Constraints of Binary Simple Homogeneous Structures

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Abstract

We show that every member of a subclass of the binary simple homogeneous structures has only finitely many constraints, that is to say there are only finitely many minimal finite structures that cannot be embedded into such a structure. This is done by relating constraints to extension problems of types and determining under which conditions extension problems have solutions.

Sammanfattning

Vi visar att varje medlem i en delklass av de binära enkla homogena strukturerna är ändligt villkorad, alltså att det finns endast ändligt många minimala ändliga strukturer som inte kan inbäddas i en sådan struktur. Vi gör detta genom att undersöka relationen mellan villkor och typers förlängningsproblem och att bestämma under vilka antaganden förlängningsproblem har lösningar.

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1 Introduction

Model theory is a subfield of mathematical logic. It studies mathematical structures like graphs, groups, fields and partial orders. But model theory can study huge classes of structures in a uniform fashion, that is to say without assuming that a particular structure is for example a graph or a partial order. How is this possible?

The answer is that structures (and their first order theories) also have “language independent” properties, properties that do not depend on whether we consider graphs or partial orders. Using such properties, one can obtain broad classifications of classes of structures. These classifications can be rather coarse or very fine, depending on the specificity of the properties involved.

Homogeneous structures are a broad and interesting class of structures. They have very elegant model-theoretic descriptions, as will be recalled in Section 3. They also appear naturally in the study of permutation groups (Cameron 1990), Ramsey theory (Nešetřil 2005) and topological dynamics (Kechris et al. 2005) in mathematics, and constraint satisfaction problems in computer science (Bodirsky and Nešetřil 2006). Macpherson (2011) gives a comprehensive survey on homogeneous structures and their applications.

Some classes of homogeneous structures have been completely classified, as homogeneous partial orders (Schmerl 1979) and homogeneous undirected graphs (Lachlan and Woodrow 1980) and directed graphs (Cherlin 1998). But few results are known about homogeneous structures in general. Homogeneous structures seem to be too diverse to be well understood in a fully unified approach.

Meaningful subclasses of the homogeneous structures can be found by using notions from classification theory such as stable and simple structures. There have been developed powerful methods to deal with these sorts of structures, and homogeneous stable structures are quite well understood (cf. Lachlan 1997). The class of simple structures extends the stable structures, and there exist interesting non-stable simple homogeneous structures such as the random graph.

Still the class of simple homogeneous structures is big and does not seem to allow for an easy classification. However, if we restrict ourselves to binary structures (essentially graphs with a finite number of different edge relations), a different picture emerges. Vera Koponen has produced a lot of results about the fine structure of binary simple homogeneous structures (see 2016, 2017a, 2017b) using various methods such like coordinatisations and the characterisation of dividing by definable equivalence relations.

One problem that remains is to determine the number of the constraints of a binary simple homogeneous structure \( \mathcal{M} \), i.e. the number of minimal finite structures \( \mathfrak{A} \) which cannot be embedded into \( \mathcal{M} \). Another question is which extension problems have solutions in \( \mathcal{M} \), i.e. which pairs of types can be sim-

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1 Unfortunately, simple structures are not at all “structures which are easy to understand”. The term simple was introduced by Shelah (1980) and has got stuck since.

2 Which is introduced in Example 3.2.
ultaneously realised in $\mathfrak{M}$. These questions are connected (as one can see in Section 10 of this thesis).

It is known that all homogeneous stable structures are finitely constrained (see Lachlan 1988). It follows that there are only countable many homogeneous stable structures. A corresponding result for binary simple homogeneous structures would be a big step forward. “Ternary” simple homogeneous structures (structures with 3-ary relation symbols) do not need to be finitely constrained, which is one of the reasons that restricting ourselves to binary structures seems to be a good approach.

In this thesis we show that all members of a subclass of the binary simple homogeneous structures are finitely constrained. This will be the binary simple homogeneous structures satisfying a technical assumption which we call “the algebraic closure property”. This thesis should be seen as an intermediate step on the way to proving the result for all binary simple homogeneous structures. Several of the arguments used here should be applicable to the general case.

This thesis is organised as follows: In Section 2 we introduce some notation and recall some basic definitions. In Sections 3–7 we give known results on binary simple homogeneous structures, most proofs are omitted. In Sections 8 and 9 we introduce some new notions (most results are due to Vera Koponen). Finally, Section 10–12 give the main arguments of this thesis, which are original work.

2 Notation and Terminology

We follow the following notational conventions. Most of it is standard notation.

- $\sigma$ denotes a signature, $\mathcal{L}$ a language (i.e. the set of formulas that can be build from some signature $\sigma$, possibly with free variables) and $\mathcal{T}$ a theory (a complete consistent set of formulas without free variables).
- $\mathfrak{M}$ and $\mathfrak{N}$ denote infinite structures, while $\mathfrak{A}$ and $\mathfrak{B}$ denote finite ones. Their universes are denoted by $M, N, A$ and $B$, respectively.
- Small Latin letters $a, b, c, \ldots$ denote elements in the universe of some structure $\mathfrak{M}$. $\bar{a}, \bar{b}, \bar{c}, \ldots$ denote finite tuples. For a tuple $\bar{a}$, $\text{rng}(\bar{a})$ denotes the set of elements occurring in $\bar{a}$. When we write $\bar{a} \in D$ for some set $D$ we mean $a \in D$ for all $a \in \text{rng}(\bar{a})$.
- $x, y, z$ and $\bar{x}, \bar{y}, \bar{z}$ denote always variables/tuples of variables.
- If $\bar{a}$ and $\bar{b}$ are tuples (or single elements), $\bar{a}\bar{b}$ denotes their concatenation.
- If $f$ is a function on single elements, $f(\bar{a})$ denotes the tuple $(f(a))_{a \in \text{rng}(\bar{a})}$.
- $C$ and $D$ (and often also $A, B, E$ etc.) denote sets. The notation $CD$ denotes the union of the sets $C$ and $D$.
- Finite sets and tuples are often used interchangeably. This simplifies notation significantly, but the reader has to be sensitive to the context at hand. Three examples of this point:

3The examples constructed in Section 7.3 of Koponen (2017c) are not finitely constrained.
The expression $D = \bar{a}$ means that $\bar{a}$ is an enumeration of $D$.

The “mixed” notation $D\bar{a}$ denotes concatenation of tuples, where it is assumed that the set $D$ is enumerated in some (most often arbitrary) way.

The expression $\bar{a} \cap D$ stands for the intersection of the sets $\text{rng}(\bar{a})$ and $D$, and similarly for unions.

- If $\mathcal{M}$ is an $\mathcal{L}$-structure and $D \subseteq M$ then $\mathcal{L}(D)$ is the set of all formulas which can be build from $\mathcal{L}$’s signature with parameters in $D$ (with free variables). If $\bar{x}$ is a tuple then $\mathcal{L}_{\bar{x}}(D)$ is the set of all formulas in $\mathcal{L}(D)$ with free variables in $\bar{x}$. If $n$ is an integer, then $\mathcal{L}_n(D)$ denotes $\mathcal{L}_{\bar{x}}(D)$ where $\bar{x}$ is some tuple of variables of length $n$. A sentence is a formula without free variables.

- If $\mathcal{M}$ is a model, $\text{Th}(\mathcal{M})$ denotes $\mathcal{M}$’s theory, i.e. the set of sentences in $\mathcal{L}$ that hold true in $\mathcal{M}$. If $D \subseteq M$, $\text{Th}(\mathcal{M}, D)$ denotes the set of sentences in $\mathcal{L}(D)$ that hold true in $\mathcal{M}$.

- An $n$-type is usually denoted $p$ or $q$ and is defined as a complete consistent set of formulas in $L_n(D)$. Here $D$ is the domain of the type, also denoted $\text{dom}(p)$. A type is an $n$-type, for some $n$.

- If $\bar{a} \in M$, $D \subseteq M$ then $\text{tp}(\bar{a}/D)$ denotes $\bar{a}$’s type over $D$. So $\text{tp}(\bar{a}/D)$ is the set of all formulas $\varphi(\bar{x}, \bar{d}) \in \mathcal{L}_{\bar{x}}(D)$ such that $\mathcal{M} \models \varphi(\bar{a}, \bar{d})$ (where the tuple $\bar{x}$ has the same length as $\bar{a}$). $\text{tp}^{qf}(\bar{a}/D)$ denotes the set of all quantifier-free formulas in $\text{tp}(\bar{a}/D)$. Note that the notion $\text{tp}$ is always relative to some model.

- If $\mathcal{M}$ is a model and $\varphi(\bar{x}, \bar{d}) \in \mathcal{L}(D)$, then $\varphi(\mathcal{M}, \bar{d})$ denotes the set defined by $\varphi(\bar{x}, \bar{d})$ in $\mathcal{M}$, i.e.

$$\varphi(\mathcal{M}, \bar{d}) = \{ \bar{a} \in M \mid \mathcal{M} \models \varphi(\bar{a}, \bar{d}) \}.$$

A set is $C$ is called $D$-definable if there is some formula with parameters in $D$ that defines it.

3 Homogeneous Structures

In this section we study homogeneous structures, the primary object of interest in this thesis, in some more detail. Our exposition is based on the ones in Tent and Ziegler (2012) and Horowitz (2008).

**Definition 3.1 (Homogeneous structure).** A structure $\mathcal{M}$ is called homogeneous if every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

**Example 3.2 (The random graph).** Let $\mathcal{L}$ be the language of graphs (i.e. with a 2-ary relation symbol). Construct an $\mathcal{L}$-structure $\mathcal{G}$ as follows: Let the universe of $\mathcal{G}$ be countable, and for every two vertices in $\mathcal{G}$ flip a coin to
decide whether or not they are connected by an edge. It turns out that with probability one, all such constructed $R$ are isomorphic. Furthermore, $R$ is homogeneous: Assume $\mathfrak{A}, \mathfrak{B}$ are finite substructures of $R$ and that $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is an isomorphism. Let $\bar{a}$ be an enumeration of $A$ and $\bar{b}$ an enumeration of $B$ such that $f(\bar{a}) = \bar{b}$, and let $c \in M \setminus A$. Then with probability 1, there is $d \in M \setminus B$ such that the edge relations of $d$ to $b_1, \ldots, b_n$ are the same as the edge relations of $c$ to $a_1, \ldots, a_n$. By a “back-and-forth-argument”, $f$ can be successively extended to an isomorphism of $M$. See Erdős and Rényi (1963) for the details of our argument, and see Horowitz (2008) for a slightly different but related construction of the random graph.

In the following we will see that homogeneous structures are essentially defined by the properties of the collection of their finitely generated substructures, their “age”. Since we want the notion of age to be stable under isomorphism issues, our formal definition of age is a bit more technical.

**Definition 3.3 (Age).** For any structure $M$, an age of $M$ is a collection $K$ of finitely generated $L$-structures, such that every structure in $K$ is isomorphic to a finitely generated substructure of $M$, and any finitely generated substructure of $M$ is isomorphic to an element of $K$.

Under certain conditions, a collection of finitely generated structures determines an infinite homogeneous structure:

**Theorem 3.4 (Fraïssé’s theorem).** Let $\mathcal{K}$ be a countable set of finitely generated $L$-structures. There is a countable homogeneous $L$-structure $M$ with age $\mathcal{K}$ if and only if the following properties hold:

(i) (Heredity) If $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{B} \subseteq \mathfrak{A}$, there is $\mathfrak{B}' \cong \mathfrak{B}$ with $\mathfrak{B}' \in \mathcal{K}$.

(ii) (Joint embedding) For $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}$ there is $\mathfrak{B} \in \mathcal{K}$ such that both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be embedded into $\mathfrak{B}$.

(iii) (Amalgamation) If $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ and $f_1 : \mathfrak{A} \hookrightarrow \mathfrak{B}_1$, $f_2 : \mathfrak{A} \hookrightarrow \mathfrak{B}_2$ are embeddings, there is $\mathfrak{D} \in \mathcal{K}$ and two embeddings $g_1 : \mathfrak{B}_1 \hookrightarrow \mathfrak{D}$, $g_2 : \mathfrak{B}_2 \hookrightarrow \mathfrak{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

In the case that $M$ exists, it is unique up to isomorphism and we call $M$ the Fraïssé limit of $\mathcal{K}$.

**Proof reference.** Theorem 4.4.4 in Tent and Ziegler (2012).

For the following theorems, it is necessary to restrict ourselves to finite relational signatures.
Definition 3.5 (Quantifier-elimination). A theory $\mathcal{T}$ is said to have quantifier-elimination, if for every $\varphi(\bar{x}) \in \mathcal{L}$ there is a quantifier-free formula $\psi(\bar{x}) \in \mathcal{L}$ such that $\mathcal{T} \models \forall \bar{x} \varphi(\bar{x}) \iff \psi(\bar{x})$.

There are three important ways to describe the theory of a homogeneous structure:

Theorem 3.6 (Characterisations of homogeneous structures). Let $\mathcal{T}$ be a complete theory in a finite relational signature $\sigma$. The following are equivalent:

(i) $\mathcal{T}$ has a countable homogeneous model $\mathfrak{M}$.
(ii) $\mathcal{T}$ has a countable model $\mathfrak{M}$, which is the Fraïssé limit of the collection $\mathcal{K}$ of its finite substructures.
(iii) $\mathcal{T}$ has quantifier-elimination.

Proof. (i) $\implies$ (ii): Let $\mathfrak{M}$ be a countable homogeneous model, and $\mathcal{K}$ be the collection of its finite substructures. It is easy to see that $\mathcal{K}$ satisfies heredity and joint embedding (in fact that is true for any age and does not require that $\mathfrak{M}$ is homogeneous). Now we want to show that $\mathcal{K}$ satisfies the amalgamation property. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ and $f_1 : \mathfrak{A} \to \mathfrak{B}_1$, $f_2 : \mathfrak{A} \to \mathfrak{B}_2$ be embeddings. First we will replace $\mathfrak{A}$ with a substructure $\mathfrak{A}'$ of $\mathfrak{B}_1$: Let $r := g_1 \upharpoonright \mathfrak{A}$ and let $\mathfrak{D} := g_2 \upharpoonright \mathfrak{A}$, $f'_1 := \text{id}_\mathfrak{D}$, and $f_2 := f_2 \circ r^{-1}$. Then the following diagram commutes:

It follows that an amalgamation for $\mathfrak{A}', \mathfrak{B}_1, \mathfrak{B}_2, f'_1, f_2$ gives us an amalgamation.

Let $h$ be an automorphism of $\mathfrak{M}$ that extends $f'_2$. Let $\mathfrak{D}$ be the substructure of $\mathfrak{M}$ with universe $h[B_1] \cup B_2$ and let $g_1 := h \upharpoonright \mathfrak{B}_1$, $g_2 := \text{id}_{\mathfrak{B}_2}$. Then

$$g_1 \circ f'_1 = h \upharpoonright \mathfrak{A}' = g_2 \circ f'_2.$$ We have found an amalgamation for $\mathfrak{A}', \mathfrak{B}_1, \mathfrak{B}_2, f'_1, f'_2$, and proved that $\mathcal{K}$ has the amalgamation property. Consequently $\mathcal{K}$ has a Fraïssé limit $\mathfrak{N}$ and $\mathcal{K}$ is the age of $\mathfrak{N}$.

(ii) $\implies$ (i): Is clear from Theorem 3.4.

(i) $\implies$ (iii): Let $\varphi(\bar{x}) \in \mathcal{L}$. We want to show that $\varphi(\bar{x})$ is equivalent to a quantifier-free formula. Let $\mathfrak{M}$ be a countable homogeneous model of $\mathcal{T}$. Since $\sigma$ is finite, there are only finitely many quantifier-free types of tuples $\bar{a} \in M$ that satisfy $\varphi(\bar{x})$, and every such type is isolated. Let $\xi_1(\bar{x}), \ldots, \xi_n(\bar{x})$ isolate these quantifier-free types, and choose witnesses $\bar{a}_1, \ldots, \bar{a}_n \in M$ with $\mathfrak{M} \models \varphi(\bar{a}_i) \land \xi_i(\bar{a}_i)$ for every $i$. Assume now that $\bar{b}$ satisfies $\xi_i(\bar{x})$, for some
Then $f : a_i \mapsto b$ is an isomorphism of finite $\mathcal{L}$-structures. Since $M$ is homogeneous, $f$ can be extended to an automorphism $g$ of $M$. So

$$M \models \varphi(a_i) \implies M \models \varphi(g(a_i)) \implies M \models \varphi(b).$$

We have shown that $T \models \varphi(x) \leftrightarrow \bigvee_{i=1}^{n} \xi_i(x)$. So $\varphi(x)$ is equivalent to a quantifier-free formula. We have shown that $T$ has quantifier-elimination.

