k=3}^n g_{n,k}(x)) \ E_{n,2,q} \right)^* (x). \right.$$  

(34)

### 4 The $q$-Lie group of $q$-Appell polynomial matrices

We first repeat and extend some definitions from [9].

**Definition 12.** A $q$-Lie group $(G_{n,q}, \cdot_q, I_q) \supseteq \mathbb{E}_q(g_q)$, is a possibly infinite set of matrices $\in GL_q(n, \mathbb{R})$, and a manifold, with two multiplications: $\cdot$, the usual matrix multiplication, and the twisted $\cdot_q$, which is defined separately. Each $q$-Lie group has a unit, denoted by $I_q$, which is the same for both multiplications. Each element $\Phi \in G_{n,q}$ has an inverse $\Phi^{-1}$ with the property $\Phi \cdot_q \Phi^{-1} = I_q$.

**Definition 13.** If $(G_1, \cdot_1, \cdot_1_q)$ and $(G_2, \cdot_2, \cdot_2_q)$ are two $q$-Lie groups, then $(G_1 \times G_2, \cdot_q)$ is a $q$-Lie group called the product $q$-Lie group. This has group operations defined by

$$(g_{11}, g_{21}) \cdot_q (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}).$$  

(35)

and

$$(g_{11}, g_{21}) \cdot_q (g_{12}, g_{22}) = (g_{11} \cdot_1 q g_{12}, g_{21} \cdot_2 q g_{22}).$$  

(36)

**Definition 14.** If $(G_{n,q}, \cdot_1, \cdot_1_q)$ is a $q$-Lie group and $H_{n,q}$ is a nonempty subset of $G_{n,q}$, then $(H_{n,q}, \cdot_1, \cdot_1_q)$ is called a $q$-Lie subgroup of $(G_{n,q}, \cdot_1, \cdot_1_q)$ if

1. $\Phi \cdot \Psi \in H_{n,q}$ and $\Phi \cdot_q \Psi \in H_{n,q}$ for all $\Phi, \Psi \in H_{n,q}$.
2. $\Phi^{-1} \in H_{n,q}$ for all $\Phi \in H_{n,q}$.
3. $H_{n,q}$ is a submanifold of $G_{n,q}$.

**Definition 15.** An invertible mapping $f : (G_{n,q}, \cdot_1, \cdot_1_q) \to (H_{n,q}, \cdot_2, \cdot_2_q)$ is called a $q$-Lie group morphism between $(G_{n,q}, \cdot_1, \cdot_1_q)$ and $(H_{n,q}, \cdot_2, \cdot_2_q)$ if

$$f(\phi \cdot_1 \psi) = f(\phi) \cdot_2 f(\psi), \text{ and } f(\phi \cdot_{1,q} \psi) = f(\phi) \cdot_{2,q} f(\psi).$$  

(39)

It is obvious that $(\mathbb{Z}, +)$ is a $q$-Lie group with only one operation. We will use this fact in formula (61).

The most general form of polynomial in this article is the $q$-Appell polynomial, which we will now define.
**Definition 16.** Let \( \mathcal{A}_q \) denote the set of real sequences \( \{ u_{v,q} \}_{v=0}^{\infty} \) such that
\[
\sum_{v=0}^{\infty} |u_{v,q}| \frac{r^v}{q^v} < \infty,
\]
for some \( q \)-dependent convergence radius \( r = r(q) > 0 \), where \( 0 < q < 1 \).

**Definition 17.** Assume that \( h(t, q), h(t, q)^{-1} \in \mathbb{R}[[t]] \). For \( f_n(t, q) = h(t, q)^n \), let \( \Phi_{v,q} \in \mathcal{A}_q \) and let \( \Phi_{v,q}^{(n)} \) denote the multiplicative \( q \)-Appell number of degree \( v \) and order \( n \) given by the generating function
\[
f_n(t, q) = \sum_{v=0}^{\infty} \frac{t^v}{(v)_q!} \Phi_{v,q}^{(n)}, \quad \Phi_{0,q} = 1, \quad n \in \mathbb{Z}.
\]

**Definition 18.** For every formal power series \( f_n(t, q) \) given by (41), the multiplicative \( q \)-Appell polynomials or \( \Phi_q \) polynomials of degree \( v \) and order \( n \) have the following generating function:
\[
f_n(t, q)E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v}{(v)_q!} \Phi_{v,q}^{(n)}(x), \quad n \in \mathbb{Z}.
\]

**Theorem 4.1.** In the following formula we assume that \( M \) and \( K \) are the x-order and y-order, respectively.
\[
\Phi_{v,q}^{(M+K)}(x \oplus_q y) = \sum_{k=0}^{v} \binom{v}{k}_q \Phi_k^{(M)}(x) \Phi_{v-k,q}^{(K)}(y).
\]

**Proof.** This is proved in the same way as in [4, 4.242, p. 136].

The following vector forms for \( q \)-Appell polynomials and numbers will be used in formulas (80), (81), (101) and (102).

