UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2018:35

Auction theory

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Examensarbete i matematik, 15 hp
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Augusti 2018

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August 20, 2018

Abstract

We first study equilibrium bidding strategies in first-price auctions and second-price auctions by assuming that the number of bidders are known. We derive the revenue equivalence principle, i.e. that the expected revenue of the seller is the same in both auction types. We then study bidding strategies in the case of uncertainty about the number of bidders, and we show that the revenue equivalence principle extends to this case. Furthermore, in point of view of the seller, we derive a strategy how many bidders to invite in order to maximize the revenue given that each invited bidder costs c.

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1 Introduction

The first auction known to history happened before Christ. It was reported by Herodotus in Babylon in 500 BC. Perhaps one of the most famous antique auctions was when Didius Julianus bought the entire Roman Empire! Items ranging from fish to tobacco and long-term securities are sold by auctions but perhaps in some other setting than the typical auction house, where more traditional items such as antiques and arts are sold. The most recent type of auctions are the Internet auctions where people from all over the world can sell items online using standard auction rules.

Perhaps the oldest and most current auction type is the open ascending price. The sale is led by an auctioneer who starts the auction with an opening price and raises it until there is only one bidder left, then the auction stops. The winner pays the second-highest bid. This type of auction is called an *English* auction.

A *Dutch* auction is the opposite of an English auction where the auctioneer opens the auction with a (very) high price and lowers it until the price is matched with an interested bidder, then the auction stops. The bidder pays what he bids.

There are many other types of auctions involving more complex frames with different combinations. However, in this paper, we are typically interested in common auction forms such as the seal-bid auctions.

In a *first-price* sealed bid auction, each bidder submits a bid in a sealed envelope. The winner of the object is the bidder who submitted the highest bid. The winner pays the amount he had bid.

In a second-price sealed bid auction, each bidder submits a bid in a sealed envelope and the winner of the object is the bidder who submitted the highest bid. Unlike in a first price auction, the winner pays the second-highest bid.

We will study how to bid in an equilibrium strategy where all bidders follow the same strategy, given that the bidders are symmetric. By symmetric equilibrium we mean that all bidders follow the same distribution function and distribution of values. We will derive the equilibrium strategies in the first-price auction and the second-price auction. We may ask which auction type to choose if we want to maximize our profits. Furthermore, we will study the point of view of the seller and move over to introducing reserve prices. Lastly, we will study the equilibrium bidding strategy which considers the uncertainty about the number of bidders in an auction. How should we bid if we do no know how many competitors we are facing?

2 Private Value Auctions

The material presented in the following sections is based on [1].

2.1 The Symmetric Model

Consider an auction where an object is for sale with N potential buyers. Let X_i be the maximum amount bidder i is willing to pay for the object and let x_i be the realized value of X_i. We assume that all X_i's are independent and identically distributed (i.i.d) on $[0, \omega]$ according to some non-decreasing and continuous distribution function F. No bidder has an infinite amount of money, so $E[X_i]$ is finite. We also assume that all bidders strictly value the object by its future expected profit and do not care about the risk involved. Hence, they are risk neutral. Each bidder has enough money is willing to pay the seller up to his realized value x_i. The only piece of information bidder i does not have is the realized values of other bidders. It is important to remind ourselves all bidders have the same distribution of values. Therefore, we call them *symmetric* bidders. Moreover, let $\beta_i : [0, \omega] \to \mathbb{R}_+$ be the bidding strategy of bidder i which determines how much he will bid.

Now that we have set some ground rules, we can study sealed bid auctions. We begin with the second-price auction for convenience since the setup is easier to work with.

2.2 Second-Price Auctions

A sealed bid b_i is submitted by each bidder. The pay-off function is given by

$$\Pi_i = \begin{cases} x_i - \max_{j \neq i} b_j, & \text{if } b_i > \max_{j \neq i} b_j \\ 0, & \text{if } b_i < \max_{j \neq i} b_j. \end{cases}$$

If there is a tie, i.e., $b_i = \max_{j \neq i} b_j$, then every bidder has equal probability of winning the object.

Proposition 2.1. *Bidding according to the strategy $\beta^{II}(x) = x$ in a second-price sealed bid auction gives a pay-off at least as high as any other strategy, and strictly higher than some.*

Proof. Take bidder 1 and suppose that he has the highest bid, $p_1 = \max_{j \neq 1} b_j$. Bidder 1 bids x_1 and wins if $x_1 > p_1$ and loses if $x_1 < p_1$. When $x_1 = p_1$, he is indifferent.

Suppose he instead bids $z_1 < x_1$. If $p_1 \leq z_1 < x_1$ he wins with profit $x_1 - p_1$. He loses if $z_1 < x_1 < p_1$ or $z_1 < p_1 < x_1$. However, if he had bid x_1 he would have made a positive profit. Thus, he can never earn more money by bidding less than x_1. ■

Remark. *Bidding according to Proposition 2.1 is called a weakly dominant strategy*[1].

[1] Source: http://www.gametheory.net/dictionary/WeaklyDominantStrategy.html

Now that we have established an equilibrium bidding strategy, it may be interesting to think about how much each bidder is expected to pay in equilibrium.

