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# The *p*-Laplace equation – general properties and boundary behaviour

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**Abstract**

In this thesis we investigate the properties of the solutions to the *p*-Laplace equation, which is the Euler-Lagrange equation of the *p*-Dirichlet integral, a generalization of the well known Dirichlet integral. It turns out that many of the properties of the harmonic functions also hold for the so called *p*-harmonic functions. After giving a comprehensive introduction to the subject, where we establish the existence of weak solutions on bounded domains, we discuss general properties such as the Harnack inequality, Hölder continuity, differentiability, and Perron's method. In the last part of the thesis we study boundary behaviour and in particular the behaviour of the ratio of two *p*-harmonic functions near a portion of the boundary where they both vanish.

**Acknowledgements**

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# Contents

# 1 Introduction

One of the central problems of modern analysis has been the Dirichlet problem, that is, given an open connected set $\Omega \subset \mathbb{R}^n$, and a real-valued function $f$, continuous on the boundary $\partial\Omega$, to find a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$, such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

The literature on the problem is extensive and goes deep into the realms of many mathematical subjects, such as complex analysis (see e.g. [5]) and probability theory (see e.g. [24]). Another approach to the Dirichlet problem is via the Dirichlet energy integral. The Dirichlet energy integral of a function $u : \Omega \to \mathbb{R}$ is defined as

$$E(u) = \int_\Omega |\nabla u|^2 \, dx = \int_\Omega \left(\frac{\partial u}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2 \, dx, \tag{1}$$

(sometimes written as multiplied by $\frac{1}{2}$). The Euler-Lagrange equation for $E(u)$ is easily shown to be the Laplace equation $\Delta u = 0$, that is, solving the Dirichlet problem with boundary conditions $f$ is the same as minimizing the Dirichlet integral over functions with boundary data $f$.

In this thesis, we consider a generalized version of (1), namely the $p$-Dirichlet integral

$$I(u) = \int_\Omega |\nabla u|^p \, dx,$$

of which the Euler-Lagrange equation is the $p$-Laplace equation

$$\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

## 1.1 Preliminaries

### 1.1.1 Notation

In the following we let $\Omega$ be a domain in $\mathbb{R}^n$, which is not necessarily bounded. $G$ and $D$ will always denote open sets in $\mathbb{R}^n$. We let $\langle \cdot, \cdot \rangle$ and $dx$ denote the inner product and the Lebesgue measure on $\mathbb{R}^n$, respectively. We will express the Euclidean distance between the two points $x_1$ and $x_2$ by $d(x_1, x_2)$. $c$ will always be a positive constant, depending on at most $p$ and $n$ unless otherwise stated, such that $c \geq 1$. The value of $c$ may vary from line to line and between occurrences. Furthermore, we will use the notation $c(a_1, a_2, \ldots, a_s)$ when $c$ also depends on the the additional constants $a_1, a_2, \ldots a_s$. For $x \in \mathbb{R}^n$ and $r > 0$ we define the open ball $B(x,r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. When the center of the ball is arbitrary or clear from the context we will denote the ball by $B_r$, and balls with different radii are always assumed to be concentric unless otherwise stated. The average of a function $U$ over the set $E$ is defined as

$$(u)_E = \fint_E u(x) \, dx = \frac{1}{|E|} \int_E u(x) \, dx.$$

The reader is assumed to be familiar with basic facts of Sobolev spaces, roughly corresponding to the content in chapter 7 in [6] or chapter 5 in [1]. We now, however, repeat the most basic definitions and properties.

Let $u$ be a locally summable function, i.e., $u \in L^1_{\text{loc}}(\Omega)$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex of order $|\alpha| = \alpha_1 + \dots + \alpha_n$. Then $v \in L^1_{\text{loc}}(\Omega)$ is the $\alpha^{\text{th}}$ weak derivative of $u$ if the equation

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v\varphi \, dx,$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots, \partial x_n^{\alpha_n}}$$

holds for all $\varphi \in C_0^\infty(\Omega)$, i.e., all smooth functions in $\Omega$ with compact support. We use the notation $D^\alpha u = v$. The Sobolev space $W^{k,p}(\Omega)$ is defined as the normed space that consists of equivalence classes of locally summable functions $u : \Omega \to \mathbb{R}$ such that $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$. The norm of $u \in W^{k,p}(\Omega)$ is defined as

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} & 1 \leq p < \infty, \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u| & p = \infty. \end{cases}$$

Unless otherwise stated, $\nabla u = (u_{x_1}, \dots, u_{x_n})$ will always denote the distributional gradient. Furthermore, $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ and we note that both $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are Banach spaces. The space $W^{k,p}_{\text{loc}}(\Omega)$ is defined analogously to $L^p_{\text{loc}}(\Omega)$: $u \in W^{k,p}_{\text{loc}}(\Omega)$ if and only if $u \in W^{k,p}(D)$ for each open set $D \subset\subset \Omega$. In the following we will almost exclusively deal with the case $k = 1$ and $1 < p < \infty$. For $u \in W_0^{1,p}(\Omega)$ and $\Omega$ bounded we recall the Poincaré inequality

$$\|u\|_{L^p(\Omega)} \leq \left( \frac{|\Omega|}{\omega_n} \right)^{1/n} \|\nabla u\|_{L^p(\Omega)},$$

where $\omega_n$ is the volume of a unit ball in $\mathbb{R}^n$. We will sometimes use the notation $A \approx B$ which means that the ratio of $A$ and $B$ is bounded from above and below by constants. The dependence of the constants will be specified at each occurrence.

### 1.1.2 Weighted Sobolev spaces

We will now give a very brief introduction to weighted Sobolev spaces. These function spaces will only occur in Section 7.4 but it actually turns out that most of the results we discuss in this thesis are also valid for functions that belong to the weighted Sobolev spaces. For a better and deeper introduction to the subject, see [4] or [9]. We consider a non-negative real-valued function $\lambda \in L^1_{\text{loc}}(\mathbb{R}^n)$ and define the Radon measure $\mu$ by

$$\mu(E) = \int_E \lambda(x) \, dx$$

whenever $E \subset \mathbb{R}^n$. We will denote the $L^p$-space corresponding to the measure $\mu$ by $\tilde{L}^p(\Omega)$ or $L^p(\Omega; \mu)$. We have the following definition:

**Definition 1.1.** A weight $\lambda$ is called $p$-admissible if the following conditions hold:

(i) $\lambda(x) \in (0, \infty)$ a.e. in $\mathbb{R}^n$ and the corresponding measure $\mu$ is a doubling measure, i.e. $\mu(B_{2r}) \leq c\mu(B_r)$ for all $B_r \subset \mathbb{R}^n$.

(ii) If $v \in \tilde{L}^p(G)$ is a vector-valued function and $\{\varphi_i\} \subset C^\infty(G)$ such that $\int_G |\varphi_i|^p d\mu \to 0$ and $\int_G |\nabla\varphi_i - v|^p d\mu \to 0$ as $i \to \infty$, then $v = 0$.

(iii) The weighted Sobolev inequality holds, i.e., there exists $\kappa > 1$ such that for all $B_r \subset \mathbb{R}^n$ it holds that

$$\left(\frac{1}{\mu(B_r)}\int_{B_r} |\varphi|^{\kappa p}\, d\mu\right)^{1/\kappa p} \le cr\left(\frac{1}{\mu(B_r)}\int_{B_r} |\nabla\varphi|^p\, d\mu\right)^{1/p},$$

where $\varphi \in C_0^\infty(B_r)$.

(iv) The weighted Poincaré inequality holds, i.e.,

$$\int_{B_r} |\varphi - (\varphi)_{B_r}|^p\, d\mu \le cr^p \int_{B_r} |\nabla\varphi|^p\, d\mu$$

for all bounded $\varphi \in C^\infty(B_r)$. Here $(\varphi)_{B_r}$ is the average over the ball $B_r$ using the weighted measure $\mu$.

We note from condition *(i)* that the Lebesgue measure $dx$ is absolutely continuous with respect to $\mu$. In the following we assume that $\lambda(x)$ is a $p$-admissible weight. The weighted Sobolev space $\widetilde{W}^{1,p}(\Omega)$ or $W^{1,p}(\Omega;\mu)$ is defined as the closure of smooth functions in $\Omega$, with respect to the weighted Sobolev norm

$$\|\varphi\tilde{\|} = \left(\int_\Omega |\varphi|^p\, d\mu\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p\, d\mu\right)^{1/p}$$

Thus, a function $u$ is in $\widetilde{W}^{1,p}(\Omega)$ if and only if $u \in \widetilde{L}^p(\Omega)$ and there exists a vector-valued function $v \in \tilde{L}^p(\Omega)$ such that for some sequence of smooth functions $\{\varphi_i\}$ such that the following conditions are satisfied:

$$\lim_{i\to\infty}\int_\Omega |\varphi_i - u|^p\, d\mu = 0$$
$$\lim_{i\to\infty}\int_\Omega |\nabla\varphi_i - v|^p\, d\mu = 0.$$

We say that $v$ is the gradient of $u$ in $\widetilde{W}^{1,p}(\Omega)$ and use the notation $v = \nabla u$. The spaces $\widetilde{W}_0^{1,p}(\Omega)$ and $\widetilde{W}_{\text{loc}}^{1,p}(\Omega)$ are defined analogous to the unweighted cases. Note that an element in $\widetilde{W}^{1,p}(\Omega)$ is not necessarily in $L^1_{\text{loc}}(\Omega)$ and therefore $\nabla u$ does not necessarily have to be the distributional gradient of $u$.

We now turn to a special class of $p$-admissible weights, called the Muckenhoupt class $A_p$, for which the distributional gradients of the elements in the corresponding Sobolev space $W^{1,p}_{\text{loc}}(\Omega,\mu)$ exist.

**Definition 1.2.** Assume that $1 < p < \infty$. We say that a locally summable function $\lambda : \mathbb{R}^n \to [0,\infty)$ is an $A_p(\mathbb{R}^n)$-weight if

$$\left(⨍_{B_r} \lambda\, dx\right)\left(⨍_{B_r} \lambda^{1/(1-p)}\, dx\right)^{p-1} \le \gamma$$

for a constant $\gamma = \gamma(p,\lambda)$ whenever $B_r \subset \mathbb{R}^n$.

We end this subsection by proving the existence of the distributional gradient, $\nabla u$, for $u \in \widetilde{W}^{1,p}(\Omega)$. We first show that $\widetilde{L}^p(D) \subset L^1(D)$ for $D \subset\subset \Omega$. This can be seen as follows:

$$\begin{aligned}\int_D |u|\, dx &= \int_D |u|\lambda^{1/p}\lambda^{-1/p}\, dx \\ &\le \left(\int_D \lambda^{1/(1-p)}\, dx\right)^{1-1/p}\left(\int_D |u|^p\, d\mu\right)^{1/p}\end{aligned}$$

where we used the Hölder inequality. Therefore, if $\varphi_j \to u$ in $\widetilde{W}^{1,p}(\Omega)$, it follows that $\varphi_j \to u$ and $\partial_i \varphi_j \to \partial_i u$ in $L^1(D)$ and thus we see that

$$\left| \int_\Omega u \partial_i \varphi + \varphi \partial_i u \, dx \right| = \left| \int_{\mathrm{spt}\varphi} (u - \varphi_j) \partial_i \varphi + (\partial_i u - \partial_i \varphi_j) \varphi \ \ dx \right| \to 0 \text{ as } j \to \infty$$

whenever $\varphi \in C_0^\infty(\Omega)$ which implies that the gradient $\nabla u$ exists in distributional sense. We will return to the weighted Sobolev spaces and the Muckenhoupt class $A_p$ in Section 7.4.

## 1.2 Variational integrals and the p-Dirichlet integral

Let $d\mu(x) = \lambda(x)\, dx$ for a $p$-admissable weight $\lambda(x)$. In the general setting, and using the notation from [9], we consider

$$I_F(u, \Omega) = \int_\Omega F(x, \nabla u(x))\, dx \tag{2}$$

where $F(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a mapping called the variational kernel satisfying the following conditions:

- $x \mapsto F(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$ for a.e. $x \in \mathbb{R}^n$,
- $\beta \lambda(x) |\xi|^p \le F(x, \xi) \le \delta \lambda(x) |\xi|^p$, for $0 < \beta \le \delta < \infty$ and $\xi \in \mathbb{R}^n$,
- the mapping $\xi \mapsto F(x, \xi)$ is strictly convex and continuously differentiable, and
- $F(x, \lambda \xi) = |\lambda|^p F(x, \xi)$, $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^n$.

From the second assumption we see that the integral in (2) is finite if and only if $F(x, \nabla u(x)) \in \widetilde{L}^1(\Omega)$. We are interested in minimizing the integral in (2) among a certain class of functions with given boundary values. We have the following definition.

**Definition 1.3.** We say that a function $u \in \widetilde{W}^{1,p}(\Omega)$ is an $F$-extremal in $\Omega$ with boundary values $g \in \widetilde{W}^{1,p}(\Omega)$ if $u - g \in \widetilde{W}_0^{1,p}(\Omega)$ and

$$I_F(u, \Omega) \le I_F(v, \Omega),$$

whenever $v - g \in \widetilde{W}_0^{1,p}(\Omega)$. Furthermore, we say that a function $u \in \widetilde{W}_{\mathrm{loc}}^{1,p}(\Omega)$ is a (free) $F$-extremal in $\Omega$ if $u$ is an $F$-extremal with boundary values $u$ in each open set $D \subset\subset \Omega$.

The following theorem characterizes all $F$-extremals in $\Omega$ (Theorem 5.18 in [9]).

**Theorem 1.1.** *A function $u \in \widetilde{W}_{\mathrm{loc}}^{1,p}(\Omega)$ is an $F$-extremal in $\Omega$ if and only if*

$$-\mathrm{div}\, \nabla_\xi F(x, \nabla u) = 0$$

*in $\Omega$, that is,*

$$\int_\Omega \langle \nabla_\xi F(x, \nabla u), \nabla \varphi \rangle \, dx = 0$$

*for all $\varphi \in C_0^\infty(\Omega)$.*

For $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ and $1 < p < \infty$ we consider the $p$-Dirichlet integral,

$$I(u) = \int_\Omega |\nabla u|^p \, dx, \tag{3}$$

i.e., $F(x,\xi) = |\xi|^p$, so $\nabla_\xi F(x,\xi) = p|\xi|^{p-2}\xi$ and thus $\nabla_\xi F(x,\nabla u) = p|\nabla u|^{p-2}\nabla u$. In order for $u$ to be an $F$-extremal, it must satisfy

$$\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \tag{4}$$

by the first condition of the theorem. This equation is called the $p$-Laplace equation in $\Omega$. From now on we only consider the $p$-Dirichlet integral and the $p$-Laplace equation. If we assume that $u \in C^2(\Omega)$ and that $\nabla u \neq 0$ in $\Omega$ we can formally carry out the differentiation in (4) which yields

$$\frac{\partial}{\partial x_j}\left(\left(\sum_{i=1}^n u_{x_i}^2\right)^{\frac{p-2}{2}} u_{x_j}\right) = (p-2)|\nabla u|^{p-4}\left(\sum_{i=1}^n u_{x_i}u_{x_ix_j}\right)u_{x_j} + |\nabla u|^{p-2}u_{x_jx_j},$$

and thus we obtain

$$\begin{aligned}
\nabla\cdot(|\nabla u|^{p-2}\nabla u) &= \sum_{j=1}^n\left(\frac{\partial}{\partial x_j}\left(\left(\sum_{i=1}^n u_{x_i}^2\right)^{\frac{p-2}{2}} u_{x_j}\right)\right)\\
&= \sum_{j=1}^n (p-2)|\nabla u|^{p-4}\left(\sum_{i=1}^n u_{x_i}u_{x_ix_j}\right)u_{x_j} + |\nabla u|^{p-2}u_{x_jx_j}\\
&= (p-2)|\nabla u|^{p-4}\sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^{p-2}\sum_{j=1}^n u_{x_jx_j}\\
&= |\nabla u|^{p-4}\{|\nabla u|^2\Delta_2 u + (p-2)\Delta_\infty u\},
\end{aligned}$$

where

$$\Delta_\infty u = \sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j},$$

and $\Delta_2$ denotes the Laplace operator. We see that for $p = 2$ we obtain the Laplace operator $\Delta_2 u = \sum(u_{x_ix_i})^2$. From the calculations above we see that we can write $\Delta_p u$ as a partial differential equation in non-divergence form,

$$Lu := \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}(x),$$

with

$$a_{ij}(x) = (p-2)|\nabla u|^{p-4}u_{x_i}u_{x_j} + |\nabla u|^{p-2}\delta_{ij}.$$

We note that $a_{ij} = a_{ji}$ and it follows that

$$\begin{aligned}
\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j &= \sum_{i,j=1}^n (p-2)|\nabla u|^{p-2}u_{x_i}\xi_j u_{x_j}\xi_i + |\nabla u|^{p-2}\delta_{ij}\xi_i\xi_j\\
&= (p-2)|\nabla u|^{p-4}\langle\nabla u,\xi\rangle^2 + |\nabla u|^{p-2}|\xi|^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality we see immediately that for $p \geq 2$

$$|\nabla u|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq (p-1)|\nabla u|^{p-2}|\xi|^2,$$

and for $1 < p < 2$ we have that

$$(p-1)|\nabla u|^{p-2}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le |\nabla u|^{p-2}|\xi|^2.$$

Thus, on compact sets $D$ where $\nabla u \neq 0$ on $D$ we see that $L$ satisfies the uniform ellipticity condition. This can also be seen by using that the Laplace equation is invariant under rotations, which we show in Lemma A.1. Using the rotation invariance it is possible to derive the following fundamental solution for the $p$-Laplace equation

$$\Phi(x) = \begin{cases} -\log|x| & p = n, \\ (n-p)|x|^{\frac{p-n}{p-1}} & p < n, \end{cases}$$

which is shown in a calculation following Lemma A.1.

### 1.3 Structure of Thesis

In the following sections, we first introduce the basic definitions (Section 2), before moving on to regularity of weak solutions (Section 3), differentiability (Section 4), $p$-superharmonic functions (Section 5) and Perron's method (Section 6). In these chapters, we closely follow [21], although we try to provide more details and on occasion we also provide theory and results from [9]. In the last section, we proceed with boundary behaviour of weak solutions to the $p$-Laplace equation (Section 7). There we concern ourselves with the theory and problems from [23], which are based on the results in [13] and [14]-[20]. Many of the statements of the theorems, lemmas, proposition and definitions are taken word for word from sources mentioned above.

## 2 Basic definitions and existence of weak solutions

In this section we concern ourselves with basic properties of weak solutions to the $p$-Laplace equation and discuss some fundamental results that will be used throughout this thesis, such as Caccioppoli's inequality and the well-known comparison principle. Furthermore, we prove the existence of a $p$-harmonic function in a bounded domain with boundary values given in Sobolev sense. We end this section by discussing regular points and the Wiener criterion.

We begin by defining a weak solution to the $p$-Laplace equation.

**Definition 2.1.** We say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution of the $p$-Laplace equation in $\Omega$, if

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, dx = 0, \tag{5}$$

for each $\varphi \in C_0^\infty(\Omega)$. If, in addition, $u$ is continuous, we say that $u$ is a $p$-harmonic function. Furthermore, we say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak supersolution of the $p$-Laplace equation in $\Omega$, if

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, dx \ge 0, \tag{6}$$

for each non-negative $\varphi \in C_0^\infty(\Omega)$. A function $u$ is a subsolution if $-u$ is a supersolution or equivalently stated that (6) holds for each non-positive $\varphi \in C_0^\infty(\Omega)$.

If we refer to $u$ as a solution of (4) it will always mean that $u$ is a solution in the weak sense unless otherwise stated. We note that if $u$ is a (super)solution and $\lambda, \tau \in \mathbb{R}$ and $\lambda \geq 0$ we have that $\lambda u+\tau$ is also a (super)solution. Since $W_0^{1,p}(\Omega)$ is the closure of smooth functions in $W^{1,p}(\Omega)$ we have the following lemma.

**Lemma 2.1.** *If $u \in W^{1,p}(\Omega)$ is a solution of* (4) *in $\Omega$, then*

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx = 0$$

*for all $\varphi \in W_0^{1,p}(\Omega)$.*

*Proof.* Since $\varphi \in W_0^{1,p}(\Omega)$ we choose functions $\varphi_i \in C_0^\infty(\Omega)$ such that $\varphi_i \to \varphi$ in $W^{1,p}(\Omega)$. By Hölder's inequality we deduce that

$$\begin{aligned} &\left| \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx - \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi_i \rangle \, dx \right| \\ &\leq \int_\Omega |\nabla u|^{p-1} |\nabla \varphi - \nabla \varphi_i| \; dx \\ &\leq \left( \int_\Omega |\nabla u|^p \, dx \right)^{(p-1)/p} \left( \int_\Omega |\nabla \varphi - \nabla \varphi_i|^p \; dx \right)^{1/p}. \end{aligned}$$

The last integral tends to zero as $i \to \infty$ so it follows that

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx = \lim_{i \to \infty} \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi_i \rangle \, dx = 0.$$

□

This lemma will be used extensively. From the proof it follows that if $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a weak solution, then (5) holds for all $\varphi \in W_0^{1,p}(\Omega)$ with compact support. Moreover, we note that (5) holds if $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a weak solution with $\nabla u \in L^p(\Omega)$. Furthermore, it is also clear from the proof that the analogous version holds for supersolutions as long as $\varphi$ is non-negative and we pick a non-negative approximating sequence.

In the more general setting of Theorem 1.1 we saw that minimizers are the same as weak solutions, which is also established in the following theorem.

**Theorem 2.2.** *The following conditions are equivalent for $u \in W^{1,p}(\Omega)$:*

*(i) $u$ is minimizing:*

$$\int_\Omega |\nabla u|^p \, dx \leq \int_\Omega |\nabla v|^p \, dx, \text{ when } v - u \in W_0^{1,p}(\Omega)$$

*(ii) The first variation vanishes:*

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx = 0, \text{ when } \varphi \in W_0^{1,p}(\Omega)$$

*Proof.* Suppose that *(i)* holds and take $v(x) = u(x) + \varepsilon \varphi(x)$ for $\varepsilon \in \mathbb{R}$ and $\varphi \in W_0^{1,p}(\Omega)$ so that $v - u \in W_0^{1,p}(\Omega)$. Since $u$ is minimizing, the integral

$$J(\varepsilon) = \int_\Omega |\nabla(u + \varepsilon\varphi)|^p \, dx,$$

attains its minimum for $\varepsilon = 0$ so the first variation vanishes at $\varepsilon = 0$, i.e., $J'(0) = 0$. Furthermore, we have that

$$\frac{d}{d\varepsilon}\left(|\nabla(u+\varepsilon\varphi)|^p\right) = p|\nabla(u+\varepsilon\varphi)|^{p-2}\left(\sum \Big\langle \frac{\partial}{\partial x_i}(u+\varepsilon\varphi), \frac{\partial}{\partial x_i}\varphi\Big\rangle\right),$$

and, since $|\nabla(u+\varepsilon\varphi)|^p$ is continuously differentiable a.e., with respect to $\varepsilon$ this implies that

$$J'(0) = \int_\Omega |\nabla(u)|^{p-2}\langle \nabla u, \nabla\varphi\rangle\, dx,$$

so *(i)* ⇒ *(ii)*. Next we assume that *(ii)* holds and that $v-u \in W_0^{1,p}(\Omega)$. Note that this implies that $v \in W^{1,p}(\Omega)$. Then

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla(v-u)\rangle\, dx = 0,$$

and using Cauchy Schwartz and the Hölder inequality we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^p\, dx &= \int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle\, dx\\ &\le \int_\Omega |\nabla u|^{p-1}|\nabla v|\, dx\\ &\le \left(\int_\Omega |\nabla u|^p\, dx\right)^{1-1/p}\left(\int_\Omega |\nabla v|^p\, dx\right)^{1/p},\end{aligned}$$

from which *(i)* follows immediately. □

We note that *(ii)* ⇒ *(i)* also holds when $u$ is a supersolution, provided that $v-u \in W_0^{1,p}(\Omega)$ with $v \ge u$.

Next, we define the so called obstacle problem and for this we assume that $\Omega$ is bounded and that $\nu \in W^{1,p}(\Omega)$. Let

$$\mathcal{F}_{\psi,\nu}(\Omega) = \{v \in W^{1,p}(\Omega) : v \ge \psi \text{ a.e. in } \Omega, v-\nu \in W_0^{1,p}(\Omega)\}.$$

**Definition 2.2.** We say that a function $u$ in $\mathcal{F}_{\psi,\nu}(\Omega)$ is a solution to the obstacle problem with obstacle $\psi$ and boundary value $\nu$ if

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla(v-u)\rangle\, dx \ge 0,$$

whenever $v \in \mathcal{F}_{\psi,\nu}(\Omega)$. Then we say that $u$ is a solution to the obstacle problem in $\mathcal{F}_{\psi,\nu}(\Omega)$.

From the definition, and an application of Hölders theorem, it follows directly that a solution $u$ to the obstacle problem minimizes the $p$-energy among the functions in $\mathcal{F}_{\psi,\nu}(\Omega)$. We note that a solution $u$ to the obstacle problem is always a weak solution to the $p$-Laplace equation since $u+\varphi \in \mathcal{F}_{\psi,\nu}(\Omega)$ for a non-negative function $\varphi \in C_0^\infty(\Omega)$. Furthermore, a supersolution $u \in L^p_{\text{loc}}(\Omega)$ is a solution to the obstacle problem in $\mathcal{F}_{u,u}(E)$ where $E$ is a compactly contained set in $\Omega$, since $v-u \in W^{1,p}_{\text{loc}}(E)$ is non-negative and $u \in W^{1,p}(E)$. If $\mathcal{F}_{\psi,\nu}(\Omega)$ is nonempty it can be shown that there exists a unique solution to the obstacle problem (Theorem 3.21 in [9]). Moreover, if the obstacle is continuous the weak solution will also be continuous. We will return to a special case of the obstacle problem in Section 5, when we discuss $p$-superharmonic functions.

The next estimate is known as Caccioppoli's inequality:

**Lemma 2.3.** *If u is a weak solution in $\Omega$, then*

$$\int_\Omega \zeta^p|\nabla u|^p\, dx \le p^p\int_\Omega |u|^p|\nabla\zeta|^p\, dx,$$

*for each $\zeta \in C_0^\infty(\Omega)$, $0 \le \zeta \le 1$. In particular, if $B_{2r} \subset \Omega$, then*

$$\int_{B_r} |\nabla u|^p \le p^p r^{-p} \int_{B_{2r}} |u|^p \, dx.$$

*Proof.* Set $\varphi = \zeta^p u$ and note that $\varphi \in W_0^{1,p}(\Omega)$ and $\nabla \varphi = \zeta^p \nabla u + p\zeta^{p-1} u \nabla \zeta$. Since $u$ is a weak solution it satisfies (5) which yields

$$\begin{aligned}
\int_\Omega \zeta^p |\nabla u|^p \, dx &= -p \int_\Omega \zeta^{p-1} u \langle |\nabla u|^{p-2} \nabla u, \nabla \zeta \rangle \, dx \\
&\le p \int_\Omega |\zeta \nabla u|^{p-1} |u \nabla \zeta| \, dx \\
&\le p \left( \int_\Omega \zeta^p |\nabla u|^p \, dx \right)^{1-1/p} \left( \int_\Omega |u|^p |\nabla \zeta|^p \, dx \right)^{1/p},
\end{aligned}$$

and therefore we conclude that

$$\int_\Omega \zeta^p |\nabla u|^p \, dx \le p^p \int_\Omega |u|^p |\nabla \zeta|^p \, dx.$$

For the second statement we choose $\zeta$ as a radial function such that $\zeta = 1$ in $B_r$, $|\nabla \zeta| \le 1/r$ and $\zeta = 0$ in $\Omega \setminus B_{2r}$. The claim now follows immediately from the first statement. □

We note that a slightly modified variant of Caccioppoli's lemma holds for bounded supersolutions. Assume that $u$ is a bounded supersolution and let $L = \sup_\Omega u$ so $0 \le (L - u)$. We choose a test function $\varphi = (L - u)\zeta^p$ for a non-negative $\zeta \in C_0^\infty(\Omega)$ and calculate $\nabla \varphi = -\nabla u \zeta^p + p(L - u)\zeta^{p-1} \nabla \zeta$. Thus,

$$\begin{aligned}
\int_\Omega \zeta^p |\nabla u|^p \, dx &\le \int_\Omega p(L - u)\zeta^{p-1} |\nabla u|^{p-2} \langle \nabla u, \nabla \zeta \rangle \, dx \\
&\le p \int_\Omega |\zeta \nabla u|^{p-1} |(L - u) \nabla \zeta| \, dx \\
&\le p \left( \int_\Omega |\zeta \nabla u|^p \, dx \right)^{1-1/p} \left( \int_\Omega |L - u|^p |\nabla \zeta|^p \, dx \right)^{1/p},
\end{aligned}$$

so

$$\int_\Omega \zeta^p |\nabla u|^p \, dx \le p^p \int_\Omega |L - u|^p |\nabla \zeta|^p \, dx.$$

The next lemma also concerns supersolutions.

**Lemma 2.4.** *If $v > 0$ is a weak supersolution in $\Omega$, then*

$$\int_\Omega \zeta^p |\nabla \log v|^p \, dx \le \left( \frac{p}{p-1} \right)^p \int_\Omega |\nabla \zeta|^p \, dx$$

*whenever $\zeta \in C_0^\infty(\Omega)$ and $\zeta \ge 0$.*

*Proof.* We prove it for $u(x) = v(x) + \varepsilon$ where $\varepsilon > 0$. This is still a supersolution since $\nabla \varepsilon = 0$. Let $\varphi = \zeta^p u^{1-p}$ and note that it is well defined since $u > 0$. It follows that

$$\nabla \varphi = p\zeta^{p-1} u^{1-p} \nabla \zeta - (p-1)\zeta^p u^{-p} \nabla u,$$

and since $u$ is a supersolution we have that

$$0 \leq \int_\Omega \langle |\nabla u|^{p-2}\nabla u, p\zeta^{p-1}u^{1-p}\nabla\zeta\rangle - (p-1)\zeta^p u^{-p}|\nabla u|^p \, dx,$$

so

$$\begin{aligned}(p-1)\int_\Omega \zeta^p u^{-p}|\nabla u|^p \, dx &\leq p\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \zeta^{p-1}u^{1-p}\nabla\zeta\rangle \\ &\leq p\int_\Omega |\nabla u|^{p-1}\zeta^{p-1}u^{1-p}|\nabla\zeta| \, dx \\ &\leq p\left(\int_\Omega |\nabla u|^p\zeta^p u^{-p}\, dx\right)^{1-\frac{1}{p}}\left(\int_\Omega |\nabla\zeta|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Thus, it is clear that

$$\int_\Omega \zeta^p|\nabla u|^p u^{-p}\, dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla\zeta|^p \, dx,$$

and since $\nabla \log(v) = \frac{\nabla v}{v}$ we obtain

$$\int_\Omega \zeta^p \frac{|\nabla v|^p}{(v+\varepsilon)^p}\, dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla\zeta|^p \, dx.$$

We conclude the proof by letting $\varepsilon$ tend to 0. □

The next theorem is the comparison principle.

**Theorem 2.5.** *Suppose that u and v are p-harmonic functions in a bounded domain* $\Omega$*. If at each* $\zeta \in \partial\Omega$

$$\limsup_{x\to\xi} u(x) \leq \liminf_{x\to\xi} v(x)$$

*excluding the situation* $\infty \leq \infty$ *and* $-\infty \leq -\infty$*, then* $u \leq v$ *in* $\Omega$*.*

*Proof.* Take $\varepsilon > 0$ and consider the open set

$$D_\varepsilon = \{x | u(x) > v(x) + \varepsilon\}.$$

Due to the assumptions in the theorem either $D_\varepsilon$ is empty, and then there is nothing to prove, or $D_\varepsilon \subset\subset \Omega$. To this end we assume that $D_\varepsilon \neq \emptyset$. If we use that $u$ and $v$ are weak solutions we obtain

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla\zeta\rangle \, dx = 0,$$

where $\zeta$ is any $\zeta \in W_0^{1,p}(\Omega)$ with compact support. By using

$$\varphi(x) = \min\{v(x) - u(x) + \varepsilon, 0\},$$

and that $\mathrm{supp}(\varphi) \subset D_\varepsilon$ and $\varphi \in W_0^{1,p}(\Omega)$ it follows that

$$\int_{D_\varepsilon} \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla v - \nabla u\rangle \, dx = 0.$$

For for $1 < p < \infty$ we note that

$$\begin{aligned}
&\langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla v - \nabla u\rangle \\
&\quad = \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla v\rangle - \langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla u\rangle \\
&\quad = |\nabla v|^{p} - \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle - \langle |\nabla v|^{p-2}\nabla v, \nabla u\rangle + |\nabla u|^{p} \\
&\quad = |\nabla v|^{p} + |\nabla u|^{p} - \langle \nabla u, \nabla v\rangle(|\nabla u|^{p-2} + |\nabla v|^{p-2}) \\
&\quad \geq |\nabla v|^{p} + |\nabla u|^{p} - |\nabla u|^{p-1}|\nabla v| - |\nabla v|^{p-1}|\nabla u| \\
&\quad = |\nabla v|^{p-1}(|\nabla v| - |\nabla u|) + |\nabla u|^{p-1}(|\nabla u| - |\nabla v|) \\
&\quad = (|\nabla v| - |\nabla u|)(|\nabla v|^{p-1} - |\nabla u|^{p-1}) \\
&\quad \geq 0,
\end{aligned}$$

and therefore the integral is strictly positive if $\nabla u \neq \nabla v$ on a set of positive measure. Thus, it follows that $\nabla u = \nabla v$ a.e. in $D_\varepsilon$ which implies that $u = v + c$ in $D_\varepsilon$. Since $u = v + \varepsilon$ on $\partial D_\varepsilon$, we have $c = \varepsilon$ so $u \leq v + \varepsilon$ in $\Omega$. We let $\varepsilon$ tend to zero to obtain the result. □

Studying the proof we see immediately that the comparison principle also holds when $u$ and $v$ are weak sub- and supersolutions, respectively. Furthermore, in Section 5 we show that the comparison principle holds when $u$ is $p$-subharmonic and $v$ is $p$-superharmonic. We next establish the existence of a solution to the Dirichlet problem for the $p$-Laplace operator with boundary values in Sobolev sense. However, we first recall the definition of a weakly lower semicontinuous function on $W^{1,p}(\Omega)$.