**Definition 19.**
\[
\phi_{n,q}(x) \equiv (\Phi_{0,q}(x), \Phi_{1,q}(x), \ldots, \Phi_{n-1,q}(x))^T,
\]
\[
\phi_{n,q} \equiv \phi_{n,q}(0).
\]

**Definition 20.** Define the \( q \)-Appell polynomial matrix by
\[
\overline{T}_{n,q}(i, j) \equiv \begin{pmatrix} i \cr j \end{pmatrix}_q \Phi_{i-j,q}(x), \quad 0 \leq i, j \leq n - 1.
\]

**Definition 21.** The multiplicative \( q \)-Appell polynomial matrices \( (M_{x,q}) \) with elements \( \overline{T}_{n,q}^{(M)}(x) \) of order \( M \in \mathbb{Z} \) are defined by
\[
\overline{T}_{n,q}^{(M)}(i, j) \equiv \begin{pmatrix} i \cr j \end{pmatrix}_q \Phi_{i-j,q}^{(M)}(x), \quad 0 \leq i, j \leq n - 1.
\]

**Definition 22.** The multiplicative \( q \)-Appell number matrices or the \( q \)-transfer matrices \( (M_q) \) with elements \( \overline{T}_{n,q}^{(M)} \) of order \( M \in \mathbb{Z} \) are defined by
\[
\overline{T}_{n,q}^{(M)}(i, j) \equiv \overline{T}_{n,q}^{(M)}(0)(i, j), \quad 0 \leq i, j \leq n - 1.
\]

**Theorem 4.2.** A \( q \)-analogue of [2, (3.9), p. 432]
\[
\overline{T}_{n,q}(x) = \overline{T}_{n,q}^{(M)} \Phi_n(x), \quad \text{where}
\]
\[ \xi_n(x) \equiv (1, x, x^2, \ldots, x^{n-1})^T. \] (50)

We define a generalization of formulas (82) and (103).

**Definition 23.** The shifted \( q \)-Appell polynomial matrix \( \Phi_{n,q}(x) \) is defined by

\[ \Phi_{n,q}(x)(i, j) \equiv \Phi_{i,q}(x \oplus_q j_q), \quad 0 \leq i, j \leq n - 1. \] (51)

**Corollary 4.3.** A generalization of [10]. The shifted \( q \)-Appell polynomial matrix can be written as the product of the \( q \)-Appell number matrix and the \( q \)-Cauchy matrix.

\[ \Phi_{n,q}(x) = \Phi_{n,q} W_{n,q}(x). \] (52)

**Proof.** We show that the matrix indices are equal.

\[ \sum_{k=0}^{i} \binom{i}{k} \Phi_{i-k,q}(x \oplus_q k_q) = \Phi_{i,q}(x \oplus_q i_q). \] (53)

We remark that a special case of this equation can be found in [16, p. 1631].

In [7] we proved the formula

\[ P_{n,q}(s \oplus_q t) = P_{n,q}(s)P_{n,q}(t), \quad s, t \in \mathbb{R}. \] (54)

This can be generalized to

**Theorem 4.4.** We assume that \( M \) and \( K \) are the \( x \)-order and \( y \)-order, respectively. The formula (43) can be rewritten in the following matrix form, where \( \cdot \) on the RHS denotes matrix multiplication.

\[ \Phi_{n,q}^{(M+K)}(x \oplus_q y) = \Phi_{n,q}^{(M)}(x) \cdot \Phi_{n,q}^{(K)}(y). \] (55)

For the following proof, compare with [16, p. 1624].

**Proof.** We compute the \((i, j)\) matrix element of the matrix multiplication on the RHS.

\[ \sum_{k=0}^{i} \binom{i}{k} \Phi_{i-k,q}^{(M)}(x) \cdot \Phi_{k-j,q}^{(K)}(y) = \sum_{k=0}^{i} \binom{i}{k} \Phi_{i-k,q}^{(M)}(x) \cdot \Phi_{k-j,q}^{(K)}(y) = \Phi_{i-j,q}^{(M+K)}(x \oplus_q y) = \text{LHS}. \] (56)

By formula (47), the \( \Phi_{n,q}^{(M)}(x) \) are matrices with matrix elements \( q \)-Appell polynomials multiplied by \( q \)-binomial coefficients, and we arrive at the next crucial definition.

**Definition 24.** We define the second matrix multiplication \( \bigoplus_q \) by

\[ \Phi_{n,q}^{(M)}(x) \bigoplus_q \Phi_{n,q}^{(K)}(y) \equiv \Phi_{n,q}^{(M+K)}(x \oplus_q y). \] (57)

**Theorem 4.5.** The set \( (M_{x,q}, \ldots, q, 1_n) \) is a \( q \)-Lie group with multiplications given by (55) and (57), and inverse \( \Phi_{n,q}^{(-M)}(x) \). The unit element is the unit matrix \( 1_n \).
Proof. The set $M_{x,q}$ is closed under the two operations by (55) and (57). By (57) we have

$$
\overline{\Phi}_n^{(M)}(x) \cdot q \overline{\Phi}_n^{(-M)}(x) = \overline{\Phi}_n^{(0)}(\theta) = I_n,
$$

which shows the existence of an inverse element and a unit.