Consider bidder 1 and let $Y_1 \equiv Y_1^{N-1}$ be a random variable with the highest value of the remaining $N-1$ bidders. Let G be the distribution function of Y_1 such that $G(y) = F(y)^{N-1}$ for all y. The expected payment of a bidder with realized value x is given by

$$\begin{aligned} m^{\mathrm{II}}(x) &= P(Win) \cdot E[\text{2nd highest bid} \mid x \text{ is the highest bid}] \\ &= P(Win) \cdot E[\text{2nd highest value} \mid x \text{ is the highest value}] \\ &= G(x) \cdot E[Y_1 \mid Y_1 < x]. \end{aligned} \tag{2.1}$$

2.3 First-Price Auctions

A sealed bid b_i is submitted by each bidder. The pay-off function is

$$\Pi_i = \begin{cases} x_i - b_i, & \text{if } b_i > \max_{j \neq i} b_j \\ 0, & \text{if } b_i < \max_{j \neq i} b_j. \end{cases}$$

If there is a tie, then every bidder has equal probability of winning the object. It is a bit tricky to study the bidding behavior in equilibrium since no bidder would bid their value unless he wants a zero pay-off. The bidder faces a trade-off because if he increases his bid he has a higher chance of winning, but lowers his gains from winning at the same time. By deriving the equilibrium strategy we will see how the effects cancel out each other.

Suppose that all bidders follow the strategy function $\beta^{\mathrm{I}} \equiv \beta$ and that bidder 1 knows his realized value x, but bids b. How can we determine b so that he maximizes his profits?

It is suboptimal to bid $b > \beta(\omega)$ since he would most definitely win. By lowering his bid by just a little so that he still wins, he would increase his pay-off by paying less. Therefore, we will only study bids where $b \leq \beta(\omega)$. Note that a bidder with zero value will never place a bid since he would lose money if he won the auction. We establish the initial condition $\beta(0) = 0$. We remind ourselves that β is an increasing and continuous function, so

$$\max_{i \neq 1} \beta(X_i) = \beta(\max_{i \neq 1} X_i) = \beta(Y_1),$$

where Y_1 is the highest value of the remaining $N-1$ bidders.

Bidder 1 wins the auction if $\max_{i \neq 1} \beta(X_i) = \beta(Y_1) < b$ which is equivalent to $Y_1 < \beta^{-1}(b)$. The expected pay-off is given by

$$G(\beta^{-1}(b)) \cdot (x - b)$$

where G is the distribution function of Y_1. We want to maximize our profits, so differentiating with respect to b gives

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x - b) - G(\beta^{-1}(b)) = 0. \tag{2.2}$$

Here, we denote $g = G'$ as the density function of Y_1. If $b = \beta(x)$ we get

$$\frac{g(x)}{\beta'(x)}\big(x - \beta(x)\big) - G(x) = 0.$$

Rewriting the equation above, we have a first order differential equation of the form

$$G(x)\beta^{'}(x) + g(x)\beta(x) = xg(x), \tag{2.3}$$

which is equivalent to the derivative of the product of two functions

$$\frac{d}{dx}\big(G(x)\beta(x)\big) = xg(x).$$

Since $\beta(0) = 0$, we obtain a solution of the equilibrium bidding strategy in a first-price auction

$$\begin{aligned} \beta(x) &= \frac{1}{G(x)} \int_0^x yg(y)dy \\ &= E[Y_1 \mid Y_1 < x]. \end{aligned} \tag{2.4}$$

The next proposition verifies our conclusion.

Proposition 2.2. *In a first-price sealed bid auction, bidding according to the strategy*

$$\beta^I(x) = E[Y_1 \mid Y_1 < x] \tag{2.5}$$

is a symmetric equilibrium strategy where Y_1 is the highest value of remaining $N-1$ bidders.

Proof. Suppose that each bidder follows the strategy $\beta^{\mathrm{I}} \equiv \beta$ from (2.5). At equilibrium, the bidder with the highest bid will win the auction. Let $z = \beta^{-1}(b)$ where b is the equilibrium bid, i.e. $\beta(z) = b$. If bidder 1 bids $b \leq \beta(\omega)$ we can write his expected pay-off function by bidding $\beta(z)$ as

$$\begin{aligned} \Pi(b, x) &= G(z)(x - \beta(z)) \\ &= G(z)x - G(z)E[Y_1 \mid Y_1 < z] \\ &= G(z)x - \int_0^z yg(y)dy \\ &= G(z)x - \Big[yG(y)\Big]_0^z + \int_0^z G(y)dy \\ &= G(z)x - zG(z) + \int_0^z G(y)dy \\ &= G(z)(x - z) + \int_0^z G(y)dy \end{aligned}$$

where x is his realized value. We obtain

$$\begin{aligned}\Pi(\beta(x),x)-\Pi(\beta(z),x) &= \int_0^x G(y)dy - G(z)(x-z) - \int_0^z G(y)dy \\ &= G(z)(z-x) + \int_0^x G(y)dy - \int_0^z G(y)dy \\ &= G(z)(z-x) + \int_z^x G(y)dy \\ &= G(z)(z-x) - \int_x^z G(y)dy \geq 0.\end{aligned}$$