(iii) $\implies$ (i): Let $M$ be a countable model of $T$. We show first that any isomorphism between finite substructures of $M$ can be extended to any chosen further element. Let $f : A \rightarrow B$ be an isomorphism with $A, B \subset M$ finite. Let $a$ enumerate $A$ and $b$ enumerate $B$ such that $f : a \mapsto b$. Let $c \in M \setminus A$ and let $\xi(x,y)$ isolate the quantifier-free type of $ac$. Then $\text{tp}(a)$ contains the formula $\exists y \xi(x,y)$. Since $T$ has quantifier-elimination, an equivalent formula is already contained in $\text{tp}_{\text{qf}}(a)$. But since $f$ is a partial isomorphism this implies that $\exists y \xi(x,y)$ is contained in $\text{tp}(b)$. Choose $d$ with $\xi(b,d)$. Then $f' : ac \mapsto bd$ is an isomorphism of finite substructures of $M$ extending $f$.

If $f : A \rightarrow B$ is any isomorphism between finite substructures of $M$, we can use a back-and-forth argument to construct an automorphism $g$ of $M$ extending $f$. That shows that $M$ is homogeneous.

\textbf{Example 3.7.} It is easy to see that all finite graphs can be embedded into the random graph. So the random graph is the Fra"issé limit of the class of finite graphs.

In the following we summarise some more properties of homogeneous structures, all in essence due to the fact that the theory of a homogeneous structure has quantifier-elimination.

\textbf{Definition 3.8 (k-saturated structure).} Let $\kappa \geq \aleph_0$ be a cardinal and $M$ be a model of some theory. We say that $M$ is $\kappa$-\textit{saturated} if for any $B \subset M$ of cardinality $< \kappa$ and any set of formulas $\Pi(x)$ with a finite number of variables $x$ and parameters in $B$ the following holds: If $\Pi(x)$ is consistent with $\text{Th}(M)$, there is $a \in M$ that realises $\Pi(x)$.

A structure $M$ is called saturated if it is $|M|$-saturated.

\textbf{Definition 3.9 (k-categorical structure).} Let $\kappa \geq \aleph_0$. A theory $T$ is called $\kappa$-\textit{categorical} if $M$ has exactly one model in cardinality $\kappa$, up to isomorphism. We usually use the term “$\omega$-categorical” instead of “$\aleph_0$-categorical”.

\textbf{Theorem 3.10 (Properties of homogeneous structures).} Assume that $M$ is a homogeneous and countable model in a finite relational signature, with theory $T$.

(i) If $a \in M$ and $B \subset M$ is finite then $\text{tp}(a/B)$ is isolated.
(ii) $M$ is $\aleph_0$-saturated.
(iii) $T$ is $\omega$-categorical.
Proof. (i): Since $\sigma$ is finite $tp^f(a/B)$ is isolated. Since $\mathcal{T}$ has quantifier-elimination, $tp(a/B)$ is as well.

(ii): Let $B \subset M$ be finite and $\Pi(\bar{x})$ be a set of formulas with parameters in $B$ consistent with $Th(M)$. Since $\sigma$ is finite and $\mathcal{T}$ has quantifier-elimination $\Pi(\bar{x})$ is equivalent to a formula $\psi(\bar{x}, \bar{b})$. Take $\mathfrak{N} \supseteq \mathfrak{M}$ such that $\mathfrak{N}$ realises $\Pi(\bar{x})$. Then $\mathfrak{N} \models \exists \bar{x} \psi(\bar{x}, \bar{b})$, so $\mathfrak{N} \models \exists \bar{x} \psi(\bar{x}, \bar{b})$.

(iii): Using that $\mathcal{T}$ has quantifier-elimination, we can show for any countable model $\mathfrak{N} \models \mathcal{T}$ that $\mathfrak{M} \equiv \mathfrak{N}$. That is shown by a back-and-forth argument that is essentially the same as in the proof of Theorem 3.6 implication (iii) $\implies$ (i). □

4 Simple Structures

Simple structures are a generalisation of stable structures, the most studied objects within classification theory. Classification theory tries to classify the models of a given theory, often using cardinal invariants. Classification theory was started by Morley (1965). Shelah is responsible for many central results of classification theory, and introduced the notion of stability (1969). Stable theories are reasonably “well-behaved” and can therefore often be classified. The standard reference for stability theory is Shelah (1990).

Simple theories were introduced in Shelah (1980). They share some fundamental properties with the stable structures and have proven fruitful to be studied in their own right (see Kim and Pillay, 1998). Examples of simple unstable theories include the random graph and pseudo-finite fields (see Chapter 7.5 in Tent and Ziegler, 2012). Here we introduce some of the machinery central to the study of stable and simple structures and give important results. We follow to a large extent the presentation in Tent and Ziegler (2012), which in its turn is based on the one in Casanovas (2011).

We will start by defining the important notions of algebraic and definable closure.

Definition 4.1 (Algebraic and definable closure). Let $\mathfrak{M}$ be a structure and $A \subset M$ a set. We define the definable closure of $A$ as

$$dcl(A) := \{ b \in M \mid \exists \varphi(x, \bar{a}) \in \mathcal{L}_1(A) \text{ such that } \varphi(\mathfrak{M}, a) = \{ b \} \}.$$ 

We define the algebraic closure of $A$, denoted $acl(A)$, as

$$\{ b \in M \mid \exists \varphi(x, \bar{a}) \in \mathcal{L}_i(A) \text{ such that } b \in \varphi(\mathfrak{M}, a) \text{ and } \varphi(\mathfrak{M}, a) \text{ is finite } \}.$$ 

Note that when we write $\bar{b} \in dcl(A)$ or $\bar{b} \in acl(A)$, respectively $b \in acl(A)$, for any $b \in \text{rng}(\bar{b})$. This is consistent with our notation, but is sometimes handled differently in other treatments of this topic.

The notions of definable and algebraic closure are “robust” in respect to elementary extensions.

Proposition 4.2. Let $\mathfrak{M}$ be a structure, $A \subset M$ and $\mathfrak{N} \supseteq \mathfrak{M}$ an elementary extension of $\mathfrak{M}$. Then $dcl_\mathfrak{N}(A) = dcl_\mathfrak{M}(A)$ and $acl_\mathfrak{N}(A) = acl_\mathfrak{M}(A)$.

Proof. Straightforward. □
The following lemma states that in many situations, we can “replace” elements with other elements that they define.

Lemma 4.3. Let $\mathfrak{M}$ be a structure and let $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2 \in M$ with $\text{tp}(\bar{a}_1 \bar{b}_1) = \text{tp}(\bar{a}_2 \bar{b}_2)$. Assume that $\bar{c}_1 \in \text{dcl}(\bar{a}_1)$, $\bar{c}_2 \in \text{dcl}(\bar{a}_2)$ are defined by the same formula, i.e. there is $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}$ with $\varphi(\mathfrak{M}, \bar{a}_1) = \{\bar{c}_1\}$ and $\varphi(\mathfrak{M}, \bar{a}_2) = \{\bar{c}_2\}$. Then $\text{tp}(\bar{c}_1 \bar{b}_1) = \text{tp}(\bar{c}_2 \bar{b}_2)$.

Proof. Straightforward. \qed

Definition 4.4 (Algebraic formula, algebraic type). Let $\mathfrak{M}$ be a model, $\bar{a} \in M$ and $\varphi(\bar{x}, \bar{a}) \in \mathcal{L}(M)$. We say that $\varphi$ is algebraic if it has only finitely many realisations in $\mathfrak{M}$. We call a type $p$ (over some subset of $\mathfrak{M}$) algebraic if it contains some algebraic formula.

It is easy to see that if $\varphi(\bar{x}, \bar{a})$ is algebraic then it has the same number of realisations in every model of $\text{Th}(\mathfrak{M}, M)$. Is it also easy to see that a type $p$ is algebraic if and only if every realisation of $p$ lies in $\text{acl}(\text{dom}(p))$.

Now we want to introduce simple theories. In most treatments of this topic, all definitions are made with respect to some monster model, a very large saturated model of a given theory. A good presentation of that notion is given in Tent and Ziegler (2012). The monster model is convenient since it eliminates the need to consider many different models and elementary extensions. However, to avoid introducing a lot of additional machinery, we decided to work without a monster model.

Definition 4.5 ($k$-inconsistent). Let $\mathcal{T}$ be a theory. Let $k$ be an integer. A family $(\varphi_i(\bar{x}, \bar{a}_i) \mid i \in I)$ is called $k$-inconsistent if for any $k$-elementary $J \subset I$, the family $(\varphi_i(\bar{x}, \bar{a}_i) \mid i \in J)$ is inconsistent (with respect to $\mathcal{T}$).

Definition 4.6 (Tree property). Let $\mathcal{T}$ be a theory. A formula $\varphi(\bar{x}, \bar{y}) \in \mathcal{T}$ has the tree property with respect to $k$ if there is a model $\mathfrak{M}$ of $\mathcal{T}$ such that there is a tree of parameters $(\bar{a}_s \mid s \in \omega^\omega)$ in $M$ such that

- For all $s \in \omega^\omega$, the family $(\varphi(\bar{x}, \bar{a}_s \bar{a}_{s-n}) \mid n \in \omega)$ is $k$-inconsistent.
- For all $t \in \omega^\omega$, the set $\{\varphi(\bar{x}, \bar{a}_s) \mid s \in \omega^\omega, s \subset t\}$ is consistent.

Here $\omega^\omega$ denotes the set of all infinite sequences in $\omega$, while $\omega^\omega$ denotes the set of all countable infinite sequences in $\omega$.

Definition 4.7 (Simplicity). A theory $\mathcal{T}$ is simple if no $\varphi \in \mathcal{T}$ has the tree property (for any $k$ and partition $\varphi(\bar{x}, \bar{y})$ of parameters). A structure $\mathfrak{M}$ is simple if $\text{Th}(\mathfrak{M})$ is simple.

Simple structure are interesting since they behave well in respect to the notions of forking and dividing.

Definition 4.8 (Dividing and Forking). Let $\mathfrak{M}$ be a model and $\varphi(\bar{x}, \bar{b}) \in \mathcal{L}(M)$. We say that $\varphi(\bar{x}, \bar{b})$ divides over some $A \subset M$ if there is some model $\mathfrak{M} \succ \mathfrak{N}$ and a sequence $(\bar{b}_i \mid i \in \omega)$ in $\mathfrak{N}$ such that:
• For every $i$, $tp(\bar{b}_i/A) = tp(\bar{b}/A)$
• There is some $k$ such that $(\varphi(\bar{x}, \bar{b}_i) | i \in \omega)$ is $k$-inconsistent.

A set of formulas $\Pi(\bar{x}) \subset \mathcal{L}(M)$ (with parameters) divides over $A$ if it implies some formula $\varphi(\bar{x}, \bar{b})$ that divides over $A$. A set of formulas $\Pi(\bar{x}) \subset \mathcal{L}(M)$ forks over $A$ if it implies a disjunction $\bigvee_{i=1}^n \varphi_i(\bar{x}, \bar{b})$ such that all $\varphi_i(\bar{x}, \bar{b})$ divide over $A$.

**Theorem 4.9.** Let $\mathfrak{M}$ be simple and $A \subseteq M$. Then any set of formulas $\Pi(\bar{x}) \subset \mathcal{L}(M)$ forks over $A$ if and only if it divides over $A$.

*Proof reference.* This is Proposition 7.2.15 in Tent and Ziegler (2012).

Next we will introduce an independence relation for simple structures, using the notion of dividing.

**Definition 4.10 (Independence relation).** Let $\mathfrak{M}$ be simple, $\bar{a} \in M$ and $B, C \subseteq M$. We say that $\bar{a}$ is independent from $B$ over $C$ if $tp(\bar{a}/BC)$ does not divide over $C$, written $\bar{a} \indep_C B$. For $A \subseteq M$ we write $A \indep_C B$ if $\bar{a} \indep_C B$ for all tuples $\bar{a} \in A$.

Intuitively, $A \indep_C B$ should be understood as “$A$ does not contain more information about $B$ than what is already contained in $C$”. In some sense, simple structures are exactly the structures which allow an independence relation with reasonable properties to exist.

**Example 4.11 (Vector spaces).** Vector spaces are simple (in fact even stable, see Ziegler, 1984). If $V$ is a vector space and $A, B, C \subseteq V$, then we have $A \indep_C B$ if and only if $< A > \cap < B > < C >$, where $< A >$ denotes the linear closure of $A$. This shows that for vector spaces, the independence relation defined above coincides with linear independence: If $v_1, \ldots, v_n \in V$ then $v_1, \ldots, v_n$ are linearly independent if and only if $v_i \indep v_j$ for all $i, j \leq n$.

**Example 4.12.** The random graph is simple. If $A, B, C$ are subsets of $\mathfrak{M}$, then we have $A \indep_C B$ if and only if $A \cap B \subseteq C$. See Corollary 7.3.14 in Tent and Ziegler (2012).

The following lemma shows that independence is well-behaved with respect to elementary extensions and that it is “type-definable”.

**Lemma 4.13 (Robustness of Independence).** Let $\mathfrak{M}$ be simple.

(i) If $A \indep_C B$ holds in $\mathfrak{M}$, then it holds in every model of $Th(\mathfrak{M}, M)$, so in particular in every $\mathfrak{N} \models M$.

(ii) Assume that $tp(\bar{a}/BC) = tp(\bar{a}'/BC)$. Then $\bar{a} \indep_C B \iff \bar{a}' \indep_C B$.

*Proof.* Both follow from the definition of independence, for (i) one has to go through the definition of dividing as well.

In the following we give some “arithmetical rules” for the independence relation.
Theorem 4.14 (Properties of Independence). Let \( \mathfrak{M} \) be simple, and assume that all sets and tuples are in \( M \).

(i) For all \( \bar{a}, B \) we have \( \bar{a} \downarrow_B B \).
(ii) (Side monotonicity). Let \( B' \subset B \). Then \( \bar{a} \downarrow_C B \) implies \( \bar{a} \downarrow_C B' \).
(iii) (Lower monotonicity). Let \( B' \subset B \). Then \( \bar{a} \downarrow_C B \) implies \( \bar{a} \downarrow_{CB'} B \).
(iv) (Upper monotonicity). Let \( C' \subset C \). Then \( \bar{a} \downarrow_C B \) implies \( \bar{a} \downarrow_{C'BC} B \).
(v) (Transitivity). Let \( \bar{a} \) and \( B \subset C \subset D \) be given with \( \bar{a} \downarrow_B C \) and \( \bar{a} \downarrow_C D \). Then \( \bar{a} \downarrow_B D \).
(vi) (Finite character). \( \bar{a} \downarrow_C B \iff \bar{a} \downarrow_C b \) for all finite tuples \( \bar{b} \in B \).
(vii) (Symmetry). \( \bar{a} \downarrow_C b \) implies \( b \downarrow_C \bar{a} \).
(viii) (Invariance under algebraic closure). For any \( \bar{a}, B \) and \( C \) we have that \( \bar{a} \downarrow_C B \iff \bar{a} \downarrow_{\text{acl}(C)} B \).

Proof references.
(i): This follows from Corollary 7.2.6 in Tent and Ziegler (2012).
(ii)-(iv): Follow quite straight-forwardly from the definition.
(v): This is part of Corollary 7.2.17 in Tent and Ziegler (2012).
(vi): Follows from Corollary 7.1.9 in Tent and Ziegler (2012).
(vii): This is Proposition 7.2.16 in Tent and Ziegler (2012).
(viii): Follows with 5. of Remark 4.4 in Casanovas (and lower monotonicity).

Remark 4.15 (Lascar strong type). For a structure \( \mathfrak{M} \), \( \bar{a} \in \mathfrak{M} \) and \( B \subset \mathfrak{M} \) there is a notion of Lascar strong type of \( \bar{a} \) over \( B \), written \( \text{Lstp}(\bar{a}/B) \).