The associative law reads:

$$
(\overline{\Phi}_n^{(M)}(x) \cdot \overline{\Phi}_n^{(K)}(y)) \cdot q \overline{\Phi}_n^{(f)}(z) = \overline{\Phi}_n^{(M)}(x) \cdot (\overline{\Phi}_n^{(K)}(y) \cdot q \overline{\Phi}_n^{(f)}(z)) \quad (59)
$$

which is equivalent to

$$
\overline{\Phi}_n^{(M+K+f)}((x \oplus y) \oplus q z) = \overline{\Phi}_n^{(M+K+f)}(x \oplus y \oplus q z).
$$

(60)

However, formula (60) follows from the associativity of the two $q$-additions.

\[ \square \]

**Definition 25.** In the definition of product $q$-Lie group, put

$$
(R_q, \cdot, \cdot q) \equiv (\mathbb{R}_q, \oplus q, \ominus q) \times (\mathbb{Z}, +) \quad (61)
$$

It is clear that formulas (55) and (57) defines a $q$-Lie group morphism from $R_q$ to $M_{x,q}$.

Let

$$
(\overline{\Phi}_n^{(M)}(x))^k \equiv \overline{\Phi}_n^{(M)}(x) \cdot \overline{\Phi}_n^{(M)}(x) \cdot \ldots \cdot \overline{\Phi}_n^{(M)}(x),
$$

where the right hand side denotes the product of $k$ equal matrices $\overline{\Phi}_n^{(M)}(x)$.

In [7] we proved the formula

$$
P_{n,q}^k = P_{n,q}(\overline{R}_q).
$$

(63)

This can be generalized to

$$
(\overline{\Phi}_n^{(M)}(x))^k = \overline{\Phi}_n^{(M)}(\overline{R}_q x).
$$

(64)

Furthermore, the formulas in [16, p. 1624] can be generalized to the special cases

$$
(\overline{\Phi}_n^{(M)})^k = \overline{\Phi}_n^{(M)}, \quad (\overline{\Phi}_n^{(1)})^k = \overline{\Phi}_n^{(k)}.
$$

(65)

### 4.1 Two factorizations

We show that our $q$-Appell polynomials allow simple extensions to factorizations by Fibonacci number matrices. The first Fibonacci numbers $F_n$ have the following values:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

**Definition 26.** [12] We use the following notation for the matrix form of the Fibonacci numbers: $F_n(i, j) \equiv F_{i-j}$.

It follows that [12, p. 205]

$$
F_n^{-1}(i, i) \equiv 1, i = 0, \ldots, n - 1, \quad F_n^{-1}(i + 1, i) \equiv -1, i = 0, \ldots, n - 2,
$$

$$
F_n^{-1}(i + 2, i) \equiv -1, i = 0, \ldots, n - 3, \quad F_n^{-1}(i, j) = 0 \text{ otherwise.}
$$

(66)

**Definition 27.** The matrix $M_{n,q}(x)$ has matrix elements

$$
m_{i,j} = \binom{i}{j}_q \Phi_{i-j,q}(x) - \binom{i-1}{j}_q \Phi_{i-j-1,q}(x) - \binom{i-2}{j}_q \Phi_{i-j-2,q}(x).
$$

(67)
For the following formula, compare with [16, p. 1627], where the corresponding formula for the Bernoulli matrix was given. Note that we do not need the order of the polynomials.

**Theorem 4.6.** The $q$-Appell polynomial matrix can be factorized as

$$\mathfrak{T}_{n,q}(x) = \mathcal{T}_n \mathcal{M}_{n,q}(x).$$  \hfill (68)

**Proof.** It would suffice to prove that

$$\mathcal{T}_n^{-1} \mathfrak{T}_{n,q}(x) = \mathcal{M}_{n,q}(x).$$  \hfill (69)

The matrix index of the left hand side is given by

$$\mathcal{T}_n^{-1} \mathfrak{T}_{n,q}(x)(i,j) = \sum_{k=0}^{n-1} \mathcal{T}_n^{-1}(i,k) \Phi_{k-j,q}(x)$$

$$= \left( \begin{array}{c} i \\ j \end{array} \right)_q \Phi_{i-j,q}(x) - \left( \begin{array}{c} i - 1 \\ j \end{array} \right)_q \Phi_{i-j-1,q}(x) - \left( \begin{array}{c} i - 2 \\ j \end{array} \right)_q \Phi_{i-j-2,q}(x) = \text{RHS}. \hfill (70)$$

We shall now prove a similar formula.

**Definition 28.** The matrix $\mathcal{R}_{n,q}(x)$ has matrix elements

$$r_{i,j} = \left( \begin{array}{c} i \\ j \end{array} \right)_q \Phi_{i-j,q}(x) - \left( \begin{array}{c} i \\ j + 1 \end{array} \right)_q \Phi_{i-j-1,q}(x) - \left( \begin{array}{c} i \\ j + 2 \end{array} \right)_q \Phi_{i-j-2,q}(x).$$  \hfill (71)

For the following formula, compare with [17, p. 2372], where the corresponding formula for Pascal matrices was given.