**Definition 2.3.** We say that a function $I(\cdot)$ is weakly lower semicontinuous on $W^{1,p}(\Omega)$ if

$$I(u) \leq \liminf_{k\to\infty} I(u_k)$$

whenever $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.

We are now ready to state and prove the existence theorem.

**Theorem 2.6.** *Suppose that $g \in W^{1,p}(\Omega)$ where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and define*

$$\mathcal{A} = \{v \in W^{1,p}(\Omega) : v - g \in W_0^{1,p}(\Omega)\}.$$

*There exists a unique $u \in \mathcal{A}$ such that*

$$\int_\Omega |\nabla u|^p\, dx \leq \int_\Omega |\nabla v|^p\, dx$$

*for all $v \in \mathcal{A}$. This $u$ is a weak solution to the p-Laplace equation.*

*Proof.* At first we show the uniqueness and thus we assume that there exist two different minimizers $u_1$ and $u_2$ s.t. the set $\{\nabla u_1 \neq \nabla u_2\}$ has positive measure. Let $v = (u_1 + u_2)/2$. By the convexity of $|x|^p$ we know that

$$\left|\frac{\nabla u_1 + \nabla u_2}{2}\right|^p \leq \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2},$$

with strict inequality for $\{\nabla u_1 \neq \nabla u_2\}$. Since $v \in \mathcal{A}$,

$$
\begin{aligned}
\int_\Omega |\nabla u_2|^p \, dx &\leq \int_\Omega \left| \frac{\nabla u_1 + \nabla u_2}{2} \right|^p \, dx \\
&< \frac{1}{2} \int_\Omega |\nabla u_1|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u_2|^p \, dx \\
&\leq \int_\Omega |\nabla u_2|^p \, dx,
\end{aligned}
$$

where the first inequality follows from $u_2$ being a minimizer and the third inequality from $u_1$ being a minimizer. We have arrived at a contradiction and hence $\nabla u_1 = \nabla u_2$ a.e. in $\Omega$ so $u_1 = u_2 + c$ a.e. in $\Omega$. Since $u_2 - u_1 \in W_0^{1,p}(\Omega)$ it follows that $c = 0$. Next we establish the existence of a $p$-harmonic function with the given boundary values in $\Omega$. Let

$$
I_0 = \inf_{v \in \mathcal{A}} I(v) = \inf_{v \in \mathcal{A}} \int_\Omega |\nabla v|^p \, dx.
$$

It is clear that

$$
0 \leq I_0 \leq \int_\Omega |\nabla g|^p \, dx < \infty,
$$

since $g \in \mathcal{A}$. We proceed by choosing a minimizing sequence $\{v_j\}_{j=1}^\infty$ such that $v_j \in \mathcal{A}$ for all $j$ and

$$
\int_\Omega |\nabla v_j|^p \, dx < I_0 + \frac{1}{j}, \quad j = 1, 2, 3, \ldots
$$

We next show that $\{v_j\}$ is bounded in $W^{1,p}(\Omega)$, i.e.,

$$
\int_\Omega |v_j|^p \, dx + \int_\Omega |\nabla v_j|^p \, dx \leq M. \tag{7}
$$

Since $\Omega$ is bounded, we can employ the Poincaré inequality to assert that

$$
\int_\Omega |w|^p \, dx \leq c \int |\nabla w|^p \, dx,
$$

where $c = c(p, n, \Omega)$, holds for all $w \in W_0^{1,p}(\Omega)$. Applying this to $v_j - g$ yields

$$
\begin{aligned}
\int_\Omega |v_j - g|^p \, dx &\leq c \left( \int_\Omega |\nabla v_j|^p \, dx + \int_\Omega |\nabla g|^p \, dx \right) \\
&\leq c \left( (I_0 + 1) + \int_\Omega |\nabla g|^p \, dx \right),
\end{aligned}
$$

so by using the revered triangle inequality it follows that

$$
\int_\Omega |v_j|^p \, dx \leq M_1,
$$

for all $j$ which gives us (7). Since $1 < p < \infty$, $W^{1,p}(\Omega)$ is reflexive and therefore the sequence $\{v_j\}_{j=1}^\infty$ is weakly precompact in $\Omega$, which implies that there exists a $u \in W^{1,p}(\Omega)$ and a subsequence $\{v_{j_\nu}\}$ such that

$$
v_{j_\nu} \rightharpoonup u, \qquad \nabla v_{j_\nu} \rightharpoonup \nabla u,
$$

weakly in $L^p(\Omega)$ so $v_{j_\nu} \rightharpoonup u \in W^{1,p}(\Omega)$. Thus it follows that $v_{j_\nu} - g \rightharpoonup u - g$ in $W_0^{1,p}(\Omega)$. By definition, $W_0^{1,p}(\Omega)$ is a closed linear subspace of $W^{1,p}(\Omega)$ so it is convex and therefore it follows from Mazurs lemma that $W_0^{1,p}(\Omega)$ is closed under weak convergence. Hence $u - g \in W_0^{1,p}(\Omega)$, so $u$ is an admissible function, i.e., $u \in \mathcal{A}$. It is left to show that $u$ is the desired minimizer. $I(\cdot)$ is weakly lower semicontinuous on $W^{1,p}(\Omega)$ (see e.g. Theorem 8.2.1 in [1]), and thus

$$I(u) \leq \liminf_{j_\nu \to \infty} I(v_{j_\nu}) = I_0.$$

Since $u \in \mathcal{A}$ we conclude that $I(u) = I_0$. □

We next consider the Dirichlet problem for the $p$-Laplace operator. If the boundary function $g$ is continuous, i.e., $g \in C(\bar{\Omega})$ and $\Omega$ is regular, then $u \in C(\bar{\Omega})$ and $u|\partial\Omega = g|\partial\Omega$. By regular we mean the following:

**Definition 2.4.** Assume that $\Omega$ is a bounded domain. We say that $x_0 \in \partial\Omega$ is a regular point for the $p$-Dirichlet problem if for each $g \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, and the unique $p$-harmonic function $u$ such that $u - g \in W_0^{1,p}(\Omega)$ it holds that

$$\lim_{x \to x_0} u(x) = g(x_0).$$

$\Omega$ is regular if each boundary point is regular.

Examples of regular sets are balls and polyhedra. Furthermore, every open domain has a so called exhaustion of $\Omega$ with regular domains, i.e., there exist domains $D_1 \subset D_2 \subset \ldots$ such that $D_j \subset\subset \Omega$ is regular for each $j$ and $\Omega = \cup D_j$. This can be seen as follows: We first find domains $G_1 \subset\subset G_2 \subset\subset \ldots \subset\subset \Omega$ and then cover each $\overline{G}_j$ with a finite union $Q$ of open cubes and let $D_j = \operatorname{int} \overline{Q}$, see [9] and Corollary 6.32 in particular. In 1924 Wiener developed a nifty method to determine if a boundary point $w \in \partial\Omega$ is regular, known as the Wiener criterion. Before stating that we need to define the $p$-capacity of a set.

**Definition 2.5.** Let $K \subset\subset B(x, r)$ be a compact set and define

$$W(K, B(x, r)) = \{\varphi \in C_0^\infty(\Omega) : 0 \leq \varphi \leq 1 \text{ and } \varphi = 1 \text{ in } K\}.$$

We define the $p$-capacity of $K$ as

$$\operatorname{cap}_p(K, B(x, r)) = \inf_{W(K, B(x,r))} \int_{B(x,r)} |\nabla\varphi|^p \, dx.$$

We now formulate the Wiener criterion.

**Theorem 2.7.** *The set $\Omega$ is regular at $w \in \partial\Omega$ for the p-Dirichlet problem if and only if the following condition holds:*

$$\int_0^1 \left( \frac{\operatorname{cap}_p(\overline{B(w,t)} \cap (\mathbb{R}^n \setminus \Omega), B(w, 2t))}{\operatorname{cap}_p(\overline{B(w,t)}, B(w, 2r))} \right)^{1/(p-1)} \frac{dt}{t} = \infty.$$

We will return to the $p$-Dirichlet problem and regular points in Section 6, where we discuss Perron's method.

# 3 Regularity of weak solutions

In this section we will prove that the weak solutions to the $p$-Laplace equation are locally Hölder continuous. In order to do so one can use the fact that weak solutions to the $p$-Laplace equation satisfy the Harnack inequality, i.e., that the maximum of the function in a ball is bounded by a constant times the minimum in the same ball, where the constant only depends on $p$ and $n$. The theorems are stated in the beginning and proved for different values of $p$ throughout the section.

**Theorem 3.1.** *Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to the p-Laplace equation. Then there exists constants $\alpha > 0$ and $L$ such that $\alpha = \alpha(p, n)$ and $L = L(p, n, \|u\|_{L^p(B_{2r})})$ such that*

$$|u(x) - u(y)| \leq L|x - y|^\alpha$$

*for a.e. $x, y \in B(x_0, r)$ whenever $B(x_0, 2r) \subset\subset \Omega$.*

We next formulate Harnack's inequality from which the above theorem follows. This inequality will be very useful later in Section 7 when discussing the boundary behaviour of $p$-harmonic functions. In particular, we will then prove that the ratio of two $p$-harmonic functions that vanish on a portion of the boundary satisfies a Harnack inequality close to that part of the boundary.

**Theorem 3.2.** *Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution and that $u \geq 0$ in $B_{2r} \subset \Omega$. We define the essential minimum and essential supremum as follows:*

$$m(r) = \operatorname*{ess\,inf}_{B_r} u, \quad M(r) = \operatorname*{ess\,sup}_{B_r} u.$$

*Then there exists $c = c(n, p)$ such that*

$$M(r) \leq cm(r).$$

Later in this section the Harnack inequality will be proved for $n < p < \infty$ and for $1 < p < n$ (see [9] for a proof that holds for $1 < p < \infty$). We almost immediately obtain the strong maximum principle.

**Corollary 3.3.** *If a p-harmonic function attains its maximum at an interior point, then the function is constant.*

*Proof.* We suppose that $u(x_0) = \max_{x\in\Omega} u(x)$ for $x_0 \in \Omega$ and apply Harnack's inequality to the non-negative $p$-harmonic function $v(x) = u(x_0) - u(x)$. Thus, the minimum of $v(x)$ is $m(r) = 0$ and by applying the Harnack inequality again it follows that $M(r) = 0$ when $2|x - x_0| < \text{dist}(x_0, \partial\Omega)$. Therefore $v(x)$ is constant and it follows directly that $u(x)$ is also constant. In order to show that $u$ is constant in the whole domain we can either use a chain of intersecting balls or note that the set $\{x \in \Omega | u(x) = s\}$ is both open and closed in $\Omega$ and thus equals $\Omega$ since a domain is connected. □

We next show that the Harnack inequality implies Hölder continuity. This type of iterative argument is a standard technique and will be referred to several times throughout this thesis.

*Proof of Theorem 3.1.* We begin by choosing $r$ sufficiently small such that $B_{2r} \subset \Omega$ and applying Harnack's inequality to the non-negative weak solutions $u(x) - m(2r)$ and $M(2r) - u(x)$ to obtain

$$\begin{aligned} M(r) - m(2r) &\leq c(m(r) - m(2r)) \\ M(2r) - m(r) &\leq c(M(2r) - M(r)). \end{aligned}$$

After adding them we see that

$$\begin{aligned} &M(r) - m(2r) + M(2r) - m(r) \leq c(m(r) - m(2r) + M(2r) - M(r)) \\ \Leftrightarrow\ &M(r) - m(r) + M(2r) - m(2r) \leq -c(M(r) - m(r)) + c(M(2r) - m(2r)) \\ \Leftrightarrow\ &(M(r) - m(r))(1 + c) \leq (c - 1)(M(2r) - m(2r)) \end{aligned}$$

so

$$\omega(r) \le \Lambda \omega(2r), \quad \Lambda \equiv \frac{c-1}{c+1} \tag{8}$$

where $\omega(r) = M(r) - m(r)$ is the essential oscillation of $n$ over $B(x_0, r)$. From Harnack's inequality it follows that $c \ge 1$ but if $c = 1$ the function is constant and in that case there is nothing to prove. For the remaining part of the proof we thus assume that $c > 1$. We may assume that $c \ge 3$ so that $\kappa = -\log(\Lambda)/\log(2) \le 1$. By iterating (8) it follows that $\omega(2^{-(m-1)}r) \le \Lambda^{(m-1)}\omega(r)$. Choose $m \ge 1$ such that $2^{m-1} < R/r \le 2^m$. Then

$$\left(\frac{r}{R}\right)^{\kappa} \ge (2^{-m})^{\kappa} = 2^{-\kappa}(2^{m-1})^{-\kappa} = 2^{-\kappa}\Lambda^{m-1},$$

and thus

$$\omega(r) \le \Lambda^{(m-1)}\omega(2^{(m-1)}r) \le \Lambda^{(m-1)}\omega(2^{(m-1)}R) \le 2^{\kappa}\left(\frac{r}{R}\right)^{\kappa}\omega(R),$$

so we see that

$$\operatorname{osc}(u, B(r)) \le 2^{\kappa}\left(\frac{r}{R}\right)^{\kappa}\operatorname{osc}(u, B(R)), \quad 0 < r < R,$$

so $u \in C^{\alpha}_{\text{loc}}(\Omega)$. □

We will prove the Harnack inequality for the $1 < p < \infty, p \ne n$. For $p = n$ we will instead prove the Hölder continuity using Morrey's lemma. However, we note that Harnack's inequality also holds for the case where $p = n$ and refer to [9] for a proof.

### 3.0.1 The case $1 < p < n$

We will prove the Harnack inequality for a non-negative weak solution $u$. In order to do so we will use the interpretations

$$\operatorname*{ess\,sup}_{B_r} = \lim_{q\to\infty}\left(\int_{B_r} u^q \, dx\right)^{1/q}$$

$$\operatorname*{ess\,inf}_{B_r} = \lim_{q\to-\infty}\left(\int_{B_r} u^q \, dx\right)^{1/q}$$

Furthermore, we will use the notation

$$\|u\|_{q,s} = \left(\int_{B_s} u^q \, dx\right)^{\frac{1}{q}}$$

On more than one occasion in this section we will use one of the Sobolev inequalities, which states that for $p < n$ there exists a constant $c = c(n, p)$ such that for any $u \in W_0^{1,p}(\Omega)$ the following inequality holds:

$$\|u\|_{L^{\kappa p}(\Omega)} \le c\|\nabla u\|_{L^p(\Omega)} \tag{9}$$

where $\kappa = n(n-p)$. The proof of the Harnack inequality requires several lemmas and the first one concerns subsolutions.

**Lemma 3.4.** *Let $u \in W_0^{1,p}(\Omega)$ be a weak subsolution and $B_R \subset\subset \Omega$. Then for $\beta > p-1$ there exists $c = c(n, p, \beta)$*

$$\operatorname*{ess\,sup}_{B_r}(u_+) \le c\left(\frac{1}{(R-r)^n}\int_{B_R} u_+^{\beta} \, dx\right)^{1/\beta}$$

*where $r < R$ and $u_+ = \max\{u(x), 0\}$.*

*Proof.* We start by defining $v = u_+$. Next we set $\varphi = \zeta^p v^{\beta-(p-1)}$ for some non-negative $\zeta \in C_0^\infty(\Omega)$. Since $v \in W^{1,p}_{\text{loc}}(\Omega)$ we see that $\varphi \in W_0^{1,p}(\Omega)$. For convenience we set $\alpha = \beta - (p-1)$ which is positive by assumption. Furthermore, we see that

$$\nabla\varphi = p\zeta^{p-1}\nabla\zeta v^\alpha + \alpha v^{\alpha-1}\zeta^p\nabla v.$$

Since $u$ is a weak subsolution and $\nabla v = \nabla u_+ = \nabla u$ a.e on $\{u \geq 0\}$ it follows that

$$\alpha\int_\Omega \zeta^p v^{\alpha-1}|\nabla v|^p\,dx \leq -p\int_\Omega \zeta^{p-1}v^\alpha\langle|\nabla v|^{p-2}\nabla v, \nabla\zeta\rangle\,dx.$$

Note that

$$\alpha = \frac{(\alpha-1)(p-1)}{p} + \frac{\alpha+p-1}{p}. \tag{10}$$

By the Cauchy-Schwarz and Hölder inequalities we obtain

$$\begin{aligned}\alpha\int_\Omega \zeta^p v^{\alpha-1}|\nabla v|^p\,dx &\leq p\int_\Omega \zeta^{p-1}v^{(\alpha-1)(p-1)/p}|\nabla v|^{p-1}u^{\beta/p}|\nabla\zeta|\,dx\\ &\leq p\left(\int_\Omega \zeta^p v^{\alpha-1}|\nabla v|^p\,dx\right)^{1-\frac{1}{p}}\left(\int_\Omega |v^{\beta/p}\nabla\zeta|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

and therefore

$$\int_\Omega \zeta^p v^{\alpha-1}|\nabla v|^p\,dx \leq \left(\frac{p}{\alpha}\right)^p\int_\Omega v^\beta|\nabla\zeta|^p\,dx. \tag{11}$$

Next, we use that

$$|\nabla(v^{\beta/p})|^p = \left|\frac{\beta}{p}v^{\frac{\beta-p}{p}}\nabla v\right|^p = \left(\frac{\beta}{p}\right)^p|v^{\alpha-1}||\nabla v|^p \tag{12}$$

and combined with (11) we obtain

$$\int_\Omega |\zeta\nabla v^{\beta/p}|^p\,dx \leq \left(\frac{\beta}{\beta-(p-1)}\right)^p\int_\Omega |v^{\beta/p}\nabla\zeta|^p\,dx.$$

We note that

$$|\nabla(\zeta v^{\beta/p})| \leq |\zeta\nabla v^{\beta/p}| + |v^{\beta/p}\nabla\zeta|$$

and use this together with Minkowski's inequality as follows:

$$\begin{aligned}\left(\int_\Omega |\nabla(\zeta v^{\beta/p})|^p\,dx\right)^{\frac{1}{p}} &\leq \left(\int_\Omega \left(|\zeta\nabla v^{\beta/p}| + |v^{\beta/p}\nabla\zeta|\right)^p\right)^{\frac{1}{p}}\\ &\leq \left(\int_\Omega |\zeta\nabla v^{\beta/p}|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |v^{\beta/p}\nabla\zeta|^p\,dx\right)^{\frac{1}{p}}\\ &\leq \left(1+\frac{\beta}{\beta-p+1}\right)\left(\int_\Omega |v^{\beta/p}\nabla\zeta|^p\,dx\right)^{\frac{1}{p}},\end{aligned}$$

so

$$\int_\Omega |\nabla(\zeta v^{\beta/p})|^p\,dx \leq \left(\frac{2\beta-p+1}{\beta-p+1}\right)^p\int_\Omega |v^{\beta/p}\nabla\zeta|^p\,dx.$$

Since $\zeta v^{\beta/p} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ has compact support we can apply the Sobolev inequality which yields:

$$\left(\int_\Omega |\zeta v^{\beta/p}|^{\kappa p}\, dx\right)^{\frac{1}{\kappa}} \leq c^p \int_\Omega |\nabla(\zeta v^{\beta/p})|^p\, dx,$$

where $c = c(n,p)$ and $\kappa = n/(n-p)$. We choose a test function $\zeta \in C_0^\infty(\Omega)$ such that $0 \leq \zeta \leq 1$, $|\nabla\zeta| \leq 1/(R-r)$, $\zeta = 1$ in $B_r$ and 0 outside $B_R$ so that

$$\begin{aligned}\left(\int_{B_r} |v|^{\kappa\beta}\, dx\right)^{1/\kappa\beta} &\leq \left(\int_{B_R} |\zeta v^{\beta/p}|^{\kappa p}\, dx\right)^{1/\beta} \\ &\leq \left(\left(c\frac{2\beta-p+1}{\beta-p+1}\frac{1}{R-r}\right)^p \int_{B_R} v^\beta\, dx\right)^{\frac{1}{\beta}}.\end{aligned}$$

We fix $\beta_0 > p-1$ and note that

$$\frac{2\beta-p+1}{\beta-p+1} \leq \frac{2\beta_0-p+1}{\beta_0-p+1} = b,$$

when $\beta \geq \beta_0$ since

$$\frac{2\beta-p+1}{\beta-p+1} = 2 + \frac{p-1}{\beta-(p-1)}.$$

We are now going to iterate the estimate and we start with the radii $r_0 = R$ and $r_1 = r + (R-r)/2$. Since $r_0 - r_1 = (R-r)/2$ the corresponding test function that will be used has $|\nabla\zeta| \leq 2/(R-r)$ and it follows that

$$\|v\|_{\kappa\beta_0, r_1} \leq (cb)^{p\ \beta_0}\left(\frac{2}{R-r}\right)^{p/\kappa\beta_0} \|v\|_{\beta_0, r_0}.$$

Then we use $r_1$ and $r_2 = r + 2^{-2}(R-2)$ which results in

$$\|v\|_{\kappa^2\beta_0, r_2} \leq (cb)^{\frac{p}{\beta_0}+\frac{p}{\kappa\beta_0}} \frac{2^{\frac{p}{\beta_0}+\frac{2p}{\kappa\beta_0}}}{(R-r)^{\frac{p}{\beta_0}+\frac{\beta}{\kappa\beta_0}}} \|v\|_{\beta_0, r_0}.$$

Proceeding in this manner, i.e., using radii $r_j = r + 2^{-j}(R-r)$ yields

$$\|v\|_{\kappa^{j+1}\beta_0, r_{j+1}} \leq \left(\frac{cb}{R-r}\right)^{\frac{p}{\beta_0}\sum \kappa^{-l}} 2^{\frac{p}{\beta_0}\sum l\kappa^{-l+1}} \|v\|_{\beta_0, r_0}.$$

Since $|\kappa^{-1}| < 1$, the two sums in the exponents are convergent and

$$\sum_{l=0}^{j} \kappa^{-l} = \frac{1-\kappa^{-(j+1)}}{1-\kappa^{-1}} \longrightarrow \frac{n}{p} \quad \text{as } j \to \infty,$$

$$\sum_{l=1}^{j+1} l\kappa^{-l+1} \longrightarrow \frac{1}{(1-\kappa^{-1})^2} = \frac{n^2}{p^2} \quad \text{as } j \to \infty.$$

Finally, since $\|v\|_{\kappa^{j+1}\beta,r} \leq \|v\|_{\kappa^{j+1}\beta_0,R}$ we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{B_r}(v) &= \lim_{j\to\infty} \|v\|_{\kappa^{j+1}\beta_0,r} \\
&\leq \lim_{j\to\infty} \|v\|_{\kappa^{j+1}\beta_0,R} \\
&= \left(\frac{cb}{R-r}\right)^{\frac{n}{\beta_0}} 2^{\frac{n^2}{\beta_0 p}} \left(\int_{B_R} |v|^{\beta_0}\, dx\right)^{1/\beta_0} \\
&= c\left(\frac{1}{(R-r)^n}\int_{B_R} |v|^{\beta_0}\, dx\right)^{\frac{1}{\beta_0}}.
\end{aligned}$$

□

**Lemma 3.5.** *Suppose that $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a non-negative supersolution. Then for $\beta < 0$ and $B_R \subset\subset \Omega$ there exists a constant $c = c(n,p,\beta) = c(n,p)^{-1/\beta}$ such that*

$$\left(\frac{1}{(R-r)^n}\int_{B_R} v^\beta\, dx\right)^{1/\beta} \leq c \operatorname*{ess\,inf}_{B_r} v.$$

*Proof.* We note that we can assume that $v(x) \geq \varepsilon > 0$ since otherwise we prove inequality for $v(x)+\varepsilon$ instead of $v(x)$ and let $\varepsilon$ tend to zero. In this proof we use the function $\varphi = \zeta^p v^{\beta-(p-1)}$ and set $\alpha = \beta - (p-1)$ and note that $\alpha$ is negative. Since $v$ is a supersolution we obtain:

$$\int_\Omega \zeta^p \alpha v^{\alpha-1}|\nabla v|^p\, dx \geq -p\int_\Omega \zeta^{p-1} v^\alpha \langle |\nabla v|^{p-2}\nabla v, \nabla\zeta\rangle\, dx$$

and we use (10) and the Hölder inequality to obtain

$$\begin{aligned}
\int_\Omega (-\alpha)\zeta^p v^{\alpha-1}|\nabla v|^p\, dx &\leq p\int_\Omega \zeta^{p-1} v^\alpha \langle |\nabla v|^{p-2}\nabla v, \nabla\zeta\rangle\, dx \\
&\leq p\left(\int_\Omega \zeta^p v^{\alpha-1}|\nabla v|^p\, dx\right)^{1-\frac{1}{p}} \left(\int_\Omega v^\beta |\nabla\zeta|^p\, dx\right)^{1/p}
\end{aligned}$$

so

$$\int_\Omega |\zeta\nabla u^{\beta/p}|^p\, dx \leq \left(\frac{p}{-\alpha}\right)^p \int_\Omega v^\beta|\nabla\zeta|^p\, dx.$$

Using (12) we obtain

$$\int_\Omega |\zeta\nabla v^{\beta/p}|^p\, dx \leq \left(\frac{|\beta|}{-\alpha}\right)^p \int_\Omega |v^{\beta/p}\nabla\zeta|^p\, dx. \tag{13}$$

If we now repeat the same procedure using Minkowski's inequality, as in Lemma 3.4, and note that $|\beta| = -\beta$ we deduce that

$$\begin{aligned}
\int_\Omega |\nabla(\zeta v^{\beta/p})|^p\, dx &\leq \left(1 - \frac{\beta}{-\alpha}\right)^p \int_\Omega v^\beta|\nabla\zeta|^p\, dx \\
&= \left(\frac{p-1-2\beta}{p-1-\beta}\right)^p \int_\Omega v^\beta |\nabla\zeta|^p\, dx.
\end{aligned}$$

If we use (9) for $\zeta v^{\beta/p}$ we see that

$$\begin{aligned}\left(\int_\Omega \zeta^{\kappa p} v^{\kappa\beta}\right)^{1/\kappa} &\le c^p \int_\Omega |\nabla(\zeta u^{\beta/p})|^p\, dx \\ &\le (2c)^p \int_\Omega v^\beta |\nabla\zeta|^p\, dx\end{aligned}$$

since

$$\frac{p-1-2\beta}{p-1-\beta} = \frac{p-1-\beta}{p-1-\beta} - \frac{\beta}{p-1-\beta} \le 2$$

If we choose $\zeta$ as in Lemma 3.4 we see that

$$\left(\int_{B_r} v^{\kappa\beta}\, dx\right)^{1/\kappa} \le \left(\frac{2c}{R-r}\right)^p \int_{B_R} v^\beta\, dx,$$

and if we continue to use the same technique as in Lemma 3.4 it follows that

$$\begin{aligned}\left(\int_{B_{r_2}} (v^{\kappa\beta})^\kappa\, dx\right)^{1/\kappa^2} &\le \left(\frac{2c}{R-r}\right)^{p/\kappa} (2^2)^{p/\kappa} \left(\int_{B_{r_1}} v^{\beta\kappa}\, dx\right)^{1/\kappa} \\ &\le \left(\frac{2S}{R-r}\right)^{p(1+\frac{1}{\kappa})} 2^{p(1+\frac{1}{\kappa})} \int_{B_{r_0}} v^\beta\, dx\end{aligned}$$

and eventually we arrive at the estimate

$$\left(\int_{B_{r_j}} (v^\beta)^{k_j}\, dx\right)^{1/k^j} \le \left(\frac{2c}{R-r}\right)^{p\sum \kappa^{-l}} 2^{p\sum l\kappa^{1-l}} \int_{B_R} v^\beta\, dx.$$

As we let $j \to \infty$ we have that

$$\begin{aligned}\operatorname*{ess\,sup}_{B_r} v^\beta &\le \left(\frac{2c}{R-r}\right)^n 2^{\frac{n^2}{p}} \int_{B_R} v^\beta\, dx \\ &\le c\left(\frac{1}{R-r}\right)^n \int_{B_R} v^\beta\, dx,\end{aligned}$$

and therefore

$$\frac{1}{\operatorname{ess\,inf}_{B_r} v} \le \left(c\left(\frac{1}{R-r}\right)^n \int_{B_R} v^\beta\, dx\right)^{1/|\beta|},$$

and hence we obtain

$$\operatorname*{ess\,inf}_{B_r} v \ge \left(\left(\frac{1}{R-r}\right)^n \int_{B_R} v^\beta\, dx\right)^{-1/|\beta|}.$$

□

**Lemma 3.6.** *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a non-negative weak supersolution. For $0 < \varepsilon < \beta < \kappa(p-1) = n(p-1)/(n-p)$ and $B_R \subset\subset \Omega$ there exists $c = c(\varepsilon, \beta, n, p)$ such that*

$$\left(\frac{1}{(R-r)^n} \int_{B_r} v^\beta\, dx\right)^{1/\beta} \le c\left(\frac{1}{(R-r)^n} \int_{B_R} v^\varepsilon\, dx\right)^{1/\varepsilon}.$$

*Proof.* We start by assuming that $1 < \beta < p-1$. By following the proof of Lemma 3.5 but using that $|\beta| = \beta$ in (13) we arrive at the estimate

$$\left(\int_{B_r} v^{\kappa\beta}\right)^{1/\kappa\beta} \leq \left(c\left(\frac{p-1}{p-1-\beta}\right)\right)^{p/\beta} \frac{1}{(R-r)^{p/\beta}} \left(\int_{B_R} v^{\beta}\, dx\right)^{1/\beta}$$

By iterating the estimate as in the previous lemmas we obtain

$$\|v\|_{\kappa^{j+1}\beta, r_{j+1}} \leq \left(\frac{cb}{R-r}\right)^{\frac{p}{\beta}\sum \kappa^{-l}} 2^{\frac{p}{\beta}\sum l\kappa^{-l+1}} \|v\|_{\beta, r_0}$$

It is clear that we can iterate the result an appropriate number of steps to obtain an exponent $\kappa^j\beta > p-1$. By eventually using Hölder's inequality to obtain the right exponent and multiply with a suitable constant the conclusion of the lemma follows. □

If we combine the the lemmas above we obtain the following bounds for non-negative weak supersolutions:

$$\operatorname*{ess\,sup}_{B_r} u \leq c(\beta, n, p)\left(\frac{1}{(R-r)^n}\int_{B_r} u^{\beta}\, dx\right)^{1/\beta}$$

$$\operatorname*{ess\,inf}_{B_r} u \geq c(\beta, n, p)\left(\frac{1}{(R-r)^n}\int_{B_r} u^{-\beta}\, dx\right)^{-1/\beta}$$

where $\beta > 0$. In order to prove the Harnack inequality it suffices to show that

$$\left(\int_{B_r} u^{\beta}\, dx\right)^{1/\beta} \leq c\left(\int_{B_r} u^{-\beta}\, dx\right)^{-1/\beta}. \tag{14}$$

for some value of $\beta$. In order to do so we will use the John-Nirenberg theorem.