This notion is stronger than the notion of type, in the sense that \( \text{Lstp}(\bar{a}/B) = \text{Lstp}(\bar{c}/B) \) implies \( \text{tp}(\bar{a}/B) = \text{tp}(\bar{c}/B) \).

Reference. A formal definition is given in Tent and Ziegler (2012), Definition 7.4.1.

The notion of Lascar strong type is used in the statement of the following important theorem. However, since we will use only a special case of this theorem, we can live without a definition.

Theorem 4.16 (Independence Theorem). Assume that \( \mathfrak{M} \) is simple and let \( \bar{a}_1, \bar{a}_2 \in M \) and \( B_1, B_2, C \subset M \) with \( B_1 \downarrow_C B_2 \), \( \bar{a}_1 \downarrow_C B_1 \), \( \bar{a}_2 \downarrow_C B_2 \) and \( \text{Lstp}(\bar{a}_1/C) = \text{Lstp}(\bar{a}_2/C) \). Then there is \( \mathfrak{N} \supset \mathfrak{M} \) and \( \bar{d} \in N \) with \( \text{Lstp}(\bar{d}/B_1) = \text{Lstp}(\bar{a}_1/B_1) \), \( \text{Lstp}(\bar{d}/B_2) = \text{Lstp}(\bar{a}_2/B_2) \) and \( \bar{d} \downarrow_C B_1B_2 \).

Proof reference. This is a special case of Corollary 7.4.7 in Tent and Ziegler (2012).

The independence theorem is a very powerful tool for solving so-called extension problem, which will be introduced later on. The notion of independence allows us to define to define a notion of rank of types, which will be a central tool in this thesis.
Definition 4.17 (Forking extension). Let $\mathfrak{M}$ be simple, let $p$ be a type with domain $A$ and $q$ a type with domain $B$. We say that $q$ is a forking extension of $p$ if $q$ is an extension of $p$ (i.e. $q \supset p$) and $q$ forks over $A$. In other notation, $\text{tp}(\bar{b}/B)$ is a forking extension of $\text{tp}(\bar{a}/A)$ if $\text{tp}(\bar{b}/B) \supset \text{tp}(\bar{a}/A)$ and $\bar{b} \downarrow_A B$.

Definition 4.18 (SU-rank). Let $\mathcal{T}$ be a simple theory. For any model $\mathfrak{M}$ of $\mathcal{T}$, for any type $p$ in $\mathfrak{M}$, we define the SU-rank of $p$ by recursion over Ord. Let

- $\text{SU}(p) \geq 0$ for any $p$.
- $\text{SU}(p) \geq \beta + 1$ if there is some $\mathfrak{N} \succ \mathfrak{M}$ and a type $q$ in $\mathfrak{N}$ such that $\text{SU}(q) \geq \beta$ and $q$ is a forking extension of $p$.
- If $\lambda$ is a limit ordinal, $\text{SU}(p) \geq \lambda$ if $\text{SU}(p) \geq \beta$ for all $\beta < \lambda$.

Let $\text{SU}(p)$ be defined as the maximal $\alpha \in \text{Ord}$ such that $\text{SU}(p) \geq \alpha$. If there is no such maximal $\alpha$, let $\text{SU}(p) := \infty$.

If $p = \text{tp}(\bar{a}/B)$, we usually use the notation $\text{SU}(\bar{a}/B)$ instead of $\text{SU}(\text{tp}(\bar{a}/B))$.

Definition 4.19 (Supersimple). A simple structure $\mathfrak{M}$ is called supersimple, if all SU-ranks $\text{SU}(\bar{a}/B)$ are ordinal-valued (for $\bar{a} \in \mathfrak{M}$, $B \subset \mathfrak{M}$).

We summarise some important properties of the SU-rank.

Proposition 4.20 (Properties of SU-rank).

(i) $\text{SU}(\bar{a}/D) = 0$ if and only if $\text{tp}(\bar{a}/D)$ is algebraic.

(ii) Let $\bar{a} \in \mathfrak{M}$ and $\bar{a}'$ be a permutation of the tuple $\bar{a}$. Then $\text{SU}(\bar{a}) = \text{SU}(\bar{a}')$.

(iii) (Antimonotonicity). Let $\bar{a} \in \mathfrak{M}$ and $B, C \subset \mathfrak{M}$ with $C \supset B$. Then $\text{SU}(\bar{a}/C) \leq \text{SU}(\bar{a}/B)$.

(iv) Let $\bar{a} \in \mathfrak{M}$, $B, C \subset \mathfrak{M}$ with $\bar{a} \downarrow_B C$. Then $\text{SU}(\bar{a}/B) = \text{SU}(\bar{a}/BC)$.

(v) Let $\bar{a} \in \mathfrak{M}$, $B, C \subset \mathfrak{M}$ such that $\text{SU}(\bar{a}/B) = \text{SU}(\bar{a}/BC)$ and $\text{SU}(\bar{a}/B)$ is ordinal-valued. Then $\bar{a} \downarrow_B C$.

Proofs and proof references. [i] This is part of exercise 8.6.1 in Tent and Ziegler (2012). See the solution given there.

[ii] This is a straightforward (but tedious) walk through the definitions made. In essence it is due to the fact that the order of parameters does not determine whether or not a formula forks.

[iii] Assume that $\text{SU}(\bar{a}/C) \geq \alpha + 1$ for some $\alpha$. Then there is $\mathfrak{N} \succ \mathfrak{M}$ and $D \subset N$ with $D \supset C$, $\bar{a} \downarrow_C D$ and $\text{SU}(\bar{a}/D) \geq \alpha$. This implies $\bar{a} \downarrow_B D$, so $\text{SU}(\bar{a}/B) \geq \alpha + 1$. Using the definition of the SU-rank, $\text{SU}(\bar{a}/C) \leq \text{SU}(\bar{a}/B)$.

(iv) [v] Follow from Lemma 8.6.2 in Tent and Ziegler (2012).

The SU-rank satisfies some useful inequalities.

Theorem 4.21 (Lascar inequalities). Let $\mathfrak{M}$ be simple and $\bar{a}, \bar{b} \in \mathfrak{M}$, $C \subset \mathfrak{M}$. Then

$$\text{SU}(\bar{a}/\bar{b}C) + \text{SU}(\bar{b}/C) \leq \text{SU}(\bar{a}\bar{b}/C) \leq \text{SU}(\bar{a}/\bar{b}C) \oplus \text{SU}(\bar{b}/C).$$

Here “$+$” denotes usual ordinal addition while “$\oplus$” denotes the natural or commutative sum of ordinals. See Wagner (2000), p. 149 for a definition.
For finite SU-ranks we can get more, applying the fact that for for natural numbers, usual addition and natural sum coincide.

**Corollary 4.22** (Lascar equation for finite SU-ranks). Let $\mathfrak{M}$ be simple such that all SU-ranks in $\mathfrak{M}$ are finite. Then for all $\bar{a}, \bar{b} \in M$, $C \subset M$

$$SU(\bar{a}/\bar{b}C) + SU(\bar{b}/C) = SU(\bar{a}\bar{b}/C).$$

**Lemma 4.23.** Assume that $\mathfrak{M}$ is simple, and let $\bar{a}, \bar{b} \in M$ with $\bar{b} \in acl(\bar{a})$. Then $SU(\bar{a}) = SU(\bar{a}\bar{b})$.

**Proof.** By Proposition 4.20 (i), $SU(\bar{b}/\bar{a}) = 0$. So using Theorem 4.21 $SU(\bar{a}) = SU(\bar{b}\bar{a}) = SU(\bar{a}\bar{b})$. □

**Definition 4.24** (SU-rank of a structure). Let $\mathfrak{M}$ be simple structure. Let

$$SU(\mathfrak{M}) := \sup \{SU(\bar{a}) \mid M \succeq \mathfrak{M}, a \in N\}.$$

Note that this a supremum over the SU-ranks of 1-types only. By antimonicity, for every $B \subset M$, $\mathfrak{M} \succeq \mathfrak{M}$, $a \in N$ we have $SU(\bar{a}/B) \leq SU(\bar{a})$. This means that the SU-rank of a structure bounds the SU-ranks of 1-types over parameter sets as well.

**Lemma 4.25.** Assume that $\mathfrak{M}$ is simple with finite SU-rank. Then for all tuples $\bar{a} \in M$, for all $D \subset M$ the rank $SU(\bar{a}/D)$ is finite.

**Proof.** By induction on the length of the tuple $\bar{a}$, where the assumption that $\mathfrak{M}$ has finite SU-rank gives us the base case of length one. So assume $\bar{a} = \bar{b}c$ is a tuple of length $n + 1$ and that the claim holds for tuples of length $n$. By the Lascar inequalities,

$$SU(\bar{b}c/D) \leq SU(\bar{b}/cD) \oplus SU(\bar{c}/D).$$

Both SU-ranks on the right-hand side are finite by the inductive assumption, and therefore their natural sum is just the usual addition giving a finite number. We conclude that $SU(\bar{b}c/D)$ is finite as well. This finishes the inductive step and the claim follows. □

Note that the converse the lemma does not necessarily hold - even if all SU-ranks in a structure are finite, that structure may not have finite SU-rank as SU-ranks can be unbounded in $\omega$.

## 5 $\mathfrak{M}^{eq}$ and Imaginary Elements

Some questions about the reals can be better studied by extending the field of real numbers by “imaginary numbers”. Similarly, in model theory it is often useful to extend the model $\mathfrak{M}$ of a given theory by *imaginary elements*. The
resulting structure $M^{eq}$ turns out to be “closed” in some fundamental respects (similarly $\mathbb{C}$ is an algebraically closed field, while $\mathbb{R}$ is not). Moreover, many of the important properties of $\mathcal{M}$ will carry over to $M^{eq}$.

There are some slightly different ways to define $M^{eq}$, see for example [Hodges 1997] and [Tent and Ziegler 2012]. Our presentation of $M^{eq}$ is largely taken from [Jin 2013] who introduces $M^{eq}$ as a many-sorted structure.

**Definition 5.1 (Many-sorted language).** A many-sorted signature $\sigma^*$ consists of the following items:

(i) A list of sorts $S_1, \ldots, S_n$.

(ii) Relation symbols, where every relation symbol $R$ is associated with a finite tuple of sorts $(S_1, \ldots, S_m)$.

(iii) Function symbols, where every function symbol $f$ is associated with a finite tuple of sorts (of length $\geq 2$) $(S_1, \ldots, S_m, T)$ (we think of $(S_1, \ldots, S_m)$ as the domain sorts of $f$ and $T$ as the target sort of $f$).

(iv) Constant symbols, every constant symbol $c$ is associated with a sort $S$.

We assume that we have access to a countable set of variables $(x^i_{S_j} \mid i \in \omega)$ for every sort $S^i$. The language $L^*$ that is associated with $\sigma^*$ is obtained as follows: Formulas are build from the symbols in $\sigma^*$, variables, connectives and quantifiers as in unsorted first-order logic. However, every variable belongs now to a specific sort, and the variables or constants used in the scope of function or relation symbols must agree with the sort tuple associated to the symbol.

**Definition 5.2 (Many-sorted structure).** A many-sorted structure $\mathcal{M}$ in the signature $\sigma^*$ is a function with domain $\sigma^*$ with the following properties:

(i) For every sort $S^i \in \sigma^*$, its interpretation $\mathcal{M}(S^i)$ is a set.

(ii) If $R$ is relation symbol in the sorts $(S_1, \ldots, S_m)$, its interpretation $\mathcal{M}(R) \subset \mathcal{M}(S_1) \times \ldots \times \mathcal{M}(S_m)$ is a relation.

(iii) If $f$ is a function symbol $f$ in the sorts $(S_1, \ldots, S_m, T)$, then its interpretation is a function $\mathcal{M}(f) : \mathcal{M}(S_1) \times \ldots \times \mathcal{M}(S_m) \to \mathcal{M}(T)$.

(iv) If $c$ is a constant symbol in the sort $S$ then $\mathcal{M}(c) \in \mathcal{M}(S)$.

For any formula $\varphi(\mathcal{N}) \in L^*$, $\mathcal{M}(\varphi)$ is defined accordingly. Note that every quantifier ranges only over the interpretation of a sort.

Now we are ready to define the structure $\mathcal{M}^{eq}$. The imaginary elements $\mathcal{M}^{eq}$ extends $\mathcal{M}$ with will be the equivalence classes of definable equivalence relations.

**Definition 5.3 ($\mathcal{M}^{eq}$).** Fix a language $L$ and an $L$-structure $\mathcal{M}$ with theory $T$. We define a many-sorted signature $\sigma^{eq}$ (with corresponding language $L^{eq}$), and an $L^{eq}$-structure $\mathcal{M}^{eq}$ (with corresponding theory $T^{eq}$) as follows:

(i) The sorts of $\sigma^{eq}$ are as follows: For every $\emptyset$-definable, distinct equivalence relation $E \subseteq M^i \times M^j$ (for some $l$), $S^i_E$ is a sort.

(ii) For every $E$ as above, $\sigma^{eq}$ contains a function symbol $f^i_E$ with domain sort $S^i_E$ and target sort $S^j_E$. 

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(iii) Otherwise \( \sigma^{eq} \) exactly contains the symbols from \( \sigma \). If \( R \) is an \( l \)-ary relation symbol in \( \sigma \) then the sort of \( R \) in \( \sigma^{eq} \) is \( S_l \), similarly for function and constant symbols.

(iv) \( \mathfrak{M}^{eq}(S_E) \) is the set of equivalence classes of \( E \). We identify \( \mathfrak{M}^{eq}(S_\Xi) \) with \( M \).

(v) \( \mathfrak{M}^{eq} \models \sigma = \mathfrak{N} \) (so the symbols from \( \sigma \) are interpreted as in \( M \)).

(vi) For \( l \)-ary \( f \in \sigma^{eq} \): \( \mathfrak{M}^{eq}(f_E) : M^l \to M^l / E \) is the function \( (a_1, \ldots, a_l) \mapsto [a_1, \ldots, a_l]_E \).

One of the fundamental properties of \( \mathfrak{M}^{eq} \) is that any set that is definable in \( \mathfrak{M}^{eq} \) is “canonically definable” or definable by a “code”.

**Definition 5.4 (Code).** Let \( \mathfrak{M} \) be a model and assume that \( X \subset M \) is definable (possibly with parameters). We say that \( c \in M \) is a code for \( X \) if there is an \( L \)-formula \( \phi(x, y) \) such that \( \mathfrak{M} \models \phi(c) = X \) and if for any other \( c' \in M \) we have that if \( \mathfrak{M} \models \phi(c) = X \) then \( c = c' \).

The following is a collection of properties of \( \mathfrak{M}^{eq} \).

**Theorem 5.5 (Properties of \( \mathfrak{M}^{eq} \)).**

(i) For any \( L \)-formula \( \psi(x) \) and \( \bar{a} \in M \) we have \( \mathfrak{M} \models \psi(\bar{a}) \) if and only if \( \mathfrak{M}^{eq} \models \psi(\bar{a}) \).

(ii) Let \( \varphi(x) \) be a \( L^{eq} \)-formula such that every \( x_i \) is of sort \( S_\Xi \). Then there is an \( L \)-formula \( \psi(x) \) such that for every \( \bar{a} \in M \) we have that \( \mathfrak{M}^{eq} \models \varphi(\bar{a}) \leftrightarrow \psi(\bar{a}) \).

(iii) Let \( \bar{a}, \bar{b} \in M \) and \( D \subset M \). Then \( tp_{\mathfrak{M}}(\bar{a}/D) = tp_{\mathfrak{M}^{eq}}(\bar{b}/D) \) if and only if \( tp_{\mathfrak{M}^{eq}}(\bar{a}/D) = tp_{\mathfrak{M}^{eq}}(\bar{b}/D) \).

(iv) For any \( \bar{a} \in M^{eq} \) there is \( \bar{a}' \in M \) with \( \bar{a} \in dcl(\bar{a}') \).

(v) Assume that \( \mathfrak{M} \) is \( |T|^+ \)-saturated. Then every set in \( X \subset M^{eq} \) that is definable (possibly with parameters) has a code in \( M^{eq} \).

(vi) Assume \( \mathfrak{M} \) is arbitrary (not necessarily saturated) and let \( X \subset M^{eq} \) be finite. Then \( X \) has a code in \( M^{eq} \).