**Theorem 4.7.** The $q$-Appell polynomial matrix can be factorized as

$$\mathfrak{T}_{n,q}(x) = \mathcal{R}_{n,q}(x) \mathcal{T}_n.$$  \hfill (72)

**Proof.** It suffices to prove that

$$\mathfrak{T}_{n,q}(x) \mathcal{T}_n^{-1} = \mathcal{R}_{n,q}(x).$$  \hfill (73)

The matrix index of the left hand side is given by

$$\mathfrak{T}_{n,q}(x) \mathcal{T}_n^{-1}(i,j) = \sum_{k=0}^{n-1} \mathfrak{T}_{n,q}(x) \mathcal{T}_n^{-1}(k,j)$$

$$= \left( \begin{array}{c} i \\ j \end{array} \right)_q \Phi_{i-j,q}(x) - \left( \begin{array}{c} i \\ j + 1 \end{array} \right)_q \Phi_{i-j-1,q}(x) - \left( \begin{array}{c} i \\ j + 2 \end{array} \right)_q \Phi_{i-j-2,q}(x) = \text{RHS}. \hfill (74)$$

\[\square\]

### 4.2 $q$-Bernoulli and $q$-Euler polynomials

We will also consider the special cases $q$-Bernoulli and $q$-Euler polynomials.

**Definition 29.** There are two types of $q$-Bernoulli polynomials, called $B_{\text{NWA},v,q}(x)$, NWA $q$-Bernoulli polynomials, and $B_{\text{JHC},v,q}(x)$, JHC $q$-Bernoulli polynomials. They are defined by the two generating functions

$$\frac{t}{(E_q(t) - 1)} E_q(xt) = \sum_{\nu=0}^{\infty} t^\nu B_{\text{NWA},v,q}(x) \frac{\nu!}{q^\nu}, \quad |t| < 2\pi.$$  \hfill (75)
and
\[
\frac{t}{(E_q(t) - 1)} E_q(x t) = \sum_{v=0}^{\infty} \frac{t^v B_{v,q}(x)}{(v)_q!}, \quad |t| < 2\pi. \tag{76}
\]

**Definition 30.** The Ward $q$-Bernoulli numbers are given by
\[
B_{\text{NWA},n,q} \equiv B_{\text{NWA},n,q}(0). \tag{77}
\]
The Jackson $q$-Bernoulli numbers are given by
\[
B_{\text{JHC},n,q} \equiv B_{\text{JHC},n,q}(0). \tag{78}
\]
The following table lists some of the first Ward $q$-Bernoulli numbers.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$(1 + q)^{-1}$</td>
<td>$q^2((3)_q!)^{-1}$</td>
<td>$(1 - q)q^3((2)_q)^{-1}((4)_q)^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To save space, we will use the following abbreviation in equations (80) - (84), (87), (88), (91), (95), (97), (98), (101)-(105), (108)-(109), (112)-(114).

\[
\text{NWA} = \text{NWA} \lor \text{JHC}. \tag{79}
\]

We will use the following vector forms for the $q$-Bernoulli polynomials corresponding to $q$-analogues of [1, p. 239].

\[
b_{\text{NWA},n,q}(x) \equiv (B_{\text{NWA},0,q}(x), B_{\text{NWA},1,q}(x), \ldots, B_{\text{NWA},n-1,q}(x))^T. \tag{80}
\]
The corresponding vector forms for numbers are

\[
b_{\text{NWA},n,q} \equiv (B_{\text{NWA},0,q}, B_{\text{NWA},1,q}, \ldots, B_{\text{NWA},n-1,q})^T. \tag{81}
\]

Let us introduce the NWA and JHC shifted $q$-Bernoulli matrices.

**Definition 31.**
\[
\mathcal{B}_{\text{NWA},n,q}(x) \equiv (b_{\text{NWA},q}(x), E(\oplus_q) b_{\text{NWA},q}(x) \cdots E(\oplus_q)^{n-1} b_{\text{NWA},q}(x)), \tag{82}
\]
where $E(\oplus_q)^{n-1} b_{\text{NWA},q}(x) \equiv (x \oplus_q n^{-1} q)^n$.

We will need two similar matrices based on the $B_{\text{NWA}}$ and $B_{\text{JHC}}$ polynomials and numbers.

**Definition 32.** Two $q$-analogues of [3, p. 193]. The NWA and JHC $q$-Bernoulli polynomial matrices are defined by

\[
\mathcal{B}_{\text{NWA},n,q}(x)(i,j) \equiv \binom{i}{j}_q B_{\text{NWA},i-j,q}(x), 0 \leq i,j \leq n-1. \tag{83}
\]

**Definition 33.** The NWA and JHC $q$-Bernoulli number matrices are defined by

\[
\mathcal{B}_{\text{NWA},n,q}(i,j) \equiv \binom{i}{j}_q B_{\text{NWA},i-j,q}, 0 \leq i,j \leq n-1. \tag{84}
\]
Definition 34. The matrix $D_{NWA,n,q}$ has matrix elements

$$d_{NWA,i,j} = \begin{cases} \frac{1}{(i-j+1)_q}{j \choose i}_q & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (85)

Definition 35. The matrix $D_{JHC,n,q}$ has matrix elements

$$d_{JHC,i,j} = \begin{cases} \frac{q^{i-j}}{(i-j+1)_q}{j \choose i}_q & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (86)

Theorem 4.8. The inverses of the $q$-Bernoulli number matrices are given by

$$(B_{NWA,n,q})^{-1} = D_{NWA,n,q}.$$ \hspace{1cm} (87)

This implies that

$$B_{NWA,n,q}^{-k} = D_{NWA,n,q}^k.$$ \hspace{1cm} (88)

The following proof is very similar to [16, p. 1624].