**Theorem 3.7.** *Let $w \in L^1_{\mathrm{loc}}(\Omega)$. Suppose that there is a constant $K$ such that*

$$\fint_{B_r} |w(x) - w_{B_r}|\, dx \leq K \tag{15}$$

*holds whenever $B_{2r} \subset \Omega$. Then there exists a constant $\nu = \nu(n) > 0$ such that*

$$\fint_{B_r} e^{\nu|w(x)-w_{B-r}|/K}\, dx \leq 2$$

*whenever $B_{2r} \subset \Omega$.*

For a proof of the theorem, see e.g. [7]. We start proving (14) by assuming that $u$ is a positive weak solution and that $B_{2r} \subset\subset \Omega$. Since $u > 0$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ it follows that $w = \log(u) \in W^{1,p}_{\mathrm{loc}}(\Omega)$. We next use the Poincaré inequality for a ball to obtain

$$\int_{B_r} |w - (w)_{B_r}|^p\, dx \leq cr^p \int_{B_r} |\nabla w|^p\, dx, \tag{16}$$

where $c = c(p, n)$ and if we combine this with Lemma 2.4 and a suitable test function, e.g., a radial test function $\zeta \in C_0^{\infty}(\Omega)$ s.t. $0 \leq \zeta \leq 1, \zeta = 1$ in $B_r$, $|\nabla\zeta| \leq r^{-1}$ and $\zeta = 0$ in $\Omega \setminus B_{2r}$, we see that

$$\int_{B_r} |\nabla w|^p\, dx \leq c(p)\int_{B_{2r}} |\nabla\zeta|^p\, dx \leq cr^{(n-p)}. \tag{17}$$

Combining (16) and (17) yields

$$\fint_{B_r} |w - (w)_{B_r}|^p \, dx \leq K$$

and by applying the Hölder inequality and using that $w$ is locally summable, we see that condition (15) is satisfied. Thus, we may apply the John-Nirenberg theorem from which it follows that

$$\fint_{B_r} e^{\pm\nu(w-(w)_{B_r})/K} \, dx \leq 2$$

and thus,

$$\begin{aligned} 4 &\geq \fint_{B_r} e^{\nu(w-(w)_{B_r})/K} \, dx \fint_{B_r} e^{-u(w-(w)_{B_r})/K} \, dx \\ &= \fint_{B_r} e^{\nu w/K} \, dx \fint_{B_r} e^{-\nu w/K} \, dx \\ &= \fint_{B_r} u^{\nu/K} \, dx \fint_{B_r} u^{-\nu/K} \, dx. \end{aligned}$$

We next set $\beta = \nu/K$ to obtain

$$\left( \fint_{B_r} u^{\beta} \, dx \right)^{1/\beta} \leq c \left( \fint_{B_r} u^{-\beta} \, dx \right)^{-1/\beta} \tag{18}$$

which proves Theorem 3.2 for $1 < p < n$ and $u > 0$. In order to show that the theorem holds for non-negative $u$ we note that if (18) holds for $u + \varepsilon$ it will continue to hold when we let $\varepsilon$ tend to 0.

### 3.0.2 The case $p = n$

For this case we will not prove Harnack's inequality but instead that $u$ satisfies the conditions for Morrey's lemma which in turn will imply the Hölder continuity.

**Lemma 3.8.** *Assume that $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Suppose that*

$$\int_{B_r} |\nabla u|^p \, dx \leq K r^{n-p+p\alpha}$$

*whenever $B_{2r} \subset \Omega$. Here $0 < \alpha \leq 1$ and $K$ are independent of the ball $B_r$. Then $u \in C^{\alpha}_{\text{loc}}(\Omega)$. In fact,*

$$\operatorname*{osc}_{B_r}(u) \leq \frac{4}{\alpha} \left( \frac{K}{\omega_n} \right)^{\frac{1}{p}} r^{\alpha}, \quad B_{2r} \subset \Omega.$$

For a proof see [11]. Next we show that a weak solution $u$ satisfies these conditions when $p = n$, using a method known as the hole filling technique. We select $B_{2r} \subset\subset \Omega$ and $\varphi = \zeta^n(u - a)$ for an arbitrary constant $a \in \mathbb{R}$. Thus, we have

$$\nabla\varphi = n\zeta^{n-1}(u - a)\nabla\zeta + \zeta^n \nabla u.$$

Since $u$ is a weak solution and $\varphi \in W_0^{1,p}(\Omega)$ with compact support we obtain

$$\begin{aligned}\int_\Omega \zeta^n |\nabla u|^n \, dx &= -n \int_\Omega \zeta^{n-1}(u-a)\langle |\nabla u|^{n-2}\nabla u, \nabla\zeta\rangle \, dx \\ &\le n \int_\Omega |\zeta \nabla u|^{n-1} |(u-a)\nabla\zeta| \, dx \\ &\le n \left(\int_\Omega \zeta^n |\nabla u|^n \, dx\right)^{1-1/n} \left(\int_\Omega |u-a|^n |\nabla\zeta|^n \, dx\right)^{1/n}\end{aligned}$$

so

$$\int_\Omega \zeta^n |\nabla u|^n \, dx \le n^n \left(\int_\Omega |u-a|^n |\nabla\zeta|^n \, dx\right).$$

We intend to use the above calculations in combination with a test function $\zeta$ such that $0 \le \zeta \le 1$, $\zeta = 1$ in $B_r$, $\zeta = 0$ outside of $B_{2r}$ and $|\nabla\zeta| < r^{-1}$. Since $\nabla\zeta = 0$ on $B_r$ it follows that

$$\begin{aligned}\int_{B_r} |\nabla u|^n \, dx &\le n^n \left(\int_{B_{2r}} |u-a|^n |\nabla\zeta|^n \, dx\right) \\ &\le n^n \frac{1}{r^n} \int_{H(r)} |u-a|^n \, dx\end{aligned}$$

where $H(r) = B_{2r} \setminus B_r$. We choose the constant $a$ as

$$a = \frac{1}{|H(r)|} \int_{H(r)} u(x) \, dx.$$

Using the Poincaré inequality

$$\int_{H(r)} |u-a|^n \, dx \le cr^n \int_{H(r)} |\nabla u|^n \, dx,$$

it follows that

$$\int_{B_r} |\nabla u|^n \, dx \le cn^n \int_{H(r)} |\nabla u|^n \, dx.$$

We next add $cn^n \int_{B_r} |\nabla u|^n \, dx$ to both sides in the above equation in order to fill in the hole in the annulus which results in

$$(1 + Cn^n) \int_{B_r} |\nabla u|^n \, dx \le cn^n \int_{B_{2r}} |\nabla u|^n \, dx.$$

Thus, for the $p$-energy

$$D(r) = \int_{B_r} |\nabla u|^n \, dxj$$

it follows that

$$D(r) \le \Lambda D(2r), \quad \Lambda = \frac{cn^n}{1 + cn^n} < 1.$$

By iterating the estimate above we see that $D(2^{-(m-1)}) \le \Lambda^{(m-1)} D(r)$, with $m \ge 1$. By using the same technique as in the proof of Theorem 3.1 it follows that

$$D(r) \le 2^\kappa \left(\frac{r}{R}\right)^\kappa D(R), \quad 0 < r < R,$$

with $\kappa = -\log(\Lambda)/\log(2)$ when $B_{2r} \subset \Omega$ and thus we can invoke Morrey's lemma.

### 3.0.3 The case n<p<∞

When $p > n$ the local Hölder continuity is well-known. If $u \in W^{1,p}(B_r)$ where $B_r \subset\subset \Omega$ it follows that $u$ has a version $u^* \in C^{0,\alpha}(\bar{B}_r)$ for $\alpha = 1 - \frac{n}{p}$ (see e.g. Theorem 5.6.5 in [1]). Nevertheless we show the Harnack inequality for a positive weak solution $u$. At first we choose $r$ such that $B_{2r} \subset \Omega$ and apply Theorem 7.17 in [6] which states that for $x, y \in B_r$ we have the following bound on the oscillation:

$$|u(x) - u(y)| \leq cr^{1-n/p} \|\nabla u\|_{L^p(B_r)}.$$

where $c = c(p, n)$. Next we want to apply Lemma 2.4 and therefore we choose $\zeta \in W_0^{1,p}(\Omega)$ to be a radial function satisfying $\zeta = 1$ in $B_r$, $|\nabla \zeta| \leq 1/r^{(n-2p)/p}$ and $\zeta = 0$ outside $B_{2r}$ which yields $\|\nabla \log u\|_{L^p(B_r)} \leq \widetilde{c} r^{\frac{n-p}{p}}$. We define $v = \log u$ and conclude that

$$\left|\log \frac{u(y)}{u(x)}\right| \leq c\widetilde{c}$$

and therefore

$$-c\widetilde{c} \leq \log \frac{u(y)}{u(x)} \leq c\widetilde{c}$$

so

$$e^{-c\widetilde{c}} u(x) \leq u(y) \leq e^{c\widetilde{c}} u(x)$$

which proves that $u$ satisfies the Harnack inequality.

# 4 Differentiability

In this section we continue to explore the regularity of the solutions to the $p$-Laplace equation but we now consider the gradients. For $1 < p \leq 2$ we prove that $u$ has second Sobolev derivatives, i.e., $u \in W^{2,p}_{\text{loc}}(\Omega)$, and for $p \geq 2$ we prove that $|\nabla u|^{(p-2)/2} \nabla u \in W^{1,2}_{\text{loc}}(\Omega)$. However, a much stronger result holds, namely that the gradients are locally Hölder continuous for $1 < p < \infty$. This fact is stated, but not proved, in Lemma 4.4 and it will be used later when we consider the boundary behaviour of $p$-harmonic functions in Section 7. Apart from Lemma 4.4 this section is completely independent from the rest of the thesis. The proofs in this section are based on integrated difference quotients. We define

$$DF = \left|\frac{F(x+h) - F(x)}{h}\right|$$

where

$$F(x) = |\nabla u(x)|^{\frac{p-2}{2}} \nabla u(x).$$

We are now ready to state the first theorem.

**Theorem 4.1.** *Let $p \geq 2$. If $u$ is $p$-harmonic in $\Omega$, then $F \in W^{1,2}_{\text{loc}}(\Omega)$. For each subdomain $G \subset\subset \Omega$,*

$$||DF||_{L^2(G)} \leq \frac{c||F||_{L^2(\Omega)}}{\text{dist}(G, \partial\Omega)} \tag{19}$$

*where $c = c(p, n)$.*

*Proof.* We let $\zeta \in C_0^\infty(\Omega)$ be a cutoff function, satisfying $0 \le \zeta \le 1$, $\zeta \equiv 1$ on $G$ and $|\nabla\zeta| \le \frac{c(n)}{\mathrm{dist}(G,\partial\Omega)}$ (replacing $\Omega$ by some $\Omega_1$ such that $G \subset\subset \Omega_1 \subset\subset \Omega$ if necessary). Let $h$ be some constant vector such that $|h| < \mathrm{dist}(\mathrm{supp}\,\zeta, \partial\Omega)$. Furthermore, $u_h(x) = u(x+h)$ is $p$-harmonic in $\{x : x+h \in \Omega\}$. We denote by $\varphi$ the function

$$\varphi(x) = \zeta(x)^2(u(x+h) - u(x)).$$

Then,

$$\nabla\varphi(x) = \zeta(x)^2(\nabla u(x+h) - \nabla u(x)) + 2\zeta(x)\nabla\zeta(x)(u(x+h) - u(x)),$$

and since $u(x+h)$ and $u(x)$ satisfies (5),

$$\int_\Omega \left\langle |\nabla u(x+h)|^{p-2}\nabla u(x+h) - |\nabla u(x)|^{p-2}\nabla u(x), \nabla\varphi(x) \right\rangle \, dx = 0,$$

so,

$$\begin{aligned}
&\int_\Omega \zeta^2 \left\langle |\nabla u(x+h)|^{p-2}\nabla u(x+h) - |\nabla u(x)|^{p-2}\nabla u(x), \nabla u(x+h) - \nabla u(x) \right\rangle \, dx \\
&= -2\int_\Omega \zeta(x)(u(x+h) - u(x)) \left\langle |\nabla u(x+h)|^{p-2}\nabla u(x+h) - |\nabla u(x)|^{p-2}\nabla u(x), \nabla\zeta \right\rangle \, dx \\
&\le 2\int_\Omega \zeta(x)|u(x+h) - u(x)| \left| |\nabla u(x+h)|^{p-2}\nabla u(x+h) - |\nabla u(x)|^{p-2}\nabla u(x) \right| |\nabla\zeta| \, dx
\end{aligned}$$

Employing the inequalities *(iv)* and *(v)* of Lemma B.1,

$$\begin{aligned}
\frac{4}{p^2}\left| |b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a \right|^2 &\le \left\langle |b|^{p-2}b - |a|^{p-2}a, b-a \right\rangle, \\
\left| |b|^{p-2}b - |a|^{p-2}a \right| &\le (p-1)\left( |a|^{\frac{p-2}{2}} + |b|^{\frac{p-2}{2}} \right)\left| |b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a \right|,
\end{aligned}$$

we get (recall that $F(x) = |\nabla u(x)|^{\frac{p-2}{2}}\nabla u(x)$)

$$\begin{aligned}
&\frac{4}{p^2}\int_\Omega \zeta^2 |F(x+h) - F(x)|^2 \, dx \\
&\le \int_\Omega \zeta^2 \langle |\nabla u(x+h)|^{p-2}\nabla u(x+h) - |\nabla u(x)|^{p-2}\nabla u(x), \nabla u(x+h) - \nabla u(x) \rangle \, dx \\
&\le 2\int_\Omega \zeta|u(x+h) - u(x)| \left| |\nabla u(x+h)|^{p-2}\nabla u(x+h) - |\nabla u(x)|^{p-2}\nabla u(x) \right| |\nabla\zeta| \, dx \\
&\le 2(p-1)\int_\Omega \zeta|u(x+h) - u(x)| \left( |\nabla u(x+h)|^{\frac{p-2}{2}} + |u(x)|^{\frac{p-2}{2}} \right) |F(x+h) - F(x)||\nabla\zeta| \, dx \\
&\le 2(p-1)\left\{ \int_\Omega |u(x+h) - u(x)|^p |\nabla\zeta|^p \, dx \right\}^{\frac{1}{p}} \left\{ \int_\Omega \zeta^2 |F(x+h) - F(x)|^2 \, dx \right\}^{\frac{1}{2}} \\
&\qquad \times \left\{ \int_{\mathrm{supp}\,\zeta} \left( |\nabla u(x+h)|^{\frac{p-2}{2}} + |\nabla u(x)|^{\frac{p-2}{2}} \right)^{\frac{2p}{p-2}} \, dx \right\}^{\frac{p-2}{2p}}, \qquad (20)
\end{aligned}$$

where we used *(iv)* in the first inequality, *(v)* in the third and the generalized Hölder inequality with constants $p$, 2 and $2p/(p-2)$ in the fourth. Note that the last integral is finite, since $\nabla u \in L^p_{\text{loc}}(\Omega)$. We bound it using Minkowski's inequality:

$$\begin{aligned}
&\left\{ \int_{\operatorname{supp}\zeta} \left( |\nabla u(x+h)|^{\frac{p-2}{2}} + |\nabla u(x)|^{\frac{p-2}{2}} \right)^{\frac{2p}{p-2}} dx \right\}^{\frac{p-2}{2p}} \\
\leq &\left\{ \int |\nabla u(x+h)|^p \, dx \right\}^{\frac{p-2}{2p}} + \left\{ \int |\nabla u(x)|^p \, dx \right\}^{\frac{p-2}{2p}} \\
&\leq \frac{5}{2} \left\{ \int_\Omega |\nabla u(x)|^p \, dx \right\}^{\frac{p-2}{2p}} = \frac{5}{2} \left\{ \int_\Omega |F|^2 \, dx \right\}^{\frac{p-2}{2p}} \qquad (21)
\end{aligned}$$

for $|h|$ small enough. Multiplying both sides of (20) by $\frac{1}{|h|} \left\{ \int_\Omega \zeta^2 |F(x+h) - F(x)|^2 \, dx \right\}^{-\frac{1}{2}}$, and using (21), we obtain

$$\begin{aligned}
&\frac{4}{p^2} \left\{ \int_\Omega \zeta^2 \left| \frac{F(x+h) - F(x)}{h} \right|^2 dx \right\}^{\frac{1}{2}} \\
&\leq 5(p-1) \left\{ \int_\Omega |F|^2 \, dx \right\}^{\frac{p-2}{2p}} \left\{ \int_\Omega \left| \frac{u(x+h) - u(x)}{h} \right|^p |\nabla \zeta|^p \, dx \right\}^{\frac{1}{p}}.
\end{aligned}$$

By Theorem 5.8.3 in [1],

$$\left\{ \int_\Omega \left| \frac{u(x+h) - u(x)}{h} \right|^p |\nabla \zeta|^p \, dx \right\}^{\frac{1}{p}} \leq \frac{c(n)}{\operatorname{dist}(G, \partial\Omega)} \left\{ \int_\Omega |\nabla u(x)|^p \, dx \right\}^{\frac{1}{p}},$$

and thus, since $|F(x)|^2 = |\nabla u(x)|^p$, (21) implies that

$$\begin{aligned}
\left\{ \int_G \left| \frac{F(x+h) - F(x)}{h} \right|^2 dx \right\}^{\frac{1}{2}} &\leq \frac{c(n,p)}{\operatorname{dist}(G, \partial\Omega)} \left\{ \int_\Omega |F|^2 \, dx \right\}^{\frac{p-2}{2p} + \frac{1}{p}} \\
&= \frac{c(n,p)}{\operatorname{dist}(G, \partial\Omega)} \left\{ \int_\Omega |F|^2 \, dx \right\}^{\frac{1}{2}}.
\end{aligned}$$

By using Theorem 5.8.3 in [1] again, it follows that that $F \in W^{1,2}(G)$. □

The following lemma will be used in the proof of the next theorem.
**Lemma 4.2.** *Let* $f \in L^1_{\text{loc}}(\Omega)$. *Then*

$$\int_\Omega \varphi(x) \frac{f(x + he_k) - f(x)}{h} \, dx = -\int_\Omega \frac{\partial \varphi}{\partial x_k} \left( \int_0^1 f(x + the_k) dt \right) dx$$

*holds for all* $\varphi \in C_0^\infty(\Omega)$.

*Proof.* For a smooth function $f$ we use the Leibniz rule to obtain

$$\begin{aligned}\frac{\partial}{\partial x_k}\int_0^1 f(x+the_k)dt &= \int_0^1 \frac{\partial}{\partial x_k} f(x+the_k)dt \\ &= \int_0^1 \left(\frac{1}{h}\frac{\partial}{\partial t} f(x+the_k)\right) dt \\ &= \frac{f(x+he_k)-f(x)}{h}\end{aligned}$$

and by partial integration and since $\varphi$ has compact support it follows that

$$\begin{aligned}\int_\Omega \varphi \frac{f(x+he_k)-f(x)}{h}\, dx &= \int_\Omega \varphi \frac{\partial}{\partial x_k}\left(\int_0^1 f(x+the_k)dt\right) dx \\ &= -\int_\Omega \frac{\partial \varphi}{\partial x_k}\left(\int_0^1 f(x+the_k)dt\right) dx\end{aligned}$$

Since smooth functions are dense in $L^1_{\text{loc}}(\Omega)$ the result follows by approximation. □

**Theorem 4.3.** *Let $1 < p \le 2$. If $u$ is p-harmonic in $\Omega$, then $u \in W^{2,p}_{\text{loc}}(\Omega)$. Moreover, there exists $c = C(D)$ such that*

$$\int_D \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^p dx \le c \int_\Omega |\nabla u|^p \, dx$$

*whenever $D \subset\subset \Omega$.*

*Proof.* We use the notation

$$\Delta^h f(x) = \frac{f(x+he_k)-f(x)}{h}.$$

From Lemma 4.2 we have the formula

$$\Delta^h(|\nabla u|^{p-2}\nabla u) = \frac{\partial}{\partial x_k}\int_0^1 |\nabla u(x+the_k)|^{p-2}\nabla u(x+the_k)dt$$

(in Sobolev sense). Using this (in the second equality) and the calculations from Theorem 7.19 (in the first inequality)

$$\begin{aligned}&\int \zeta^2 \langle \Delta^h(|\nabla u|^{p-2}\nabla u), \Delta^h(\nabla u)\rangle\, dx = -2\int \zeta \Delta^h u \langle \Delta^h(|\nabla u|^{p-2}\nabla u), \nabla\zeta\rangle\, dx \\ &= -2\int \left\langle \frac{\partial}{\partial x_k}\left[\int_0^1 |\nabla u(x+the_k)|^{p-2}\nabla u(x+the_k)dt\right], (\Delta^h u)\zeta\nabla\zeta\right\rangle dx \\ &= 2\int \left\langle \int_0^1 |\nabla u(x+the_k)|^{p-2}\nabla u(x+the_k)dt, \frac{\partial}{\partial x_k}(\Delta^h u)\zeta\nabla\zeta\right\rangle dx, \qquad (22)\end{aligned}$$

where we used the equality

$$\frac{\partial}{\partial x}\langle f(x), g(x)\rangle = \left\langle \frac{\partial}{\partial x} f(x), g(x)\right\rangle + \left\langle f(x), \frac{\partial}{\partial x} g(x)\right\rangle,$$

and that $\zeta$ has compact support.

**Claim 1.**

$$\frac{\partial}{\partial x_k}(\Delta^h u)\zeta\nabla\zeta = \zeta\nabla\zeta\Delta^h u_{x_k} + \Delta^h u(\zeta_{x_k}\nabla\zeta + \zeta\nabla\zeta_{x_k})$$

*Proof of claim.*

$$\begin{aligned}
&\frac{\partial}{\partial x_k}\left(\frac{u(x+he_k)-u(x)}{h}\zeta\frac{\partial\zeta}{\partial x_1},\dots,\frac{u(x+he_k)-u(x)}{h}\zeta\frac{\partial\zeta}{\partial x_n}\right)\\
&=\left(\Delta^h u_{x_k}\zeta\frac{\partial\zeta}{\partial x_1}+\Delta^h u\frac{\partial}{\partial x_k}\left(\zeta\frac{\partial\zeta}{\partial x_1}\right),\dots,\Delta^h u_{x_k}\zeta\frac{\partial\zeta}{\partial x_n}+\Delta^h u\frac{\partial}{\partial x_k}\left(\zeta\frac{\partial\zeta}{\partial x_n}\right)\right)\\
&=\zeta\nabla\zeta\Delta^h u_{x_k}+\Delta^h u\left(\frac{\partial\zeta}{\partial x_k}\frac{\partial\zeta}{\partial x_1}+\zeta\frac{\partial^2\zeta}{\partial x_k\partial x_1},\dots,\frac{\partial\zeta}{\partial x_k}\frac{\partial\zeta}{\partial x_n}+\zeta\frac{\partial^2\zeta}{\partial x_k\partial x_n}\right)\\
&=\zeta\nabla\zeta\Delta^h u_{x_k}+\Delta^h u\left(\frac{\partial\zeta}{\partial x_k}\nabla\zeta+\zeta\nabla\frac{\partial\zeta}{\partial x_k}\right).
\end{aligned}$$

□

Thus, by the claim, (22) equals

$$2\int\left\langle\int_0^1|\nabla u(x+the_k|^{p-2}\nabla u(x+the_k)dt,\zeta\nabla\zeta\Delta^h u_{x_k}+(\Delta^h u)(\zeta_{x_k}\nabla\zeta+\zeta\nabla\zeta_{x_k})\right\rangle\,dx. \tag{23}$$

Next, we choose a ball $B_{3R}\subset\subset\Omega$ of radius $3R$ and a smooth $\zeta$ such that $\zeta=1$ in $B_R$, $0\le\zeta\le1$ in $B_{2R}$ and 0 outside of $B_{2R}$, and such that

$$|\nabla\zeta|\le R^{-1},\ |D^2\zeta|\le CR^{-2}$$

for some positive constant $c$. We write

$$Y(x)=\int_{B_{2R}}|\nabla u(x+the_k)|^{p-1}dt,$$

and hence, (23) is smaller than

$$\begin{aligned}
&2\int_\Omega Y\zeta|\nabla\zeta|\Delta^h u_{x_k}\,dx+2\int_\Omega Y|\Delta^h u|(|\zeta_{x_k}\nabla\zeta|+|\zeta\nabla\zeta_{x_k}|)\,dx\\
&\le\frac{2}{R}\int_\Omega\zeta Y|\Delta^h u_{x_k}|\,dx+\frac{c}{R^2}\int_{B_{2R}}|\Delta^h u|Y\,dx.
\end{aligned}\tag{24}$$

So to summarize we have that

$$\int\zeta^2\langle\Delta^h(|\nabla u|^{p-2}\nabla u),\Delta^h(\nabla u)\rangle\,dx\le\frac{2}{R}\int_\Omega\zeta Y|\Delta^h u_{x_k}|\,dx+\frac{C}{R^2}\int_{B_{2R}}|\Delta^h u|Y\,dx.$$

Letting $W(x)^2=1+|\nabla u(x)|^2+|\nabla u(x+he_k)|^2$ and using that $|\Delta^h(\nabla u)|\ge|\Delta^h(u_{x_k})|$, together with the inequality *(vi)* of Lemma B.1,

$$\langle|b|^{p-2}b-|a|^{p-2}a,b-a\rangle\ge(p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}},$$

(since $1 < p \le 2$) we have that

$$
\begin{aligned}
&(p-1)\int_\Omega \zeta^2 W^{p-2}|\Delta^h(\nabla u)|^2\,dx\\
&\le \int_\Omega \zeta^2\left\langle \frac{|\nabla u(x+he_k)|^{p-2}\nabla u(x+he_k)-|\nabla u(x)|^{p-2}\nabla u(x)}{h}, \frac{(\nabla u(x+he_k)-\nabla u(x))}{h}\right\rangle dx\\
&= \int_\Omega \zeta^2\langle \Delta^h(|\nabla u|^{p-2}\nabla u), \Delta^h(\nabla u)\rangle\,dx\\
&\le \frac{2}{R}\int_\Omega \zeta Y|\Delta^h(\nabla u)|\,dx + \frac{c}{R^2}\int_{B_{2R}}|\Delta^h u|Y\,dx.
\end{aligned}
$$

Using Cauchy's inequality with $\varepsilon$, i.e., that for $a, b > 0$, $\varepsilon > 0$

$$
ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon},
$$

with $a = \zeta W^{\frac{p-2}{2}}|\Delta^h(\nabla u)|$ and $b = 2W^{\frac{2-p}{2}}YR^{-1}$, we get that

$$
\begin{aligned}
&2R^{-1}\zeta Y|\Delta^h(\nabla u)| = (\zeta W^{\frac{p-2}{2}}|\Delta^h(\nabla u)|)(2W^{\frac{2-p}{2}}YR^{-1})\\
&\le \varepsilon\zeta^2 W^{p-2}|\Delta^h(\nabla u)|^2 + \varepsilon^{-1}R^{-2}W^{2-p}Y^2
\end{aligned}
$$

Choosing $\varepsilon = (p-1)/2$, we get (note that the integral inequality is still valid with $\Omega$ replaced by $B_R$)

$$
\begin{aligned}
&(p-1)\int_{B_R}\zeta^2 W^{p-2}|\Delta^h(\nabla u)|^2\,dx\\
&\le \frac{p-1}{2}\int_{B_R}\zeta^2 W^{p-2}|\Delta^h(\nabla u)|^2\,dx + \frac{2}{p-1}\int_{B_{2R}}R^{-2}W^{2-p}Y^2\,dx\\
&+\frac{c}{R^2}\int_{B_{2R}}|\Delta^h u|Y\,dx,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{p-1}{2}\int_{B_R}W^{p-2}|\Delta^h(\nabla u)|^2\,dx\\
&\le \frac{2}{p-1}\int_{B_{2R}}R^{-2}W^{2-p}Y^2\,dx + \frac{c}{R^2}\int_{B_{2R}}|\Delta^h u|Y\,dx, \qquad (25)
\end{aligned}
$$

since $\zeta = 1$ on $B_R$. Next, note that

$$
|\Delta^h(\nabla u)|^p \le W^{p-2}|\Delta^h(\nabla u)|^2 + W^p, \qquad (26)
$$

$$
W^{2-p}Y^2 \le W^p + Y^{\frac{p}{p-1}}, \qquad (27)
$$

$$
|\Delta^h u|Y \le |\Delta^h u|^p + Y^{\frac{p}{p-1}}. \qquad (28)
$$

Using first (26), then (25) and lastly (27) and (28), we get

$$\int_{B_R} |\Delta^h(\nabla u)|^p \, dx \le \int_{B_R} W^{p-2} |\Delta^h(\nabla u)|^2 \, dx + \int_{B_{2R}} W^p \, dx$$
$$\le c \int_{B_{2R}} W^{2-p} Y^2 \, dx + \frac{c}{R^2} \int_{B_{2R}} |\Delta^h u| Y \, dx + \int_{B_{2R}} W^p \, dx$$
$$\le c \int_{B_{2R}} W^p \, dx + c \int_{B_{2R}} Y^{\frac{p}{p-1}} \, dx + c \int_{B_{2R}} |\Delta^h u|^p \, dx,$$

where $c = c(p, R)$, and hence we need to bound each of these integrals as $h \to 0$. The first one is immediate:

$$\int_{B_{2R}} W^p \, dx = \int_{B_{2R}} (1 + |\nabla u(x)|^2 + |\nabla u(x + he_k)|^2)^{p/2} \, dx$$
$$\le c \int_{B_{3R}} dx + c \int_{B_{3R}} |\nabla u(x)|^p \, dx.$$

For the second integral,

$$\int_{B_{2R}} Y^{\frac{p}{p-1}} \, dx = \int_{B_{2R}} \left[ \left( \int_0^1 |\nabla u(x + the_k)|^{p-1} dt \right)^{\frac{1}{p-1}} \right]^p \, dx$$
$$\le \int_{B_{2R}} \left[ \left( \int_0^1 |\nabla u(x + the_k)|^p dt \right)^{\frac{p-1}{p}} \left( \int_0^1 1^p dt \right)^p \right]^{\frac{p}{p-1}} \, dx$$
$$= \int_{B_{2R}} \int_0^1 |\nabla u(x + the_k)|^p dt \, dx \le \int_{B_{3R}} |\nabla u|^p \, dx$$

for small enough $h$. The integral bound

$$\int_{B_{2R}} |\Delta^h u|^p \, dx \le c \int_{B_{3R}} |\nabla u|^p \, dx$$

is obtained by applying results from Section 5.8 of [1]. Hence,

$$\int_{B_R} |\Delta^h(\nabla u)|^p \, dx \le c \int_{B_{3R}} |\nabla u|^p \, dx,$$

where $c = c(p, n, R)$ and the result follows since $\nabla u \in L^p_{\text{loc}}(\Omega)$ and the fact that any compact set can be covered by a finite number of balls. □

We finish this section by stating the following lemma, the proof of which can be found in [12].

**Lemma 4.4.** *Let $1 < p < \infty$, let $u$ be $p$-harmonic in $B(z, 4r)$. Then $u$ has a representative in $W^{1,p}(B(z, 4r))$ with Hölder continuous partial derivatives in $B(z, 4r)$. In particular, there exist constants $c = c(p, n) \in [1, \infty)$ and $\sigma = \sigma(p, n) \in (0, 1]$ such that if $x, y \in B(\hat{z}, \hat{r}/2)$, $B(\hat{z}, 4\hat{r}) \subset B(z, 4r)$, then*

$$c^{-1} |\nabla u(x) - \nabla u(y)| \le \left( \frac{|x - y|}{\hat{r}} \right)^\sigma \max_{B(\hat{z}, \hat{r})} |\nabla u|.$$

*Moreover, if $u$ is non-negative, then*

$$\max_{B(\hat{z}, \hat{r})} |\nabla u| \le c \hat{r}^{-1} \max_{B(\hat{z}, 2\hat{r})} u.$$

## 5 The $p$-superharmonic functions and their properties

In this section we will study the properties of $p$-superharmonic functions. They will be of great importance when constructing Perron's solution for the $p$-Dirichlet problem in Section 6. At first we establish some of their basic properties and later we will explain the connection between $p$-superharmonic functions and weak supersolutions to the $p$-Laplace equation. We start by recalling the definition of a lower semi-continuous function.

**Definition 5.1.** We say that $v : \Omega \to (-\infty, \infty]$ is lower semi-continuous (l.s.c) at $x \in \Omega$ if

$$\liminf_{y \to x} v(y) \geq v(x).$$

We say that $v$ is lower semi-continuous in $\Omega$ if $v$ is lower semi-continuous at every point in $\Omega$.

We note that it follows directly from the definition that a lower semi-continuous function $v$ attains its infimum on compact sets. Furthermore, following the procedure outlined in [9], one can show that $v$ can be approximated from below with smooth functions, which will be used in the proof of Theorem 5.6. We now turn to the definition of a $p$-superharmonic function.

**Definition 5.2.** We say that a function $v : \Omega \to (-\infty, \infty]$ is $p$-superharmonic in $\Omega$ if the following conditions are satisfied:

(i) $v$ is lower semi-continuous in $\Omega$,

(ii) $v \not\equiv \infty$ in $\Omega$

(iii) For each compactly contained domain, $D \subset\subset \Omega$, the comparison principle holds: if $h \in C(\bar{D})$ is $p$-harmonic in $D$ and $h|_{\partial D} \leq v|_{\partial D}$, then $h \leq v$ in $D$.

We say that a function $u : \Omega \to [-\infty, \infty)$ is $p$-subharmonic if $v = -u$ is $p$-superharmonic.

The following three results can be found Chapter 7 in [9].