Since all other bidders follow the strategy β, he cannot profit from bidding anything other than the symmetric equilibrium strategy $\beta(x)$, and we are done. ∎

We can rewrite the equilibrium bid from (2.4) as

$$\begin{aligned}\beta^{\mathrm{I}}(x) &= \frac{1}{G(x)}\left[yG(y)\Big|_0^x - \int_0^x G(y)dy\right] \\ &= x - \int_0^x \frac{G(y)}{G(x)}dy\end{aligned} \tag{2.6}$$

which means that the bid is less than the realized value x. The second term in (2.6) is equal to

$$\frac{G(y)}{G(x)} = \left(\frac{F(y)}{F(x)}\right)^{N-1} \tag{2.7}$$

which is dependent on the number of bidders. Since we have assumed that all bidders follow the same bidding strategy and that y is at most x, (2.7) goes to zero as N increases. Then, the equilibrium bidding strategy $\beta^{\mathrm{I}}(x)$ goes to the realized value x.

Example 2.1. *Suppose we have values uniformly distributed on [0,1]. If the distribution function is $F(x) = x$ then $G(x) = F(x)^{N-1} = x^{N-1}$. The equilibrium bid is*

$$\begin{aligned}\beta^I(x) &= x - \int_0^x \frac{F(y)}{F(x)}dy \\ &= x - \frac{1}{x^{N-1}}\int_0^x y^{N-1}dy \\ &= x - \frac{1}{x^{N-1}}\frac{x^N}{N} \\ &= x - \frac{x}{N} \\ &= \frac{N-1}{N}x.\end{aligned}$$

As N increases, the equilibrium bid goes to the realized value x.

Example 2.2. *Suppose there are two bidders with exponentially distributed values on* $[0, \infty]$. *The distribution function is* $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, $G(x) = F(x)$ *(*$N = 2$*) and we have the equilibrium bidding strategy*

$$\begin{aligned}
\beta^I(x) &= x - \int_0^x \frac{F(y)}{F(x)} dy \\
&= x - \frac{1}{F(x)} \int_0^x \big(1 - e^{-\lambda y}\big) dy \\
&= x - \frac{1}{F(x)} \left[y + \frac{e^{-\lambda y}}{\lambda} \right]_0^x \\
&= x - \frac{1}{F(x)} \left(x + \frac{e^{-\lambda x}}{\lambda} - \frac{1}{\lambda} \right) \\
&= \frac{1}{F(x)} \Big(xF(x) - x \Big) + \frac{1 - e^{-\lambda x}}{\lambda F(x)} \\
&= \frac{1}{1 - e^{-\lambda x}} \Big(x - xe^{-\lambda x} - x \Big) + \frac{1 - e^{-\lambda x}}{\lambda \big(1 - e^{-\lambda x}\big)} \\
&= \frac{1}{\lambda} - \frac{xe^{-\lambda x}}{1 - e^{-\lambda x}}.
\end{aligned}$$

The winner in a first-price auction pays what he actually bid so the expected payment of a bidder with value x is then

$$\begin{aligned}
m^{\mathrm{I}}(x) &= P(\mathrm{Win}) \cdot \mathrm{Amount\ bid} \\
&= G(x) \cdot E[Y_1 \mid Y_1 < x].
\end{aligned} \tag{2.8}$$

It turns out that the expected payment in a first-price auction is the same as in a second-price auction. In the following section, we will study how the auction types affect the expected revenue to the seller.

2.4 Revenue Comparison

The expected revenue of the seller is the amount that the bidder expects to pay. We showed in the previous section that the expected payment of a bidder in a first-price auction and in a second-price auction is the same. It must then hold that the expected revenue is equal regardless of the auction type. To see this, let $A = \mathrm{I}$ or II be either a first-price auction or a second-price auction.

Then the *ex-ante*[2] expected payment of a bidder in A is given by

$$\begin{aligned} E[m^A(X)] &= \int_0^{\omega} m^A(x) f(x) dx \\ &= \int_0^{\omega} \left(\int_0^{x} y g(y) dy \right) f(x) dx \\ &= \int_0^{\omega} \left(\int_y^{\omega} f(x) dx \right) y g(y) dy \\ &= \int_0^{\omega} y(1 - F(y)) g(y) dy. \end{aligned} \tag{2.9}$$

It follows that the expected revenue accumulating to the seller is given by

$$\begin{aligned} E[R^A] &= N \cdot E[m^A(X)] \\ &= N \int_0^{\omega} y(1 - F(y)) g(y) dy, \end{aligned} \tag{2.10}$$

which is the number of bidders times the *ex-ante* expected payment of a bidder. Note that the integrand in (2.10) is the density of the second highest values of N bidders, $Y_2^{(N)}$. For a more in-depth derivation of the second highest order of statistics we refer to [1]. Since

$$f_2^{(N)}(y) = N(1 - F(y)) f_1^{(N-1)}(y)$$

where $f_1^{(N-1)} = g(y)$, (2.10) can be written as

$$\begin{aligned} E[R^A] &= \int_0^{\omega} y f_2^{(N)}(y) dy \\ &= E[Y_2^{(N)}]. \end{aligned} \tag{2.11}$$

Regardless of the type of auction, we see that the expected revenue of the seller is the expected value of the second highest value of N bidders. Therefore, we can agree that the expected revenue of the seller is the same in both auctions. The result is presented in the following proposition.