Proof. For the NWA case, take away the factor $q^{(i+j)}$ and corresponding $q$-powers in the following equations. We show that $B_{JHC,n,q}D_{JHC,n,q}$ is equal to the unit matrix. We know that

$$\sum_{k=0}^{n} \frac{q^{i-k}}{(k+1)_q}{n \choose k}_q B_{JHC,n-k,q} = \delta_{n,0}.$$ \hspace{1cm} (89)

Then we have

$$\sum_{k=0}^{i} \frac{q^{i-k}}{(k+1-j)_q}{i \choose k}_q B_{JHC,i-k,q} \frac{k}{j}_q = \left(\sum_{k=0}^{i} \frac{q^{i-k}}{(k+1-j)_q}{i \choose k}_q B_{JHC,i-k,q}\right) \frac{k}{j}_q$$

$$= \left(\sum_{k=0}^{i} \frac{q^{i-k}}{(k+1-j)_q}{i-j \choose k}_q B_{JHC,i-j-k,q}\right) \frac{k}{j}_q$$

$$= \left(\sum_{k=0}^{i} \frac{q^{i-k}}{(k+1-j)_q}{i-j \choose k}_q B_{JHC,i-j-k,q}\right) \frac{k}{j}_q = \delta_{i-j,0}.$$ \hspace{1cm} (90)

In [10] we considered the following $q$-analogues of [16, p. 1625]

$$B_{NWA,n,q}(x \oplus_q y) = P_{n,q}(x)B_{NWA,n,q}(y).$$ \hspace{1cm} (91)

These can be generalized to

Theorem 4.9.

$$\mathcal{B}_{n,q}(x \oplus_q y) = P_{n,q}(x)\mathcal{B}_{n,q}(y).$$ \hspace{1cm} (92)

In particular,

$$\mathcal{B}_{n,q}(x) = P_{n,q}(x)\mathcal{B}_{n,q}.$$ \hspace{1cm} (93)

For the following proof one should compare with [16, p. 1625].
Proof.

\[
\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q x^{i-k} \Phi_{k-j,q}(y) = \left( \sum_{k=j}^{i} \binom{i}{j}_q \right) x^{i-j} \Phi_{k-j,q}(y)
\]

(94)

\[
= \left( \sum_{k=0}^{i-j} \binom{i}{j}_q \binom{k-1}{j}_q x^{i-j-k}(x) \Phi_{k,q}(y) = \left( \sum_{k=0}^{i-j} \binom{i}{j}_q \right) \Phi_{i-j,q}(x \otimes_q y) = \text{LHS.}
\]

\[\square\]

**Theorem 4.10.** Two q-analogues of [16, p. 1626] The inverses of the q-Bernoulli polynomial matrices are given by

\[
(\mathcal{B}_{\text{NWA},n,q}(x))^{-1} = (\mathcal{B}_{\text{NWA},n,q})^{-1} \mathcal{P}_{n,q}(x)^{-1} = D_{\text{NWA},n,q} \mathcal{P}_{n,q}(x)^{-1}.
\]

(95)

When the order is increased, for \( y = 0 \) in (55), we multiply the q-transfer matrix by \( \mathcal{T}(M)_{n,q}(x) \). When the order is constant, in (93), we multiply the q-transfer matrix by the q-Pascal matrix.

We can now find a factorization of the q-Appell polynomial matrix and its inverse.

**Theorem 4.11.** A q-analogue and generalization of [16, p. 1626].

\[
\mathcal{T}_{n,q}(x) = \prod_{k=n}^{3} \left( (n,k,q)(x) \mathcal{G}_{n,k}(x) \right) \mathcal{G}_{n,2,q} \mathcal{P}_{n,2,q}^{-1} (x) \mathcal{D}_{n,2,q} \mathcal{T}_{n,q}(x),
\]

(96)

where the product is taken in decreasing order of \( k \).

A factorization of the two q-Bernoulli matrices.

\[
(\mathcal{B}_{\text{NWA},n,q}(x))^{-1} = D_{\text{NWA},n,q} \mathcal{P}_{n,2,q} \mathcal{G}_{n,2,q}^{-1} (x) \prod_{k=3}^{n} \left( (n,k,q)(x) \mathcal{G}_{n,k,q}(x) \right).
\]

(97)

Proof. Use formulas (33), (34) and (93).

\[\square\]

We now return to the q-Lie groups. We find that

**Theorem 4.12.** The functions of q-Bernoulli polynomial matrices

\( (\mathcal{B}_{\text{NWA},q}, \cdot, q, I_n) \) and \( (\mathcal{B}_{\text{HHC},q}, \cdot, q, I_n) \) with elements

\[
\mathcal{B}_{\text{NWA},n,q}(x)
\]

(98)

are q-Lie subgroups of \( M_{k,q} \).