**Lemma 5.1.** *Suppose that $v_i$ is $p$-superharmonic in $\Omega$ for $i \in \{1, 2\}$. Then the minimum,* $\min\{v_1, v_2\}$ *is $p$-superharmonic.*

This lemma is not hard to show and can be be naturally generalized, using induction, to hold for

$$\tilde{v} = \min\{v_1, v_2, \dots, v_n\},$$

where $v_i$ is $p$-superharmonic for each $i$.

**Lemma 5.2.** *Suppose that $\{v_i\}$ is an increasing sequence of $p$-superharmonic functions in $\Omega$. Then the limit function $v = \lim_{i\to\infty} v_i$ is either $p$-superharmonic or $v \equiv \infty$.*

The next theorem is the comparison principle for $p$-sub- and $p$-superharmonic functions.

**Theorem 5.3.** *Let $\Omega$ be a bounded domain. Furthermore, let $v_1$ and $v_2$ be $p$-superharmonic and $p$-subharmonic, respectively. If*

$$\limsup_{y \to x} v_2(y) \leq \liminf_{y \to x} v_1(y) \tag{29}$$

*for each $x \in \Omega$ and both sides of* (29) *are not $\infty$ or $-\infty$ at the same time, then $v_2 \leq v_1$ in $\Omega$.*

*Proof.* Take $\varepsilon > 0$ and let $x \in \Omega$. We recall the discussion about regular sets in the end of Section 2, and note that there exists a regular open set $D \subset\subset \Omega$ such that $x \in D$ and $v < u + \varepsilon$ on $\partial D$. Since $v$ is $p$-subharmonic there exists a decreasing sequence $\{\varphi_i\}$ of smooth functions in $\Omega$ which converges to $v$ on $\bar{D}$. Since $u + \varepsilon$ attains its infimum on compact domains there exists an $N$ such that $\varphi_j \leq u + \varepsilon$ on $\partial D$ for $j \geq N$. Next we let $h$ denote the the unique $p$-harmonic function in $D$ with $h|_{\partial D} = \varphi_N$. Thus, $v \leq h = \varphi_N \leq u + \varepsilon$ on $\partial D$ and it now follows from Theorem 2.5 that

$$v \leq h \leq u + \varepsilon \text{ in } D.$$

The conclusion of the lemma follows by letting $\varepsilon$ tend to zero. □

The next proposition states that the set where a $p$-superharmonic function is finite, $\{x \in \Omega : v(x) < \infty\}$, is dense in $\Omega$.

**Proposition 5.4.** *If $v$ is $p$-superharmonic in $\Omega$, then the set where $v = \infty$ does not contain any ball.*

*Proof.* At first we show the result for $v \geq 0$. Let $x_0 \in \Omega$ and assume that there exists $r < R$ such that $v \equiv \infty$ in $\overline{B(x_0, r)} \subset B(x_0, R) \subset\subset \Omega$. Next we consider the function

$$h(x) = \frac{\int_{|x-x_0|}^{R} t^{-\frac{n-1}{p-1}}\,dt}{\int_{r}^{R} t^{-\frac{n-1}{p-1}}\,dt}$$

which is $p$-harmonic for $x \neq x_0$. Thus $h(x)$ is $p$-harmonic in the annulus $A = \{x \in \Omega : r < |x - x_0| < R\}$ and $h|_{\partial B_R} = 0$ and $h|_{\partial B_r} = 1$. The $p$-harmonicity of $h$ and the fact that it is continuous on $\bar{A}$, $v|_{\partial B_R} \geq 0$ and $v|_{\partial B_r} = \infty$ allows us to use the comparison principle to conclude that

$$v(x) \geq kh(x), \quad k = 1, 2, 3, \dots$$

in $A$ which implies that $v|_A = \infty$. Thus, $v \equiv \infty$ in $B_R$ and we can use a covering argument with a chain of intersecting balls to conclude that $v \equiv \infty$ in $\Omega$ which contradicts the fact that $v$ is $p$-superharmonic. In order to prove the lemma for an arbitrary $p$-superharmonic function $v$ we can apply the argument above to $\tilde{v} = v - \inf_{B_R} v$. □

The next theorem connects the $p$-superharmonic functions to weak supersolutions of the $p$-Laplace equation, in the case where the function is continuous.

**Theorem 5.5.** *Suppose that $v \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$. Then the following conditions are equivalent:*

*(i)* $\int_D |\nabla v|^p\,dx \leq \int_D |\nabla(v + \varphi)|^p\,dx$ *whenever* $\varphi \in C_0^\infty(D)$,

*(ii)* $\int \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle\,dx \geq 0$ *whenever* $\varphi \in C_0^\infty(\Omega)$ *is non-negative,*

*(iii)* *$v$ is $p$-superharmonic in $\Omega$.*

*Proof.* We assume that *(i)* holds and consider $D \subset\subset \Omega$ such that supp $\varphi \subset D$. Take $\varepsilon \geq 0$. From the assumption it follows that $J(0) \leq J(\varepsilon)$ where

$$J(\varepsilon) = \int_D |\nabla(v(x) + \varepsilon\varphi(x)|^p\,dx.$$

From the proof of Theorem 2.2 we know that $J'(0) = \int_D |\nabla v|^{p-2}\langle\nabla u, \nabla\varphi\rangle\,dx$. By dividing the equation $J(\varepsilon) - J(0) \geq 0$ with $\varepsilon$ and let $\varepsilon$ tend zero it follows that $J'(0) \geq 0$, which is *(iii)*. We next assume that *(ii)* holds. Then *(i)* follows directly since $|\nabla(v+\varphi)|^p \geq |\nabla v|^p + p\langle|\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle$, by Lemma B.1 *(viii)*. We next show that *(ii)* implies *(iii)*. Since $v \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ we need to show that the comparison principle holds.

We thus assume that there exists a function $h$ which satisfies the conditions in Definition 5.2 *(iii)*. We use the function $\varphi = \max\{h - v, 0\}$. Using *(ii)* and Hölder's inequality we obtain for $G = \{x \in \Omega : v(x) \le h(x)\}$

$$\int_G |\nabla v|^p \, dx \le \int_G \langle |\nabla v|^{p-2} \nabla v, \nabla h \rangle \, dx \le \left( \int_G |\nabla v|^p \, dx \right)^{1-1/p} \left( \int_G |\nabla h|^p \, dx \right)^{1/p}$$

so $\int_G |\nabla v|^p \, dx \le \int_G |\nabla h|^p \, dx$. Thus, $v$ is the minimizer in $G$ with boundary vales $v = h$. According to the proof of Theorem 2.6 the minimizer is unique so $v = h$ in $G$. Thus $v$ satisfies the comparison principle and is therefore $p$-superharmonic. That *(iii)* implies *(ii)* follows from Corollary 5.9. □

In the theorem above we assumed that $v \in C(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega)$, but we can do better. In Corollary 5.9 we will show that a locally bounded $p$-superharmonic function is always a weak supersolution. In order to prove this we need some more theory. A very nice feature of the $p$-superharmonic functions is that they can be approximated from below by continuous functions on compactly contained domains as follows:

**Theorem 5.6.** *Suppose that $v$ is a p-superharmonic functions in the domain $\Omega$. Given a subdomain $D \subset\subset \Omega$ there exists and increasing sequence of p-superharmonic functions $\{v_j\}$ such that $v_j \in C(\bar{D}) \cap W^{1,p}(D)$ for each $v_j$ and*

$$v_1 \le v_2 \le \dots \text{ and } v = \lim_{j\to\infty} v_j$$

*at each point in $D$.*

In order to prove this theorem we need to return to the obstacle problem that was defined in the introduction but this time we consider a slightly less general situation. Assume that $\Omega \subset \mathbb{R}^n$ is bounded and consider the obstacle problem for the class

$$\mathcal{F}_\psi(\Omega) = \{v \in C(\Omega) \cap W^{1,p}(\Omega) | v \ge \psi \text{ in } \Omega \text{ and } v - \psi \in W^{1,p}_{\text{loc}}(\Omega)\},$$

where $\psi$ is now acting as both the boundary function and the obstacle. Just as in the case for $p$-harmonic functions there exists a unique minimizer of the $p$-Dirichlet integral among the functions in $\mathcal{F}_\psi(\Omega)$.

**Theorem 5.7.** *Given $\psi \in C(\Omega) \cap W^{1,p}(\Omega)$, there exists a unique minimizer $v_\psi$ in the class $\mathcal{F}_\psi(\Omega)$ , i.e.,*

$$\int_\Omega |\nabla v_\psi|^p \, dx \le \int_\Omega |\nabla v|^p \, dx \tag{30}$$

*for all similar $v$. The function $v_\psi$ is p-superharmonic in $\Omega$ and p-harmonic in the open set $S = \{x \in \Omega : v_\psi(x) > \psi(x)\}$. If, in addition, $\Omega$ is regular enough and $\psi \in C(\bar{\Omega})$, then also $v_\psi \in C(\bar{\Omega})$ and $v_\psi = \psi$ on $\partial\Omega$.*

*Proof.* We will not prove the continuity and the boundary value property but instead refer the reader to [21] and the references therein. In order to prove the existence and uniqueness of $v_\psi$ we proceed as in Theorem 2.6 with $\mathcal{A} = \mathcal{F}_\psi(\Omega)$. However, we must now prove that $u \ge \psi$ almost everywhere. Since $v_{j_\nu} \rightharpoonup v_\psi$ in $L^p(\Omega)$ it is possible to apply Mazurs lemma which states that there exists a sequence of convex combinations of $v_{j_\nu}$, denoted $\{\tilde{v}_i\}$, which converges strongly to $v_\psi$ in $L^p(\Omega)$. By extracting a subsequence of $\{\tilde{v}_j\}$ that converges to $v_\psi$ a.e., we conclude that $v_\psi \ge \psi$ a.e. Since $\mathcal{F}_\psi(\Omega)$ is closed under addition with non-negative smooth functions with compact support we can use Theorem 5.5 to assert that $v_\psi$ is $p$-superharmonic. We next assume that $v_\psi \in C(\Omega)$. It remains to show that $v_\psi$ is $p$-harmonic in $S$. If $\tilde{v} = v_\psi(x) + \varepsilon\varphi(x) \ge \psi(x)$ for $\varphi \in C_0^\infty(\Omega)$ we have that $\tilde{v} \in \mathcal{F}_\psi(\Omega)$ so (30) and the proof of Theorem 5.5 implies that

$$\int_\Omega \langle |\nabla v_\psi|^{p-2} \nabla v_\psi, \nabla\varphi \rangle \ge 0 \tag{31}$$

whenever $\varphi \in C_0^\infty(\Omega)$. We next consider $\varphi \in C_0^\infty(S)$ and $\varepsilon$ small enough for $\varepsilon\|\varphi\|_\infty \leq \min_{\text{supp}\,\varphi}(v_\psi - \psi)$ to hold which is possible due to the definition of $S$. Thus $v_\psi(x) + \varepsilon\varphi(x) \geq \psi(x)$ which implies that (31) holds with $\Omega$ replaced by $S$. □

*Proof of Theorem 5.6.* We begin by choosing a regular domain $D_1$ such that $D \subset\subset D_1 \subset\subset \Omega$. Since $v$ is lower semi-continuous there exists an increasing sequence of smooth functions $\{\psi_j\}$ such that for all $x \in \Omega$ it holds that

$$\psi_1(x) \leq \psi_2(x) \leq \ldots \leq v(x) \text{ and } \lim_{j\to\infty} \psi_j(x) = v(x).$$

Since $\psi_j \in C(\Omega) \cap W^{1,p}(\Omega)$, we now solve the obstacle for each $j$ in $\mathcal{F}_{\psi_j}(D_1)$, i.e., we find the minimizer $v_j := v_{\psi_j}$ in $D_1$ with $\psi_j$ as obstacle. We next show that

$$v_1(x) \leq v_2(x) \leq \ldots, \quad v_j(x) \leq v(x).$$

for each $x \in \Omega$. Since $\psi_j \leq v$ we see that $v_j \leq v$ in the set $\{v_j = \psi_j\}$ and we therefore consider the open set $A_j = \{v_j > \psi_j\}$ and assume that $A_j \neq \emptyset$. $v_j$ is $p$-harmonic in $A_j$ according to Theorem 5.7 and since $v_j|_{\partial A} = \psi_j|_{\partial A} \leq v|_{\partial A}$ we can use the comparison principle to conclude that $v_j \leq v$ in $A_j$ and thus $v_j \leq v$ in $D_1$. The same argument can be applied to show that $v_j \leq v_{j+1}$ for $j = 1, 2, \ldots$. In $\{v_j = \psi_j\}$ it holds directly since $v_j = \psi_j \leq \psi_{j+1} \leq v_{j+1}$. Thus we consider $A_j$ and note that $v_j|_{\partial A_j} \leq v_{j+1}|_{\partial A_j}$. Since $v_{j+1}$ is $p$-harmonic according to Theorem 5.7 we may again use the comparison principle to conclude that $v_j \leq v_{j+1}$ in $D_1$. □

**Theorem 5.8.** *If $v$ $p$-superharmonic and locally bounded from above in $\Omega$, then $v \in W^{1,p}_{\text{loc}}(\Omega)$, and the approximants $v_j$ from Theorem 5.6 can be chosen so that*

$$\lim_{j\to\infty} \int_D |\nabla(v - v_j)|^p \, dx = 0.$$

*Proof.* We use the construction in the proof of Theorem 5.6. We note that since $v$ is locally bounded it holds that

$$C = \sup_{D_1} v - \inf_{D_1} \psi_1 < \infty.$$

We may therefore use the modified version of Caccioppoli's inequality, that was discussed after Lemma 2.3, to bound our sequence $\{v_j\}$ such that we obtain

$$\int_D |\nabla v_j|^p \, dx \leq p^p C^p \int_{D_1} |\nabla \zeta|^p \, dx = M, \quad j = 1, 2, \ldots$$

Thus, since $\psi_1 \leq v_j \leq v$, $\{v_j\}$ is a bounded sequence in $W^{1,p}(D)$ and $v_j \to v$ pointwise a.e. It follows (see e.g., Theorem 13.44 in [10]) that $v_j \rightharpoonup v$ in $L^p(D)$ and we can use pre-compactness and Mazur's lemma to conclude that $v \in W^{1,p}(D)$, that $\|\nabla v\|_{L^p(D)} \leq M$ and that there exists a subsequence $v_{j_\nu}$ such that $\nabla v_{j_\nu} \rightharpoonup \nabla v$ in $L^p(D)$. From now on we let $v_j$ denote the subsequence.

Next we turn to the proof of proving the strong $L^p$-convergence of the gradients. We note that is sufficient to show that

$$\lim_{j\to\infty} \int_{B_r} |\nabla v - \nabla v_j|^p \, dx = 0$$

whenever $B_r \subset D$ such that $B_{2r} \subset D_1$. Moreover, we consider the test function $\zeta \in C_0^\infty(B_{2r}), 0 \leq \zeta \leq 1$, and $\zeta = 1$ in $B_r$. We next define $\varphi_j = \zeta(v - v_j), \varphi_j \geq 0$ and note that $\nabla\varphi_j = \nabla\zeta(v - v_j) + \zeta(\nabla v - \nabla v_j)$. Since $v_j$ is a supersolution it holds that

$$\int_{B_{2r}} \langle |\nabla v_j|^{p-2}\nabla v_j, \nabla\varphi_j\rangle \, dx \geq 0$$

so

$$\begin{aligned} J_j &= \int_{B_{2r}} \langle |\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j, \nabla(\zeta(v - v_j))\rangle \, dx \\ &\leq \int_{B_{2r}} \langle |\nabla v|^{p-2}\nabla v, \nabla(\zeta(v - v_j))\rangle \, dx. \end{aligned}$$

Using Hölder's inequality and the fact that $\nabla(\zeta v_j) \rightharpoonup \nabla(\zeta v)$ in $L^p_{B_{2r}}$ it follows that $\limsup_{j\to\infty} J_j \leq 0$ for a subsequence. Furthermore,

$$J_j = \int_{B_{2r}} \zeta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j, \nabla v - \nabla v_j\rangle \, dx + \int_{B_{2r}} (v - v_j)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j, \nabla\zeta\rangle \, dx$$

We use Hölder's and Minowski's inequalities to obtain an upper bound of the second integral as follows:

$$\begin{aligned} &\left(\int_{B_{2r}} (v - v_j)^p \, dx\right)^{1/p} \left(\left(\int_{B_{2r}} |\nabla v|^p \, dx\right)^{1-1/p} + \left(\int_{B_{2r}} |\nabla v_j|^p \, dx\right)^{1-1/p}\right) \max|\nabla\zeta| \\ &\leq 2M^{1-1/p} \max|\nabla\zeta| \left(\int_{B_{2r}} (v - v_j)^p \, dx\right)^{1/p} \end{aligned}$$

which tends to zero as $j \to \infty$, by dominated convergence. Thus,

$$0 \leq \lim_{j\to\infty} \int_{B_{2r}} \zeta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j, \nabla v - \nabla v_j\rangle \, dx \leq 0$$

In order to complete the proof for $p \geq 2$ one may use inequality *(i)* in Lemma B.1. We skip the proof for $1 < p < 2$. □

We can use this theorem to prove that locally bounded $p$-superharmonic functions are weak supersolutions.

**Corollary 5.9.** *Suppose that $v$ is $p$-superharmonic and locally bounded in $\Omega$. Then $v \in W^{1,p}_{\text{loc}}(\Omega)$ and $v$ is a weak supersolution, i.e.,*

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle \, dx \geq 0$$

*for all $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 0$.*

*Proof.* By Theorem 5.6 there exists an increasing sequence $\{v_j\}$ of $p$-superharmonic functions such that $v_j \in C(\bar{D}) \cap W^{1,p}(D)$ and $\lim_{j\to\infty} v_j = v$. Take $\varphi \in C_0^\infty(\Omega)$. We need to show that

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle \, dx = \lim_{j\to\infty} \int_\Omega \langle |\nabla v_j|^{p-2}\nabla v_j, \nabla\varphi\rangle \, dx \geq 0$$

One can easily show that (see Chapter 10 in [21])

$$||\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j| \leq (p-1)|\nabla v - \nabla v_j|(|\nabla v|^{p-2} + |\nabla v_j|^{p-2})$$

when $p \geq 2$. Thus,

$$\begin{aligned}
&\left|\lim_{j\to\infty}\int_\Omega \langle |\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j, \nabla\varphi\rangle\, dx\right| \\
&\leq \max|\nabla\varphi| \lim_{j\to\infty}\int_\Omega ||\nabla v|^{p-2}\nabla v - |\nabla v_j|^{p-2}\nabla v_j|\, dx \\
&\leq \max|\nabla\varphi|(p-1)\lim_{j\to\infty}\int_\Omega |\nabla v - \nabla v_j|(|\nabla v|^{p-2} + |\nabla v_j|^{p-2})\, dx \\
&\leq \max|\nabla\varphi|(p-1)\lim_{j\to\infty}\left(\int_\Omega |\nabla v - \nabla v_j|^p\, dx\right)^{1/p}\left(\int_\Omega \left(|\nabla v|^{p-2} + |\nabla v_j|^{p-2}\right)^{p/(p-1)}\, dx\right)^{1-1/p}
\end{aligned}$$

which tends to zero by the strong convergence of the gradients. We note that the integral in the last parenthesis is finite since if $\alpha = (p-2)/p - 1$ we have that $1/2 \leq \alpha \leq 1$ so $\alpha p \geq p/2$ and

$$\int_\Omega |\nabla v|^{\alpha p}\, dx \leq |\Omega|^{1/2}\left(\int_\Omega |\nabla v|^{2\alpha p}\, dx\right)^{1/2}$$

which is finite since $2\alpha p \geq p$. This also holds for $v_j, j = 1, 2, \ldots$. In the case where $1 < p \leq 2$ we use inequality *(iii)* in Lemma B.1 and one application of Hölder's inequality. □

With a little more work it is actually possible to show that a $p$-superharmonic function $v$ such that $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a supersolution (Corollary 7.21, [9]). The following theorem is known as Harnack's convergence theorem and states that the limit of an increasing sequence of $p$-harmonic functions is either infinite or a $p$-harmonic function. It will be of great importance when we discuss solutions to the $p$-Dirichlet problem in Section 6.

**Theorem 5.10.** *Suppose that $u_j$ is $p$-harmonic and that*

$$0 \leq u_1 \leq u_2 \leq \ldots,\; u = \lim_{j\to\infty} u_j$$

*pointwise in $\Omega$. Then either $u \equiv \infty$ or $u$ is a $p$-harmonic function in $\Omega$.*

*Proof.* According to Harnack's inequality for $p$-harmonic functions it holds that

$$u_j(x) \leq c u_j(x_0), \quad j = 1, 2, 3, \ldots$$

where $c = c(p, n)$, whenever $x \in B(x_0, r)$ such that $B(x_0, 2r) \subset\subset \Omega$. From this it follows that if $u(x_0) < \infty$ at some $x_0 \in \Omega$, then $u(x)$ is locally bounded in $\Omega$. By Cacciopoli's lemma for $p$-harmonic functions we obtain

$$\int_{B_r} |\nabla u_j|^p\, dx \leq c r^{-p}\int_{B_{2r}} |u_j|^p\, dx \leq \int_{B_{2r}} |u|^p\, dx \leq c^p r^{n-p} u(x_0)^p$$

so $u \in W^{1,p}_{\text{loc}}(\Omega)$. We can now repeat the argument in Corollary 5.9 but use that each $u_j$ is $p$-harmonic to conclude that

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\, dx = \lim_{j\to\infty}\int_\Omega \langle |\nabla u_j|^{p-2}\nabla u_j, \nabla\varphi\rangle\, dx = 0$$

whenever $\varphi \in C_0^\infty(\Omega)$. □

Note that it follows from the proof that the theorem is valid even for non-positive functions $u_j$. This can be seen by adding a constant to the sequence since $u$ is locally bounded.

We next describe a method known as the Poisson modification which provides a way to locally smooth a $p$-superharmonic function. Let $v$ be a $p$-superharmonic function in $\Omega$, $D \subset\subset \Omega$ is a regular domain and $h$ the $p$-harmonic function in $D$ such that $h|_{\partial D} = v$. In the case where $v$ is continuous we define the Poisson modification as the function

$$V = P(v, D) = \begin{cases} v, & \text{in } \Omega \setminus D \\ h, & \text{in } D \end{cases}$$

$V$ is clearly $p$-superharmonic and it follows directly from the comparison principle that $V \leq v$. In the case where $v$ is not continuous we can define the Poisson modification as the limit of an increasing sequence of continuous and $p$-harmonic functions on $D$ as follows: We use the construction explained in Theorem 5.6 to approximate $v$ from below with continuous functions $v_j$ and define the Poisson approximation through

$$V = \lim_{j\to\infty} V_j = \lim_{j\to\infty} P(v_j, D),$$

which is possible since $v_j \in C(\bar{D}) \cap W^{1,p}(D)$. We have the following proposition:

**Proposition 5.11.** *Suppose that $v$ is $p$-superharmonic in $\Omega$ and that $D \subset\subset \Omega$. Then the Poisson modification $V = P(v, D)$ is $p$-superharmonic in $\Omega$, $p$-harmonic in $D$ and $V \leq v$. Moreover, if $v$ is locally bounded, then*

$$\int_G |\nabla V|^p \, dx \leq \int_G |\nabla v|^p \, dx$$

*for $D \subset G \subset\subset \Omega$.*

*Proof.* We assume that $v$ is not continuous and note that from the definition of the Poisson modification we get a corresponding sequence of increasing $p$-harmonic functions $h_j, j = 1, 2, \ldots$. Since $h_j \leq v_j \leq v \not\equiv \infty$ the limit function is $p$-harmonic in $D$ according to Theorem 5.10. $V$ is $p$-superharmonic according to Lemma 5.2. Furthermore, since $p$-harmonic functions minimize the $p$-Dirichlet integral it follows that

$$\int_D |\nabla V_j|^p \, dx \leq \int_D |\nabla v_j|^p \, dx$$

if $v$ is locally bounded, and by using monotone convergence we obtain

$$\int_D |\nabla V|^p \, dx \leq \int_D |\nabla v|^p \, dx.$$

□

There exists another equivalent definition of the Poisson modification, which highlights the fact that it is a local smoothing procedure. For a regular set $D \subset\subset \Omega$ we define

$$v_D = \inf\{\tilde{v} : \tilde{v} \text{ is } p\text{-harmonic, } \liminf_{y\to x} \tilde{v}(y) \geq v(x)\}$$

and

$$V = P(v, D) = \begin{cases} v, & \text{in } \Omega \setminus D, \\ v_D, & \text{in } D. \end{cases}$$

For more details, see [9]. We next use the Poisson modification to show that $p$-superharmonic functions have the so called essential limit inferior property.

**Theorem 5.12.** *If $v$ is p-superharmonic in $\Omega$, then*

$$v(x) = \operatorname*{ess\,inf}_{y \to x} v(y)$$

*at each point $x$ in $\Omega$.*

In order to prove the theorem we will need two additional lemmas.
**Lemma 5.13.** *Suppose that $v$ is p-superharmonic in $\Omega$. If $v(x) \le \lambda$ at each $x \in \Omega$ and if $v(x) = \lambda$ for a.e. $x \in \Omega$, then $v(x) = \lambda$ at each $x \in \Omega$.*

*Proof.* If $v$ is continuous there is nothing to prove. Since $p$-harmonic functions are continuous we choose an arbitrary regular domain, e.g., an open ball, $D \subset\subset \Omega$ and use the Poisson modification $V = P(v, D)$. Since $V \le v \le \lambda$ at each $x \in \Omega$ it is sufficient to show that $V(x) = \lambda$ at each $x \in D$. $v$ is locally bounded by assumption, and thus we can use Proposition 5.11 to conclude that

$$\int_G |\nabla V|^p \, dx \le \int_G |\nabla v|^p \, dx = \int_G |\nabla \lambda|^p \, dx = 0$$

where $D \subset G \subset\subset \Omega$. Thus $\nabla V = 0$ so $V = \lambda$ almost everywhere in $G$. However, since $V$ is $p$-harmonic and thus continuous in $D$ we conclude that $V(x) = \lambda$ for each $x \in D$. □

**Lemma 5.14.** *If $v$ is p-superharmonic in $\Omega$ and if $v(x) > \lambda$ for a.e. $x \in \Omega$, then $v(x) \ge \lambda$ for every $x \in \Omega$.*

*Proof.* We note that we may assume that $\lambda < \infty$ because otherwise there is nothing to prove. Next we consider the function $\tilde{v} = \min\{v(x), \lambda\}$. Since $v(x) > \lambda$ for a.e. $x$ by assumption it holds that $\tilde{v}(x) = \lambda$ for a.e. $x \in \Omega$. The conclusion of the lemma follows by applying Lemma 5.13 . □

*Proof of Theorem 5.12.* Since $v$ is lower semi-continuous it follows directly that $v(x) \le \liminf_{y \to x} v(y) \le$ ess lim inf $_{y \to x} v(y)$. We next prove that $\lambda =$ ess lim inf$_{y \to x} v(y) \le v(x)$. Furthermore, $v$ is $p$-superharmonic and thus we can, given $\varepsilon > 0$, find a $\delta > 0$ s.t.

$$v(y) > \lambda - \varepsilon \quad \text{for a.e. } y \in B(x, \delta).$$

From Lemma 5.14 it follows that $v(x) \ge \lambda - \varepsilon$. Since $\varepsilon$ is arbitrary the proof of the lemma is complete. □

We finish this section by stating a theorem which provides the final connection between supersolutions and $p$-harmonic functions (Theorem 7.16 in [9]).
**Theorem 5.15.** *Let $v$ be a weak supersolution of the p-Laplace equation in $\Omega$. If*

$$v(x) = \operatorname*{ess\,inf}_{y \to x} v(y)$$

*holds for each $x \in \Omega$, then $v$ is p-harmonic.*

# 6 Perron's method

In this section we return to the $p$-Dirichlet boundary value problem. In particular we consider Perron's method, which is a technique to solve the Dirichlet problem for the Laplace operator. It is, however, possible to generalize this method for other partial differential operators including the $p$-Laplace operator. Throughout this section we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $g : \partial\Omega \to [-\infty, \infty]$.

**Definition 6.1.** We say that a function $v : \Omega \to (-\infty, \infty]$ is a member of the upper class $\mathcal{U}_g$ if the following conditions are satisfies

1. $v$ is $p$-superharmonic in $\Omega$,
2. $v$ is bounded from below,
3. $\liminf_{x \to w} v(x) \geq g(w)$ when $w \in \partial\Omega$.

We define the lower class the lower class $\mathcal{L}_g$ symmetrically: a function $u$ is in $\mathcal{L}_g$ if

1. $u$ is $p$-subharmonic in $\Omega$,
2. $u$ is bounded from above,
3. $\limsup_{x \to w} u(x) \leq g(w)$ when $w \in \partial\Omega$.

Furthermore, for each $x \in \Omega$ we define

$$\text{the upper Perron solution } \overline{H}_g(x) = \inf_{v \in \mathcal{U}_g} v(x),$$
$$\text{the lower Perron solution } \underline{H}_g(x) = \sup_{u \in \mathcal{L}_g} u(x).$$

When the boundary function $g$ is clear from context, we sometimes omit the subsript.

It follows from the comparison principle that $\underline{H} \leq \overline{H}$. In Theorem 6.3 we will show that if $g$ is continuous the upper and the lower solution will coincide i.e., $\underline{H} = \overline{H} =: H$. Furthermore, under these circumstances $H$ will agree with the unique $p$-harmonic function in $\Omega$ with $g$ as boundary values in Sobolev sense. In order to simplify the calculations, we will from now on assume that $g$ is bounded, i.e., $m \leq g(x) \leq M$ for each $x \in \Omega$. Since $m \in \mathcal{L}_g$ and $M \in \mathcal{U}_g$ it follows that $m \leq \underline{H} \leq \overline{H} \leq M$. Due to Lemma 5.1 and a corresponding version for the maximum of $p$-subharmonic functions we note that we can cut of all the functions and therefore assume that all functions are bounded from below my $m$ and from above by $M$. There is an obvious symmetry between the upper and the lower Perron solution. Just as in [21] we will focus on proving the statements concerning $\overline{H}$. We have the following useful lemma.

**Lemma 6.1.** *If $g$ is bounded, $\overline{H}$ and $\underline{H}$ are continuous in $\Omega$.*

*Proof.* We will show the lemma for $\overline{H}$. Take $\varepsilon > 0$. We fix $x_0 \in \Omega$ and consider $B(x_0, R) \subset\subset \Omega$ and $0 < r < R$. For $x_1, x_2 \in B(x_0, r)$ there exist sequences of functions $\{v_i^1\}$ and $\{v_i^2\}$ such that $v_i^k \in \mathcal{U}_g$ for each $i = 1, 2 \ldots$ and $k \in \{1, 2\}$. Furthermore, $v_i^1(x_1) \to \overline{H}(x_1)$ and $v_i^2(x_2) \to \overline{H}(x_2)$ as $i \to \infty$. We define $v_i = \min\{v_i^1, v_i^2\}$ and note that

$$\lim_{i \to \infty} v_i(x_1) = \overline{H}(x_1), \quad \lim_{i \to \infty} v_i(x_2) = \overline{H}(x_2)$$

We next define the Poisson modification $V_i = P(v_i, B(x_0, R))$ and note that $V_i \in \mathcal{U}_g$ since $V_i$ is $p$-superharmonic by Proposition 5.11 and

$$\liminf_{x \to w \in \partial\Omega} V_i(x) = \liminf_{x \to w \in \partial\Omega} v_i(x) \geq g(w).$$

Thus, it follows that $\overline{H} \leq V_i \leq v_i$ in $\Omega$. Moreover, for $i$ sufficiently large and $k \in \{1, 2\}$, it holds that $v_i(x_k) < \overline{H}(x_k) + \varepsilon$, so $\overline{H}(x_k) > v_i(x_k) - \varepsilon > V_i(x_k) - \varepsilon$, which implies that

$$\overline{H}(x_2) - \overline{H}(x_1) \leq V_i(x_2) - V_i(x_1) + \varepsilon \leq \operatorname*{osc}_{B(x_0, r)} V_i + \varepsilon.$$

Since $V_i$ is $p$-harmonic in $B(x_0, R)$, and therefore Hölder continuous, we obtain

$$\operatorname*{osc}_{B(x_0,r)} V_i \leq L\left(\frac{r}{R}\right)^{\alpha} \operatorname*{osc}_{B(x_0,R)} V_i \leq L\left(\frac{r}{R}\right)^{\alpha}(M-m).$$

Hence we can find $r > 0$ such that $|\overline{H}(x_1) - \overline{H}(x_2)| \leq 2\varepsilon$ for $x_1, x_2 \in B(x_0, r)$, which completes the proof of the lemma. □

This lemma will be used in order to prove that Perron's solutions are either $p$-harmonic of infinite.