Proposition 2.3. *With i.i.d private values, the expected revenues in a first-price auction and in a second-price auction are equal.*

Remark. *In auctions with specific realized values, the end price of the object may be higher in a first-price auction, or vice versa. Consequently, the expected revenue may be higher in one auction over another.*

Example 2.3. *If there are only two bidders with values according to a uniform distribution, the equilibrium strategy in a first-price auction is* $\beta^1(x) = \frac{1}{2}x$*. The revenue in a first-price auction is bigger than in a second-price auction if* $\frac{1}{2}x_1 > x_2$*. The opposite is true if* $\frac{1}{2}x_1 < x_2 < x_1$*. Therefore we say that the revenues to the seller in a first-price auction and in a second-price auction are equal, on average.*

[2] The definition of *ex-ante*: "based on anticipated changes or activity in an economy." Source: http://www.dictionary.com/browse/ex-ante

2.5 Reserve Prices

We have assumed that the seller sells the object at the price the auction ends with. However, it is common that the seller does not want to sell the object if the end price is lower than some predetermined amount that the seller has set. A predetermined amount, $r > 0$ is called a *reserve price*. In this section, we will study how the reserve price affects the bidders and the seller in the two auctions.

2.5.1 Reserve Prices in Second-Price Auctions

We assume that the seller reserves a price $r > 0$. The object will not be sold if the price is less than r. Bidders with realized value less than the reserve price cannot profit from the auction. In a second-price auction, we know that the winner pays the second highest bid so the reserve price does not affect the bidding strategy if the second highest bid is less than r. Therefore, Proposition 2.1 still holds. The expected payment of a bidder with realized value x is given by

$$m^{\mathrm{II}}(x) = \begin{cases} rG(r), & \text{if } x = r \\ rG(r) + \displaystyle\int_r^x yg(y)dy, & \text{if } x \geq r. \end{cases} \tag{2.12}$$

2.5.2 Reserve Prices in First-Price Auctions

As in the second-price auction, suppose that the seller sets a reserve price, $r > 0$. Bidders with realized values less than the reserve price cannot make a profit. Moreover, it must hold that $\beta^{\mathrm{I}}(r) = r$ since any bidder can win by bidding r only if the rest have realized values smaller than r. This implies that Proposition 2.2 still holds. The symmetric equilibrium bidding strategy for a bidder with realized value $x \geq r$ is given by

$$\begin{aligned} \beta^{\mathrm{I}}(x) &= E\big[\max\{Y_1, r\} \mid Y_1 < y\big] \\ &= r\frac{G(r)}{G(x)} + \frac{1}{G(x)}\int_r^x yg(y)dy. \end{aligned}$$

Similarly, the expected payment for $x \geq r$ is then

$$\begin{aligned} m^{\mathrm{I}}(x, r) &= G(x) \cdot \beta^{\mathrm{I}}(x) \\ &= rG(r) + \int_r^x yg(y)dy. \end{aligned} \tag{2.13}$$

We see that the expected payment of a bidder in a first-price auction is the same as the expected payment in a second-price auction.

Since the expected payment is the same in both auction types, the expected revenue of the seller must also be the same. Proposition 2.3 is also valid with reserve prices.

2.5.3 Revenue Effects of Reserve Prices

It is, of course, interesting to study how the reserve prices affects the expected revenue of the seller. Let A be a first-price auction or a second-price auction. The expected payment of a bidder with realized value r in A is $rG(r)$. The *ex-ante* expected payment of a bidder is then

$$\begin{aligned} E[m^A(X,r)] &= \int_r^\omega m^A(x,r)f(x)dx \\ &= r(1-F(r))G(r) + \int_r^\omega y(1-F(y))g(y)dy \end{aligned}$$

which is derived by similar calculations to (2.9). We want to choose the optimal r such that the seller maximizes his expected revenue. Assume that the seller has a value $x_0 \in [0,\omega)$ attached to the object. If the object does not get sold the seller would obtain a value x_0 from the auction. It is then clear that the seller would not set a reserve price $r < x_0$. Setting a reserve price $r \geq x_0$ the expected pay-off of the seller is

$$\Pi_0 = N \cdot E\big[m^A(X,r)\big] + F(r)^N x_0.$$

By differentiating with respect to r we get

$$\frac{dT_0}{dr} = N[1-F(r)-rf(r)]G(r) + NG(r)f(r)x_0.$$

Since the distribution function F has support on $[0,\omega]$, we define the *hazard rate* of F as function $\lambda : [0,\omega) \to \mathbb{R}_+$ given by

$$\lambda(x) = \frac{f(x)}{1-F(x)}.$$

We refer to [1] for a more in-depth definition of the hazard rate function. Substituting $\lambda(x)$ in the equation above we get

$$\frac{dT_0}{dr} = N[1-(r-x_0)\lambda(x)](1-F(r))G(r). \tag{2.14}$$

If $x_0 > 0$, then $\Pi_0'(r = x_0) > 0$ which implies that the seller should set a reserve price $r > x_0$.
If $x_0 = 0$, then $\Pi_0'(r = x_0) = 0$, the expected payment has a local minimum at zero if λ is bounded. This means that a small reserve price will increase the revenue.