Proof. The sets \( \mathcal{B} \) are closed under the two operations by (55) and (57). The existence of inverses follows as for \( M_{k,q} \).

\[\square\]

**Definition 36.** There are two types of q-Euler polynomials, called \( F_{\text{NWA},n,q}(x) \), NWA q-Euler polynomials, and \( F_{\text{HHC},n,q}(x) \), HHC q-Euler polynomials. They are defined by the following two generating functions:

\[
\frac{2E_q(xt)}{E_q(t) + 1} = \sum_{v=0}^{\infty} \frac{t^v}{[v]_q!} F_{\text{NWA},n,q}(x), \ |t| < \pi,
\]

(99)

and

\[
\frac{2E_q(xt)}{E_q(xt) + 1} = \sum_{v=0}^{\infty} \frac{t^v}{[v]_q!} F_{\text{HHC},n,q}(x), \ |t| < \pi.
\]

(100)
**Definition 37.** We will use the following vector forms for these polynomials.

\[ f_{\text{NWA}, n, q}(x) \equiv (F_{\text{NWA}, 0, q}(x), F_{\text{NWA}, 1, q}(x), \ldots, F_{\text{NWA}, n-1, q}(x))^T. \]  
(101)

The corresponding \( q \)-Euler number vectors are

\[ f_{\text{NWA}, n, q} \equiv (F_{\text{NWA}, 0, q}, F_{\text{NWA}, 1, q}, \ldots, F_{\text{NWA}, n-1, q})^T. \]  
(102)

Let us introduce the two shifted \( q \)-Euler matrices.

**Definition 38.**

\[ \mathcal{F}_{\text{NWA}, n, q}(x) \equiv (f_{\text{NWA}, q}(x) \ E(\oplus_q f_{\text{NWA}, q}(x)) \cdots E(\oplus_q)^{n-1} f_{\text{NWA}, q}(x)). \]  
(103)

We will need two similar matrices, based on the \( F_{\text{NWA}} \) polynomials.

**Definition 39.** The two \( q \)-Euler polynomial matrices are defined by

\[ F_{\text{NWA}, n, q}(i, j) \equiv \binom{i}{j}_q f_{\text{NWA}, i-j, q}(x). \]  
(104)

**Definition 40.** The NWA and JHC \( q \)-Euler matrices are defined by

\[ F_{\text{NWA}, n, q}(i, j) \equiv \binom{i}{j}_q F_{\text{NWA}, i-j, q}, \quad 0 \leq i, j \leq n-1. \]  
(105)

**Definition 41.** The matrix \( C_{\text{NWA}, n, q} \) has matrix elements

\[ c_{\text{NWA}, i, j} \equiv \begin{cases} \frac{1}{2} \left[ 1 + \delta_{i-j, 0} \right] \binom{i}{j}_q & \text{if } i \geq j, \\ 0 & \text{otherwise}. \end{cases} \]  
(106)

**Definition 42.** The matrix \( C_{\text{JHC}, n, q} \) has matrix elements

\[ c_{\text{JHC}, i, j} \equiv \begin{cases} \frac{1}{2} \left[ q^{(i)} \binom{i}{j}_q + \delta_{i-j, 0} \right] & \text{if } i \geq j, \\ 0 & \text{otherwise}. \end{cases} \]  
(107)

**Theorem 4.13.** The inverses of the \( q \)-Euler number matrices are given by

\[ (F_{\text{NWA}, n, q})^{-1} = C_{\text{NWA}, n, q}. \]  
(108)

This implies that

\[ F_{\text{NWA}, n, q}^{-k} C_{\text{NWA}, n, q} \]  
(109)

**Proof.** For the NWA case, replace the factor \( q^{(i)} \) by 1. We show that \( F_{\text{JHC}, n, q} C_{\text{JHC}, n, q} \) is equal to the unit matrix. We know that

\[ \sum_{k=0}^{n} \binom{n}{k}_q F_{\text{JHC}, n-k, q} + F_{\text{JHC}, n, q} = 2 \delta_{n, 0}. \]  
(110)
Introduce a function $G(k)$. Then we have
\[
\sum_{k=j}^{i} \binom{i}{k} q_{i,k}G(k-j) \binom{k}{j}
= \binom{i}{j} \sum_{k=j}^{i} \binom{i-j}{k} q_{i,k}G(k-j)
= \left( \begin{array}{c} i \\ j \end{array} \right) \sum_{k=0}^{i-j} \binom{i-j}{k} q_{i-k,j}G(k) \text{ by } (110)
= \binom{i}{j} \delta_{i-j,0}.
\]

It is now obvious that $G(k) = \frac{1}{2} \left[ q^{(k)} + \delta_{k,0} \right]$ solves this equation for JHC and similar for NWA. \hfill \Box

**Theorem 4.14.** Compare with [16, p. 1626] The inverses of the $q$-Euler polynomial matrices are given by
\[
(F_{\text{NWA},n,q}(x))^{-1} = (F_{\text{NWA},n,q})^{-1} P_{n,q}(x)^{-1} = c_{\text{NWA},n,q} P_{n,q}(x)^{-1}.
\]

**Theorem 4.15.** A factorization of the two $q$-Euler matrices.
\[
(F_{\text{NWA},n,q}(x))^{-1} = c_{\text{NWA},n,q} F_{n,2,q} \ast (x) \prod_{k=3}^{n} (F_{n,k}(x) E_{n,k,q}(x)).
\]

We now return to the $q$-Lie groups.