(vii) Assume that \( \mathfrak{M} \) is arbitrary, \( A \subset M^{eq} \) and that \( E(\bar{x}, \bar{y}) \) is an \( A \)-definable equivalence relation with finitely many classes, and assume that \( X \) is an equivalence class of \( E \). Then \( X \) is acl(A)-definable.

*Proofs and proof references.*

- (i): By induction over the definition of \( \mathfrak{M}^{eq} \).
- (ii), (iii): Folklore. Fact 2.8 in Jin (2013) contains statements of these facts.
- (iv): Straightforward from the definition of \( \mathfrak{M}^{eq} \).
- (v): This is Proposition 2.11 in Jin (2013).
- (vi): Let \( \mathfrak{N} \models \mathfrak{M}^{eq} \) be \( |T|^+ \)-saturated. By (v) there is a code \( c \in \mathfrak{N} \) for \( X \) (every finite set is definable with parameters). So there is a formula \( \varphi(x, y) \in L^{eq} \) with \( \varphi(\mathfrak{N}, c) = X \) and for every \( c' \in \mathfrak{N} \) with \( \varphi(\mathfrak{N}, c') = X \) we have \( c = c' \).
Say $n$ is the cardinality of $X$, and let $\bar{d}$ be an enumeration of $X$. Now $c$ satisfies the formula

$$\psi(c) := \forall x. \varphi(x, c) \leftrightarrow \bigvee_{i=1}^{n} x = d_i.$$ 

Since $M^{eq}$ is an elementary substructure of $N$ containing $\bar{d}$, we can choose $c' \in M^{eq}$ satisfying $\psi$. Then $\varphi(N, c') = X$ and by the the properties of codes it follows that $c = c'$. So $c$ already was in $M^{eq}$.

\[\text{(vii) Follows by Lemma 8.4.4 in Tent and Ziegler (2012).}\]

Property (v) of the previous theorem is often called “elimination of imaginaries”. In particular it states that if $M^{eq}$ is sufficiently saturated, any equivalence class of a definable equivalence relation in $M^{eq}$ is determined by a code lying in $M^{eq}$. In this sense $M^{eq}$ is closed - it is usually not necessary to pass from $M^{eq}$ to $(M^{eq})^{eq}$.

From now on, if not stated otherwise, we will always work in $M^{eq}$, and all the definitions referring to models are interpreted thereafter. For example, “$\text{tp}$” will mean “$\text{tp}_{M^{eq}}$”.

6 Binary Simple Homogeneous Structures

As mentioned in the introduction, binary simple homogeneous structures were extensively studied by Vera Koponen. In this section we list some results which we will need later on. In the following let $M$ be countable, binary, simple and homogeneous.

**Theorem 6.1.** $M$ is supersimple with finite SU-rank.

*Proof reference.* This is the main result of [Koponen (2016)].

Often we will not work in $M$, but in its extension $M^{eq}$. Therefore the exact properties of $M^{eq}$ are very important.

**Lemma 6.2** (Independence in $M^{eq}$).

(i) $M^{eq}$ is simple.

(ii) Let $\bar{a} \in M$ and $B, C \subset M$. Then $\bar{a} \perp_{B} C$ holds in $M$ if and only if it holds in $M^{eq}$.

(iii) Let $\bar{a} \in M$, $B \subset M$. Then $SU_{M}(\bar{a}/B) = SU_{M^{eq}}(\bar{a}/B)$ (so the SU-rank of $tp_{M}(\bar{a}/B)$ taken in $M$ as the SU-rank of $tp_{M^{eq}}(\bar{a}/B)$ taken in $M^{eq}$).

*Proof references.* (ii) Follows by Remark 2.27 in [Casanovas (2011)] or alternatively from Corollary 2.8.11 in [Wagner (2000)].

(iii) Folklore.

In $M$ the SU-ranks of 1-types are bounded by a finite number. In $M^{eq}$ this is not true, but the statement of the following theorem will be enough for our purposes.
Theorem 6.3. All SU-ranks in $\mathfrak{M}^\text{eq}$ are finite. In particular, $\mathfrak{M}^\text{eq}$ is supersimple.

Proof. The previous lemma shows that $\mathfrak{M}^\text{eq}$ is simple. To prove the statement, we will make use of the Lascar inequalities (Theorem 4.21).

Let $\bar{a} \in M^\text{eq}$ and $B \subset M^\text{eq}$. By the properties of $\mathfrak{M}^\text{eq}$ there are $\bar{a}' \in M$ and $B' \subset M$ with $\bar{a} \in \text{dcl}(\bar{a}')$ and $B \subset \text{dcl}(B')$. By Theorem 4.14 (i) $\bar{a} \downarrow_{\text{acl}(B')} B'$, and by (viii) of the same theorem $\bar{a} \downarrow_{B} B'$ by monotonicity. Using Proposition 4.20 (iii) and (iv) this implies $SU(\bar{a}/B') \geq SU(\bar{a}/BB') = SU(\bar{a}/B)$.

So it is enough to show that $SU(\bar{a}/B')$ is finite. The type $\text{tp}(\bar{a}/B')$ is algebraic (it has only one realisation), so by Proposition 4.20 (i) $SU(\bar{a}/\bar{a}'B') = 0$. By the Lascar inequalities it follows $SU(\bar{a}/B') = SU(\bar{a}/B'B)$.

We conclude this section with a list of other useful properties of $\mathfrak{M}^\text{eq}$.

Proposition 6.4 (Types and sets in $\mathfrak{M}^\text{eq}$).

(i) $\mathfrak{M}^\text{eq}$ is countable.

(ii) Let $\bar{a} \in M^\text{eq}$ and $D \subset M^\text{eq}$ be finite. Then $\text{tp}(\bar{a}/D)$ and $\text{tp}(\bar{a}/\text{acl}(D))$ are both isolated.

(iii) Assume $\bar{a}, \bar{b}, \bar{b}' \in M^\text{eq}$ with $\text{tp}(\bar{b}) = \text{tp}(\bar{b}')$. Then there is $\bar{a}' \in M^\text{eq}$ with $\text{tp}(\bar{a}'\bar{b}') = \text{tp}(\bar{a}\bar{b})$.

(iv) Let $D \subset M^\text{eq}$ be finite and $C \subset M^\text{eq}$ some set containing only finitely many sorts. Then $\text{acl}(D) \cap C$ is finite. If $C$ is $\emptyset$-definable, then $\text{acl}(D) \cap C$ is definable in $D$ (as a set).

Proofs and proof references.

(i) Since $L$ is countable, there are at most countable many $n$-ary $\emptyset$-definable equivalence relations on $\mathfrak{M}$, for any $n$. Since $\mathfrak{M}$ is countable, every equivalence relation gives rise to at most countable many equivalence classes. So by the construction of $\mathfrak{M}^\text{eq}$ it is countable.

(ii) That $\text{tp}(\bar{a}/\text{acl}(D))$ is isolated is Fact 2.14 (ii) in Ahlman and Koponen (2015), and that $\text{tp}(\bar{a}/D)$ is as well is given in the proof of this fact.

(iii) By (ii) $\text{tp}(\bar{a}/\bar{b})$ is isolated, say by a formula $\varphi(\bar{x}, \bar{b})$. Then it holds $\exists \bar{x}\varphi(\bar{x}, \bar{b})$ and since $\text{tp}(\bar{b}) = \text{tp}(\bar{b}')$ we have $\exists \bar{x}\varphi(\bar{x}, \bar{b}')$. Choose $\bar{a}'$ with $\varphi(\bar{a}', \bar{b}')$. By the choice of $\varphi$ it follows $\text{tp}(\bar{a}'\bar{b}') = \text{tp}(\bar{a}/\bar{b})$.

(iv) The first statement is Fact 2.11 (ii) in Ahlman and Koponen (2015), the second one is a straight-forward corollary.  \qed
7 Coordinatisations

Coordinatisations, in the sense used here, were introduced in Djordjevic (2006), and our presentation is largely borrowed from Koponen (2017b). They are a powerful machinery which we will make extensive use of in this thesis. Coordinatisations allow us to neatly describe the properties of an element in \( M \) via the properties of its finitely many coordinates. As before, let \( M \) be a countable binary simple homogeneous structure.

Definition 7.1 (Self-coordinatised). We say that \( C \subset M^{eq} \) is self-coordinatised if the following conditions hold for any \( a \in C \) with \( SU(a) > 1 \):

- There is \( b \in acl(a) \cap C \) such that \( SU(a/b) = 1 \).
- For any \( b \in acl(a) \cap C \) such that \( SU(a/b) = 1 \): If there is \( c \in acl(a) \setminus acl(b) \) with \( a \notin acl(c) \), then such \( c \) exists in \( C \).

Theorem 7.2. There is \( C \) with \( M \subset C \subset M^{eq} \) and \( (C_i \mid 0 \leq i \leq h) \) with \( \emptyset = C_0 \subset C_1 \subset \ldots \subset C_h \subset C \) such that:

(i) Only finitely many sorts are represented in \( C \).
(ii) \( C \) is self-coordinatised and \( C \subset acl(C_h) \).
(iii) \( C \) is \( \emptyset \)-definable, and every \( C_i \) as well.
(iv) For every \( i < h \) and \( c \in C_{i+1} \), \( SU(c/C_i) = 1 \) and \( acl(c) \cap C_i \neq \emptyset \).

We refer to \( (C, (C_i \mid i \leq h)) \) as a coordinatisation of \( M \). \( h \) is called the height of the coordinatisation.

Proof reference. This is a special case of Fact 3.2 in Koponen (2017b), which in turn derives from the construction in Djordjevic (2006).

From now on we fix \( C \) and \( (C_i \mid 0 \leq i \leq h) \) with minimal height. We say that \( h \) is the height of \( M \).

Definition 7.3 (Levels of coordinatisation). Let \( L_0 := C_0 \), and for \( i \in \{1, \ldots, h\} \) let \( L_i := C_i \setminus C_{i-1} \). We refer to \( L_i \) as the level \( i \) of the coordinatisation.

Definition 7.4 (Coordinates). Let \( \bar{a} \subset C \):

- For any \( i \leq h \), let \( crd_i(\bar{a}) := acl(\bar{a}) \cap C_i \). We refer \( crd_i(\bar{a}) \) as the set of \( \bar{a} \)'s coordinate up to level \( i \).
- We let \( crd(\bar{a}) := crd_h(\bar{a}) \) and refer to \( crd(\bar{a}) \) as the set of coordinates of \( \bar{a} \).

Definition 7.5 (crd-closed). Let \( A \subset C_h \). \( A \) is called crd-closed if \( crd(A) = A \).

In the following we list some important properties of coordinatisations.

Theorem 7.6 (Properties of coordinatisations).

(i) For every \( a_1 \ldots a_n \subset C \) and \( i \leq h \), \( crd_i(a_1 \ldots a_n) = \bigcap_{j=1}^n crd_i(a_j) \). In particular it follows \( crd(a_1 \ldots a_n) = \bigcap_{j=1}^n crd(a_j) \).
(ii) For every \( \bar{a} \in C \), \( \bar{a} \in acl(crd(\bar{a})) \). It follows \( acl(\bar{a}) = acl(\bar{a}) \).
Let \( i < h \): For any \( a \in L_{i+1} \) we have \( SU(a/crd_i(a)) = 1 \).

(iv) For \( \bar{a}, \bar{b} \in C, D \subseteq C \) finite: \( \bar{a} \downarrow_D \bar{b} \) if and only if \( acl(D) \supset crd(\bar{a}) \cap crd(\bar{b}) \).

(v) If \( a \in C_h \) and \( \bar{d} \in M^a \) such that \( a \in acl(\bar{d}) \), there is \( d_i \in rng(\bar{d}) \) such that \( a \in acl(d_i) \).

Proof references. (i)-(iii) are contained in Fact 3.5 in Koponen (2017b), confer the proof references given there. (iv) is Lemma 3.7 in Koponen (2017b). (v) is Lemma 3.16 in Djordjevic (2006).

For our purposes later on, it is necessary to require some extra properties from our coordinatisation.

**Definition 7.7** (Minimality property). The coordinatisation \( C \) satisfies the minimality property if for all \( 1 \leq i \leq h \) and \( c \in L_i \), \( crd(c) \cap L_i = \{c\} \).

**Definition 7.8** (Definability property). The coordinatisation \( C \) satisfies the definability property if for all \( a \in C \) and any \( b \in crd(a) \), \( b \) is definable by \( a \).

**Theorem 7.9.** There is a coordinatisation \( (C', (C'_{i} \mid i \leq h)) \) of \( M \) of the same height as \( C \) that satisfies the minimality and the definability property.

**Proof.** This is Proposition 14 in Koponen (2018).

In the following we will assume that our coordination \( C \) satisfies the minimality and the definability property.

**Lemma 7.10** (Definability of coordinates).

(i) For any \( 1 \leq l \leq h \), the relation \( R(x, y) := x \in crd_l(y) \) is \( \emptyset \)-definable. In particular, the relation \( x \in crd_l(y) \) is definable.

(ii) Assume \( a, b \in C \) with \( tp(a) = tp(b) \). Then \( |crd_l(a)| = |crd_l(b)| \) for any \( l \). Furthermore, the coordinates of \( a \) and \( b \) are defined by the same formulas, so if \( \varphi_1(x, a), \ldots, \varphi_n(x, a) \) define the \( n \) distinct coordinates of \( a \), then \( \varphi_1(x, b), \ldots, \varphi_n(x, b) \) define the \( n \) coordinates of \( b \).

**Proof.**

(i) By definition, \( x \in crd_l(y) \iff x \in acl(y) \cap C_l \). Since any \( C_l \) is \( \emptyset \)-definable, the claim follows from Theorem 6.4 (iv).

(ii) If \( |crd_l(a)| = n \), then the formula \( \exists^n x : x \in crd_l(y) \) is in \( tp(a) \), which proves the first claim. If \( \varphi_i(x, a) \) defines a coordinate of \( a \), then the formulas \( \forall x \varphi_i(x, a) \rightarrow x \in crd(y) \) and \( \exists^1 x \varphi_i(x, a) \) lie in \( tp(a) \), this proves the second claim.

The previous lemma shows that if \( a \) and \( b \) behave the same types, their coordinates behave the same.

**Definition 7.11** (Cognate coordinate). Let \( \bar{a}, \bar{b} \in C \) with \( tp(\bar{a}) = tp(\bar{b}) \) and let \( c \in crd(\bar{a}) \). Then the cognate of \( c \) (with respect to \( \bar{b} \)) is the unique \( d \in crd(\bar{b}) \) with \( tp(\bar{a}, c) = tp(\bar{b}, d) \). We also say that \( c \) and \( d \) are cognate or that \( c \) is cognate with \( d \).
Note that \( c \in \text{crd}(\overline{a}) \) and \( b \in \text{crd}(\overline{b}) \) are cognate if and only if they are defined by the same formula in \( \overline{a} \) and \( \overline{b} \), so there is \( \varphi(x, \overline{y}) \in \mathcal{L} \) with \( \varphi(\mathfrak{M}^\text{eq}, \overline{a}) = \{c\} \) and \( \varphi(\mathfrak{M}^\text{eq}, \overline{b}) = \{d\} \).

**Definition 7.12** (Arranging coordinate tuples). Let \( \overline{a}, \overline{b} \in \mathcal{C} \) with \( \text{tp}(\overline{a}) = \text{tp}(\overline{b}) \) and let \( \overline{c} \in \text{crd}(\overline{a}) \), \( \overline{d} \in \text{crd}(\overline{b}) \) be tuples of the same length. We say that \( \overline{c} \) and \( \overline{d} \) are arranged by cognates or arranged in the same way if \( \text{tp}(\overline{a}, \overline{c}) = \text{tp}(\overline{b}, \overline{d}) \).

It is easy to see that \( \overline{c} \) and \( \overline{d} \) are arranged by cognates if only if for any index \( i \) we have that \( c_i \) and \( d_i \) are cognate, using Theorem 7.6(i) and Lemma 7.10.

In many of our arguments it is vital that coordinate tuples are arranged by cognates. We often suppress this assumption, but whenever reasonable coordinate tuples are assumed to be arranged in the same way.

The coordinates of some element \( a \in M \) should be seen as an ordered set of witnesses describing the behaviour of \( a \). In the following we will make this more concrete.