Therefore, if a seller wants to maximize his revenue he should always set a reserve price $r > x_0$. We may ask why $r > x_0$ gives an increase in revenue. For instance, in a second-price auction with two bidders and $x_0 = 0$, setting $r > 0$, the seller takes a risk that the object does not get sold if the highest value of the bidders Y_1 is less than r. If the second highest value $Y_2 < r$, the object will be sold for r and the seller has mitigated any potential bigger loss.

Furthermore, it turns out that the expected gain is bigger than the expected loss by setting a small r by a fact called the *exclusion principle.* This implies that it is optimal to exclude bidders with values below r, even if their values are bigger than x_0.

The optimal reserve price r^* from (2.14) must satisfy the equation

$$(r^* - x_0)\lambda(r^*) = 1$$

which can be rewritten as

$$r^* - \frac{1}{\lambda(r^*)} = x_0. \tag{2.15}$$

If λ is increasing we see that the optimal reserve price is independent of the number of bidders. A reserve price only affects auctions when there is only a bidder with a realized value bigger than the reserve price.

3 The Revenue Equivalence Principle

3.1 Uncertain number of bidders

We have assumed that each bidder knows their value but not the value of others. We have also assumed that the number of bidders and the distribution of values are known to all. In reality, the number of other competing bidders may be unknown in a sealed-bid auction. We will now include this uncertainty to our standard auction models.

Let $\mathcal{N} = \{1, 2, ..., N\}$ be the set of possible bidders and let $\mathcal{A} \subseteq \mathcal{N}$ be the subset of participating bidders.

The type of auction is a standard one (either a first-price auction or second-price auction) with equilibrium strategy function β. Note that β does not depend on the number of competitors n. Recall F as the distribution function of the maximum amount bidder i is willing to pay for the object. Suppose that bidder 1 has realized value x but bids $\beta(z)$. The total probability that the wins if the bid is $\beta(z)$ is

$$G(z) = \sum_{n=0}^{N-1} p_n G^{(n)}(z) \tag{3.1}$$

where p_n is the probability that he competes with n other bidders, $G^{(n)}(z) = F(z)^n$ is the probability that he wins if the highest value of n remaining bidders $Y_1^{(n)}$ is less than z. Note that p_n is independent of bidders identities and their values.

3.1.1 Uncertain number of bidders in a second-price auction

Recall the expression in (2.1) and substitute with (3.1). Then the expected payment of a participating bidder with realized value x is given by

$$\begin{aligned} m^{\mathrm{II}}(x) &= G(x)E\Big[Y_1^{(n)}\Big|Y_1^{(n)} < x\Big] \\ &= \sum_{n=0}^{N-1} p_n G^{(n)}(x)E\Big[Y_1^{(n)}\Big|Y_1^{(n)} < x\Big]. \end{aligned} \tag{3.2}$$

3.1.2 Uncertain number of bidders in a first-price auction

Let $X_i \in [0, \omega]$ be i.i.d random variables as the maximum amount bidder i is willing to pay for the object in the auction. Suppose that all bidders follow the symmetric and increasing equilibrium strategy function β. Fix bidder 1 who knows his realized value x but bids $b < \beta(\omega)$.

It is suboptimal to bid $b > \beta(\omega)$ since he would most definitely win. By lowering his bid by just a little so that he still wins, he would increase his pay-off by paying less. Therefore, we will only study bids where $b \leq \beta(\omega)$. Note that a bidder with zero value will never place a bid since he would lose money if he won the auction. We establish the initial condition $\beta(0) = 0$. We remind ourselves that β is an increasing and continuous function, so

$$\max_{i\neq 1}\beta(X_i) = \beta(\max_{i\neq 1}X_i) = \beta(Y_1),$$

where Y_1 is the highest value of the remaining $N-1$ bidders.

Bidder 1 wins the auction if $\max_{i\neq 1}\beta(X_i) = \beta(Y_1) < b$ which is equivalent to $Y_1 < \beta^{-1}(b)$. The expected pay-off is given by

$$G\big[\beta^{-1}(b)\big]\cdot(x-b) = \sum_{n=0}^{N-1} p_n G^{(n)}\big[\beta^{-1}(b)\big](x-b)$$

where $G^{(n)}$ is the the distribution function of $Y_1^{(n)}$. Maximizing with respect to b gives

$$\sum_{n=0}^{N-1}\frac{p_n g^{(n)}\big[\beta^{-1}(b)\big]}{\beta'[\beta^{-1}(b)]}x - \left[\sum_{n=0}^{N-1}\frac{p_n g^{(n)}\big[\beta^{-1}(b)\big]}{\beta'[\beta^{-1}(b)]}b + \sum_{n=0}^{N-1}\frac{p_n G^{(n)}\big[\beta^{-1}(b)\big]}{\beta'[\beta^{-1}(b)]}\right] = 0$$

where $g^{(n)} = [G^{(n)}]'$ is the density function of $Y_1^{(n)}$. If $b = \beta(x)$ we get

$$\sum_{n=0}^{N-1}\frac{p_n g^{(n)}(x)}{\beta'(x)}x = \sum_{n=0}^{N-1}\frac{p_n g^{(n)}(x)}{\beta'(x)}\beta(x) + \sum_{n=0}^{N-1}p_n G^{(n)}(x).$$