**Theorem 4.16.** The sets of $q$-Euler polynomial matrices $(\mathcal{F}_{\text{NWA},q}, \cdot, \cdot, I_n)$ and $(\mathcal{F}_{\text{JHC},q}, \cdot, \cdot, I_n)$ with elements
\[
F_{\text{NWA},n,q}(x)
\]
are $q$-Lie subgroups of $\mathcal{M}_{x,q}$.

## 5 A related ring of matrices

We first recall the commutative ring of $q$–Appell polynomials.

**Definition 43.** [11] We denote the set of all $q$–Appell polynomials (in the variable $x$) by $A_{x,q}$.

Let $\Phi_{n,q}(x)$ and $\Psi_{n,q}(x)$ be two elements in $A_{x,q}$. Then the operations $\oplus$ and $\ominus$ are defined as follows:
\[
(\Phi_{q}(x) \oplus \Psi_{q}(x))_{n} \equiv (\Phi_{q}(x) + \Psi_{q}(x))_{n},
\]
\[
(\Phi_{q}(x) \ominus \Psi_{q}(x))_{n} \equiv (\Phi_{q}(x) \ominus q \Psi_{q}(x)_{n} = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \Phi_{q-n,k}(x) \Psi_{k,q}(x).
\]

We keep the usual priority between $\oplus$ and $\ominus$.

**Theorem 5.1.** [8], [11] $(A_{x,q}, \oplus, \ominus)$ is a commutative ring.

In the following, for clarity, we sometimes write $f(t)$ for $f(t, q)$.

**Definition 44.** Let $f(t) \in \mathbb{R}[[x]]$. The $q$-deformed Leibniz functional matrix is given by
\[
(L_{n,q})(f(t, q))_{i,j} \equiv \begin{cases} \frac{D_{q}^{(i-j)}f(t, q)}{(t-j)^{i-j}} & \text{if } i \geq j; \\ 0, & \text{otherwise, } i, j = 0, 1, 2 \ldots, n-1. \end{cases}
\]
Let the operator \( \epsilon : \mathbb{R}[[x]] \to \mathbb{R}[[x]] \) be defined by
\[
cef(x) \equiv f(qx).
\]
(118)

We infer that by the \( q \)-Leibniz formula [5]
\[
(\mathcal{L}_n,q)[f(t, q)g(t, q)] = (\mathcal{L}_n,q)[f(t, q)] \cdot (\mathcal{L}_n,q)[g(t, q)],
\]
(119)

where in the matrix multiplication for every term which includes \( D_q^k f \), we operate with \( e^k \) on \( g \). We denote this by \( \cdot \). This operator can also be iterated, compare with [5].

For the following considerations, compare with [6], [13, p. 232] and [14, p. 67].

**Definition 45.** Assume that \( f(t) \in \mathbb{R}[[x]] \). The \( i, j \) entries of the generalized \( q \)-Pascal functional matrix \( P_{n,q}[f(t, q)] \) of size \( n \times n \) are
\[
P_{n,q}[f(t, q)](i, j) \equiv \begin{cases} \binom{i}{j} P_{q,i}^{-1}f(t, q) & \text{if } i \geq j; \\ 0 & \text{otherwise} \end{cases} \quad i, j = 0, 1, 2, \ldots, n - 1.
\]
(120)

The function \( f(t) \) is called invertible if \( f(0) \neq 0 \).

**Example 1.** When \( f(t) \equiv E_q(at) \) we have
\[
P_{n,q}[f(t, q)] = P_{n,q}[E_q(at)] = P_{n,q}(a).
\]
(121)


\[
P_{n,q}[f(t) + g(t)] = P_{n,q}[f(t)] + P_{n,q}[g(t)],
\]
\[
P_{n,q}[f(t)] \cdot \cdot P_{n,q}[g(t)] = P_{n,q}[f(t)g(t)].
\]
(122)

**Definition 46.** [6] Assume that \( f(t) \) is invertible. If the inverse \( (f(t)^{-1})^k \) exists for \( k \leq n \), we can define the \( q \)-inverse of the generalized \( q \)-Pascal functional matrix as
\[
[P_{n,q}[f(t, q)]]^{-1} \equiv P_{n,q}[f(t, q)]^{-1}.
\]
(123)

Inspired by this, we make the following definition:

**Definition 47.** The operations \( \oplus \) and \( \odot \) are defined as follows:
\[
P_{n,q}[f(t)] \oplus P_{n,q}[g(t)] \equiv P_{n,q}[f(t) + g(t)].
\]
(124)
\[
P_{n,q}[f(t)] \odot P_{n,q}[g(t)] \equiv P_{n,q}[f(t) \cdot \cdot P_{n,q}[g(t)] = P_{n,q}[f(t)g(t)].
\]
(125)

We keep the usual priority between \( \oplus \) and \( \odot \).