**Definition 7.13** (\( \langle C \rangle \)). We define a relation \( \langle C \rangle \) on \( \mathcal{C}_h \) as follows:

\[
a \langle C \rangle b : \iff a \in \text{acl}(b) \wedge b \notin \text{acl}(a).
\]

It is straightforward to see that that \( \langle C \rangle \) is irreflexive, anti-symmetric and transitive, i.e. a strict partial order. The following are some characterisations \( \langle C \rangle \). Note that the equivalence of (v) to the other requires that the coordinatisation \( \mathcal{C} \) is minimal.

**Proposition 7.14** (Characterisations of \( \langle C \rangle \)). Let \( a, b \in \mathcal{C}_h \). The following are equivalent:

(i) \( a \langle C \rangle b \)
(ii) \( a \in \text{acl}(b) \) and \( \text{SU}(a) < \text{SU}(b) \)
(iii) \( a \in \text{acl}(b) \) and \( \text{SU}(b/a) > 0 \)
(iv) \( a \in \text{acl}(b) \), \( a \in \mathcal{L}_i \), \( b \in \mathcal{L}_j \) where \( i < j \)
(v) \( a \in \text{crd}(b) \) and \( a \neq b \).

**Proof.** Note first that in every statement we have that \( a \in \text{acl}(b) \), using the definitions of \( \langle C \rangle \) and \( \text{crd} \). From this it follows by Lemma 4.23 that \( \text{SU}(b) = \text{SU}(ba) \). So using the Lascar equation (Corollary 4.22) we always have

\[
\text{SU}(b) = \text{SU}(ba) = \text{SU}(b/a) + \text{SU}(a). \tag{1}
\]

(i) \( \implies \) (ii): We have \( a \in \text{acl}(b) \) and \( b \notin \text{acl}(a) \). From the latter it follows that \( \text{tp}(b/a) \) is not algebraic, so by Proposition 4.20(i) we have \( \text{SU}(b/a) \geq 1 \). Using equation (1) it follows \( \text{SU}(b) > \text{SU}(a) \).

(ii) \( \implies \) (iii): From \( \text{SU}(b) > \text{SU}(a) \) it follows that \( \text{SU}(b/a) > 0 \) by equation (1).

(iii) \( \implies \) (iv): Choose \( i \) and \( j \) with \( a \in \mathcal{L}_i \), \( b \in \mathcal{L}_j \). From the definition of \( \text{crd} \) it follows that \( a \in \text{crd}(b) \), and by (iii) it follows that \( a \neq b \). By the minimality property \( i \neq j \) (in fact this can also be derived without using the minimality
property, but by a more involved argument). Assume \( i > j \). Then \( SU(a/C_{i-1}) = 1 \) (by Theorem 7.2), so by antimonotonicity (Proposition 4.20 (iii)) \( SU(a/b) \geq 1 \). Using the Lascar equation (Corollary 4.22) we have \( SU(ab) = SU(a/b) + SU(b) \). Together with equation (1) this gives

\[
SU(b) = SU(ba) = SU(ab) = SU(a/b) + SU(b),
\]

contradicting \( SU(a/b) \geq 1 \), so \( i < j \).

(iv) \( \Rightarrow \) (v): Obvious.

(v) \( \Rightarrow \) (iii): We have to show that \( SU(b/a) > 0 \). Let \( i, j \) be given with \( a \in L_i, b \in L_j \). By the minimality property \( i \neq j \). Now, by exactly the same argument as in our proof of (iii) \( \Rightarrow \) (iv) we get \( i < j \). Then \( SU(b/C_{j-1}) = 1 \), so by antimonotonicity \( SU(b/a) \geq 1 \).

(iii) \( \Rightarrow \) (i): If \( SU(b/a) > 0 \), \( b /a \in acl(D) \) follows by Proposition 4.20 (i).

We conclude this section with a lemma about \( <C \)-maximal elements, which will be useful later on.

**Lemma 7.15.** Let \( D \subset C_h \) be finite and \( c \in D \) such that \( crd(c) \subset D \) and \( c \) is \( <C \)-maximal in \( D \). Then \( SU(c/(D \setminus \{ c \})) = 1 \).

**Proof.** Choose \( j \) such that \( c \in L_j \). Using Theorem 7.6 (iii) we have that \( SU(c/crd_{j-1}(c)) = 1 \). Since \( crd_{j-1}(c) \subset D \setminus \{ c \} \), by antimonotonicity it follows that \( SU(D \setminus \{ c \}) \leq 1 \).

It remains to show that \( SU(c/(D \setminus \{ c \})) \geq 1 \). Assume \( SU(c/D \setminus \{ c \}) = 0 \). Then by Proposition 4.20 (i) we have \( c \in acl(D \setminus \{ c \}) \). Using Theorem 7.6 (v) there is \( d \in D \setminus \{ c \} \) with \( c \in acl(d) \). In that case \( c \in crd(d) \), and we also have \( c \neq d \). Using Proposition 7.14 we have \( c <C d \), which contradicts the maximality of \( c \).

8 The Algebraic Closure Property

For the proof of the main result of the thesis we have to make an extra assumption on our binary simple homogeneous \( \mathfrak{M} \):

**Definition 8.1** (Algebraic closure property). Some structure \( \mathfrak{M} \) has the **algebraic closure property**, if for \( \bar{a}, \bar{b} \in M^{eq} \) and finite \( D \subset M^{eq} \) we have that \( tp(\bar{a}/D) = tp(\bar{b}/D) \) implies \( tp(\bar{a}/acl_{eq}(D)) = tp(\bar{b}/acl_{eq}(D)) \).

**Example 8.2.** The random graph satisfies the algebraic closure property.

**Proof reference.** In the random graph, algebraic closure is “degenerate”, meaning that \( acl_{eq}(D) = D \) for all \( D \subset R \). This is due to the fact that for all formulas \( \varphi(\bar{x}, \bar{a}) \) there are either infinitely many realisations in \( R \), or all realisations lie in \( \bar{a} \). This is not too hard to see (using that \( \mathcal{R} \) has quantifier-elimination).

The random graph is also a “random structure” in the sense of Definition 2.1 in Koponen (2017c). Using Corollary 6.2 and Example 6.4 (i) of the same paper, it follows \( acl_{eq}(D) = dcl_{eq}(D) \) for all \( D \subset R \). This implies the result.

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Background assumptions
From now on and through the rest of this thesis assume that \( \mathfrak{M} \) is a fixed countable binary simple homogeneous structure satisfying the algebraic closure property, and that \( C \) is a fixed coordinatisation of \( \mathfrak{M} \) with the minimality and definability property.

The algebraic closure property will allow us to use a stronger form of the independence theorem. We will also need it for a result on the definability of elements in \( C \). The results on 2-type we will give in the next section also make use of it.

**Definition 8.3 (Strong type).** For \( \mathfrak{M} \), \( \bar{a} \in M^{\text{eq}} \) and \( B \subset M^{\text{eq}} \), define the strong type of \( \bar{a} \) over \( B \) by

\[
\text{stp}(\bar{a}/B) := \{ [\bar{a}]_E \mid E(\bar{x}, \bar{y}) \text{ B-definable finite equivalence relation} \}
\]

Here “finite” means that \( E \) has only finitely many classes. So we have that \( \text{stp}(\bar{a}'/B) = \text{stp}(\bar{a}/B) \) if and only if \( \bar{a}' \) is equivalent to \( \bar{a} \) under all \( B \)-definable finite equivalence relations \( E \).

Often the strong type is defined in a different way, but the definition using equivalence relations will be more handy in our applications. The usual definition turns out to be equivalent to ours, see Exercise 8.4.9 in Tent and Ziegler (2012).

It is easy to see that \( \text{stp}(\bar{a}/B) = \text{stp}(\bar{a}'/B) \) implies \( \text{tp}(\bar{a}/B) = \text{tp}(\bar{a}'/B) \):
If \( \varphi(\bar{x}, \bar{y}) \in \text{tp}(\bar{a}/B) \), then the formula \( E(\bar{x}, \bar{y}) := \varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{y}, \bar{b}) \) gives a \( B \)-definable equivalence relation with at most two classes, and it follows \( \varphi(\bar{a}'/\bar{b}) \leftrightarrow \varphi(\bar{a}/\bar{b}) \).

With the algebraic closure property we can get the other direction in the case of finite parameter sets.

**Lemma 8.4.** Let \( \bar{a}_1, \bar{a}_2 \in M^{\text{eq}} \) and \( B \subset M^{\text{eq}} \) be finite. Then \( \text{tp}(\bar{a}_1/B) = \text{tp}(\bar{a}_2/B) \) implies \( \text{stp}(\bar{a}_1/B) = \text{stp}(\bar{a}_2/B) \).

**Proof.** Assume that \( \text{tp}(\bar{a}_1/B) = \text{tp}(\bar{a}_2/B) \). Then with the algebraic closure property, \( \text{tp}(\bar{a}_1/\text{acl}(B)) = \text{tp}(\bar{a}_2/\text{acl}(B)) \). Let \( E(\bar{x}, \bar{y}) \) be a \( B \)-definable finite equivalence relation. Then the set \( E(\mathfrak{M}^{\text{eq}}, \bar{a}_1) \) is \( \text{acl}(B) \)-definable by Theorem 5.3[vii]. So \( E(\bar{a}_1, \bar{a}_1) \) and \( \text{tp}(\bar{a}_1/\text{acl}(B)) = \text{tp}(\bar{a}_2/\text{acl}(B)) \) imply \( E(\bar{a}_2, \bar{a}_1) \).
Hence \( \text{stp}(\bar{a}_1/B) = \text{stp}(\bar{a}_2/B) \).

**Theorem 8.5.** Let \( \bar{a}_1, \bar{a}_2 \in M^{\text{eq}} \) and \( B \subset M^{\text{eq}} \) be finite. Then \( \text{stp}(\bar{a}_1/B) = \text{stp}(\bar{a}_2/B) \) implies \( \text{Lstp}(\bar{a}_1/B) = \text{Lstp}(\bar{a}_2/B) \).

**Proof reference.** By Fact 2.14 (iii) in Ahlman and Koponen (2013), every type over a finite set in \( \mathfrak{M}^{\text{eq}} \) is realised in \( \mathfrak{M}^{\text{eq}} \). By Proposition 6.4 (i) \( \mathfrak{M}^{\text{eq}} \) is countable, so there can only be countably many such types. That means that \( \text{Th}(\mathfrak{M}^{\text{eq}}) \) is “small” in the sense of Kim (2013), so the result follows by Corollary 5.3.5 in Kim.
Now we are finally able to give the desired form of the independence theorem mentioned earlier.

**Theorem 8.6** (Independence Theorem for finite sets in $\mathfrak{M}^\aleph_0$). Let $\bar{c}_1, \bar{c}_2, \bar{a}_1, \bar{a}_2 \in M^\aleph_0$ and $D \subset M^\aleph_0$ be finite with $\bar{a}_1 \sqsubseteq_D \bar{a}_2$, $\bar{c}_1 \sqsubseteq_D \bar{a}_1$ and $\bar{c}_2 \sqsubseteq_D \bar{a}_2$. If $\text{tp}(\bar{c}_1/D) = \text{tp}(\bar{c}_2/D)$, then there is $\bar{c} \in \mathfrak{M}^\aleph_0$ with $\text{tp}(\bar{c}/\bar{a}_1) = \text{tp}(\bar{c}_1/\bar{a}_1)$, $\text{tp}(\bar{c}/\bar{a}_2) = \text{tp}(\bar{c}_2/\bar{a}_2)$ and $\bar{c} \sqsubseteq_D \bar{a}_1 \bar{a}_2$.

**Proof.** Let the assumption of the theorem be true. By Lemma 8.4 and Theorem 8.5 we have $\text{Lstp}(\bar{c}/\bar{a}_1) = \text{Lstp}(\bar{c}_1/\bar{a}_1)$ and $\text{Lstp}(\bar{c}/\bar{a}_2) = \text{Lstp}(\bar{c}_2/\bar{a}_2)$. By the Independence theorem 4.13, there are $\mathfrak{M} \models \mathfrak{M}^\aleph_0$ and $\bar{c} \in N$ with $\text{tp}(\bar{c}/\bar{a}_1) = \text{tp}(\bar{c}_1/\bar{a}_1)$, $\text{tp}(\bar{c}/\bar{a}_2) = \text{tp}(\bar{c}_2/\bar{a}_2)$ and $\bar{c} \sqsubseteq_D \bar{a}_1 \bar{a}_2$. Now $\text{tp}(\bar{c}/\bar{a}_2D)$ is isolated by Proposition 6.4(ii) (note that it is here we use that $D$ is finite), hence realised in $\mathfrak{M}^\aleph_0$. So there is $\bar{c}' \in \mathfrak{M}^\aleph_0$ with $\text{tp}(\bar{c}'/\bar{a}_1) = \text{tp}(\bar{c}_1/\bar{a}_1)$, $\text{tp}(\bar{c}'/\bar{a}_2) = \text{tp}(\bar{c}_2/\bar{a}_2)$ and $\bar{c}' \sqsubseteq_D \bar{a}_1 \bar{a}_2$ (using that the independence relation is type-definable, see Lemma 4.13). This finishes the proof.

The following theorem is the other main result of this section:

**Theorem 8.7** (Definability by coordinates). For any $a \in M$ we have that $a \in \text{acl}(\text{crd}(a))$.

**Proof.** Let $D := \text{acl}(\text{crd}(a)) \cap C$. By Theorem 7.6(ii) we have $a \in D$. By Theorem 6.4(iv) $D$ is finite, and definable by $\text{crd}(a)$. Choose $\psi$ with $\psi(\mathfrak{M}^\aleph_0, \text{crd}(a)) = D$. By Theorem 6.4(ii) $\text{tp}(a)$ is isolated, say by a formula $\varphi(x)$. If $a$ is the only element in $D$ with type $\text{tp}(a)$, then $a$ is definable by $\varphi(x) \land \psi(x, \text{crd}(a))$. Let us assume that there is $b$ with $\text{tp}(a) = \text{tp}(b)$. Then we have

$$\text{crd}(b) = \text{acl}(b) \cap C \subset \text{acl}(a) \cap C = \text{crd}(a),$$

using $b \in \text{acl}(\text{crd}(a))$ and $\text{acl}(\text{crd}(a)) \subset \text{acl}(a)$ for the subset relation. Using that $|\text{crd}(b)| = |\text{crd}(a)|$ (Lemma 7.10(iii)), it follows that $\text{crd}(a) = \text{crd}(b)$.

Let $D' := \{d \in D \mid \text{tp}(d) = \text{tp}(a)\}$. Then $D' \subset D$, so $D'$ is finite. Furthermore $D$ is definable by $\varphi(x) \land \psi(x, \text{crd}(a))$.

By Theorem 5.5(iv) there is a code $c \in M^\aleph_0$ for the set $D'$. So there is a formula $\xi(x, y) \in \mathcal{L}$ with $\xi(\mathfrak{M}^\aleph_0, c) = D'$ and for any other $c'$ with $\xi(\mathfrak{M}^\aleph_0, c') = D'$ we have $c = c'$. By the definability property $\text{crd}(a)$ is pointwise definable by $a$, so now $c$ is definable by $a$ by

$$\eta(y, a) := \forall x \varphi(x) \land \psi(x, \text{crd}(a)) \leftrightarrow \xi(x, y).$$

Furthermore, $c$ is definable by $\eta(y, b)$ as well (using that $\text{crd}(a) = \text{crd}(b)$). So from $\text{tp}(a) = \text{tp}(b)$ it follows that $\text{tp}(a/c) = \text{tp}(b/c)$ (using Lemma 4.3).

Using the Algebraic closure property this implies $\text{tp}(a/\text{acl}(c)) = \text{tp}(b/\text{acl}(c))$, and in particular $\text{tp}(a/D') = \text{tp}(b/D')$. But $D'$ contains both $a$ and $b$, so $\text{tp}(a/D')$ contains the formula $x = b$. It follows that $a = b$. This concludes the proof.

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9 2-Types

In $\mathfrak{M}$, types are fully determined by their 2-subtypes (types containing only two variables or parameters). In $\mathfrak{M}^\text{eq}$ this is not true, but some partial results hold. They will be important ingredients in the arguments of the following sections.