Multiplying with $\beta'(x)$ on both sides, the right hand sides is equivalent to the derivative of the product of two functions

$$\sum_{n=0}^{N-1}p_n g^{(n)}(x)x = \frac{d}{dx}\left[\sum_{n=0}^{N-1}p_n G^{(n)}(x)\beta(x)\right].$$

The equation above is a first-order differential equation. Since $\beta(0) = 0$ we get

$$\begin{aligned}\beta(x) &= \frac{\int_0^x \sum_{n=0}^{N-1} p_n y g^{(n)}(y) dy}{\sum_{n=0}^{N-1} p_n G^{(n)}(x)} \\ &= E\Big[Y_1^{(n)} \Big| Y_1^{(n)} < x\Big].\end{aligned}$$

The expected payment of a participating bidder with realized value x in a first-price auction is

$$\begin{aligned}m^{\mathrm{I}}(x) &= G(x)\beta(x) \\ &= \sum_{n=0}^{N-1} p_n G^{(n)}(x) E\Big[Y_1^{(n)} \Big| Y_1^{(n)} < x\Big]\end{aligned} \tag{3.3}$$

which is the same expression as in (3.2). Thus the expected payment of a participating bidder in a first-price auction and in a second-price auction is the same for all x. In the case when we know how many bidders we are competing against, we showed that the expected revenue to the seller also were the same in both auctions. It must then also hold that the expected revenue to the seller is the same in a first-price auction or in a second-price auction with uncertain number of bidders.

Example 3.1. *Suppose we are in a standard auction with one or two participating bidders. Each bidder is unaware of how many other bidders he might compete against. Let x be the realized value of the first bidder and y be the submitted bid. Let p be the probability that only one bidder is invited. Then $1-p$ is the probability that another bidder is invited, given that the first bidder is invited. The expected pay-off of the first bidder is*

$$\Pi(y,x) = p(x-y) + (1-p)(x-y)P\big(\textit{The other bidder's bid} \leq y\big).$$

Let Z be the the value of the other bidder. The probability that his value is less than or equal to y is

$$\begin{aligned}P\big(\textit{The other bidder's bid} \leq y\big) &= P\big(\beta(Z) \leq y\big) \\ &= P\big(Z \leq \beta^{-1}(y)\big) \\ &= F\big(\beta^{-1}(y)\big)\end{aligned}$$

where he bids z according to some increasing distribution function F. We have

$$\Pi(y,x) = p(x-y) + (1-p)(x-y)F\big(\beta^{-1}(y)\big)$$

Differentiating with respect to y gives

$$\begin{aligned}\frac{\partial \Pi}{\partial y} &= -p + (1-p)\left[x\frac{f(\beta^{-1}(y))}{\beta'(\beta(y))} - \left(F(\beta^{-1}(y)) + y\frac{f(\beta^{-1}(y))}{\beta'(\beta(y))}\right)\right] \\ &= -p + (1-p)(x-y)\frac{f(\beta^{-1}(y))}{\beta'(\beta(y))} - (1-p)F(\beta^{-1}(y)) = 0.\end{aligned}$$

At equilibrium, $x = \beta^{-1}(y)$, so

$$\frac{\partial \Pi}{\partial y} = -p + (1-p)(x - \beta(x))\frac{f(x)}{\beta'(x)} - (1-p)F(x) = 0.$$

Multiplying with $\beta'(x)$ on both sides and rearranging, we get

$$f(x)\beta(x) + F(x)\beta'(x) = -\frac{p}{1-p}\beta'(x) + xf(x).$$

The left hand side is the derivative of the product of two functions. The equation becomes a first order differential equation. We know that if a bidder's value is zero, he will bid zero, so $\beta(0) = 0$. Integrating both sides gives

$$F(x)\beta(x) = -\frac{p}{1-p}\beta(x) + \int_0^x sg(s)ds.$$

Rearranging some terms, we end up with the equilibrium bidding strategy for a bidder with value x

$$\beta(x) = \frac{1-p}{F(x)(1-p)+p}\int_0^x sg(s)ds. \tag{3.4}$$

Example 3.2. *Standard auction with one or two bidders. We have values uniformly distributed on [0,1] with $F(x) = x$. The probability that the first bidder is the only bidder is $p_0 = p$. The probability that the first bidder is competing with another bidder is $p_1 = 1 - p_0 = 1 - p$. If x is the realized value, the symmetric equilibrium bid is then*

$$\begin{aligned}\beta(x) &= \frac{1-p}{F(x)(1-p)+p}\int_0^x sg(s)ds.\\ &= \frac{1-p}{(1-p)x+p}\frac{x^2}{2}.\end{aligned}$$