**Theorem 6.2.** *The upper solution $\overline{H}$ satisfies one of the following properties:*

1. *$\overline{H}$ is $p$-harmonic in $\Omega$,*
2. *$\overline{H} \equiv \infty$ in $\Omega$,*
3. *$\overline{H} \equiv -\infty$ in $\Omega$.*

*A similar result holds for $\underline{H}$.*

*Proof.* We need to show that $\overline{H} \not\equiv \infty$ is a solution to the $p$-Laplace equation. At first we construct a sequence of functions $\{\omega_i\}$ such that $\omega_i \in \mathcal{U}_g$ and the sequence converges to $\overline{H}$ at the rational points. After that we use Posson modification to smooth the sequence and show that its limit coincides with Perron's solution on a dense subset. Since $\overline{H}$ is continuous, this implies that $\overline{H}$ is $p$-harmonic.

Thus, we let $q_1, q_2, \ldots \in \mathbb{Q}^n \cap \Omega$. From the definition of $\overline{H}$ there exists $\{v_i^k\}_{i=1}^{\infty}$ such that

$$\overline{H}(q_k) \leq v_i^k(q_k) \leq \overline{H}(q_k) + \frac{1}{i}, \quad i = 1, 2, 3, \ldots$$

for each $q_k \in \mathbb{Q}^n \cap \Omega$. Next we set

$$\omega_i = \min\{v_1^1, v_2^1, \ldots, v_i^1, v_1^2, v_2^2, \ldots, v_i^2, \ldots, v_1^i, v_2^i, \ldots, v_i^i\}$$

Clearly, $\omega_i \in \mathcal{U}_g, \omega_1 \geq \omega_2 \geq \ldots$ and $\overline{H}(q_k) \leq \omega_i(q_k) \leq v_i^k(q_k)$ when $i \geq k$, so

$$\lim_{i\to\infty} w_i(q_k) = \overline{H}(q_k) \text{ for } q_k \in \mathbb{Q}^n \cap \Omega.$$

Set $W_i = P(w_i, B)$ for a ball $B \subset\subset \Omega$. From the proof of Lemma 6.1 we know that $W_i \in \mathcal{U}_g$, so $\overline{H} \leq W_i \leq w_i$. Hence $W_i(q_k) \to \overline{H}(q_k)$ as $i \to \infty$ for $q_k \in \mathbb{Q}^n \cap \Omega$. Consider the limit

$$W = \lim_{i\to\infty} W_i.$$

Since $W_1 \geq W_2 \geq W_2 \geq, \ldots$ we can use Theorem 5.10 to conclude that the $W$ is $p$-harmonic in $B$. Since $W_i \geq \overline{H}$ for $i = 1, 2, \ldots$ it follows that $W \geq \overline{H}$ and in addition $W(q_k) = \overline{H}(q_k)$ for each $q_k \in \mathbb{Q}^n \cap B$. Both $W$ and $\overline{H}$ are continuous in $B$ and they agree on $\mathbb{Q}^n \cap B$. Thus they agree everywhere on $B$ and therefore it follows that $\overline{H} = W$ so $\overline{H}$ is $p$-harmonic in $B$. We conclude the proof by noting that $B$ is arbitrary. □

The next theorem is known as Wiener's resolutivity theorem.

**Theorem 6.3.** *Suppose that $g : \partial\Omega \to \mathbb{R}$ is continuous. Then $\overline{H} = \underline{H}$ in $\Omega$.*

*Proof.* We claim that it is sufficient to prove the theorem when $g \in C^\infty(\Omega)$. By approximating $g$ with smooth functions such that, given $\varepsilon$, there is a function $\varphi \in C^\infty(\Omega)$ such that

$$\varphi(\xi) - \varepsilon < g(\xi) < \varphi(\xi) + \varepsilon.$$

Thus, assuming that the theorem holds for smooth functions, i.e., $\underline{H}_\varphi = \overline{H}_\varphi$ it follows that

$$H_\varphi - \varepsilon = H_{\varphi-\varepsilon} \le \underline{H}_g \le \overline{H}_g \le H_{\varphi+\varepsilon} = H_\varphi + \varepsilon$$

which imples that $\underline{H}_g = \overline{H}_g$. From now on we thus assume that $g \in C^\infty(\mathbb{R}^n)$. Let $u \in C(\Omega) \cap W^{1,p}(\Omega)$ be the unique $p$-harmonic function with $u - g \in W_0^{1,p}(\Omega)$. Furthermore, let $v$ be the solution to the obstacle problem in $\mathcal{F}_g(\Omega)$. Since $v \ge g$ and since $v$ is $p$-superharmonic, $v \in \mathcal{U}_g$. Since $\Omega$ is a domain, we consider the regular sets $D_1 \subset D_2 \subset \dots$ such that $\Omega = \cup D_j$ (see the discussion after Definition 2.4) and the sequence $\{V_i\}$ such that $V_i = P(v, D_i)$. Since $V_{i+1} = P(v, D_{i+1}) = P(V_i, D_{i+1})$ it holds that $V_1 \ge V_2 \ge, \dots$ and $V_j \in \mathcal{U}_g$ for each $j = 1, 2, \dots$. By construction $V_j - g \in W_0^{1,p}(\Omega)$ and from Theorem 5.7 and Proposition 5.11 it follows that

$$\int_\Omega |\nabla V_j|^p \, dx \le \int_\Omega |\nabla v|^p \, dx \le \int_\Omega |\nabla g|^p \, dx \tag{32}$$

According to Theorem 5.10 the limit function $V = \lim_{j\to\infty} V_j$ is $p$-harmonic in $D_j$ for $j = 1, 2, \dots$ and we conclude that $V$ is $p$-harmonic in $\Omega$. By using (32) together with a standard compactness argument, similar to the one in the proof of Theorem 3.1, it follows that $V - g \in W_0^{1,p}(\Omega)$. Since $p$-harmonic functions in bounded domains with given boundary values are unique, it follows that $V = u$. In addition, $\overline{H}_g \le V_j$ for each $j$ so we see that $\overline{H}_g \le \lim_{j\to\infty} V_j = u$. One can also show that $\underline{H}_g \ge u$, which implies that $u \le \underline{H}_g \le \overline{H}_g \le u$. $\square$

By studying the proof we see that as long as $g \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, the solution $u$ obtained in Theorem 3.1 agrees with Perron's solution $H_g$. This is so important that it is summarized in the proposition below.

**Proposition 6.4.** *If $g \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, then the p-harmonic function with boundary values in Sobolev sense coincides with the Perron solution $H$.*

We recall the discussion about regular points in Section 2 and restate the definition of regular boundary points in terms of Perron's solution.

**Definition 6.2.** We say that $w \in \partial\Omega$ is regular if

$$\lim_{x\to w} H_g(x) = g(w)$$

whenever $g : \partial\Omega \to \mathbb{R}$ is continuous. A point which is not regular is called irregular.

We saw in Section 2 that it is possible to use the Wiener criterion to characterize the regular boundary points of $\partial\Omega$. We will now present another method, which uses so called barrier functions.

**Definition 6.3.** A point $w_0 \in \partial\Omega$ has a barrier if there exists a function $v : \Omega \to \mathbb{R}$ such that

1. $v$ is $p$-superharmonic in $\Omega$,
2. $\liminf_{x\to w} v(x) > 0$ for all $w \ne w_0, w \in \Omega$,
3. $\lim_{x\to w_0} v(x) = 0$.

**Theorem 6.5.** *Let $\Omega$ be a bounded domain. The point $w_0 \in \partial\Omega$ is regular if and only if there exists a barrier at $w_0$.*

*Proof.* We first show the if direction. We assume that $g \in C(\partial\Omega)$ and set $M = \sup|g|$. Using the minimum principle we know that the barrier $v$ is positive inside the domain. Given $\varepsilon > 0$ and $x \in \Omega$, we can find $\delta > 0$ and $\lambda > 0$ such that $|g(w_0) - g(w)| < \varepsilon$ whenever $|w_0 - w| < \delta$ and $\lambda v(x) \geq 2M$ whenever $|x - w_0| \geq \delta$. We consider the function $\tilde{v} = g(w_0) + \varepsilon + \lambda v(x)$. $\tilde{v}$ is clearly $p$-superharmonic and bounded from below. Furthermore, for $\xi \in \partial\Omega$ we obtain

$$\liminf_{x \to \xi} \tilde{v}(x) - g(\xi) \geq \liminf_{x \to \xi} \lambda v(x) = 0$$

so $\tilde{v} \in \mathcal{U}_g$. Similarly we see that $\tilde{u}(x) = g(w_0) - \varepsilon - \lambda v(x)$ belongs to $\mathcal{L}_g$. It follows that

$$g(w_0) - \varepsilon - \lambda v(x) \leq \underline{H}_g(x) = \overline{H}_g(x) \leq g(w_0) + \varepsilon + \lambda v(x).$$

so $|H_g(x) - g(w_0)| \leq \varepsilon + \lambda v(x)$. Thus $H_g(x) \to g(w_0)$ as $x \to w_0$ so $w_0$ is a regular point. We next show the existence of a barrier when $w_0$ is a regular boundary point. Consider the function $g(x) = |x - w_0|^{p/(p-1)}$. When $x \neq w_0$ we see that $\Delta_p g = \operatorname{div}(|\nabla g|^{p-2}\nabla g) = (p/(p-1))^{p-1} n$. We note that

$$\int_\Omega \langle |\nabla g|^{p-2}\nabla g, \nabla\varphi\rangle = -\int_\Omega \operatorname{div}(|\nabla g|^{p-2}\nabla g)\varphi = -\int_\Omega \left(\frac{p}{p-1}\right)^{p-1} \varphi \, dx \leq 0$$

for all non-negative $\varphi \in C_0^\infty(\Omega)$ and by Theorem 5.5 we conclude that $g$ is $p$-subharmonic in $\Omega$. It follows from the comparison principle that $\overline{H}_g \geq g$ in $\Omega$. By the properties of a regular boundary point it holds that $\lim_{x \to w_0} \overline{H}_g(x) = g(w_0) = 0$, so we can use $\overline{H}_g$ as a barrier function. □

# 7 Boundary behaviour

We now discuss the boundary behaviour of $p$-harmonic functions vanishing in a special type of Lipschitz domains, known as $C^{1,\alpha}$-domains. The whole section is devoted to proving Theorem 7.4 which is a special case of the theory developed in [13] and [14]-[20]. We let $u$ and $v$ be two non-negative $p$-harmonic functions vanishing on a portion of the boundary. The conclusion of the theorem is that close to that part of the boundary the ratio of $u$ and $v$ is Hölder continuous and bounded from above and below by constants. Before formulating the theorem we first discuss some fundamental properties of Lipschitz- and $C^{1,\alpha}$-domains.

A Lipschitz domain is a domain in $\mathbb{R}^n$ where locally the boundary is a graph of a Lipschitz continuous function. That is, after a possible rotation, the boundary can be described by the graph $\{(w', \phi(w') : w' \in \mathbb{R}^{n-1}\}$ where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, i.e., $||\,|\nabla\phi_i|\,||_\infty < \infty$. More formally, we say that $\Omega$ is a Lipschitz domain if for each $x_i \in \partial\Omega$, there exists $r_i > 0$ and a Lipschitz function $\phi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ with Lipschitz constant $M_i = ||\,|\nabla\phi_i|\,||_\infty$ such that

$$\begin{aligned} \Omega \cap B(w_i, r_i) &= \{x = (x', x_n) \in \mathbb{R}^n : x_n > \phi_i(x')\} \cap B(w_i, r_i) \\ \partial\Omega \cap B(w_i, r_i) &= \{x = (x', x_n) \in \mathbb{R}^n : x_n = \phi_i(x')\} \cap B(w_i, r_i) \end{aligned} \tag{33}$$

hold in an appropriate coordinate system. In the following we assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain i.e., $\partial\Omega$ is compact and we can find Lipschitz constants $M$ and a $r_0$ that hold for all $w \in \partial\Omega$. This can be seen as follows: let $\{B(w_i, \frac{r_i}{2})\}$ be an open cover of a neighbourhood of $\partial\Omega$. Since $\partial\Omega$ is compact we can choose a finite subcover $\{B(w_i, \frac{r_i}{2})\}_{i=1}^N$. Next we define $2r_0 = \min_i r_i$ and $M = \max_i M_i$ and note that for these constants the conditions in (33) hold for all $x \in \partial\Omega$. In addition, if $\phi_i \in C^{k,\alpha}(\mathbb{R}^n)$ for each $i$ where $k$ is a non-negative integer and $\alpha \in (0,1]$, we define $\Omega$ to be a bounded $C^{k,\alpha}$-domain.

We say that a domain $\Omega$ satisfies the corkscrew condition if there exists constants $M > 1$ and $r_0 > 0$ such that for any $w \in \partial\Omega$ and $r \in (0, r_0)$, there exists a point $a_r(w) \in \Omega$, such that

$$\frac{r}{M} < |a_r(w) - w| \leq r, \quad \text{and} \quad \text{dist}(a_r(w), \partial\Omega) \geq \frac{r}{M}.$$

Furthermore, we say that $\Omega$ satisfies the uniform condition if for each $w \in \partial\Omega$, $r \in (0, r_0)$, and $x_1, x_2 \in B(w,r) \cap \Omega$, there exists a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x_1$, $\gamma(1) = x_2$, such that

$$L(\gamma) \leq c(M)|x_1 - x_2|, \tag{34}$$

$$\min\{L(\gamma([0,t])), L(\gamma([t,1]))\} \leq c(M)\text{dist}(\gamma(t), \partial\Omega), \tag{35}$$

where $L(\gamma)$ denotes the length of $\gamma$.

**Lemma 7.1.** *Let $\Omega$ be a bounded Lipschitz domain with Lipschitz constant $M \geq 2$ and $r_0 > 0$. Then*

*(i)* *$\Omega$ satisfies the corkscrew condition,*

*(ii)* *$\mathbb{R}^n \setminus \bar{\Omega}$ satisfies the corkscrew condition,*

*(iii)* *$\Omega$ satisfies the uniform condition.*

*Proof.* For part *(i)*, if $w = (w', \phi(w')) \in \partial\Omega$ is given and $0 < r < r_0$, then (using the local coordinate system)

$$a_r(w) = (w', \phi(w') + r) \tag{36}$$

does the job. In order to show that $\text{dist}(a_r(w), \partial\Omega) \geq M^{-1}r$ we note that using an appropriate local coordinate system we may assume that $w = (0^{n-1}, 0)$ and that $a_r(w) = (0^{n-1}, r)$. Using the Lipschitz continuity of $\phi$ we see that the distance between $a_r(w)$ and the boundary is greater than the distance between the point $(0^{n-1}, r)$ and the line $L = 0^n + t(0^{n-2}, 1, M)$ for $t \in \mathbb{R}$. The orthogonal projection of $a_r(w)$ on $L$ is given by $Mr(0^{n-2}, 1, M)/(1 + M^2)$ and thus it follows that

$$d(a_r(w), L) = \left|(0^{n-1}, r) - \frac{Mr(0^{n-2}, 1, M)}{1 + M^2}\right| = \frac{r}{1 + M^2}(M^4 - M^2 + 1)^{1/2} \geq r/M$$

for $M \geq 2$. The proof of part *(ii)* is analogous. In the proof of part *(iii)* we need to consider different cases, see Figure 1 for an illustration. For $i \in \{1,2\}$ we set $x_i = (x_i', \phi(x_i') + s_i)$ where $s_i > 0$ and let $d_i = d(x_i, \partial\Omega)$ denote the distance from $x_i$ to the boundary.

1. Here we assume that $d_i \leq |x_1 - x_2|$ for $i \in \{1,2\}$. Next we define

$$\tilde{x}_1 = \left(x_1', \phi(x_1') + \frac{|x_1 - x_2|}{2}\right),$$

$$\tilde{x}_2 = \left(x_2', \phi(x_2') + \frac{|x_1 - x_2|}{2}\right),$$

and note that $\tilde{x}_1, \tilde{x}_2 \in \Omega$, since $|x_1 - x_2|/2 < r$. The first part, $\gamma_1$, is the straight line segment from $x_1$ to $\tilde{x}_1$. Clearly $L^1(\gamma_1) \leq M|x_1 - x_2|$. The second curve $\gamma_2$ follows the path between $(x_1', \phi(x_1'))$ to $(x_2', \phi(x_2'))$ but translated $|x_1 - x_2|/2$ in the $x_n$-direction so it becomes a path between $\tilde{x}_1$ and $\tilde{x}_2$. Thus,

$\gamma_2(t) = (\eta(t), \phi(\eta(t)) + |x_1 - x_2|/2)$ where $\eta(t)$ is the straight line between $x'_1$ and $x'_2$. Furthermore, we can bound the length of $\gamma_2$ by

$$
\begin{aligned}
L^1(\gamma_2) &\le \sup_{|t_i - t_{i-1}| \to 0} \left\{ \sum_{i=1}^{n} |\gamma_2(t_i) - \gamma_2(t_{i-1})| \right\} \\
&\le \sup_{|t_i - t_{i-1}| \to 0} \left\{ \sum_{i=1}^{n} |\eta(t_i) - \eta(t_{i-1}), \phi(\eta(t_i)) - \phi(\eta(t_{i-1}))| \right\} \\
&\le \sup_{|t_i - t_{i-1}| \to 0} \left\{ \sum_{i=1}^{n} |\eta(t_i) - \eta(t_{i-1})| + |\phi(\eta(t_i)) - \phi(\eta(t_{i-1}))| \right\} \\
&\le (1 + M)|x'_1 - x'_2| \\
&\le (1 + M)|x_1 - x_2|.
\end{aligned}
$$

The last part, $\gamma_3$, is the straight line between $\tilde{x}_2$ to $x_2$ and $L^1(\gamma_3) \le M|x_1 - x_2|$. Thus we define $\gamma$ to be the curve $\gamma = \cup \gamma_i$ and clearly (34) is satisfied. We will make a short comment on why (35) holds. We note that we can make sure that the condition holds if we are considering a point on $\gamma_1$, since either we are moving away from the boundary or the distance to the boundary is greater than $|x_1 - x_2|/2$. On $\gamma_2$ we have control because of the corkscrew condition. By symmetry we have control on $\gamma_3$.

2. Now, $d = |x_1 - x_2| \le d_i$ for $i \in \{1, 2\}$. For this case we take $\gamma$ to be the straight line from $x_1$ to $x_2$. Obviously (34) is satisfied. For the second condition we note that we obtain the smallest distance $d(\gamma(t), \partial\Omega)$ if $d_1 = d_2 = |x_1 - x_2|$. In this case it is clear that $d(\gamma(t), \partial\Omega) \ge \sqrt{3}|x_1 - x_2|/2$ so the second condition is also satisfied.

3. Assume that $d_1 \le |x_1 - x_2| \le d_2$. Again, we use the notation $d = |x_1 - x_2|$. We consider $\gamma$ to be the straight line between $x_1$ and $x_2$, i.e., $\gamma(t) = x_1 + t(x_2 - x_1)$. Let $B_{d_1} = B(x_1, d_1)$ and $B_d = B(x_2, d)$. We need to make sure that (35) is satisfied. We first consider $d(\gamma(t), \partial\Omega)$ for $0 \le t \le \min\{d_1/d, 1/2\}$ and note that $d(\gamma(t), \partial\Omega) \ge d(\gamma(t), \partial B_{d_1} \cap \partial B_d) = s$. Let $\tilde{x}$ be the point on $\gamma(t)$ such that $s = d(\tilde{x}, \partial B_{d_1} \cap \partial B_d)$ and $y_0 = d(x_1, \tilde{x})$. By solving the following equation system

$$
\begin{aligned}
y_0^2 + s^2 &= d_1^2 \\
(d - y_0)^2 + s^2 &= d^2
\end{aligned}
$$

we see that $y_0 = d_1^2/2d$ and $s = d_1(1 - d_1^2/4d^2)^{1/2}$ so $s/d_1 = (1 - d_1^2/4d^2)^{1/2}$. Furthermore, we note that $s/d_1$ is as small as possible for $d_1 = d$ and $s/d_1 = \sqrt{3}/2$. Thus, for $0 \le t \le \min\{d_1/d, 1/2\}$ it follows that

$$d(\gamma(t), \partial\Omega) \ge \sqrt{3}d_1/2 \ge (\sqrt{3}/2) \min\{L(\gamma([0, t])), L(\gamma([t, 1]))\}.$$

For $\min\{d_1/d, 1/2\} < t \le 1$ we obtain

$$
\begin{aligned}
d(\gamma(t), \partial\Omega) &\ge d(x_1, \gamma(t)) - d_1^2/2d \ge d(x_1, \gamma(t)) - d_1/2 \\
&\ge d(x_1, \gamma(t))/2 \\
&= \min\{L(\gamma([0, t])), L(\gamma([t, 1]))\}/2
\end{aligned}
$$

4. The case when $d_2 \le |x_1 - x_2| \le d_1$ follows by symmetry.

□

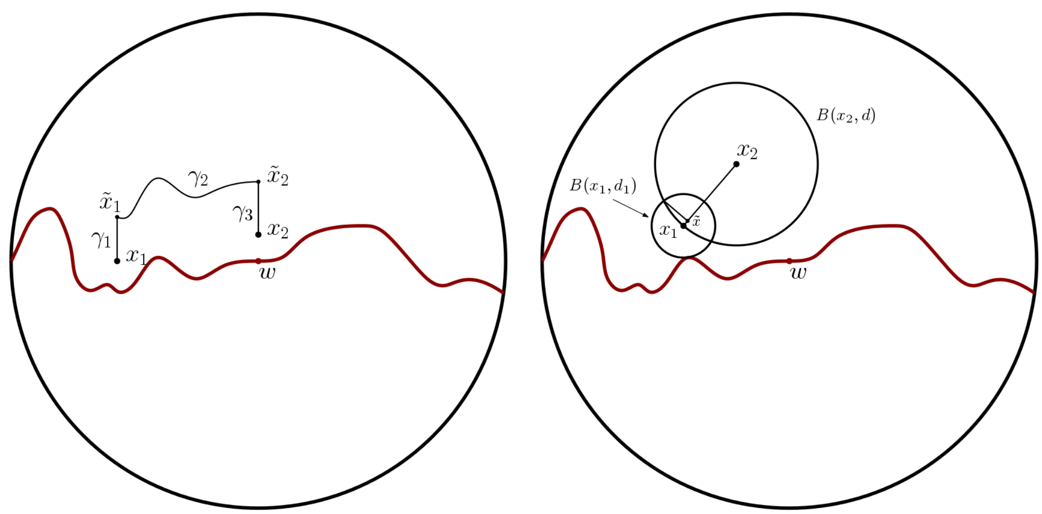


Figure 1: Illustrations for the proof of Lemma 7.1. *Left*: case 1. *Right*: case 3.

From now on, we will denote by $a_r(w)$, the point given by (36). We note that it follows from Theorem 6.31 in [9] that a Lipschitz domain is a regular domain. Furthermore, the previous lemma can also be used to show that bounded Lipschitz domains satisfy the so called Harnack chain condition.

**Lemma 7.2.** *Let $\Omega$ be a Lipschitz domain with Lipschitz constant $M \geq 2$, and $r_0 > 0$, and let $1 < p < \infty$. Let $u$ be a $p$-harmonic function in $\Omega$. Assume that $w \in \partial\Omega$, $0 < r < r_0$, $x_1, x_2 \in B(w, r) \cap \Omega$, $\operatorname{dist}(x_1, \partial\Omega) > \varepsilon$, $\operatorname{dist}(x_2, \partial\Omega) > \varepsilon$, and $|x_1 - x_2| \leq A\varepsilon$, for some $A \geq 1$ and some $\varepsilon > 0$. Then there exists a constant $c = c(p, n, M, A)$ such that*

$$u(x_1) \leq cu(x_2).$$

*Proof.* The existence of a curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x_1$, $\gamma(1) = x_2$, satisfying (34) and (35) follows from Lemma 7.1. We note that from (34) it follows that $L(\gamma) \leq c(M)A\varepsilon$. Clearly, $B(x_i, \varepsilon/8) \subset \Omega$ for $i \in \{1, 2\}$. We note that if $x_2 \in B(x_1, \varepsilon/16)$ the conclusion of the lemma follows directly by applying the Harnack inequality for $p$-harmonic functions, and to this end we therefore assume that $x_2 \notin B(x_1, \varepsilon/16)$. Next we define $\tilde{t}_1$ to be the smallest $t$ s.t. $\gamma(t) \in \partial B(x_1, \varepsilon/32)$ and $\tilde{t}_2$ to be the largest $t$ s.t. $\gamma(t) \in \partial B(x_2, \varepsilon/32)$, i.e., $\tilde{t}_1$ denotes the first time the curve leaves $B(x_1, \varepsilon/32)$ and $\tilde{t}_2$ denotes the last time the curve enters $B(x_2, \varepsilon/32)$. It follows that $\min\{L(\gamma([0, t])), L(\gamma([t, 1]))\} \geq \varepsilon/32$ whenever $t \in [\tilde{t}_1, \tilde{t}_2]$ and from (35) we see that $c(M)\operatorname{dist}(\gamma(t), \partial\Omega) \geq \varepsilon/32$ whenever $t \in [\tilde{t}_1, \tilde{t}_2]$. Since $c(M) > 1$ it follows that $\operatorname{dist}(\gamma(t), \partial\Omega) \geq \varepsilon/(32c(M))$. This will allow us to create a chain of balls with appropriate radius along the curve for which it will be possible to use Harnack's inequality. We proceed as follows: let $\tilde{r} = \varepsilon/(4 \cdot 32c(M))$, $z_0 = x_1, t_0 = 0$ and set $z_{j+1} = \gamma(t_{j+1})$ where $t_{j+1}$ is defined to be the smallest $t > t_j$ such that $\gamma(t) \in \partial B(z_j, \tilde{r})$ for all $j = 0, \ldots, k-1$. Here $k \geq 1$ denotes the greatest $k$ such that $z_k \notin B(x_2, \tilde{r})$, *i.e.* at time $t_k$ the curve has not yet entered $B(x_2, \tilde{r})$. However, by construction $z_{k+1} \in B(x_2, \tilde{r})$. We may now use Harnack's inequality to conclude that

$$u(x_1) = u(z_0) \leq cu(z_1) \leq \ldots \leq c^{k+1}u(z_{k+1}) \leq c^{k+2}u(x_2).$$

In order to conclude the proof we need to make sure that $k = k(p, n, M, A)$. However, using the first property of the curve we have that

$$k\tilde{r} \leq L(\gamma) \leq c(M)A\varepsilon$$

which implies that $k \leq \tilde{c}(p, n, M, A)$ and so the conclusion of the lemma follows. □

Another very useful property of $C^{1,\alpha}$-domains is that they are relatively flat in the sense that it is possible to locally approximate their boundaries with hyperplanes. In particular, the more we zoom in on the boundary, the flatter it will look. This is summarized in the following lemma.

**Lemma 7.3.** *Let $\Omega$ be a bounded $C^{1,\alpha}$-domain for some $\alpha \in (0,1]$. Given $\delta << 1$, there exists $\bar{r}_0 = \bar{r}_0(n,\alpha) << 1$ such that for all $w \in \partial\Omega$ and $0 < r < \bar{r}_0$ there exists a hyperplane $\Lambda = \Lambda(w)$ containing $w$ such that*

*(i)* $h(\partial\Omega \cap B(w,r), \Lambda \cap B(w,r)) \leq \delta r$

*(ii)* $\{x \in \Omega \cap B(w,r/2) : d(x,\partial\Omega) \geq 2\delta r\} \subset$ *one component of* $\mathbb{R}^n \setminus \Lambda$.

*where $h$ denotes the Hausdorff distance between two sets $E, F \subset \mathbb{R}^n$, i.e.,*

$$h(E,F) = \max(\sup\{d(x,E) : x \in F\}, \sup\{d(x,F) : x \in E\}).$$

*Proof.* We assume the setting of a local coordinate chart containing $w$ with a $C^{1,\alpha}$-function $\phi$, such that $w = (w', \phi(w'))$, and we consider the hyperplane

$$\Lambda = \{y = (y', y_n) \in \mathbb{R}^n : y_n = \phi(w') + \langle \nabla\phi(w'), (y' - w')\rangle\}.$$

Note that this is the tangent plane for $\Omega$ at the point $(w', \phi(w'))$. We claim that for $y' \in \mathbb{R}^{n-1}$ it holds that

$$|\phi(y') - \phi(w') - \langle \nabla\phi(w'), (y' - w')\rangle| \leq c|y' - w'|^{1+\alpha}$$

which is just another way to express the $C^{1,\alpha}$-regularity, i.e., that the magnitude of $|\nabla\phi|$ is not to large. For $\tilde{c} \in (0,1)$ it follows from the mean value theorem that

$$\begin{aligned} |\phi(y') - \phi(w') - \langle \nabla\phi(w'), (y' - w')\rangle| &= |\langle \nabla\phi(\tilde{c}w' + (1-\tilde{c})y') - \nabla\phi(w'), y' - w'\rangle| \\ &\leq c|\tilde{c}w' + (1-\tilde{c}|w' - y'|)|^{\alpha}|y' - w'| \\ &\leq c(|y' - w'| + \tilde{c}|w' - y'|)^{\alpha}|y' - w'| \\ &\leq c|y' - w'|^{\alpha+1}. \end{aligned}$$

Since $\Omega$ is a bounded domain we can apply a covering argument to prove the existence of a uniform constant $c$ that works for each $w \in \partial\Omega$. From the claim it follows that

$$h(\partial\Omega \cap B(w,r), \Lambda \cap B(w,r)) \leq cr^{1+\alpha}.$$

Next we define $\delta = \delta(\bar{r}_0, \alpha) = \bar{r}_0^{\alpha}/c$ and note that for $0 < r < \bar{r}_0$ we obtain *(i)*. The second claim, *(ii)*, follows directly from the first. □

We are now ready to state the main theorem of this section.

**Theorem 7.4.** *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\alpha}$-domain for some $\alpha > 0$. Given $p$, $1 < p < \infty$, there exists $\bar{r}_0 = \bar{r}_0(p,n,\alpha) > 0$ such that the following is true. Let $w \in \partial\Omega, 0 < r < \bar{r}_0$, suppose that $u, v$ are non-negative $p$-harmonic functions in $\Omega \cap B(w,4r)$, $u, v$ are continuous in $\bar{\Omega} \cap B(w,4r)$, and $u = 0 = v$ on $\partial\Omega \cap B(w,4r)$. Then there exists $c = c(p,n,\alpha), 1 \leq c < \infty$, and $\sigma = \sigma(p,n,\alpha) \in (0,1)$, such that if $0 < r < \bar{r}_0$, then*

$$c^{-1}\frac{u(a_{r/c}(w))}{v(a_{r/c}(w))} \leq \frac{u(y)}{v(y)} \leq c\frac{u(a_{r/c}(w))}{v(a_{r/c}(w))},$$

*whenever $y \in \Omega \cap B(w,r/c)$ and*

$$\left|\frac{u(y_1)}{v(y_1)} - \frac{u(y_2)}{v(y_2)}\right| \leq c\frac{u(a_{r/c}(w))}{v(a_{r/c}(w))}\left(\frac{|y_1 - y_2|}{r}\right)^{\sigma}$$

*whenever $y_1, y_2 \in \Omega \cap B(w,r/c)$.*

The proof is divided into a number of steps as follows:

- **Step 1.** In Section 7.2 we begin by proving Theorem 7.4 in the special case of a half space, see Theorem 7.8. One of the key steps of the proof is to make a Schwartz reflection and show that the extended function is $p$-harmonic. This will allow us to apply Lemma 4.4, and in combination with some boundary inequalities and barrier estimates the conclusion of the theorem will follow.
- **Step 2.** In step 2 we will work with estimates for the gradient of a $p$-harmonic function $u$ in a $C^{1,\alpha}$-domain vanishing on $\partial(\Omega \cap B(w, 4r))$, for some $r$, and where $w \in \partial\Omega$. In particular we will prove the so called fundamental inequality which states that for some constants $c$ and $\beta$

$$\beta^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \beta \frac{u(y)}{d(y, \partial\Omega)}$$

  whenever $y \in \Omega \cap B(w, r/c)$. This will be proved by approximating the boundary by a hyperplane according to Lemma 7.3 and then apply Theorem 7.8. We do this in Section 7.3.
- **Step 3.** In this step (Section 7.4) we consider degenerate elliptic operators in weighted Sobolev spaces, for weights belonging to the Muckenhoupt class $A_2$. Under certain assumptions it is possible to derive estimates similar to those in Theorem 7.4 for this class of differential operators. In addition, if $u$ is a $p$-harmonic function vanishing on a portion of the boundary containing $w$ we may extend $|\nabla u|^{p-2}$ to an $A_2(B(w, 2\hat{r}))$-weight for some small $\hat{r}$. We prove this using the estimates obtained in step 2.
- **Step 4.** In this step (Section 7.5) we prove Theorem 7.4. We introduce an elliptic operator $L$ of the type discussed in step 3 such that $u - v$ is a (local) solution to to the equation $Lf = 0$. Using estimates for the gradients obtained in step 3 and 4, $L$ can be locally reduced to a linear and uniformly elliptic operator. This can be used in order prove Theorem 7.4.

## 7.1 Boundary estimates

We begin by stating some boundary estimates which will be used throughout the remaining sections. For the statements and references to the proofs, see Lemma 2.1-2.3 in [13]. The first lemma is a boundary variant of Caccioppoli's inequality.