**Definition 9.1 (2-type).** Let $\overline{a}$ be a tuple of length $n$. $\text{tp}^2(\overline{a})$ is the subset of all formulas in $\text{tp}(\overline{a})$ that contains at most two variables. In other words,

$$\text{tp}^2(\overline{a}) = \bigcup_{i,j \leq n} \text{tp}(a_i, a_j),$$

where variables have been renamed properly. If $D$ is a set, we define

$$\text{tp}^2(\overline{a}/D) := \text{tp}^2(\overline{a}) \cup \bigcup_{i \leq n} \text{tp}(a_i/d).$$

**Lemma 9.2 (Types in $\mathfrak{M}$ are determined by 2-subtypes).** Let $\overline{a}, \overline{b} \in \mathfrak{M}$ and $D \subset \mathfrak{M}$. Then

$$\text{tp}(\overline{a}/D) = \text{tp}(\overline{b}/D) \iff \text{tp}^2(\overline{a}/D) = \text{tp}^2(\overline{b}/D).$$

**Proof.** Using the properties of $\mathfrak{M}^\text{eq}$ (Theorem 5.5 (iii)) it is enough to show that $\text{tp}^2_M(\overline{a}/D) = \text{tp}^2_M(\overline{b}/D)$ implies $\text{tp}_M(\overline{a}/D) = \text{tp}_M(\overline{b}/D)$. This is a straightforward induction, using that $\mathfrak{M}$ has quantifier-elimination and contains only binary relation symbols.

For $\mathfrak{M}^\text{eq}$ the situation is more complicated, but if restrict ourselves to co-ordinates of elements in $\mathfrak{M}$ we can get some strong results.

**Theorem 9.3.** Suppose that $b_1 \in \mathfrak{M}$, $X \subset \text{crd}(b_1)$ and that $\bar{c} \in \text{crd}(b_1)$ is a tuple of length $m$. Assume that $b_2^1, \ldots, b_2^m \in \mathfrak{M}$ and that $\bar{d}$ is a tuple of length $m$ with $d_i \in \text{crd}(b_2^i)$ for every $i \leq m$. Assume furthermore $\text{SU}(c_i/X) = 1$ for every $i \leq m$. If we have

$$\text{tp}(b_1/X) = \text{tp}(b_2^i/X), \text{for all } i \leq m$$

$$\text{tp}(c_i c_j/X) = \text{tp}(d_i d_j/X) \text{ for all } i, j \leq m,$n

$$\text{tp}(c_i a_j/X) = \text{tp}(d_i a_j/X) \text{ for all } i \leq m, j \leq n,$n

then $\text{tp}(\bar{c}/\bar{a}X) = \text{tp}(\bar{d}/\bar{a}X)$.

**Proof reference.** This is Proposition 1 in Koponen (2018).

The existence of the different $b_2^i$ (for $i \leq m$) is a very specific assumption and can in many cases be strengthened to the existence of some $b_2$ with $\bar{d} \subset \text{crd}(b_2)$ and $\text{tp}(b_1/X) = \text{tp}(b_2/X)$.

The following corollary shows that some parameters from the theorem can be chosen more freely:

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Corollary 9.4. Suppose that $b_1, b_2 \in M$, that $\vec{e} \in \text{crd}(b_1)$ and $\vec{e}' \in \text{crd}(b_2)$ are tuples of the same length, that $\vec{e} \in \text{crd}(b_1)$ and $\vec{d} \in \text{crd}(b_2)$ have length $m$, and that $\vec{a}$ and $\vec{a}'$ are tuples of length $n$ in $M$. Assume furthermore $\text{SU}(e_i/\vec{e}) = 1$ for every $i \leq m$. If we have

$$
\begin{align*}
\text{tp}(b_1\vec{e}) &= \text{tp}(b_2\vec{e}'), \\
\text{tp}(\vec{e}\vec{a}) &= \text{tp}(\vec{e}'\vec{a}''), \\
\text{tp}(c_i\vec{c}_j\vec{e}) &= \text{tp}(d_i\vec{d}_j\vec{e}') \text{ for all } i, j \leq m, \text{ and} \\
\text{tp}(c_i\vec{c}_j\vec{e}) &= \text{tp}(d_i\vec{a}_j\vec{e}') \text{ for all } i \leq m, j \leq n,
\end{align*}
$$

then $\text{tp}(\vec{e}\vec{a}) = \text{tp}(\vec{d}\vec{a}'\vec{e}')$.

**Proof.** Assume that all the premises of the corollary are satisfied, but that the conclusion does not hold, i.e. $\text{tp}(\vec{e}\vec{a}) \neq \text{tp}(\vec{d}\vec{a}'\vec{e}')$. Using $\text{tp}(\vec{e}\vec{a}) = \text{tp}(\vec{e}'\vec{a}'')$ and Proposition 6.4 (iii) we can find $b_1^* \in M$, $\vec{e}'' \in \mathbb{N}^m$ with

$$
\text{tp}(b_1^*\vec{e}''\vec{a}) = \text{tp}(b_2\vec{d}\vec{a}'\vec{e}').
$$

Then we have

$$
\begin{align*}
\text{tp}(b_1^*\vec{e}) &= \text{tp}(b_2\vec{e}') = \text{tp}(b_1\vec{e}), \\
\text{tp}(c_i^*\vec{c}_j\vec{e}) &= \text{tp}(d_i\vec{d}_j\vec{e}') = \text{tp}(c_i\vec{c}_j\vec{e}) \text{ for all } i, j \leq m, \\
\text{tp}(c_i^*\vec{c}_j\vec{e}) &= \text{tp}(d_i\vec{a}_j\vec{e}') = \text{tp}(c_i\vec{a}_j\vec{e}) \text{ for all } i \leq m, j \leq n,
\end{align*}
$$

so by Theorem 9.3 it follows $\text{tp}(\vec{e}\vec{a}) = \text{tp}(\vec{e}\vec{a})$. By (2) this implies $\text{tp}(\vec{d}\vec{a}'\vec{e}') = \text{tp}(\vec{d}\vec{a}'\vec{e}')$, contradicting our assumption. \qed

We conclude this section with two propositions which are some “more concrete” consequences of Theorem 9.3 and Corollary 9.4.

**Proposition 9.5.** Let $b_1, b_2 \in M$ and $D \subset C_h$ be crd-closed such that there is $a \in M$ with $D \subset \text{crd}(a)$. Then

$$
\text{tp}(b_1/D) = \text{tp}(b_2/D) \iff \text{tp}^2(b_1/D) = \text{tp}^2(b_2/D).
$$

**Proof.** Assume that $\text{tp}^2(b_1/D) = \text{tp}^2(b_2/D)$ but $\text{tp}(b_1/D) \neq \text{tp}(b_2/D)$. Let $E$ be a minimal crd-closed subset of $D$ such that $\text{tp}(b_1/E) \neq \text{tp}(b_2/E)$. So for any crd-closed $E' \subset E$ we have $\text{tp}(b_1/E') = \text{tp}(b_2/E')$.

We would like to show that $E = \emptyset$ or that there is $e \in E$ such that $E \subset \text{crd}(e)$. In the first case we get $\text{tp}(b_1) \neq \text{tp}(b_2)$ which contradicts $\text{tp}^2(b_1/D) = \text{tp}^2(b_2/D)$, in the second case we get $\text{tp}(b_1/e) \neq \text{tp}(b_2/e)$ (since then $E \subset \text{dcl}(e)$) which contradicts $\text{tp}^2(b_1/D) = \text{tp}^2(b_2/D)$ as well.

So assume that $E \neq \emptyset$ and there is no $e \in E$ with $E \subset \text{crd}(e)$. Let $e_1, \ldots, e_m \in E$ be the $\prec_C$-maximal elements of $E$. Then $E \subset \text{crd}(e_1, \ldots, e_m)$ and for any $i \neq j$ we have $e_i \notin \text{crd}(e_j)$ (using Proposition 7.14).

By our assumption, $m \geq 2$. Let $f$ enumerate $E \setminus \{e_1, \ldots, e_m\}$. Note that for every $i \leq m$ we have $\text{crd}(e_i) \setminus \{e_i\} \subset f$. Furthermore $\text{SU}(e_i/f) = 1$ by Lemma 7.14.
7.15 (every \( e_i \) is maximal in the set \( e_i \)). Moreover for any \( i \) the set \( \bar{f} \cup \text{crd}(e_i) \) is a proper and \( \text{crd} \)-closed subset of \( E \), so \( \text{tp}(e_i b_1 \bar{f}) = \text{tp}(e_i b_2 \bar{f}) \) and in particular \( \text{tp}(b_1 \bar{f}) = \text{tp}(b_2 \bar{f}) \). Using Corollary 9.4 with parameters \( \bar{a} := (b_1), \bar{a}' := (b_2), \bar{c} := \bar{d} := (e_1, \ldots, e_m) \) and \( \bar{c}' = \bar{e} := \bar{f} \) we get

\[
\text{tp}(e_1 \ldots e_m b_1 \bar{f}) = \text{tp}(e_1 \ldots e_m b_2 \bar{f}).
\]

This is a contradiction to the choice of \( E \), so there is \( e \in E \) with \( E \subset \text{crd}(e) \). This finishes the proof.

**Proposition 9.6.** Suppose \( b_1, b_2 \in M \), and that \( \bar{f} \in \text{crd}(b_1), \bar{g} \in \text{crd}(b_2) \) are \( \text{crd} \)-closed with \( \text{tp}(b_1 \bar{f}) = \text{tp}(b_2 \bar{g}) \). Let \( \bar{a} \in M \). Then

\[
\text{tp}(\bar{f}/\bar{a}) = \text{tp}(\bar{g}/\bar{a}) \iff \text{tp}^2(\bar{f}/\bar{a}) = \text{tp}^2(\bar{g}/\bar{a}).
\]

**Proof.** We prove that by induction on the height of \( \bar{f} \) and \( \bar{g} \), that is to say on \( l \) such that \( \bar{f}, \bar{g} \in C_l \). For \( l = 0 \) there is nothing to prove. Assume (as inductive assumption) that \( l \geq 1 \) and that the claim holds for all \( \bar{f}', \bar{g}' \in C_{l-1} \). Suppose now that \( \bar{f}, \bar{g} \in C_l \) and that \( \text{tp}^2(\bar{f}/\bar{a}) = \text{tp}^2(\bar{g}/\bar{a}) \).

Without loss of generality we can assume that \( \bar{f} = \bar{f}_1 \bar{f}_2, \bar{g} = \bar{g}_1 \bar{g}_2 \) where \( \bar{f}_1, \bar{g}_1 \in L_l \) and \( \bar{f}_2, \bar{g}_2 \in C_{l-1} \). By our inductive assumption, \( \text{tp}(\bar{f}_2/\bar{a}) = \text{tp}(\bar{g}_2/\bar{a}) \).

Furthermore \( \text{SU}(\bar{f}_1/\bar{f}_2) = 1 \) for all \( \bar{f}_1 \in \text{rng}(\bar{f}_1) \) (using Lemma 7.15). Now all the requirements of Corollary 9.4 are satisfied, with parameters \( \bar{c} := \bar{f}_2, \bar{c}' := \bar{g}_2, \bar{d} := \bar{f}_1, \bar{d}' := \bar{g}_1, \bar{a} = \bar{a}' := \bar{a} \). It follows that \( \text{tp}(\bar{f}_1 \bar{a} \bar{f}_2) = \text{tp}(\bar{g}_1 \bar{a} \bar{g}_2) \), so \( \text{tp}(\bar{f}/\bar{a}) = \text{tp}(\bar{g}/\bar{a}) \). \( \square \)

## 10 From Constraints to Extension Problems

In this section we finally introduce constraints, and link them to extension problems. Recall that \( \mathcal{M} \) is a countable binary simple homogeneous structure with the algebraic closure property.

**Definition 10.1 (Constraint).** A constraint of \( \mathcal{M} \) is a finite \( L \)-structure \( \mathcal{A} \) such that \( \mathcal{A} \) cannot be embedded into \( \mathcal{M} \), but every proper substructure of \( \mathcal{A} \) can.

The following is our main theorem:

**Theorem 10.2.** \( \mathcal{M} \) has (up to isomorphism) only finitely many constraints.

If the theorem holds, there are only countable many \( \mathcal{M} \) satisfying our background assumptions:

**Proposition 10.3.** Assume that \( \mathcal{H} \) is a class of pairwise non-isomorphic countable homogeneous structures in a finite relational language, such that every structure in \( \mathcal{H} \) is finitely constrained. Then \( \mathcal{H} \) is countable.
Proof. Up to isomorphism, there are only countable many finite sets of finite structures in a fixed finite relational language. Let \( \mathfrak{M} \in \mathcal{H} \). It is easy to see that the set of constraints of \( \mathfrak{M} \) determines \( \mathfrak{M} \)'s age. But any two countable homogeneous structures with the same age are isomorphic by Fraïssé’s theorem (Theorem 3.4). So \( \mathcal{H} \) has to be countable.

To show Theorem 10.2 we will show the slightly stronger:

**Theorem 10.4.** If \( \mathfrak{A} \) is a constraint of \( \mathfrak{M} \), then \( \mathfrak{A} \) has at most cardinality 3.

**Example 10.5.** By Example 3.7 it follows that the random graph has in fact no constraints.

Let us first prove some useful lemmas about embeddings:

**Lemma 10.6** (Determining embeddings with types). Let \( \mathfrak{A} \) be a finite \( \mathcal{L} \)-structure, and \( \bar{a} \) be some enumeration of its universe \( A \). Then \( \mathfrak{A} \) can be embedded into \( \mathfrak{M} \) if and only if \( \text{tp}_{\mathfrak{A}}(\bar{a}) \) (a set of \( \mathcal{L} \)-formulas) is realised in \( \mathfrak{M} \). Furthermore if \( f : \mathfrak{A} \to \mathfrak{M} \) is an embedding, then \( \text{tp}_{\mathfrak{A}}(\bar{a}) = \text{tp}_{\mathfrak{M}}(f(\bar{a})) \).

*Proof.* Follows immediately since embeddings are exactly the maps that preserve quantifier-free formulas. \( \square \)

**Lemma 10.7.** Let \( \mathfrak{A} \) be a finite \( \mathcal{L} \)-structure, and \( \bar{a} \) be some enumeration of its universe \( A \). Assume that \( f \) and \( g \) are two embeddings of \( \mathfrak{A} \) into \( \mathfrak{M} \). Then \( \text{tp}(f(\bar{a})) = \text{tp}(g(\bar{a})) \).

*Proof.* By Lemma 10.6 we have \( \text{tp}_{\mathfrak{M}}(f(\bar{a})) = \text{tp}_{\mathfrak{M}}(g(\bar{a})) \). Since \( \mathfrak{M} \) has quantifier-elimination this implies \( \text{tp}_{\mathfrak{M}}(f(\bar{a})) = \text{tp}_{\mathfrak{M}}(g(\bar{a})) \). By the properties of \( \mathfrak{M}^{eq} \) it follows \( \text{tp}(f(\bar{a})) = \text{tp}(g(\bar{a})) \). \( \square \)

**Lemma 10.8** (Sharing of embeddings). Assume that \( \mathfrak{A} \) is a finite structure that can be embedded into \( \mathfrak{M} \), with universe \( A = \bar{a}b \) and that \( \bar{a} \in M \cap A \) with \( M \upharpoonright \bar{a} = \mathfrak{A} \upharpoonright \bar{a} \). Then there is \( b' \subset M \) such that \( f : \mathfrak{A} \to \mathfrak{M}, \bar{a}b \mapsto \bar{a}b' \) is an embedding.

*Proof.* Assume that \( g : \mathfrak{A} \to \mathfrak{M}, \bar{a}b \mapsto \bar{a}_s \bar{b}_s \) is an embedding. Then the map \( \bar{a}_s \mapsto \bar{a} \) a partial automorphism of \( \mathfrak{M} \). Since \( \mathfrak{M} \) is homogeneous, it can be extended to an automorphism \( h \) of \( \mathfrak{M} \), with \( h(\bar{a}_s) = \bar{a} \). Let \( b' := h(\bar{b}_s) \). Then \( f := h \circ g \) is an embedding of \( \mathfrak{A} \) into \( \mathfrak{M} \), and \( f(\bar{a}b) = \bar{a}b' \). \( \square \)

**Notation 10.9** (Notation for potential constraints). Assume that \( \mathfrak{A} \) is a constraint of \( \mathfrak{M} \) of cardinality \( \geq 4 \). In the following we will fix some notation for \( \mathfrak{A} \) (ultimately, we want to show that \( \mathfrak{A} \) cannot exist, showing Theorem 10.4). Since \( \mathfrak{A} \) is a constraint, all proper substructures of \( \mathfrak{A} \) are embeddable into \( \mathfrak{M} \).