3.2 The point of view of the seller in a standard auction type

Consider a seller in a standard auction type with two bidders, bidder 1 and bidder 2. Suppose the seller invites both bidders with probability q, and one bidder with probability $1 - q$. The probability that the seller invites another bidder, given that the first invited bidder is

$$\begin{aligned}P(\exists \text{ another bidder} \mid \text{the first bidder is invited}) &= \frac{P(\text{two bidders})}{q + \frac{1}{2}(1-q)}\\ &= \frac{q}{\frac{1}{2}+\frac{q}{2}}\\ &= \frac{2q}{1+q}\\ &= p\end{aligned} \tag{3.5}$$

The seller pays a fee c for each invited bidder. An invited bidder bids x according to some increasing distribution $F_x^q(x)$. The seller must be indifferent between inviting one bidder and two bidders. We want to find the optimal number of competing bidders so that the seller can maximize his revenue. In the case of two invited bidders, the expected revenue of the seller is then

$$E\big[\beta(X)\big] - c = E\Big[max\{\beta(X_1), \beta(X_2)\}\Big] - 2c. \tag{3.6}$$

.

Example 3.3. *In the case where values are U(0,1) we have*

$$E\big[\beta(X)\big] = \frac{1}{2}\int_0^1 \frac{(1-p)x^2}{(1-p)x+p} \cdot 1dx.$$

To solve the integral, add and subtract the term $\frac{px}{(1-p)x+p}$ *so that the integral becomes*

$$\begin{aligned} E\big[\beta(X)\big] &= \frac{1}{2}\int_0^1 \bigg[\frac{(1-p)x^2+px}{(1-p)x+p} - \frac{px}{(1-p)x+p}\bigg]dx \\ &= \frac{1}{2}\bigg[\int_0^1 x\frac{(1-p)x+p}{(1-p)x+p}dx - \int_0^1 \frac{px}{(1-p)x+p}dx\bigg]. \end{aligned}$$

In the second term, we can rewrite the function as

$$\begin{aligned} \frac{px}{(1-p)x+p} &= \frac{\frac{p}{1-p}\big[(1-p)x+p-p)\big]}{(1-p)x+p} \\ &= \frac{p}{1-p}\bigg[\frac{(1-p)x+p}{(1-p)x+p} - \frac{p}{(1-p)x+p}\bigg] \\ &= \frac{p}{1-p}\bigg[1 - \frac{p}{(1-p)x+p}\bigg]. \end{aligned} \tag{3.7}$$

Thus, we get

$$\begin{aligned} E\big[\beta(X)\big] &= \frac{1}{2}\int_0^1 \Bigg[x - \frac{p}{1-p}\bigg[1 - \frac{p}{(1-p)x+p}\bigg]\Bigg]dx \\ &= \frac{1}{2}\bigg[\frac{1}{2} - \frac{p}{1-p} + \frac{p^2}{1-p}\int_0^1 \frac{1}{(1-p)x+p}dx\bigg] \\ &= \frac{1}{2}\bigg[\frac{1}{2} - \frac{p}{1-p} + \frac{p^2}{(1-p)^2}\Big[ln\big((1-p)x+p\big)\Big]_0^1\bigg] \\ &= \frac{1}{2}\bigg[\frac{1}{2} - \frac{p}{1-p} - \frac{p^2}{(1-p)^2}ln(p)\bigg] \\ &= \frac{1}{4} - \frac{1}{2}\frac{p}{1-p} - \frac{1}{2}\frac{p^2}{(1-p)^2}ln(p). \end{aligned}$$

We know that β is an increasing function so $max\{\beta(X_1), \beta(X_2)\} = \beta\big(max\{X_1, X_2\}\big)$. The distribution function of $max\{X_1, X_2\}$ is

$$\begin{aligned}
F_{max\{X_1,X_2\}}(a) &= P\big(max\{X_1, X_2\} \leq a\big) \\
&= P\big(X_1 \leq, X_2 \leq a\big) \\
&\overset{\text{indep.}}{=} P\big(X_1 \leq a\big) \cdot P\big(X_2 \leq a\big) \\
&\overset{(\text{X}_1,\text{X}_2)\in\text{U}(0,1)}{=} a^2.
\end{aligned}$$

Then, the probability density function is $f_{max\{X_1,X_2\}}(a) = 2a$. Thus, we get

$$\begin{aligned}
E\Big[max\{\beta(X_1), \beta(X_2)\}\Big] &= \frac{1}{2}\int_0^1 \frac{(1-p)x^2}{(1-p)x+p} f_{max\{X_1,X_2\}}(x)dx \\
&= \frac{1}{2}\int_0^1 \frac{(1-p)x^2}{(1-p)x+p} 2xdx \\
&= \int_0^1 \frac{(1-p)x^3}{(1-p)x+p} dx.
\end{aligned}$$

We add and subtract the term $\frac{px^2}{(1-p)x+p}$ so that

$$\begin{aligned}
E\Big[max\{\beta(X_1), \beta(X_2)\}\Big] &= \int_0^1 \frac{(1-p)x^3 + px^2 - px^2}{(1-p)x+p} dx \\
&= \int_0^1 x^2 \frac{(1-p)x+p}{(1-p)x+p} dx - \int_0^1 \frac{px^2}{(1-p)x+p} dx \\
&= \frac{1}{3} - \int_0^1 \frac{px^2}{(1-p)x+p} dx.
\end{aligned}$$