The generalized \( q \)-Pascal polynomial functional matrix \( (P_n, f(t), \oplus, \odot, O_n, I_n) \) is the set of all \( P_{n,q}[f] \) with the operations \( \oplus \) and \( \odot \) and units \( O_n \) and \( I_n \).

**Theorem 5.3.** Assume that \( O_n \) is the unit for \( \oplus \), and that \( I_n \) is the unit for \( \odot \). The generalized \( q \)-Pascal polynomial functional matrix
\[
(P_n, f(t), \oplus, \odot, O_n, I_n)
\]
(126)
is a commutative ring.
Proof. We presume that \( \mathcal{P}_{n,q}[f], \mathcal{P}_{n,q}[g] \) and \( \mathcal{P}_{n,q}[h] \) are three elements in \( \mathcal{P}^F \) corresponding to \( f(t), g(t) \) and \( h(t) \) respectively.

We first show that \( \oplus \) is well-defined. Assume that \( f(t) + g(t) \neq 0 \). Then
\[
\mathcal{P}_{n,q}[f(t)] \oplus \mathcal{P}_{n,q}[g(t)] \in \mathcal{P}^F,
\]
(127)
The associative law for \( \oplus \) reads:
\[
\mathcal{P}_{n,q}[f(t)] \oplus \left( \mathcal{P}_{n,q}[g(t)] \oplus \mathcal{P}_{n,q}[h(t)] \right) = \left( \mathcal{P}_{n,q}[f(t)] \oplus \mathcal{P}_{n,q}[g(t)] \right) \oplus \mathcal{P}_{n,q}[h(t)]).
\]
(128)
This follows from the associativity of \( + \).
The commutative law for \( \oplus \) reads:
\[
\mathcal{P}_{n,q}[f(t)] \oplus \mathcal{P}_{n,q}[g(t)] = \mathcal{P}_{n,q}[g(t)] \oplus \mathcal{P}_{n,q}[f(t)].
\]
(129)
This follows from the commutativity of \( + \). The identity element with respect to \( \oplus \) is the zero matrix \( O_n \). We have
\[
\mathcal{P}_{n,q}[f(t)] \oplus O_n = \mathcal{P}_{n,q}[f(t)].
\]
(130)
There exists \( -\mathcal{P}_{n,q}[f(t)] \) such that
\[
\mathcal{P}_{n,q}[f(t)] \oplus -\mathcal{P}_{n,q}[f(t)] = O_n.
\]
(131)
This follows from the corresponding property of real numbers.
Then we show that \( \odot \) is well-defined.
Assume that \( f(t)g(t) \neq 0 \). Then we have
\[
\mathcal{P}_{n,q}[f(t)] \odot \mathcal{P}_{n,q}[g(t)] \quad \text{by (125)} \in \mathcal{P}^F.
\]
(132)
The associative law for \( \odot \) reads:
\[
\mathcal{P}_{n,q}[f(t)] \odot \left( \mathcal{P}_{n,q}[g(t)] \odot \mathcal{P}_{n,q}[h(t)] \right) = \left( \mathcal{P}_{n,q}[f(t)] \odot \mathcal{P}_{n,q}[g(t)] \right) \odot \mathcal{P}_{n,q}[h(t))].
\]
(133)
This follows from the associativity of the multiplication \( \cdot \).
The commutative law for \( \odot \) reads:
\[
\mathcal{P}_{n,q}[f(t)] \odot \mathcal{P}_{n,q}[g(t)] = \mathcal{P}_{n,q}[g(t)] \odot \mathcal{P}_{n,q}[f(t)].
\]
(134)
This follows from the commutativity of the multiplication \( \cdot \).
The identity element with respect to \( \odot \) is the identity matrix \( I_n \). We have
\[
\mathcal{P}_{n,q}[f(t)] \odot I_n = \mathcal{P}_{n,q}[f(t)].
\]
(135)
The distributive law reads:
\[
\mathcal{P}_{n,q}[f(t)] \odot \left( \mathcal{P}_{n,q}[g(t)] \oplus \mathcal{P}_{n,q}[h(t)] \right) = \mathcal{P}_{n,q}[f(t)(g(t) + h(t))] = \mathcal{P}_{n,q}[f(t)g(t) + f(t)h(t)]
\]
(136)
\[
= \left( \mathcal{P}_{n,q}[f(t)] \odot \mathcal{P}_{n,q}[g(t)] \right) \oplus \left( \mathcal{P}_{n,q}[f(t)] \odot \mathcal{P}_{n,q}[h(t)] \right).
\]
This follows from the distributive law of real numbers.
\( \square \)
6 Conclusion

We have united and $q$-deformed formulas from papers by Arponen, Aceto et al., Yang et al. and Zhang et al. to give a first synthesis of $q$-Appell polynomial matrices, which were previously only known in special cases. Some of the formulas for $q$-Pascal matrices and their factorizations are generalized as well as formulas for $q$-Bernoulli and $q$-Euler matrices. We have given the first concrete examples of $q$-Lie subgroups, and constructed a similar ring of matrices with many pleasant properties.

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References