Let \( b \in A \) be arbitrary, and let \( a_1 \in A \setminus \{b\} \) be arbitrary too. Let \( \bar{a}_2 \) be a tuple enumerating \( A \setminus \{a_1, b\} \). So we have \( A = ba_1\bar{a}_2 \).

Since \( \mathfrak{A} \upharpoonright a_1\bar{a}_2 \) can be embedded into \( \mathfrak{M} \), we can assume \( a_1\bar{a}_2 \subset M \) with \( M \upharpoonright a_1\bar{a}_2 = \mathfrak{A} \upharpoonright a_1\bar{a}_2 \). The structures \( \mathfrak{A} \upharpoonright ba_1 \) and \( \mathfrak{A} \upharpoonright b\bar{a}_2 \) can also be embedded.
into \( M \). Using Lemma \([10.8]\) there are \( b_1, b_2 \in M \) such that \( b a_1 \mapsto b_1 a_1 \) is an embedding of \( A \upharpoonright b a_1 \) into \( \mathfrak{M} \) and \( b \bar{a}_2 \mapsto b_2 \bar{a}_2 \) is an embedding of \( A \upharpoonright b \bar{a}_2 \) into \( \mathfrak{M} \). Note that we have \( \text{tp}(b_1) = \text{tp}(b_2) \) (follows from Lemma \([10.7]\)).

The following theorem will give us a way to transform the question whether a certain structure is embeddable into \( \mathfrak{M} \) in a so-called extension problem.

**Definition 10.10 (Extension problem).** For any two types \( p \) and \( q \), we say that the extension problem of \( p \) and \( q \), short \( \mathcal{E}(p, q) \), has a solution if and only if there is \( c \in M^{eq} \) such that \( \text{tp}(c/\text{dom}(p)) = p \) and \( \text{tp}(c/\text{dom}(q)) = q \).

Let \( \bar{c}_1, \bar{c}_2 \in M \) and \( D_1, D_2 \subset M \). Then is is easy to see that the extension problem \( \mathcal{E}(\text{tp}(\bar{c}_1/D_1),\text{tp}(\bar{c}_1/D_1)) \) has a solution (in \( \mathfrak{M} \)) if and only if \( \mathcal{E}(\text{tp}(\bar{c}_1/D_1),\text{tp}(\bar{c}_1/D_1)) \) has a solution (in \( M^{eq} \)).

**Theorem 10.11.** Assume that \( A \) is a finite structure with universe \( A = ba_1 \bar{a}_2 \), where \( a_1, a_2 \in M \) and \( A \upharpoonright a_1 a_2 = \mathfrak{M} \upharpoonright a_1 a_2 \). Assume that \( b_1 \) and \( b_2 \) are in \( M \) such that \( ba_1 \mapsto b_1 a_1 \) is an embedding of \( A \upharpoonright ba_1 \) into \( \mathfrak{M} \) and \( b\bar{a}_2 \mapsto b_2 \bar{a}_2 \) is an embedding of \( A \upharpoonright b \bar{a}_2 \) into \( \mathfrak{M} \). Then the following are equivalent:

(i) \( A \) can be embedded into \( \mathfrak{M} \).

(ii) There is \( b' \in M \) such that \( \text{tp}(b'/a_1) = \text{tp}(b_1/a_1) \) and \( \text{tp}(b'/\bar{a}_2) = \text{tp}(b_2/\bar{a}_2) \).

In other words, \( \mathcal{E}(\text{tp}(b_1/a_1),\text{tp}(b_2/\bar{a}_2)) \) has a solution.

**Proof.** Assume that \( A \) can be embedded into \( \mathfrak{M} \). Using Lemma \([10.8]\) there is \( b' \in M \) such that \( ba_1 \bar{a}_2 \mapsto b'a_1 \bar{a}_2 \) is an embedding of \( A \) into \( \mathfrak{M} \). Using Lemma \([10.7]\) we have \( \text{tp}(b'a_1) = \text{tp}(b_1a_1) \) and \( \text{tp}(b'\bar{a}_2) = \text{tp}(b_2\bar{a}_2) \). It follows that \( b' \) is a solution to the extension problem.

Conversely, if \( b' \) is a solution to the extension problem we have

\[
\text{tp}_{\mathfrak{M}}(b', a_1) = \text{tp}_{\mathfrak{M}}(b_1, a_1) = \text{tp}_{A}(b, a_1),
\]

\[
\text{tp}_{\mathfrak{M}}(b', \bar{a}_2) = \text{tp}_{\mathfrak{M}}(b_2, \bar{a}_2) = \text{tp}_{A}(b, \bar{a}_2).
\]

Since our vocabulary is binary it follows that \( \text{tp}_{\mathfrak{M}}(b'a_1 \bar{a}_2) = \text{tp}_{A}(ba_1 \bar{a}_2) \), so by Lemma \([10.6]\) the map \( ba_1 \bar{a}_2 \mapsto b'a_1 \bar{a}_2 \) is an embedding of \( A \) into \( \mathfrak{M} \). \( \square \)

Now our proof of Theorem \([10.4]\) will proceed as follows:

1. Assume that \( A \) is a constraint of cardinality \( \geq 4 \) like in Notation \([10.9]\). Then by Theorem \([10.11]\) \( \mathcal{E}(\text{tp}(b_1/a_1),\text{tp}(b_2/\bar{a}_2)) \) has no solution.

2. Show that this implies that there is \( a_2' \) in the tuple \( \bar{a}_2 \) such that already \( \mathcal{E}(\text{tp}(b_1/a_1),\text{tp}(b_2/a_2')) \) has no solution.

3. Using the other direction of Theorem \([10.11]\) this implies that the structure \( A \upharpoonright ba_1 a_2' \) cannot be embedded into \( \mathfrak{M} \). Since \( |A| \geq 4 \), this is a proper substructure of \( A \). This contradicts the assumption that \( A \) is a constraint.

Together (since \( A \) was chosen arbitrarily) this will show Theorem \([10.4]\). The open part in this proof outline is step 2. To show this critical step it will be necessary to determine exactly under which circumstances an extension problem has a solution. This will be the content of the two sections, which contain the main part of this work.
11 Conditions for Extension Problems

From now on, fix $\mathfrak{A}$, $a_1, a_2, b, b_1, b_2$ like in Notation 10.9. In the previous subsection, we reduced our problem (showing Theorem 10.2) to showing the following theorem:

Theorem 11.1. If $\mathcal{E}P(\text{tp}(b_1/a_1), \text{tp}(b_2/\bar{a}_2))$ has no solution, there is $a_2^j$ in the tuple $\bar{a}_2$ such that already $\mathcal{E}P(\text{tp}(b_1/a_1), \text{tp}(b_2/a_2^j))$ has no solution.

We will find an equivalent description of of the question “when has a certain extension problem a solution” that will allow us to prove this implication. Consider the following conditions on $a_1, b_1, b_2, \bar{a}_2$:

<table>
<thead>
<tr>
<th>Conditions A1 and A2</th>
</tr>
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<tbody>
<tr>
<td><strong>A1</strong> $\text{tp}(b_1/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = \text{tp}(b_2/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2))$</td>
</tr>
<tr>
<td><strong>A2</strong> Let $l \leq h$ and $b_1^l, b_2^l \in M$, $\bar{f} \in M^m$ such that $\text{tp}(b_1^l/a_1) = \text{tp}(b_1/a_1)$, $\text{tp}(b_2^l/\bar{a}<em>2) = \text{tp}(b_2/\bar{a}<em>2)$, $\text{crd}</em>{-1}(b_1^l) = \text{crd}</em>{-1}(b_2^l) = \bar{f}$ and $\text{tp}(b_1^l \bar{f}) = \text{tp}(\bar{b}_2^l \bar{f})$. Suppose $c \in \text{crd}(b_1^l) \cap L_l$, and $d \in \text{crd}(b_2^l) \cap L_l$ are cognates. Then:</td>
</tr>
<tr>
<td>- If $c \in \text{crd}(a_1)$ then $\text{tp}(c/\bar{f} a_2) = \text{tp}(d/\bar{f} \bar{a}_2)$.</td>
</tr>
<tr>
<td>- If $d \in \text{crd}(\bar{a}_2)$ then $\text{tp}(d/\bar{f} a_1) = \text{tp}(c/\bar{f} a_1)$.</td>
</tr>
</tbody>
</table>

The following theorem is central:

**Theorem 11.2.** $\mathcal{E}P(\text{tp}(b_1/a_1), \text{tp}(b_2/\bar{a}_2))$ has a solution if and only if A1 and A2 are satisfied.

This theorem will allow us to prove Theorem 11.1.

**Proof of Theorem 11.1 from Theorem 11.2.** Assume that the extension problem $\mathcal{E}P(\text{tp}(b_1/a_1), \text{tp}(b_2/\bar{a}_2))$ has no solution. Then either A1 or A2 (for $b_1, b_2, a_1, \bar{a}_2$) are not satisfied. Assume A1 is not satisfied, so we have

$$\text{tp}(b_1/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) \neq \text{tp}(b_2/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)).$$

Note that $\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)$ is crd-closed, so using Proposition 9.5 it follows

$$\text{tp}^2(b_1/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) \neq \text{tp}^2(b_2/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)).$$

Using now Theorem 7.6(i) there is $a_2^j \in \text{rng}(\bar{a}_2)$ and $c \in \text{crd}(a_1) \cap \text{crd}(a_2^j)$ such that $\text{tp}(b_1/c) \neq \text{tp}(b_2/e)$. So $\text{tp}(b_1/\text{crd}(a_1) \cap \text{crd}(a_2^j)) \neq \text{tp}(b_2/\text{crd}(a_1) \cap \text{crd}(a_2^j))$, so A1 is not satisfied for $b_1, b_2, a_1, a_2^j$. So with Theorem 11.2 the extension problem $\mathcal{E}P(\text{tp}(b_1/a_1), \text{tp}(b_2/a_2^j))$ has no solution.

Now assume A2 is not satisfied. Let $1 \leq l \leq h$ be such that A2 fails for $l$, and choose witnesses $b_1^l, b_2^l \in M$ and $\bar{f} \in M^m$. Then $\text{tp}(b_1^l/a_1) = \text{tp}(b_1/a_1)$, $\text{tp}(b_2^l/\bar{a}_2) = \text{tp}(b_2/\bar{a}_2)$, $\text{crd}_{-1}(b_1^l) = \text{crd}_{-1}(b_2^l) = \bar{f}$ and $\text{tp}(b_1^l \bar{f}) = \text{tp}(\bar{b}_2^l \bar{f})$ and one of the two cases holds true:
• **Case 1:** There is $c \in \text{crd}(b'_1)$ on level $l$ with $c \in \text{crd}(b'_1) \cap \text{crd}(a_1)$ but $\text{tp}(c/f\bar{a}_2) \neq \text{tp}(d/f\bar{a}_2)$ (where $d$ is $c$’s cognate).

• **Case 2:** There is $d \in \text{crd}(b'_2)$ on level $l$ with $d \in \text{crd}(b'_2) \cap \text{crd}(\bar{a}_2)$ but $\text{tp}(d/f\bar{a}_1) \neq \text{tp}(c/f\bar{a}_1)$ (where $c$ is $d$’s cognate).

In the second case, $d \in \text{crd}(b'_2) \cap \text{crd}(a'_2)$ for some $a'_2 \in \text{rng}(\bar{a}_2)$, and it is easy to see that then A2 does not hold for $b_1, b_2, a_1, a'_2$. So assume that the first case holds true. Using Proposition 9.6 (with the crd-closed tuples $c, f$ and $d, \bar{f}$ as parameters) $\text{tp}(c/f\bar{a}_2) \neq \text{tp}(d/f\bar{a}_2)$ implies $\text{tp}^2(c/f\bar{a}_2) \neq \text{tp}^2(d/f\bar{a}_2)$. Hence there must be $j$ with $\text{tp}(c/f\bar{a}_2^j) \neq \text{tp}(d/f\bar{a}_2^j)$. Then A2 is violated for $b_1, b_2, a_1, a'_2$, so with Theorem 11.2 $\mathcal{E}(\text{tp}(b_1/a_1), \text{tp}(b_2/a'_2))$ has no solution. □

Let us start with the easier direction of Theorem 11.2

**Proposition 11.3.** If $\mathcal{E}(\text{tp}(b_1/a_1), \text{tp}(b_2/a'_2))$ has a solution, then A1 and A2 are satisfied.

**Proof.** Assume that $b$ is a solution to $\mathcal{E}(\text{tp}(b_1/a_1), \text{tp}(b_2/a'_2))$. Then $\text{tp}(b/a_1) = \text{tp}(b_1/a_1)$ and $\text{tp}(b/a'_2) = \text{tp}(b_2/a'_2)$. By Lemma 4.3 (and the fact that $\mathcal{C}$ satisfies the definability property) it follows $\text{tp}(b/\text{crd}(a_1)) = \text{tp}(b_1/\text{crd}(a_1))$ and $\text{tp}(b/\text{crd}(a_2)) = \text{tp}(b_2/\text{crd}(a_2))$, which together imply A1.

Now we aim to show A2: Let $l \leq h$ and $b'_1, b'_2, f$ be given with $\text{tp}(b'_1/a_1) = \text{tp}(b'_1/a_1)$ and $\text{tp}(b'_2/a'_2) = \text{tp}(b'_2/a'_2)$, $\text{crd}_{l-1}(b'_1) = \text{crd}_{l-1}(b'_2) = f$ and $\text{tp}(b'_1 f) = \text{tp}(b'_2 f)$. Let $\bar{r} := \text{crd}_{l-1}(b), \bar{g} := \text{crd}_{l-1}(b_1), \bar{h} := \text{crd}_{l-1}(b_2)$, all arranged in the same way as $f$.

**Claim:** $\text{tp}(\bar{r}/a_1 \bar{a}_2) = \text{tp}(\bar{f}/a_1 \bar{a}_2)$.

**Proof of Claim.** We have

$$\text{tp}(\bar{r}/a_1) = \text{tp}(\bar{g}/a_1) = \text{tp}(\bar{f}/a_1),$$

$$\text{tp}(\bar{r}/a_2) = \text{tp}(\bar{h}/a_2) = \text{tp}(\bar{f}/a_2),$$

(using the definability of coordinates and Lemma 4.3 several times). So in particular $\text{tp}^2(\bar{r}/a_1 \bar{a}_2) = \text{tp}^2(\bar{f}/a_1 \bar{a}_2)$. Also we have $\text{tp}(br) = \text{tp}(b_1 \bar{g}) = \text{tp}(b'_1 \bar{f})$.

Now using Proposition 9.6 with parameters $b, b'_1, \bar{r}$ and $f$, the claim follows. qed(Claim)

Let now $c$ be on level $l$ with $c \in \text{crd}(b'_1) \cap \text{crd}(a_1)$. Then $c$ is definable by $a_1$. Let $d$ be $c$’s cognate with respect to $b'_2$ and $e$ be $c$’s cognate with respect to $b$. From $\text{tp}(b'_1/a_1) = \text{tp}(b_1/a_1) = \text{tp}(b/a_1)$ it follows that $c = e$. Using our claim and that $c \in \text{dcl}(a_1)$ we can now conclude that

$$\text{tp}(\bar{f}c \bar{a}_2) = \text{tp}(\bar{r}c \bar{a}_2) = \text{tp}(\bar{e}c \bar{a}_2).$$

From $\text{tp}(b'_2/a'_2) = \text{tp}(b_2/a'_2)$ it follows $\text{tp}(d\bar{f} \bar{a}_2) = \text{tp}(e\bar{r} \bar{a}_2)$, so together with the above we get $\text{tp}(e\bar{f} \bar{a}_2) = \text{tp}(d\bar{f} \bar{a}_2)$. In the case where $d$ is on level $l$ with $d \in \text{crd}(b'_2) \cap \text{crd}(\bar{a}_2)$ and $c$ is $d$’s cognate, we get $\text{tp}(d\bar{f} \bar{a}_1) = \text{tp}(e\bar{f} \bar{a}_1)$ by an analogous argument. This shows A2. □
It only remains to show the other direction of Theorem 11.2, which will be done in the next section. Let us conclude this section by a diagram which summarises the structure of our argument.