We rewrite the second integral with similar techniques as in (3.7)

$$\begin{aligned}
\int_0^1 \frac{px^2}{(1-p)x+p} dx &= \int_0^1 \frac{\frac{p}{1-p}\Big[(1-p)x^2 + px - px\Big]}{(1-p)x+p} dx \\
&= \frac{p}{1-p}\int_0^1 \left[x\frac{(1-p)x+p}{(1-p)x+p} - \frac{px}{(1-p)x+p}\right] dx \\
&= \frac{p}{1-p}\left[\int_0^1 xdx - \int_0^1 \frac{px}{(1-p)x+p}\right] dx \\
&= \frac{p}{1-p}\left[\frac{1}{2} - \int_0^1 \frac{px}{(1-p)x+p}\right] dx.
\end{aligned}$$

Thus, we get

$$
\begin{aligned}
E\Big[max\{\beta(X_1),\beta(X_2)\}\Big] &= \frac{1}{3} - \frac{p}{1-p}\left[\frac{1}{2} - \int_0^1 \frac{px}{(1-p)x+p}\right]dx \\
&= \frac{1}{3} - \frac{p}{1-p}\left[\frac{1}{2} - \frac{p}{1-p}\int_0^1 \left[1 - \frac{p}{(1-p)x+p}\right]dx\right] \\
&= \frac{1}{3} - \frac{p}{1-p}\left[\frac{1}{2} - \frac{p}{1-p}\left[1 - \frac{p}{(1-p)}\Big[ln\big((1-p)x+p\big)\Big]_0^1\right]\right] \\
&= \frac{1}{3} - \frac{p}{1-p}\left[\frac{1}{2} - \frac{p}{1-p}\left[1 + \frac{p}{(1-p)}ln(p)\right]\right] \\
&= \frac{1}{3} - \frac{p}{1-p}\left[\frac{1}{2} - \frac{p}{1-p} - \frac{p^2}{(1-p)^2}ln(p)\right] \\
&= \frac{1}{3} - \frac{1}{2}\frac{p}{1-p} + \frac{p^2}{(1-p)^2} + \frac{p^3}{(1-p)^3}ln(p).
\end{aligned}
$$

We want to find the optimal number of bidders that the seller should invite such that he can maximize his expected revenue. Therefore, (3.6) becomes

$$
\begin{aligned}
c &= E\Big[max\{\beta(X_1),\beta(X_2)\}\Big] - E\big[\beta(X)\big] \\
&= \frac{1}{3} - \frac{1}{2}\frac{p}{1-p} + \frac{p^2}{(1-p)^2} + \frac{p^3}{(1-p)^3}ln(p) - \left[\frac{1}{4} - \frac{1}{2}\frac{p}{1-p} - \frac{1}{2}\frac{p^2}{(1-p)^2}ln(p)\right] \\
&= \frac{1}{12} + \frac{p^2}{(1-p)^2} + \frac{1}{2}\frac{p^2}{(1-p)^2}ln(p) + \frac{p^3}{(1-p)^3}ln(p).
\end{aligned}
\tag{3.8}
$$

Recall the relation $p = \frac{2q}{1+q}$ *from (3.5), then* $1 - p = \frac{1-q}{1+q}$*. We write (3.8) in terms of* q*, i.e from the seller's point of view*

$$
\begin{aligned}
c &= \frac{1}{12} + \left(\frac{2q/(1+q)}{(1-q)/(1+q)}\right)^2 + \frac{1}{2}\left(\frac{2q/(1+q)}{(1-q)/(1+q)}\right)^2 ln\left(\frac{2q}{1+q}\right) \\
&\quad + \left(\frac{2q/(1+q)}{(1-q)/(1+q)}\right)^3 ln\left(\frac{2q}{1+q}\right) \\
&= \frac{1}{12} + \frac{4q^2}{(1-q)^2} + \frac{1}{2}\frac{4q^2}{(1-q)^2}ln\left(\frac{2q}{1+q}\right) + \frac{8q^3}{(1-q)^3}ln\left(\frac{2q}{1+q}\right) \\
&= \frac{1}{12} + \frac{4q^2}{(1-q)^2} + ln\left(\frac{2q}{1+q}\right)\left[\frac{1}{2}\frac{4q^2}{(1-q)^2} + \frac{8q^3}{(1-q)^3}\right] \\
&= \frac{1}{12} + \frac{4q^2}{(1-q)^2} + \frac{1}{2}ln\left(\frac{2q}{1+q}\right)\left[\frac{4q^2 + 12q^3}{(1-q)^3}\right] \\
&= \frac{1}{12} + \frac{4q^2}{(1-q)^2} + 2ln\left(\frac{2q}{1+q}\right)\left[\frac{q^2 + 3q^3}{(1-q)^3}\right].
\end{aligned}
\tag{3.9}
$$

References

[1] Krishna, V. (2002). *Auction Theory*. 4th ed. California: Academic Press.