**Lemma 7.5.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and suppose that $1 < p < \infty$. Let $x \in \partial\Omega, 0 < r < r_0$, and suppose that $u$ is a non-negative continuous $p$-harmonic function in $\bar{\Omega} \cap B(w, 2r)$ and that $u = 0$ on $\partial\Omega \cap B(w, 2r)$. Then*

$$r^{p-n} \int_{\Omega \cap B(w, r/2)} |\nabla u|^p \, dx \leq c \left( \max_{\Omega \cap B(w,r)} u \right)^p .$$

The second lemma states that a $p$-harmonic function that vanished on the boundary is Hölder continuous close to that part of the boundary.

**Lemma 7.6.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and suppose that $1 < p < \infty$. Let $x \in \partial\Omega, 0 < r < r_0$, and suppose that $u$ is a non-negative continuous $p$-harmonic function in $\bar{\Omega} \cap B(w, 2r)$ and that $u = 0$ on $\partial\Omega \cap B(w, 2r)$. Then there exists $\alpha = \alpha(p, n, M) \in (0, 1]$ such that if $x, y \in \Omega \cap B(w, r)$, then*

$$|u(x) - u(y)| \leq c \left( \frac{|x - y|}{r} \right)^\alpha \max_{\Omega \cap B(w,2r)} u$$

**Lemma 7.7.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and let $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $u$ is a non-negative continuous $p$-harmonic function in $\bar{\Omega} \cap B(w, 2r)$ and that $u = 0$ on $\Delta(w, 2r)$. Then* *there exists a constant* $c = c(p, n, M) \in [1, \infty)$ *such that if* $\tilde{r} = r/c$*, then*

$$\max_{\Omega \cap B(w, \tilde{r})} u \leq c u(a_{\tilde{r}}(w)).$$

From the last lemma we note that if $u$ is a non-negative continuous $p$-harmonic function vanishing on a portion of $\partial\Omega$ we are now able to obtain an upper bound of $u$ close to that part of the boundary. This is obviously of great importance since we cannot use the Harnack inequality arbitrarily close to the boundary.

## 7.2 Halfspace

This section is devoted entirely to proving Theorem 7.4 for the special case where $\Omega$ is a halfspace. The key observation for this is that in the case of a completely flat boundary it is possible to make a Schwarz reflection that preserves the $p$-harmonicity of the $p$-harmonic functions. This will make it possible to apply Lemma 4.4 which provides valuable estimates for the gradients of the $p$-harmonic functions. We begin this section by restating Theorem 7.4 tailored to the situation of a halfspace.

**Theorem 7.8.** *Let* $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ *and fix* $1 < p < \infty$*, then there exists a constant* $\tilde{r}_0 = \tilde{r}_0(p, n) > 0$ *such that the following holds. Fix a point* $w \in \partial\mathbb{R}^n_+$*, i.e., such that* $w_n = 0$*, let* $r \in (0, \tilde{r}_0)$ *and suppose that* $u$ *and* $v$ *are non-negative p-harmonic functions in* $\mathbb{R}^n_+ \cap B(w, 4r)$*, continuous in* $\overline{\mathbb{R}^n_+} \cap B(w, 4r)$ *and* $u = 0 = v$ *on* $\partial\mathbb{R}^n_+ \cap B(w, 4r)$*. Then there exist constants* $c = c(p, n) \in [1, \infty)$ *and* $\sigma = \sigma(p, n) \in (0, 1)$*, such that if* $r \in (0, \tilde{r}_0)$*, then*

$$c^{-1} \frac{u(a_{r/c}(w))}{v(a_{r/c}(w))} \leq \frac{u(y)}{v(y)} \leq c \frac{u(a_{r/c}(w))}{v(a_{r/c}(w))},$$

*for* $y \in \mathbb{R}^n_+ \cap B(w, r/c)$ *and*

$$\left| \frac{u(y_1)}{v(y_1)} - \frac{u(y_2)}{v(y_2)} \right| \leq c \frac{u(a_{r/c}(w))}{v(a_{r/c}(w))} \left( \frac{|y_1 - y_2|}{r} \right)^{\sigma}$$

*for* $y_1, y_2 \in \mathbb{R}^n_+ \cap B(w, r/c)$*.*

Given $1 < p < \infty$, $r > 0$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define the rectangles

$$\begin{aligned} Q_r(y) &= \{x : |x_i - y_i| < r, i \in \{1, \dots, n\}\}, \\ Q_r^+(y) &= \{x : |x_i - y_i| < r, i \in \{1, \dots, n-1\}, 0 < |x_n - y_n| < r\}. \end{aligned}$$

Let $u$ be a non-negative $p$-harmonic function in $Q_1^+(0)$, continuous on $\overline{Q_1^+(0)}$ and $u = 0$ on $\partial Q_1^+(0) \cap \{x_n = 0\}$. We extend $u$ to $Q_1(0)$ by a Schwarz reflection:

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n), & x_n \geq 0, \\ -u(x', -x_n), & x_n < 0. \end{cases}$$

for $x = (x', x_n) \in Q_1(0)$. Note that $\tilde{u}$ is continuous in $Q_1(0)$.

**Claim 2.** *$\tilde{u}$ is p-harmonic in* $Q_1(0)$*.*

*Proof of claim.* We note that $\tilde{u} \in W^{1,p}(Q_1(0))$. Next, let $\varphi \in W_0^{1,p}(Q_1(0))$,

$$\phi(x', x_n) = \frac{\varphi(x', x_n) + \varphi(x', -x_n)}{2}$$

and $\theta = \phi + \psi$, i.e.,

$$\psi(x', x_n) = \frac{\varphi(x', x_n) - \varphi(x', -x_n)}{2}.$$

Then $\psi \in W_0^{1,p}(Q_1^+(0))$ and $\psi(x) = 0$ on $Q_1(0) \cap \{x_n = 0\}$ in Sobolev sense. Note that,

$$\nabla \tilde{u}(x) = \begin{cases} (u_{x_1}, \dots, u_{x_{n-1}}, u_{x_n}), & x_n \geq 0, \\ (-u_{x_1}, \dots, -u_{x_{n-1}}, u_{x_n}), & x_n < 0. \end{cases}$$

and

$$\nabla \psi = \frac{1}{2}\left[\varphi_{x_1}(x', x_n) - \varphi_{x_1}(x', -x_n), \dots, \varphi_{x_{n-1}}(x', x_n) - \varphi_{x_{n-1}}(x', -x_n), \varphi_{x_n}(x', x_n) + \varphi_{x_n}(x', -x_n)\right].$$

Hence,

$$\langle \nabla \tilde{u}, \nabla \psi \rangle = \begin{cases} \frac{1}{2}\left[\sum_{i=1}^{n} u_{x_i}\varphi_{x_i}(x', x_n) - \sum_{i=1}^{n-1} u_{x_i}\varphi_{x_i}(x', -x_n) + u_{x_n}\varphi_{x_n}(x', -x_n)\right], x_n \geq 0, \\ \frac{1}{2}\left[-\sum_{i=1}^{n-1} u_{x_i}\varphi_{x_i}(x', x_n) + u_{x_n}\varphi_{x_n}(x', x_n) + \sum_{i=1}^{n} u_{x_i}\varphi_{x_i}(x', -x_n)\right], x_n < 0, \end{cases}$$

which implies that

$$\int_{Q_1(0)\setminus Q_1^+(0)} |\nabla \tilde{u}|^{p-2}\langle \nabla \tilde{u}, \nabla \psi \rangle \, dx = \int_{Q_1^+(0)} |\nabla \tilde{u}|^{p-2}\langle \nabla \tilde{u}, \nabla \psi \rangle \, dx = 0,$$

since $u$ is $p$-harmonic and $\psi \in W_0^{1,p}(Q_1^+(0))$. Thus,

$$\int_{Q_1(0)} |\nabla \tilde{u}|^{p-2}\langle \nabla \tilde{u}, \nabla \psi \rangle \, dx = 0. \tag{37}$$

What is left is to show is the corresponding equality for $\phi$. Note that $|\nabla \tilde{u}|$ and $\tilde{u}_{x_n}$ are even functions of $x_n$ and that $\tilde{u}_{x_1}, \dots, \tilde{u}_{x_{n-1}}$ are odd functions of $x_n$. Furthermore, $\phi_{x_1}, \dots, \phi_{x_{n-1}}$ are odd and $\phi_{x_n}$ is even, as functions of $x_n$. Thus, $|\nabla \tilde{u}|^{p-2}\langle \nabla u, \nabla \phi \rangle$ is an odd function of $x_n$, and noting that the domain $Q_1(0)$ is symmetric around $\{x_n = 0\}$, we get

$$\int_{Q_1(0)} |\nabla \tilde{u}|^{p-2}\langle \nabla \tilde{u}, \nabla \phi \rangle \, dx = 0. \tag{38}$$

Thus, by (37) and (38), $\tilde{u}$ is $p$-harmonic in $Q_1(0)$. □

We first prove the boundary Harnack inequality in Theorem 7.8.

**Lemma 7.9.** *Let $1 < p < \infty$ and suppose that $u$ and $v$ are non-negative $p$-harmonic functions in $Q_1^+(0)$, continuous on $\overline{Q_1^+(0)}$ and that $u = 0 = v$ on $\partial Q_1^+(0) \cap \{x_n = 0\}$. Then there exists a constant $c = c(p, n) \in [1, \infty)$ such that*

$$c^{-1}\frac{u(e_n/8)}{v(e_n/8)} \leq \frac{u(x)}{v(x)} \leq c\frac{u(e_n/8)}{v(e_n/8)}$$

*for $x \in Q_{1/8}^+(0)$, where $e_n$ is the unit vector in the $x_n$-direction.*

*Proof.* Note that if the lemma holds for a $p$-harmonic function $u(x)$ and $v(x) = x_n$ (note that then $v$ is $p$-harmoinic as well), then for any $p$-harmonic functions $u$ and $\omega$, we have

$$c^{-2}\frac{u(e_n/8)}{\omega(e_n/8)} \le \frac{u(x)}{\omega(x)} \le c^2\frac{u(e_n/8)}{\omega(e_n/8)},$$

by writing $\frac{u(x)}{\omega(x)} = \frac{u(x)/x_n}{\omega(x)/x_n}$ and using the bounds on $u(x)/x_n$ and $\omega(x)/x_n$. Thus, we are done if we prove the lemma with $v(x) = x_n$.
Given $\tilde{x} = (x', x_n) \in Q^+_{1/4}(0)$, such that $x_n \le 1/100$, we let $\hat{x} = (x', 1/8)$ and let $\hat{u}$ denote the unique $p$-harmonic function in $D = B(\hat{x}, 1/8) \setminus \overline{B(\hat{x}, 1/100)}$ with boundary values $\hat{u} = u(e_n/8)$ on $\partial B(\hat{x}, 1/100)$ and $\hat{u} = 0$ on $\partial B(\hat{x}, 1/8)$. We next prove the following claim.

**Claim 3.**

$$\hat{u}(x) = \begin{cases} a\,|x-\hat{x}|^{(p-n)/(p-1)} + b, & p \ne n \\ a\ln|x-\hat{x}| + b, & p = n, \end{cases}$$

*for some constants a and b. Furthermore, there exists a constant $c = c(n, p)$ such that*

$$c\hat{u}(x) \ge u(e_n/8)x_n, \tag{39}$$

*for $x \in D$, such that $x_n \le 1/100$.*

*Proof of claim.* This is done by comparing with the fundamental solution. We begin with the case $p = n$. Let $a$ and $b$ be such that

$$\begin{cases} a\ln(1/8) + b = 0, \\ a\ln(1/100) + b = u(e_n/8). \end{cases}$$

Then $a = \frac{u(e_n/8)}{\ln(8/100)}$ and $b = -\frac{\ln(1/8)u(e_n/8)}{\ln(8/100)}$.
For the case $p \ne n$, we solve

$$\begin{cases} a\left(\frac{1}{100}\right)^{\frac{p-n}{p-1}} + b = u(e_n/8), \\ a\left(\frac{1}{100}\right)^{\frac{p-n}{p-1}} + b = u(e_n/8). \end{cases}$$

Then,

$$a = \frac{u(e_n/8)}{\left(\frac{1}{100}\right)^{\frac{p-n}{p-1}} - \left(\frac{1}{8}\right)^{\frac{p-n}{p-1}}}, \quad b = \frac{-\left(\frac{1}{8}\right)^{\frac{p-n}{p-1}}u(e_n/8)}{\left(\frac{1}{100}\right)^{\frac{p-n}{p-1}} - \left(\frac{1}{8}\right)^{\frac{p-n}{p-1}}}.$$

We only do the computation for the constant in the case $p < n$, the other case is similar.

$$\hat{u}(x) = \frac{u(e_n/8)}{\left(\frac{1}{100}\right)^{\frac{p-n}{p-1}} - \left(\frac{1}{8}\right)^{\frac{p-n}{p-1}}}\left[|x-\hat{x}|^{\frac{p-n}{p-1}} - \left(\frac{1}{8}\right)^{\frac{p-n}{p-1}}\right]$$

Letting $x = (x_1, \dots, x_n) \to (x', 0)$ in $D$, both $\hat{u}(x)$ and $x_n$ tend to 0, we thus have to check that $\hat{u}$ does not decrease too quickly (at most to the same order as $x_n$). As $\hat{u}$ clearly decreases the fastest along the line $t \mapsto (x', t)$, we need only consider the decay there. Let

$$f(x_n) = \left(\frac{1}{8} - x_n\right)^{\frac{p-n}{p-1}} - \left(\frac{1}{8}\right)^{\frac{p-n}{p-1}}$$

for $x_n \le 1/100$. Then,

$$f'(x_n) = \frac{n-p}{p-1}\left(\frac{1}{8} - x_n\right)^{\frac{1-n}{p-1}} \ge c_1(p,n)$$

for $x \le 1/100$. Thus, it is clear that (39) holds. □

Using the Harnack inequality and that $u$ is non-negative, we have that $\hat{u} \le c_1 u$ on $\partial B(\hat{x}, 1/100)$. Hence, $\hat{u} \le c_1 u$ on $\partial D$, so by the comparison principle,

$$\hat{u}(x) \le c_1 u(x), \quad x \in D.$$

Thus,

$$u(e_n/8) \le c\frac{\hat{u}(x)}{x_n} \le cc_1\frac{u(x)}{x_n} = c\frac{u(x)}{x_n} \tag{40}$$

for $x \in D$ such that $x_n \le 1/100$. Since $x \in Q^+_{1/4}(0) \cap \{x_n \le 1/100\}$ the estimate

$$u(e_n/8) \le c\frac{u}{x_n}$$

in fact holds for $x \in Q^+_{1/4}(0)$ such that $x_n \le 1/100$. Using the Harnack inequality, it follows that (40) holds in $Q^+_{1/8}(0)$. Thus we have proved the lower bound.

We now extend $u$ to $Q_1(0)$ by Schwarz reflection, as above. Then, $u$ is $p$-harmonic in $Q_1(0)$, and by Lemma 7.7,

$$\max_{Q_{1/4}(0)} u \le c' u(e_n/8).$$

In combination with Lemma 4.4 and the Harnack inequality we obtain

$$\max_{Q_{1/8}(0)} |\nabla u| \le c \max_{Q_{1/4}(0)} u \le cc' u(e_n/8), \tag{41}$$

where the constant only depend on $n$ and $p$. By an application of the mean value theorem we see that

$$\frac{u(x)}{x_n} \le cu(e_n/8), \tag{42}$$

which together with (40) implies the result. □

**Lemma 7.10.** *Let $1 < p < \infty$ and suppose that $u$ and $v$ are non-negative $p$-harmonic functions in $Q^+_1(0)$, continuous on $\overline{Q^+_1(0)}$ and that $u = 0 = v$ on $\partial Q^+_1(0) \cap \{x_n = 0\}$. Then there exist constants $c = c(p,n) \in [1,\infty)$ and $\sigma = \sigma(p,n) \in (0,1)$ such that*

$$\left|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}\right| \le c\frac{u(e_n/8)}{v(e_n/8)}|x-y|^\sigma$$

*whenever $x, y \in Q^+_{1/32}(0)$.*

*Proof.* As above, we will prove the lemma for $v(x) = x_n$ and derive the general result from that. Note that

$$\frac{u(x)}{x_n} - \frac{u(y)}{y_n} = \int_0^1 u_{x_n}(x', tx_n) - u_{x_n}(y', ty_n)dt,$$

for $x, y \in Q^+_{1/32}(0)$. We want to apply Lemma 4.4 again, and therefore we extend $u$ to $Q_1(0)$ by Schwarz reflection. Thus,

$$\begin{aligned}
\left|\frac{u(x)}{x_n} - \frac{u(y)}{y_n}\right| &\le \int_0^1 |\nabla u(x', tx_n) - \nabla u(y', ty_n)|dt \\
&\le c \max_{B(0,1/4)} |\nabla u| \int_0^1 |(x', tx_n) - (y', ty_n)|^\sigma dt \\
&\le \tilde{c} u(e_n/8) \int_0^1 \left[|x' - y'|^2 + t(x_n - y_n)^2\right]^{\frac{\sigma}{2}} dt \\
&\le \hat{c} \frac{u(e_n/8)}{1/8} |x - y|^\sigma,
\end{aligned}$$

where we used Lemma 4.4 in the second inequality and (41) in the third.
Next, we show that this implies the result.

$$\begin{aligned}
\left|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}\right| &= \left|\frac{\frac{u(x)}{x_n}}{\frac{v(x)}{x_n}} - \frac{\frac{u(y)}{y_n}}{\frac{v(y)}{y_n}}\right| = \left|\frac{\frac{u(x)}{x_n} - \frac{u(y)}{y_n}}{\frac{v(x)}{x_n}} + \frac{\frac{u(y)}{y_n}}{\frac{v(x)}{x_n}} - \frac{\frac{u(y)}{y_n}}{\frac{v(y)}{y_n}}\right| \\
&\le \frac{1}{\frac{v(x)}{x_n}} \left|\frac{u(x)}{x_n} - \frac{u(y)}{y_n}\right| + \frac{u(y)}{y_n} \left|\frac{1}{\frac{v(x)}{x_n}} - \frac{1}{\frac{v(y)}{y_n}}\right| \\
&\le \frac{1}{\frac{v(x)}{x_n}} c \frac{u(e_n/8)}{1/8} |x - y|^\sigma + \frac{u(y)}{y_n} \frac{\left|\frac{v(y)}{y_n} - \frac{v(x)}{x_n}\right|}{\frac{v(x)}{x_n}\frac{v(y)}{y_n}} \\
&\le c \frac{u(e_n/8)}{v(e_n/8)} |x - y|^\sigma + \frac{\frac{u(y)}{y_n}}{\frac{v(x)}{x_n}\frac{v(y)}{y_n}} \cdot c \frac{v(e_n/8)}{1/8} |x - y|^\sigma \\
&\le c \frac{u(e_n/8)}{v(e_n/8)} |x - y|^\sigma + \tilde{c} \frac{u(e_n/8)}{v(e_n/8)} |x - y|^\sigma \\
&= C \frac{u(e_n/8)}{v(e_n/8)} |x - y|^\sigma,
\end{aligned}$$

where we used Lemma 7.9 for the third and fourth inequality. □

*Proof of Theorem 7.8.* This is an immediate consequence of Lemma 7.9, Lemma 7.10 and Harnack's inequality. □

## 7.3 The fundamental inequality for the gradient of a $p$-harmonic function

In this section we will establish an inequality for the gradient of a positive $p$-harmonic function, $u$, vanishing on a portion of the boundary. Let $\Omega$ be a $C^{1,\alpha}$-domain for $\alpha \in (0, 1]$ and $u$ be $p$-harmonic in $\Omega \cap B(w, 4r)$ and continuous up to and vanishing on $\partial(\Omega \cap B(w, 4r))$. Then, for some constants $c$ and $\beta$ it holds that

$$\beta^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \beta \frac{u(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap B(w, r/c)$. The idea is to first prove the inequality for the case where $\Omega$ is a half-space and then prove the general case by approximating the boundary with hyperplanes according to Lemma 7.3 and use the translation and rotation invariance of the $p$-Laplace operator. In combination with the rotation and translation invariance of the $p$-Laplace operator will result in a situation similar to the half space. These estimates will be used in the next section when we connect the boundary behaviour of $p$-harmonic functions to linear degenerate elliptic operators in weighted Sobolev spaces. We begin this section by stating and proving a technical lemma (Lemma 3.18 in [13]).

**Lemma 7.11.** *Let $G \subset \mathbb{R}^n$ be an open set, suppose that $1 < p < \infty$. Also, suppose that $u_1$ and $u_2$ are non-negative $p$-harmonic functions in $G$. Let $\tilde{a} \geq 1$, $y \in G$ and assume that*

$$\frac{1}{\tilde{a}} \frac{u_1(y)}{d(y, \partial G)} \leq |\nabla u_1(y)| \leq \tilde{a} \frac{u_1(y)}{d(y, \partial G)}. \tag{43}$$

*Let $\tilde{\varepsilon}^{-1} = (c\tilde{a})^{(1+\sigma)/\sigma}$ where $\sigma$ is as in Lemma 4.4. Then the following is true for $c = c(p, n)$ sufficiently large. If*

$$(1 - \tilde{\varepsilon})L \leq \frac{u_2}{u_1}(1 + \tilde{\varepsilon})L \text{ in } B(y, \frac{1}{100} d(y, \partial G)) \tag{44}$$

*for some $L$ such that $0 < L < \infty$, then*

$$\frac{1}{c\tilde{a}} \frac{u_2(y)}{d(y, \partial G)} \leq |\nabla u_2(y)| \leq c\tilde{a} \frac{u_2(y)}{d(y, \partial G)}.$$

*Proof.* Let $z_1, z_2 \in \bar{B}(y, td(y, \partial G))$ where $0 < t \leq 10^{-3}$. By first applying Lemma 4.4 and thereafter Harnack's inequality it follows that

$$|\nabla u_i(z_1) - \nabla u_i(z_2)| \leq \frac{ct^\sigma}{d(y, \partial G)} \max_{B(y, 2td(y, \partial G))} u_i(y)/d(y, \partial G) \tag{45}$$

$$\leq c^2 t^\sigma u_i(y) \tag{46}$$

for $i \in \{1, 2\}$ and $c = c(p, n)$. From this the upper bound of the gradient of $u_2$ at $y$ follows directly. To establish the lower bound we will argue by contradiction. Thus, we assume that

$$|\nabla u_2(y)| \leq \delta u_2(y)/d(y, \partial G) \tag{47}$$

for some $\delta > 0$ which will be chosen later. We use (45) with $z_1 = z, z_2 = y$, (47) and the reversed triangle inequality to obtain

$$|\nabla u_2(z)| \leq (\delta + c^2 t^\sigma) u_2(y)/d(y, \partial G)$$

whenever $z \in B(y, td(y, \partial G))$ and $c = c(p, n)$. For $\tilde{y} \in \partial B(y, td(y, \partial G))$ we define the line segment from $y$ to $\tilde{y}$ as $r(s) = y + (\tilde{y} - y)s$ for $s \in [0, 1]$. Note that $|r'(s)| = |\tilde{y} - y| = td(y, \partial G)$. Integrating along $r(s)$ with $t = \delta^\sigma$ we see that

$$\begin{aligned} |u_2(\tilde{y}) - u_2(y)| &\leq \left| \int_0^1 \frac{d}{ds} \left( u_2(r(s)) \right) ds \right| \\ &\leq \int_0^1 |\nabla u_2(r(s))||r'(s)| ds \\ &\leq c' \delta^{1+1/\sigma} u_2(y) \end{aligned}$$

where $c' = c'(p,n)$. We let $\eta = \nabla u_1(y)/|\nabla u_1(y)|$ and use the Cauchy-Schwarz inequality, (45), and (43) to conclude that

$$\begin{aligned}\langle \nabla u_1(z), \eta\rangle &= \langle \nabla u_1(z) - \nabla u_1(y), \eta\rangle + |\nabla u_1(y)| \\ &\geq |\nabla u_1(y)| - (|\nabla u_1(z) - \nabla u_1(y)|) \\ &\geq |\nabla u_1(y)| - c^2\delta \frac{u_1(y)}{d(y,\partial G)} \\ &\geq (1 - c\tilde{a}\delta)|\nabla u_1(y)|,\end{aligned}$$

so

$$\langle \nabla u_1(z), \eta\rangle \geq (1 - c\tilde{a}\delta)|\nabla u_1(y)| \tag{48}$$

for $c = c(p,n)$ whenever $z \in \bar{B}(y, \delta^{1/\sigma} d(y,\partial G))$. Next, we let $\tilde{y} = y + \delta^{1/\sigma} d(y,\partial G)\eta$ and let

$$r(s) = y + \delta^{1/\sigma} d(y,\partial G)\eta s \text{ for } s \in [0,1] \tag{49}$$

denote the line segment from $y$ to $\tilde{y}$ and note that $r'(s) = \delta^{1/\sigma} d(y,\partial G)\eta$. By integrating along the line segment and using (48) and (43) we obtain

$$\begin{aligned}u_1(\tilde{y}) - u_1(y) &= \int_0^1 \frac{d}{ds}(u_1(r(s)))ds \\ &= \delta^{1/\sigma} d(y,\partial G) \int_0^1 \langle \nabla u_1(r(s)), \eta\rangle dt \\ &\geq \delta^{1/\sigma} d(y,\partial G) \int_0^1 \langle \nabla u_1(r(s)), \eta\rangle ds \\ &\geq \delta^{1/\sigma} d(y,\partial G) \int_0^1 (1 - c\tilde{a}\delta)|\nabla u_1(y)| ds \\ &\geq \delta^{1/\sigma} d(y,\partial G)(1 - c\tilde{a}\delta)\frac{u_1(y)}{\tilde{a} d(y,\partial G)}.\end{aligned}$$

If we let $\delta \leq (2c\tilde{a})^{-1}$ we find that

$$(u_1(\tilde{y}) - u_1(y)) \geq \frac{\delta^{1/\sigma}}{2\tilde{a}}. \tag{50}$$

From (48) and (50) we see that

$$\begin{aligned}u_2(\tilde{y}) &\leq (1 + c'\delta^{1+1/\sigma})u_2(y) \\ u_1(\tilde{y}) &\geq (1 + \delta^{1/\sigma}/(2\tilde{a}))u_1(y)\end{aligned}$$

and in combination with (44) we obtain

$$(1-\tilde{\varepsilon})L \leq \frac{u_2(\tilde{y})}{u_1(\tilde{y})} \leq \frac{\left(1 + c'\delta^{1+1/\sigma}\right)}{\left(1 + \delta^{1/\sigma}/(2\tilde{a})\right)}(1+\tilde{\varepsilon})L < (1-\tilde{\varepsilon})L \tag{51}$$

if $1/(\tilde{a}c)^{1/\sigma} \geq \delta^{1/\sigma} \geq \tilde{a}c\tilde{\varepsilon}$ for $c = c(p,n)$ sufficiently large. This holds since

$$\begin{aligned}\frac{\left(1 + c'\delta^{1+1/\sigma}\right)}{\left(1 + \delta^{1/\sigma}/(2\tilde{a})\right)}(1+\tilde{\varepsilon}) - (1-\tilde{\varepsilon}) &= \frac{\tilde{\varepsilon}\left(2 + c'\delta^{1+1/\sigma} + \delta^{1/\sigma}/(2\tilde{a})\right) + c'\delta^{1+1/\delta} - \delta^{1/\sigma}/(2\tilde{a})}{\left(1 + \delta^{1/\sigma}/(2\tilde{a})\right)} \\ &\leq \frac{\frac{\delta^{1/\sigma}}{\tilde{a}}\left(\frac{1}{c}\left(2 + \delta^{1+1/\sigma} + \delta^{1/\sigma}/(2\tilde{a}) + c'\right) - \frac{1}{2}\right)}{\left(1 + \delta^{1/\sigma}/(2\tilde{a})\right)} \\ &< 0\end{aligned}$$

for a sufficiently large value of $c$. Next we choose $\tilde{\varepsilon}^{-1} = (c\tilde{a})^{(1+\sigma)/\sigma}$ and $\delta^{-1} = c\tilde{a}$ and note that these choices satisfy the earlier claims. Thus, we have reached a contradiction which implies that (47) is false and the conclusion of the lemma follows. □

This lemma will be used in order to prove the fundamental inequality in a half space.

**Lemma 7.12.** *Let $1 < p < \infty$. Suppose that $u$ is a $p$-harmonic function in $Q_1^+(0)$, continuous on the closure of $Q_1^+(0)$ and that $u = 0$ on $\partial Q_1^+(0) \cap \{y_n = 0\}$. Then there exists $\tilde{c} = \tilde{c}(p,n)$ and $\beta = \beta(p,n)$ such that*

$$\beta^{-1}\frac{u(y)}{y_n} \leq |\nabla u(y)| \leq \beta\frac{u(y)}{y_n} \quad \text{when} \quad y \in Q^+_{1/\tilde{c}}(0) \tag{52}$$

*Proof.* We begin by applying Lemma 7.9 and Lemma 7.10 to the $p$-harmonic functions $u_1 = y_n$ and $u_2 = u$. It follows that

$$\left|\frac{u_1(y_1)}{u_2(y_1)} - \frac{u_1(y_2)}{u_2(y_2)}\right| \leq c'\frac{u_1(y_2)}{u_2(y_2)}|y_1 - y_2|^\sigma \tag{53}$$

for $y_1, y_2 \in Q^+_{1/4}(0)$ and $\sigma = \sigma(p,n) \in (0,1)$. We note that the claim in Lemma 7.10 is for $y_1, y_2 \in Q^+_{1/32}(0)$ but due to scale invariance and the fact that we are looking for an estimate for $Q^+_{1/\tilde{c}}$ it will not make any difference. For simplicity we denote $Q^+_{1/4}(0)$ by $D$. If $y_2 \in Q^+_{1/8}(0)$ we note that $d(y_2, \partial D)$ is always attained for $x \in \partial D \cap \{x_n = 0\}$, Therefore it follows that

$$\frac{1}{\tilde{a}}\frac{u_1(y_2)}{d(y_2,\partial D)} \leq |\nabla u_1(y_2)| \leq \tilde{a}\frac{u_1(y_2)}{d(y_2,\partial D)} \tag{54}$$

where $\tilde{a} = \tilde{a}(n)$ since $u_1(y_2)$ will go to zero at the same speed as $d(y_2,\partial D)$. We restrict $y_2$ according to

$$d(y_2,\partial D) \leq 100(\tilde{\varepsilon}/(2c'))^{1/\sigma}, \tag{55}$$

where $\tilde{\varepsilon}$ is as in Lemma 7.11. We claim that

$$(1-\tilde{\varepsilon}/2)\frac{u_1(y_2)}{u_2(y_2)} \leq \frac{u_1(y_1)}{u_2(y_1)} \leq (1+\tilde{\varepsilon}/2)\frac{u_1(y_2)}{u_2(y_2)} \tag{56}$$

whenever $y_1 \in B(y_2, d(y_2,\partial D)/100)$. From (53) it follows that

$$\left(1 - c'|y_1 - y_2|^\sigma\right)\frac{u_1(y_2)}{u_2(y_2)} \leq \frac{u_1(y_1)}{u_2(y_1)} \leq \left(1 + c'|y_1 - y_2|^\sigma\right)\frac{u_1(y_2)}{u_2(y_2)}. \tag{57}$$

Furthermore, from the restriction in (55) we note that

$$\left(\frac{d(y_2,\partial D)}{100}\right)^\sigma \leq \tilde{\varepsilon}/(2c')$$

and for $|y_1 - y_2| \leq d(y_2,\partial D)/100$ we see that $c'|y_1 - y_2|^\sigma \leq \tilde{\varepsilon}/2$. The claim follows by combining this estimate with (57). Finally, we note that by (54) and (56) we can apply Lemma 7.11 and thus we see that (52) holds at $y_2$, so we have indeed proven the existence of the constant $\tilde{c}$ in Lemma 7.12 such that (52) holds for $y \in Q^+_{1/\tilde{c}}(0)$. □

We next prove the fundamental inequality in $C^{1,\alpha}$-domains (see Lemma 3.35 in [13]).