12 Solving the Extension Problem

In this section we will close the last remaining gap in our argument chain and show the following theorem:

**Theorem 12.1.** If A1 and A2 are satisfied then $\mathcal{E}(\text{tp}(b_1/a_1), \text{tp}(b_2/a_2))$ has a solution.

**Proof.** Let $\bar{u} := \text{crd}(b_1)$ and $\bar{v} := \text{crd}(b_2)$ (where $\bar{u}$ and $\bar{v}$ are arranged by cognates). Using the Theorems 8.7 and 4.3, it is enough for us to show that $\mathcal{E}(\text{tp}(\bar{u}/a_1), \text{tp}(\bar{v}/a_2))$ has a solution. We will construct a solution to this extension problem level-wise, using an induction argument.

Let $1 \leq l \leq h$ and let $\bar{s} := \text{crd}_{l-1}(b_1)$ and $\bar{t} := \text{crd}_{l-1}(b_2)$ (where $\bar{s}$ and $\bar{t}$ are arranged by cognates). The following is our inductive assumption, which we will assume throughout the proof:

**(IA)** $l = 1$ or $\mathcal{E}(\text{tp}(\bar{s}/a_1), \text{tp}(\bar{t}/a_2))$ has a solution.

If $l = 1$, let $\bar{f} := \emptyset$, otherwise let $\bar{f}$ be the solution to $\mathcal{E}(\text{tp}(\bar{s}/a_1), \text{tp}(\bar{t}/a_2))$. Let $\bar{c} := \text{crd}(b_1) \cap L_1$ and $\bar{d} := \text{crd}(b_2) \cap L_1$ (where $\bar{c}$ and $\bar{d}$ are arranged by cognates). The induction step is to show the following:

**(IS)** $\mathcal{E}(\text{tp}(\bar{c}\bar{s}/a_1), \text{tp}(\bar{d}\bar{t}/a_2))$ has a solution.

The following claim shows that we can assume $b_1$ and $b_2$ to have the same coordinates up to level $l-1$.

**Claim 1:** There are $b'_1, b'_2 \in M$ and $c', d' \in L_l$ with

$$
\text{tp}(b'_1 \bar{c}' \bar{f} a_1) = \text{tp}(b_1 \bar{c} \bar{s} a_1),
$$

$$
\text{tp}(b'_2 d' \bar{f} a_2) = \text{tp}(b_2 d \bar{t} a_2),
$$

$$
\text{tp}(b'_1 \bar{c}' \bar{f}) = \text{tp}(b'_2 d' \bar{f}),
$$

$$
\text{crd}_{l-1}(b'_1) = \text{crd}_{l-1}(b'_2) = \bar{f}.
$$

**Proof of Claim.** If $l = 1$, then $\bar{f} = \emptyset$ and we can simply set $\bar{c}' := \bar{c}, \bar{d}' := \bar{d}, b'_1 := b_1$ and $b'_2 := b_2$. Assume $l > 1$. Since $\bar{f}$ is a solution to $\mathcal{E}(\text{tp}(\bar{s}/a_1), \text{tp}(\bar{t}/a_2))$, we have $\text{tp}(\bar{f} a_1) = \text{tp}(\bar{s} a_1)$ and $\text{tp}(\bar{f} a_2) = \text{tp}(\bar{t} a_2)$. Using Lemma 6.4(iii) there are $b'_1, b'_2, c', d'$ satisfying (3) and (4).
From \( \text{tp}(b_1) = \text{tp}(b_2) \) (and the right arrangement of the coordinates) it follows that \( \text{tp}(b_1 \bar{a}s) = \text{tp}(b_2 \bar{d}f) \). So together with (3) and (4) also (5) follows. Now the statement \( \text{crd}_{d-1}(b'_1) = \bar{f} \) follows since \( \text{tp}(b'_1 \bar{f}) = \text{tp}(b_1 \bar{s}) \) and the relation "\( x \in \text{crd}_{d-1}(\bar{y}) \)" is definable, and similar for \( b'_2 \).

Let \( b'_1, b'_2, c', d' \) be like in the claim. Using (3) and (4), to show (IS), it is enough to show that \( \text{EP}(\text{tp}(c' \bar{f}/a_1), \text{tp}(d'/\bar{f} a_2)) \) has a solution, or equivalently, that the problem \( \text{EP}(\text{tp}(c' /fa_1), \text{tp}(d'/fa_2)) \) has. In the following we will construct a solution \( \bar{g} \) of the latter.

Let \( n := \text{length}(\bar{c}') \) and let \( \mathcal{I} := \{1, \ldots, n\} \). We will separate the indices in \( \mathcal{I} \) in different classes:

\[
\mathcal{I}_0 := \{i \in \mathcal{I} \mid c'_i \notin \text{crd}(a_1) \land d'_i \notin \text{crd}(a_2)\}, \\
\mathcal{I}_1 := \{i \in \mathcal{I} \mid c'_i \in \text{crd}(a_1)\}, \\
\mathcal{I}_2 := \{i \in \mathcal{I} \mid d'_i \in \text{crd}(a_2)\}.
\]

For each class \( \mathcal{I}_k \) the coordinates \( g_i \) with \( i \in \mathcal{I}_k \) will have to be chosen in a different way. Note that \( \mathcal{I}_0 \cap \mathcal{I}_1 = \emptyset \) and \( \mathcal{I}_0 \cap \mathcal{I}_2 = \emptyset \). For \( i \in \mathcal{I}_1 \) and \( i \in \mathcal{I}_2 \), there is no "degree of freedom" in how to choose \( g_i \), as the following argument shows:

**Claim 2:** Assume \( \bar{h} \) is a solution to \( \text{EP}(\text{tp}(c' /fa_1), \text{tp}(d'/\bar{f} a_2)) \). Then:

(i) If \( i \in \mathcal{I}_1 \), \( h_i = c'_i \).

(ii) If \( i \in \mathcal{I}_2 \), \( h_i = d'_i \).

**Proof of Claim.** For \( i \in \mathcal{I}_1 \), \( \text{tp}(c' /fa_1) \) contains the formula \( x_i = c'_i \) (since \( c'_i \) is definable by \( a_1 \)). Since \( \bar{h} \) is a solution to the independence problem, \( \text{tp}(\bar{h}/fa_1) = \text{tp}(c' /fa_1) \), so it follows \( h_i = c'_i \). Analogously, for \( i \in \mathcal{I}_2 \), \( \text{tp}(d'/\bar{f} a_2) \) contains the formula \( x_i = d'_i \), and we need to have \( h_i = d'_i \).

\( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) can have a non-empty intersection. However, A2 ensures that (i) and (ii) in the above claim can be simultaneously satisfied in the critical cases:

**Claim 3:** For \( i \in \mathcal{I}_1 \cap \mathcal{I}_2 \) we have \( c'_i = d'_i \).

**Proof of Claim.** Since \( i \in \mathcal{I}_1 \), we have \( c'_i \in \text{crd}(a_1) \). By A2 it follows that

\[
\text{tp}(c'_i /\bar{f} a_2) = \text{tp}(d'_i /\bar{f} a_2).
\]

But from \( i \in \mathcal{I}_2 \) it follows \( d'_i \in \text{crd}(a_2) \), so it is definable by \( a_2 \), and from the above equation it follows \( \text{tp}(c'_i, d'_i) = \text{tp}(d'_i, d'_i) \). This implies \( c'_i = d'_i \).

So let us now define for \( i \in \mathcal{I}_1 \) that \( g_i := c'_i \), and for \( i \in \mathcal{I}_2 \) that \( g_i := d'_i \). To find \( g_i \) for \( i \in \mathcal{I}_0 \), we will use the independence theorem. Let

\[
D := \bar{f} \cup (\text{crd}(a_1) \cap \text{crd}(a_2))
\]

**Claim 4:** We have

(i) \( \bar{f} a_1 \perp_D \bar{f} a_2 \),
(ii) \( (c'_i)_{i \in \mathcal{I}_0} \perp_D \bar{f} a_1 \),
(iii) \((d'_i)_{i \in J_0} \downarrow D \bar{f} \bar{a}_2\),
(iv) \(\text{tp}((c'_i)_{i \in J_0}/D) = \text{tp}((d'_i)_{i \in J_0}/D)\).

Proof of Claim. (i): By Proposition 7.6 (i) we have
\[
\text{crd}(\bar{f}a_1) \cap \text{crd}(\bar{f}\bar{a}_2) = \text{crd}(\bar{f}) \cup (\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = D.
\]
The statement follows by (iv) of the same proposition.
(ii): Note that \(\text{crd}((c'_i)_{i \in J_0}) \subset (c'_i)_{i \in J_0} f\). Since \((c'_i)_{i \in J_0} \cap \text{crd}(a_1) = \emptyset\) we get
\[
(\text{crd}((c'_i)_{i \in J_0}) \cap \text{crd}(\bar{f}a_1)) \subset \bar{f} \subset D.
\]
The statement follows by Proposition 7.6 (iv).
(iii): Analogous to (ii).
(iv): By A1 \(\text{tp}(b_1/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = \text{tp}(b_2/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2))\). This implies
\[
\text{tp}(\bar{c}s/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = \text{tp}(\bar{d}t/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)).
\]
Using the equations (3) and (4) we have
\[
\text{tp}(\bar{c}s/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = \text{tp}(\bar{d}t/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = \text{tp}(\bar{c}s/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)),
\]
so \(\text{tp}(\bar{c}s/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2)) = \text{tp}(\bar{d}t/\text{crd}(a_1) \cap \text{crd}(\bar{a}_2))\), which implies the statement.

By Claim 1 we can use the Independence Theorem 8.6 to find \((g_i)_{i \in J_0}\) with
\[
\begin{align*}
\text{tp}((g_i)_{i \in J_0}/\bar{f}a_1) &= \text{tp}((c'_i)_{i \in J_0}/\bar{f}a_1), \\
\text{tp}((g_i)_{i \in J_0}/\bar{f}\bar{a}_2) &= \text{tp}((d'_i)_{i \in J_0}/\bar{f}\bar{a}_2).
\end{align*}
\]
In the following we will show that \((g_i)_{i \in J_0}\) “has the right types” in relation to the other \(g_i\) which were chosen without the Independence Theorem.

Claim 5: For any \(i \in J\) we have that \(\text{tp}(g_i, \bar{f}a_1) = \text{tp}(c'_i, \bar{f}a_1)\) and \(\text{tp}(g_i, \bar{f}\bar{a}_2) = \text{tp}(d'_i, \bar{f}\bar{a}_2)\).

Proof of Claim. We only prove \(\text{tp}(g_i, \bar{f}a_1) = \text{tp}(c'_i, \bar{f}a_1)\) since the other statement is proven in an analogous way. If \(i \in J_0\), then \(\text{tp}(g_i, \bar{f}a_1) = \text{tp}(c'_i, \bar{f}a_1)\) follows by (7). If \(i \in J_1\), then \(g_i = c'_i\) and the statement follows trivially. If \(i \in J_2\), then \(g_i = d'_i\). Furthermore in that case \(d'_i \in \text{crd}(\bar{a}_2)\), so by A2 \(\text{tp}(d'_i, \bar{f}a_1) = \text{tp}(c'_i, \bar{f}a_1)\) and the statement follows again.

Claim 6: For all \(i, j \in J\) we have that \(\text{tp}(g_i, g_j, \bar{f}) = \text{tp}(c'_i, c'_j, \bar{f}) = \text{tp}(d'_i, d'_j, \bar{f})\).

Proof of Claim. Note first that by equation (5) we have \(\text{tp}(c'_i, c'_j, \bar{f}) = \text{tp}(d'_i, d'_j, \bar{f})\), for any choice of \(i\) and \(j\). So it is enough to show that either \(\text{tp}(g_i, g_j, \bar{f}) = \text{tp}(c'_i, c'_j, \bar{f})\) or that \(\text{tp}(g_i, g_j, \bar{f}) = \text{tp}(d'_i, d'_j, \bar{f})\).

We proof the statement for all \(i\) and \(j\) by making a case distinction. We omit the cases that can be solved by swapping \(i\) and \(j\).

• Case 1: \(i, j \in J_0\). In this case the the statement follows by equation (7).
• Case 2: $i \in \mathcal{I}_0$, $j \in \mathcal{I}_1$. In this case $g_j = c'_j$ is definable by $a_1$, so the statement follows also by (7).

• Case 3: $i \in \mathcal{I}_0$, $j \in \mathcal{I}_2$. In this case $g_j = d'_j$ is definable by $\bar{a}_2$, so the statement follows by (8).

• Case 4: $i, j \in \mathcal{I}_1$. In this case $g_i = c'_i$ and $g_j = c'_j$, so the statement is trivial.

• Case 5: $i, j \in \mathcal{I}_2$. In this case $g_i = d'_i$ and $g_j = d'_j$, so the statement is trivial.

• Case 6: $i \in \mathcal{I}_1$, $j \in \mathcal{I}_2$: In this case $g_i = c'_i$ and $g_j = d'_j$. Furthermore $c'_i \in \text{crd}(a_1)$ and $d'_j \in \text{crd}(\bar{a}_2)$. Using A2

$$\text{tp}(g_i/f\bar{a}_2) = \text{tp}(c'_i/f\bar{a}_2) = \text{tp}(d'_i/f\bar{a}_2).$$

Since $g_j = d'_j$ is definable by $\bar{a}_2$ it follows that $\text{tp}(g_i g_j \bar{f}) = \text{tp}(d'_i d'_j \bar{f})$.

\[\text{qed}(6)\]

For every $i \in \mathcal{I}$, the type $\text{tp}(c'_i \bar{f})$ contains the formula:

$$\exists x \ (c'_i \in \text{crd}(x) \land \text{tp}(x \bar{f}) = \text{tp}(b'_i \bar{f}))$$

This used that $\text{tp}(b'_i \bar{f})$ is isolated and that being a coordinate is a definable relation. Thus Claim 5 implies in particular for every $i \in \mathcal{I}$ that $\text{tp}(g_i \bar{f}) = \text{tp}(c'_i \bar{f})$, so for every $i$ there is $b'_i$ such that

$$g_i \in \text{crd}(b'_i), \quad \text{(9)}$$

$$\text{tp}(b'_i / \bar{f}) = \text{tp}(b'_i / \bar{f}) = \text{tp}(b'_i / \bar{f}), \quad \text{(10)}$$

the second equation in (10) follows from equation (6). Furthermore, for every $i$ we have that $c'_i$ is $<_{c}$-maximal in $f c'_1$, so $\text{SU}(c'_i / \bar{f}) = 1$ (using Lemma 7.15).

Now we can use Theorem 9.3 with the parameters $\bar{c} := \bar{c}'$, $\bar{d} := \bar{g}$, $X := \bar{f}$ and $\bar{a} := a_1 - 1$ by the Claims 3 and 6 and equations (9) and (10), all its requirements are satisfied. We conclude

$$\text{tp}(\bar{g}/a_1 \bar{f}) = \text{tp}(\bar{c}/a_1 \bar{f}) = \text{tp}(\bar{e}/a_1 s). \ (11)$$

In the same vein, for every $i$ we have that $d'_i$ is $<_{c}$-maximal in $\bar{f} d'_1$, so $\text{SU}(d'_i / \bar{f}) = 1$. So using Theorem 9.3 again, this time with parameters $\bar{c} := \bar{d}'$, $\bar{d} := \bar{g}$, $X := \bar{f}$ and $\bar{a} := a_2$ we get

$$\text{tp}(\bar{g}/a_2 \bar{f}) = \text{tp}(\bar{d} / a_2 \bar{f}) = \text{tp}(\bar{d} / a_2 \bar{f}). \ (12)$$

Together (11) and (12) show that $\bar{g}$ is a solution to $\mathcal{E}\mathcal{P}(\text{tp}(\bar{c}/f a_1), \text{tp}(\bar{d}/f a_2))$ and thus also a solution to $\mathcal{E}\mathcal{P}(\text{tp}(\bar{c}/f a_1), \text{tp}(\bar{d}/f a_2))$. Thus we have derived (IS) from (IA). By induction, Theorem 12.1 follows.

\[\square\]
References


Cameron, Peter J. (1990), Oligomorphic permutation groups. Cambridge University Press.