**Lemma 7.13.** *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\alpha}$-domain for some $\alpha \in (0,1]$. Given $p, 1 < p < \infty$ there exists $\bar{r}_0 = \bar{r}_0(p,n,\alpha) > 0$ such that the following is true. Let $w \in \partial\Omega$, and $0 < r < \bar{r}_0$. Suppose that $u$ is a positive $p$-harmonic function in $\Omega \cap B(w,4r)$, that $u$ is continuous in $\bar{\Omega} \cap B(w,4r)$, and $u = 0$ on $\partial\Omega \cap B(w,4r)$. There exists $\tilde{c} = \tilde{c}(p,n,\alpha)$ and $\bar{\beta} = \bar{\beta}(p,n,\alpha)$ such that*

$$\bar{\beta}^{-1} \frac{u(y)}{d(y,\partial\Omega)} \leq |\nabla u(y)| \leq \bar{\beta} \frac{u(y)}{d(y,\partial\Omega)}$$

*whenever $y \in \Omega \cap B(w, r/\tilde{c})$.*

*Proof.* We begin by continuously extending $u$ to $B(w,4r)$ by letting $u = 0$ in $B(w,4r) \setminus \Omega$. The idea of the proof is to use the $C^{1,\alpha}$- regularity of the domain to obtain a situation similar to when the domain is a halfplane so that Lemma 7.12 is applicable. Therefore we let $c_1$ be a constant which will later be used as $\tilde{c}$ in Lemma 7.12. Due to Lemma 7.7 and Harnack's inequality it is possible to find $c' \geq 100c_1$ such that for each $\tilde{y} \in \Omega \cap B(w, r/c')$ and $s = 4c_1 d(\tilde{y},\partial\Omega)$ and $z \in \partial\Omega$ with $|\tilde{y} - z| = d(\tilde{y},\partial\Omega)$ it holds that

$$\max_{B(z,4s)} u \leq c u(\tilde{y}) \tag{58}$$

where $c = c(p,n)$. By Lemma 7.3 we have that for $\delta << 1$, there exists $0 < \bar{r}_0 << 1$ such that for $4s \leq \bar{r}_0$ there exists a hyperplane $\Lambda$ containing $z$ such that

$$h(\partial\Omega \cap B(z,4s), \Lambda \cap B(z,4s)) \leq 4\delta s \tag{59}$$

$$\{y \in \Omega \cap B(z,4s) : d(y,\partial) \geq 8\delta s\} \subset \text{ one component of } \mathbb{R}^n \setminus \Lambda. \tag{60}$$

Using the invariance properties of the $p$-Laplace operator we may assume that $\Lambda = \{(y', y_n) : y' \in \mathbb{R}^{n-1}, y_n = 0\}$ and that

$$\{y \in \Omega \cap B(z,4s) : d(y,\partial\Omega) \geq 8\delta s\} \subset \{y \in \mathbb{R}^n : y_n > 0\}$$

We next define another hyperplane $\Lambda'$ and a domain $\Omega'$ as follows:

$$\Lambda' = \{(y',0) + 20\delta s e_n, y' \in \mathbb{R}^{n-1}\},\ \Omega' = \{y \in \mathbb{R}^n : y_n > 20\delta s\}$$

and clearly $\Omega' \cap B(z,2s) \subset \Omega \cap B(z,2s)$. For an illustration, see Figure 2. We define $v$ to be the $p$-harmonic function in $\Omega' \cap B(z,2s)$ with continuous boundary values

$$v(y) = u(y) \text{ whenever } y \in \partial[\Omega' \cap B(z,2s)] \text{ and } y_n > 40\delta s$$
$$v(y) = 0 \text{ whenever } y \in \partial[\Omega' \cap B(z,2s)] \text{ and } y_n < 30\delta s$$

such that $v \leq u$ on $\partial[\Omega' \cap B(z,2s)]$. From the comparison principle it follows that $v \leq u$ in $\Omega' \cap B(z,2s)$. We note that the points on the boundary of $\partial[\Omega' \cap B(z,2s)]$ where $u \neq v$ lie within $80\delta s$ from $\partial\Omega$ where $u = 0$ and therefore we can use Lemma 7.6 and (58) to conclude that

$$u \leq v + c\delta^{\tilde{\alpha}} u(\tilde{y}) \text{ in } \partial\Omega' \cap B(z,2s).$$

where we have used $\tilde{\alpha}$ to denote the exponent from Lemma 7.6. From the maximum principle for $p$-harmonic functions it follows that

$$v \leq u \leq v + c\delta^{\alpha} u(\tilde{y}) \text{ in } \Omega' \cap B(z,2s).$$

By applying the Harnack inequality we see that

$$1 \leq \frac{u(y)}{v(y)} \leq (1 - c\delta^{\alpha})^{-1} \text{ in } \Omega' \cap B(z,2s).$$

We next apply Lemma 7.12 and use that $d(\tilde{y}, \partial\Omega) \approx d(\tilde{y}, \partial\Omega')$ which gives us

$$\beta^{-1} \frac{v(\tilde{y})}{d(\tilde{y}, \partial\Omega)} \leq |\nabla v(\tilde{y})| \leq \beta \frac{v(\tilde{y})}{d(\tilde{y}, \partial\Omega)}$$

for $\beta = \beta(p, n)$. We note that if we choose $\delta = \delta(\bar{r}_0)$ (i.e. $\bar{r}_0$) small enough, we can apply Lemma 7.11 with $G = \Omega' \cap B(z, 2s)$ and $\tilde{a} = \beta$. Using that it follows that

$$\bar{\beta}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{\beta} \frac{u(y)}{d(y, \partial\Omega)}$$

for $\bar{\beta} = \bar{\beta}(p, n, \alpha)$. □

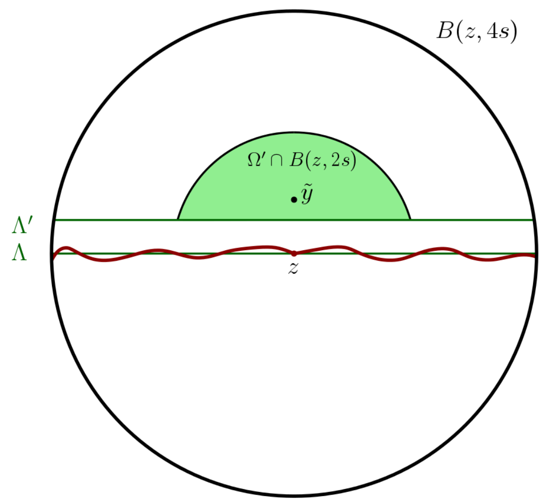


Figure 2: Lemma 7.13.

We finish this section by stating and proving a lemma which will be used when we extend $|\nabla u|^{p-2}$ to an $A_2$-weight in the next section.

**Lemma 7.14.** *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\alpha}$-domain for some $\alpha \in (0, 1]$. Given, $p, 1 < p < \infty$, and $0 < \varepsilon < 1$, there exists $\bar{r}_0 = \bar{r}_0(p, n, \alpha, \varepsilon) > 0$ such that the following is true. Let $w \in \partial\Omega, 0 < r < \bar{r}_0$, suppose that $u$ is a non-negative $p$-harmonic function in $\Omega \cap B(w, r)$, $u$ is continuous in $\bar{\Omega} \cap \bar{B}(w, r)$, and that $u = 0$ on $\partial[\Omega \cap B(w, r)]$. Then there exists $c = c(p, n, \alpha, \varepsilon), 1 \leq c < \infty$, such that*

$$c^{-1} \left(\frac{\tilde{r}}{r}\right)^{1+\varepsilon} \leq \frac{u(a_{\tilde{r}}(w))}{u(a_r(w))} \leq c \left(\frac{\tilde{r}}{r}\right)^{1-\varepsilon}$$

*whenever $0 < \tilde{r} < r/4$.*

*Proof.* As a first step we extend $u$ continuously to $B(w, r)$ by defining $u = 0$ in $B(w, r) \subset \Omega$. Note that $u$ is not $p$-harmonic in $B(w, r)$. By applying Lemma 7.3 we know that given $\tilde{\delta} << 1$, there exists $\bar{r}_0 = \bar{r}_0(n, \alpha) << 1$ such that for $r < \bar{r}_0$, there exists a hyperplane $\Lambda$ containing $w$ such that

$$h(\partial\Omega \cap B(w, r), \Lambda \cap B(w, r)) \leq \tilde{\delta} r.$$

We note that it is sufficient to prove the lemma for $r = \bar{r}_0$. Using the invariance properties of the $p$-Laplace equation we, without loss of generality, may assume that $u(a_4(0)) = 1, r = 4$ and $w = 0$ so that

$$h(\Lambda \cap B(0,4), \partial\Omega \cap B(0,4)) \leq 4\tilde{\delta}$$

where $\Lambda = \{y \in \mathbb{R}^n : y_n = 0\}$. We let $\delta = 4\tilde{\delta}$ and note that we may assume that

$$B(0,4) \cap (x', x_n) : x_n \geq 2\delta\} \subset \Omega \tag{61}$$

$$B(0,4) \cap (x', x_n) : x_n \leq -2\delta\} \subset \mathbb{R}^n \setminus \Omega. \tag{62}$$

Thus, due to the invariance properties it is enough to prove that

$$c^{-1}\tilde{r}^{1+\epsilon} \leq u(a_{\tilde{r}}(0)) \leq c\tilde{r}^{1-\epsilon} \text{ when } 0 < \tilde{r} < 1. \tag{63}$$

The proof will be conducted using barrier type estimates. We begin by defining the rectangles

$$Q^+ = \{x : |x'| < 1, 2\delta < x_n < 1\}$$
$$Q^- = \{x : |x'| < 1, -2\delta < x_n < 1\}$$

and introduce two $p$-harmonic functions, $u^+$ and $u^-$ on $Q^+$ and $Q^-$, respectively, as follows. We let $u^+$ be the $p$-harmonic function on $Q^+$ with the (continuous) boundary values

$$\begin{cases} u^+(x) = u(x), & \text{if } x \in \partial Q^+ \cap \{x : x_n \geq 4\delta\} \\ u^+(x) = \frac{(x_n - 2\delta)}{2\delta} u(x) & \text{if } x \in \partial Q^+ \cap \{x : 2\delta < x_n < 4\delta\} \\ u^+(x) = 0, & \text{if } x \in \partial Q^+ \cap \{x_n = \delta\} \end{cases}$$

Furthermore, we let $u^-$ denote the $p$-harmonic function in $Q^-$ with $u^-(x) = u(x)$ on $\partial Q^-$. It follows directly from the comparison principle that $u^+ \leq u \leq u^-$. In addition, we also define the $p$-harmonic functions $v^+$ and $v^-$ in $\mathbb{R}^n$ such that $v^+(x) = x_n - 2\delta$ and $v^-(x) = x_n + 2\delta$.

Take $0 < \hat{r} < 1$. We first show the upper bound in (63). This will be achieved using induction on the scale of the rectangle. We want to show that

$$u(a_{\delta^k(0)}) \leq (c\delta)^k \tag{64}$$

for $c = c(p, n)$ and every integer $k \geq 1$. Since $\delta << 1$ we can use Lemma 7.9 on $u^-$ and $v^-$ to conclude that

$$u^-(x) \leq cu^-(e_n/8)v^-(x), \text{ when } x \in \Omega \cap B(0, 1/16).$$

We see that

$$u(x) \leq u^-(x) \leq \max_{\partial Q^-} u^-(x) = \max_{\partial Q^-} u(x) \leq cu(a_4(0)) = c \tag{65}$$

whenever $x \in Q^-$, where we have used the comparison principle for the first inequality and Lemma 7.7 and the Harnack inequality for the last inequality. Thus, $u^-(e_n/8) \leq c$ and in combination with (65) we obtain

$$u(x) \leq u^-(x) \leq c(x_n + 2\delta) \text{ whenever } x \in \Omega \cap B(0, 1/16).$$

Hence it follows that $u(a_\delta(0)) \leq c\delta$. Next, we proceed with the induction step and therefore assume that (64) holds. Using Lemma 7.3 we see that there exists a hyperplane $\Lambda'$ containing $w = 0$ such that

$$h(\Lambda \cap B(0, 4\delta^k), \partial\Omega \cap B(0, 4\delta^k)) \leq 4\tilde{\delta}\delta^k = \delta^{k+1}$$

If we now replace $\Lambda$ and 4 with $\Lambda'$ and $4\delta^k$, respectively, and repeat the above argument on a smaller scale $Q_k^-$ with $u_k^-$ defined analogous to $u^-$ but on $Q_k^-$, and $v_k^- = x_n + 2\delta^{k+1}$ we find that

$$u(x) \le u_k^-(x) \le c\frac{u_k^-(a_{\delta^k}(0))}{v_k^-(a_{\delta^k}(0))}v_k^-(x) \le c\frac{(x_n + 2\delta^{k+1})}{\delta^k + 2\delta^{k+1}}u_k^-(a_{\delta^k}(0)) \le c\frac{(x_n + 2\delta^{k+1})}{\delta^k + 2\delta^{k+1}}u(a_{\delta^k}(0))$$

whenever $x \in \Omega \cap B(0, 1/\tilde{c}(k))$. Thus, we can conclude that $u(a_{\delta^{k+1}}(0)) \le \hat{c}\delta u(a_{\delta^k}(0)) \le (\hat{c}\delta)^k$, which finishes the induction argument. We choose $\delta$ such that $\delta^{-\varepsilon} \ge \hat{c}$ where $\hat{c}$ is the same constant as in (64). Furthermore, let $k$ be the smallest integer so that $\delta^k \le \hat{r}$ holds. For a sufficiently small value of $\delta$ we can apply the Harnack inequality to $a_{\hat{r}}(0)$ and $a_{\delta^k}(0)$. The length of the Harnack chain depends on $\delta$ which in turn depends on $\bar{r}_0$ so we conclude that for some $c = c(p, n, \alpha, \varepsilon)$ it holds that

$$u(a_{\hat{r}}(0)) \le cu(a_{\delta^k}(0)) \le c\tilde{r}^{1-\varepsilon}$$

and so the right hand inequality i (63) is proved.

We begin the proof of the left hand inequality by applying Lemma 7.9 to $u^+, v^+$ in $Q^+$ from which it follows that, for a sufficiently small value of $\delta$,

$$\frac{u^+(a_{4M\delta})(0)}{v^+(a_{4M\delta})(0)} \approx \frac{u^+(a_{1/8}(0))}{v^+(a_{1/8}(0))} \approx u^+(a_{1/8}(0))$$

where we have used that $v^+(a_{1/8}(0)) \approx 1$. Assuming that $\delta$ is sufficiently small it holds that

$$cu^+(a_{1/8)}(0) \ge \min_{\partial Q^+ \cap \{x_n \ge 4\delta\} \cup \{x_n = 4\delta\}} u^+(x) \ge c'^{-1} \min_{\partial Q^+ \cap \{x_n \ge 4\delta\} \cup \{x_n = 4\delta\}} u(x) \ge \tilde{c}^{-1}u(a_4(0))$$

so $u^+(a_{1/8}(0)) \ge c^{-1}$ where $c = c(p, n)$. Using this inequality in combination with Harnack's inequality and the fact that $v^+(a_{4M\delta}(0)) \approx \delta$ we find that

$$u(a_{8\delta}(0)) \ge c^{-1}u(a_{4M\delta}(0)) \ge c^{-1}u^+(a_{4M\delta}(0)) \ge \hat{c}^{-1}v^+(a_{4M\bar{\delta}}(0)) \ge \bar{c}^{-1}\bar{\delta}$$

for $\bar{c} = \bar{c}(p, n)$. By repeating the induction argument for the right hand inequality we see that

$$u(x) \ge u_k^+(x) \ge c^{-1}\frac{u_k^+(a_{(8\delta)^k}(0))}{v_k^+(a_{(8\delta)^k}(0))}v_k^+(x) \text{ whenever } x \in B(0, 1/\tilde{c}(k))$$

for some small $\tilde{c}(k)$ which implies that $u(a_{(8\delta)^{k+1}}(0)) \ge \bar{c}^{-1}\delta u(a_{(8\delta)^k}(0)) \ge (\bar{c}^{-1}\delta)^k$. Choose $\delta = \delta(p, n, \alpha, \varepsilon)$ small enough for $\bar{c}^{-1}\delta \ge (8\delta)^{1+\varepsilon}$ to hold and let $k$ be the integer such that $\hat{r} \in [(8\delta)^{k+1}, (8\delta)^k]$. We can now use Harnack's inequality and the estimate above to conclude that for some $c = c(p, n, \alpha, \varepsilon)$

$$u(a_{\tilde{r}}(0)) \ge c^{-1}u(a_{(8\delta)^k}(0)) \ge c^{-1}(8\delta)^{k(1+\varepsilon)} \ge c^{-1}\tilde{r}^{(1+\varepsilon)}$$

and so the conclusion of the lemma follows. □

## 7.4 Estimates for degenerate elliptic equations in weighted Sobolev spaces

In this section we will work with a certain class of linear degenerate elliptic differential operators discussed in [4], [2] and [3]. We consider $Lv = \operatorname{div}(B(x)\nabla v)$ where $L$ is the operator and $B(x)$ is a symmetric matrix whose elements satisfy a degenerate ellipticity conditon involving a non-negative function $\lambda(x)$. When the authors to [4] established the Harnack inequality and interior Hölder continuity for weak solutions to the equation $Lv = 0$, they found that the Poincaré inequality and the Sobolev embedding theorem were needed (cf. Section 1.1.2). Therefore, they imposed the restriction that $\lambda(x)$ has to belong to the Muckenhoupt class $A_2$ and showed that these theorems hold for $A_p$-weights. This section is organized as follows: in the first part we state some general results and properties obtained in [4]- [3]. After that we continue by discussing some boundary Harnack inequalities for the ratio of two $L$-harmonic functions, i.e., weak solutions to the equation $Lv = 0$. We end this section by explaining the connection between the theory of linear elliptic differential operators and the theory for $p$-harmonic functions. In particular we prove that if a $p$-harmonic function $u$ satisfies certain assumptions, then $|\nabla u|^{p-2}$ can be extended to an $A_2$-weight. The last part of the section is based on the work in [13] and [14]-[20]. Much of the theory needed in order to understand the proofs is outside the scope of this thesis and therefore most of the proofs will either be left out or sketched.

Throughout this section we assume that $w \in \partial\Omega$ and that $\Omega$ is a Lipschitz domain with Lipschitz constants $M$ and $r_0$. For $0 < r < r_0$, we let $\lambda(x)$ be a non-negative Lebesgue measurable function defined a.e. on $B(w, 2r)$ such that $\lambda \in A_2(B(w, r))$, i.e.,

$$\tilde{r}^{-2n} \sup \left( \int_{B(z,\tilde{r})} \lambda dy \cdot \int_{B(z,\tilde{r})} \lambda^{-1} dy \right) \leq \gamma$$

where $\gamma = \gamma(p, \lambda)$ and the supremum is taken over all balls $B(z, \tilde{r})$ such that $z \in B(w, r)$ and $0 < \tilde{r} \leq r$. Furthermore, let $B(x) = \{b_{ij}(x)\}$ be a symmetric matrix where each $b_{ij}(x)$ is a Lebesgue measurable function defined a.e. on $B(w, 2r)$ for $i, j \in \{1, \ldots, n\}$. In addition, we assume that the coefficients satisfy an ellipticity condition, namely that for $\xi \in \mathbb{R}^n$

$$\beta^{-1} \lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\xi_j \leq \beta\lambda(x)|\xi|^2$$

for almost every $x \in B(w, 2r)$. We note that since $\lambda(x)$ may vanish or be infinite the expression above is a degenerate ellipticity condition. For $x \in \Omega \cap B(w, 2r)$ we define the operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( b_{ij}(x) \frac{\partial}{\partial x_j} \right). \tag{66}$$

We are now ready to state the definition of a weak solution.

**Definition 7.1.** Let $G$ be an open set such that $G \subset B(w, 2r)$. We say that $v \in \widetilde{W}^{1,2}(G)$ is a weak solution to $Lv = 0$ in $G$, or that $v$ is $L$-harmonic in $G$, if the Dirichlet form vanishes, i.e.,

$$\int_G \sum_{i,j=1}^{n} b_{ij}(x) v_{x_i} \varphi_{x_j} \, dx = 0 \tag{67}$$

for all $\varphi \in C_0^\infty(G)$.

In a manner similar to the corresponding one for the $p$-Laplace operator it is possible to show the existence of solutions, the comparison principle, the Harnack inequality, local interior Hölder inequality and the maximum principle, see [4]. Furthermore, in [2] Fabes et al. proved a Wiener criterion for regular boundary points for the $L$-Dirichlet problem which is defined analogous to the $p$-Dirichlet problem but with $L$ instead of $\Delta_p$. Using this it is possible to prove that all boundary points of a Lipschiz domain are regular. The following two lemmas can be found in [13] and summarize some of the work in [4]-[3].

**Lemma 7.15.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constants $M, r_0, w \in \partial\Omega, 0 < r < r_0$, and let $\lambda$ be an $A_2(B(w, 2r))$- weight with constant $\gamma$. Suppose that $v$ is a positive weak solution to $L\omega = 0$ in $\Omega \cap B(w, 2r)$. Then there exists a constant $c \geq 1$, such that if $z \in \Omega, \tilde{r} > 0, B(z, 2\tilde{r}) \subset \Omega \cap B(w, r)$, then*

*(i)*

$$c^{-1}\tilde{r}^2 \int_{B(z,\tilde{r}/2)} |\nabla v|^2 d\mu \le c \left( \int_{B(z,\tilde{r})} d\mu \right) (\max_{B(z,\tilde{r})} v)^2 \le c \int_{B(z,2\tilde{r})} |v|^2 d\mu$$

*where $d\mu = \lambda dx$.*

*(ii) The Harnack inequality holds, i.e.,*

$$\max_{B(z,\tilde{r})} v \le \min_{B(z,\tilde{r})} v.$$

*(iii) There exists $\sigma = \sigma(n, M, \beta, \gamma) \in (0, 1)$ such that if $x, y \in B(z, \tilde{r})$, then*

$$|v(x) - v(y)| \le c \left( \frac{|x-y|}{r} \right)^{\sigma} \max_{B(z,2\tilde{r})} v.$$

**Lemma 7.16.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constants $M, r_0, w \in \partial\Omega, 0 < r < r_0$, and let $\lambda$ be an $A_2(B(w, 2r))$-weight with constant $\gamma$. Suppose that $v$ is a positive weak solution to $L\omega = 0$ in $\Omega \cap B(w, 2r)$ and that $v = 0$ on $\partial\Omega \cap B(w, 2r)$ in the weighted Sobolev sense. Then there exists $1 \le c \le \infty$, such that the following holds with $\tilde{r} = r/c$:*

*(i)*

$$r^2 \int_{\Omega\cap B(w,r/2)} |\nabla v|^2 d\mu \le \int_{\Omega\cap B(w,r)} |v|^2 d\mu$$

*where $d\mu = \lambda dx$.*

*(ii)*

$$\max_{\Omega\cap B(w,\tilde{r})} v \le c v(a_{\tilde{r}}(w))$$

*(iii) There exists $\sigma = \sigma(n, M, \beta, \gamma) \in (0, 1)$ such that if $x, y \in B(w, \tilde{r})$, then*

$$|v(x) - v(y)| \le c \left( \frac{|x-y|}{r} \right)^{\sigma} \max_{\Omega\cap B(w,2\tilde{r})} v.$$

We note that the properties described for the weak solutions to $Lv = 0$ are essentially the same as the ones described in Section 7.1. We next state a boundary Harnack inequality for the ratio of two positive $L$-harmonic functions.

**Lemma 7.17.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with constants $M, r_0$, and let $\lambda$ be an $A_2(\mathbb{R}^n)$-weight with constant $\gamma$. Suppose that $u$ and $v$ are two non-negative weak solutions to $L\omega = 0$ in $\Omega$ and that $u = v = 0$ continuously on $\partial\Omega \cap B(w, 2r)$. Then there exists $c = c(n, M, \gamma, \beta), 1 \le c \le \infty$, such that*

$$c^{-1} \frac{u(a_r(w))}{v(a_r(w))} \le \frac{u(x)}{v(x)} \le c \frac{u(a_r(w))}{v(a_r(w))}$$

*whenever $x \in \Omega \cap B(w, r/4)$.*

The proof of this lemma uses Green functions and an elliptic measures associated to $L$. The existence of those are established in [2] and [3]. We refer to Lemma 3.13 in [14]. We can use this lemma to obtain local Hölder continuity of the ratio of two $L$-harmonic functions near a portion of the the boundary where they are both vanishing.

**Lemma 7.18.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with constants $M, r_0$, and let $\lambda$ be an $A_2(\mathbb{R}^n)$-weight with constant $\gamma$. Suppose that $v_1$ and $v_2$ are two non-negative weak solutions to $L\omega = 0$ in $\Omega$ and that $v_1 = v_2 = 0$ continuously on $\partial\Omega \cap B(w, 2r)$. Then there exist $1 \leq c \leq \infty$, and $\sigma = \sigma(n, M, \gamma, \beta)$, such that*

$$\left|\frac{v_1(x_1)}{v_2(x_1)} - \frac{v_1(x_2)}{v_2(x_2)}\right| \leq c\left(\frac{|x_1 - x_2|}{r}\right)^{\sigma} \frac{v_1(a_r(w))}{v_2(a_r(w))} \tag{68}$$

*whenever $x_1, x_2 \in \Omega \cap B(w, r/16)$.*

*Proof (sketch).* We will prove the Hölder continuity using an iterative argument. We consider $\tilde{x} \in \partial\Omega \cap B(w, r/8)$ and define the functions

$$M(s) = M(s, \tilde{x}) = \sup_{x\in\Omega\cap B(\tilde{x},s)} \frac{v_1(x)}{v_2(x)}, \quad m(s) = m(s, \tilde{x}) = \inf_{x\in\Omega\cap B(\tilde{x},s)} \frac{v_1(x)}{v_2(x)},$$

for $0 < s < r/8$. We easily see that

$$\begin{aligned} \tilde{u}_1 &= v_1 - m(s)v_2 \geq 0 \\ \tilde{u}_2 &= M(s)v_2 - v_1 \geq 0 \end{aligned}$$

in $\Omega \cap B(\tilde{x}, s)$. Since these functions are $L$-harmonic we can apply the Harnack inequality from which it follows that they are either positive or identically zero in $B(\tilde{x}, s)$. First we assume that they are strictly positive and apply Lemma 7.17 two times with $u = \tilde{u}_1, v = v_2$ the first time and $u = \tilde{u}_2, v = v_2$ the second time, which gives

$$\begin{aligned} M(\tilde{s}) - m(s) &\leq \tilde{c}^2(m(\tilde{s}) - m(s)) \\ M(s) - m(\tilde{s}) &\leq \tilde{c}^2(M(s) - M(\tilde{s})) \end{aligned}$$

where $\hat{s} = s/4$ and $\tilde{c}$ is the constant in Lemma 7.17. We next set $\operatorname{osc}(t) = \operatorname{osc}(t, \tilde{x}) = M(t) - m(t)$ and add the above equations to obtain

$$\operatorname{osc}(s) + \operatorname{osc}(\tilde{s}) \leq \tilde{c}^2(\operatorname{osc}(s) - \operatorname{osc}(\tilde{s}))$$

so

$$\operatorname{osc}(\tilde{s}) \leq \frac{\tilde{c}^2 - 1}{\tilde{c}^2 + 1} \operatorname{osc}(s). \tag{69}$$

If $u_i(s) \equiv 0$ for some $i$ we note that this implies that $v_1 \equiv v_2$ and $\operatorname{osc}(s) = 0$ for each $s$ which means that (69) still holds. We need to consider three different cases. In all the cases we assume that $x_1, x_2 \in B(w, r/16)$. At first we assume that $|x_1 - x_2| > r/1000$. This is in some sense a trivial case and it follows from Lemma 7.17 and (69) that (68) holds. If $|x_1 - x_2| \leq r/1000$ and $d(x, \partial\Omega) \geq 2r/1000$ we are on a safe distance from the boundary and therefore do not need (69). Instead it is possible to obtain (68) by using the Harnack inequality and Lemma 7.15 *(iii)*. For the third case we consider $|x_1 - x_2| \leq r/1000$ and $d(x, \partial\Omega) < 2r/1000$. We choose $\tilde{w} \in \partial\Omega \cap B(w, r/8)$ such that $d(x, \tilde{w}) = d(x_1, \partial\Omega)$. By iterating (69) in an argument similar to the one described in the proof of Theorem 3.1, where we start with $s = r/16$ and end with $s \approx 4|x_1 - x_2|$, it is possible to conclude that

$$\left|\frac{v_1(x_1)}{v_2(x_1)} - \frac{v_1(x_2)}{v_2(x_2)}\right| \leq \operatorname{osc}(2|x_1 - x_2|, \tilde{x}) \leq c\left(\frac{|x_1 - x_2|}{r}\right)^{\sigma} \frac{v_1(a_r(w))}{v_2(a_r(w))}$$

for some $\sigma$ such that $0 < \sigma < 1$. □

In order to get rid of the condition that $\lambda(x)$ is a global $A_2$-weight, i.e., that $\lambda(x) \in A_2(\mathbb{R}^n)$, we state the following theorem, which is a localized version of Lemma 7.17 and Lemma 7.18.

**Theorem 7.19.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with constants $M, r_0$. Let $w \in \partial\Omega$, $0 < r < r_0$, and assume that $\lambda$ is an $A_2(B(w, 4r))$-weight with constant $\gamma$. Furthermore, suppose that $v_1$ and $v_2$ are two non-negative weak solutions to $Lv = 0$ in $\Omega \cap B(w, 2r)$ and that $v_1 = 0 = v_2$ continuously on $\partial\Omega \cap B(w, 2r)$. Then there exist $1 \leq c \leq \infty$ and $\alpha = \alpha(n, M, \gamma, \beta) \in (0, 1)$ such that*

$$c^{-1}\frac{u(a_{r/c}(w))}{v(a_{r/c}(w))} \leq \frac{u(x)}{v(x)} \leq c\frac{u(a_{r/c}(w))}{v(a_{r/c}(w))}$$

*and*

$$\left|\frac{u(x_1)}{v(x_1)} - \frac{u(x_2)}{v(x_2)}\right| \leq c\left(\frac{|x_1 - x_2|}{r}\right)^{\alpha} \frac{u(a_{r/c}(w))}{v(a_{r/c}(w))}$$

*whenever $x_1, x_2 \in \Omega \cap B(w, r/c)$.*

The following lemma relates this section to the previous theory about $p$-harmonic functions vanishing on a portion of the boundary. We note that the setting is now a $C^{1,\alpha}$-domain which will allow us to use Lemma 7.14 and Lemma 7.13. (This is Lemma 4.9 in [13].)

**Lemma 7.20.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\alpha}$-domain for some $\alpha > 0$. Given $p$, $1 < p < \infty$, there exists $\bar{r}_0 = \bar{r}_0(p, n, \alpha) > 0$ such that the following is true. Let $w \in \partial\Omega$, $0 < r < \bar{r}_0$, and suppose that $u$ is a positive $p$-harmonic function in $\Omega \cap B(w, 4r)$, $u$ is continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\partial\Omega \cap B(w, 4r)$. Then there exists $\bar{c} = \bar{c}(p, n, \alpha) \geq 1$, such that if $\hat{r} = r/\bar{c}$, then $|\nabla u|^{p-2}$ extends to an $A_2(B(w, 2\hat{r}))$-weight with constant depending on $p, n, \alpha$.*

Note that in order to show that $|\nabla u|^{p-2}$ extends to an $A_2(B(w, 2\hat{r}))$-weight we first need to extend $\nabla u$ to $B(w, 2\hat{r}) \setminus \Omega$. In order to do so we use a Whitney decomposition of $\mathbb{R}^n \setminus \bar{\Omega}$. The Whitney covering lemma states that given an open set $G \subset \mathbb{R}^n$ there exists a decomposition of $G$ into closed dyadic cubes with disjoint interior. Furthermore, the lengths of the sides of the cubes are proportional to the distance from the boundary of $G$. For more information see [8].

*Proof (sketch).* This proof is based on estimates of the gradients obtained from Lemma 7.13 and 7.14. Therefore, we begin by choosing $\bar{r}_0 = \bar{r}_0(p, n, \alpha) > 0$ sufficiently small to enable the use of these lemmas. We let the family of open cubes $\{Q(x_j, r_j)\}$ where $Q_j := Q(x_j, r_j)$ has center $x_j$ and sidelength $r_j$ be a Whitney decomposition of $\mathbb{R}^n \setminus \Omega$ such that

$$\begin{aligned} \cup_j \bar{Q}(x_j, r_j) &= \mathbb{R}^n \setminus \bar{\Omega} \\ Q(x_j, r_j) \cap Q(x_i, r_i) &= \emptyset \text{ when } i \neq j \\ 10^{-4n} d(Q_j, \partial\Omega) \leq r_j &\leq 10^{-2n} d(Q_j, \partial\Omega). \end{aligned}$$

Furthermore, if $Q_j \cap B(w, r) \neq \emptyset$ there exists a $w_j \in \Omega \cap B(w, \tilde{c}r)$ such that

$$d(w_j, \tilde{r}) \approx d(w_j, x) \approx d(x_j, \partial\Omega) \tag{70}$$

where $\tilde{c} = \tilde{c}(n)$, $1 \leq \tilde{c} < \infty$ and the proportionality constants only depend on $n$. The existence of $w_j$ and $\tilde{c}$ follows from the fact that the distances from the boundary to the centers of the cubes depends on $r$ and $n$ and the fact that $\Omega$ is a Lipschitz domain so an open neighbourhood of the boundary can be covered of balls with uniform radius. We want to apply Lemma 7.13 and therefore let $\hat{r}$ be so small that if $Q_j \cap B(w, 8\hat{r}) \neq \emptyset$ it holds that

$$c_*^{-1}\frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq c_*\frac{u(x)}{d(x, \partial\Omega)} \tag{71}$$

for $x \in B(w_j, d(w_j, \partial\Omega)/2)$ and $c_* = \bar{\beta}$ in Lemma 7.13. This can be done by setting $\hat{r} = r/\max\{\hat{c}, \tilde{c}\}$ where $\hat{c} = \hat{c}(p, n)$ is the constant $\hat{c}$ from Lemma 7.13. We note that this construction will later allow us to use Harnack's inequality on $u(x)$. We next define $\lambda(x)$ almost everywhere on $B(w, 4\hat{r}) \setminus \partial\Omega$ as follows:

$$\lambda(x) = \begin{cases} |\nabla u(x)|^{p-2}, & x \in \Omega \cap B(w, 4\hat{r}), \\ |\nabla u(w_j)|^{p-2}, & x \in Q_j \cap B(w, 4\hat{r}). \end{cases}$$

Note that $\lambda(x)$ is not defined in $\partial\Omega \cap B(w, 4\hat{r})$ which has Lebesgue measure zero. By an application of Harnack's inequality to $u(x)$ and (70), (71) it follows that

$$\lambda(x) = \lambda(w_j) \approx \lambda(z) \text{ when } x \in Q_j \text{ and } z \in B(w_j, d(w_j, \partial\Omega)/2). \tag{72}$$

In order to prove the lemma, we need to show that

$$\tilde{r}^{-2n} \int_{B(z,\tilde{r})} \lambda dx \cdot \int_{B(z,\tilde{r})} \lambda^{-1} dx \leq \gamma \tag{73}$$

where $z \in B(w, \hat{r})$ and $\tilde{r} \leq \hat{r}$. As usual we need to split the proof into different cases. If $\tilde{r} < d(z, \partial\Omega)/2$ we can use Harnack's inequality in combination with (71) and (72) to obtain (73). For the case where $d(z, \partial\Omega)/2 \leq \tilde{r} \leq \hat{r}$ we let $\hat{w} \in \partial\Omega$ denote a point such that $d(z, \partial\Omega) = d(z, \hat{w})$. The we see that

$$B(z, \partial\Omega) \subset B(\hat{w}, 3\hat{r}) \subset B(w, 8\hat{r}).$$

and set $c^* = 3\hat{r}/\tilde{r}$. We will again need to consider several cases. We note that when $p = 2$ the statement in (73) follows trivially. We next assume that $p > 2$, and note that we can use Hölder's inequality, Lemma 7.5, Lemma 7.7 and Harnack's inequality to conclude that

$$\begin{aligned} \int_{B(z,\tilde{r})} \lambda\, dx &\leq \int_{B(\hat{w},3\tilde{r})} \lambda\, dx \\ &\leq c \int_{\Omega \cap B(\hat{w},c^*\tilde{r})} |\nabla u|^{p-2}\, dx \\ &\leq c \left( \int_{\Omega \cap B(\hat{w},c^*\tilde{r})} |\nabla u|^p\, dx \right)^{(p-2)/p} \tilde{r}^{2n/p} \\ &\leq c u(a_{\tilde{r}}(\hat{w}))^{p-2} \tilde{r}^{n+2-p}. \end{aligned}$$

We turn to the task of estimating the second integral in (73). In order to do so we note that we can use Lemma 7.14 and Harnack's inequality to see that given $\varepsilon = \min\{1, |p-2|^{-1}\}/20$ it follows that

$$c u(x) \geq u(a_{\tilde{r}}(\hat{w})) \left( \frac{d(x, \partial\Omega)}{\tilde{r}} \right)^{1+\varepsilon} \tag{74}$$

when $x \in \Omega \cap B(\hat{w}, c^*\tilde{r})$. Using (71) and (74) it follows that

$$\begin{aligned} \int_{B(z,\tilde{r})} \lambda^{-1}\, dx &\leq \int_{B(\hat{w},3\tilde{r})} \lambda^{-1}\, dx \\ &\leq c \int_{\Omega \cap B(\hat{w},c^*\tilde{r})} (|\nabla u|^{p-2})^{-1}\, dx \\ &\leq c \int_{\Omega \cap B(\hat{w},c^*\tilde{r})} \left( \frac{d(x, \partial\Omega)}{u(x)} \right)^{p-2} dx \\ &\leq c \tilde{r}^{(1+\varepsilon)(p-2)} u(a_{\tilde{r}}(\hat{w}))^{2-p} \int_{\Omega \cap B(\hat{w},c^*\tilde{r})} d(x, \partial\Omega)^{-\varepsilon(p-2)}\, dx. \end{aligned}$$

Furthermore, using the $C^{1,\alpha}$-regularity of $\Omega$ it is possible to prove that

$$\int_{\Omega\cap B(\hat{w},c^*\tilde{r})} d(x,\partial\Omega)^{-\varepsilon(p-2)}\,dx \le c\tilde{r}^{n-\varepsilon(p-2)}$$

from which it follows that

$$\int_{B(z,\tilde{r})} \lambda^{-1}\,dx \le cu(a_{\tilde{r}}(\hat{w}))^{2-p}\tilde{r}^{n+p-2}$$

and (73) follows by combining the estimates for $\lambda$ and $\lambda^{-1}$ and multiplying with $\tilde{r}^{-2n}$. Due to the symmetry of (73) we can repeat the above argument when $p<2$ and instead work with $2-p<p$. □

## 7.5 Reduction to linear equation and final proof

In this section we will prove Theorem 7.4. Throughout this section we let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{1,\alpha}$-domain with Lipschitz constants $M\ge 2$ and $r_0>0$. Let $w\in\partial\Omega$ and $u$ and $v$ be two positive $p$-harmonic functions in $\Omega\cap B(w,2r)$, where $0<r<r_0$ such that $u$ and $v$ are continuous in $B(w,2r)$ and $u=v=0$ on $B(w,2r)\setminus\Omega$. Since $\Omega$ is a $C^{1,\alpha}$-domain we can use Lemma 7.13 and Lemma 7.20 to assert the validity of the following two assumptions.

**Assumption 1.** *There exists $1\le c_0<\infty$ such that for $\hat{r}=r/c_0$ and for some $\beta\ge 1$ it holds that*

$$\beta^{-1}\frac{u(x)}{d(x,\partial\Omega)} \le |\nabla u(x)| \le \beta\frac{u(x)}{d(x,\partial\Omega)},\quad \beta^{-1}\frac{v(x)}{d(x,\partial\Omega)} \le |\nabla v(x)| \le \beta\frac{v(x)}{d(x,\partial\Omega)}$$

*whenever $x\in\Omega\cap B(w,4\hat{r})$.*

**Assumption 2.** *There exists $\tilde{c}_0\ge 1$ such that for $r^*=\hat{r}/\tilde{c}_0$ we have that $(|\nabla u|+|\nabla v|)^{p-2}$ is an $A_2(B(w,4r^*))$-weight with constant $\gamma$.*

We proceed by defining the partial differential operator

$$L=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(b_{ij}(y)\frac{\partial}{\partial x_j}\right) \tag{75}$$

in $\Omega\cap B(w,4\hat{r})$, where

$$\begin{aligned} b_{ij}(x) &= \int_0^1 \tilde{b}_{ij}(x,t)dt \\ \tilde{b}_{ij}(x,t) &= |\nabla\tilde{u}(x,t)|^{p-4}((p-2)(\partial_{x_i}\tilde{u})(x,t)(\partial_{x_j}\tilde{u})(x,t)+\delta_{ij}|\nabla\tilde{u}(x,t)|^2) \\ \tilde{u}(x,t) &= tu(x)+(1-t)v(x) \end{aligned}$$

for $t\in[0,1]$. Furthermore, we define

$$e(x)=u(x)-v(x) \text{ for } x\in\Omega\cap B(w,4\hat{r}).$$

Next, we will use Assumption 1 to prove two important claims which will be used to prove Theorem 7.21.

**Claim 4.** *$e$ is a weak solution to $Lv=0$ in $\Omega\cap B(w,4\hat{r})$.*

*Proof of claim.* We need to show that $e(x)$ satisfies the conditions in Definition 7.1 in $\Omega\cap B(w,4\hat{r})$. Clearly

$$e(x)\in\widetilde{W}^{1,p}(B(w,4\hat{r}),(|\nabla u|+|\nabla v|)^{p-2})$$

and thus we only need to show that (67) holds. Take $\varphi \in C_0^\infty(B(w, 4\hat{r}))$. For $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ we define $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$ so that $\mathcal{A}_i(\xi) = |\xi|^{p-2}\xi_i$. We note that

$$\int \sum_{i,j} b_{ij} e_{x_i} \varphi_{x_j}\, dx = \int \Big\langle (\sum_i b_{i1} e_{x_i}, \dots, \sum_i b_{in} e_{x_i}), \nabla\varphi \Big\rangle\, dx.$$

Proceeding with the calculations we see that

$$\begin{aligned}
\sum_i b_{ij} e_{x_i} &= (u_{x_j} - v_{x_j}) \int_0^1 |\nabla\tilde{u}(x,t)|^{p-2} dt + (p-2) \int_0^1 |\nabla\tilde{u}(x,t)|^{p-4} \langle \nabla u - \nabla v, \nabla\tilde{u}\rangle (u_{x_j} + (1-t) v_{x_j}) dt \\
&= \int_0^1 \frac{\partial}{\partial t} \Big( |\nabla\tilde{u}|^{p-2} (t u_{x_j} + (1-t) v_{x_j}) \Big)\, dt \\
&= \int_0^1 \frac{\partial}{\partial t} (\mathcal{A}_j(\nabla\tilde{u}(x,t)) dt \\
&= \mathcal{A}_j(\nabla u(x,t)) - \mathcal{A}_j(\nabla v(x,t)) \\
&= |\nabla u|^{p-2} u_{x_j} - |\nabla v|^{p-2} v_{x_j}
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
|\xi|^{p-2}\xi - |\eta|^{p-2}\eta &= \int_0^1 \frac{d}{dt} \big( |t\xi + (1-t)\eta|^{p-2} (t\xi + (1-t)\eta) \big)\, dt \\
&= (\xi - \eta) \int_0^1 |t\xi + (1-t)(\eta)|^{p-2} dt + (p-2) \int_0^1 |t\xi + (1-t)\eta|^{p-4} \langle t\xi + (1-t)\eta, \xi - \eta \rangle dt.
\end{aligned}$$

It follows that

$$\int \sum_{i,j} b_{ij} v_{x_i} \varphi_{x_j}\, dx = \int \langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla\varphi \rangle\, dx = 0$$

due to the $p$-harmonicity of $u$ and $v$ so $e$ is indeed a weak solution to $Lv = 0$. □

By using the Cauchy-Schwarz inequality, we obtain

$$\min\{p-1, 1\} |\xi|^2 \tilde{\lambda}(x) \le \sum_{i,j=1}^n b_{ij} \xi_i \xi_j \le \max\{p-1, 1\} |\xi|^2 \tilde{\lambda}(x),$$

where

$$\tilde{\lambda}(x) = \int_0^1 |\nabla\tilde{u}(x,t)|^{p-2} dt.$$

We are now ready to state the second claim.

**Claim 5.** *L can be locally reduced to a linear and uniformly elliptic differential operator.*

*Proof.*

$$\begin{aligned}
\tilde{\lambda}(x) &= \int_0^1 |\nabla\tilde{u}(x,t)|^{p-2} dt \\
&\approx (|\nabla u(x)| + |\nabla v(x)|)^{p-2} \\
&\approx (u(x)/d(x, \partial\Omega) + v(x)/d(x, \partial\Omega))^{p-2}
\end{aligned}$$

whenever $x \in \Omega \cap B(w, 4\hat{r})$. Note that we used Assumption 1 in the calculations above. By using the Harnack inequality on $u(x)$ and $v(x)$ it follows that $\tilde{\lambda}(x)$ is (locally) proportional to a constant. Thus, it follows that locally $L$ is a linear and uniformly elliptic differential operator. □

In the setting described in the beginning of this section, we have the following theorem.

**Theorem 7.21.** *Assume that $v \le u$. Then there exists $c \ge 1$, $c = c(p, n, M, c_0, \beta, \bar{c}_0, \gamma)$ such that if $\tilde{r} = r^*/c$, then*

$$c^{-1}\frac{u(a_{\tilde{r}}(w)) - v(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \le \frac{u(x) - v(x)}{v(x)} \le c\frac{u(a_{\tilde{r}}(w)) - v(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))}$$

*whenever $x \in \Omega \cap B(w, r^*)$.*

*Proof.* We will only prove the first inequality. The proof is divided into two parts. First we show the existence of a $\Lambda$, $1 \le \Lambda < \infty$ and $\hat{c} \ge 1$ s.t. if $r' = r^*/\hat{c}$ and

$$e(x) = \Lambda\left(\frac{u(x) - v(x)}{u(a_{r^*(w)} - v(a_*(w)))}\right) - \frac{v(x)}{v(a_{r^*}(w))}$$

for $x \in \Omega \cap B(w, r^*)$, then it holds that $e(x) \ge 0$ when $x \in \Omega \cap B(w, 2r')$. From this it follows directly that

$$\frac{u(x) - v(x)}{v(x)} \ge \Lambda^{-1}\frac{u(a_{r^*}(w)) - v(a_{r^*}(w))}{v(a_{r^*}(w))}.$$

After that we set $\tilde{r} = 2r'$, and show the existence of a $c = c(p, n, M, c_0, \mu, \tilde{c}_0, \gamma)$ such that $\Lambda \le c$, which implies that

$$\frac{u(x) - v(x)}{v(x)} \ge c^{-1}\frac{u(a_{r^*}(w)) - v(a_{r^*}(w))}{v(a_{r^*}(w))},$$

and in combination with the corresponding right-hand side inequality, the conclusion of the lemma follows. We first allow $\Lambda$ and $\hat{c}$ to vary but they will be chosen properly along the proof. Let

$$u'(x) = \frac{\Lambda u(x)}{u(a_{r^*}(w) - v(a_{r^*}(w))},$$
$$v'(x) = \frac{\Lambda v(x)}{u(a_{r^*}(w) - v(a_{r^*}(w))} + \frac{v(x)}{v(a_{r^*}(w))}.$$

We note that $u'(x)$ and $v'(x)$ are both $p$-harmonic and that $e = u' - v'$ and we can therefore define an operator $\tilde{L}$ as in (75) for $x \in \Omega \cap B(w, r^*)$ with $u = u'$ and $v = v'$. Next we let $e_1$ and $e_2$ denote the two solutions to $\tilde{L}\omega = 0$ in $\Omega \cap B(w, r^*)$ with the continuous boundary values

$$e_1|_{\partial(\Omega \cap B(w,r^*))} = \frac{u(x) - v(x)}{u(a_{r^*}(w)) - v(a_{r^*}(w))},$$
$$e_2|_{\partial(\Omega \cap B(w,r^*))} = \frac{v(x)}{v(a_{r_*}(w))}.$$

Recall that the existence of $e_1$ and $e_2$ follows from the Wiener criterion in [3]. Since Assumption 2 holds, we can apply Theorem 7.19 which ensures the existence of $c_+ \ge 1$ and $r_+ = r^*/c_+$ such that

$$c_+^{-1}\frac{e_1(a_{r_+}(w))}{e_2(a_{r_+}(w))} \le \frac{e_1(x)}{e_2(x)} \le c_+\frac{e_1(a_{r_+}(w))}{e_2(a_{r_+}(w))}$$

for $x \in \Omega \cap B(w, r_+)$. We next fix the constants so that $\hat{c} = c_+, r' = r_+$, and $\Lambda = \hat{c}e_2(a_{r'}(w))/e_1(a_{r'}(w))$. We define $\tilde{e} = \Lambda e_1(x) - e_2(x)$ and note that $\tilde{e} \geq 0$ whenever $x \in \Omega \cap B(w, 2r')$. Furthermore, it follows from the local linearity of $\tilde{L}$ that both $\tilde{e}$ and $e$ are weak solutions to $L\omega = 0$ in $\Omega \cap B(w, r^*)$. In addition $\tilde{e}$ and $e$ have the same continuous boundary values by construction which means that we can use the maximum principle for $\tilde{L}$ to conclude that $\tilde{e} = e$. Thus, we are finished with the first part of the proof.

For the second part of the proof we let $L$ denote the operator constructed in (75) such that $u(x) - v(x)$ is a weak solution in $\Omega \cap B(w, 4\hat{r})$. By applying the Harnack inequality from Lemma 7.15 *(ii)* to the weak solution $u - v$ it follows that

$$1 = \frac{u(a_{r^*}(w)) - v(a_{r^*}(w))}{u(a_{r^*}(w)) - v(a_{r^*}(w))} \leq c\,\frac{u(x) - v(x)}{u(a_{r^*}(w)) - v(a_{r^*}(w))} \leq ce_1(x)$$

for $x \in \Omega \cap \partial B(w, r^*)$ such that $d(x, \partial\Omega) > r^*/c^*$ for some $c^*$. Thus, we can conclude that there exists a $\zeta \in \Omega \cap \partial B(w, r^*)$ with $d(\zeta, \partial\Omega) \geq r^*/c$ such that $e_1(x) > c^{-1}$ for $x \in \partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$. See Figure 3 for an illustration.

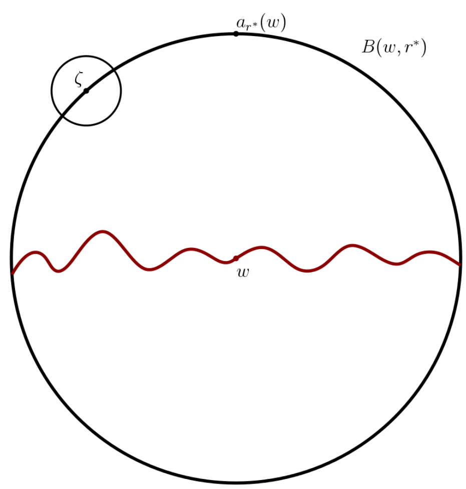


Figure 3: Lemma 7.13.

We let $\xi$ be a weak solution such that $L\xi = 0$ in $\Omega \cap B(w, r^*)$ with continuous boundary values on $\partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$ such that $\xi = 1$ on $\partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/16)$ and $\xi = 0$ on $\partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/4) \setminus B(\zeta, d(\zeta, \partial\Omega)/8)$. Note that this is possible since the points on $\partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$ are regular since they are part of the boundary of an open ball. Furthermore, by an appropriate choice of boundary values, an application of the maximum principle and the fact that $e_1 c \geq 1$ on $\partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$ we can ensure that $\xi \leq ce_1$ in $\Omega \cap B(w, r^*)$. We set $\omega = 1 - \xi$ and note that $\omega$ is also a weak solution in $\Omega \cap B(w, r^*)$ with $\omega(w_0) = 0$ on $\partial B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/16)$. By applying Lemma 7.16 *(iii)* to $\omega$ we see that there exists $s$ such that

$$|(1 - \xi(x)) - (1 - \xi(y))| = |\xi(y) - \xi(x)| \leq c\left(\frac{|y - x|}{d(\zeta, \partial\Omega)/32}\right)^{\sigma} \cdot 1$$

whenever $x, y \in B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/s)$. Thus, for $\tilde{c}$ sufficiently large and letting $y \to w_0 \in B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/\tilde{c})$ we note that

$$\xi(x) \geq 1 - c(1/\tilde{c})^{\sigma},$$

so for $\tilde{c}$ large enough we see that

$$\xi \geq \tilde{c}^{-1} \text{ in } B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/\tilde{c}).$$

By combining this with the fact that $\xi \le ce_1$ it follows that

$$e_1 \ge c^{-1}\xi \ge c^{-1}\tilde{c}^{-1} \text{ in } B(w, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/\tilde{c})$$

and by applying the Harnack inequality to $e_1$ it follows that $e_1(a_{r'}(w)) \ge \tilde{c}^{-1}$. Furthermore, we can apply the maximum principle, Harnack's inequality to $v$, and Lemma 7.7 to see that

$$e_2(x) \le \frac{\sup_{x\in\partial(\Omega\cap B(w,r^*))} v(x)}{v(a_{r^*}(w))} \le c(p, n, M, r^*)$$

and thus we conclude that $\Lambda < \bar{c}$ holds for some $\bar{c} = \bar{c}(p, n, M, c_0, \beta, \tilde{c}_0, \gamma)$. This concludes the proof of the left-hand inequality. □

**Theorem 7.22.** *Assume that $v \le u$. Then there exist $c \ge 1$ such that $c = c(p, n, M, c_0, \beta, \tilde{c}_0, \gamma)$, and $\sigma = \sigma(p, n, M, c_0, \beta, \tilde{c}_0, \gamma)$ where $\sigma \in (0, 1)$ such that if $\tilde{r} = r^*/c$, then*

$$\left|\frac{u(x_1)}{v(x_1)} - \frac{u(x_2)}{v(x_2)}\right| \le c\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))}\left(\frac{|x_1 - x_2|}{r}\right)^{\sigma}$$

*whenever $x_1, x_2 \in \Omega \cap B(w, \tilde{r})$.*

*Proof (sketch).* As usual when we want to prove Hölder continuity from a Harnack type inequality, we use an iterative approach, and thus this proof is similar to the proof of Lemma 7.18. We start by an application of Theorem 7.21 from which it follows that

$$c^{-1}\frac{u(x)}{v(x)} + 1 - c^{-1} \le \frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \le (1 - c) + c\frac{u(x)}{v(x)}$$

whenever $x \in B(w, \tilde{r})$, and since $c \ge 1$ we see that

$$\frac{u(x_1)}{v(x_1)} \le c\frac{u(x_2)}{v(x_2)} \text{ whenever } x_1, x_2 \in \Omega \cap B(w, \tilde{r}). \tag{76}$$

For $\tilde{w} \in \partial\Omega \cap B(w, \tilde{r}/8)$ and $0 < \rho < \tilde{r}$ we define

$$M(\rho) = \sup_{\Omega\cap B(\tilde{w},\rho)} \frac{u}{v}, \quad m(\rho) = \inf_{\Omega\cap B(\tilde{w},\rho)} \frac{u}{v}.$$

We recall that $p$-harmonicity is preserved for addition and multiplication by a constant. Thus, while keeping $\rho$ fixed we can use Theorem 7.21 with $u = u$ and $v = m(\rho)v$ to assert the existence of constants $c^*$ and $c_*$ such that

$$M(\tilde{\rho}) - m(\rho) \le c_*(m(\tilde{\rho}) - m(\rho)) \tag{77}$$

where $\tilde{\rho} = \rho/c^*$. By another application of the theorem but this time with $u = M(\rho)v$ and $v = u$ we obtain

$$(M(\rho)v - u)/u \approx \text{constant on } \Omega \cap B(w, \tilde{\rho})$$

where the constant depends on many variables. By combining this estimate with (76) it follows that

$$(M(\rho)v - u)/v \approx \text{constant on } \Omega \cap B(w, \tilde{\rho})$$

and thus we can conclude that

$$M(\rho) - m(\tilde{\rho}) \le \tilde{c}_*(M(\rho) - M(\tilde{\rho})). \tag{78}$$

We next set $\tilde{c} = \max(c_*, \tilde{c}_*)$ and by replacing the occurrences of $c_*$ and $\tilde{c}_*$ with $\tilde{c}$ and adding (77) and (78), we see that

$$\operatorname{osc}(\tilde{\rho}) \leq \frac{c_* - 1}{c_* + 1}\operatorname{osc}(\rho), \tag{79}$$

where $\operatorname{osc}(t) := M(t) - m(t)$. By iterating (79) it follows that

$$\operatorname{osc}(s) \leq c(s/t)^{\sigma}\operatorname{osc}(t) \text{ whenever } 0 < s < t \leq r/2 \tag{80}$$

for some $\sigma > 0, c \geq 1$. Similar to the proof of Lemma 7.18 we can use (76), (79), (80), the interior Hölder continuity and Harnack's inequality for $p$-harmonic functions to complete the proof. □

We are finally ready to prove Theorem 7.4.

*Proof of Theorem 7.4.* We let $\hat{u}, \hat{v}$ be the $p$-harmonic functions in $\Omega \cap B(w, 2r)$ with

$$\hat{u} = \max\{u, v\} \text{ and } \hat{v} = \min\{u, v\} \text{ on } \partial[\Omega \cap B(w, 2r)]$$

It follows directly from the maximum principle that $\hat{u} \geq \hat{v}$. Moreover, both $\hat{u}$ and $\hat{v}$ satisfy Assumption 1 and Assumption 2, and thus we can apply Theorem 7.21 to obtain a boundary Harnack inequality for the ratio of $\hat{u}$ and $\hat{v}$ so

$$c^{-1}\frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))} \leq \frac{\hat{u}(x)}{\hat{v}(x)} \leq c\frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))} \tag{81}$$

for $x \in \Omega \cap B(w, \tilde{r})$. Furthermore, by another application of the maximum principle we conclude that

$$\hat{v} \leq v \leq \hat{u}, \quad \hat{v} \leq u \leq \hat{u}$$

and thus it holds that

$$\frac{u}{v} = \frac{\hat{u}}{\hat{v}} \cdot \frac{\hat{v}}{v} \cdot \frac{u}{\hat{u}} \leq \frac{\hat{u}}{\hat{v}},$$
$$\frac{\hat{u}}{\hat{v}} = \frac{u}{v} \cdot \frac{\hat{u}}{u} \cdot \frac{v}{\hat{v}} \leq \frac{u}{v}.$$

By applying (81) we note that

$$c^{-1}\frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))} \leq \frac{u(x)}{v(x)} \leq c\frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))}.$$

Thus, we have shown that

$$\frac{u(x_1)}{v(x_1)} \leq c\frac{u(x_2)}{v(x_2)} \text{ whenever } x_1, x_2 \in \Omega \cap B(w, \tilde{r}).$$

We may now finish the proof of Theorem 7.4 by using the same technique as in Theorem 7.22. □

# Appendix

# A Proofs of basic properties

We first prove the rotation invariance of the $p$-Laplace equation.

**Lemma A.1.** *The p-Laplace equation is rotationally invariant.*

*Proof.* Let $O = (o_{ij})$ be an orthogonal $n \times n$-matrix, that is, for $k, l \in \{1, \dots, n\}$,

$$\sum_{i=1}^{n} o_{k_i} o_{li} = \delta_{kl} = \begin{cases} 1 \text{ if } k = l, \\ 0 \text{ if } k \neq l. \end{cases}$$

Let $u$ be a solution to the $p$-Laplace equation and define

$$v(x) = u(Ox).$$

We need to show that $\Delta_p v(x) = 0$. We use the notation

$$D_i v(x) = \frac{\partial}{\partial x_i} v(x) = v_{x_i}$$

and $v_k = D_k u(Ox)$. Then,

$$v_{x_i} = \frac{\partial}{\partial x_i} u(Ox) = \sum_{k=1}^{n} D_k u(Ox) o_{ki} = \sum_{i=1}^{n} v_k o_{ki}$$

and

$$v_{x_i x_j} = \frac{\partial^2}{\partial x_j \partial x_i} u(Ox) = \sum_{l=1}^{n} \sum_{k=1}^{n} v_{kl} o_{ki} o_{lj}.$$

Clearly,

$$\sum_{i=1}^{n} v_{x_i x_i} = \sum_{i=1}^{n} \sum_{k,l=1}^{n} v_{kl} o_{ki} o_{li} = \sum_{k,l=1}^{n} v_{kl} \sum_{i=1}^{n} o_{ki} o_{li} = \sum_{k,l=1}^{n} v_{kl} \delta_{kl} = \Delta_2 u(Ox),$$

and

$$\begin{aligned}
\sum_{i,j}^{n} v_{x_i} v_{x_j} v_{x_i x_j} &= \sum_{i,j=1}^{n} v_{x_i} v_{x_j} \left( \sum_{k,l=1}^{n} v_{kl} o_{ki} o_{lj} \right) \\
&= \sum_{i,j=1}^{n} \left( \sum_{m=1}^{n} v_m o_{mi} \right) \left( \sum_{s=1}^{n} v_s o_{sj} \right) \left( \sum_{k,l=1}^{n} v_{kl} o_{ki} o_{lj} \right) \\
&= \sum_{i,m,s,l,k} v_m o_{mi} v_s v_{kl} o_{ki} \sum_{j=1}^{n} o_{sj} o_{lj} \\
&= \sum_{i,m,k,l} v_m o_{mi} v_l v_{kl} o_{ki} = \sum_{m,k,l=1}^{n} v_m v_l v_{kl} \sum_{i=1}^{n} o_{mi} o_{ki} \\
&= \sum_{k,l=1}^{n} v_k v_l v_{kl} = \Delta_\infty u(Ox).
\end{aligned}$$

Hence, $\nabla \cdot (|\nabla v|^{p-2} \nabla v) = |\nabla u(Ox)|^{p-4} \left\{ |\nabla u(Ox)|^2 \Delta_2 u(Ox) + (p-2) \Delta_\infty u(Ox) \right\}$, and the lemma is proven. □

Next, we want to show that the fundamental solution of the $p$-Laplace equation is $(n-p)|x|^{\frac{p-n}{p-1}}$ if $p < n$ and $-\log|x|$ if $p = n$. Because of the rotation invariance, we search for a radial solution. Suppose that $u(x) = v(r)$, $r = |x|$, is a solution, then

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \qquad u_{x_i} = v'(r)\frac{x_i}{r},$$

and hence

$$\begin{aligned}
|\nabla u|^{p-2} &= \left(\frac{v'(r)^2}{r^2}\sum_{i=1}^n x_i^2\right)^{\frac{p-2}{2}} = |v'(r)|^{p-2}, \\
|\nabla u|^{p-2}\nabla u &= \frac{|v'(r)|^{p-2}v'(r)}{r}x
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\left(\frac{|v'(r)|^{p-2}v'(r)}{r}x_i\right) \\
&= \left[\frac{\partial}{\partial x_i}\left(|v'(r)|^{p-2}\right)\right]\frac{v'(r)}{r}x_i + |v'(r)|^{p-2}\left[\frac{\partial}{\partial x_i}\frac{v'(r)}{r}x_i\right] \\
&= (p-2)|v'(r)|^{p-2}\frac{v''(r)}{r^2}x_i^2 + |v'(r)|^{p-2}\left(v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right),
\end{aligned}$$

which gives

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = (p-1)|v'(r)|^{p-2}v''(r) + \frac{n-1}{r}|v'(r)|^{p-2}v'(r) = 0.$$

Assuming that $|v'(r)| \neq 0$, we have

$$(p-1)v''(r) + \frac{n-1}{r}v'(r) = 0,$$

i.e., that

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{(p-1)r}. \tag{82}$$

If $p = n$, then

$$\frac{v''(r)}{v'(r)} = -\frac{1}{r},$$

and thus $v(r) = -\log r$. If $p < n$, then (82) is solved by $v(r) = (n-p)r^{\frac{p-n}{p-1}}$.

# B Some useful inequalities

**Lemma B.1.** *Let $a, b \in \mathbb{R}^n$.*

*(i) If $p \geq 2$, then*

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle \geq \frac{1}{2}(|b|^{p-2} + |a|^{p-2})|b-a|^2 \geq 2^{2-p}|b-a|,$$

*(ii) if $p \leq 2$, then*

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle \leq \frac{1}{2}(|b|^{p-2} + |a|^{p-2})|b-a|^2,$$

*(iii) if $p \leq 2$, then*

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle \leq \gamma(p)|b-a|^p$$

*for some constant $\gamma(p)$,*

*(iv) if $p \geq 2$, then*

$$\left| |b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a \right|^2 \leq \frac{p^2}{4}\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle,$$

*(v) if $p \geq 2$, then*

$$\left| |b|^{p-2}b - |a|^{p-2}a \right| \leq (p-1)\left(|b|^{\frac{p-2}{2}} + |a|^{\frac{p-2}{2}}\right)\left| |b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a \right|,$$

*(vi) if $1 \leq p \leq 2$, then*

$$\langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle \geq (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}},$$

*(vii) and if $p \geq 2$, then*

$$|b|^p \geq |a|^p + p\langle |a|^{p-2}a, b-a\rangle + C(p)|b-a|^p,$$

*(viii) for $p \geq 1$*

$$|b|^p \geq |a|^p + p\langle |a|^{p-2}a, b-a\rangle.$$

*Proof.* The proof of this *(i)-(vii)* is contained in [21]. We only prove *(viii)*. Since $|x|^p$ is a convex function we note that for $a, b \in \mathbb{R}^n$ and $t \in (0,1)$

$$|a + t(b-a)|^p = |(1-t)a + tb|^p \leq (1-t)|a|^p + t|b|^p$$

We set $c = (b-a)$ and subtract $|a|^p$ which yields

$$|a+tc|^p - |a|^p \leq t(|b|^p - |a|^p)$$

and by subtracting $tp|a|^{p-2}\langle a, c\rangle$ and dividing by $t$ we obtain

$$\frac{|a+tc|^p - |a|^p}{t} - p|a|^{p-2}\langle a, c\rangle \leq |b|^p - |a|^p - p|a|^{p-2}\langle a, c\rangle$$

By letting $t$ tend to 0 we see that the left hand side also tends to 0 from which (viii) follows immediately. □

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